Title: Two problems in finite elasticity

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Year: 2009

Repository DOI:

2009

Two problems in finite elasticity

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Two Problems in Finite Elasticity

A thesis submitted in fulfilment of the

requirement of the award for the degree of

Masters by Research

From

University of Wollongong

By

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2009

Certification

I, Himanshuki Nilmini Padukka Withana, declare that this thesis, submitted in fulfilment of the requirements for the award of Masters by Research, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

H. Nilmini Paduuka Withana
August, 2009

Acknowledgements

I owe a great debt of thanks to my supervisor Professor Jim Hill not only for his consultation on the subject matter of this thesis but on his selfless help in solving the seemingly intractable problems that arose alongside my academic pursuits.

Not surprisingly teams with great leaders are always great and this is certainly the case with the team of Nano Technology at UOW. While too numerous to mention here by name the wonderful support of everyone is greatly appreciated. I would especially like to thank Dr. Barry Cox for always being there so cheerfully when ever help was needed. My thanks also go to Dr.Ngamta Thamwattana for kindly sharing her room with me. I would like to thank Dr.Tamsyn Hilder and Dr.Duangkamon Baowan for their support and friendship.

A special thank you to my boss and my fiend Mr. Stephen Haines for the valuable suggestions to the introduction.

Finally to my husband Sajeewa , the best thing that ever happened to me, thank you for always being there smoothening the rough road ahead of me!

Abstract

Some materials encountered in nature and used in engineering exhibit mechanical effects which cannot be adequately explained by classical linear elastic theories. For example, rubber is an elastic material that undergoes large elastic deformations, and therefore renders a non-linear mechanical behavior. An analytical investigation dealing with the problem of static deformation of such materials therefore involves highly non-linear equations leading to arduous mathematical work. Consequently there exists only a limited number of known exact solutions for such problems in the field of finite elasticity.

This thesis is concerned with two problems of finite elastic deformations of rubber blocks. Rubber has been successfully modeled as an isotropic incompressible hyperelastic material with strain energy function given by either the neo-Hookean or Mooney forms. For this class of materials, substantial reductions of the basic underlying equilibrium equations can be obtained, making the problems more tractable and for plane and axially symmetric deformations of these materials, simpler stress-strain relations can be obtained. Therefore, by combining these essentially two-dimensional stress-strain relations together with the reduced equilibrium equations it is possible to obtain comparatively tractible forms of the equations.

In this thesis the following problems for axially symmetric deformations of isotropic incompressible neo-Hookean and Mooney materials are investigated:

(i) asymptotic axially symmetric deformations describing compression of rubber cylindrical tubes with bonded metal end plates;

(ii) rippling of a long rectangular rubber block bent into a sector of a solid bounded by two circular arcs.

The above mentioned reduced equllibrium equations are employed in the context of non-linear continuum mechanics to arrive at approximate solutions. The solutions are approximate in the sense that the point-wise vanishing of the stress vector on a boundary is assumed to be replaced by the vanishing of forces in an average manner.

In the first problem, for axially symmetric deformations of the perfectly elastic neo-Hookean and Mooney materials, formal asymptotic solutions are determined in terms of expansions in appropriate powers of $1/R$, where R is the cylindrical

polar material coordinate. Remarkably, for both the neo-Hookean and Mooney materials, the first three terms of such expansions can be completely determined analytically in terms of elementary integrals. From the incompressibility condition and the equilibrium equations, the six unknown deformation functions, that appear in the first three terms can be reduced to five formal integrations involving in total, seven arbitrary constants, and a further five integration constants, making a total of twelve integration constants for the deformation field. The solutions so obtained for the neo-Hookean material are applied to the problem of the axial compression of a cylindrical rubber tube which has bonded metal end-plates. The resulting solution is approximate in two senses; namely as an approximate solution of the governing equations and for which the stress free boundary conditions are satisfied in an average manner only. The resulting deformation and load-deflection relation are shown graphically.

