

University of Wollongong - Research Online

Thesis Collection

Title: Two problems in finite elasticity

Author: Himanshuki Nilmini Padukka Withana

Year: 2009

Repository DOI:

Copyright Warning

You may print or download ONE copy of this document for the purpose of your own research or study. The University does not authorise you to copy, communicate or otherwise make available electronically to any other person any copyright material contained on this site.

You are reminded of the following: This work is copyright. Apart from any use permitted under the Copyright Act 1968, no part of this work may be reproduced by any process, nor may any other exclusive right be exercised, without the permission of the author. Copyright owners are entitled to take legal action against persons who infringe their copyright. A reproduction of material that is protected by copyright may be a copyright infringement. A court may impose penalties and award damages in relation to offences and infringements relating to copyright material.

Higher penalties may apply, and higher damages may be awarded, for offences and infringements involving the conversion of material into digital or electronic form.

Unless otherwise indicated, the views expressed in this thesis are those of the author and do not necessarily represent the views of the University of Wollongong.

Research Online is the open access repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

University of Wollongong Thesis Collections

University of Wollongong Thesis Collection

University of Wollongong

Year 2009

Two problems in finite elasticity

Himanshuki Nilmini Padukka Withana
University of Wollongong

Padukka Withana, Himanshuki N, Two problems in finite elasticity, MScRes thesis, School of Mathematics and Applied Statistics, University of Wollongong, 2009.
<http://ro.uow.edu.au/theses/869>

This paper is posted at Research Online.
<http://ro.uow.edu.au/theses/869>

NOTE

This online version of the thesis may have different page formatting and pagination from the paper copy held in the University of Wollongong Library.

UNIVERSITY OF WOLLONGONG

COPYRIGHT WARNING

You may print or download ONE copy of this document for the purpose of your own research or study. The University does not authorise you to copy, communicate or otherwise make available electronically to any other person any copyright material contained on this site. You are reminded of the following:

Copyright owners are entitled to take legal action against persons who infringe their copyright. A reproduction of material that is protected by copyright may be a copyright infringement. A court may impose penalties and award damages in relation to offences and infringements relating to copyright material. Higher penalties may apply, and higher damages may be awarded, for offences and infringements involving the conversion of material into digital or electronic form.

Two Problems in Finite Elasticity

A thesis submitted in fulfilment of the
requirement of the award for the degree of

Masters by Research

From

University of Wollongong

By

Himanshuki Nilmini Padukka Withana

Bsc (hons) University of Peradeniya Srilanka

School of Mathematics and Applied Statistics

2009

Certification

I, Himanshuki Nilmini Padukka Withana, declare that this thesis, submitted in fulfilment of the requirements for the award of Masters by Research, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

H. Nilmini Paduuka Withana

August, 2009

Acknowledgements

I owe a great debt of thanks to my supervisor Professor Jim Hill not only for his consultation on the subject matter of this thesis but on his selfless help in solving the seemingly intractable problems that arose alongside my academic pursuits.

Not surprisingly teams with great leaders are always great and this is certainly the case with the team of Nano Technology at UOW. While too numerous to mention here by name the wonderful support of everyone is greatly appreciated. I would especially like to thank Dr. Barry Cox for always being there so cheerfully when ever help was needed. My thanks also go to Dr.Ngamta Thamwattana for kindly sharing her room with me. I would like to thank Dr.Tamsyn Hilder and Dr.Duangkamon Baowan for their support and friendship.

A special thank you to my boss and my fiend Mr. Stephen Haines for the valuable suggestions to the introduction.

Finally to my husband Sajeewa , the best thing that ever happened to me, thank you for always being there smoothening the rough road ahead of me!

Abstract

Some materials encountered in nature and used in engineering exhibit mechanical effects which cannot be adequately explained by classical linear elastic theories. For example, rubber is an elastic material that undergoes large elastic deformations, and therefore renders a non-linear mechanical behavior. An analytical investigation dealing with the problem of static deformation of such materials therefore involves highly non-linear equations leading to arduous mathematical work. Consequently there exists only a limited number of known exact solutions for such problems in the field of finite elasticity.

This thesis is concerned with two problems of finite elastic deformations of rubber blocks. Rubber has been successfully modeled as an isotropic incompressible hyperelastic material with strain energy function given by either the neo-Hookean or Mooney forms. For this class of materials, substantial reductions of the basic underlying equilibrium equations can be obtained, making the problems more tractable and for plane and axially symmetric deformations of these materials, simpler stress-strain relations can be obtained. Therefore, by combining these essentially two-dimensional stress-strain relations together with the reduced equilibrium equations it is possible to obtain comparatively tractable forms of the equations.

In this thesis the following problems for axially symmetric deformations of isotropic incompressible neo-Hookean and Mooney materials are investigated:

- (i) asymptotic axially symmetric deformations describing compression of rubber cylindrical tubes with bonded metal end plates;
- (ii) rippling of a long rectangular rubber block bent into a sector of a solid bounded by two circular arcs.

The above mentioned reduced equilibrium equations are employed in the context of non-linear continuum mechanics to arrive at approximate solutions. The solutions are approximate in the sense that the point-wise vanishing of the stress vector on a boundary is assumed to be replaced by the vanishing of forces in an average manner.

In the first problem, for axially symmetric deformations of the perfectly elastic neo-Hookean and Mooney materials, formal asymptotic solutions are determined in terms of expansions in appropriate powers of $1/R$, where R is the cylindrical

polar material coordinate. Remarkably, for both the neo-Hookean and Mooney materials, the first three terms of such expansions can be completely determined analytically in terms of elementary integrals. From the incompressibility condition and the equilibrium equations, the six unknown deformation functions, that appear in the first three terms can be reduced to five formal integrations involving in total, seven arbitrary constants, and a further five integration constants, making a total of twelve integration constants for the deformation field. The solutions so obtained for the neo-Hookean material are applied to the problem of the axial compression of a cylindrical rubber tube which has bonded metal end-plates. The resulting solution is approximate in two senses; namely as an approximate solution of the governing equations and for which the stress free boundary conditions are satisfied in an average manner only. The resulting deformation and load-deflection relation are shown graphically.

The second problem examined in this thesis is that of finite elastic deformation of a long rectangular rubber block which is deformed in a perturbed cylindrical configuration. This problem is motivated from the problem of determining surface rippling that is observed in bent multi-walled carbon nano-tubes. The problem of finite elastic bending of a tube is considerably more complicated than the geometrically simpler problem of the finite elastic bending of a rectangular block. Accordingly, we examine here the simpler block problem which is assumed to be sufficiently long so that the out of plane end effects may be ignored. The general equations governing plane strain deformations of an isotropic incompressible perfectly elastic Mooney material, which models rubber like materials, are used to determine small superimposed deformations upon the well known controllable family for the deformation of rectangular blocks into a sector of a solid bounded by two circular arcs. Traction free boundary conditions are assumed to be satisfied in an average sense along the bounding circular arcs. Physically realistic rippling is found to occur and typical numerical values are used to illustrate the solution graphically.

In summary reduced equilibrium equations and simplified two-dimensional stress strain relations are used in this study to solve two problems for isotropic incompressible neo-Hookean and Mooney materials. Such deformations and the class of materials studied considerably simplify what are otherwise very complex problems

from the theory of finite elasticity.

Nomenclature

B_R	undeformed configuration
B	deformed configuration
\mathbf{C}	Green deformation tensor
\mathbf{c}	Cauchy deformation tensor
\mathbf{C}^{-1}	Piola deformation tensor
\mathbf{c}^{-1}	Finger deformation tensor
dA	element area in B_R
da	element area in B
dF	force acting on an element area in da
dS	line element in B_R
ds	line element in B
dV	element of volume in B_R
dv	element of volume in B
\mathbf{F}	deformation gradient
$\hat{\mathbf{g}}$	response function
\mathbf{G}_K	material base vectors for the curvilinear coordinate system
\mathbf{g}_i	spatial base vectors for the curvilinear coordinate system
G_{KL}	elements of material metric tensor
g_{ij}	elements of spatial metric tensor
G^{KL}	elements of conjugate material metric tensor
g^{ij}	elements of conjugate spatial metric tensor

G	determinate of material metric tensor $ \mathbf{G} $
g	determinate of spatial metric tensor $ \mathbf{g} $
\mathbf{I}_K	unit rectangular base vectors
I_1, I_2, I_3	principal invariants of the Finger deformation tensor
J	Jacobian of the rectangular Cartesian coordinate system $ \frac{\partial z^i}{\partial Z^K} $
j	Jacobian of the curvilinear coordinate system $ \frac{\partial x^i}{\partial X^K} $
K, L, M	labeling indices associated with B_R
i, j, k	labeling indices associated with B
\mathbf{n}	unit normal to da
\mathbf{n}_R	unit normal to dA
p	modified pressure function
p^*	pressure function
\mathbf{Q}	an orthogonal tensor
\mathbf{T}	stress tensor
\mathbf{T}_R	first Piola-Kirchoff stress tensor
\mathbf{t}	stress vector
$t_K^i(x, X)$	double tensor field
X^K	material curvilinear coordinates
x^i	spatial curvilinear coordinates
Z^K	material rectangular Cartesian coordinates
z^i	spatial rectangular Cartesian coordinates
$x_{,K}^i$	deformation gradients
$X_{,i}^K$	inverse deformation gradient

δ	unit tensor
δ_{ij}, δ_{KL}	Kronecker deltas
∇^2	Laplacian of a scalar with respect to X^K
Γ_{KL}^M	Christoffel symbols based on G_{KL}
Γ_{jk}^i	Christoffel symbols based on g_{ij}
μ	shear modulus
ϕ_i	response coefficients
ρ_R	density in undeformed body B_R
ρ	density in deformed body B
Σ	strain energy function

List of coordinate systems used

(X, Y, Z)	material rectangular Cartesian coordinates
(x, y, z)	spatial rectangular Cartesian coordinates
(R, Θ, Z)	material cylindrical polar coordinates
(r, θ, z)	spatial cylindrical polar coordinates

Contents

1	Overview	13
1.1	Background	13
1.2	Thesis structure	18
2	Basic Equations	20
2.1	Deformation gradients	20
2.2	Metric tensors	21
2.3	Deformation tensors	23
2.4	Changes in lengths, areas and volumes during deformation	24
2.5	Strain invariants	24
2.6	The Christoffel Symbols	24
2.7	Double tensor fields and covariant derivative	25
2.8	Stress tensor	26
2.9	Physical components of a tensor	28
2.10	Equilibrium equations	29
2.11	Hyperelastic materials	30
2.12	Isotropic incompressible hyperelastic materials	31
2.13	Mooney and neo-Hookean materials	32
3	Asymptotic axially symmetric deformations for perfectly elastic neo-Hookean and Mooney materials	33
3.1	Introduction	33
3.2	Basic equations	35
3.2.1	Metric tensors	36
3.2.2	Deformation tensors	36
3.2.3	Incompressibility condition	37

3.2.4	Strain invariants	37
3.2.5	Equilibrium equations	38
3.3	Governing equations for the deformation (3.3)	41
3.4	Solutions and integrals for the neo-Hookean material	44
3.5	Solutions and integrals for the Mooney material	46
3.6	Axial compression of a cylindrical tube with bonded metal end-plates	48
3.7	Numerical results and conclusions	57
4	Rippling of long rectangular rubber blocks under bending	62
4.1	Introduction	62
4.2	Geometry of the deformation	64
4.3	Basic equations for perfectly elastic materials	65
4.3.1	Metric tensors	66
4.3.2	Deformation tensors	66
4.3.3	Strain invariants	67
4.3.4	Incompressibility condition	67
4.3.5	Equilibrium equations	68
4.4	Governing equations for the deformation (4.1)	70
4.5	Load-deflection relations	72
4.6	Numerical results	76
5	Concluding remarks	81
5.1	Summary	81
5.1.1	Asymptotic axially symmetric deformations for perfectly elastic neo-Hookean and Mooney materials	82
5.1.2	Rippling of long rectangular rubber blocks under bending	83
A	Derivation of (3.72)	85
B	Derivation of (3.91)	87
C	Derivation of (3.94)	89
D	Derivation of (3.103)	90

E Derivation of (3.121)	93
F Tables of integrals	94
G Derivation of (4.45)	96
Bibliography	98
List of the author's publications	101

List of Figures

2.1	Analysis of deformation of a body [16]	22
2.2	Stress on an elementary tetrahedron in rectangular Cartesian coordinates [17]	27
2.3	Illustration of first Piola Kirchhoff stress [18]	28
3.1	Original and deformed body of the cylindrical tube with bonded metal plates subjected to axial compression	48
3.2	Right hand side and left hand sides of equation (3.124) versus $2kL$ for three values of B : (a) $B=0$, (b) $B=-0.640$ and (c) $B=6.415$	58
3.3	Non-dimensional force $Fk^2/2C_1$ versus non-dimensional deflection $\delta = 1 - \ell/L$ for $B=0$ and $a=L/2$: (a) $b=L$, (b) $b=2L$, (c) $b=3L$ and (d) $b=4L$	60
4.1	Original and deformed body of the rubber block subjected to bending	65
4.2	Variation of θ_0 against n for $h/t=1, 1.5$ and 2	76
4.3	Variation of $\alpha = \theta_0 tb^2/h(b^2 - a^2)$ against n	77
4.4	Original and deformed body of the rubber block subjected to bending for $t^* = t/h = 0.5$, $a = 3$ and $n = 55.1$	78
4.5	Original and deformed body of the rubber block subjected to bending for $t^* = t/h = 0.5$, $a = 3$ and $n = 50.1$	79
4.6	Original and deformed body of the rubber block subjected to bending for $t^* = t/h = 0.5$, $a = 3$ and $n = 45.1$	80

Chapter 1

Overview

1.1 Background

The theory of non-linear elasticity, also known as finite elasticity, is becoming increasingly important in modern mathematics due to its predominate use in many engineering applications. Its authority in applied mathematics in the main is due to the major role of rubber as a successful engineering material. Within the pages that follow it is the intention of the author to look into two problems related to the analysis of finite elastic deformations specifically as encountered with rubber blocks.

Natural rubber was first found as latex, and in common-use the terms are interchangeable. Latex was first discovered in South America as a naturally occurring "sap" which issued from the wound of certain trees as a milky colloidal suspension. Though Latex found employment primarily as a water proofing agent, it's commercial development did not really begin in earnest until Charles Goodyear discovered the process of Vulcanization in 1839 [1]. Generally referred to as simply 'rubber' is actually vulcanized rubber which is obtained from raw rubber by masticating, mixing with sulphur and heating. When raw rubber undergoes this chemical reaction it is converted into an elastic material, making it a versatile material. Since then rubber has been used extensively in nearly all forms of engineering due to its outstanding physical properties.

Some of vulcanized rubbers more outstanding qualities which have promoted its use are: excellent weathering resistance and general durability, high energy storage capacity, inherent damping qualities in applications where resonant vibrations are encountered, low maintenance and near to zero service requirement coupled with a comparatively high resistance to fatigue, easy installation due to its flexibility,

in-expensive nature and ease of manufacture. Furthermore, rubber lends its-self to efficient bonding, making it a readily deployable material across different materials especially metals and also protect them from rusting by acting as a cover. Finally but not definitively, is the overarching observation that its employment across both the developing and existing industrial landscapes will continue to grow, making it a preferred material over others. Its main industrial applications include: bridge bearings, bush mounting, rubber rollers, springs, rings and tyres.

As rubber is becoming an increasingly popular material in construction and manufacturing engineering, understanding its mechanical properties is proving to be significant. Owing to its high elastic energy, rubber can undergo large elastic deformations, and thus a theory aiming to describe its mechanical behavior under such conditions must take account of the non-linearities involved.

Historically the development of a descriptive theory for rubber was not achieved until the close of the 19th century. Bearing in mind that the real value in application of rubber was not fully appreciated until Goodyear's discovery (1839) of vulcanization there was, none the less, a significant gap between its emerging use and the development of a suitable theory. Although Stokes in 1845 proposed the concept of non-linear constitutive equations for viscous fluid which exhibit similar non-linear mechanical behavior like rubber, nothing significant was done in this regard for the next 100 years. However the concepts of elasticity and hyper elasticity were developed prior and during this era. To this achievement much is owed to those famous early workers in the area notably James Bernoulli (1691-1705) and Euler (1727-1778) who contributed through their important work in one dimensional non-linear elasticity. Although in the main, ideas for a general theory of elasticity were first introduced by Cauchy (1823-1828) and Green (1839-1841) who followed with suggesting the main theories of hyper-elasticity. Subsequent to Green's theory other savants such as Kirchhoff (1852-1859) and Kelvin (1863) added to his original work thus enhancing our understanding of hyper-elasticity. This area of work attracted others like St. Venant, Stokes, Boussinesq, Gibbs, Duhem, and Hadamard. In 1896, the Cosserat brothers published a exposition containing general equations of finite elasticity. However this early work was followed by a period of little significant progress. Signorini was commended in Italy for his teaching and writing on the

subject in 1930 [2], but it was not until Reiner in 1945 and then Rivlin in 1948 contributed significantly to this area, that a concrete foundation was laid in the theory [3]. Reiner is credited as providing the first general approach to the topic of rubber through constitutive equations of a non-linear nature. However, it was Rivlin who sparked a resurgence in this area through his problems of physical interest in non-linear theories providing exact solutions as distinct from the linear general solutions previously employed. Much is owed to Rivlin for rekindling interest in this work and it is reasonable to attribute much of his ingenuity to the recent advances. In the 1950's it was primarily Truesdell's contributions that further progressed research and it is worth mentioning both Erickson and Eringen as important workers from that era. A fruitful progression has been since made with researchers publishing numerous papers solving many physical problems in engineering.

Some other examples of elastic media with large deformations which admits non-linear theories are biological tissues, metals and alloys under high pressure [4], non-linear liquid polymer solutions and solid rocket propellant [5].

The obvious historical question in engineering as to why after the development of a rich set of linear theory and Stokes proposal in 1845, did a sound development in non-linear theory not achieved until after 1940 still remains ambiguous. One possible reason could be that the need for such a theory was not fully realized as not many materials exhibiting non-linear behavior were used in applications. A second reason, still applicable today, is that many believe that only a molecular-statistical theory of the structure of the materials can lead to an understanding of their behavior [2]. Another presumable cause would be the tedious and numerous mathematical complexity involved in a comprehensive nonlinear mathematical model which encourages workers to rely on linear theory for solutions. This being said it must be noted that these theoretical presentations supply engineers with satisfactory results in so far as practical applications are concerned, but they do not answer the problems that are manifest in materials that present large deformations. In such cases linear theory does not approach the accuracy required to answer these problems. Hence in such cases the governing equations involve highly non-linear terms which, due to mathematical complexity, involve analytical solutions to a precision that is not easily tenable in the real world. As a result only a limited number of exact solutions

are known to exist. Due to the mathematical tediousness involved in such cases a number of approximating theories were developed, from the exact theories, which provide a tenable "rule of thumb" to address such problems.

This leaves a number of challenging problems on the table that are yet to be satisfactorily answered. Here-in we will explore, as our first problem, axially symmetrical compression of thick cylindrical rubber tubes as well as, for our second problem, finite elastic bending of a long rectangular rubber block into a perturbed cylindrical configuration from the point of view of surface rippling.

Our first problem, that of axially symmetric deformations of the perfectly elastic neo-Hookean and Mooney materials, an asymptotic expansion in appropriate powers of $1/R$, where R is the cylindrical polar coordinate for the material coordinates is assumed for the axis-symmetric deformation field. The first three terms of the expansion are determined completely analytically in terms of elementary integrals. The solutions obtained for the neo-Hookean material are applied to the problem of the axial compression of a cylindrical rubber tube which has bonded metal end plates.

Six unknown deformation functions appear in the first three terms of the above asymptotic expansion. Using incompressibility condition and equilibrium equations they are reduced to five integrations involving in total seven arbitrary constants and a further five integration constants making a total of 12 constants.

In this problem the solutions obtained are approximate in the sense that the traction free boundary condition is not satisfied in a point wise manner but only in an average sense. As well it should be noted that the solution is followed only to the third term. Progressive terms increase both the tediousness and complexity of the algorithm without enhancing the practical value of the result. In which case the theoretical solution whilst increasing in expended effort provides diminishing returns such that even the fourth term would prove of limited if any use in a practical engineering application.

Our second problem deals with finite elastic bending of a long rectangular rubber block into a perturbed cylindrical configuration and examining from the point of view of surface rippling. Approximate solutions are sought for small superimposed deformations upon the well known controllable family for the deformations

of rectangular blocks into a sector of a solid bounded by two circular arcs. Again the solutions obtained are approximate in the sense that the traction free boundary condition are not satisfied in a point wise manner but only in average sense.

Problems arising from finite deformations of rubber blocks have been looked at by many authors before. Hill and Lee [6] study the problem of large elastic compression of a finite rectangular rubber block which has bonded metal plates to its upper and lower surfaces. An approximate load deflection relation is derived from a fully three dimensional deformation and the results of both finite and infinite relations are compared with experimental results and with that predicted by the conventional engineering approximation deduced from the 'shape factor' method. Their work indicates that if the shape factor approximation is used in conjunction with a value of the Young's modulus obtained from the linear experimental data then this generates a consistently good estimate. They also present a finite model that provides a reasonable accurate approximation irrespective of whether the Young's modulus is determined from the linear experimental data or from a hardness test. A similar approximation is given by Klingbeil and Sheild [7] for extremely long rubber blocks with rectangular cross sections bonded between two parallel rigid end-plates. They also investigate flat deformable circular disks bonded between two end-plates to which closed form of solutions are presented for materials possessing either of two extreme forms of the Mooney strain-energy function. The results are compared with the experimental results of Gent and Lindley [8]. Load deflection relationships are also presented. Experimental observations of the failure of rubber mounts in tension given by Gent and Lindley [8] are used to assess the stress distributions.

The above study was extended by Hill and Lee [9] to include torsion and compression of a cylindrical rubber pad with bonded metal end-plates. Numerical results are obtained for neo-Hookean material and experimental values for pure compression are obtained. The experimental results are compared with the theoretical results due to Klingbeil and Shield [7] and the shape factor approximation and found that they provide satisfactory agreement but only up to some percentage. Bending and stretching of a rectangular rubber blocks into a circular cylindrical tube has also investigated by earlier researchers. For example, Hill [10] describes theoretical load deflection relations for the four principle modes of bonded rubber bush mountings,

constructed from rectangular blocks of rubber namely, torsional, axial radial and conical deflections, based upon well-known exact solution due to Rivlin [11]. Triantafyllidis [12] studied the bifurcation problem of an incompressible plate under pure bending. Bifurcation problem of incompressible isotropic elastic plates subjected to a combined flexure and axial compression is also studied by Haughton [13]. Biot [30] carries theoretical analysis of similar problem of rubber blocks. Gent's [29] contribution through his experimental studies on deformations of rubber blocks has rendered valuable insight to the subject.

1.2 Thesis structure

Chapter one establishes the context of the thesis and explains where in the subject domain the problems examined lie. Some important existing researches with relevance are outlined.

Chapter two summaries the general theories that have been derived by previous workers which are useful in solving our problem.

Chapter three discusses the problem of axially symmetric deformations and axial compression of a cylindrical rubber tube which has bonded metal end plates. We present a brief introduction to the problem followed by basic equations governing the six functions associated with the problem. Then we give the analytical details for the neo-Hookean material and the corresponding details of the Mooney material. In the subsequent section we use the derived results to determine load deflection relation for the axial compression of a hollow neo-Hookean rubber cylinder. Finally we present numerical results.

Chapter four examines the problem of finite elastic bending of a long rectangular rubber block into a perturbed cylindrical configuration. We start with a introduction to the problem and then explain the geometry of the block and initial deformation. Next we present the equations governing the deformation followed by the load-deflection relationships. Finally we illustrate possible solutions with typical numerical values.

In Chapter five we present the summary and concluding remarks. The final chapter comprises the appendices that present some mathematical details and bib-

liography.

Chapter 2

Basic Equations

This chapter outlines the fundamental equations governing large deformations of homogeneous isotropic incompressible elastic materials which are useful in our study. These results are the work of previous researches and will not be repeated in full detail here. For more comprehensive reading we refer to Green and Zerna [14], Truesdell and Noll [2] and Eringen [15]. We comment that these theories follow from the theory of classical continuous media, where a material body is thought to be composed of a large collection of material particles, which are assumed to be continuous throughout the medium except, possibly, at macro-scale discontinuities [16].

The first step in defining large deformation is to define the relationship between the initial undeformed configuration of a body and its deformed configuration as different to classical theory where we analyze both configurations with respect to one reference frame neglecting the difference due to infinitesimal strains.

