Title: Fourth order geometric evolution equations

Author: Glen Wheeler

Year: 2009


Repository DOI:

2009

# Fourth Order Geometric Evolution Equations

Glen Wheeler
*University of Wollongong*

# Fourth Order Geometric Evolution Equations

A thesis submitted in fulfilment of the requirements for the award of the degree

**Doctor of Philosophy**

from

**University of Wollongong**

by

**Glen Edward Wheeler, B. Comp. Sci., B. Math. Adv. (Hons.) First Class**

School of Mathematics and Applied Statistics

2009

Abstract. In this thesis the chief object of study are hypersurface flows of fourth order, with the speed of the flow varying from the Laplacian of the mean curvature, to the more general constrained flows which include a function of time in the speed, and satisfy various conditions. Our aim is to instigate a study of the regularity of these flows, answering questions of local and global existence, and some preliminary singularity analysis. Among our results are positive lower bounds for smooth and regular existence, classification of stationary solutions, interior estimates, and blowup asymptotics. Applying these results to a certain class of constrained surface diffusion flows, we obtain long time existence and exponential convergence to spheres for initial surfaces with small $L^2$ norm of tracefree curvature. We present one application of this theorem, using it to deduce the isoperimetric inequality with optimal constant for 2-surfaces satisfying the above smallness condition. The theorem can be thought of as a stability of spheres result, as the smallness condition is an averaged distance from a standard round sphere to the initial manifold in $L^2$. This strengthens a related earlier result specialised to surface diffusion flow where the distance is small in $C^{2,\alpha}$, obtained through a completely different method. The results throughout this thesis are new contributions for both surface diffusion flow, which has been considered by many authors, and the constrained flows, which have only recently been considered.

**Certification**

I, Glen Edward Wheeler, declare that this thesis "Fourth Order Geometric Evolution Equations" submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged below. The first publication from this thesis, "Lifespan Theorem for Constrained Surface Diffusion Flows" is under supervision of Dr. McCoy and Assoc. Prof. Williams with each contributing approximately 20%. It is convention to list authors alphabetically by last name regardless of contribution. Later work in the thesis was completed with significantly less input from supervisors and the second publication "Surface Diffusion Flow Near Spheres" is wholly my own work. The document has not been submitted for qualifications at any other academic institution.

**Acknowledgements**

Throughout this study I have received support from many places, and although it is not possible for me to even recall them all, I will do my best. I would like to thank my supervisors, Dr. James McCoy and Assoc. Prof. Graham Williams, for pushing me to succeed and for helping me to understand many things which were inaccessible to me previously. I was lucky enough to have another great help from my mathematical grandfather, Prof. Dr. Klaus Ecker, who invited me for two visits to the Freie Universität in Berlin. During this time I also twice visited the beautiful Mathematisches Forschungsinstitut Oberwolfach, which truly opened my eyes to international research. While there I witnessed many great talks and even had the opportunity to discuss with Prof. Dr. Ernst Kuwert and Prof. Dr. Rainer Schätzle, who are the progenitors of the program of study which I used as a guiding light in this research. Subsequently I was invited by my de facto grandfather, Prof. Dr. Gerhard Huisken, to the Max-Planck-Instituts für Gravitationphysik (Albert-Einstein-Institut), where I had the pleasure of discussing my research with many people working in similar areas. This included an enjoyable invitation to Otto-von-Guericke-Universität Magdeburg, which I am grateful for, and enabled several enlightening conversations.

I must also thank Dr. Marty Ross for the straw which broke my computer scientist back, and for my beautiful fiancè Valentina without which none of this would be possible.

# Contents

CHAPTER 1

# Introduction

It is of inherent interest for us to understand how our environment operates and behaves over time. This is one of the key motivating factors behind experimental physics; determining so-called governing equations or physical 'laws' to explain observed phenomena. One of the most important features of a proposed 'law' is that it is *predictive* in the sense that we can determine, with some accepted degree of error, what will happen in the future. If these predictions turn out to be correct, we may even adopt this new 'law' into the accepted theory. In many cases, it is extremely difficult to determine which forces are at work in a certain situation, and further, how these forces act on objects in our world. It is often the case that experiments imply that some governing equation determines the evolution of an object under a given force, or geometrical constraint, but even with this supposed governing equation we cannot tell which properties an affected object will possess, over time. Using mathematical techniques, we can investigate extremely general formulations of these geometric flows and attack this problem.

These investigations can yield surprising results; principally concerned with categorisation of the effects a given flow or family of flows has on a class of manifolds it acts upon. Understanding the underlying properties of a flow can lead to new symmetry and intrinsic geometry techniques. As a classical example of this in history, the breakthrough discovery of new minimal surfaces by Meusnier [**46**] had far reaching consequences throughout physics and mathematics. Frequently, we can infer new relationships between special geometric objects, for example the isoperimetric inequalities. It is also common to see a major result in this area specific to a class of geometric flows infer or assist in proving another theorem in a different area of mathematics: this was witnessed recently with the very public solution of the Poincarè Conjecture enabled by the breakthrough work of Richard Hamilton [**27**] and Grisha Perelman [**51**]. Given the extreme generality of these investigations, and the extensive mathematical background required, it is often the case that great time and work is required to fully determine the behaviour and characteristics of even one geometric flow. However, despite this, research has steadily grown in intensity in past years.

The natural first target for mathematicians are the *second order* geometric heat flows. These are natural due to the availability of the maximum principle, adopted from the theory of second order partial differential equations. We see the pivotal role this plays in the papers of Hamilton [**27**] and Huisken [**28**] on the Ricci flow and mean curvature flow respectively. The mean curvature flow is motivated initially by the experimental and theoretical work of the physicist Mullins [**48, 49**], and reads

$$\text{(MC)} \qquad \frac{\partial}{\partial t} f = -H\nu,$$

where $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a smooth immersion of the $n$-dimensional manifold $M$, $H$ the mean curvature of $M_t = f(\cdot, t)$ and $\nu$ the outer unit normal vector field to $M$. This equation has been well studied by many authors, see the excellent book [**12**] and the references contained therein for a survey of results. For our purposes here we make a few elementary remarks. The flow (MC) is the steepest descent gradient flow for area, and for any initial data $M_0$ we have

$$\frac{d}{dt}\mu(M_t) = -\int_M H^2 d\mu,$$

where $\mu$ is the surface measure on $M_t$. That is, surface area is monotonically decreasing and stationary if and only if $M_t$ is a minimal surface. This leads one to suspect that mean curvature flow would be useful in studying minimal surfaces, and indeed this intuition turns out to be correct. Of particular interest to us is the proposal in Mullins' earlier paper for the *surface diffusion flow*, used there to model the formation of thermal grooves in phase interfaces. This can be written as

$$\text{(SD)} \qquad \frac{\partial}{\partial t}f = (\Delta H)\nu.$$

The flow (SD) is called *fourth order*, due to the second order differential operator $\Delta$ being applied to the mean curvature $H$, which is itself a function of up to the second order spatial derivatives of the immersion $f$. Being a gradient flow for surface area in $H^{-1}$, the surface diffusion flow enjoys two hallmark geometric characteristics: a reduction of free surface energy (or surface area) and a conservation of mass (or volume). These attributes of (SD), along with the relatively simple algebraic structure of the flow, make it a natural fourth order analog of mean curvature flow, and a model problem to be studied thoroughly before moving on to more general evolution equations. Taking the geometric properties further, one is lead to the case of *constrained surface diffusion flows*, where the immersion $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ evolves by

$$\text{(CSD)} \qquad \frac{\partial}{\partial t}f = (\Delta H + h)\nu,$$

with the constraint $h : [0, T) \to \mathbb{R}$ being a function of time chosen to coincide with a natural geometric restriction present in the problem being studied. For example, various choices of $h$ correspond to conservation of mixed volumes or a reduction in mass and increase of free surface energy.

The surface diffusion flow has attracted some interest in the literature. It has been examined from both a physical and mathematical perspective by Davi, Gurtin [**10**] and in a more general context by Cahn, Taylor [**7**]. Davi and Gurtin propose another physical motivation by recovering the surface diffusion flow (SD) when considering motion governed by mass diffusion in a phase interface. Cahn and Taylor provide further motivation by demonstrating connections that the surface diffusion flow (SD) exhibits with other kinds of motion of surfaces under various conditions. These are all of the type where total free surface energy is reduced with conservation of volume. This work is particularly notable for providing no less than thirty different approaches to solving the surface diffusion equation, although only the proposed numerical techniques have been carried out. A few years after these developments Cahn, Elliott, and Novick-Cohen [**6**] demonstrated that the surface diffusion flow can be derived under certain conditions from the Cahn-Hilliard equation, which predicts isothermal separation of a binary alloy. This gives substantial motivation for research into the behaviour of the surface diffusion flow.

To effectively analyse fourth order flows we must overcome the lack of a maximum principle. One technique is to use curvature integral estimates. We can see this employed by Kuwert and Schätzle [**36, 37**] for the Willmore flow, where the immersion evolves by

$$\frac{\partial}{\partial t} f = (\Delta H + \|A^o\|^2 H)\nu, \tag{W}$$

with $A^o$ the tracefree curvature. Willmore flow is the $L^2$ gradient flow of total squared mean curvature, and as such is the other 'most natural' fourth order flow. This means that

$$\frac{d}{dt}\int_M H^2 d\mu = -\int_M \left(\Delta H + \|A^o\|^2 H\right)^2 d\mu \le 0,$$

and the stationary solutions are elastic minima. The tracefree and full curvature tensor also exhibit this monotone decreasing property in $L^2$, and as such integral estimates where a curvature quantity is small in $L^2$ becomes a natural strategy in attacking Willmore flow.

In an abstract sense, the overall argument in [**37**] is due to work by Struwe [**59**] on harmonic mappings of Riemann surfaces. The ingredients required to carry out this argument for a general flow are evolution equations for curvature quantities, short time existence, integral estimates, and the concentration-compactness alternative. The culmination of this argument is a non-zero lower bound on time for which the total squared curvature of the evolving manifold remains bounded. This can be thought of as a time limit during which the flow remains well behaved. This is important even in light of the short time existence results (which are a prerequisite for this argument) for higher order flows due to Polden and Huisken [**52, 29**]: short time existence gives an arbitrarily small lower bound for the time in which the manifold remains smooth, whereas for many applications one requires the lower bound to be absolute, and not dependent on initial data. This is why the result in question is called a *lifespan* theorem. One of our main results in this research is establishing a lifespan theorem for constrained surface diffusion flows, which is the subject of Chapter 3.

Beyond this point lie questions of a global nature: under which conditions can one infer long time existence of the flow, where $T = \infty$, and in this situation can one obtain any qualitative information on the asymptotic behaviour of the flow? Again, we take our inspiration from Kuwert and Schätzle [**36**], attacking the problem with local and global integral estimates combined with blowup analysis. Our main adversary is the lack of a useful function space in which surface diffusion flow is a gradient flow. Willmore flow, being an $L^2$ gradient flow of total squared curvature, lends itself naturally to such an analysis. This problem lead to a popular opinion that such an approach is not appropriate for surface diffusion flow.