The second problem examined in this thesis is that of finite elastic deformation of a long rectangular rubber block which is deformed in a perturbed cylindrical configuration. This problem is motivated from the problem of determining surface rippling that is observed in bent multi-walled carbon nano-tubes. The problem of finite elastic bending of a tube is considerably more complicated than the geometrically simpler problem of the finite elastic bending of a rectangular block. Accordingly, we examine here the simpler block problem which is assumed to be sufficiently long so that the out of plane end effects may be ignored. The general equations governing plane strain deformations of an isotropic incompressible perfectly elastic Mooney material, which models rubber like materials, are used to determine small superimposed deformations upon the well known controllable family for the deformation of rectangular blocks into a sector of a solid bounded by two circular arcs. Traction free boundary conditions are assumed to be satisfied in an average sense along the bounding circular arcs. Physically realistic rippling is found to occur and typical numerical values are used to illustrate the solution graphically.

In summary reduced equilibrium equations and simplified two-dimensional stress strain relations are used in this study to solve two problems for isotropic incompressible neo-Hookean and Mooney materials. Such deformations and the class of materials studied considerably simplify what are otherwise very complex problems from the theory of finite elasticity.

Nomenclature

B_R	undeformed configuration
B	deformed configuration
$\mathbf{C}$	Green deformation tensor
$\mathbf{c}$	Cauchy deformation tensor
$\mathbf{C}^{-1}$	Piola deformation tensor
$\mathbf{c}^{-1}$	Finger deformation tensor
dA	element area in B_R
da	element area in B
dF	force acting on an element area in da
dS	line element in B_R
ds	line element in B
dV	element of volume in B_R
dv	element of volume in B
$\mathbf{F}$	deformation gradient
$\hat{\mathbf{g}}$	response function
$\mathbf{G}_K$	material base vectors for the curvilinear coordinate system
$\mathbf{g}_i$	spatial base vectors for the curvilinear coordinate system
G_{KL}	elements of material metric tensor
g_{ij}	elements of spatial metric tensor
G^{KL}	elements of conjugate material metric tensor
g^{ij}	elements of conjugate spatial metric tensor

G	determinate of material metric tensor $\mid \mathbf{G} \mid$
g	determinate of spatial metric tensor $\mid \mathbf{g} \mid$
$\mathbf{I}_K$	unit rectangular base vectors
I_1, I_2, I_3	principal invariants of the Finger deformation tensor
J	Jacobian of the rectangular Cartesian coordinate system $\mid \frac{\partial z^i}{\partial Z^K} \mid$
j	Jacobian of the curvilinear coordinate system $\mid \frac{\partial x^i}{\partial X^K} \mid$
K, L, M	labeling indices associated with B_R
i, j, k	labeling indices associated with B
$\mathbf{n}$	unit normal to da
$\mathbf{n}_R$	unit normal to dA
p	modified pressure function
p^*	pressure function
$\mathbf{Q}$	an orthogonal tensor
$\mathbf{T}$	stress tensor
$\mathbf{T}_R$	first Piola-Kirchoff stress tensor
$\mathbf{t}$	stress vector
$t^i_K(x, X)$	double tensor field
X^K	material curvilinear coordinates
x^i	spatial curvilinear coordinates
Z^K	material rectangular Cartesian coordinates
z^i	spatial rectangular Cartesian coordinates
$x^i{}_{,K}$	deformation gradients
$X^K{}_{,i}$	inverse deformation gradient

$\boldsymbol{\delta}$	unit tensor
δ_{ij}, δ_{KL}	Kronecker deltas
∇^2	Laplacian of a scalar with respect to X^K
Γ^M_{KL}	Christoffel symbols based on G_{KL}
Γ^i_{jk}	Christoffel symbols based on g_{ij}
μ	shear modulus
ϕ_i	response coefficients
ρ_R	density in undeformed body B_R
ρ	density in deformed body B
Σ	strain energy function

List of coordinate systems used

(X, Y, Z)	material rectangular Cartesian coordinates
(x, y, z)	spatial rectangular Cartesian coordinates
(R, Θ, Z)	material cylindrical polar coordinates
(r, θ, z)	spatial cylindrical polar coordinates

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