2.1 Deformation gradients

Consider a body whose points are denoted by X^K ($K = 1, 2, 3$) in a curvilinear coordinate system, occupying a region B_R in Euclidian space. If it under goes a deformation so that the deformed body occupy a another region B and if the position of points of B is denoted by a curvilinear coordinate system by x^i ($i = 1, 2, 3$) then the points are transformed according to

$$x^i = x^i(X^K) \quad \text{or} \quad X^K = X^K(x^i). \quad (2.1)$$

The coordinates X^k are called material or Lagrangian coordinates and the coordinates x^k are called spatial or Eulerian coordinates. The original body is also referred to as reference configuration. It is assumed that

$$j = \left| \frac{\partial x^i}{\partial X^K} \right| \neq 0, \quad (2.2)$$

so that the above coordinate transformation posses a one-to-one relation, where j is the Jacobian of (2.1). From (2.1) we have

$$dx^i = x_{i,K} dX_K, \quad dX_K = X_{K,i} dx_i. \quad (2.3)$$

The two quantities $x^i_{,K}$ and $X^K_{,i}$ are defined as deformation gradients and inverse gradients respectively. Here the subscript comma followed by an index indicates a partial derivative, namely

$$x^i_{,K} = \frac{\partial x^i}{\partial X^K}, \quad X^K_{,i} = \frac{\partial X^K}{\partial x^i}. \quad (2.4)$$

From the chain rule we have that

$$x_{i,K} X_{K,j} = \delta_{ij}, \quad X_{K,i} x_{i,L} = \delta_{KL}. \quad (2.5)$$

2.2 Metric tensors

If the position vector of a material point P whose coordinates are Z^M ($M = 1, 2, 3$) and X^K ($K = 1, 2, 3$) with respect to rectangular cartesian coordinate system and some general curvilinear coordinate system respectively, is \mathbf{P} , then the base vectors of the X^K system are denoted by $\mathbf{G}_K(X)$ and defined by

$$\mathbf{G}_K = \frac{\partial \mathbf{P}}{\partial X^K} = \frac{\partial Z^M}{\partial X^K} \mathbf{I}_M, \quad (2.6)$$

where \mathbf{I}_M ($M = 1, 2, 3$) are the unit rectangular base vectors and can be given by

$$\mathbf{I}_M = \mathbf{G}_K(X) \frac{\partial X^K}{\partial Z^M}. \quad (2.7)$$

Similarly for the corresponding spatial quantities base vectors $\mathbf{g}_i(x)$ are defined by

$$\mathbf{g}_i = \frac{\partial \mathbf{P}}{\partial x^i} = \frac{\partial z^m}{\partial x^i} \mathbf{I}_m. \quad (2.8)$$



Figure 2.1: Analysis of deformation of a body [16]

The metric tensors associated with the above deformation are given by

$$G_{KL} = \mathbf{G}_K \cdot \mathbf{G}_L = \frac{\partial Z^M \partial Z^M}{\partial X^K \partial X^L}, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial z^m \partial z^m}{\partial x^i \partial x^j}, \quad (2.9)$$

and we have the conjugate metric tensors G^{KL} and g^{ij} defined by

$$G^{KL} G_{LM} = \delta_M^K, \quad g^{ij} g_{jk} = \delta_j^i, \quad (2.10)$$

where δ denotes the Kronecker symbol. If the Jacobian of the deformation mapping in rectangular cartesian coordinate system is defined by

$$J = \left| \frac{\partial z^m}{\partial Z^M} \right|, \quad (2.11)$$

then we have

$$j = J \frac{\sqrt{G}}{\sqrt{g}}, \quad (2.12)$$

where G and g are scalars defined by

$$G = |G_{KL}|, \quad g = |g_{ij}|. \quad (2.13)$$

It is understood that the necessary and sufficient condition for isochoric deformation is that $J = 1$.

2.3 Deformation tensors

If dS and ds are the distances between two neighboring points in the undeformed and deformed bodies respectively, then we have the squares of the line elements in B_R and B given by

$$\begin{aligned} dS^2 &= dZ^M dZ^M = G_{KL} dX^K dX^L = c_{ij} dx^i dx^j, \\ ds^2 &= dz^m dz^m = g_{ij} dx^i dx^j = C_{KL} dX^K dX^L, \end{aligned} \quad (2.14)$$

where we have defined

$$\begin{aligned} C_{KL} &= \mathbf{C}_K \cdot \mathbf{C}_L = g_{ij} x^i_{,K} x^j_{,L}, \\ c_{ij} &= \mathbf{c}_i \cdot \mathbf{c}_j = G_{KL} X^K_{,i} X^L_{,j}. \end{aligned} \quad (2.15)$$

\mathbf{C} is called the Green deformation tensor while \mathbf{c} is known as the Cauchy deformation tensor. The reciprocal of C_{KL} which satisfies

$$C^{LK} C_{KM} = \delta^L_M, \quad (2.16)$$

is denoted by C^{-1KL} and given by

$$C^{-1KL} = g^{ij} X^K_{,i} X^L_{,j}. \quad (2.17)$$

Similarly, the reciprocal of c_{ij} which satisfies

$$c^{ji} c_{im} = \delta^j_m, \quad (2.18)$$

is denoted by c^{-1ij} and given by

$$c^{-1ij} = G^{KL} x^i_{,K} x^j_{,L}. \quad (2.19)$$

The above inverse tensors \mathbf{C}^{-1} and \mathbf{c}^{-1} are known as Piola deformation tensor and Finger deformation tensor respectively. It can be also noted that deformation tensors and their reciprocals are symmetrical. That is,

$$\begin{aligned} C_{KL} &= C_{LK}, & C^{-1KL} &= C^{-1LK}, \\ c_{ij} &= c_{ji}, & c^{-1ij} &= c^{-1ji}. \end{aligned} \quad (2.20)$$

We also note that the mixed components of \mathbf{C} and \mathbf{c} can be obtained as follows by raising indices using respective metric tensors.

$$C^K_L = C_{ML} G^{KM}, \quad c^i_j = c_{mj} g^{im}. \quad (2.21)$$

2.4 Changes in lengths, areas and volumes during deformation

Changes in lengths during deformation can be expressed by (2.14) and the areas are related by

$$da_m = JX^M_{,m} dA_M, \quad dA_M = J^{-1}x^m_{,M} da_m, \quad (2.22)$$

where dA_M is the element of area vectors of a surface which transforms to area vector element da_m after deformation. If dv is the volume into which the volume of material element dV deforms, it can be deduced that they are related by

$$dv = JdV. \quad (2.23)$$

2.5 Strain invariants

The principle invariants of Finger deformation tensor \mathbf{c}^{-1} , are denoted by I_1, I_2 and I_3 and obtained from the characteristic equation as

$$I_1 = \bar{c}^{1i}_i, \quad I_2 = |\bar{c}^{1i}_j|c^j_i, \quad I_3 = |\bar{c}^{1i}_j|. \quad (2.24)$$

Utilizing the symmetry of Finger deformation tensor \mathbf{c}^{-1} and using equations (2.2), (2.13) and (2.12) it can be shown that

$$I_3 = J^2. \quad (2.25)$$

Hence the principal invariants become

$$I_1 = \bar{c}^{1i}_i, \quad I_2 = J^2 c^i_i, \quad I_3 = J^2. \quad (2.26)$$

2.6 The Christoffel Symbols

Christoffel symbols represent partial derivatives of base vectors with respect to co-ordinate variables. From (2.6) and (2.7) we obtain

$$\frac{\partial \mathbf{G}_K}{\partial X^L} = \frac{\partial X^M}{\partial X^N} \frac{\partial^2 Z^N}{\partial X^K \partial X^L} \mathbf{G}_M, \quad (2.27)$$

which may be written as

$$\frac{\partial \mathbf{G}_K}{\partial X^L} = \Gamma_{KL}^M \mathbf{G}_M, \quad (2.28)$$

where

$$\Gamma_{KL}^M = \frac{\partial X^M}{\partial X^N} \frac{\partial^2 Z^N}{\partial X^K \partial X^L}, \quad (2.29)$$

are defined as Christoffel symbols of second kind relative to the coordinate system X^K . Similarly Christoffel symbols of second kind relative to the coordinate system x^k is given by

$$\Gamma_{ij}^m = \frac{\partial x^m}{\partial x^n} \frac{\partial^2 z^n}{\partial x^i \partial x^j} \quad (2.30)$$

2.7 Double tensor fields and covariant derivative

If points x and X belong to two spaces and the functions $t_K^i(x, X)$ and $t_{K'}^{i'}(x', X')$ transform according to

$$t_{K'}^{i'}(x', X') = t_K^i(x, X) \frac{\partial x^{i'}}{\partial x^i} \frac{\partial X^K}{\partial X'^{K'}}, \quad (2.31)$$

where

$$x' = x'(x), \quad X' = X'(X), \quad (2.32)$$

then the above functions are defined as a double tensor field in the sets of variables x, X, x' and X' . The covariant partial derivatives of a double tensor field $t_K^i(x, X)$ are denoted by $t_{K,L}^i$ and $t_{K,j}^i$ in which the differentiation is carried out with respect to the coordinates X^K with x held fixed and with respect to the coordinates x^i with X held fixed respectively. Hence we have the expressions

$$\begin{aligned} t_{K,L}^i &= \frac{\partial t_K^i}{\partial X^L} - t_M^i \Gamma_{KL}^M, \\ t_{K,j}^i &= \frac{\partial t_K^i}{\partial x^j} + t_K^m \Gamma_{mj}^i, \end{aligned} \quad (2.33)$$

where Γ_{KL}^M and Γ_{mj}^i denote Christoffel symbols based on G_{KL} and g_{ij} respectively. The total covariant derivative of a double tensor field $t_K^i(x, X)$ with $x = x(X)$ is defined by

$$\begin{aligned} t_{K;L}^i &= t_{K,L}^i + t_{K,j}^i \frac{\partial x^j}{\partial X^L}, \\ t_{K;j}^i &= t_{K,j}^i + t_{K,L}^i \frac{\partial X^L}{\partial x^j}. \end{aligned} \quad (2.34)$$

It is noted that for a double tensor $t_K^i(x, X)$ that $t_{j,K}^i \neq 0$, while for a single-point tensor field $t_K^i(x, X)$, $t_{j,K}^i = 0$. Thus it is clear that partial covariant derivatives of both metric tensors and their conjugates are zero. Since deformation gradients $x^i_{,L}$ and $X^K_{,j}$ are two point tensors, it can be shown that the total covariances are given by

$$\begin{aligned} [x^i_{,K}]_{;L} &= \frac{\partial^2 x^i}{\partial X^L \partial X^K} - \Gamma_{LK}^M \frac{\partial x^i}{\partial X^M} + \Gamma_{mj}^i x^j_{,L} x^m_{,K}, \\ [X^K_{,i}]_{;j} &= \frac{\partial^2 X^K}{\partial x^j \partial x^i} - \Gamma_{ij}^m \frac{\partial X^K}{\partial x^m} + \Gamma_{IM}^K X^I_{,i} X^M_{,j}. \end{aligned} \quad (2.35)$$

From equations (2.35), (2.17) and (2.19) it can be obtained

$$\begin{aligned} G^{KL} [x^i_{,L}]_{;K} &= \nabla^2 x^i + \Gamma_{jm}^i c^{-1jm}, \\ g^{ji} [X^K_{,i}]_{;j} &= \nabla_1^2 X^K + \Gamma_{ML}^K C^{-1ML}, \end{aligned} \quad (2.36)$$

where ∇ and ∇_1 are Laplacians defined by

$$\begin{aligned} \nabla^2 &= G^{KL} \left[\frac{\partial^2}{\partial X^K \partial X^L} - \Gamma_{KL}^M \frac{\partial}{\partial X^M} \right], \\ \nabla_1^2 &= g^{ij} \left[\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^m \frac{\partial}{\partial x^m} \right]. \end{aligned} \quad (2.37)$$

2.8 Stress tensor

If da is a small element of the deformed body and the force acting on the element at time t is dF , then the stress vector \mathbf{t} at a point x inside the element at time t is given by

$$\mathbf{t} da = d\mathbf{F}. \quad (2.38)$$

The stress tensor \mathbf{T} is defined by

$$\mathbf{t} = \mathbf{T} \mathbf{n}, \quad (2.39)$$

where \mathbf{n} is the unit normal of the element da . The first Piola-Kirchhoff stress tensor \mathbf{T}_R is defined by

$$T_R^{Kj} = J X^K_{,i} T^{ij}, \quad (2.40)$$

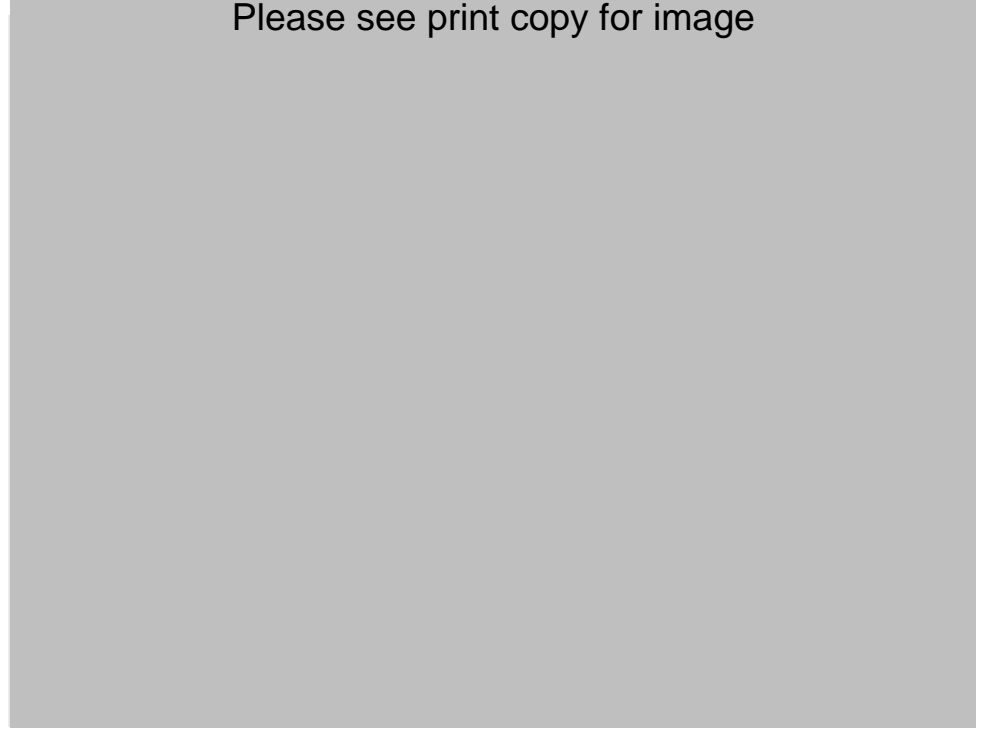


Figure 2.2: Stress on an elementary tetrahedron in rectangular Cartesian coordinates [17]

while if the undeformed area of the deformed element da is dA , then the Piola-Kirchoff stress vector \mathbf{t}_R is defined by

$$\mathbf{t}_R = \mathbf{T}_R \mathbf{n}_R, \quad t_R^j = T_R^{Kj} n_{RK}, \quad (2.41)$$

where \mathbf{n}_R is the unit normal to element dA . On using Nanson's formula for area change during deformation, namely

$$da_i = J X^K_{,i} dA_K, \quad (2.42)$$

and equations (2.40) and (2.41) it is obtained that

$$t_R^j dA = t^{ij} da_i. \quad (2.43)$$

Hence from (2.43) it is clear that

$$t_R^j dA = t^i da_i, \quad (2.44)$$

and

$$\mathbf{t}_R dA = \mathbf{t} da = d\mathbf{F}. \quad (2.45)$$

Please see print copy for image

Figure 2.3: Illustration of first Piola Kirchhoff stress [18]

Here the area vectors of deformed and un-deformed bodies are respectively,

$$\mathbf{da} = da \mathbf{n}, \quad \mathbf{dA} = dA \mathbf{n}_R. \quad (2.46)$$

2.9 Physical components of a tensor

The components of a tensor in a curvilinear coordinate system do not have the same physical dimensions as the tensor itself. Thus when those tensor components are used in solving physical problems we need to convert them to physical components. The physical components of a tensor are the tensor components possessing the same physical dimensions as the physical quantity represented by the tensor. However in nonorthogonal coordinates several different types of physical components arise. That is to say that there is no unique definition for physical components of a second-

order tensor. For example for a tensor \mathbf{T} the physical components $T_{(j)}^{(i)}$ can be given by

$$T_{(j)}^{(i)} = T_j^i \frac{\sqrt{g_{ii}}}{\sqrt{g_{jj}}}, \quad (2.47)$$

or the physical components $T^{(i)(j)}$ can be defined by

$$T^{(i)(j)} = T^{ij} \sqrt{g_{ii} g_{jj}}, \quad (2.48)$$

where underscore for indices suspends the summation. When the coordinates are orthogonal we note that

$$g^{ii} = \frac{1}{g_{ii}}. \quad (2.49)$$

Hence in the case of symmetric tensors referred to orthogonal curvilinear coordinates, the physical components $T_{(j)}^{(i)}$ and $T_{(j)}^{(i)}$ are identical. For symmetric tensors in orthogonal curvilinear coordinates we therefore have [16], [19].

$$T_{(j)}^{(i)} = \frac{T_{ij}}{\sqrt{g_{ii}g_{jj}}} = T^{ij} \sqrt{g_{ii}g_{jj}} = T_j^i \frac{\sqrt{g_{ii}}}{\sqrt{g_{jj}}}. \quad (2.50)$$

2.10 Equilibrium equations

The law of conservation of mass states that

$$\int_v \rho dv = \int_V \rho_0 dV, \quad (2.51)$$

where ρ_0 and dV denote the mass density and the element of volume respectively, before deformation while ρ and dv denote the corresponding quantities after deformation. Equation (2.23) and localizing may result in

$$\rho J = \rho_0, \quad (2.52)$$

which is the local law of conservation of mass generally known as the continuity equation. Due to local balance law of linear momentum, in the absence of body forces we have the equilibrium equation given by

$$\operatorname{div} \mathbf{T} = 0. \quad (2.53)$$

Upon using the Euler-C. Neumann identity

$$[J X^K]_{,i};_K = 0, \quad (2.54)$$

and equation (2.40) on (2.53), the equilibrium equation in terms of first Piola-Kirchoff stress tensor can be given by

$$T_R^{Kj};_K = 0. \quad (2.55)$$

Under the local balance law of moment of momentum, for nonpolar media (ie in the absence of torques or couple stresses) the stress tensor \mathbf{T} requires to be symmetric. Hence we have

$$\mathbf{T}^T = \mathbf{T}, \quad T^{ij} = T^{ji}, \quad (2.56)$$

which is known as Cauchy's second law of motion.

2.11 Hyperelastic materials

A constitutive equation relates the mechanical behavior of the material to stress and motion. The constitutive equation of an elastic material is given by

$$\mathbf{T} = \hat{\mathbf{g}}(\mathbf{F}), \quad (2.57)$$

where \mathbf{T} is the stress tensor and \mathbf{F} is the deformation gradient. The function $\hat{\mathbf{g}}$ which is referred to as response function of the elastic material satisfies the relation

$$\mathbf{Q} \hat{\mathbf{g}}(\mathbf{F}) \mathbf{Q}^T = \hat{\mathbf{g}}(\mathbf{QF}), \quad (2.58)$$

for a orthogonal tensor \mathbf{Q} [2]. An elastic material with response function $\hat{\mathbf{g}}$ is said to be isotropic if and only if $\hat{\mathbf{g}}$ satisfies the relation

$$\mathbf{Q} \hat{\mathbf{g}}(\mathbf{F}) \mathbf{Q}^T = \hat{\mathbf{g}}(\mathbf{QFQ}^T), \quad (2.59)$$

for all orthogonal \mathbf{Q} [5]. The general constitutive equation for isotropic elastic materials is given by

$$\mathbf{T} = \phi_0 \mathbf{c} + \phi_1 \delta + \phi_2 \mathbf{c}^{-1}, \quad (2.60)$$

where \mathbf{T} is the stress tensor, \mathbf{c} is the Cauchy deformation tensor and δ is the unit tensor. ϕ_i ($i = 1, 2, 3$) are scalar functions of the invariants I_1, I_2 and I_3 of inverse Cauchy deformation tensor \mathbf{c}^{-1} and is called the response coefficients of the material.

Hyperelastic materials are isotropic elastic materials whose response functions can be given by derivatives of a scalar function with respect to the strain invariants. That is, for such materials there exists a scalar function

$$\Sigma = \Sigma(I_1, I_2, I_3), \quad (2.61)$$

where I_1, I_2 and I_3 are strain invariants defined above. The scalar function Σ is called the strain energy function or stored energy function. The response coefficients are namely,

$$\begin{aligned} \phi_0 &= -2\sqrt{I_3} \frac{\partial \Sigma}{\partial I_2}, \\ \phi_1 &= \frac{2}{\sqrt{I_3}} \left(I_2 \frac{\partial \Sigma}{\partial I_2} + I_3 \frac{\partial \Sigma}{\partial I_3} \right), \\ \phi_3 &= \frac{2}{\sqrt{I_3}} \frac{\partial \Sigma}{\partial I_1}. \end{aligned} \quad (2.62)$$

Hence from equation (2.60) the Finger stress strain relation becomes [20]

$$\mathbf{T} = \frac{2}{\sqrt{I_3}} \left[\left(I_2 \frac{\partial \Sigma}{\partial I_2} + I_3 \frac{\partial \Sigma}{\partial I_3} \right) \delta + \frac{\partial \Sigma}{\partial I_1} \mathbf{c}^{-1} - I_3 \frac{\partial \Sigma}{\partial I_2} \mathbf{c} \right]. \quad (2.63)$$

2.12 Isotropic incompressible hyperelastic materials

Deformations in which volume elements remain unaltered are called isochoric deformations. A material for which only isochoric deformations are possible is said to be incompressible. According to (2.23) the necessary and sufficient condition for isochoric deformations is that $J = 1$. Hence the continuity equation for incompressible materials becomes

$$\rho_R = \rho, \quad (2.64)$$

where ρ_R and ρ are the densities in the deformed and undeformed configurations respectively. Further due to (2.25) invariant I_3 becomes unity which subsequently makes the equation (2.62)₂ indeterminate. Hence the Finger stress-strain relation for isotropic incompressible materials is given by

$$\mathbf{T} = p^* \delta + 2 \frac{\partial \Sigma}{\partial I_1} \mathbf{c}^1 - 2 \frac{\partial \Sigma}{\partial I_2} \mathbf{c}, \quad (2.65)$$

where p^* is an arbitrary function, referred to as pressure function and strain energy function $\Sigma = \Sigma(I_1, I_2)$.

2.13 Mooney and neo-Hookean materials

An incompressible material with strain energy function given by

$$\Sigma = C_1(I_1 - 3) + C_2(I_2 - 3). \quad (2.66)$$

is called a Mooney material. This was first proposed by Melvin Mooney to model certain rubber like materials and such models were in fairly good agreement with the experimental results [3]. In the special case when C_2 is zero, the theory is called neo-Hookean, viz

$$\Sigma = C_1(I_1 - 3), \quad (2.67)$$

for a neo-Hookean material [2]. I_1 and I_2 denote the first two invariants of the inverse Cauchy deformation tensor and C_1 and C_2 are positive material constants such that $2(C_1 + C_2)$ may be identified with the infinitesimal shear modulus μ , given by

$$\mu = 2(C_1 + C_2). \quad (2.68)$$

Moreover, for many rubber-like materials typically C_2 is approximately $C_1/10$. Here, following normal practice, we use Γ to denote the ratio C_2/C_1 . The Mooney model usually fits experimental data better than Neo-Hookean model does, but involves an additional material constant leading to more complex analytical results. Due to the above definition of strain energy functions we have

$$\frac{\partial \Sigma}{\partial I_1} = C_1, \quad (2.69)$$

for both type of materials and

$$\frac{\partial \Sigma}{\partial I_2} = C_2, \quad (2.70)$$

for Mooney materials.

Chapter 3

Asymptotic axially symmetric deformations for perfectly elastic neo-Hookean and Mooney materials

3.1 Introduction

In this chapter we provide a detailed treatment for the problem of axial compression of cylindrical rubber tubes of neo-Hookean and Mooney material with bonded metal end plates. For axially symmetric deformations, an asymptotic expansion in powers of $1/R$ is assumed, where R is the cylindrical polar coordinate for the material coordinates. The first three terms are completely determined analytically in terms of elementary integrals. The solution determined is approximate in two senses; namely as an approximate solution of the governing equations and for which the stress free and displacement boundary conditions are satisfied in an average manner only.

For homogeneous isotropic incompressible elastic material sometimes referred to as simply perfectly elastic materials, there are a limited number of exact analytical solutions for finite deformations. However, frequently these limited analytical results can be exploited to determine approximate analytical solutions to many problems of practical interest (see for example Hill [21]). Klingbeil and Shield [7] determined analytical expressions for the neo-Hookean and Mooney elastic materials for the axially symmetric deformation

$$r = f(Z)R, \quad \theta = \Theta, \quad z = g(Z), \quad (3.1)$$

where (R, Θ, Z) and (r, θ, z) denote material and spatial cylindrical polar coordinates respectively and f and g are functions of Z only. Further, the deformation (3.1) can be utilized to determine approximate load deflection relations for the two problems of axial tension and compression of solid rubber cylinders under conditions of axial symmetry and loaded by means of bonded metal end-plates. The deformation (3.1) specifically only applies for solid cylinders, and the question arises as to how (3.1) might be extended to deal with the corresponding problem but for the axial tension and compression of hollow rubber cylinders, again loaded by means of bonded metal end-plates. Since it is not known how to determine exact analytical solutions applicable to tubes rather than solid cylinders, it is natural to consider some analytical asymptotic expansions involving powers of R and coefficients as functions of Z only. However, if we assume such an expansion involving only positive powers of R , then it may be readily shown that an infinite number of such terms are required to properly close the system. This is essentially because ∇^2 operating on the typical term, say $f_n(Z)R^n$, produces in particular the term $n^2 f_n(Z)R^{n-2}$, which makes a contribution to the lower order term $f_{n-2}(Z)R^{n-2}$, and evidently it is not possible to properly close such a system using only a finite number of terms. In this paper we consider the axially symmetric static deformation

$$r = r(R, Z), \quad \theta = \Theta, \quad z = z(R, Z), \quad (3.2)$$

for which $r(R, Z)$ and $z(R, Z)$ admit the following asymptotic expansions

$$\begin{aligned} r &= Rf_0(Z) + \frac{f_1(Z)}{R} + \frac{f_2(Z)}{R^3} + \dots, \\ z &= g_0(Z) + \frac{g_1(Z)}{R^2} + \frac{g_2(Z)}{R^4} + \dots, \end{aligned} \quad (3.3)$$

so that now for example, ∇^2 operating on the typical term $f_n(Z)/R^{2n-1}$ produces in particular the term $(2n-1)^2 f_n(Z)/R^{2n+1}$ and this makes a contribution to the determination of the next higher order term $f_{n+1}(Z)$ rather than a lower one. Accordingly, we may properly close the system using only a finite number of terms. Here for the perfectly elastic neo-Hookean and Mooney materials, we show that for the six functions involved in (3.3) we may produce five first order integrals involving seven arbitrary constants A, B, C, D, E, H and k^2 , and from which the full solution

for all functions may be determined from a further five formal integrations. Thus, for both neo-Hookean and Mooney materials the full analytical solutions for (3.3) can be presented in terms of five formal integrations and involving in total twelve arbitrary constants.