Indeed, our results in blowup analysis are weaker than that of Kuwert and Schätzle for the Willmore flow [**36**]. Briefly, we can only guarantee a stationary blowup if the average distance in $L^2$ from a round sphere is small. Contrast this with the Willmore flow, where one obtains this *regardless* of the initial data. Thus, the initial aversion to using this program of study is in one sense confirmed: one may indeed obtain self-similar or translating solutions (as opposed to stationary solutions) from a blowup, but *only* if the initial manifold is far enough from a sphere. This behaviour is something like a mixture of Willmore flow with mean curvature flow, which is also a gradient flow of area, and closely related to surface diffusion flow.

However, this does not stop us from obtaining our long time existence result. The idea of our method is as follows. Armed with the Lifespan Theorem from Chapter

3, we know that the only obstacle to global existence is possible concentrations of curvature in $L^2$. We proceed by contradiction and assume that the initial manifold is close to a sphere in an average sense and that the curvature has concentrated in finite time. This finite time concentration begs further study, and so we blow up around the singularity. This is performed in Chapter 6. To obtain existence of a limit, we need the Interior Estimates from Chapter 5. Using ideas developed in Chapter 5, we can also prove that the average closeness to spheres stays well controlled, and this allows us to prove that the blowup constructed is a stationary, nonumbilic surface. This lends itself neatly to a contradiction with the main result of Chapter 4, the Gap Lemma, which states that precisely such a surface must indeed be umbilic. This implies that there is no concentration of curvature in finite time, and so we must have long time existence. From here most of the work is done and we use straightforward arguments to obtain exponential convergence to spheres in Chapter 7.

Thus we summarise the main contributions of this thesis as the following.

(Ch. 2) Short time existence for higher order hypersurface flows which are quasilinear and parabolic in a local chart. This chapter collects many references from the vast literature on local existence and organises them together to show short time existence for our flows under consideration.

(Ch. 3) Lifespan Theorem. Here we present a proof of the aforementioned Lifespan Theorem for constrained surface diffusion flows. This is an absolute lower bound for the maximal time of existence of the flow which depends on the concentration of curvature in the initial data.

(Ch. 4) Gap Lemma. This chapter proves a gap lemma for constrained surface diffusion flows. There the concerns are the stationary solutions to the flow equation, and their geometry under certain conditions. The Gap Lemma we prove here shows that under a small tracefree curvature condition, a growth at infinity of curvature condition, and a structure of $h$ condition the stationary solutions are indeed spheres and planes.

(Ch. 5) Curvature and Interior Estimates. This chapter investigates the consequences of local integral of curvature estimates where the speed of the flow appears on the left hand side; these are natural for a gradient flow of curvature. For us, we can still use this technique to conclude several interesting results. The first are some pointwise curvature estimates where the speed of the flow appears on the right hand side. The second are interior estimates where we control all higher derivatives of curvature so long as the curvature is already well-controlled in $L^2$. The conditions required of the constraint function for the interior estimates are the same as the conditions required for the Lifespan Theorem, and as such these two theorems enjoy a convenient synergy.

(Ch. 6) Almost Preservation and Stationary, Non-Umbilic Blowups. The first result is on the almost preservation of initially small tracefree curvature. Using a technical estimate from Chapter 5 we can show that if the tracefree curvature begins small in $L^2$ then it stays well-controlled along the flow. One of the many consequences is that using another technical estimate from Chapter 5 the blowup at an assumed finite time singularity is stationary.

Arguments from Kuwert and Schätzle [**36**] strengthen this to stationary and non-umbilic.

(Ch. 7) Long time existence and exponential convergence to spheres. This is an orchestration of all our previous results, and some additional analysis in the same vein as Chapter 6. The final conclusion is that for 2 dimensional surface diffusion flows, if the $L^2$ norm of the tracefree curvature is small at initial time then the surface diffusion flow exists for all time and converges exponentially to a round sphere. For the constrained flows, there are several structure conditions placed on $h$ and the initial condition is strengthened to smallness in $L^p$ for some $p > 4$ which depends on the constraint function.

Opportunities for further research abound. Many of the results in this thesis may be easily adapted to the case of constrained Willmore flows, and as there does already exist some work on constrained Willmore surfaces, the stationary case of the flow, this is well motivated. Determining precisely the nature of any obstructions to obtaining long time existence and convergence to spheres, and working to overcome these obstructions would be an interesting topic. In another direction, one may consider higher intrinsic dimensions, moving from two dimensional surfaces to three dimensional manifolds. We have already commented on both of these possibilities throughout the thesis, where adaptations to these situations lend themselves easily. In a more difficult vein, one may ask the following question. Given recent work on the eventual positivity of the Green's function for the parabolic bilaplacian [**18, 22, 23, 21, 26**], and the fact that eventual preservation of positivity of the mean curvature is obtained in the proof of our main theorem (Proposition 7.7), does there exist an underlying phenomena for fourth order equations analogous to the ubiquitous maximum principle? Investigations on this question are continuing, but it seems that answering this even in part would allow a solution to many unsolved problems. For example, when does surface diffusion flow or Willmore flow develop a singularity? If we have a flow with very large deviation from a sphere, does there develop a curvature singularity? In the second order case, the comparison principle allows one to determine many such situations where this will occur. For the flows in consideration here however this simple question has not been resolved. In a different topological class, the work by Blatt [**4**] on Willmore flow shows that *eventually* (perhaps in infinite time) one must obtain non-existence, however there a topological obstruction is used to obtain the result, and this does not seem to give a hint as to an underlying, deep property of the flow. It is this author's hope that one day the analysis given here is used to obtain qualitative information on surface diffusion flows with large distance from spheres, and that these problems can therefore be resolved.

## 1. Prerequisites from differential geometry

It is our immediate task to set our notation and discuss the definition and fundamental properties pertaining to our chief objects of study. Intuitively, these are somehow lower dimensional, structured subsets of $\mathbb{R}^n$, to which we can express smooth deformations via some mapping. We will characterise the 'interesting' subsets of $\mathbb{R}^n$ as objects which are in a local sense Euclidean, and intrinsically of dimension less than $n$. Further, at least for now, we will only consider those objects which are smooth everywhere.

Let $M$ be contained in some open set $U \subset \mathbb{R}^{n+1}$, determined by a smooth *embedding map* $f : \Omega \to \mathbb{R}^{n+1}$ with $F(\Omega) = M$ where $\Omega \subset \mathbb{R}^n$ is open. $M$ is called an *$n$-dimensional hypersurface* or simply a hypersurface, since the codimension of $M$ is 1. One can think of this as meaning that after the embedding map has taken $\Omega$ to $M$, there is still one dimension 'left over' in the ambient space $\mathbb{R}^{n+1}$. These 'left over' dimensions are called *codimensions*. Equivalently, one codimension also means that at every point $x = F(p) \in M$, $p \in \Omega$, the space of vectors normal to $M$ anchored at $x$ is one dimensional.

Of course there are many generalisations to this particular notion of 'interesting set', and an enormous body of work supporting them. However for us we will not consider a large number of abstractions or even very general results and constructions. Here, our most general consideration will be an isometric immersion $f : \Omega \to M$ where $\Omega \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^{n+k}$, which is an $n$-dimensional manifold with $k$ codimensions. The word 'isometric' here means that the geodesic distance on $M$ is inherited from the metric on $\mathbb{R}^{n+k}$. We will make this more precise later.

One general notion deserves mentioning here. Given a hypersurface, say the three dimensional sphere $\mathcal{S}^3$ immersed in $\mathbb{R}^4$ (this is to say that at every point $p \in \mathcal{S}^3$, the tangent space is three dimensional), we can consider now immersions with codomain $\mathcal{S}^3$. That is, let $\Sigma \subset \mathbb{R}^p$ be open with $p \in \{1, 2\}$ and consider a smooth map $g_p : \Sigma \to N$ where $N \subset \mathcal{S}^3$ is open. It seems naturally a much more general setting, to consider submanifolds of 'nice' objects such as $\mathcal{S}^3$ instead of simply isometric immersions in $\mathbb{R}^{n+k}$.

This intuition holds, and so-called submanifold theory is in fact a strictly more general setting. However, one may also consider the *intrinsic* analog. Consider a set which exhibits, in terms of its intrinsic geometry, properties which are desired by our 'interesting' sets. One may be worried that in our study of immersions or embeddings of Riemannian manifolds into Euclidean space, we are losing some portion of all possible Riemannian manifolds. In other words, perhaps there are some very strange Riemannian manifolds for which there is no embedding into Euclidean space. Can each $g_p$ be isometrically immersed in Euclidean space? The answer is a resounding 'yes', due to the celebrated result of Nash [**50**].

THEOREM 1.1 (Nash Embedding Theorem). *Any $n$-dimensional Riemannian manifold $N$ with $C^k$ metric, where $k \geq 3$, has a $C^k$ isometric embedding in an arbitrarily small portion of $\mathbb{R}^{n+l}$, for $l \geq \frac{3}{2}n^3 + 7n^2 + \frac{11}{2}n$.*

Therefore, if we can move to considering isometric immersions of arbitrary (or large enough) codimension $n + l$, we will have (up to a diffeomorphism) covered all Riemannian manifolds of dimension $n$. Obtaining results in arbitrary codimension is not always possible however, and generally range from being as difficult as one codimension to being much more difficult. We will remark throughout the thesis on which of our results carries over easily to the case of arbitrary codimension.

We organise the remaining parts of this chapter as follows. Section 2 gives an overview of the basic definitions and geometric facts associated with hypersurfaces, with some elementary results. Section 3 details some additional notation, so-called modern, intrinsic or suffix-free notation, which will be useful at times in the coming chapters. We finish the chapter with some notes and further references for the interested reader.

## 2. Geometry of hypersurfaces

Let $M$ be contained in some smooth open set $U \subset \mathbb{R}^{n+1}$ and be such that $M = f(\Omega)$, where $f : \Omega \to \mathbb{R}^{n+1}$ is a smooth mapping with everywhere injective derivative (that is, $f$ is an *immersion*) and $\Omega \subset \mathbb{R}^n$ is open. We say that $M$ is a *properly immersed hypersurface* if $f^{-1}(K) \subset \Omega$ is compact whenever $K \subset U$ is compact.

The coordinate tangent vectors $\partial_i f(p) \equiv \frac{\partial f}{\partial p_i}(p)$, $1 \le i \le n$, form a basis of the *tangent space* $T_xM$ at $x = f(p)$ for every $p \in \Omega$. Note that this means the tangent space is $n$-dimensional.