In the following section we present the basic equations for axially symmetric deformations of the incompressible neo-Hookean and Mooney perfectly elastic materials. In the subsequent section we present the basic equations governing the six functions $f_j(Z)$ and $g_j(Z)$ ($j = 0, 1, 2$) appearing in (3.3). Since the analytical details for the Mooney material are considerably more complicated than those for the neo-Hookean material, in Section 3.4 we first present those for the neo-Hookean material and then in Section 3.5 we give the corresponding results for the Mooney material. In Section 3.6 we use the solutions obtained to determine the load-deflection relation for the axial compression of a hollow neo-Hookean rubber cylinder. Such a solution is approximate both in the sense that the governing equations are satisfied asymptotically, and in the sense that the point-wise stress free boundary conditions are assumed to be replaced by average or integral conditions on every arbitrary elemental strip. However, the formal solutions obtained here are “exact” asymptotic solutions of the governing equations, and in the absence of precise analytical results might be rendered as an approximate solution for any axially symmetric problem for which the Z-axis is excluded. In Section 3.7 we exploit these solutions for the problem of the axial compression of a hollow thick walled rubber cylinder by application of equal and opposite forces applied to bonded metal end plates. In Section 3.7 we give numerical results for the approximate load deflection relation so determined. Appendices A to F record various analytical details required for the approximate load-deflection relation derived in Section 3.6.

3.2 Basic equations

The general equations outlined in Chapter 2 for perfectly elastic materials are simplified here for the case of axially symmetric deformation field given by (3.2). We briefly present here the basic equations that are useful in solving the present problem. More detailed derivation of this equations are given in the literature [20].

Let the points of the undeformed body $X^K (K = 1, 2, 3)$ and those of the deformed body $x^k, (k = 1, 2, 3)$ be referred to by the cylindrical polar coordinates (R, Θ, Z) and (r, θ, z) respectively. If the corresponding rectangular cartesian coordinates are given by $Z^M (M = 1, 2, 3)$ and $z^m (m = 1, 2, 3)$ respectively then we have

$$\begin{aligned} Z^1 &= R \cos \Theta & Z^2 &= R \sin \Theta & Z^3 &= Z, \\ z^1 &= r \cos \theta & z^2 &= r \sin \theta & z^3 &= z. \end{aligned} \quad (3.4)$$

It should be noted that in the following discussions only the values 1 and 3 are assigned to the labeling indices a,b A and B.

3.2.1 Metric tensors

Due to (2.9) the metric tensors for the deformation (3.2) are given by

$$G_{KL} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.5)$$

and

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.6)$$

From (2.10) the spatial conjugate metric tensor becomes

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.7)$$

while the material conjugate metric tensor is given by

$$G^{KL} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.8)$$

3.2.2 Deformation tensors

From (2.15)₂ Cauchy deformation tensor for the deformation (3.2) is given by

$$c_{ij} = \begin{bmatrix} R_r^2 + Z_r^2 & 0 & R_r R_z + Z_r Z_z \\ 0 & R^2 & 0 \\ R_r R_z + Z_r Z_z & 0 & R_z^2 + Z_z^2 \end{bmatrix}, \quad (3.9)$$

while on using (2.19), Finger deformation tensor is shown to become

$$c^{-1ij} = \begin{bmatrix} r_R^2 + r_Z^2 & 0 & r_R z_R + r_Z z_Z \\ 0 & 1/R^2 & 0 \\ r_R z_R + r_Z z_Z & 0 & z_R^2 + z_Z^2 \end{bmatrix}. \quad (3.10)$$

The corresponding mixed components of the above tensors are given by

$$c_j^i = \begin{bmatrix} R_r^2 + Z_r^2 & 0 & R_r R_z + Z_r Z_z \\ 0 & 1/\lambda^2 & 0 \\ R_r R_z + Z_r Z_z & 0 & R_z^2 + Z_z^2 \end{bmatrix}, \quad (3.11)$$

and

$$c^{-1i}_j = \begin{bmatrix} r_R^2 + r_Z^2 & 0 & r_R z_R + r_Z z_Z \\ 0 & \lambda^2 & 0 \\ r_R z_R + r_Z z_Z & 0 & z_R^2 + z_Z^2 \end{bmatrix}. \quad (3.12)$$

3.2.3 Incompressibility condition

From (2.2) we have

$$j = \frac{\partial(r, z)}{\partial(R, Z)}, \quad (3.13)$$

and from (2.13)

$$g = r^2, \quad G = R^2. \quad (3.14)$$

Hence due to (3.13) and (2.12) the incompressibility condition becomes

$$\frac{\partial(r, z)}{\partial(R, Z)} = r_R z_Z - r_Z z_R = \frac{R}{r}. \quad (3.15)$$

3.2.4 Strain invariants

Since for incompressible materials $J = 1$, due to (2.26), principle invariants of the Finger deformation tensor become

$$I_1 = c^{-1i}_i, \quad I_2 = c^i_i, \quad I_3 = 1. \quad (3.16)$$

We let I and II denote the principle invariants of c^{-1a}_b . Thus we have

$$I = c^{-1a}_a, \quad II = |\bar{c}^{1a}_b|. \quad (3.17)$$

Now combining (3.17) in (2.24) and utilizing (3.12) we obtain

$$I_1 = I + \lambda^2, \quad I_2 = \lambda^2 I + II, \quad I_3 = \lambda^2 II, \quad (3.18)$$

where λ is defined by

$$\lambda = r/R. \quad (3.19)$$

Due to (3.17) it is also clear that I is defined by

$$I = r_R^2 + r_Z^2 + z_R^2 + z_z^2. \quad (3.20)$$

Further the Cauchy Hamilton theorem for c^{-1a}_b yields

$$IIc_b^a = \delta_b^a I - c^{-1a}_b. \quad (3.21)$$

However it can be shown that

$$II = |c^{-1a}_b| = \frac{1}{\lambda^2}, \quad (3.22)$$

and therefore we have the strain invariants expressed by

$$I_1 = I + \lambda^2, \quad I_2 = \lambda^2 I + \frac{1}{\lambda^2}, \quad I_3 = 1. \quad (3.23)$$

3.2.5 Equilibrium equations

Since in terms of free products the stress tensor \mathbf{T} can be expressed as

$$\mathbf{T} = T_b^a g_a g^b, \quad (3.24)$$

for deformation (3.2), (2.62) becomes

$$\begin{aligned} T_b^a &= p^* \delta_b^a + 2 \frac{\partial \Sigma}{\partial I_1} c^{-1a}_b - 2 \frac{\partial \Sigma}{\partial I_2} c_b^a, \\ T_2^2 &= p^* + 2 \frac{\partial \Sigma}{\partial I_1} \lambda^2 - 2 \frac{\partial \Sigma}{\partial I_2} \frac{1}{\lambda^2}, \\ T_a^2 &= T_2^a = 0. \end{aligned} \quad (3.25)$$

Upon using (3.21) in (3.25)₁ and introducing p and ϕ defined by

$$\begin{aligned} p &= p^* + 2\lambda^2 I \frac{\partial \Sigma}{\partial I_2}, \\ \phi &= 2 \left[\frac{\partial \Sigma}{\partial I_1} + \lambda^2 \frac{\partial \Sigma}{\partial I_2} \right], \end{aligned} \quad (3.26)$$

T_b^a can be shown to become

$$T_b^a = -p\delta_b^a + \phi c^{-1a}_b. \quad (3.27)$$

Hence by substituting from (3.12) the stress components are shown to be given by

$$T_1^1 = -p + \phi (r_R^2 + r_Z^2),$$

$$T_3^3 = -p + \phi (z_R^2 + z_Z^2), \quad (3.28)$$

$$T_3^1 = T_1^3 = \phi (r_R z_R + r_Z z_Z).$$

Now by introducing

$$\psi = \left[\frac{\partial \Sigma}{\partial I_1} + \left(I - \frac{1}{\lambda^4} \right) \frac{\partial \Sigma}{\partial I_2} \right], \quad (3.29)$$

(3.25)₂ can be expressed as

$$T_2^2 = -p + \psi \lambda^2, \quad (3.30)$$

where p is called the modified pressure function while ϕ and ψ are known as response functions which due to (3.18) can also be expressed as

$$\phi = 2 \frac{\partial \Sigma}{\partial I}, \quad \psi = 2 \frac{\partial \Sigma}{\partial \lambda^2}. \quad (3.31)$$

Next on using (3.15) we deduce the following results

$$\begin{aligned} R_r &= \lambda z_Z, & R_z &= -\lambda r_Z, \\ Z_r &= -\lambda z_R, & Z_z &= \lambda r_R. \end{aligned} \quad (3.32)$$

Upon utilizing $T^{ij} = g^{ik} T_k^j$ and (2.40) and subsisting above results into (3.28) we obtain

$$\begin{aligned} T_R^{11} &= -\lambda p z_Z + \phi r_R, \\ T_R^{33} &= -\lambda p r_R + \phi z_Z, \\ T_R^{13} &= \lambda p r_Z + \phi z_R, \\ T_R^{31} &= \lambda p z_R + \phi r_Z, \\ T_R^{22} &= -\frac{p}{r^2} + \frac{\psi}{R^2}, \\ T_R^{2a} &= T_R^{a2} = 0. \end{aligned} \quad (3.33)$$

Noting that for orthogonal coordinates Christoffel symbols Γ_{ij}^r vanish for $r \neq i \neq j$, there remain only the following non zero terms for the coordinates (r, θ, Z)

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}. \quad (3.34)$$

The equilibrium equations (2.55) therefore give

$$\frac{\partial T_R^{11}}{\partial R} + \frac{T_R^{11}}{R} + \frac{\partial T_R^{31}}{\partial Z} = r T_R^{22}, \quad (3.35)$$

$$\frac{\partial T_R^{13}}{\partial R} + \frac{T_R^{13}}{R} + \frac{\partial T_R^{33}}{\partial Z} = 0. \quad (3.36)$$

It is understood that the second equilibrium equation (3.36) is automatically satisfied.

Owing to (3.19) it is easily seen that

$$\lambda_R = -\frac{r}{R^2} + \frac{r_R}{R}, \quad \lambda_Z = \frac{r_Z}{R}. \quad (3.37)$$

Upon substituting (3.33) in (3.35) and accompanying the above results the two equilibrium equations simplify to give

$$\frac{\partial p}{\partial r} = \phi \nabla^2 r + \phi_R r_R + \phi_Z r_Z - \psi \frac{r}{R^2}, \quad (3.38)$$

$$\frac{\partial p}{\partial z} = \phi \nabla^2 z + \phi_R z_R + \phi_Z z_Z, \quad (3.39)$$

where ∇^2 denotes the usual axially symmetric Laplacian, thus

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}. \quad (3.40)$$

For the Mooney strain-energy function, namely

$$\sum (I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3), \quad (3.41)$$

those relations yield

$$\phi = 2(C_1 + \lambda^2 C_2), \quad \psi = 2 \left(C_1 + \left(I - \frac{1}{\lambda^4} \right) C_2 \right). \quad (3.42)$$

In the following section we use the above results to determine the basic equations for the asymptotic deformation (3.3).

3.3 Governing equations for the deformation (3.3)

Formally, we assume that term by term differentiation of (3.3) with respect to both R and Z is permissible any number of times. Thus for example

$$r_R = f_0(Z) - \frac{f_1(Z)}{R^2} - \frac{3f_2(Z)}{R^4} + \dots, \quad (3.43)$$

$$r_Z = Rf'_0(Z) + \frac{f'_1(Z)}{R} + \frac{f'_2(Z)}{R^3} + \dots,$$

and

$$z_R = -\frac{2g_1(Z)}{R^3} - \frac{4g_2(Z)}{R^5} + \dots, \quad (3.44)$$

$$z_Z = g'_0(Z) + \frac{g'_1(Z)}{R^2} + \frac{g'_2(Z)}{R^4} + \dots,$$

where throughout primes denote differentiation with respect to Z . It is also convenient to use ε for $1/R^2$, and subsequently we generally omit any indication of higher order terms. Noting simply that the analysis given here is consistently accurate to order ε^2 .

We first deal with the incompressibility condition. From (3.3), (3.15) and formulae such as (3.43), we obtain respectively on equating coefficients of $\varepsilon^0, \varepsilon$ and ε^2 ,

$$\begin{aligned} g'_0 &= 1/f_0^2, \\ f_0g'_1 + 2f'_0g_1 &= 0, \\ f_0g'_2 - f_1g'_1 - 3f_2g'_0 + 4f'_0g_2 + 2g_1f'_1 &= \frac{f_1^2}{f_0^3} - \frac{f_2}{f_0^2}. \end{aligned} \quad (3.45)$$

By rearranging terms and integrating, from (3.45)₂ it can be shown that

$$g_1 = \frac{A}{f_0^2}, \quad (3.46)$$

where A denotes a constant of integration. Upon substituting for g'_0 , g_1 and g'_1 in (3.45)₃ we may readily deduce

$$(f_0^4g_2 + 2Af_0f_1)' = f_1^2 + 2f_0f_2. \quad (3.47)$$

Thus, assuming $f_0(Z)$, $f_1(Z)$ and $f_2(Z)$ are known functions, the functions $g_0(Z)$, $g_1(Z)$ and $g_2(Z)$ can be formally determined by two integrations.

Next, by substituting $\varepsilon = \frac{1}{R^2}$ in equation (3.3)₁, squaring it and neglecting the terms of order ε^3 and higher we obtain

$$\lambda^2 = f_0^2 + 2\varepsilon f_0 f_1 + \varepsilon^2 (f_1^2 + 2f_0 f_2). \quad (3.48)$$

Using the above relationship and employing the binomial expansion we yield

$$\frac{1}{\lambda^4} = \frac{1}{f_0^4} - 4\varepsilon \frac{f_1}{f_0^5} + \varepsilon^2 \frac{(10f_1^2 - 4f_0 f_2)}{f_0^6}. \quad (3.49)$$

It is understood that again we have neglected the terms of order ε^3 and higher in equation (3.49). Now carrying on the expressions (3.43) and (3.44) into (3.20) we have I given by

$$I = \frac{f_0'^2}{\varepsilon} + \left(f_0^2 + 2f_0' f_1' + g_0'^2\right) + \varepsilon \left(f_1'^2 - 2f_0 f_1 + 2f_0' f_2' + 2g_0' g_1'\right),$$

noting that the order ε^2 contribution in the equation has not been included, because subsequently we need ψ/R^2 (ie $\varepsilon\psi$) and therefore the order ε^2 terms in both (3.50) and (3.49) are not required. From (3.48), (3.49) and (3.42) we obtain

$$\begin{aligned} \phi &= 2(C_1 + f_0^2 C_2) + 4\varepsilon C_2 f_0 f_1 + 2\varepsilon^2 C_2 (f_1^2 + 2f_0 f_2), \\ \frac{\psi}{R^2} &= 2C_2 f_0'^2 + 2C_1 \varepsilon + 2C_2 \varepsilon (f_0^2 + 2f_0' f_1') \\ &\quad + 2C_2 \varepsilon^2 \left(f_1'^2 + 2f_0' f_2' + 2g_0' g_1' - 2f_0 f_1 + \frac{4f_1}{f_0^5}\right), \end{aligned} \quad (3.50)$$

noting that in deriving the latter expression we have utilized (3.45)₁.

Next for the convenience of calculations we use a, b, c, α, β and γ as shorthand for the coefficients arising in (3.50), thus

$$\begin{aligned} \phi &= a + b\varepsilon + c\varepsilon^2, \\ \frac{\psi}{R^2} &= \alpha + \beta\varepsilon + \gamma\varepsilon^2, \end{aligned} \quad (3.51)$$

where it is readily seen from (3.50) that

$$\begin{aligned} a^* &= 2(C_1 + f_0^2 C_2), \\ b^* &= 4C_2 f_0 f_1, \\ c^* &= 2C_2 (f_1^2 + 2f_0 f_2), \\ \alpha &= 2C_2 f_0'^2, \\ \beta &= 2C_1 + 2C_2 (f_0^2 + 2f_0' f_1'), \\ \gamma &= 2C_2 \left(f_1'^2 + 2f_0' f_2' + 2g_0' g_1' - 2f_0 f_1 + \frac{4f_1}{f_0^5}\right). \end{aligned} \quad (3.52)$$

Now due to our assumption that deformation (3.3) is differentiable any number of times with respect to R and Z , from (3.43) and (3.44) we have

$$\begin{aligned}
r_{RR} &= \frac{2f_1(Z)}{R^3} + \frac{12f_2(Z)}{R^5} + \dots, \\
r_{ZZ} &= Rf_0''(Z) + \frac{f_1''(Z)}{R} + \frac{f_2''(Z)}{R^3} + \dots, \\
z_{RR} &= \frac{6g_1(Z)}{R^4} + \frac{20g_2(Z)}{R^6} + \dots, \\
z_{ZZ} &= g_0''(Z) + \frac{g_1''(Z)}{R^2} + \frac{g_2''(Z)}{R^4} + \dots,
\end{aligned} \tag{3.53}$$

where prime denotes differentiation with respect to Z . Applying the Laplacian given by (3.40) on (3.3) and utilizing equations (3.53) and (3.43) it is easily obtained that

$$\begin{aligned}
\nabla^2 r &= Rf_0'' + R\varepsilon(f_0 + f_1'') + R\varepsilon^2(f_2'' + f_1), \\
\nabla^2 z &= g_0'' + \varepsilon g_1'' + \varepsilon^2(4g_1 + g_2'').
\end{aligned} \tag{3.54}$$

Now using equations (3.51) and substituting (3.54) in the two equilibrium equations given by (3.38) we deduce the following results

$$\begin{aligned}
\frac{\partial p}{\partial r} &= RF_0(Z) + \frac{F_1(Z)}{R} + \frac{F_2(Z)}{R^3} + \dots, \\
\frac{\partial p}{\partial z} &= G_0(Z) + \frac{G_1(Z)}{R^2} + \frac{G_2(Z)}{R^4} + \dots,
\end{aligned} \tag{3.55}$$

where the six functions $F_j(Z)$ and $G_j(Z)$ ($j = 0, 1, 2$) are given by

$$\begin{aligned}
F_0 &= (a^* f_0')' - \alpha f_0, \\
F_1 &= (a^* f_1' + b^* f_0')' + a^* f_0 - \alpha f_1 - \beta f_0, \\
F_2 &= (a^* f_2' + b^* f_1' + c^* f_0')' + a^* f_1 - b^* f_0 - \alpha f_2 - \beta f_1 - \gamma f_0,
\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}
G_0 &= (a^* g_0')', \\
G_1 &= (a^* g_1' + b^* g_0')', \\
G_2 &= (a^* g_2' + b^* g_1' + c^* g_0')' + 4a^* g_1,
\end{aligned} \tag{3.57}$$

Now on eliminating the pressure function p from (3.55), namely using the equation

$$\frac{\partial(\partial p/\partial r, r)}{\partial(R, Z)} + \frac{\partial(\partial p/\partial z, z)}{\partial(R, Z)} = 0, \quad (3.58)$$

we may deduce from (3.3), (3.55) and (3.58)

$$\begin{aligned} F_0 f'_0 - F'_0 f_0 &= 0, \\ F_0 f'_1 - F_1 f'_0 + F'_0 f_1 - F'_1 f_0 &= 0, \\ F_0 f'_2 - F'_2 f_0 + F'_1 f_1 - F_1 f'_1 + 3F'_0 f_2 - 3F_2 f'_0 + 2g_1 G'_0 - 2G_1 g'_0 &= 0, \end{aligned} \quad (3.59)$$

the first two of which readily integrate to yield

$$F_0 = -2C_1 k^2 f_0, \quad F_1 = -2C_1 k^2 f_1 + 2C_1 B/f_0, \quad (3.60)$$

where B and k^2 denote real constants of integration, noting that k itself can be real or pure imaginary. Next on using (3.45) and (3.60) in equation (3.59)₃ we may deduce

$$[f_0^3(F_2 + 2C_1 k^2 f_2) + 2C_1 B f_0 f_1]' = 2(AG'_0 - G_1), \quad (3.61)$$

and from (3.57) we have

$$AG'_0 - G_1 = \left[\frac{4C_2}{f_0} (Af'_0 - f_1) \right]'. \quad (3.62)$$

Therefore on integration we obtain

$$f_0^3(F_2 + 2C_1 k^2 f_2) + 2C_1 B f_0 f_1 = \frac{8C_2}{f_0} (Af'_0 - f_1) + 2C_1 C, \quad (3.63)$$

where C denotes the constant of integration.

In the following section we deduce from (3.60) and (3.63) the greatly simplified equations which apply for the special case of the neo-Hookean material ($C_2 = 0$). The analysis for $C_2 \neq 0$ is presented in the section thereafter.

3.4 Solutions and integrals for the neo-Hookean material

For the case $C_2 = 0$, we have simply $a^* = \beta = 2C_1$ and $b^* = c^* = \alpha = \gamma = 0$ and the three determining equations for $f_0(Z)$ and $f_1(Z)$ are obtained from (3.60) as

$$f''_0 + k^2 f_0 = 0, \quad f''_1 + k^2 f_1 = B/f_0. \quad (3.64)$$

Due to (3.56)₃ it is clear that

$$F_2 = 2C_1 f_2'' . \quad (3.65)$$

Hence from (3.63) we obtain

$$f_2'' + k^2 f_2 = \frac{C}{f_0^3} - \frac{B f_1}{f_0^2} , \quad (3.66)$$

again noting that B , C and k^2 are all constants of integration. In summary for neo-Hookean Material we therefore have

$$\begin{aligned} F_0 &= 2C_1 f_0'' , & F_1 &= 2C_1 f_1'' , & F_2 &= 2C_1 f_2'' , \\ G_0 &= 2C_1 g_0'' , & G_1 &= 2C_1 g_1'' , & G_2 &= 2C_1 g_2'' + 8C_1 g_1 . \end{aligned} \quad (3.67)$$

Multiplication of (3.64)₁ by f_0' gives

$$f_0'^2 + k^2 f_0^2 = D , \quad (3.68)$$

where D is a further constant of integration. Similarly, multiplication of (3.64)₁ by f_1' and (3.64)₂ by f_0' and by addition gives the integral

$$f_0' f_1' + k^2 f_0 f_1 = B \log f_0 + E , \quad (3.69)$$

where E denotes the constant of integration. The formal integration of (3.66) is achieved by multiplication of this equation by f_0' and adding f_1' times equation (3.64)₂ plus f_2' times equation (3.64)₁, thus

$$f_0'(f_2'' + k^2 f_2) + f_1'(f_1'' + k^2 f_1) + f_2'(f_0'' + k^2 f_0) = \frac{B}{f_0} f_1' - \frac{B f_1}{f_0^2} f_0' + \frac{C f_0'}{f_0^3} ,$$

which may be readily integrated to yield

$$f_0' f_2' + \frac{f_1'^2}{2} + k^2 \left(f_0 f_2 + \frac{f_1^2}{2} \right) = \frac{B f_1}{f_0} - \frac{C}{2 f_0^2} + H , \quad (3.70)$$

where H denotes the constant of integration.

The calculation for the pressure function p is quite arduous even for the neo-Hookean material. We need to integrate (3.55) using (3.60) and (3.63) with $C_2 = 0$ and using

$$\frac{\partial p}{\partial R} = \frac{\partial p}{\partial r} r_R + \frac{\partial p}{\partial z} z_R , \quad \frac{\partial p}{\partial Z} = \frac{\partial p}{\partial r} r_Z + \frac{\partial p}{\partial z} z_Z , \quad (3.71)$$

we may eventually deduce as shown in Appendix A

$$p = -2C_1 \left\{ \frac{k^2}{2} \left(Rf_0 + \frac{f_1}{R} + \frac{f_2}{R^3} \right)^2 + \frac{2Af'_0}{R^2 f_0^5} + \frac{C}{2R^2 f_0^2} - B \log(Rf_0) - \frac{Bf_1}{R^2 f_0} - \frac{1}{2f_0^4} \right\} + 2C_1 p_0, \quad (3.72)$$

where p_0 denotes the constant of integration and the term r^2 appearing in (3.72) means

$$\left(Rf_0 + \frac{f_1}{R} + \frac{f_2}{R^3} \right)^2 = R^2 f_0^2 + 2f_0 f_1 + \frac{(f_1^2 + 2f_0 f_2)}{R^2} + \dots \quad (3.73)$$

Thus the functions appearing in equation (3.3) can be determined completely from five further formal integrations, namely (3.47)₁, (3.47)₃, (3.68), (3.69) and (3.70). These five equations involve in total seven arbitrary constants (A, B, C, D, E, H and k^2) which together with the five further constants of integration makes in total twelve arbitrary constants. The stresses will generally involve all these constants plus the constant p_0 which arises in the integration of the pressure function. The same picture applies for the Mooney material, but the integrals corresponding to (3.68), (3.69) and (3.70) are far more complicated. These results are obtained in the following section

3.5 Solutions and integrals for the Mooney material

For C_2 non-zero we have from (3.50) and (3.51)

$$\begin{aligned} a^* &= 2(C_1 + f_0^2 C_2), \\ b^* &= 4C_2 f_0 f_1, \\ c^* &= 2C_2(f_1^2 + 2f_0 f_2), \\ \alpha &= 2C_2 f_0'^2, \\ \beta &= 2C_1 + 2C_2(f_0^2 + 2f_0' f_1'), \\ \gamma &= 2C_2(f_1'^2 + 2f_0' f_2' + 2g_0' g_1' - 2f_0 f_1 + 4f_1/f_0^5), \end{aligned} \quad (3.74)$$

and from the equations (3.56), (3.60) and (3.63) we may eventually deduce the following expressions for $f_0(Z)$, $f_1(Z)$ and $f_2(Z)$, thus

$$(1 + \Gamma f_0^2) f_0'' + \Gamma f_0 f_0'^2 + k^2 f_0 = 0, \quad (3.75)$$

$$(1 + \Gamma f_0^2)f_1'' + 2\Gamma f_0 f_0' f_1' + \Gamma(2f_0 f_0'' + f_0'^2)f_1 + k^2 f_1 = B/f_0, \quad (3.76)$$

$$(1 + \Gamma f_0^2)f_2'' + 2\Gamma f_0 f_0' f_2' + \Gamma(2f_0 f_0'' + f_0'^2)f_2 + k^2 f_2 \\ + \Gamma(f_1^2 f_0'' + 2f_1 f_1' f_0') + \Gamma(2f_1 f_1'' + f_1'^2)f_0 = C/f_0^3 - Bf_1/f_0^2, \quad (3.77)$$

where Γ denotes this ratio C_2/C_1 . Formal integrals for these three equations are obtained precisely as described in the previous section for the neo-Hookean material, although of course the details are more complicated.

Multiplication of (3.75) by f_0' and integrating yields

$$(1 + \Gamma f_0^2)f_0'^2 + k^2 f_0^2 = D. \quad (3.78)$$

This nonlinear first order differential equation was first derived by Klingbiel and Shield [7] and has a solution which may be expressed in terms of elliptic functions. Similarly, multiplication of (3.75) by f_1' and (3.76) by f_0' , adding the two equations and integrating yields

$$(1 + \Gamma f_0^2)f_0' f_1' + \Gamma f_0 f_0' f_1^2 + k^2 f_0 f_1 = B \log f_0 + E, \quad (3.79)$$

Although complicated, It is noted that by dividing by $(1 + \Gamma f_0^2)f_0'$ this equation is formally only a first-order linear ordinary differential equation whose integrating factor is given by e^Λ where

$$\Lambda = \int_0^\zeta \frac{(\Gamma f_0'^2 + k^2)f_0 d\zeta}{(1 + \Gamma f_0^2)f_0'}, \quad (3.80)$$

which on using (3.75) can be shown to become

$$\Lambda = -\log f_0'. \quad (3.81)$$

Hence the integrating factor becomes f_0' and the differential equation (3.79) becomes

$$\left(\frac{f_1}{f_0'}\right)' = \frac{B \log f_0 + E}{D - k^2 f_0^2}, \quad (3.82)$$

which for known $f_0(Z)$ can usually be integrated directly. Following the same recipe for the determination of (3.70), we multiply (3.75) by f_2' , (3.76) by f_1' and (3.77) by f_0' , and an addition the resulting equation may be integrated to yield

$$(1 + \Gamma f_0^2)f_0' f_2' + \Gamma f_0 f_0' f_2^2 + k^2 f_0 f_2 + \frac{\Gamma}{2} \{(f_1 f_0' + f_0 f_1')^2 + 2f_0 f_0' f_1 f_1'\} \\ + \frac{f_1'^2}{2} + \frac{k^2}{2} f_1^2 = Bf_1/f_0 - C/2f_0^2 + H, \quad (3.83)$$

and again, although more complicated, this is merely a first-order linear ordinary differential equation with the same integrating factor as (3.79), and simplifies to give

$$\begin{aligned} \left(\frac{f_2}{f_0'}\right)' + \frac{\Gamma \{(f_1 f_0' + f_0 f_1')^2 + 2f_0 f_0' f_1 f_1'\} + f_1'^2 + k^2 f_1^2}{2(D - k^2 f_0^2)} \\ = \frac{B f_1 / f_0 - C / 2 f_0^2 + H}{(D - k^2 f_0^2)}, \end{aligned} \quad (3.84)$$

and again, in principle for known $f_0(Z)$ and $f_1(Z)$, this equation may be formally integrated. Thus for the deformation (3.3) we have again reduced the problem to five formal integrations, namely (3.47)₁, (3.47)₃, (3.78), (3.82) and (3.84). In the following section we use the results of the previous section to determine an approximate solution to the problem of the axially symmetric compression of a rubber tube with bonded metal end-plates.

3.6 Axial compression of a cylindrical tube with bonded metal end-plates

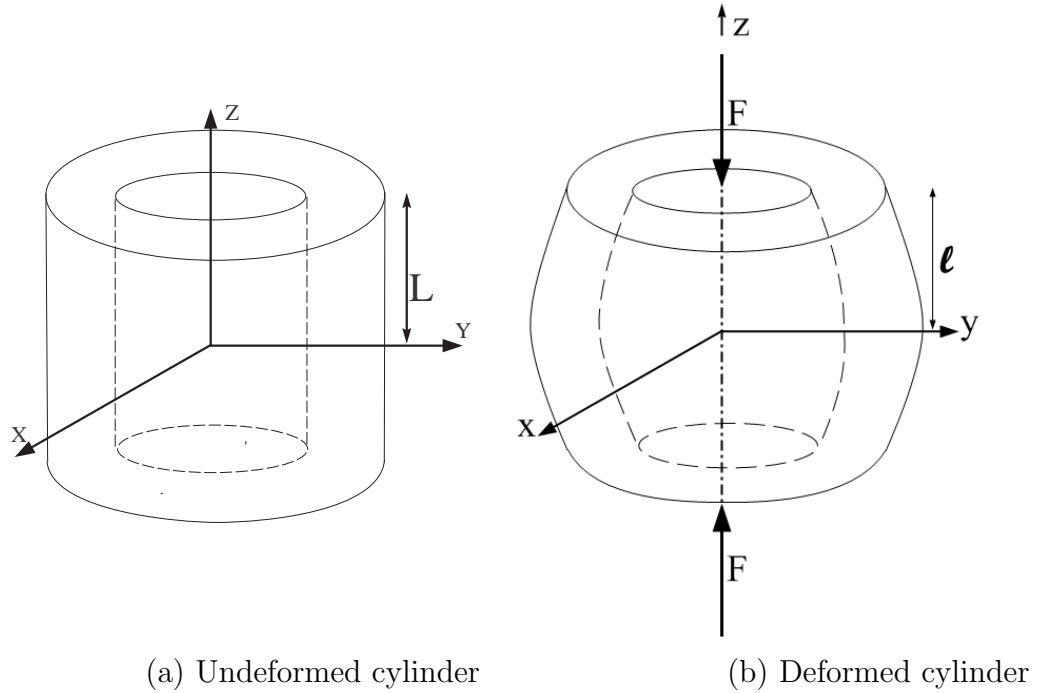


Figure 3.1: Original and deformed body of the cylindrical tube with bonded metal plates subjected to axial compression

In this section we determine an approximate solution to the problem of the axial

compression of a cylindrical tube as shown in Figure 3.1. We assume inner and outer radii a and b respectively, and originally of length $2L$ and deformed symmetrically as shown in the figure to have final overall length 2ℓ . We suppose that the actual deformation may be approximated by a deformation of the form (3.3), and ideally we seek functions $f_j(Z)$ and $g_j(Z)$ ($j = 0, 1, 2$) such that they are even and odd respectively and that the following boundary values apply, namely

$$f_0(\pm L) = 1, \quad f_1(\pm L) = f_2(\pm L) = 0,$$

$$g_0(\pm L) = \pm\ell, \quad g_1(\pm L) = g_2(\pm L) = 0, \quad (3.85)$$

and since $g_j(Z)$ ($j = 0, 1, 2$) are assumed to be odd functions of Z , we have automatically $g_j(0) = 0$ ($j = 0, 1, 2$). Subsequently, we find that it is not possible to satisfy (3.85)₆ in a pointwise manner and we have to satisfy the condition $g_2(\pm L) = 0$ in an average sense. Now from (3.68), (3.45)₁ and the above boundary values, we may readily deduce

$$f_0(Z) = \frac{\cos kZ}{\cos kL}, \quad g_0(Z) = \frac{\cos^2 kL}{k} \tan kZ, \quad (3.86)$$

which is the solution for the axial compression for the solid rubber cylinder, and first given by Klingbeil and Shield [7].

Now with $f_0(Z)$ defined by (3.86)₁ and (3.69) we may obtain

$$\left(\frac{f_1}{\sin kZ} \right)' = - \frac{\left[B \log \left(\frac{\cos kZ}{\sin kL} \right) + E \right] \cos kL}{k \sin^2 kZ}, \quad (3.87)$$

which can readily be integrated to give

$$f_1(Z) = \frac{B \cos kL}{k^2} \left[\cos kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin kZ \right] + \frac{E \cos kL}{k^2} \cos kZ + I \sin kZ, \quad (3.88)$$

where I denotes the integration constant. From the boundary condition (3.86)₂ we may deduce $I = 0$ and $E = BkL \tan kL$, and we obtain

$$f_1(Z) = \frac{B \cos kL}{k^2} \left[\cos kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin kZ - kL \tan kL \cos kZ \right]. \quad (3.89)$$

Since $g_1(Z)$ is assumed to be odd, we have immediately from (3.46) that the constant A is zero and thus $g_1(Z)$ is identically zero.

Now again with $f_0(Z)$ defined by (3.86)₁ and with $f_1(Z)$ defined by (3.89), equation (3.70) may be integrated to give

$$\begin{aligned}
f_2(Z) = & \frac{B^2 \cos^3 kL}{k^4} \left\{ -\frac{\cos kZ}{2} \log^2 \left(\frac{\cos kZ}{\cos kL} \right) \right. \\
& + (1 + kL \tan kL) \left[\cos kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin kZ \right] \\
& - kZ \sin kZ \log \left(\frac{\cos kZ}{\cos kL} \right) - \frac{k^2}{2} Z^2 \cos kZ \\
& \left. - kL \tan kL \left(1 + \frac{kL \tan kL}{2} \right) \cos kZ - 2kI_1(Z) \sin kZ \right\} \\
& - \frac{C \cos^3 kL}{2k^2} \frac{\cos 2kZ}{\cos kZ} + \frac{H \cos kL}{k^2} \cos kZ + J \sin kZ, \tag{3.90}
\end{aligned}$$

where J denotes the further constant of integration. From the boundary values we may deduce J is zero and $H = B^2 \{ (L^2 \cos 2kL)/2 + I_1(L) \sin 2kL \} + (C/2) \cos 2kL$ and we obtain the following expression which is derived in Appendix B,

$$\begin{aligned}
f_2(Z) = & \frac{B^2 \cos^3 kL}{k^4} \left\{ -\frac{\cos kZ}{2} \log^2 \left(\frac{\cos kZ}{\cos kL} \right) \right. \\
& + (1 + kL \tan kL) \left[\cos kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin kZ \right] \\
& \left. - kZ \sin kZ \log \left(\frac{\cos kZ}{\cos kL} \right) - \frac{k^2}{2} Z^2 \cos kZ + T \cos kZ - 2k^2 I_1(Z) \sin kZ \right\} \\
& + \frac{C \cos kL}{2k^2 \cos kZ} (\cos^2 kL - \cos^2 kZ), \tag{3.91}
\end{aligned}$$

where T is a constant defined by

$$T = (kL)^2/2 - k^2 L^2 \tan^2 kL - kL \tan kL + 2k^2 I_1(L) \tan kL, \tag{3.92}$$

and $I_1(Z)$ is defined by

$$I_1(Z) = \int_0^Z \xi \tan k\xi d\xi = -\frac{Z}{k} \log kZ + \frac{1}{k} \int_0^Z \log \cos k\xi d\xi. \tag{3.93}$$

Further, with $f_0(Z)$, $f_1(Z)$ and $f_2(Z)$ defined by (3.86), (3.89)₁ and (3.91), as shown

in Appendix C we may integrate equation (3.47) to give $g_2(Z)$ as

$$\begin{aligned}
g_2(Z) = & \frac{B^2 \cos^6 kL}{k^5 \cos^4 kZ} \left[\frac{1}{2} \sin 2kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin^2 kZ \right. \\
& - \frac{1}{2} k^2 Z^2 \sin 2kZ + k^2 I_1(Z) (1 + 2 \cos^2 kZ) + \frac{T^*}{2} \left(kZ + \frac{\sin 2kZ}{2} \right) \Big] \\
& + \frac{C}{2k^2} \left(\frac{\cos kL}{\cos kZ} \right)^4 \left[Z \cos 2kL - \frac{\sin 2kZ}{2k} \right], \tag{3.94}
\end{aligned}$$

where T^* is the constant defined by $T^* = 2T + k^2 L^2 \tan^2 kL$. Here we have utilized the fact that $g_2(Z)$ is an odd function, to determine the integral constant as zero. We observe that while $g_2(Z)$ is correctly an odd function, we are not able to impose the additional requirement that $g_2(\pm L) = 0$. Accordingly, we assume that we may replace the assumption of a bonded metal plate, by an averaged approximation for this requirement.

At this point there remains four as yet undetermined constants, namely B, C, k^2 and p_0 . These constants are obtained by prescribing the resultant applied force F to the bonded metal end-plates, and by requiring that every elemental strip subtended by an arbitrary angle $d\Theta$ on both surfaces originally given by $R = a$ and $R = b$ respectively be subject to an uniform vertical force. This procedure follows that first exploited by Klingbeil and Shield [7] and later used by Hill and Lee [9]. Given the approximate nature of both the assumed deformation and the average boundary conditions, it is not a simple matter to deduce conditions under which such a solution might apply. We point out that for the series solution (3.3) to be a good approximate analytical solution, it is necessary that R is large. For the problem considered in this section, this implies that a should be large, where of course the largeness or smallness of a quantity is in a relative sense. Here, we might expect what is required is that a is much larger than the characteristic axial length, that is $2L$ or indeed L . Under such a condition, it would then be reasonable to replace the point-wise traction free conditions on the two lateral surfaces, by requiring the resultant to be zero. However numerical results for the final load deflection relation (3.127) show that a physically sensible relation is only obtained for $a < L$. For $a \geq L$ the relation (3.127) does not display the characteristics one might expect such as achieving the maximum force in the vicinity of $\delta = 1 - \ell/L$ equals unity, namely $\ell \approx 0$. Accordingly, in the numerical

example of the following section we restrict our attention to $a < L$.

Due to the axial symmetry and the mid-plane symmetry, the only non-zero resultant force acting on every elemental strip subtended by the arbitrary angle $d\Theta$ is given by

$$dF_r = d\Theta \int_{-L}^L T_R^{11} R dZ, \quad (3.95)$$

where T_R^{Kj} ($K, j = 1, 2, 3$) denotes the first Piola-Kirchoff stress tensor which is given explicitly in Hill and Lee [9]. In particular for the neo-Hookean material we have

$$T_R^{11} = -p \frac{r}{R} z_Z + 2C_1 r_R, \quad T_R^{33} = -p \frac{r}{R} r_R + 2C z_Z, \quad (3.96)$$

and therefore from (3.3) with $g_1(Z) \equiv 0$ and (3.88) we have

$$T_R^{11} = -p \left(f_0 + \frac{f_1}{R^2} + \frac{f_2}{R^4} \right) \left(g'_0 + \frac{g'_2}{R^4} \right) + 2C_1 \left(f_0 - \frac{f_1}{R^2} - \frac{3f_2}{R^4} \right), \quad (3.97)$$

$$T_R^{33} = -p \left(f_0 + \frac{f_1}{R^2} + \frac{f_2}{R^4} \right) \left(f_0 - \frac{f_1}{R^2} - \frac{3f_2}{R^4} \right) + 2C_1 \left(g'_0 + \frac{g'_2}{R^4} \right), \quad (3.98)$$

where for the neo-Hookean material from (3.72) the pressure function p is given by

$$\begin{aligned} p = & -2C_1 \left[\frac{R^2 k^2}{2} f_0^2 + \left(k^2 f_0 f_1 - B \log f_0 - \frac{1}{2f_0^4} \right) \right. \\ & \left. + \frac{(f_1^2 + 2f_0 f_2)k^2 + C/f_0^2 - 2Bf_1/f_0}{2R^2} - B \log R \right] + 2C_1 p_0. \end{aligned} \quad (3.99)$$

If we include the three leading order terms in the expressions for (3.97) and (3.98) then we obtain

$$\begin{aligned} \frac{T_R^{11}}{2C_1} = & R^2 \frac{k^2}{2} f_0 + \left(f_0 - \frac{1}{2f_0^5} - \frac{p_0}{f_0} - \frac{B \log f_0}{f_0} + \frac{3k^2 f_1}{2} \right) - B \log R \left(\frac{1}{f_0} + \frac{f_1}{R^2 f_0^2} \right) + \\ & \frac{1}{R^2} \left[\frac{k^2}{2} (5f_2 - 4f_0^2 f'_0 g_2) + \frac{C}{2f_0^3} + \frac{2k^2 f_1^2}{f_0} \right. \\ & \left. - (B + B \log f_0 + p) \frac{f_1}{f_0^2} - f_1 - \frac{f_1}{2f_0^6} \right] + \dots, \end{aligned} \quad (3.100)$$

$$\begin{aligned} \frac{T_R^{33}}{2C_1} = & R^2 \frac{k^2}{2} f_0^4 - \left(p_0 f_0^2 - \frac{1}{2f_0^2} - k^2 f_0^3 f_1 + B f_0^2 \log f_0 \right) + \frac{1}{R^2} \left(\frac{C}{2} - B f_1 f_0 \right) \\ & - B f_0^2 \log R + \dots \end{aligned} \quad (3.101)$$

From the expressions (3.86), (3.91) and (3.94) we have from (3.100)

$$\begin{aligned}
\frac{T_R^{11}}{2C_1} = & R^2 \frac{k^2 \cos kZ}{2 \cos kL} + \left[\frac{\cos kZ}{\cos kL} - \frac{\cos^5 kL}{2 \cos^5 kZ} - p_0 \frac{\cos kL}{\cos kZ} + \right. \\
& B \cos kL \left(\frac{3}{2} \cos kZ \log f_0 + \frac{3}{2} kZ \sin kZ - \frac{3}{2} kL \tan kL \cos kZ - \frac{\log f_0}{\cos kZ} \right) \Big] \\
& + \frac{1}{R^2} \left\{ \frac{B^2 \cos^3 kL}{k^2} \left[\frac{3}{4} \log^2 f_0 \cos kZ - \frac{\log^2 f_0}{\cos kZ} + \frac{kZ \log f_0 \sin kZ}{\cos^2 kZ} \right. \right. \\
& + \frac{1}{2} (1 - 3kL \tan kL) (\log f_0 \cos kZ + kZ \sin kZ) + \frac{3}{2} kZ \log f_0 \sin kZ \\
& + (1 + kL \tan kL) \frac{\log f_0}{\cos kZ} - \frac{5}{4} k^2 Z^2 \cos kZ + (1 + T^*) \frac{kZ \sin kZ}{\cos^2 kZ} + \\
& \left. \frac{1}{4} (T^* + 3k^2 L^2 \tan^2 kL) \cos kZ + \frac{(T^* + kL \tan kL)}{\cos kZ} \right. \\
& \left. + k^2 I_1(Z) \sin kZ \left(\frac{2}{\cos^2 kZ} - 1 \right) \right] \\
& - \frac{B}{k^2} (\cos kL \cos kZ \log f_0 + kZ \cos kL \sin kZ - kL \sin kL \cos kZ \\
& + \frac{\cos^7 kL \log f_0}{2 \cos^5 kZ} + \frac{kZ \cos^7 kL \sin kZ}{2 \cos^6 kZ} - \frac{kL \cos^6 kL \sin kL}{2 \cos^5 kZ} \Big) \\
& - \frac{p_0 B \cos^3 kL}{k^2} \left(\frac{\log f_0}{\cos kZ} + \frac{kZ \sin kZ}{\cos^2 kZ} - \frac{kL \tan kL}{\cos kZ} \right) \\
& + \frac{C \cos kL}{4} \left(\frac{4kZ \tan kZ \cos 2kL + \sin^2 kZ - 5 \sin^2 kL}{\cos kZ} + 2 \frac{\cos^2 kL}{\cos^3 kZ} \right) \Big\} + \dots \quad .
\end{aligned} \tag{3.102}$$

Now on using (3.102) as shown in Appendix D, the condition (3.95) becomes

$$\begin{aligned}
p_0 = & -B \log R + (kR)^2 \frac{\kappa_1}{\kappa_7 + B/(kR)^2 \kappa_8} + \frac{\kappa_2 + B\kappa_3}{\kappa_7 + B/(kR)^2 \kappa_8} \\
& + \frac{1}{(kR)^2} \frac{(B^2 \kappa_4 + B\kappa_5 + k^2 C \kappa_6)}{\kappa_7 + B/(kR)^2 \kappa_8},
\end{aligned} \tag{3.103}$$

$R = a$ and b , and where κ_j ($j = 1..8$) are constants not involving R and defined by

$$\begin{aligned}
\kappa_1 &= \frac{s}{c}, \quad \kappa_2 = \frac{2s}{c} - \frac{sc}{4} - \frac{3}{8}sc^3 - \frac{3}{8}c^5w, \quad \kappa_3 = 3cw - 3kL - ckI_2, \\
\kappa_4 &= kL(2T^*c^2 - c^2sw - c^4) + k^2Lc^2sI_2 + wc^3 + \frac{T^*c^3s}{2} \\
&\quad + \frac{c^3}{2}(3kI_2 - 4kI_3) + \frac{k^2L^2}{2}(c^3s + 3s^3c) + 2k^2I_1(L)c^2(c^2 + 2), \\
\kappa_5 &= kL(2 - \frac{c^2}{5} + \frac{c^2s^2}{4} + \frac{3}{8}c^4s^2 + \frac{3}{8}c^6sw) \\
&\quad - (2cw + \frac{c^3s}{80} + \frac{13}{160}c^5s - \frac{47}{160}c^7w + \frac{3}{16}kc^7I_2), \\
\kappa_6 &= 2kL(2c^2 - 1) - wc^3, \quad \kappa_7 = 2cw, \\
\kappa_8 &= 2kLc^2(1 - sw) - 2c^2w + kc^3I_2.
\end{aligned} \tag{3.104}$$

Here for convenience we adopt the abbreviations

$$\begin{aligned}
c &= \cos kL, \quad s = \sin kL, \quad w = \log[\sec kL + \tan kL], \\
I_2 &= \int_{-L}^L \log(\sin kZ + 1) \tan kZ dZ, \\
I_3 &= k \int_{-L}^{+L} \int_{-L}^Z \log(\sin kx + 1) \tan kx \tan kZ dx dZ.
\end{aligned} \tag{3.105}$$

Since (3.103) applies at both $R = a$ and $R = b$ we have

$$\begin{aligned}
p_0 &= (ka)^2 \frac{\kappa_1}{\kappa_7 + B/(ka)^2\kappa_8} + \frac{\kappa_2 + B\kappa_3}{\kappa_7 + B/(ka)^2\kappa_8} + \frac{1}{(ka)^2} \frac{(B^2\kappa_4 + B\kappa_5 + k^2C\kappa_6)}{\kappa_7 + B/(ka)^2\kappa_8} \\
&\quad - B \log a,
\end{aligned} \tag{3.106}$$

$$\begin{aligned}
p_0 &= (kb)^2 \frac{\kappa_1}{\kappa_7 + B/(kb)^2\kappa_8} + \frac{\kappa_2 + B\kappa_3}{\kappa_7 + B/(kb)^2\kappa_8} + \frac{1}{(kb)^2} \frac{(B^2\kappa_4 + B\kappa_5 + k^2C\kappa_6)}{\kappa_7 + B/(kb)^2\kappa_8} \\
&\quad - B \log b,
\end{aligned} \tag{3.107}$$

and therefore we may deduce

$$\begin{aligned}
C &= \frac{\kappa_1}{\kappa_6} k^2 a^2 b^2 + \frac{B}{k^2} \left(\frac{\kappa_1 \kappa_8}{\kappa_7 \kappa_6} k^2 (a^2 + b^2) + \frac{\kappa_2 \kappa_8}{\kappa_7 \kappa_6} - \frac{\kappa_5}{\kappa_6} - \frac{\kappa_7}{\kappa_6} \frac{k^2 a^2 b^2}{(b^2 - a^2)} \log(b/a) \right) \\
&\quad - \frac{B^2}{k^2} \left(\frac{\kappa_3 \kappa_8}{\kappa_7 \kappa_6} - \frac{\kappa_4}{\kappa_6} - \frac{\kappa_8}{\kappa_6} \frac{(b^2 + a^2)}{(b^2 - a^2)} \log(b/a) \right) - \frac{B^3}{k^4} \frac{k_8^2 \log(b/a)}{k_7 k_6 (b^2 - a^2)},
\end{aligned} \tag{3.108}$$

$$\begin{aligned}
p_0 &= \frac{\kappa_1}{\kappa_7} k^2 (a^2 + b^2) + \frac{\kappa_2}{\kappa_7} + B \left(\frac{\kappa_3}{\kappa_7} - \frac{(b^2 \log b - a^2 \log a)}{(b^2 - a^2)} \right) - \frac{B^2}{k^2} \frac{k_8 \log(b/a)}{k_7 (b^2 - a^2)},
\end{aligned} \tag{3.109}$$

as the two determining equations for C and p_0 .