The components of the *metric* on $M$ are given by

$$g_{ij} = (\partial_i f \,|\, \partial_j f)$$

for $1 \le i, j \le n$, where $(\cdot \,|\, \cdot)$ is the regular Euclidean inner product in $\mathbb{R}^{n+1}$. When the metric is specified in this way, we call $f$ an *isometric immersion.* For an expanded introduction to these especially nice immersions, please see [**11**, Chapter 6]. Let $(g_{ij})$ denote the matrix with elements $g_{ij}$; then the components of the *inverse metric* are given by inverting $(g_{ij})$; that is

$$(g^{ij}) = (g_{ij})^{-1}.$$

The natural induced area element of $M$ is

$$\sqrt{g} = \sqrt{\det\,(g_{ij})}.$$

We can integrate compactly supported functions $h : M \to \mathbb{R}$ over a properly immersed hypersurface. The integral is defined by

$$\int_M h d\mu \equiv \int_M h d\mathcal{H}^n \equiv \int_M h(x) d\mathcal{H}^n(x) \equiv \int_\Omega h(f(p))\sqrt{g(p)}dp.$$

Here $d\mu$ is the measure on $M$, which we will always choose to be $n$-dimensional Hausdorff measure $\mathcal{H}^n$ on $M$. Note that we always have

$$\mathcal{H}^n(M \cap K) < \infty$$

for any compact $K \subset U$.

We may consider any function $h : M \to \mathbb{R}$ as a function on $\Omega$ via the immersion:

$$h : f(\Omega) \to \mathbb{R}, \qquad h \circ f : \Omega \to \mathbb{R}.$$

The *tangential* or *surface gradient* is defined by

$$\nabla^M h = g^{ij} \partial_j h \partial_i F$$

where we sum over repeated indices from 1 to $n$. Let $X$ be a smooth mapping which takes any point $p \in M$ to a vector $X(p) \in \mathbb{R}^n$ which is tangent to $M$ at $p$; such an $X$ is called a *smooth tangent vector field* and can be decomposed as

$$X = X^i \partial_i f = g^{ij} X_j \partial_i f, \quad \text{where} \quad X_i = (X \,|\, \partial_i f).$$

It is natural to question if one has a suitable notion of differentiation on $M$. There is indeed, and one approach is to simply define the covariant derivative $\nabla^M : \mathcal{X}(M) \to TM$ as the ambient derivative projected back onto the tangent bundle of $M$. That is,

$$\nabla^M X = (DX)^\top.$$

When there is no chance for confusion we will in further chapters omit the $M$ superscript. The components of the *covariant derivative* are given by

$$\nabla_i^M X^j = \partial_i X^j + \Gamma_{ik}^j X^k = g^{jl}(\partial_i X_l - \Gamma_{il}^k X_k)$$

where the *Christoffel symbols* $\Gamma$ are

$$(\partial_{ij} f)^\top = \Gamma_{ij}^k \partial_k f,$$

where the superscript $\top$ denotes the tangential component of a vector. A good exercise is to show that the components of the Christoffel symbols are also given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{1}$$

We define the *tangential divergence of $X$ on $M$* by

$$\mathrm{div}_M X = \nabla_i^M X^i = g^{ij} \nabla_i^M X_j = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} X_j).$$

The last equality follows from our earlier definitions. This expression for the divergence is useful for proving the *divergence theorem*, which will appear later.

The *Laplace-Beltrami operator* of $h$ on $M$ is given by

$$\Delta_M h = \mathrm{div}_M \nabla^M h = g^{ij}(\partial_{ij} h - \Gamma_{ij}^k \partial_k h) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j h).$$

For a smooth vector field $X : M \to \mathbb{R}^{n+1}$ which is not necessarily tangent to $M$, we can also define the divergence with respect to $M$ by the projection

$$\mathrm{div}_M X = g^{ij} \left( \partial_i X \,|\, \partial_j f \right).$$

One can check that this reduces to the previous expression for tangent vector fields.

We now move towards defining the curvature of $M$. First, let $\nu$ be a choice of *unit normal vector field* to $M$. In particular, this satisfies

$$(\nu \,|\, \partial_i f) = 0, \quad \text{and} \quad (\nu \,|\, \nu) = 1$$

on $M$ for $1 \leq i \leq n$. Note that the second identity implies that any derivative of $\nu$ is a tangent vector field to $M$. We make extensive use of the *second fundamental form* $A : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ associated with $M$, with components defined by

$$A_{ij} = (\partial_i \nu \,|\, \partial_j f) = -(\nu \,|\, \partial_{ij} f).$$

The second equality follows from the product rule and the fact that $\partial_i f$ is a tangent vector.

The eigenvalues $\kappa_1, \ldots, \kappa_n$ of the *Weingarten map* given by

$$A^i_j = g^{ik} A_{kj}, \quad (A^i_j) : TM \to TM,$$

are called the *principal curvatures of* $M$. The *mean curvature* $H$ can then be expressed in various ways,

$$H = \sum_{i=1}^{n} \kappa_i = A^i_i = g^{ij} A_{ij} = g^{ij} (\partial_i \nu \,|\, \partial_j f) = \mathrm{div}_M \nu.$$

Combining these we can define the *tracefree second fundamental form*, sometimes called the *tracefree curvature*, as the tensor $A^o$ with components

$$A^o_{ij} = A_{ij} - \frac{1}{n} g_{ij} H.$$

One may immediately check that $\mathrm{tr}\, A^o = g^{ij} A^o_{ij} = 0$. The *mean curvature vector of* $M$ is given by

$$\vec{H} = -H\nu.$$

Using the previous identities, we therefore have

$$\Delta_M f = \vec{H}, \quad \text{and} \quad \Delta^2_M f = \Delta_M \vec{H}.$$

This is relevant since in the case of the *mean curvature flow*, where $f : M \times [0, T) \to \mathbb{R}^{n+1}$ evolves by

$$\partial_t f = \vec{H} = \Delta_M f$$

we see immediately that the structure of the equation is similar to that of the regular heat equation. One may guess that techniques for this equation (such as the maximum principle) will be useful in analysing the flow, and indeed this is very much the case. Contrast this to the case of the *surface diffusion flow*,

$$\partial_t f = \Delta_M \vec{H} = \Delta_M^2 f,$$

and one can guess that here the tools which are of so much help in the analysis of mean curvature flow are not applicable to the surface diffusion flow. In fact this is only half true; some techniques will prove useful, but we will always be working to overcome the deficit. We will greatly expand on this in the coming chapters.

Finally we come to define the *Riemann curvature tensor* of $M$. This is a precise measurement of how well or how poorly the covariant derivatives on $M$ commute. The components of $R$ are defined by

$$\nabla^M_{ij} X_k - \nabla^M_{ji} X_k = R^M_{ijkl} X^l,$$

where $X$ is a tangent vector field on $M$ and

$$\nabla^M_{ij} \equiv \nabla^M_{\tau_i} \nabla^M_{\tau_j} - \nabla^M_{\nabla^M_{\tau_i} \tau_j}$$

denotes the Hessian operator in a local orthonormal frame $\tau_1, \ldots, \tau_n$. Note that for Euclidean space, all the covariant derivatives coincide with partial derivatives and so they commute. Therefore $R^{\mathbb{R}^n}_{ijkl} \equiv 0$, and we call $\mathbb{R}^n$ *flat*.

The Riemann tensor satisfies the symmetry relations

$$R^M_{ijkl} = -R^M_{jikl}, \quad \text{and} \quad R^M_{ijkl} = -R^M_{klij}.$$

The *Gauss equations* express this tensor in terms of the second fundamental form of $M$ by

$$R^M_{ijkl} = A_{ik}A_{jl} - A_{jk}A_{il}.$$

Importantly, this means that in our case (this is not true in general) it is sufficient to study the second fundamental form instead of the full Riemann curvature tensor.

The *Codazzi equations* state that the 3-tensor of covariant derivatives of the second fundamental form

$$\nabla^M A = (\nabla^M_i A_{jk})$$

is totally symmetric.

It will be important for us to consider not only vector fields on $M$, but tensor fields also. The definitions of covariant derivative, Hessian and Laplacian operators are performed analogously. In an orthonormal frame $\tau_1, \ldots, \tau_n$ we denote the component of $\nabla^M_{ij} A$ with respect to $\tau_k$ and $\tau_l$ by $\nabla^M_{ij} A_{kl}$. Note that all of the quantities spoken of thus far are geometric invariants, and so the choice of basis is not important.

We will be making extensive use of the consequences of the Codazzi equations and the interchange of covariant derivatives. We define the *inner product on $M$*, which operates on tensors of similar type, as being the trace over the induced metric

$$\left\langle S^{j_1\cdots j_q}_{i_1\cdots i_p}, T^{j_1\cdots j_q}_{i_1\cdots i_p} \right\rangle = g^{i_1k_1}\cdots g^{i_pk_p} g_{j_1l_1}\cdots g_{j_ql_q} S^{j_1\cdots j_q}_{i_1\cdots i_p} T^{l_1\cdots l_q}_{k_1\cdots k_p},$$

where the summation convention is understood. Note that as all covariant derivatives of the metric are identically zero we have in any local frame

$$\nabla_i^M \langle S, T\rangle = \left\langle \nabla_i^M S, T\right\rangle + \left\langle S, \nabla_i^M T\right\rangle.$$

We define the norm of a tensor as

$$\|T\|^2 = \langle T, T\rangle .$$

In particular, this gives the norm of the second fundamental form as

$$\|A\|^2 = A_i^j A_j^i = g^{ij} g^{kl} A_{ik} A_{jl}.$$

At times we will need to perform analysis with large convoluted contractions, where the exact algebraic structure of each contraction is not critical. To this end, we follow Hamilton [**27**] in using for tensors $T$ and $S$ the notation $T * S$ to denote a new tensor formed by summations of contractions of pairs of indices from $T$ and $S$ by the metric $g$, with possible multiplication by a universal constant. The resultant tensor will have the same type as the other quantities in the equation it appears. Keeping these in mind we also denote polynomials in the iterated covariant derivatives of these terms by

$$P_j^i(T) = \sum_{k_1+\ldots+k_j=i} c\nabla_{(k_1)} T * \cdots * \nabla_{(k_j)} T,$$

where the constant $c \in \mathbb{R}$ is absolute and may vary from one term in the summation to another. In the above we have used $\nabla_{(n)} T$ to denote the tensor with components $\nabla_{i_1\ldots i_n} T_{j_1\ldots}^{k_1\ldots}$. As is common for the $*$-notation, we slightly abuse the absolute constant when certain subterms do not appear in our $P$-style terms. For example

$$\begin{aligned}\|\nabla A\|^2 &= \langle \nabla A, \nabla A\rangle \\ &= 1 \cdot \left(\nabla_{(1)} A * \nabla_{(1)} A\right) + 0 \cdot \left(A * \nabla_{(2)} A\right)\end{aligned}$$

$$= P_2^2(A).$$

To simplify the notation, we will always work in a local orthonormal frame. This means that we can use lower indices only. If we need to work with an arbitrary basis we can raise repeated occurences of each index, relabel and multiply by the metric in that basis.