It remains only to determine an expression for the total applied vertical force, which may be deduced from (3.96)₂ and the following equation, namely

$$F = \int_0^{2\pi} \int_a^b T_R^{33}(Z = \pm L) R dR d\Theta. \quad (3.110)$$

From (3.101) noting that $f_0(\pm L) = 1$, we may readily deduce from (3.110)

$$\begin{aligned} \frac{F}{2C_1} = & 2\pi \left\{ \frac{k^2}{8}(b^4 - a^4) - \left(p_0 - \frac{1}{2}\right) \frac{1}{2}(b^2 - a^2) + \frac{C}{2} \log(b/a) \right. \\ & \left. - \frac{B}{2} \left[b^2 \log b - a^2 \log a - \frac{(b^2 - a^2)}{2} \right] \right\}, \end{aligned} \quad (3.111)$$

and on using (3.108) and (3.109) we obtain

$$\begin{aligned} \frac{F}{2C_1} = & 2\pi \left[\frac{k^2}{8}(b^4 - a^4) - \left(k^2(a^2 + b^2) \frac{\kappa_1}{\kappa_7} + \frac{\kappa_2}{\kappa_7} - \frac{1}{2} \right) \frac{1}{2}(b^2 - a^2) \right. \\ & \left. + \frac{k^2}{2} a^2 b^2 \log(b/a) \frac{\kappa_1}{\kappa_6} + B\mu_1 + B^2\mu_2 + B^3\mu_3 \right], \end{aligned} \quad (3.112)$$

where μ_1 , μ_2 and μ_3 are constants defined by

$$\begin{aligned} \mu_1 = & \left[k^2(a^2 + b^2) \frac{\kappa_1 \kappa_8}{\kappa_7 \kappa_6} + \frac{\kappa_2 \kappa_8}{\kappa_6 \kappa_7} - \frac{\kappa_5}{\kappa_6} - \frac{k^2 a^2 b^2 \log(b/a)}{(b^2 - a^2)} \frac{\kappa_7}{\kappa_6} \right] \frac{\log(b/a)}{2k^2} \\ & + \frac{(b^2 - a^2)}{2} \left(\frac{1}{2} - \frac{\kappa_3}{\kappa_7} \right), \end{aligned} \quad (3.113)$$

$$\mu_2 = \frac{\log(b/a)}{2k^2} \left[\frac{\kappa_3 \kappa_8}{\kappa_7 \kappa_6} - \frac{\kappa_4}{\kappa_6} - \frac{\kappa_8}{\kappa_7} - \frac{\kappa_8}{\kappa_6} \frac{(b^2 + a^2)}{(b^2 - a^2)} \log(b/a) \right],$$

$$\mu_3 = -\frac{k_8^2 \log(b/a)}{2k^4 k_7 k_6 (b^2 - a^2)}.$$

In the above we have determined a one-parameter family of deformations characterized by the parameter B . However, if we impose the additional requirement that the value zero of the applied force F corresponds to the absence of any deformation, then, as shown below, we obtain three allowable values of B . However we comment that only the value $B = 0$ gives a physically reasonable response.

In the limit k tending to zero we have from (3.64) and (3.66),

$$f_0 \simeq 1, \quad f_1 \simeq \frac{B}{2}(Z^2 - L^2), \quad f_2 \simeq \frac{1}{2}(Z^2 - L^2) \left[C - \frac{B^2}{12}(Z^2 - 5L^2) \right], \quad (3.114)$$

and from (3.45), (3.46) and (3.47) we have,

$$g_0 \simeq Z, \quad g_1 = 0, \quad g_2 \simeq \frac{B^2}{6} \left(\frac{Z^5}{5} - L^4 Z \right) + \frac{C}{2} \left(\frac{Z^3}{3} - L^2 Z \right). \quad (3.115)$$

We note that in this limit, from (3.114) and (3.115) the deformation (3.3) becomes simply

$$\begin{aligned} r &= R + \frac{B}{2R}(Z^2 - L^2) + \frac{1}{2R^3}(Z^2 - L^2) \left[C - \frac{B^2}{12}(Z^2 - 5L^2) \right], \\ z &= Z + \frac{1}{2R^4} \left[C \left(\frac{Z^2}{3} - L^2 \right) + \frac{B^2}{3} \left(\frac{Z^4}{5} - L^4 \right) \right], \end{aligned} \quad (3.116)$$

giving rise to displacements which are asymptotic linear incompressible elastic deformations. On using (3.114) and (3.115) in (3.100) we can show that as k tends to zero we have from the condition (3.95),

$$p_0 \simeq \frac{1}{2} - B \frac{(b^2 \log b - a^2 \log a)}{b^2 - a^2} + B^2 \frac{L^2 \log(b/a)}{3(b^2 - a^2)}, \quad (3.117)$$

and

$$\begin{aligned} C &\simeq -2B \left(\frac{a^2 b^2 \log(b/a)}{b^2 - a^2} + \frac{2}{3} L^2 \right) + B^2 \frac{2L^2}{3} \left(\frac{b^2 + a^2}{b^2 - a^2} \log b/a - 1 \right) \\ &\quad - B^3 \frac{2L^4 \log(b/a)}{9(b^2 - a^2)}. \end{aligned} \quad (3.118)$$

Therefore to leading order we have

$$\begin{aligned} \frac{F}{2C_1} &\simeq 2\pi B \left[\left(\frac{b^2 - a^2}{4} - \frac{2L^2}{3} \log(b/a) - \frac{a^2 b^2}{b^2 - a^2} \log^2(b/a) \right) \right. \\ &\quad \left. - BL^2 \log(b/a) \left(\frac{1}{2} - \frac{b^2 + a^2}{3(b^2 - a^2)} \log(b/a) \right) - B^2 \frac{L^4 \log^2(b/a)}{9(b^2 - a^2)} \right], \end{aligned} \quad (3.119)$$

and hence F tends to zero with k only when $B = 0$ or when B is a root of the quadratic equation

$$\begin{aligned} B^2 \frac{L^4 \log^2(b/a)}{9(b^2 - a^2)} + BL^2 \log(b/a) \left(\frac{1}{2} - \frac{b^2 + a^2}{3(b^2 - a^2)} \log(b/a) \right) \\ - \left(\frac{b^2 - a^2}{4} - \frac{2L^2}{3} \log(b/a) - \frac{a^2 b^2}{b^2 - a^2} \log^2(b/a) \right) = 0. \end{aligned} \quad (3.120)$$

As shown in Appendix E , equation (3.120) has real solutions when

$$\frac{b}{a} > \frac{\sqrt{13}+3}{2}, \quad L^2 \leq \frac{3}{8} \left[(\sqrt{13}-3)b^2 - (\sqrt{13}+3)a^2 \right], \quad (3.121)$$

where $2L$ is the height and a and b are inner and outer radii respectively of the undeformed body. We comment that subsequently for the problem under consideration, numerical results indicate that only the case $B = 0$ provides a meaningful physical result.

3.7 Numerical results and conclusions

From the assumption that the bonded metal plate boundary conditions (3.85)₄, (3.85)₅ and (3.85)₆ may be replaced by the average requirement, namely

$$\int_A^B \left[\ell - g_0(L) - \frac{g_2(L)}{R^4} \right] R dR = 0, \quad (3.122)$$

which gives on simplification

$$\ell = g_0(L) + \frac{g_2(L)}{(ab)^2}, \quad (3.123)$$

we obtain the new deformed height 2ℓ . By substituting for $g_0(L)$ and $g_2(L)$ from (3.86)₂ and (3.94) respectively in (3.123) we obtain,

$$\begin{aligned} 2k\ell = & \frac{B^2 \cos^2 kL}{k^4 a^2 b^2} \left[(kL)^2 (kL - 3 \tan kL - kL \tan^2 kL) + 2k^2 I_1(L) (3 + 2kL \tan kL) \right] \\ & + \frac{C}{4k^2 a^2 b^2} (2kL \cos 2kL - \sin 2kL) + \sin 2kL, \end{aligned} \quad (3.124)$$

where $I_1(Z)$ is the integral defined by equation (3.93) and again we note that the constant C is given by (3.108), and either B is zero or given as a root of (3.120). On using the conditions (3.121) by taking the original dimensions of the cylinder as $a = b/4$ and $L = b/3$, from (3.120) we get the non zero values of B as 4.667 and -0.774. Now in order to plot the variation of the applied load $Fk^2/2C_1$ as a function of the displacement $\delta = (1 - \ell/L)$ for $B = 0, 4.667$ and -0.774 , we seek a set of values for kL from equation (3.124) corresponding to the deflections from $0.1L$ to $0.9L$. Figure 3.2 shows the right hand side of equation (3.124) and the left hand side of equation (3.124) corresponding to the deflections from $0.1L$ to $0.9L$ plotted as two separate functions of $2kL$ for the three value of B . The straight

lines shown represent the left hand side of equation (3.124) using $2k\ell = 2kL(1 - \delta)$ where δ is the deflection. We see that the physically acceptable values are given only when $B = 0$. Hence we conclude that the constant B is zero. For the two non-zero values of B , we find that the first term of the right hand side of equation (3.124) is considerably larger than the left hand side, and accordingly we have the situation shown in Figures 3.2b and 3.2c. The curves shown in these two figures are actually non-constant, but the variation is only apparent outside the shown range.

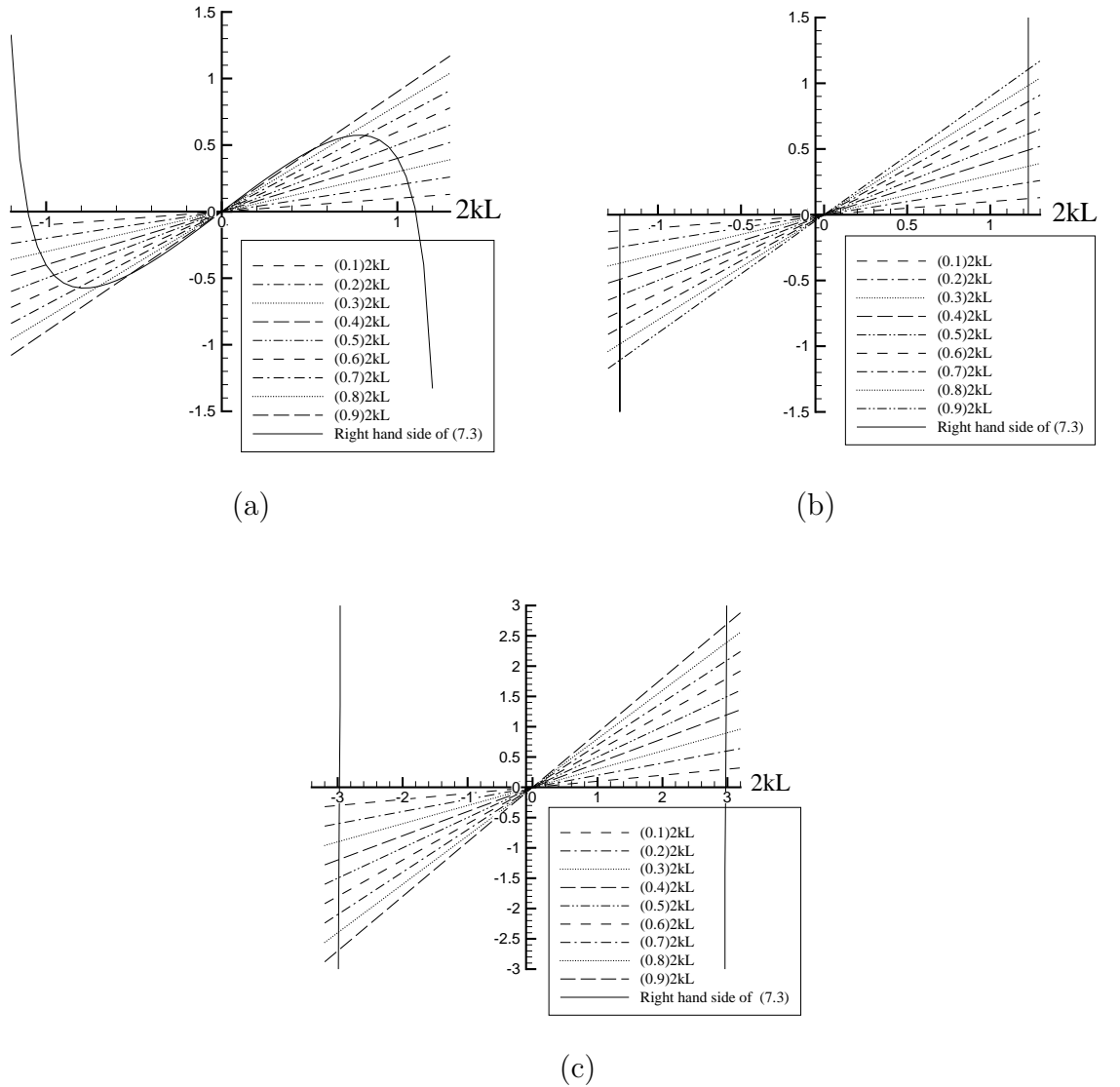


Figure 3.2: Right hand side and left hand sides of equation (3.124) versus $2kL$ for three values of B : (a) $B=0$, (b) $B=-0.640$ and (c) $B=6.415$

Therefore, in summary for the problem of the axial squashing of a hollow circular cylindrical tube with $B = 0$, we have from equations (3.3), (3.86), (3.89), (3.91) and

(3.94) that the deformation is approximated by

$$r = Rf_0(Z) + \frac{f_1(Z)}{R} + \frac{f_2(Z)}{R^3} + \dots, \quad (3.125)$$

$$z = g_0(Z) + \frac{g_1(Z)}{R^2} + \frac{g_2(Z)}{R^4} + \dots,$$

where the functions $f_i(Z)$ and $g_j(Z)$ ($j = 0, 1, 2$) are even and odd, respectively, and are given explicitly by

$$\begin{aligned} f_0(Z) &= \frac{\cos kZ}{\cos kL}, \quad f_1(z) = 0, \quad f_2(Z) = \frac{C \cos kL}{2k^2 \cos kZ} (\cos^2 kL - \cos^2 kZ), \\ g_0(Z) &= \frac{\cos^2 kL}{k} \tan kZ, \quad g_1(z) = 0, \\ g_2(Z) &= \frac{C}{2k^2} \left(\frac{\cos kL}{\cos kZ} \right)^4 \left[Z \cos 2kL - \frac{\sin 2kZ}{2k} \right], \end{aligned} \quad (3.126)$$

and from equation (3.112) the load-deflection relation becomes

$$F = \pi \left[\frac{k^2}{8} (b^4 - a^4) - \frac{1}{2} \left(k^2 (a^2 + b^2) \frac{\kappa_1}{\kappa_7} + \frac{\kappa_2}{\kappa_7} - \frac{1}{2} \right) (b^2 - a^2) + k^2 a^2 b^2 \frac{\kappa_1}{\kappa_6} \log(b/a) \right], \quad (3.127)$$

where $\kappa_1, \kappa_2, \kappa_6$ and κ_7 are constants defined in (6.17), namely

$$\begin{aligned} \kappa_1 &= \frac{s}{c}, \quad \kappa_2 = \frac{2s}{c} - \frac{sc}{4} - \frac{3}{8} sc^3 - \frac{3}{8} c^5 w, \quad \kappa_3 = 3cw - 3kL - ckI_2, \\ \kappa_4 &= kL(2T^*c^2 - c^2sw - c^4) + k^2 Lc^2 sI_2 + wc^3 + \frac{T^*c^3 s}{2} \\ &\quad + \frac{c^3}{2} (3kI_2 - 4kI_3) + \frac{k^2 L^2}{2} (c^3 s + 3s^3 c) + 2k^2 I_1(L) c^2 (c^2 + 2), \\ \kappa_5 &= kL(2 - \frac{c^2}{5} + \frac{c^2 s^2}{4} + \frac{3}{8} c^4 s^2 + \frac{3}{8} c^6 sw) \\ &\quad - (2cw + \frac{c^3 s}{80} + \frac{13}{160} c^5 s - \frac{47}{160} c^7 w + \frac{3}{16} kc^7 I_2), \\ \kappa_6 &= 2kL(2c^2 - 1) - wc^3, \quad \kappa_7 = 2cw, \quad \kappa_8 = 2kLc^2(1 - sw) - 2c^2 w + kc^3 I_2, \end{aligned}$$

and c, s , and w are defined by

$$c = \cos kL, \quad s = \sin kL, \quad w = \log(\sec kL + \tan kL),$$

and in the above formulae a and b denote the inner and outer radii, respectively, $2L$ is the original height of the tube and for an assumed deformed height 2ℓ , the constant k is determined from equation (3.124) with $B \equiv 0$, thus

$$2k\ell = C(2kL \cos 2kL - \sin 2kL)/2k^2 a^2 b^2 + \sin 2kL, \quad (3.128)$$

where $C = k_1^2 k^2 a^2 b^2 / k_6$. Figure 3.3 shows the variation of non-dimensional force against the deflection for $B = 0$ for $a = L/2$ and $b = L, 2L, 3L$ and $4L$. The shown curves have an initial linear-quadratic response and achieve a maximum in the vicinity of the maximum deflection. Although nonlinear the response appears not to predict an intermediate maximum such as occurs when blowing up a balloon. In the absence of any accurate experimental data, the predicted response appears to be physically reasonable.

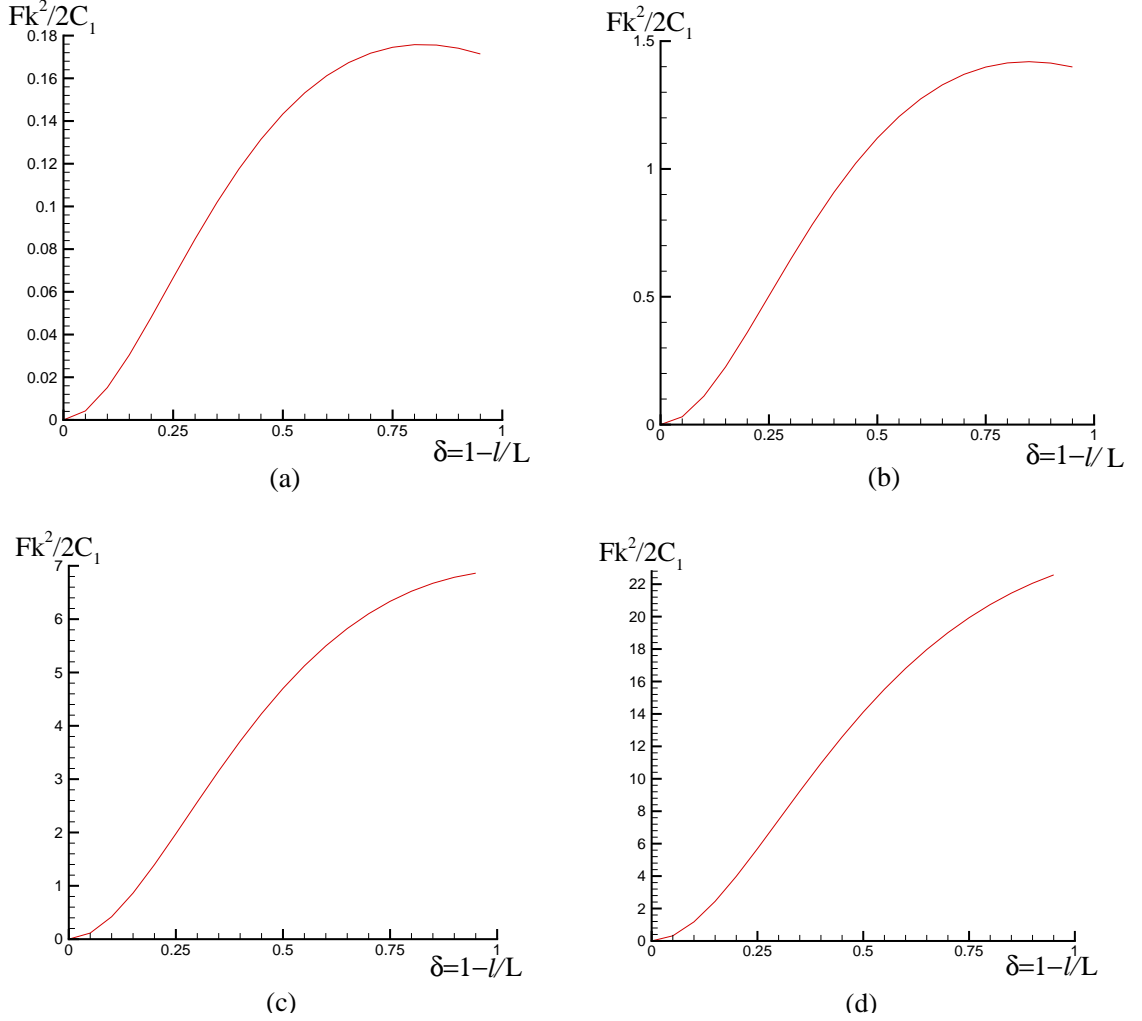


Figure 3.3: Non-dimensional force $Fk^2/2C_1$ versus non-dimensional deflection $\delta = 1 - \ell/L$ for $B=0$ and $a=L/2$: (a) $b=L$, (b) $b=2L$, (c) $b=3L$ and (d) $b=4L$

In conclusion, we have determined asymptotic axially symmetric deformations applicable for hollow cylinders and applying to the neo-Hookean and Mooney perfectly elastic incompressible materials. From the incompressibility condition and the equilibrium equations, we show that formal solutions for the first three terms

can be expressed in terms of seven integration constants and five formal integrations, making a total of twelve integration constants. For the particular case of the neo-Hookean strain-energy function we exploit these solutions to determine an approximate load-deflection relation for the squashing under axially symmetric conditions of a thick walled hollow cylinder by equal and opposite forces applied to bonded metal end plates. The resulting load-deflection curves are shown graphically in Figure 3.3.

Chapter 4

Rippling of long rectangular rubber blocks under bending

4.1 Introduction

In this chapter we investigate the problem of finite elastic deformation of a long rectangular rubber block which is deformed in a perturbed cylindrical configuration with the point of view of surface rippling. Motivated by the surface rippling observed in bent multi-walled carbon nanotubes, we examine here the simpler block problem which is assumed to be sufficiently long so that the out of plane end effects may be ignored. The general equations governing plane strain deformations of an isotropic incompressible perfectly elastic Mooney material, which models rubber like materials, are used to determine small superimposed deformations upon the well known controllable family for the deformation of rectangular blocks into a sector of a solid bounded by two circular arcs. Traction free boundary conditions are assumed in an average sense along the bounding circular arcs.

Studies have shown that due to their superior mechanical properties, carbon nanotubes exhibit many new mechanical phenomena, and in particular exhibit almost perfect elasticity and extraordinarily high strength. Moreover, from experiments surface rippling is observed when the nanotube is subjected to bending [22]. Further, Liu et al [23] use a model nanobeam subjected to pure bending to confirm a transition from the classical bending mode predicted by linear theory, to a rippling bending mode under severe bending. Their analysis indicates that the bending moment and the bending curvature have a bilinear relation, in which the transition corresponds to the emergence of a rippling mode. In another study they present a nonlinear vibration analysis which suggests that the effective Young's modulus

drops sharply as the diameter increases upon the emergence of a rippling mode [22], which is also confirmed by calculations in [24]. A rippling bending mode is also observed by Ruoff and Lorents [25], Kuzumaki et al [26] and Poncharal et al [27]. The analysis presented by Mahadevan et al [28] indicates that the rippling instability is not unique to the bent multi-walled nanotubes, but also when a rubber tube made by rolling a thin sheet of rubber into a scroll is bent, a similar rippling instability is formed. Mahadevan et al [28] verify their results experimentally using materials as disparate as rubber and graphite, for varying lengths ranging from millimeters to nanometers. Their study concludes that the rippling pattern is independent of the material properties but depends on the effective anisotropy of the system generated by the layered structure, suggesting similar behavior can be expected in macroscopic systems with layered structures. Since carbon nanotubes are known to be perfectly elastic, in the present study we examine the related problem which is mathematically much simpler, of the finite deformations of a rectangular rubber block.

The surface rippling of blocks and rods under bending has received attention in the literature. For example, Gent et al [29] studied the instabilities of thick rubber cuboids subjected to bending. They compared the experimental values of the critical degree of bending, at which the sharp folds appear, with Biot's [30] theoretical predictions. Gent et al [29] show that the on-set of surface instability occurs at a much less severe degree of bending than predicted by Biot [30]. Haughton [13] considered the bifurcation problem for three-dimensional, incompressible elastic plates subjected to a combined flexure and axial compression. By forming a right circular cylinder, the pure bending mode as well as the interaction of the buckling and barreling modes due to the curvature are studied. The bifurcation problem of an incompressible plate under pure bending is studied by Traintafyllidis [12]. In a recent study Ghatak et al [31] carry out an experimental analysis of soft elastic cylinders of different diameters subjected to bending and axial compression. They observe a sharp fold on the compressive side at a critical radius which increases linearly with the diameter, but remains independent of the material properties.

In the present study we consider a rectangular rubber block which is initially deformed into a circular arc, and then subjected to a small superimposed deformation. For material rectangular Cartesian coordinates (X, Y, Z) , spatial cylindrical

polar coordinates (r, θ, z) , and a positive constant λ we consider the plane strain deformation

$$r = r(X, Y), \quad \theta = \theta(X, Y), \quad z = \lambda Z, \quad (4.1)$$

for an isotropic incompressible hyperelastic Mooney material. We suppose that a long rectangular rubber block, assumed to be an isotropic incompressible hyperelastic Mooney material, is bent into a circular arc by symmetrically applied loads and moments at the ends. The block is assumed to be sufficiently long in the Z -direction so that the longitudinal end effects may be ignored. We then superimpose a small deformation on the initial exact deformation such that the angle subtended at the center is fixed, and we look for solutions of the resulting equations.

In the following section we detail the geometry of the block and describe the initial deformation, and in the section thereafter we present the basic equations governing the deformation (4.1). The equations specific to the deformation (4.34) and the radial deformation that is superimposed upon (4.34) are described in the subsequent section. In Section 4.5 we use the load deflection relationships to determine the constants appearing in the above equations which are useful in solving our problem. We finally illustrate our solution with typical numerical values in Section 4.6. Appendix G presents certain analytical details required for the solution derived in Section 4.5.

4.2 Geometry of the deformation

An undeformed rectangular rubber block of length L , height $2h$ and thickness t which is assumed to be an isotropic incompressible perfectly elastic Mooney material, is deformed into a circular arc as shown in Figure 4.1. We suppose that the $X = \text{constant}$ planes become $r = \text{constant}$ and $Y = \text{constant}$ planes become $\theta = \text{constant}$ after the deformation. If the arc subtends an angle $2\theta_0$ at the center, and the inner and outer radii are given by a and b respectively, then the deformation is described by (4.1) for which $r(X, Y)$ and $\theta(X, Y)$ admit the well known form,

$$r = \left[b^2 - (b^2 - a^2) \frac{X}{t} \right]^{1/2}, \quad \theta = \pi - \frac{\theta_0}{h} Y. \quad (4.2)$$

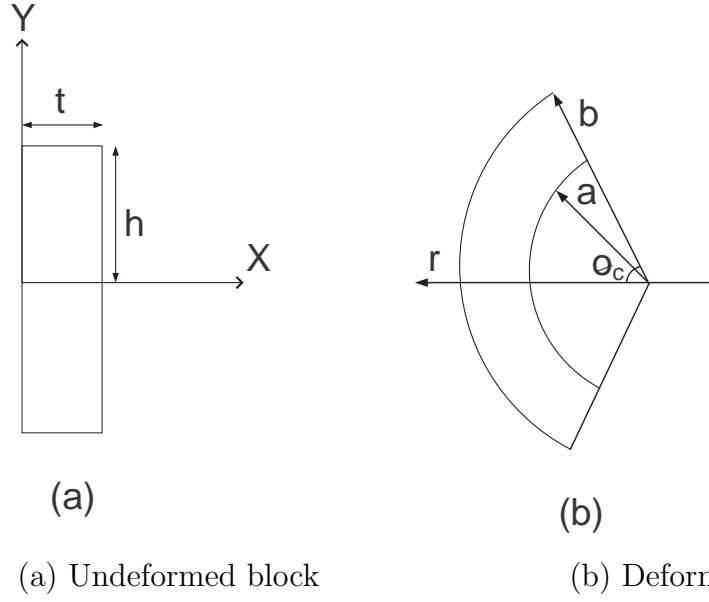


Figure 4.1: Original and deformed body of the rubber block subjected to bending

From the incompressible constraint we find that the new length is λL where λ is given by

$$\lambda = \frac{2th}{\theta_0(b^2 - a^2)}. \quad (4.3)$$

4.3 Basic equations for perfectly elastic materials

This section summarizes the basic equations governing the deformation field given by equation (4.1) that are utilized here in solving the present problem. The basic theory presented in Chapter 2 for perfectly elastic incompressible material are simplified here for the problem of axially symmetric plane strain deformation and are available in literature for a more informative derivation. For more details on these equations, the reader is referred to Hill [32] .

The coordinates of the undeformed body $X^K (K = 1, 2, 3)$ are the rectangular coordinates X, Y, Z and those of the deformed body $x^i (i = 1, 2, 3)$ are the cylindrical polar coordinates (r, θ, z) , viz:

$$\begin{aligned} X^1 &= X, & X^2 &= Y, & X^3 &= Z, \\ x^1 &= r \cos \theta, & x^2 &= r \sin \theta, & x^3 &= z, \end{aligned} \quad (4.4)$$

where the corresponding rectangular cartesian coordinates are given by $Z^M (M = 1, 2, 3)$ and $z^m (m = 1, 2, 3)$ respectively.

It is understood that in the following discussions the labeling indices a,b A and B represent the values 1 and 2 only.

4.3.1 Metric tensors

From (2.9), for the deformation (4.1) the material and spatial metric tensors are given respectively by

$$G_{KL} = G^{KL} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (4.5)$$

and

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.6)$$

Further due to (2.10)₂ spatial conjugate metric tensor is given by

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.7)$$

4.3.2 Deformation tensors

Cauchy deformation tensor is obtained for the deformation (4.1) from (2.15)₂ to give

$$c_{ij} = \begin{bmatrix} X_r^2 + Y_r^2 & X_r X_\theta + Y_r Y_\theta & 0 \\ X_r X_\theta + Y_r Y_\theta & X_\theta^2 + Y_\theta^2 & 0 \\ 0 & 0 & 1/\lambda^2 \end{bmatrix}, \quad (4.8)$$

while the Green deformation tensor is given by

$$C_{KL} = \begin{bmatrix} r_X^2 + r^2 \theta_X^2 & r_X r_Y + r^2 \theta_X \theta_Y & 0 \\ r_Y r_X + r^2 \theta_Y \theta_X & r_Y^2 + r^2 \theta_Y^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}. \quad (4.9)$$

Due to (2.19) Finger deformation tensor can be shown to become

$$c^{-1ij} = \begin{bmatrix} r_X^2 + r_Y^2 & r_X \theta_X + r_Y \theta_Y & 0 \\ r_X \theta_X + r_Y \theta_Y & \theta_X^2 + \theta_Y^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \quad (4.10)$$

and from (2.21) the corresponding mixed tensor is given by

$$\bar{c}^{-1i}_{\ j} = \begin{bmatrix} r_X^2 + r_Y^2 & r^2(r_X\theta_X + r_Y\theta_Y) & 0 \\ r_X\theta_X + r_Y\theta_Y & r^2(\theta_X^2 + \theta_Y^2) & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}. \quad (4.11)$$

4.3.3 Strain invariants

The perfectly elastic Mooney material has strain-energy function $\sum(I_1, I_2)$ given by

$$\sum(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3), \quad (4.12)$$

where I_1 and I_2 denote the first two invariants of the inverse Cauchy deformation tensor. Noting that for incompressible materials $J = 1$, due to (2.26), as shown in Section 3.2.3 the principle invariants I_1, I_2 and I_3 become

$$I_1 = I + \lambda^2, \quad I_2 = \lambda^2 I + \frac{1}{\lambda^2}, \quad I_3 = 1, \quad (4.13)$$

where I is defined by

$$I = r_X^2 + r_Y^2 + r^2(\theta_X^2 + \theta_Y^2), \quad (4.14)$$

and subscripts denote partial derivatives.

4.3.4 Incompressibility condition

Owing to (2.2)

$$j = \lambda \frac{\partial(r, \theta)}{\partial(X, Y)}, \quad (4.15)$$

and as before from (2.13)

$$g = r^2, \quad G = 1. \quad (4.16)$$

Hence from (4.15) and (2.12) the incompressibility condition becomes

$$\frac{\partial(r, \theta)}{\partial(X, Y)} = r_X\theta_Y - r_Y\theta_X = \frac{1}{\lambda r}. \quad (4.17)$$

4.3.5 Equilibrium equations

For deformation (4.1) due to (2.65) the stress tensor becomes

$$\begin{aligned} T^{ab} &= p^* g^{ab} + 2 \frac{\partial \Sigma}{\partial I_1} c^{-1ab} - 2 \frac{\partial \Sigma}{\partial I_2} c^{ab} \\ T^{33} &= p^* + 2 \frac{\partial \Sigma}{\partial I_1} \lambda^2 - 2 \frac{\partial \Sigma}{\partial I_2} \frac{1}{\lambda^2} \end{aligned} \quad (4.18)$$

$$T^{3a} = T^{a3} = 0.$$

where p^* is an arbitrary function referred to in Chapter 2. Now as shown in Chapter 3 Section 2.4 we obtain

$$\begin{aligned} T^{ab} &= -p \delta^{ab} + \phi c^{-1ab} \\ T^{33} &= -p + \psi \lambda^2 \\ T^{3a} &= 0 \end{aligned} \quad (4.19)$$

where we define the functions

$$\begin{aligned} p &= p^* + 2\lambda^2 I \frac{\partial \Sigma}{\partial I_2}, \\ \phi &= 2 \left(\frac{\partial \Sigma}{\partial I_1} + \lambda^2 \frac{\partial \Sigma}{\partial I_2} \right), \\ \psi &= \left[\frac{\partial \Sigma}{\partial I_1} + \left(I - \frac{1}{\lambda^4} \right) \frac{\partial \Sigma}{\partial I_2} \right]. \end{aligned} \quad (4.20)$$

We therefore have the stress components for deformation (4.1) given by

$$\begin{aligned} T^{11} &= -p^* + 2C_1(r_X^2 + r_Y^2) - 2C_2(X_r^2 + Y_r^2), \\ T^{22} &= -\frac{p^*}{r^2} + 2C_1(\theta_X^2 + \theta_Y^2) - 2C_2 \frac{(X_\theta^2 + Y_\theta^2)}{r^4}, \\ T^{33} &= -p^* + 2C_1 \lambda^2 - \frac{2C_2}{\lambda^2}, \\ T^{12} &= T^{21} = 2C_1(r_X \theta_X + r_Y \theta_Y) - \frac{2C_2}{r^2} (X_r X_\theta + Y_r Y_\theta), \\ T^{31} &= T^{13} = 0, \quad T^{32} = T^{23} = 0. \end{aligned} \quad (4.21)$$

Now owing to (2.40) for incompressible material we have the first Piola-Kirchoff stress tensors given by

$$T_R^{Ab} = X^A{}_{;a} T^{ab}, \quad T_R^{33} = X^3 T^{33} = T^{33}/\lambda, \quad (4.22)$$

and from the equilibrium equation (2.55), since $T^{3a} = 0$ we have

$$T_R^{Aa}{}_{;A} = 0, \quad T_R^{33}{}_{;3} = 0. \quad (4.23)$$

Further from (4.23)₂ it is noted that

$$p = p(x^1, x^2). \quad (4.24)$$

Next due to Euler-C.Nuemann identity (2.54) we obtain

$$[X^A{}_{;a}]_{;A} = 0 \quad [X^3{}_{;3}]_{;3} = 0, \quad (4.25)$$

and on using (4.19) in (4.22)₁ we may readily deduce

$$T_R^{Ab} = -pX^A{}_{;a} g^{ab} + \phi G^{AB} x^b{}_{;B}. \quad (4.26)$$

Now carrying the above result into (4.23)₁ yields

$$-p_{;A} X^A{}_{;a} g^{ab} + \phi_{;A} G^{AB} x^b{}_{;B} + \phi G^{AB} [x^b{}_{;B}]_{;A} = 0. \quad (4.27)$$

However (2.36) is reduced to give

$$G^{AB} [x^b{}_{;B}]_{;A} = \nabla^2 x^b + \Gamma_{ad}^b c^{-1ad}, \quad (4.28)$$

and therefore combining (4.28) and (4.27) we obtain

$$p_{;a} g^{ab} = \phi (\nabla^2 x^b + \Gamma_{ad}^b c^{-1ad}) + G^{AB} \phi_{;A} x^b{}_{;B}, \quad (4.29)$$

where

$$\nabla^2 = G^{AB} \left(\frac{\partial^2}{\partial X^A \partial X^B} - \Gamma_{AB}^D \frac{\partial}{\partial X^D} \right). \quad (4.30)$$

Noting that for orthogonal coordinates the only non zero Christoffel symbols Γ_{ij}^r are

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad (4.31)$$

the equilibrium equations simplify to give

$$p_r = \mu [\nabla^2 r - r(\theta_X^2 + \theta_Y^2)], \quad p_\theta = \mu r^2 \left[\nabla^2 \theta + \frac{2}{r}(r_X \theta_X + r_Y \theta_Y) \right], \quad (4.32)$$

where $\mu = 2(C_1 + \lambda^2 C_2)$, ∇^2 is the two-dimensional Laplacian given by

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}, \quad (4.33)$$

and p is the modified pressure function defined above.

Although the terms r_Y and θ_X are readily zero for deformation (4.2) we have included them here as they are non-zero for deformation (4.39).

4.4 Governing equations for the deformation (4.1)

For convenience we write the deformation (4.2) as

$$r = (AX + B)^{1/2}, \quad \theta = CY + D, \quad z = \lambda Z, \quad (4.34)$$

where A, B, C and D are constants which are readily identified from (4.2), namely

$$A = -\frac{(b^2 - a^2)}{t}, \quad B = b^2, \quad C = -\frac{\theta_o}{h}, \quad D = \pi. \quad (4.35)$$

It is easily seen that the pressure function can be expressed as

$$p_r = \lambda C r p_X, \quad p_\theta = \frac{1}{C} p_Y, \quad (4.36)$$

and we also have from (4.32)₁ and (4.32)₂ that the pressure function given by

$$p_r = -\mu \left(\frac{A^2}{4r^3} + rC^2 \right), \quad p_\theta = 0. \quad (4.37)$$

Hence combining the above results (4.36) and (4.37) we obtain

$$p_0 = \frac{\mu A}{2} \left\{ \frac{A}{4(AX + B)} - C^2 X \right\} + \sigma, \quad (4.38)$$

where σ is a constant. Now, if due to a further small end moment $\epsilon \Delta M$ the curve which was originally given by the straight line $X = \text{constant}$ is further displaced by a small distance $\epsilon u(x)$ in the direction of $\theta = 0$, we suppose that the deformation (4.34) and the pressure function (4.38) can be modified to become [10]

$$\begin{aligned} r &= (AX + B)^{1/2} + \epsilon u(x) \cos n(CY + D), \\ \theta &= CY + D + \epsilon v(x) k^{1/2} \sin n(CY + D), \\ p &= p_0 + \epsilon q(x) A^2 k^{3/2} \cos n(CY + D), \end{aligned} \quad (4.39)$$

where $k = C/A$, n is a positive number and u, v and q are functions of x only, which is defined by

$$x = k(AX + B). \quad (4.40)$$

The equations (4.39) form the basis of an incremental stability analysis, in the spirit of Biot [30] and many other authors. By substituting (4.39)₁ in (4.17) and neglecting the higher order terms of ϵ we may deduce

$$x^{1/2}u' + \frac{u}{2x^{1/2}} + \frac{nv}{2} = 0. \quad (4.41)$$

It is noted that due to (4.40) $X' = 1/C$, where primes denote differentiation with respect to x . Again by expressing the pressure functions in the form

$$\begin{aligned} p_r &= \lambda r \frac{\partial(p, \theta)}{\partial(X, Y)}, \\ p_\theta &= -\lambda r \frac{\partial(p, r)}{\partial(X, Y)}, \end{aligned} \quad (4.42)$$

and combining with equation (4.32) and substituting from (4.39) we may yield

$$\begin{aligned} 2x^{1/2}q' &= \mu \left[u'' + (1 - n^2)u + \left(2x - \frac{1}{2x} \right) u' \right], \\ qn &= -\mu x \left[v'' + \frac{v'}{x} + \frac{nu}{4x^{5/2}} + 2nx^{1/2}u' \right]. \end{aligned} \quad (4.43)$$

It is understood that the higher order terms of ϵ are neglected when deriving the above expressions. Next on using (4.41) and (4.43) we eventually obtain a fourth order differential equation for $u(x)$, namely

$$xu'''' + 4u''' - \left\{ \left(\frac{n^2}{4} - \frac{3}{2} \right) \frac{1}{x} + n^2x \right\} u'' - 2n^2u' - \frac{n^2(1 - n^2)}{4x}u = 0. \quad (4.44)$$

As shown in Appendix G, the solution for $u(x)$ can be given by

$$u(x) = \sum_{i=1}^4 c_i x^{m_i} u_i(x), \quad (4.45)$$

where for $i = 1, 2, 3, 4$, c_i denote four arbitrary constants, m_i are numbers given by

$$m_1 = 0, \quad m_2 = 1, \quad m_3 = \frac{1}{2}(1 - N), \quad m_4 = \frac{1}{2}(1 + N), \quad (4.46)$$

and $u_i(x)$ are functions of x defined by

$$\begin{aligned}
u_1(x) &= {}_2F_3 \left(\frac{1}{4}(1-n), \frac{1}{4}(1+n); \frac{1}{2}, \frac{1}{4}(3-N), \frac{1}{4}(3+N); \frac{1}{4}x^2n^2 \right), \\
u_2(x) &= {}_2F_3 \left(\frac{1}{4}(3-n), \frac{1}{4}(3+n); \frac{3}{2}, \frac{1}{4}(5-N), \frac{1}{4}(5+N); \frac{1}{4}x^2n^2 \right), \\
u_3(x) &= {}_2F_3 \left(\frac{1}{4}(2-n-N), \frac{1}{4}(2+n-N); \right. \\
&\quad \left. \frac{1}{4}(2-N), \frac{1}{4}(3-N), \frac{1}{4}(5-N); \frac{1}{4}x^2n^2 \right), \\
u_4(x) &= {}_2F_3 \left(\frac{1}{4}(2-n+N), \frac{1}{4}(2+n+N); \right. \\
&\quad \left. \frac{1}{4}(2+N), \frac{1}{4}(3+N), \frac{1}{4}(5+N); \frac{1}{4}x^2n^2 \right), \tag{4.47}
\end{aligned}$$

with $N = (3+n^2)^{1/2}$ and ${}_2F_3$ denotes the usual generalization of the hypergeometric function [33] which is formally defined by the series

$${}_2F_3(\alpha_1, \alpha_2; \rho_1, \rho_2, \rho_3; z) = \sum_{n=1}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n z^n}{(\rho_1)_n (\rho_2)_n (\rho_3)_n n!}. \tag{4.48}$$

Here $\alpha_1, \alpha_2, \rho_1, \rho_2$ and ρ_3 are parameters and $(\alpha)_n$ is the Pochhammer symbol which is defined by $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$, for $n = 1, 2, \dots$. We note that for $n = 1$ (4.6) agrees with the solutions given in Haughton [13] and Hill [10].

4.5 Load-deflection relations

For the deformation (4.2) we have the stress components given by

$$\begin{aligned}
T_1^1 &= -p^* + C_1 \frac{A^2}{2r^2} - \frac{8C_2 r^2}{A^2} \\
T_2^2 &= -p^* + 2C_1 C^2 r^2 - \frac{2C_2}{c^2 r^2} \\
T_3^3 &= -p^* + 2C_1 \lambda^2 - \frac{2C_2}{\lambda^2} \\
T_2^1 &= T_1^2 = 0 \\
T_1^3 &= T_3^1 = 0 \quad T_2^3 = T_3^2 = 0 \tag{4.49}
\end{aligned}$$

We suppose that the deformation (4.2) is produced by symmetrical applied end loadings of force F and moment M . The midplane symmetry and equilibrium require that the curved surfaces to be traction free, that is,

$$t_1^1(a) = 0, \quad t_1^1(b) = 0, \tag{4.50}$$

where t_j^i ($i, j = 1, 2, 3$) are the physical components of the stress tensor in the usual notation. Now due to (4.14) we have I given by

$$I = \frac{A^2}{4r^2} + r^2 C^2. \quad (4.51)$$

Carrying this into (4.20)₁ we have modified pressure function p related to pressure function p^* by

$$p = p^* + 2\lambda^2 \left(\frac{A^2}{4r^2} + r^2 C^2 \right). \quad (4.52)$$

On using boundary condition (4.50)₁ and substituting for p^* from (4.52) and (4.38) the constant σ can be shown to be given by

$$\sigma = \mu A^2 / 8B. \quad (4.53)$$

Further, the boundary condition (4.50)₂ requires that the constants A and C to be related by

$$A = 2abC. \quad (4.54)$$

The condition (4.50)₁ also reveals that the forces F at the surfaces $\theta = \pi \pm \theta_0$ vanish and we see that the equation

$$F(\pi \pm \theta_0) = \int_a^b t_2^2 dr, \quad (4.55)$$

is correctly satisfied. The moment M applied at the ends is given by

$$M = \int_a^b t_2^2 r dr, \quad (4.56)$$

from which we may obtain

$$M = \frac{\mu C^2}{2} \left[\frac{(b^4 - a^4)}{4} - a^2 b^2 \log \left(\frac{b}{a} \right) \right]. \quad (4.57)$$

Next we suppose that a further small moment $\epsilon \Delta M$ is applied at the ends of the block and the corresponding deformation is given by (4.39). Again since the deformation is produced by end loads alone, the curved surfaces initially given by $r = a$ (or $X = t$) and $r = b$ (or $X = 0$) are to be traction free. Hence we have the first Piola-Kirchoff stress tensors denoted by t_R^{Xx} and t_R^{Xy} vanishing on the boundaries $X = 0$ and $X = t$. Subsequently, we find that it is not possible to satisfy the above boundary conditions in a point-wise manner. Therefore, we attempt to satisfy these conditions in an average sense, by replacement with the following integral requirements, namely

$$\int_0^L \int_{-h}^h t_R^{Xx}(X = 0, t) dY dZ = 0, \quad \int_0^L \int_{-h}^h t_R^{Xy}(X = 0, t) dY dZ = 0. \quad (4.58)$$

From (4.17) we deduce the results

$$\begin{aligned} X_r &= \lambda r \theta_Y, & X_\theta &= -\lambda r r_Y, \\ Y_r &= -\lambda r \theta_X, & Y_\theta &= \lambda r r_X, \end{aligned} \quad (4.59)$$

and the physical stress components are related by

$$\begin{aligned} T_{(r)}^{(X)} &= T^{Xr}, & T_{(\theta)}^{(X)} &= r T^{X\theta}, \\ T_{(\theta)}^{(Y)} &= r T^{Y\theta}, & T_{(r)}^{(Y)} &= T^{Yr}. \end{aligned} \quad (4.60)$$

On using equations (4.26) and the results given in (3.32), from (2.40) the first Piolar-Kirchoff stress components are obtained to give

$$\begin{aligned} T_{R(r)}^{(X)} &= -\lambda p r \theta_Y + \mu r_X, \\ T_{R(\theta)}^{(X)} &= \lambda p r_Y + \mu r \theta_X, \\ T_{R(r)}^{(Y)} &= \lambda p r \theta_X + \mu r_Y, \\ T_{R(\theta)}^{(Y)} &= -p \lambda r_X + \mu r \theta_Y, \end{aligned} \quad (4.61)$$

which by substituting from (4.39) yields

$$\begin{aligned} T_{R(r)}^{(X)} &= \mu C \left[\left(-f \sqrt{\frac{x}{k}} + \frac{1}{2\sqrt{xk}} \right) - \varepsilon (2p\sqrt{x} - 2fxu' - u') \cos n\theta^* \right], \\ T_{R(\theta)}^{(X)} &= \mu C \varepsilon (v' \sqrt{x} - fnu) \sin n\theta^*, \\ T_{R(r)}^{(Y)} &= \mu C \varepsilon (fv' \sqrt{x} - nu) \sin n\theta^*, \\ T_{R(\theta)}^{(Y)} &= \mu C \left[\left(\frac{-f}{2\sqrt{xk}} + \sqrt{\frac{x}{k}} \right) + \varepsilon \left(\frac{-p}{\mu\sqrt{x}} - u'(f + 2x) \right) \cos n\theta^* \right], \end{aligned} \quad (4.62)$$

where $f = (1/4x - x - \alpha + 1/4\alpha)$ and $\theta^* = CY + D$. Further we have the first Piolar-Kirchoff stress components expressed by

$$\begin{aligned} T_R^{Xx} &= T_{R(r)}^{(X)} \cos \theta - T_{R(\theta)}^{(X)} \sin \theta, \\ T_R^{Xy} &= T_{R(\theta)}^{(X)} \cos \theta + T_{R(r)}^{(X)} \sin \theta. \end{aligned} \quad (4.63)$$

Now upon substituting from equations (4.39) and (4.62) in (4.63) and utilizing the four boundary conditions (4.58) we obtain

$$A^*(\alpha) = 0, \quad A^*(\beta) = 0, \quad B^*(\alpha) = 0, \quad B^*(\beta) = 0, \quad (4.64)$$

where $A^*(x)$ and $B^*(x)$ are functions defined by

$$A^*(x) = u' - \frac{q\sqrt{x}}{\mu}, \quad B^*(x) = \frac{nu}{2x} - v'\sqrt{x}, \quad (4.65)$$

$c_i (i = 1, 2, 3, 4)$ are constants and α and β are defined by

$$\alpha = \frac{\theta_0 tb^2}{h(b^2 - a^2)}, \quad \beta = \frac{\theta_0 ta^2}{h(b^2 - a^2)}. \quad (4.66)$$

Here we have used equations (4.54) and (4.53) respectively to substitute for constants A and σ appearing in (4.64). We remark that the integrals involve in the above evaluation vanish only for the non integer values of n and subsequently the results in (4.64) hold true for non integer values of n only. It is also noted that from (4.66) and (4.54) that

$$\beta = 1/4\alpha. \quad (4.67)$$

By substituting for v and q from (4.41) and (4.43) respectively, into (4.64) and carrying the value of u given in (4.45), we obtain the homogeneous system of equations given by

$$\mathbf{A} \cdot \mathbf{c} = \mathbf{0}, \quad (4.68)$$

where

$$\mathbf{A} = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & A_3(\alpha) & A_4(\alpha) \\ A_1(\beta) & A_2(\beta) & A_3(\beta) & A_4(\beta) \\ B_1(\alpha) & B_2(\alpha) & B_3(\alpha) & B_4(\alpha) \\ B_1(\beta) & B_2(\beta) & B_3(\beta) & B_4(\beta) \end{pmatrix}, \quad (4.69)$$

and

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}. \quad (4.70)$$

Here A_i and B_i are the terms associated with c_i in (4.65)₁ and (4.65)₂ respectively for $i = 1, 2, 3, 4$. For non trivial solutions of $c_i (i = 1, 2, 3, 4)$ we require that

$$\det \mathbf{A} = 0. \quad (4.71)$$

In the next section we present some typical numerical values that have been calculated using the mathematical software MAPLE.