Using the Gauss equation and definition of the Riemann curvature tensor, we have

$$\nabla^M_{kl} A_{ij} - \nabla^M_{lk} A_{ij} = A_{im} R^M_{mjkl} + A_{mj} R^M_{imkl}.$$

Computing, with the use of the above identity, the Gauss equation and the Codazzi equations:

$$\begin{aligned}
\Delta_M A_{ij} &= \nabla^M_{kk} A_{ij} = \nabla^M_{ki} A_{kj} = \nabla^M_{ki} A_{jk} \\
&= \nabla^M_{ik} A_{jk} + A_{km} R^M_{mijk} + A_{mi} R^M_{mkjk} \\
&= \nabla^M_{ik} A_{jk} + A_{km}(A_{mj} A_{ik} - A_{ij} A_{mk}) + A_{mi}(A_{mj} A_{kk} - A_{kj} A_{mk}) \\
&= \nabla^M_{ij} H - \|A\|^2 A_{ij} + H A_{ik} A_{kj}. \qquad (2)
\end{aligned}$$

This is generally referred to as *Simons' identity* [**56**]. This also implies

$$\Delta_M \|A\|^2 = 2A_{ij} \nabla^M_{ij} H + 2\|\nabla^M A\|^2 + 2H A_{ij} A_{ik} A_{kj} - 2\|A\|^4. \qquad \text{(SI)}$$

We will need to control the second derivatives of the unit normal vector field, and for that purpose we derive an expression for the Laplacian of $\nu$. In the computations to follow it is convenient to work with geodesic normal coordinates on $M$. (Note that we can always do this in a small neighbourhood on $M$, by solving a system of ODEs.) That is, at a point $x = f(p) \in M$ we have

$$g_{ij} = \delta_{ij}, \text{ and } (\partial_{ij} f)^\top = 0.$$

We will perform our calculations at this point $x$. At this point we have

$$\Delta_M \nu = \partial_{ii}\nu.$$

Moreover,

$$A_{ij} = -\left(\partial_{ij} f \,|\, \nu\right), \quad \text{and} \quad \partial_i \nu = A_{ij}\partial_j f,$$

where the second equality follows from $\partial_i \nu$ being a tangent vector field and therefore expressible in the basis $\partial_j f$. The Codazzi equations imply

$$\partial_i A_{ij} = \partial_j A_{ii}.$$

Using these identities we compute

$$\begin{aligned}
\partial_{ij}\nu &= \partial_i \left(A_{ij}\partial_j f\right) \\
&= \partial_i A_{ij}\partial_j f + A_{ij}\partial_{ij} f \\
&= \partial_j A_{ii}\partial_j f + A_{ij}A_{ij}\nu,
\end{aligned}$$

which is

$$\Delta_M \nu = -\|A\|^2\nu + \nabla^M H.$$

This identity is often referred to as the *Jacobi field equation.*

Since we do not have access to tools such as the maximum principle, we must make the most of what we do have, and apart from some special circumstances the only tool available is integration by parts. The *divergence theorem* for smooth, properly embedded hypersurfaces with smooth boundary gives us a way of integrating by parts on manifolds; it states that for any $C^1$ vectorfield $X : \overline{M} \to \mathbb{R}^{n+1}$ the identity

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^n = -\int_M \left(\vec{H} \,\middle|\, X\right) d\mathcal{H}^n + \int_{\partial M} \left(X \,|\, \gamma\right) d\mathcal{H}^{n-1}$$

holds where $\gamma$ denotes the outer unit normal vector field to $\partial M$ which is tangent to $M$ at all boundary points (note that in particular this is *not* the vectorfield $\nu$). If $X$ has compact support or if $\partial M = \emptyset$ this reduces to

$$\int_M \mathrm{div}_M X \, d\mathcal{H}^n = -\int_M \left( \vec{H} \,\middle|\, X \right) d\mathcal{H}^n;$$

and if in addition $X$ is a tangent vector field we have

$$\int_M \mathrm{div}_M X \, d\mathcal{H}^n = 0.$$

For a function $\phi \in C_0^2(\mathbb{R}^{n+1})$ the divergence theorem implies

$$\int_M \mathrm{div}_M D\phi d\mu = -\int_M \left( \vec{H} \,\middle|\, D\phi \right) d\mu, \quad \text{and} \quad \int_M \Delta_M \phi d\mu = 0.$$

Let $\eta \in C^2(\mathbb{R}^{n+1})$ and then we also have

$$\int_M \phi \Delta_M \eta d\mu = -\int_M \left( \nabla^M \phi \,\middle|\, \nabla^M \eta \right) d\mu = \int_M \eta \Delta_M \phi d\mu.$$

If $\phi$ does not vanish on the boundary of $M$ then

$$\int_M \mathrm{div}_M D\phi d\mu = -\int_M \left( \vec{H} \,\middle|\, D\phi \right) d\mu + \int_{\partial M} \left( \nabla^M \phi \,\middle|\, \gamma \right) d\sigma$$

where we used that

$$\left( D\phi \,|\, \gamma \right) = \left( \nabla^M \phi \,\middle|\, \gamma \right)$$

since $\gamma$ is tangent to $M$.

## 3. Suffix free notation

We now come to introduce the notation which we will use quite often in this thesis, following Kuwert and Schätzle [**36, 37, 38**] who used this notation in their analysis of the Willmore flow. For a reader unfamiliar with the terminology used here the most complete reference is Kobayashi and Nomizu [**32, 33**]. Although at times extraordinarily terse, these books give all the details in full rigour.

The main advantages of this notation are that, especially in arbitrary codimension, there are far fewer indices to be confused with each other, and that the computations are often much quicker.

Let $f : M \to \mathbb{R}^{n+k}$ be an immersion as before. Let $X, Y, Z$ be tangent vector fields and $\phi$ a normal vector field. All vector fields are defined along $f$. The induced metric is given by

$$g(X, Y) = (Df \cdot X \,|\, Df \cdot Y) = (D_X f \,|\, D_Y f)\,.$$

Here $Df$ denotes the matrix of partial derivatives of $f$ and the notation $Df \cdot X$ denotes multiplication of the matrix $Df$ with the vector $X$; this is exactly the familiar notion of a directional derivative, which is expressed in the second equality. The passage from the previous notation to this is straightforward: for a tensor $T$ $p$ times covariant and an orthonormal basis $\{\tau_i\}$ of $TM$,

$$T_{i_1 i_2 \cdots i_p} = T(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_p).$$

Then, evaluating $T$ with more general vector fields $X_i$ reduces to knowledge of the components of $T$ since

$$\begin{aligned} T(X_1, X_2, \ldots, X_p) &= T\Big(\tau_{i_1} \langle X_1, \tau_{i_1}\rangle, \tau_{i_2} \langle X_2, \tau_{i_2}\rangle, \ldots, \tau_{i_p} \left\langle X_p, \tau_{i_p}\right\rangle\Big) \\ &= \langle X_1, \tau_{i_1}\rangle \langle X_2, \tau_{i_2}\rangle \cdots \tau_{i_p} \left\langle X_p, \tau_{i_p}\right\rangle T_{i_1 i_2 \cdots i_p}. \end{aligned}$$

Note that we used the linearity of $T$ here and of course there are many summations. Note that this expression is well defined since tensors are invariant under change of coordinates. We will define the covariant derivative of functions, vectors and tensors as follows. Let $X, X_i, Y$ be vector fields on $f$. For functions $h : M \to \mathbb{R}^{n+k}$,

$$\nabla_X h = (D_X h)^\top = (Dh \cdot X)^\top\,;$$

for vector fields,

$$\nabla_X Y = (D_X Y)^\top = (DY \cdot X)^\top ;$$

and for covariant tensors $T$ of degree $s$,

$$(\nabla T)(X_1, \ldots, X_s; X) = (\nabla_X T)(X_1, \ldots, X_s).$$

For the normal bundle we also have a covariant derivative, which we will in fact use more often than the covariant derivative on $M$. Let $\phi$ be a normal vector field on $M$ and $S$ a normal covariant tensor field of degree $s$. Then we have

$$\begin{aligned} \nabla^\perp_X h &= (D_X h)^\perp = (Dh \cdot X)^\perp , \\ \nabla_X \phi &= (D_X \phi)^\perp = (D\phi \cdot X)^\perp , \text{ and} \\ (\nabla^\perp S)(X_1, \ldots, X_s; X) &= (\nabla^\perp_X S)(X_1, \ldots, X_s). \end{aligned}$$

It is easy to see from the above formulae that the regular derivative in $\mathbb{R}^{n+k}$ splits into the covariant derivative in the tangent bundle $TM$ and the covariant derivative in the normal bundle $TM^\perp$.

We shall need to take repeated covariant derivatives of tensors and so will expand upon the latter formula. For the details of these computations we refer to chapters 2 and 7 of Kobayashi and Nomizu [**32, 33**]. To avoid duplication, let $\nabla$ denote both the covariant derivative and covariant derivative in the normal bundle.

For a covariant tensor $S$ of degree $s$ we form a new tensor $(\nabla S)$ of degree $(s+1)$ defined as above. An alternative, more useful expression for $(\nabla S)$ is

$$(\nabla S)(X_1, \ldots, X_s; X) = \nabla_X (S(X_1, \ldots, X_s)) - \sum_{i=1}^{s} S(X_1, \ldots, \nabla_X X_i, \ldots, X_s).$$

Note that $S(X_1, \ldots, X_s)$ is a function.

The second covariant differential of $S$, $\nabla_{(2)}S$, is a covariant tensor field of degree $(s+2)$ defined as $\nabla(\nabla S)$, which is

$$(\nabla_{(2)}S)(X_1,\dots,X_s;X;Y) = (\nabla_Y(\nabla S))(X_1,\dots,X_s;X).$$

A similar expression to that given above for $(\nabla S)$ is rather complicated and long. A shortened form is

$$(\nabla_{(2)}S)(;X;Y) = \nabla_Y(\nabla_X S) - \nabla_{\nabla_Y X}S.$$

In general, $\nabla_{(m)}S$ is defined inductively to be $\nabla(\nabla_{(m-1}S)$, and

$$(\nabla_{(m)}S)(X_1,\dots,X_s;Y_1;\dots;Y_m) = (\nabla_{Y_m}(\nabla_{(m-1)}S))(X_1,\dots,X_s;Y_1;\dots;Y_{m-1}).$$

## 4. Notes

The overview given here falls short of satisfactory for a large number of reasons, and the reader inexperienced in differential geometry must be aware of this. Our primary goal here is to set our notation, as in some places it is nonstandard. At times we will switch between index and suffix free notation, using whichever is most convenient for a given computation.