4.6 Numerical results

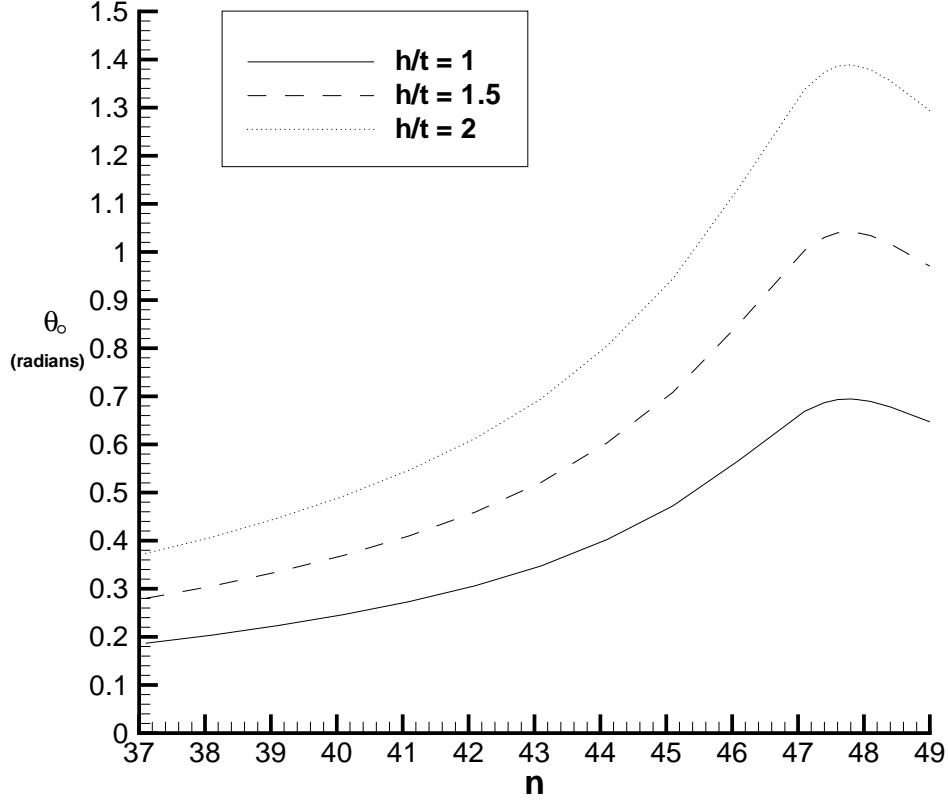


Figure 4.2: Variation of θ_0 against n for $h/t=1, 1.5$ and 2

By substituting from (4.67) for β we seek solutions for α satisfying equation (4.71). Computation is performed by MAPLE by using certain numerical values for n . We note that the condition $\alpha > \beta$ requires α to be greater than 0.5. Figure 4.2 shows a graph of the values of θ_0 against the number n for different dimensions of the block with the view of finding the critical degree of bending. Figure 4.3 shows the variation of α with n for physically acceptable values. In fact we can obtain more than one solution for α . However based on the behavior of function $u(x)$ we neglect those solutions as they do not have a physical interpretation. We observe that the smaller values of n do not satisfy the equation (4.71) for any geometry of the block and therefore we assume that no rippling occurs at these smaller values of n . We also note that α exhibits a relative maximum for $n \simeq 47.8$. Thus we may conclude that rippling start to occur at a very small degree of bending, which may be as small as

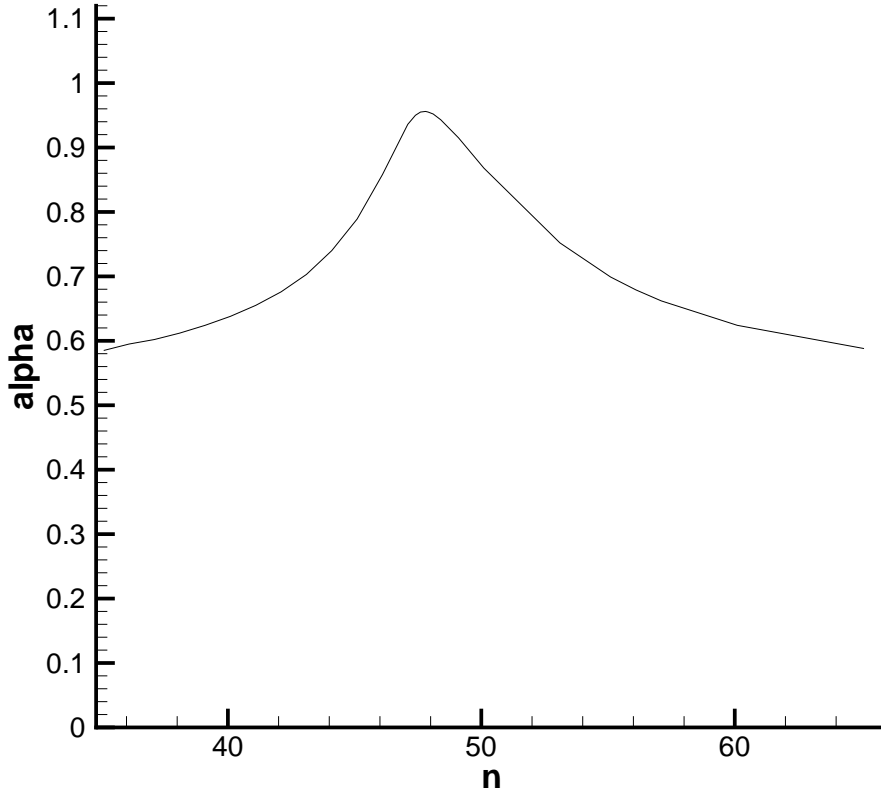


Figure 4.3: Variation of $\alpha = \theta_0 t b^2 / h(b^2 - a^2)$ against n

20 times h/t degrees corresponding to $n = 37.1$ and continue to have more ripples until around 80 times h/t degrees of bending at $n = 47.8$. We assume that the solutions of larger n values (greater than 47.8) are only mathematical possibilities that are not physically reliable.

In order to illustrate the deformation (4.39) graphically the four constants c_i ($i = 1, 2, 3, 4$) are determined from (4.68) and some typical numerical values are used for t^* , a and ϵ where $t^* = t/h$. We have $b^* = 2\alpha$ where $b^* = b/a$ and γ given by $\gamma = \alpha - 1/4\alpha$ where $\gamma = \theta_0 t^*$. Figures 4.4, 4.5 and 4.6 show the deformed profile of a surface initially deformed according to $r = (AX + B)^{1/2}$ at intervals of one tenth of t , and corresponding to the numerical values $t^* = 0.5$, $a = 3$ and $\epsilon = 0.001$ for $n = 55.1$, 50.1 and 45.1 respectively. It is noted that equation (4.71) is satisfied for non integral values for n . For integral values of n we get more complex set of equations making it untractable to carry out the numerical work using MAPLE, because for both odd and even integers of n , the hypergeometric functions may

terminate and exhibit a numerical difficulty.

In summary, our analysis shows that when a long rectangular rubber block is bent in the form of an approximately circular arc, a small superimposed end moment can exhibit wavelike distortions or ripples. As might be anticipated the ripples are formed along the circular arc with decreasing wave lengths and amplitudes from the inner to the outer boundary. The number of ripples increases with number n appearing in (4.39), but as can be seen from Figure 4.2, α reaches maximum for $n = 47.8$. Further our predictions are independent of material properties agreeing with [28]. As also noted in [28], the rippling phenomena is certainly not limited to the nano-world. Our results constitute a first step to a better understanding of the rippling deformations observed for carbon nanotubes [22]-[28].

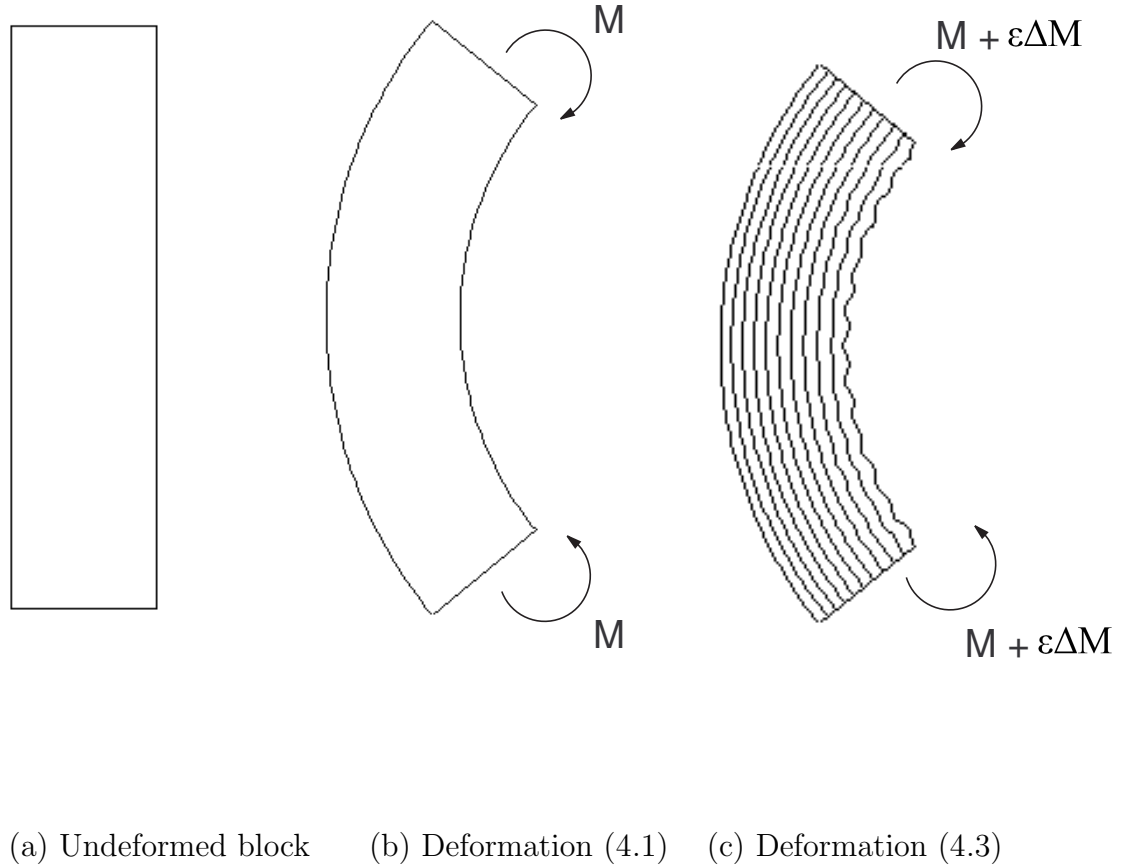


Figure 4.4: Original and deformed body of the rubber block subjected to bending for $t^* = t/h = 0.5$, $a = 3$ and $n = 55.1$

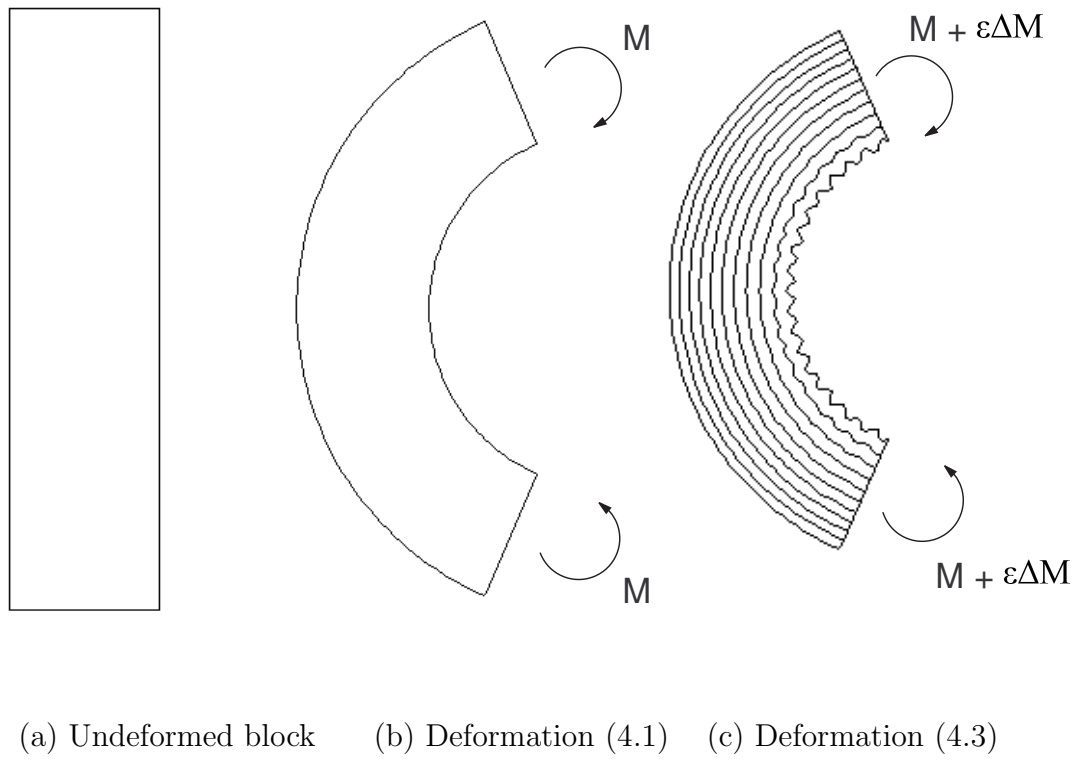
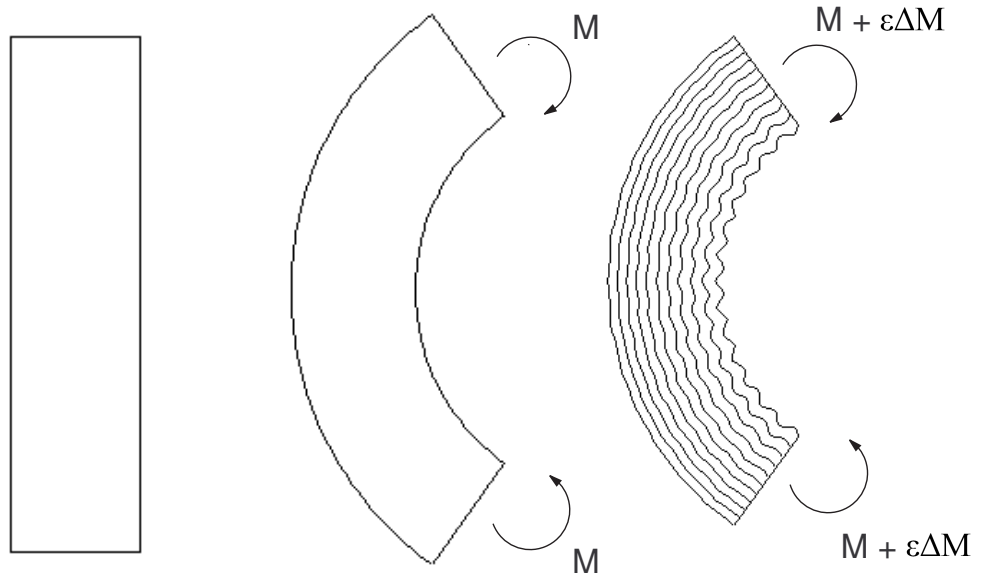


Figure 4.5: Original and deformed body of the rubber block subjected to bending for $t^* = t/h = 0.5$, $a = 3$ and $n = 50.1$



(a) Undeformed block (b) Deformation (4.1) (c) Deformation (4.3)

Figure 4.6: Original and deformed body of the rubber block subjected to bending for $t^* = t/h = 0.5$, $a = 3$ and $n = 45.1$

Chapter 5

Concluding remarks

5.1 Summary

In this thesis two problems involving large elastic deformations have been investigated for isotropic incompressible perfectly elastic material. Finite elastic theories have been employed in the context of non-linear continuum mechanics to determine approximate analytical solutions to the physical phenomena.

The two problems examined are:

- (i) asymptotic axially symmetric deformations for perfectly elastic neo-Hookean and Mooney materials,
- (ii) rippling of long rectangular rubber blocks under bending.

The theory of continuum mechanics is essentially made up of basic principles (axioms) and constitutive theory. The term 'non-linear continuum mechanics' refers to the continuum mechanics related to non-linear constitutive equations. The following postulates are implied whenever the foregoing theory is dealt with in both of the problems discussed:

- (1) mutual body forces within a body are negligible,
- (2) couple stresses and body couples can be ignored, and

The present study is devoted to isotropic incompressible perfectly elastic materials which have been shown to successfully model rubber-like materials. Elastic materials are defined as depending only on the present configuration and not on the history of deformation. An elastic material whose response function is the derivative of the strain energy function with respect to the deformation gradient is defined

as hyperelastic, and sometimes loosely known as perfectly elastic. A homogeneous material is one whose constitutive equation is the same throughout the material, and finally an isotropic material is defined as one whose isotropy group is the full orthogonal group [5]. These properties permit us to confine our analysis to a comparatively much simpler account in the theory, making the problem more tractable. Further, by considering either axially symmetric or plane strain deformations -as a first step- we arrive at an essentially simpler problem. The strain energy function employed is assumed to be of the Mooney and neo-Hookean forms.

Much of this thesis is concerned with the application of theories, modifying and extending the deduced results to tackle the present problems and carrying out comprehensive mathematical deductions. Recourse has been made to the mathematical software MAPLE where computationally more extensive numerical calculations are involved.

5.1.1 Asymptotic axially symmetric deformations for perfectly elastic neo-Hookean and Mooney materials

Chapter 3 presents a detailed treatment for the problem of axial compression of neo-Hookean and Mooney-Rivlin cylindrical tubes with bonded metal end plates. It successfully tackles a difficult analytical problem by methods of asymptotic expansions and boundary conditions.

Formal series solutions are determined in terms of expansions in appropriate powers of $1/R$, where R is the cylindrical polar coordinate for the material coordinates. ie; an expansion of the form

$$\begin{aligned} r &= Rf_0(Z) + \frac{f_1(Z)}{R} + \frac{f_2(Z)}{R^3} + \dots, \\ z &= g_0(Z) + \frac{g_1(Z)}{R^2} + \frac{g_2(Z)}{R^4} + \dots, \end{aligned}$$

is assumed for the deformation field

$$r = r(R, Z), \quad \theta = \Theta, \quad z = z(R, Z).$$

For both neo-Hookean and Mooney-Rivlin materials, the first three terms of the above expansions namely, $f_i(Z)$, $g_i(Z)$ ($i = 0, 1, 2$) are completely determined

analytically in terms of elementary integrals. From the incompressibility condition and the equilibrium equations, the above six unknown deformation functions are reduced to five formal integrations involving in total seven arbitrary constants A, B, C, D, E, H and k^2 , and a further five integration constants. The solutions obtained for the neo-Hookean material are applied to determine an approximate load-deflection relation for the squashing under axially symmetric conditions of a thick walled hollow cylinder by equal and opposite forces applied to bonded metal end plates. The resulting load deflection curves are presented graphically in Figure (3.3) for the case of $a = L/2$ for $b = L, 2L, 3L$ and $4L$, where a and b are inner and outer radii respectively and $2L$ is the original length of the tube. The presented curves suggest an initial linear-quadratic response and achieve a maximum in the vicinity of the maximum deflection. Although nonlinear the response appears not to predict an intermediate maximum such as occurs when blowing up a balloon. The results further recover known linear elastic solutions and extend foundational large deformation results of Klingbeil and Sheild [7] for the same problem.

The solution so determined is approximate in two senses; namely as an approximate solution of the governing equations and for which the stress free boundary conditions are satisfied in an average manner only. However in the absence of any accurate experimental data, the predicted response appears to be physically reasonable.

5.1.2 Rippling of long rectangular rubber blocks under bending

In Chapter 4 the problem of finite elastic deformation of a long rectangular rubber block which is deformed in a perturbed cylindrical configuration is examined with the point of view of surface rippling.

A rectangular rubber block which is initially deformed into a circular arc is then subjected to a further small superimposed deformation. The resulting deformation field is described by

$$\begin{aligned} r &= (AX + B)^{1/2} + \epsilon u(x) \cos n(CY + D), \\ \theta &= CY + D + \epsilon v(x) k^{1/2} \sin n(CY + D), \\ p &= p_0 + \epsilon q(x) A^2 k^{3/2} \cos n(CY + D), \end{aligned}$$

where $k = C/A$, n is a positive number and u, v and q are functions of x only, which is defined by

$$x = k(AX + B).$$

(r, θ, z) and (X, Y, Z) are spatial and material coordinates respectively and constants A, B, C and D are as defined in Chapter 4. The block is assumed to be sufficiently long so that the out of plane end effects may be ignored. Again the solutions are approximate due to the fact that traction free boundary conditions are assumed in an average sense along the bounding circular arcs. Typical numerical values are used to illustrate the solution graphically.

Our analysis suggest that when a long rectangular rubber block is bent in the form of an approximately circular arc, a small superimposed end moment can exhibit wavelike distortions or ripples. As might be anticipated the ripples are formed along the circular arc with decreasing wave lengths and amplitudes from the inner to the outer boundary. The number of ripples increases with number n appearing in (4.39) and reaches a maximum for $n = 47.8$. We may conclude that rippling start to occur at a very small degree of bending as small as 20 times h/t degrees corresponding to $n = 37.1$ and continue to have more ripples until around 80 times h/t degrees of bending at $n = 47.8$ where t and $2h$ are the original thickness and height of the block, respectively. Further our predictions are independent of material properties agreeing with [28]. We note that our solutions recover previous results for the case of $n = 1$ given by Haughton [13] and Hill [10]. Our predictions suggest that as also noted in [28], the rippling phenomena is certainly not limited to the nano-world. Our results constitute a first step to a better understanding of the rippling deformations observed for carbon nanotubes [22]-[28].

As the importance of non-linear mechanics are becoming increasingly appreciated, more materials exhibiting nonlinear behavior are created and more uses of these materials are found. We believe that the foregoing analysis is not only fruitful for the present problems, but may be also amenable to similar problems of interest in nonlinear elastostatics, and that the work provides potential solutions to similar problems in both theoretical and engineering applications.

Appendix A

Derivation of (3.72)

On using (3.43), (3.44) and (3.55) in (3.71) we obtain

$$\begin{aligned} \frac{\partial p}{\partial R} = & 2C_1 \left[Rf_0''f_0 + \frac{B}{R} - \frac{1}{R^3} \left(3f_0''f_2 + f_1''f_1 - f_2''f_0 - \frac{4Af_0'}{f_0^5} \right) \right. \\ & \left. - \frac{1}{R^5} \left(3f_1''f_2 + f_2''f_1 - \frac{8}{f_0^3} - \frac{4Af_0'}{f_0^5} \right) \right], \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \frac{\partial p}{\partial Z} = & 2C_1 \left[\frac{R^2}{2}(f_0'^2)' + \left(f_0'f_1' + \frac{g_0'}{2} \right)' + \frac{1}{R^2} \left(f_2'f_0' + \frac{f_1'}{2} + g_0'g_1' \right)' \right. \\ & \left. + \frac{1}{R^4} \left(f_1'f_2' + g_0'g_2' + \frac{g_1'}{2} \right)' \right]. \end{aligned} \quad (\text{A.2})$$

Equation (A.1) may be readily integrated to give

$$\begin{aligned} p = & 2C_1 \left[\frac{R^2}{2}f_0''f_0 + B \log R + \frac{1}{2R^2} \left(3f_0''f_2 + f_1''f_1 - f_2''f_0 - \frac{4Af_0'}{f_0^5} \right) \right. \\ & \left. + \frac{1}{4R^4} \left(3f_1''f_2 + f_2''f_1 - \frac{8}{f_0^3} - \frac{4Af_0'}{f_0^5} \right) + P_1 + G^*(Z) \right], \end{aligned} \quad (\text{A.3})$$

where P_1 is an integration constant and $G^*(Z)$ is a function of Z . Now using equation (3.64) and (3.66) we simplify and rearrange the above equation (A.3) to give

$$\begin{aligned} p = & 2C_1 \left[-\frac{R^2k^2f_0^2}{2} + B \log R + \frac{1}{2R^2} \left(-2k^2f_0f_2 + \frac{2Bf_1}{f_0} - k^2f_1^2 - \frac{C}{f_0^2} - \frac{4Af_0'}{f_0^5} \right) \right. \\ & \left. + P_1 + G^*(Z) \right]. \end{aligned} \quad (\text{A.4})$$

Next we integrate equation (A.2) and using equations (3.68), (3.69) and (3.70) simplify the result to yield

$$p = 2C_1 \left[-\frac{R^2 k^2 f_0^2}{2} + \frac{1}{R^2} \left[-k^2 f_0 f_2 + \frac{B f_1}{f_0} - \frac{C}{2 f_0^2} - \frac{f_1^2 k^2}{2} - \frac{2 A f_0'}{f_0^5} \right] \right. \\ \left. + \frac{R^2 D}{2} + \frac{H}{R^2} + \frac{1}{2 f_0^4} + B \log f_0 - k^2 f_0 f_1 + E + P_2 + F^*(R) \right], \quad (\text{A.5})$$

where P_2 denotes the constant of integration and $F^*(R)$ is a function of R . From the above equations (A.3) and (A.4) we observe that

$$G^*(Z) = \frac{1}{2 f_0^4} + B \log f_0 - k^2 f_0 f_1. \quad (\text{A.6})$$

Now by letting $P1 = p_0$ from (A.4) we obtain

$$p = -2C_1 \left[\frac{k^2}{2} \left(R f_0 + \frac{f_1}{R} + \frac{f_2}{R^3} \right)^2 + \frac{2 A f_0'}{R^2 f_0^5} + \frac{C}{2 R^2 f_0^2} - B \log(R f_0) - \frac{B f_1}{R^2 f_0} - \frac{1}{2 f_0^4} \right] \\ + 2C_1 p_0, \quad (\text{A.7})$$

where the term r^2 appearing in (A.7) means

$$\left(R f_0 + \frac{f_1}{R} + \frac{f_2}{R^3} \right)^2 = R^2 f_0^2 + 2 f_0 f_1 + \frac{(f_1^2 + 2 f_0 f_2)}{R^2} + \dots \quad (\text{A.8})$$

Appendix B

Derivation of (3.91)

From (3.86)₁ and we get,

$$f'_0(Z) = \frac{-k \sin kZ}{\cos kL}, \quad (\text{B.1})$$

and from (3.89) we have

$$f'_1(Z) = \frac{B \cos kL}{k^2} (k^2 Z \cos kZ + k^2 L \tan kL \sin kZ - k \sin kZ \log f_0), \quad (\text{B.2})$$

Equation (3.70) is rearranged to give

$$f'_2 + \frac{k^2 f_0 f_2}{f'_0} = \frac{B f_1}{f_0 f'_0} - \frac{C}{2 f_0^2 f'_0} + \frac{H}{f'_0} - \frac{(f_1'^2 + k^2 f_1^2)}{2 f'_0}. \quad (\text{B.3})$$

Now by substituting from (B.1) and (B.2) in (B.3) and after some simplification we may deduce

$$\begin{aligned} f'_2 + \frac{k^2 f_0 f_2}{f'_0} &= \frac{B^2}{k^3} \cos^3 kL \left[-\frac{\log f_0}{\sin kZ} - \frac{kZ}{\cos kZ} + \frac{kL \tan kL}{\sin kZ} \right. \\ &\quad \left. \frac{(\log f_0 - kL \tan kL)^2}{2 \sin kZ} + \frac{k^2 Z^2}{2 \sin kZ} \right] \\ &\quad \frac{C \cos^3 kL}{2k \cos^2 kZ \sin kZ} - \frac{H \cos kL}{k \sin kZ} \end{aligned} \quad (\text{B.4})$$

This is a first-order linear ordinary differential equation with an integrating factor which on using (3.86) can be shown to be equal to $1/\sin kZ$ and thus (B.4) becomes

$$\begin{aligned} \left(\frac{f_2}{\sin kZ} \right)' &= \frac{B^2 \cos^3 kL}{k^3} \left[\frac{1}{2 \sin^2 kZ} \log^2 f_0 - (1 + kL \tan kL) \frac{\log f_0}{\sin^2 kZ} \right. \\ &\quad \left. + kL \tan kL \left(1 + \frac{kL}{2} \tan kL \right) \frac{1}{\sin^2 kZ} + \frac{k^2}{2} \frac{Z^2}{\sin^2 kZ} - \frac{kZ}{\sin kZ \cos kZ} \right] \\ &\quad + \frac{C \cos^3 kL}{2k \cos^2 kZ \sin^2 kZ} - \frac{H \cos kL}{k \sin^2 kZ}. \end{aligned} \quad (\text{B.5})$$

Using the elementary integrals given in Table F.1 of Appendix F, from (B.5) we may eventually deduce equation (3.91) given by

$$\begin{aligned}
f_2(Z) = & \frac{B^2 \cos^3 kL}{k^4} \left\{ -\frac{\cos kZ}{2} \log^2 \left(\frac{\cos kZ}{\cos kL} \right) \right. \\
& + (1 + kL \tan kL) \left[\cos kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin kZ \right] \\
& \left. - kZ \sin kZ \log \left(\frac{\cos kZ}{\cos kL} \right) - \frac{k^2}{2} Z^2 \cos kZ + T \cos kZ - 2k^2 I_1(Z) \sin kZ \right\} \\
& + \frac{C \cos kL}{2k^2 \cos kZ} (\cos^2 kL - \cos^2 kZ). \tag{B.6}
\end{aligned}$$

Appendix C

Derivation of (3.94)

Due to (3.47), since $A = 0$ we have

$$(f_0^4 g_2)' = f_1^2 + 2f_0 f_2. \quad (\text{C.1})$$

By substituting from (3.86)₁, (3.89) and (3.91) for f_0, f_1 and f_2 respectively in (C.1) and after some simplification we obtain

$$\begin{aligned} \left[\left(\frac{\cos kZ}{\cos kL} \right)^4 g_2(Z) \right]' &= \frac{B^2 \cos^2 kL}{k^4} \left[2 \cos^2 kZ \log \left(\frac{\cos kZ}{\cos kL} \right) - k^2 Z^2 \cos 2kZ \right. \\ &\quad \left. + kZ \sin 2kZ - 2k^2 I_1(Z) \sin 2kZ + T^* \cos^2 kZ \right] \\ &\quad + \frac{C}{k^2} (\cos^2 kL - \cos^2 kZ). \end{aligned} \quad (\text{C.2})$$

Using the elementary integrals given in Table F.2 of Appendix F, from (C.2) we may deduce equation (3.94) namely,

$$\begin{aligned} g_2(Z) &= \frac{B^2 \cos^6 kL}{k^5 \cos^4 kZ} \left[\frac{1}{2} \sin 2kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \log \left(\frac{\cos kZ}{\cos kL} \right) + kZ \sin^2 kZ \right. \\ &\quad \left. - \frac{1}{2} k^2 Z^2 \sin 2kZ + k^2 I_1(Z) (1 + 2 \cos^2 kZ) + \frac{T^*}{2} \left(kZ + \frac{\sin 2kZ}{2} \right) \right] \\ &\quad + \frac{C}{2k^2} \left(\frac{\cos kL}{\cos kZ} \right)^4 \left(Z \cos 2kL - \frac{\sin 2kZ}{2k} \right). \end{aligned} \quad (\text{C.3})$$

Appendix D

Derivation of (3.103)

We may express (3.100) in the form

$$\frac{T^{11}}{2C_1} = \lambda_1 R^2 + \lambda_2 + \lambda_3 \frac{1}{R^2} + \lambda_4 \log R, \quad (\text{D.1})$$

where $\lambda_i (i = 1..4)$ can be readily identified as:

$$\begin{aligned} \lambda_1 &= \frac{k^2}{2} f_0, \\ \lambda_2 &= f_0 - \frac{1}{2f_0^5} - \frac{p_0}{f_0} - \frac{B \log f_0}{f_0} + \frac{3k^2 f_1}{2}, \\ \lambda_3 &= \frac{k^2}{2} (5f_2 - 4f_0^2 f_0' g_2) + \frac{C}{2f_0^3} + \frac{2k^2 f_1^2}{f_0} \\ &\quad - (B + B \log f_0 + p) \frac{f_1}{f_0^2} - f_1 - \frac{f_1}{2f_0^6} + \dots, \\ \lambda_4 &= -B \left(\frac{1}{f_0} + \frac{f_1}{R^2 f_0^2} \right). \end{aligned} \quad (\text{D.2})$$

Next by substituting from (3.86)₁, (3.89) and (3.91) for f_0, f_1 and f_2 in (D.2), simplifying and rearranging the terms we may deduce

$$\begin{aligned}
\lambda_1 &= \frac{k^2 \cos kZ}{2 \cos kL}, \\
\lambda_2 &= \frac{\cos kZ}{\cos kL} - \frac{\cos^5 kL}{2 \cos^5 kZ} - p_0 \frac{\cos kL}{\cos kZ} \\
&\quad + B \cos kL \left(\frac{3}{2} \cos kZ \log f_0 + \frac{3}{2} kZ \sin kZ - \frac{3}{2} kL \tan kL \cos kZ - \frac{\log f_0}{\cos kZ} \right), \\
\lambda_3 &= \frac{B^2 \cos^3 kL}{k^2} \left[\frac{3}{4} \log^2 f_0 \cos kZ - \frac{\log^2 f_0}{\cos kZ} + \frac{kZ \log f_0 \sin kZ}{\cos^2 kZ} \right. \\
&\quad + \frac{1}{2} (1 - 3kL \tan kL) (\log f_0 \cos kZ + kZ \sin kZ) + \frac{3}{2} kZ \log f_0 \sin kZ \\
&\quad + (1 + kL \tan kL) \frac{\log f_0}{\cos kZ} - \frac{5}{4} k^2 Z^2 \cos kZ + (1 + T^*) \frac{kZ \sin kZ}{\cos^2 kZ} \\
&\quad + \frac{1}{4} (T^* + 3k^2 L^2 \tan^2 kL) \cos kZ + \frac{(T^* + kL \tan kL)}{\cos kZ} \\
&\quad + k^2 I_1(Z) \sin kZ \left(\frac{2}{\cos^2 kZ} - 1 \right) \left. \right] - \frac{B}{k^2} (\cos kL \cos kZ \log f_0 + kZ \cos kL \sin kZ \\
&\quad - kL \sin kL \cos kZ + \frac{\cos^7 kL \log f_0}{2 \cos^5 kZ} + \frac{kZ \cos^7 kL \sin kZ}{2 \cos^6 kZ} - \frac{kL \cos^6 kL \sin kL}{2 \cos^5 kZ} \Big) \\
&\quad - \frac{p_0 B \cos^3 kL}{k^2} \left(\frac{\log f_0}{\cos kZ} + \frac{kZ \sin kZ}{\cos^2 kZ} - \frac{kL \tan kL}{\cos kZ} \right) \\
&\quad + \frac{C \cos kL}{4} \left(\frac{4kZ \tan kZ \cos 2kL + \sin^2 kZ - 5 \sin^2 kL}{\cos kZ} + 2 \frac{\cos^2 kL}{\cos^3 kZ} \right), \\
\lambda_4 &= -\frac{B \cos kL}{\cos kZ} - \frac{B^2 \cos^3 kL}{R^2 k^2} \left(\frac{\log f_0}{\cos kZ} + \frac{kZ \sin kZ}{\cos^2 kZ} - \frac{kL \tan kL}{\cos kZ} \right). \tag{D.3}
\end{aligned}$$

Next upon using (D.2) in the condition (3.95) and since the only applied force is the vertical force F , by making the right hand side of (3.95) zero, we obtain the following equation

$$\int_{-L}^L \left(\lambda_1 R^3 + \lambda_2 R + \lambda_3 \frac{1}{R} + \lambda_4 R \log R \right) dZ = 0. \quad (\text{D.4})$$

Hence we integrate (D.3) and use the elementary definite integrals in Table F.2 of Appendix F to evaluate the resulting expression. After some simplifications, rearranging the terms and using the abbreviations in (3.105), we deduce the results

$$\begin{aligned} \int_{-L}^L \lambda_1 dZ &= \frac{ks}{c}, \\ \int_{-L}^L \lambda_2 dZ &= \frac{B}{k} (3cw - 3kL - CkI_2) - \frac{2p_0cw}{k} + \frac{1}{k} \left(\frac{2s}{c} - \frac{cs}{4} - \frac{3c^3s}{8} - \frac{3c^5w}{8} \right), \\ \int_{-L}^L \lambda_3 dZ &= \frac{B^2}{k^3} \left[kL(2T^*c^2 - c^2sw - c^4) + k^2Lc^2sI_2 + wc^3 + \frac{T^*c^3s}{2} \right. \\ &\quad \left. + \frac{c^3}{2}(3kI_2 - 4kI_3) + \frac{k^2L^2}{2}(c^3s + 3s^3c) + 2k^2I_1(L)c^2(c^2 + 2) \right] \\ &\quad - \frac{p_0B}{k^3} [2kLc^2(1 - sw) - 2c^2w + kc^3I_2] \\ &\quad + \frac{B}{k^3} \left[kL(2 - \frac{c^2}{5} + \frac{c^2s^2}{4} + \frac{3}{8}c^4s^2 + \frac{3}{8}c^6sw) \right. \\ &\quad \left. - (2cw + \frac{c^3s}{80} + \frac{13}{160}c^5s - \frac{47}{160}c^7w + \frac{3}{16}kc^7I_2) \right], \\ &\quad + \frac{C}{k} [2kL(2c^2 - 1) - wc^3], \\ \int_{-L}^L \lambda_4 dZ &= -\frac{2Bcw}{k} - \frac{B^2}{R^2k^3} [2kLc^2(1 - sw) - 2c^2w + kc^3I_2]. \end{aligned} \quad (\text{D.5})$$

By substituting (D.5) into (D.4), we deduce the equation (3.103) namely,

$$\begin{aligned} p_0 &= -B \log R + (kR)^2 \frac{\kappa_1}{\kappa_7 + B/(kR)^2\kappa_8} + \frac{\kappa_2 + B\kappa_3}{\kappa_7 + B/(kR)^2\kappa_8} \\ &\quad + \frac{1}{(kR)^2} \frac{(B^2\kappa_4 + B\kappa_5 + k^2C\kappa_6)}{\kappa_7 + B/(kR)^2\kappa_8}. \end{aligned} \quad (\text{D.6})$$

Appendix E

Derivation of (3.121)

From equation (3.120) the condition for real roots for B gives,

$$\lambda^2(b^2 - a^2)^2 - \left(3(b^2 + a^2) + \frac{8}{3}L^2\right)\lambda + \frac{13}{4} \geq 0, \quad (\text{E.1})$$

where λ is defined by

$$\lambda = \frac{\log(b/a)}{b^2 - a^2}. \quad (\text{E.2})$$

In order to satisfy the condition (E.1), the quadratic equation in λ must have imaginary roots and therefore

$$\sqrt{13}(b^2 - a^2) - 3(b^2 + a^2) \geq \frac{8}{3}L^2. \quad (\text{E.3})$$

Since right hand side of (E.3) is greater than zero we have the condition (3.121)₁ and from (E.3) we have the condition (3.121)₂,

$$L^2 \leq \frac{3}{8} \left[(\sqrt{13} - 3)b^2 - (\sqrt{13} + 3)a^2 \right]. \quad (\text{E.4})$$

Appendix F

Tables of integrals

Table F.1: Table of indefinite integrals

$\int (Z/\sin^2 kZ) dZ$	$-Z/k \tan kZ + k^{-2} \log \sin kZ$
$\int (Z^2/\sin^2 kZ) dZ$	$-Z^2/k \tan kZ + \frac{2}{k^2} Z \log \sin kZ - \frac{2}{k^2} \int \log \sin kZ dZ$
$\int (Z/\sin 2kZ) dZ$	$(Z/2k) \log \tan kZ - \frac{1}{2k} \int \log \tan kZ dZ$
$\int (\log f_0/\sin^2 kZ) dZ$	$-\log f_0/k \tan kZ - Z$
$\int (\log^2 f_0/\sin^2 kZ) dZ$	$-\log^2 f_0/k \tan kZ - 2Z \log f_0 - 2k \int Z \tan kZ dZ$
$\int \cos^2 kZ dZ$	$Z/2 + \sin 2kZ/4k$
$\int Z^2 \cos 2kZ dZ$	$Z \cos 2kZ/2k^2 + Z^2 \sin 2kZ/2k - \sin 2kZ/4k^3$
$\int Z \sin 2kZ dZ$	$\sin 2kZ/4k^2 - Z \cos 2kZ/2k$
$\int \cos^2 kZ \log f_0 dZ$	$\frac{Z}{2} \log f_0 + \frac{1}{4k} \log f_0 \sin 2kZ + \frac{k}{2} I_1(Z) + Z/4 - \sin 2kZ/8k$
$\int I_1(Z) \sin 2kZ dZ$	$-I_1(Z) \cos 2kZ/2k - Z \cos 2kZ/4k^2 - I_1(Z)/2k + \sin 2kZ/8k^3$

where $f_0 = \cos kZ/\cos kL$ and $I_1(Z)$ is defined by the integral

$$I_1(Z) = \int_0^Z \xi \tan k\xi d\xi = -\frac{Z}{k} \log kZ + \frac{1}{k} \int_0^Z \log \cos k\xi d\xi.$$

In the following table c , s , and w are defined by

$$c = \cos kL, \quad s = \sin kL, \quad w = \log[\sec kL + \tan kL]. \quad (\text{F.1})$$

Table F.2: Table of definite integrals

$\int_{-L}^L \cos kZ dZ$	$2s/k$
$\int_{-L}^L dZ / \cos kZ$	$2w/k$
$\int_{-L}^L dZ / \cos^3 kZ$	$s/kc^2 + w/k$
$\int_{-L}^L dZ / \cos^5 kZ$	$s/2kc^4 + 3s/4kc^2 + 3w/4k$
$\int_{-L}^L (kZ \sin kZ / \cos^2 kZ) dZ$	$2L/c - 2w/k$
$\int_{-L}^L kZ \sin kZ dZ$	$2s/k - 2Lc$
$\int_{-L}^L (kZ \sin kZ / \cos^6 kZ) dZ$	$2L/5c^5 - (2s/c^4 + 3s/c^2 + 3w)/20k$
$I_2 = \int_{-L}^L (\log f_0 / \cos kZ) dZ$	$\int_{-L}^L \log(\sin kZ + 1) \tan kZ dZ$
$I_3 = \frac{1}{2} \int_{-L}^L (\log^2 f_0 / \cos kZ) dZ$	$k \int_{-L}^L \int_{-L}^Z \log(\sin kx + 1) \tan kx \tan kZ dx dZ$
$\int_{-L}^L (\log f_0 / \cos^5 kZ) dZ$	$s/k8c^4 + 5s/k16c^2 - 7w/k16 + 3I_2/8$
$\int_{-L}^L \cos kZ \log f_0 dZ$	$2(w - s)/k$
$\int_{-L}^L k^2 Z^2 \cos kZ dZ$	$2(kL^2 - 2/k)s + 4Lc$

Appendix G

Derivation of (4.45)

Upon substituting $t = \log x$ in (4.44) and by substituting $u = x^m w$ where m is a constant, we obtain

$$\begin{aligned} & w_{ttt} + (4m - 2)w_{tt} + \left[6m^2 - 6m + \left(\frac{1}{2} - \frac{n^2}{4}\right)\right] u_{tt} \\ & + \left[4m^3 - 6m^2 + 2m\left(\frac{1}{2} - \frac{n^2}{4}\right) + \left(\frac{1}{2} + \frac{n^2}{4}\right)\right] w_t \\ & + \left[m^4 - 2m^3 + m^2\left(\frac{1}{2} - \frac{n^2}{4}\right) + m\left(\frac{1}{2} + \frac{n^2}{4}\right)\right] w \\ & - e^{2t} n^2 \left\{ w_{tt} + (2m + 1)w_t + \left[m^2 + m + \left(\frac{1 - n^2}{4}\right)\right] w \right\} = 0. \end{aligned} \tag{G.1}$$

Now if we choose m to satisfy

$$m^4 - 2m^3 + m^2\left(\frac{1}{2} - \frac{n^2}{4}\right) + m\left(\frac{1}{2} + \frac{n^2}{4}\right) = 0, \tag{G.2}$$

and define a new variable $z = n^2 e^{2t}/4$ and an operator $\delta = z d/dz$, then equation (G.1) can be shown to be equivalent to the form

$$[\delta(\delta + \rho_1 - 1)(\delta + \rho_2 - 1)(\delta + \rho_3 - 1) - z(\delta + \alpha_1)(\delta + \alpha_2)]w = 0, \tag{G.3}$$

for which ρ_i ($i = 1, 2, 3$) and α_j ($j = 1, 2$) satisfy the following:

$$\begin{aligned} \alpha_1 &= \frac{1}{4}(2m + 1 + n), & \alpha_2 &= \frac{1}{4}(2m + 1 - n), \\ \rho_1 &= 1 + \frac{N}{2}, & \rho_2 &= \frac{1}{4}(5 + N), & \rho_3 &= \frac{1}{4}(3 + N). \end{aligned} \tag{G.4}$$

Here N is defined by $N = (3 + n^2)^{1/2}$. Thus $w = {}_2F_3(\alpha_1, \alpha_2; \rho_1, \rho_2, \rho_3; z)$ is a solution to the equation (G.1), where ${}_2F_3$ is the generalized hypergeometric series [33]. From

(G.2) we have the four solutions for m which are given by (4.46). Hence we obtain the four solutions of (G.2) corresponding to each value in (4.46) giving rise to (4.45). This analysis is also in agreement with the results generated by MAPLE.

Bibliography

- [1] P.B.Lindley, (1970) *Engineering Design with Natural Rubber*, The Natural Rubber Producers Research Association, London, 3rd edition.
- [2] C.Truesdell and W.Noll, (1992) *The Non-Linear Field Theories of Mechanics*, Springer-Verlag, Berlin, 2nd edition.
- [3] A.E.Green and J.E. Adkins, (1970) *Large elastic deformations*, Oxford University Press, London, 2nd edition.
- [4] A.D. Deozdov, (1996) *Finite Elasticity and Viscoelasticity*, World Scientific Publishing, Singapore.
- [5] D.C.Leigh, (1992) *Nonlinear Continuum Mechanics*, McGraw-Hill, New York.
- [6] J.M.Hill and A.I Lee (1988) "Large elastic compression of finite rectangular blocks of rubber," *Journal of Mechanics and Applied Mathematics*, **42**, 267-287.
- [7] W.W.Klingbeil and R.T Shield (1966) "Large deformation analysis of bonded elastic mounts," *Journal of Mathematics and Physics (ZAMP)*, **17**, 281-305.
- [8] A.N.Gent and P.B.Lindley (1959) "The compression of bonded rubber blocks" *Proceedings of the Institution of Mechanical Engineering*, **173**, 111-117.
- [9] J.M.Hill and A.I Lee (1989) "Combined compression and torsion of circular cylindrical pads of rubber," *Journal of the Mechanics and the Physics of Solids*, **37**, 175-190.
- [10] J.M.Hill, (1977) "Load-deflection relations of long cylindrical rubber bush mountings constructed from rectangular blocks", *Applied Polymer Science*, **21**, 1459-1467.

- [11] R.S.Rivlin (1949) “Large elastic deformations of isotropic materials,” *Philosophical Transactions of the Royal Society of London*, **A242**, 173-195.
- [12] N.Triantafyllidis, (1980) “Bifurcation phenomena in pure bending”, *Journal of the Mechanics and the Physics of Solids*, **28**, 221.
- [13] D.M.Haughton, (1999) “Flexure and compression of incompressible elastic plates”, *International Journal of Engineering Science*, **37**, 1693.
- [14] A.E.Green and W. Zerna, (1968) *Theoretical Elasticity*, Oxford University Press, London, 2nd edition.
- [15] A.C.Eringen, (1967) *Mechanics of Continua*, John Wiley and Sons, New York.
- [16] M.N.Narasimhan, (1993) *Mechanics of Continua*, John Wiley and Sons, New York.
- [17] Brown University, Division of Engineering, USA *Advanced Mechanics of Solids*, www.engin.brown.edu
- [18] University of Michigan , Division of Engineering, USA *Alternate Definitions of Stress*, www.engin.umich.edu
- [19] A.C.Eringen, (1962) *Nonlinear Theory of Continuous Media*, McGraw-Hill, New York.
- [20] J.M.Hill, (1972) *Thesis*, University of Queensland, Australia
- [21] J.M. Hill, (2001) “A review of partial solutions of finite elasticity and their applications.”, *International Journal of Non-Linear Mechanics*, **36** 447-463 .
- [22] J.Z.Liu, Q.Zheng and Q.Jiang, (2001) “Effect of rippling mode on resonances of carbon nanotubes”, *Physical Review Letters*, **86**, 4843.
- [23] J.Z.Liu, Q.Zheng and Q.Jiang, (2003) “Effect of bending instabilities on the measurements of mechanical properties of multiwalled carbon nanotubes”, *Physical Review B*, **67**, 075414.

- [24] M.Arroyo and T. Belytschko, (2003) “Nonlinear mechanical response and rippling of thick multiwalled carbon nanotubes”, *Physical Review Letters*, **91**, 215505.
- [25] R.S.Ruoff and D.C. Lorents, (1995) “Mechanical and thermal properties of carbon nanotubes”, *Carbon*, **33**, 925.
- [26] T.Kuzumaki, T. Hayashi, K. Miyazawa, H. Ichinose, K.Ito and Y. Ishida, (1998) “In-situ observed deformation of carbon nanotubes”, *Philosophical Magazine*, **A77**, 1461.
- [27] P.Poncharal, Z.L. Wang, D. Ugrate and W.A.Heer, (1999) “Electrostatic deflections and electromechanical resonances of carbon nanotubes”, *Science*, **283**, 1513.
- [28] L.Mahadevan, J. Bico and G. McKinley, (2004) “Popliteal rippling of layered elastic tubes and scrolls”, *Europhysics Letters*, **65**, 323-329.
- [29] A.N.Gent and I.S.Cho, (1999) “Surface instabilities in compressed or bent rubber blocks”, *Rubber Chemistry and Technology*, **72**, 253.
- [30] M.A.Biot, (1965) “Mechanics of Incremental Deformations”, *Wiley, New York*, **161**, 216.
- [31] A.Ghatak and A.L.Das, (2007) “Kink instability of a highly deformable elastic cylinder”, *Physical Review Letters*, **99**, 076101.
- [32] J. M. Hill, (1973) “Partial solutions of finite elasticity-plane deformations”, *Journal of Mathematics and Physics (ZAMP)*, **24**, 401-408.
- [33] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, (1953) *Higher Transcendental Functions*, volume 1, McGraw-Hill, New York, **4**, 182-184

List of the author's publications

- J. M. Hill, N. Padukka and H. H. Dai (2007) “Asymptotic axially symmetric deformations for perfectly elastic neo-Hookean and Mooney materials”, *Journal of Elasticity*, **86**, 113-137
- N. Padukka and J. M. Hill, (2007) “Rippling of long rectangular rubber blocks under bending”, *Journal of Mathematics and Mechanics of solids*, accepted for publication.