The treatment of the geometry of hypersurfaces given here is heavily inspired by Appendix A of Ecker's excellent book [**12**]. For an introduction to singularity analysis for the mean curvature flow, and regularity theory for hypersurfaces, this book comprises a clear and succinct treatment with sufficient but minimal geometric measure theory.

There are mountains of books on differential geometry, and it would be impossible to give a mention of them all. We will give comment on a small selection and invite the interested reader to find and enjoy more references of their own. The hard hitting reference for specialists are the two volumes of Nomizu and Kobayashi [**32, 33**]. These are essentially self-contained, but are not suitable for use as introductory texts. As references however they are irreplaceable. For a more gentle and geometric introduction, including the suffix free notation, do Carmo [**11**] is recommended. Beware however of the mistake with the chain rule and as always take note of sign conventions.

A classic but still relevant reference is the series by Spivak [**58**], which if you can forgive the typesetting is quite readable. For references more on geometric heat flows, such as mean curvature flows, some calculus of variations is useful. In most modern differential geometry texts there are small sections devoted to this. The collection of articles [**9**] is a good resource for the beginner.

Unfortunately, higher order flows are not so well studied and there are no standard references. We will be continuing the framework of Kuwert and Schätzle [**36, 37, 38**] who analysed the Willmore flow. However the case for second order flows is significantly better, and many of the notions used in the analysis of fourth order flows are completely analogous to the second order case (even if the techniques are required to be different). For the mean curvature flow, the aforementioned treatment given by Ecker [**12**] is a good source. We also mention the following paper of Huisken and Polden [**29**] which gives a survey of some results for various geometric evolution equations. The references contained within these two sources provide a comprehensive survey of the field.

CHAPTER 2

# Short time existence for higher order hypersurface flows

## 1. Introduction

The question of short time existence for higher order hypersurface flows has a colourful history. Several contemporary papers will quote Polden [**52**], or Huisken and Polden [**29**], such as Kuwert and Schätzle [**36, 37, 38**], Mantegazza [**44**], and others. We began with this, as the statement for local existence in [**52**] is extremely general. However, as is in fact common knowledge (being known at the very least by Bartnik, Huisken, Ecker, Kuwert, Schätzle, Andrews, and so on), there are some mistakes such as the usage of the linearisation of the quasilinear equation, and in the usage of the smoothness assumptions on the coefficients of the differential operator. Despite this, it is also common knowledge of how to fix these issues. It is in fact the case that there are a large variety of methods and techniques to obtain short time existence. Thus we do not claim any fraction of originality or ingenuity in proof: our arguments and presentation in this chapter all essentially belong to standard theory. We recommend that the reader not interested or already well-versed in short time existence skip ahead to Chapter 3.

It is worth noting that Sharples [**53**] fixed the mistakes in Polden, however there the focus is on second order equations and the higher order case is only remarked upon briefly in the introduction to the paper. We will be performing a similar feat, in that we will attempt to recover a similarly useful short time existence theorem to Polden, however we will not pursue the same technique.

Our technique is inspired by personal communication with Kuwert [**35**], where it is suggested that to prove short time existence it is sufficient to write the evolution as a graph and then use standard parabolic existence and uniqueness theory. This is essentially our procedure, although it appears the treatment of higher order parabolic equations is exclusively limited to linear equations (and our equation is very much *quasi*linear when written as a graph) and we must provide the standard alterations to upgrade the linear theory to the quasilinear context. Once we obtain the existence and uniqueness theorem for the quasilinear case, we use the method of applying a tangential diffeomorphism at each time step to our flow to ensure that the domain of our graph function is independent of time. Combining these results, and tiling the time interval as much as possible, we obtain the required theorem.

Finally we must note that the treatment given here is classical in nature, and there is an alternative: semigroup theory. Indeed, this is the technique used by Escher, Mayer and Simonett [**16**], where they quote recent results due to Amann to obtain short time existence for surface diffusion flow. However, they do not state any theorem to this effect and regardless the referred to theorem due to Amann does not appear to be published anywhere. It has recently come to light that there is now a reference which treats short time existence thoroughly: Koch and Lamm [**34**]. There a technique is employed which is very similar to ours here, however there they work in a slightly altered function space to obtain uniqueness in a slightly different way. Their approach also allows one to obtain short time existence for initial data which is only $C^1$.

Due to the sometimes difficult-to-find nature of the required references for our chosen approach, we have chosen to use major results from a variety of sources. Due to this, our exposition involves the equivalent reformulations of the problem at hand in a variety of styles. It is also because of this aspect that our discussion below is restricted to signposting the various facts required and theorems necessary for the proof. Our goal is that an interested reader will be able to use this chapter as a roadmap to a complete, rigorous proof. We collect the major references now. The linear estimates we have used follow the treatment of Friedman [**20**], and Eidelman, Zhitarashu [**14**]. We refer the reader to the papers referenced therein for the historical development of the linear theory. The fixed point argument can be found in many good books on parabolic or elliptic partial differential equations, such as Taylor [**60, 61**], Gilbarg and Trudinger [**25**], Lieberman [**42**], and of course Ladyzhenskaya, Solonnikov and Ural'ceva [**39**], among many others. The usage of tangential diffeomorphisms to fix the domain of our graph function can be found in Ecker [**12**] and Ecker, Huisken [**13**]. We have presented our results in a sufficiently general manner to be hopefully useful for further applications.

## 2. Linear theory

The classical theory of existence and uniqueness for solutions to linear higher order parabolic partial differential equations is really a large work, and more suited to a textbook than a section in a thesis. Further, the basic ideas and techniques are all very standard, in that even though we consider the higher order setting the techniques and machinery involved are essentially the same as the second order case. Therefore our strategy here is to state the main result in detail and then provide a sketch of the proof with references; more than a summary but far from complete detail. The main references for this section are the monographs Friedman [**20**] and Eidelman, Zhitarashu [**14**]. There the problem under consideration is higher order parabolic systems and higher order parabolic systems with boundary, respectively. As there is not much more difficulty, we will also pursue the boundary value problem. We see this as possibly being useful for future applications, for example a constrained Willmore flow with Neumann boundary is often used in applied mathematics to solve problems in image processing and computer vision, see [**17**] for example. There, the question of short time existence is essentially ignored; nonetheless it is still important for us. We will not however consider the case of general parabolic systems, or equations parabolic in the sense of Solonnikov and Shirota, since the changes are straightforward and serve to obscure the underlying argument and notation. The interested reader may enjoy the discussion in [**14**] which includes all the changes required.

Our linear problem is

$$\mathcal{L}(x,t;\partial,\partial_t)u \equiv \partial_t u - \sum_{|\alpha|\le 2b} a_\alpha(x,t)\partial^\alpha u = f(x,t) \tag{3}$$

$$u\Big|_{t=0} = \psi(x) \tag{4}$$

$$\mathcal{B}(x,t;D,D_t)u\Big|_S = \sum_{|\alpha|+2b\alpha_0\le r_q} b^q_{\alpha\alpha_0}(x,t)\partial^\alpha\partial_t^{\alpha_0}u\Big|_S = \phi_q \text{ for } q=1,\dots,b, \tag{5}$$

in the cylinder $\Omega = G \times (0, T]$, where $S = \partial G \times (0, T]$, and the coefficients $a_\alpha$, $b^q_{\alpha\alpha_0}$ are functions indexed by $\alpha_0$ and the multi-index $\alpha$, which vary with each term in the summation. In the systems context each of these coefficients is an $(m \times m)$ matrix, with $m$ being the number of equations. The system is of $2b$-th order. (It is impossible for an odd ordered equation to be parabolic.) The operators $\mathcal{L}$ and $\mathcal{B}$ are linear, and $\mathcal{L}$ is parabolic in the sense of Petrovski. This means that at every point $(x, t)$ in $\Omega$ the $p$-zeroes of the polynomial in the principal part $L_0(x, t; i\xi, p)$ of $\mathcal{L}$ satisfy the inequality

$$\text{Re } p(x, t, \xi) \leq -\delta_0(x, t)|\xi|^2, \quad \text{for some } \delta_0(x, t) > 0.$$

We do not require *uniform* parabolicity, so $\delta_0$ is allowed to vary with the choice of point $(x, t) \in \Omega$. Note that the function $f(x, t)$ is a known function of time and space.

We have some notation and conditions to get through before stating the main theorem. Let $C^l_0(\overline{\Omega})$ and $C^l_0(\overline{S})$ be the set of functions $f$ belonging to $C^l(\overline{\Omega})$ and $C^l(\overline{S})$ respectively which also satisfy

$$\partial_t^{\alpha_0 - 1} f \Big|_{t=0} = 0, \text{ for } \alpha_0 = 1, \ldots, \frac{l}{2b} + 1.$$

We will need the following conditions:

$(\beta_1) : a_\alpha(x, t) \in C^l(\overline{\Omega})$

$(\beta_2) : b^q_{\alpha\alpha_0}(x, t) \in C^{l+2b-r_q}(\overline{S})$

$(\beta_3) : \partial G \in C^{l+2b}$

$(\beta_4)$ : The right hand side of (3)-(5) satisfies

$$f \in C^l(\overline{\Omega}), \quad \psi \in C^{l+2b}(\overline{G}), \quad \phi_q \in C^{l+2b-r_q}(\overline{S}),$$

$$\text{for } q = 1, \dots, b, \text{ and } l > l_0 = \max_q \{0, r_q - 2b\}$$

$$(\beta_5) : \partial_t^{\alpha_0} \phi_q \Big|_{t=0} = 0, \text{ for } \alpha_0 = 0, \dots, \frac{l + 2b - r_q}{2b}, \text{ on } \partial G.$$

The last of these conditions are called *compatibility* conditions of order $\frac{l+2b}{2b}$. The degree to which these are satisfied determines the regularity of the solution we obtain.

We can now state the main theorem, due primarily to V. A. Solonnikov [**57**].

THEOREM 2.1 (Solonnikov). *Consider the parabolic boundary value problem* (3)-(5) *satisfying* $\beta_1 - \beta_4$ *and compatibility conditions* $\beta_5$ *of order* $\frac{l+2b}{2b}$. *Then for any non-integer* $l > l_0 = \max\{0, r_1 - 2b, \dots, r_b - 2b\}$ *the problem* (3)-(5) *has a unique solution* $u(x,t)$ *in the space* $C^{l+2b}(\overline{\Omega})$ *and the following estimate holds*

$$|u, \Omega|_{l+2b} \leq C\left( |f, \Omega|_l + |\psi, G|_{l+2b} + \sum_{q=1}^{b} |\phi_q, S|_{l+2b-r_q} \right),$$

*where* $C$ *is a positive constant not depending on* $u, t, \psi, \phi_q$.

Recall that the norm in the conclusion above is defined for some positive real number $s$ by

$$\begin{aligned} |u(x,t), \Omega|_s = & \sum_{|\alpha|+2b\mu \leq [s]} \sup_{(x,t)\in\Omega} |\partial^\alpha \partial_t^\mu u(x,t)| \\ & + \sum_{|\alpha|+2b\mu = [s]} \sup_{(x,t),(y,t)\in\Omega} \frac{\partial^\alpha \partial_t^\mu u(x,t) - \partial^\alpha \partial_t^\mu u(y,t)}{|x-y|^{s-[s]}} \\ & + \sum_{0 < s - 2b\mu - |\alpha| < 2b} \sup_{(x,t),(x,\tau)\in\Omega} \frac{\partial^\alpha \partial_t^\mu u(x,t) - \partial^\alpha \partial_t^\mu u(x,\tau)}{|t-\tau|^{(s-2b\mu-|\alpha|)/2b}}. \end{aligned}$$

The proof makes extensive usage of higher order analogues of the parabolic fundamental solution. One of the estimates obtained during the course of this proof will be useful for the purposes of uniqueness later. The proof is contained in [**57**]. A version (with proof) in the terminology of semigroups can be found in [**41**]. In the below we use the notation

$$A_i = \sum_{|\alpha| \le 2b} a_\alpha(x, t, u_i)\partial^\alpha$$

for the spatial part of the linear operator. The norm on this operator is the natural induced operator norm.

THEOREM 2.2. *Let $0 < \xi < \eta \le 1$. Under the conditions of Theorem 2.5 above, the difference of two solutions $u_1, u_2$ with initial data $\omega_1, \omega_2$ respectively may be estimated by*

$$\|u_1(t) - u_2(t)\|_\xi \le c\Big(t^{\eta-\xi}\|A_1 - A_2\|\|\omega_1\|_\eta + \|\omega_1 - \omega_2\|_\xi\Big),$$

*for some $c > 0$ not depending on $u_i$.*

The norm in use here is in an interpolation space, which will be discussed in the second part of the next section.

As mentioned earlier, the proof of Theorem 2.1 above in detail is a long and complicated process. We will instead outline the overall idea; familiarity with the second order case will be invaluable.

**2.1. Overview of proof of Theorem 2.1.** The most difficult step in the proof is to show that the problem (3)-(5) is well-posed in a cylinder $\Omega_h$ of small height $h \le T$ in the case where $\psi = 0$, $f \in C_0^l(\overline{\Omega_h})$, $\phi_q \in C_0^{l+2b-r_q}(\overline{S_h})$, $q = 1, \ldots, b$. Once we have local existence for this problem then we can consider $u - \psi$ for non-zero initial data, and begin the process again at $t = h$, tiling the cylinder $\Omega$.

To solve the problem in a small cylinder $\Omega_h$, we consider the equivalent formulation of finding a solution to the operator equation

$$Au = F \tag{6}$$

where $A$ is the linear operator from the linear space $B_1 = C_0^{l+2b}(\overline{\Omega_h})$ to $B_2 = C_0^l(\overline{\Omega_h}) \times \prod_{q=0}^b C_0^{l+2b-r_q}(\overline{S_h})$ which assigns to each function $u(x,t) \in B_1$ the function $F = (f, \phi_1, \dots, \phi_b)$. $B_1$ and $B_2$ are Banach spaces with the norms

$$|u|_{B_1} = |u, \Omega_h|_{l+2b}, \qquad |F|_{B_2} = |f, \Omega_h|_l + \sum_{q=1}^b |\phi_q, S_h|_{l+2b-r_q}.$$

Well-posedness of (6) is equivalent to the operator $A$ possessing a bounded inverse operator $A^{-1}$, acting from the whole of $B_2$ onto $B_1$. We find $A^{-1}$ as follows. We need to first construct a *regulariser* of $A$. For our problem this is an operator $R : B_2 \to B_1$ such that

$$AR = I + V; \quad RA = I + W, \tag{7}$$

where $I$ is the identity operator in $B_2$ in the first equation and $B_1$ in the second, and where $V, W$ are bounded operators of norm less than 1 in $B_2$ and $B_1$ respectively.

Due to the last property we can use the contraction mapping principle to imply that the operators $I+V$ and $I+W$ have bounded inverses $(I+V)^{-1}$ and $(I+W)^{-1}$. Therefore, using (7),

$$AR(I+V)^{-1} = I \quad \text{and} \quad (I+W)^{-1}RA = I.$$

That is, the operator $A$ possesses both a right bounded inverse $R(I+V)^{-1}$ and a left bounded inverse $(I+W)^{-1}R$. These inverses must necessarily coincide and so

$$A^{-1} = R(I+V)^{-1} = (I+W)^{-1}R.$$

Therefore in the small cylinder $\Omega_h$ the problem reduces to the construction of a regulariser. Unfortunately, this is by no means straightforward. We will provide a brief summary of the steps required.

**2.2. Building the regulariser.** We construct the regulariser by pasting together, with the help of a suitable partition of unity, operators which solve model problems of two types. The first is the Cauchy problem for the parabolic equation

$$\mathcal{L}_0(x^0, 0; \partial, \partial_t)u = f,$$

where the coefficients are 'frozen' at a point $(x^0, 0)$, $x^0 \in G$. Solving this problem reduces to finding *fundamental solutions* with very nice properties. These are essentially higher order analogues of heat kernels. Please refer to [**14**], Chapter IV for details. The second model problems are the boundary value problems in $\mathbb{R}^n_+ \times (0, T]$, obtained by passing to a local coordinate system with origin $x^0 \in \partial G$ in equations (3)-(5) and in the initial conditions $\mathcal{L}_0(x^0, 0; \partial, \partial_t)$ and $\mathcal{B}^0_q(x^0, 0; \partial, \partial_t)$ for $q = 1, \ldots, b$. We can think of these second type of problems as zooming in on a point on the boundary of $G$ and considering the problem as being formed in a half space by transforming the boundary locally to the tangent plane at $(x^0, 0)$. The details for the solution to these problems can also be found in [**14**], Chapter V.

To show that the operator $R$ has the desired properties we must resort to sharp estimates of solutions of the model problems above. These estimates are in turn obtained with the help of formulae representing the solutions of these problems in a form convenient for analysis: Poisson kernels and so on. We will present an overview of these briefly below.

**2.3. Estimates for the model problem.** In $\mathbb{R}^{n+1}_{++} = \mathbb{R}^n_+ \times (0, \infty)$ consider the following model parabolic problem:

$$\begin{aligned} \mathcal{L}_0(\partial_{x'}, \partial_{x_n}, \partial_t)u &= f \\ u\Big|_{t=0} &= \psi \\ \mathcal{B}^0_q(\partial_{x'}, \partial_{x_n}, \partial_t)u\Big|_{x_n=0} &= \phi_q, \text{ for } q = 1, \dots, b, \end{aligned} \tag{8}$$

where $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is smooth with compact support and where $x' = (x_1, \dots, x_{n-1})$. We will use the prime similarly throughout this subsection to denote deletion of the last coordinate. Our goal is to show that the solution to (8) above satisfies nice estimates which will allow the construction of a regulariser, as outlined earlier. We first represent the solution to (8) as a combination of elements of the *Poisson basis*, which is obtained through integral transformations. Then, the estimates we require come from estimating these 'kernels' of the solution, which will itself also rely on estimates found in basic treatments for functions which are very similar to heat kernels (called the *fundamental solutions*).

We first consider the problem:

$$\begin{aligned} \mathcal{L}_0(\partial_{x'}, \partial_{x_n}, \partial_t)u &= 0 \\ u\Big|_{t=0} &= 0 \\ \mathcal{B}^0_q(\partial_{x'}, \partial_{x_n}, \partial_t)u\Big|_{x_n=0} &= \phi_q, \text{ for } q = 1, \dots, b. \end{aligned} \tag{9}$$

Using the Laplace transformation in $x$ and the Fourier transformation in $x_1, \dots, x_{n-1}$ we obtain

$$\hat{u}(x_n, \xi', p) = \int_0^\infty e^{-pt} \int_{\mathbb{R}^{n-1}} e^{-i(x',\xi')} u(\xi', x_n, p) d\xi' dt.$$

So the problem (9) is transformed to

$$\begin{aligned}\mathcal{L}_0(i\xi', \frac{d}{dx_n}, p)\hat{u} &= 0 \\ \mathcal{B}_q(i\xi', \frac{d}{dx_n}, p)\hat{u} &= \hat{\phi}_q(\xi, p), \text{ for } q = 1, \ldots, b,\end{aligned} \tag{10}$$

and

$$|\hat{u}(x_n, \xi', p)| \to 0 \text{ as } x_n \to \infty.$$

Since (10) is an ODE we can solve it by a number of techniques; for example the method of residues will suffice, see Chapter I in [**14**] for an exposition of this approach. We obtain the solution $\hat{u}$ written in the form

$$\hat{u}(x_n, \xi', p) = \sum_{q=1}^{b} \hat{G}_q(x_n; \xi', p)\hat{\phi}_q(\xi', p).$$

The functions $\hat{G}_q$ are called the *elements of the Poisson basis.* They can be represented as contour integrals in the complex plane. We will not pursue that here. The interested reader can find the details in Vladimirov [**64**].

Using the inversion formulae from the theory of Laplace and Fourier integral transformations, in particular the fact that a product is transformed into a convolution, we obtain the following very useful expression for $u$:

$$u(x, t) = \sum_{q=1}^{b} \int_0^t \int_{\mathbb{R}^{n-1}} G_q(x - y', t - \tau)\phi_q(y', \tau)dy'd\tau, \tag{11}$$

where

$$G_q(x, t) = -i(2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{i(x',\xi')} \int_{a-i\infty}^{a+i\infty} e^{pt}\hat{G}_q(\xi; x_n, p)dpd\xi', \tag{12}$$

where $a > 0$ is arbitrary. The functions $G_q(x, t)$ are called the *Poisson kernels* of problem (9).

Our goal now is to investigate the smoothness and regularity of the solution to (9), which as we can see from the above translates to estimates of the derivatives of the Poisson kernels $G_q$. The technique used in [**57**] is via a novel transformation of the aforementioned representation of $\hat{G}$ in the complex plane. Details of this method can be found in Eidelman and Zhitarashu [**14**], Chapter VII. The result obtained is:

THEOREM 2.3 (Solonnikov). *The Poisson kernels $G_q(x,t)$ are defined and infinitely differentiable in $\mathbb{R}^n_{++}$. Their derivatives satisfy the following inequalities, where $q = 2b/(2b-1)$:*

$$|D^\alpha D_t^{\alpha_0} G_q(x,t)| \leq C_{\alpha\alpha_0} t^{-(n+2b-r_q+|\alpha|+2b\alpha_0)/2b} e^{-c|x|^q t^{1-q}}.$$

These estimates suffice for the construction of the regulariser as given in Eidelman and Zhitarashu [**14**], Chapter IV.

We have two tasks remaining. The first is that we still need to deal with the 'other half' of problem (8), again on $\mathbb{R}^{n+1}_{++}$:

$$\begin{aligned} \mathcal{L}_0(\partial_{x'}, \partial_{x_n}, \partial_t)u &= f \\ u\Big|_{t=0} &= \psi \\ \mathcal{B}_q(\partial_{x'}, \partial_{x_n}, \partial_t)u\Big|_{x_n=0} &= 0. \end{aligned} \tag{13}$$

The second is to indicate how one obtains the *sharp* estimates required to prove that the operator $R$ we construct possesses the desired properties of a regulariser; in particular, the norm property which allows us to use the contraction mapping principle (as mentioned earlier).

To find the solution of (13) we use a homogeneous Green function, defined below.

Definition. *A homogeneous Green function $G_0(x, y, t)$ of the problem* (8) *is determined by the relationship*

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} G_0(x, y, t-\tau) f(y, \tau) dy d\tau,$$

*where $u(x, t)$ is a solution of the problem* (8) *with null initial and boundary conditions, and the function $f(y, t)$ is smooth with compact support.*

We desire a homogeneous Green function of the form

$$G_0(x, y, t) = \Gamma_0(x - y, t) - V(x, y, t), \tag{14}$$

where $\Gamma_0(x, t)$ is a fundamental solution of the Cauchy problem

$$\mathcal{L}_0(\partial, \partial_t) u = 0.$$

To solve (8) we must then require $V(x, y, t)$ to be a solution of the boundary value problem

$$\begin{aligned} \mathcal{L}_0(\partial, \partial_t) u &= 0 \\ u\Big|_{t=0} &= 0 \\ \mathcal{B}_0(\partial, \partial_t) u\Big|_{x_n=0} &= \mathcal{B}_0(\partial, \partial_t)\Gamma_0\Big|_{x_n=0}. \end{aligned} \tag{15}$$

The problem (15) above is solved by (11) and (12), as with problem (9) earlier.

The three part work of Gel'fand and Shilov [**24**] gives the following estimates for $\Gamma_0$. Note that there are many similar estimates earlier in the literature (such as Friedman [**20**]) but those estimates are not sharp. The technique used in [**24**] is the Fourier integral transform of entire functions which satisfy certain nice inequalities, and so we only state a special case of the result obtained there.

Theorem 2.4. *The fundamental solution $\Gamma_0(x, t)$, regarded as a function of $(x_1 t^{-1/2b}, \ldots, x_n t^{-1/2b})$, is an entire analytic function having decay of order $q =$*

*$2b/(2b-1)$, where $x$ is real. The complex continuation $\Gamma_0(z,t)$ has growth of order $q = 2b/(2b-1)$ for imaginary values of $z$, and satisfies the estimates*

$$|\partial^\alpha \partial_t^{\alpha_0} \Gamma_0(z,t)| \leq C_{\alpha\alpha_0} t^{-(n+|\alpha|+2b\alpha_0)/2b} \exp\Big(t^{1-q}(-c_3|Re\ z|^q + c_4|Im\ z|^q)\Big).$$

Remark. For the simplest case of the heat equation

$$\partial_t u = a^2 \Delta u + f(x,t), \quad u|_{t=0} = \psi(x),$$

the fundamental solution is

$$\Gamma_0'(x,t) = (2a\sqrt{t\pi})^{-n} \exp\Big(-|x|^2/4a^2t\Big),$$

which we rewrite for comparison purposes as

$$\Gamma_0'(x,t) = (2a\sqrt{t\pi})^{-n} \exp\Bigg(-(1/4a^2)\sum_{j=1}^n (x_j t^{-1/2})^2\Bigg).$$

It is clear by differentiating that the function $\Gamma_0'$ satisfies the bounds from the theorem above, and thus one of the key properties of the 'heat kernel' above is carried over in an analogous manner to the general fundamental solution of our model problem. This is the chief reason why we claimed that the general fundamental solution is similar to the heat kernel.

For interest we note also that for the $(2b)$-th order analogue to the heat equation

$$\partial_t u + (-1)^b \Delta u = 0,$$

we have the following as fundamental solutions:

$$\Gamma_0''(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp\Big(i(x,\xi) - |\xi|^{2b}t\Big) dt.$$

Note how similar the structure of $\Gamma_0''$ is compared with $\Gamma_0'$. An alternative proof of the bounds in Theorem 2.4 for the function $\Gamma_0''$ above can be found in Eidelman and

Zhitarashu [**14**], where inequalities from the theory of Bessel functions are used as the main tool.

Using Theorem 2.4 above, we can obtain the desired estimates for the homogeneous Green function $G_0(x,y,t)$. The theorem stated below is a combination of work by Solonnikov [**57**], to whom the credit for the first statement is due, and Eidelman and Ivasishen [**31**], whose method of proving the first statement gives the expression for the solution given in the second statement.

THEOREM 2.5. *There exists a homogeneous Green function $G_0(x,y,t)$ of problem* (8)*, which is given by formula* (14)*, is infinitely differentiable with respect to all its arguments for $\{x,y\} \subset \mathbb{R}^n_+$, $t>0$, and satisfies the following estimates (in which $q = 2b/(2b-1)$):*

$$|\partial_y^l \partial_x^\alpha \partial_t^{\alpha_0} G(x,y,t)| \le C_{l\alpha\alpha_0} t^{-(n+|l|+|\alpha|+2b\alpha_0)/2b} \exp\Big(-c|x-y|^q t^{1-q}\Big),$$

*and*

$$|\partial_y^l \partial_x^\alpha \partial_t^{\alpha_0} V(x,y,t)| \le C_{l\alpha\alpha_0} t^{-(n+|l|+|\alpha|+2b\alpha_0)/2b} \exp\Big(-c(|x-y|^q + y_n^q) t^{1-q}\Big).$$

*The solution $u(x,t)$ of problem* (8)*, constructed for smooth functions $f, \psi, \phi_q, q = 1, \ldots, b$ with compact support, is given by the formula*

$$u(x,t) = \int_{\mathbb{R}^n} G_0(x,y,t)\psi(y)dy + \sum_{q=1}^{b} \int_0^t \int_{\mathbb{R}^{n-1}} G_q(x-y', t-\tau)\phi_q(y',\tau)dy'd\tau$$
$$+ \int_0^t \int_{\mathbb{R}^n} G_0(x,y,t-\tau) f(y,\tau) dy d\tau.$$

The second statement above gives the existence of the solution and control on its regularity via the Green's function. Although we do not know a priori that the

Green's function is unique, and therefore can not conclude that the solution is unique from the second statement above, the estimates in this Theorem 2.5 combined with the regulariser argument mentioned earlier do allow us to obtain uniqueness of the solution. This means that when the main argument from the linear theory has been completed, we do eventually prove that the Green's function above is unique, but this may be misleading since it comes as a consequence of the uniqueness of the solution rather than a reason for it.

Theorem 2.5 above and Theorem 2.3 from earlier are thus the key estimates used to drive the linear theory. Using these we can gain control of the norms of the relevant linear operators which we will use to construct the regulariser.

To finish this section we state the regulariser existence theorem. Although we have no desire to give a treatise on functional analysis, we have a little more notation and terminology to get through before we can properly state the final theorem before we can properly state the final theorem.

A sufficient condition for a linear operator $\mathcal{L}$ to satisfy the so-called Lopatinskii condition is for the linear operator to be parabolic in the sense of Petrovski, satisfy the compatibility conditions at the boundary, and that the number of independent boundary conditions be at least $b$, half the order of the system. There are weaker conditions, for systems parabolic in the sense of Solonnikov, detailed in [**14**], Chapter I.

The set $\Omega_+ = G \times [0, \infty)$ is a semi-infinite cylinder with lateral surface $S_+ = \Gamma \times [0, \infty)$. The domain $G$ is bounded, with a smooth $(n-1)$-dimensional boundary $\Gamma = \partial G$. Our linear operators will live in the spaces $\tilde{\mathcal{K}}^s_+$ and $\tilde{\mathcal{H}}^s_+$, which take a little effort to set up. To begin, consider the spaces $H^s(\overline{\Omega}_+)$, which are Hilbert spaces of distributions $v(x)$ with the norm

$$\|v, \overline{\Omega}_+\|_s^2 = \int_{\overline{\Omega}_+} (1 + |\xi|^2)^s |\tilde{v}(\xi)|^2 d\xi$$

and inner product

$$[v, \psi]_s = \int_{\overline{\Omega}_+} (1 + |\xi|^2)^s \tilde{v}(\xi) \cdot \overline{\tilde{\psi}(\xi)} d\xi,$$

where $\tilde{v}$ is the Fourier transform of $v$. Now, we wish to modify the spaces $H^s$ to the spaces $\mathcal{H}^{s,r}$ of distributions which are 'weighted' of order $s$ in $x$ and $t$, and are 'additionally smooth' by a factor $r$ in the 'tangent' (to the boundary) variables $(x', t)$. Let $b > 0$ be a fixed integer. Let $p = \gamma + i\xi_0$. Set

$$\rho(1, \xi, p) = (1 + |\xi|^2 + |\gamma|^{1/b} + |\xi_0|^{1/b})^{1/2}, \rho_1(1, \xi', p) = \rho(1, \xi', 0, p).$$

Then given any real $s, r$, let $\mathcal{H}^{s,r}(\overline{\Omega}_+)$ be the set of distributions which decay at infinity twice as fast as any polynomial, whose Fourier transforms are locally Lebesgue integrable functions for which the following norm is finite

$$\|u(x,t), \overline{\Omega}_+\|_{s,r}^2 = \int_{\overline{\Omega}_+} \rho^{2s}(1, \xi, p) \cdot \rho_1^{2r}(1, \xi', p) \cdot |\tilde{u}(\xi, \xi_0)|^2 d\xi d\xi_0,$$

and is made a Hilbert space by the following inner product

$$(u, v)_{s,r} = \int_{\overline{\Omega}_+} \tilde{u}(\xi, \xi_0) \cdot \overline{\tilde{v}(\xi, \xi_0)} \rho^{2s} \rho_1^{2r} d\xi d\xi_0.$$

Notice that these possibly unfamiliar spaces are isomorphic to the familiar $L_2(\overline{\Omega}_+)$ with the appropriate weight; in the case of $\mathcal{H}^{s,r}(\overline{\Omega}_+)$ this is $\rho^s \rho_1^r$. The addition of the tilde denotes a specific adaptation of these spaces to parabolic boundary value problems. Let $\gamma > 0$ be a given real number. The space $\tilde{\mathcal{H}}_+^s$ is the completion of

$C^\infty(\overline{G})$ with respect to the norm

$$|\{v(x), G\}|_s^2 = \|v_0, G\|_s^2 + \sum_{k\in K} \ll e^{-\gamma t}\omega_k(x'), \Gamma \gg_{s-k+1/2}^2$$

where

$$\ll \phi(x',t), \Gamma \gg_q^2 = \int_\Gamma \rho_1^{2q}(1,\xi',p) \cdot |\tilde{\phi}(\xi',\xi_0)|^2 d\xi' d\xi_0,$$

$v_0(x) = v(x)|_{\overline{G}}$, and $\omega_k(x') = \partial_\nu^{k-1} v(x)|_\Gamma$, where $\nu$ is the inward pointing unit normal at the point $x' \in \Gamma$. The set $K \subset \mathbb{Z}$ is constructed by taking all the 'sizes' of the various orders of differentiation in the operators $\mathcal{L}$ and $\mathcal{B}$; for every multi-index $\alpha$ such that $l_\alpha$ is a coefficient of $\mathcal{L}$, $\alpha \in K$. Note that this forces $2b \in K$. There is one final additional requirement for each space: all elements of $\tilde{\mathcal{K}}_+^s$ elements must also satisfy the compatibility conditions

$$\partial_\nu^{k-1} v_0(x)|_\Gamma = \omega_k(x')$$

for all $x' \in \Gamma$ and all $k \leq s'$.

THEOREM 2.6 (Eidel'man and Zhitarashu [**14**]). *Let the operator $\mathcal{L}$ be uniformly parabolic in $\overline{\Omega}_+$, let the operator $\mathcal{B}$ on $S_+$ satisfy the Lopatinskii condition uniformly in $(x',t) \in S_+$, and let $\Gamma$ be of class $C^{s+r}$. Assume that the coefficients of the operators $\mathcal{L}$ and $\mathcal{B}$ belong to the following classes:*

$$l_{ij\alpha\beta}(x,t) \in C^{|s|+t_j+\sigma_0+\epsilon}(\overline{\Omega}_+), \quad \forall \epsilon > 0,$$

$$b_{qj\alpha\beta}(x,t) \in C^{|s|+t_j-\sigma_q+\epsilon}(\overline{\Omega}_+), \quad \forall \epsilon > 0$$

*and do not depend on $t$ for $t > T_0$. Finally, let $l_{ij\alpha\beta}(x,t)$ be constant for $|x| > R_0$ if $G$ is an unbounded domain. Then there exists a number $\gamma_0$, which depends on $s$ and the Hölder norms of the coefficients of $\mathcal{L}$, $\mathcal{B}$, such that for $\gamma > \gamma_0$, $s + t_j \notin \mathbb{Z}_{1,2b}$,*

$$URF = (I_K + \Phi)F,$$

$$RUu = (I_H + Q)u,$$

*where $I_K$ and $I_H$ are the identity operators in $\tilde{\mathcal{K}}^s_+$, $\tilde{\mathcal{H}}^s_+$ and $\Phi$, $Q$ are operators in $\tilde{\mathcal{K}}^s_+$, $\tilde{\mathcal{H}}^s_+$ of norm less than 1.*

The above theorem gives us the existence of a suitable regulariser (recall (7)) which allows us to proceed with the argument given in Section 2.1 earlier.

## 3. Quasilinear theory

Here we will use a standard fixed point theorem argument to upgrade our existence results for the linear equation to the case of a quasilinear equation.

THEOREM 2.7 (Schauder fixed point theorem). *Let $\mathcal{I}$ be a compact, convex subset of a Banach space $\mathcal{B}$ and let $J$ be a continuous map of $\mathcal{I}$ into itself. Then $J$ has a fixed point.*

PROOF. We give a proof which is an expansion of that found in Lieberman [**42**]. For a positive integer $k$ let $\{B_i\}_{i=1}^N$ be a finite collection of balls of radius $1/k$ covering $\mathcal{I}$. We write $x_i$ for the centre of $B_i$ and $\mathcal{I}_k$ for the convex hull of the points $x_1, \dots, x_N$. Now define a mapping $I_k : \mathcal{I} \to \mathcal{I}_k$ by

$$I_k(x) = \frac{\sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i) x_i}{\sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i)}.$$

Since the distance in $\mathcal{B}$ is continuous, the map $I_k$ is continuous. We also have the bound

$$\begin{aligned} \|I_k(x) - x\| &= \left\| \frac{\sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i) x_i}{\sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i)} - x \right\| \\ &= \left\| \frac{\sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i) x_i - \sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i) x}{\sum_{i=1}^N \mathrm{dist}(x, \mathcal{I} \sim B_i)} \right\| \end{aligned}$$

$$
\begin{aligned}
&= \frac{\|\sum_{i=1}^N \operatorname{dist}(x, \mathcal{I} \sim B_i)(x_i - x)\|}{\sum_{i=1}^N \operatorname{dist}(x, \mathcal{I} \sim B_i)} \\
&\leq \frac{\sum_{i=1}^N \operatorname{dist}(x, \mathcal{I} \sim B_i)\|x_i - x\|}{\sum_{i=1}^N \operatorname{dist}(x, \mathcal{I} \sim B_i)} \\
&< \frac{\sum_{i=1}^N \operatorname{dist}(x, \mathcal{I} \sim B_i)}{\sum_{i=1}^N \operatorname{dist}(x, \mathcal{I} \sim B_i)} \frac{1}{k} \leq \frac{1}{k}
\end{aligned}
$$

for any $x \in \mathcal{I}$. The second last inequality is due to the $i$-th term in the sum in the numerator or denominator being zero when $\|x_i - x\| \geq 1/k$. Thus $\mathcal{I}_k$ is homeomorphic to a closed ball in $\mathbb{R}^N$ and it follows that $J_k$, the restriction of $I_k \circ J$ to $\mathcal{I}_k$, has a fixed point $y_k$. The compactness of $\mathcal{I}$ implies that there is a convergent subsequence $\{y_{k(m)}\}$ with

$$
\|y_{k(m)} - J\big(y_{k(m)}\big)\| = \|J_{k(m)}\big(y_{k(m)}\big) - J\big(y_{k(m)}\big)\| \leq \frac{1}{k(m)}.
$$

Therefore

$$
x = \lim_{k(m)\to\infty} y_{k(m)}
$$

is a fixed point of $J$. □

As an application of this and Theorem 2.5 we derive the following local existence result. Consider the problem

$$
(16) \quad Pu = -\partial_t u + a^{ijkl}(X, u, Du, D^2u, D^3u)D_{ijkl}u + a(X, u, Du, D^2u, D^3u) = 0,
$$

where $u : \Omega \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^{n+1}$. We think of $\Omega$ as including a time direction and as such $X \in \Omega$ has components $(x_1, \ldots, x_n, t)$. In the case where $\Omega$ is cylindrical, we define the following following associated sets:

$B\Omega = \{X \in \Omega : t = 0\}$, the bottom of $\Omega$,

$S\Omega = \partial\Omega \times (0, T)$, the side of $\Omega$,

$C\Omega = \partial\Omega \times \{0\}$, the corner of $\Omega$, and

$$\mathcal{P}\Omega = B\Omega \cup C\Omega \cup S\Omega, \text{ the parabolic boundary of } \Omega,$$

where $\partial\Omega$ is the topological boundary of $\Omega$. For the case where $\Omega$ is not cylindrical, we define $\mathcal{P}\Omega$ analogously, although in a slightly more complicated way. Consider cylinders

$$Q(X_0, R) = \{X \in \mathbb{R}^{n+1} : |X - X_0| < R, t < t_0\},$$

where the norm is weighted in the time direction by

$$|X| = \max\{|x|, |t|^{1/4}\}.$$

The set $\mathcal{P}\Omega$ then consists of all points $X_0 \in \partial\Omega$ such that for every $\epsilon > 0$ the intersection $Q(X_0, \epsilon) \cap (\mathbb{R}^{n+1} \sim \Omega)$ is non-empty. We also assume that our problem is parabolic in the sense of Petrovski, as defined previously. Note that if the coefficients $a^{ijkl}$ and $a$ are independent of $u$ and the derivatives of $u$, then the operator $P$ may be considered as a linear operator and Theorem 2.5 gives the existence and uniqueness of a solution. This is the underlying idea of the following proof. We consider a smaller domain $\Omega_\epsilon$ defined by

$$\Omega_\epsilon = \{X \in \Omega : t < \epsilon\},$$

and the spaces $H_{k+\alpha}$ are standard Hölder spaces, as defined in [**42**] for example.

THEOREM 2.8. *Suppose $\mathcal{P}\Omega \in H_\delta$ and $\phi \in H_\delta(\mathcal{P}\Omega)$ for some $\delta \in (1,2)$. Then there is a positive constant $\epsilon$ such that the problem*

$$Pu = 0, \text{ in } \Omega_\epsilon, \quad u = \phi \text{ on } \mathcal{P}\Omega_\epsilon, \tag{17}$$

*has a solution $u \in H^{(-\delta)}_{4+\alpha}$. If $\mathcal{P}\Omega \in H_{4+\alpha}$, $\phi \in H_{4+\alpha}$ and $P\phi = 0$ on $C\Omega$, then $u \in H_{4+\alpha}$.*

PROOF. Let $\theta \in (1, \delta)$, set $m_0 = 1 + |\phi|_\theta$ and for $\epsilon > 0$ to be chosen, set

$$\mathcal{I} = \{v \in H_\theta(\Omega_\epsilon) : |v|_\theta \le m_0\}.$$

We then define the map $J : \mathcal{I} \to H_\theta$ by $u = Jv$ if

$$-\partial_t u + a^{ijkl}(X, v, Dv, D^2v, D^3v) D_{ijkl} u + a(X, v, Dv, D^2v, D^3v) = 0, \text{ in } \Omega_\epsilon, \ u = \phi \text{ on } \mathcal{P}\Omega_\epsilon,$$

noting that, for each $v$, this problem has a unique solution in $H^{(-\delta)}_{4+\alpha(\theta-1)}$ by Theorem 2.5. Now

$$|u|_1 \le |u|_\delta \le C|u|^{(-\delta)}_{4+\alpha(\theta-1)} \le C(m_0).$$

It follows that $|u - \phi| \le C\epsilon$ in $\Omega_\epsilon$ and then $|u - \phi|_\theta \le C\epsilon^{(\delta-\theta)/\delta}$ by interpolation. Therefore $|u|_\theta \le m_0$ if $\epsilon$ is sufficiently small, and hence $J$ maps $\mathcal{I}$ into itself for such an $\epsilon$. Since $\mathcal{I}$ is a convex, compact subset of $H_1$, it follows that $J$ has a fixed point in $u$, which is in $H^{(-\delta)}_{4+\alpha(\theta-1)}$ and hence solves (17). Theorem 2.5 now gives $u \in H^{(-\delta)}_{4+\alpha}$. □

Unfortunately, while the previous argument gives existence, it does not allow us to conclude uniqueness. Indeed, without any additional assumptions one cannot expect uniqueness in general. To remedy this situation we will use the following argument which one can find in [**41**]. Regrettably, to give even a summary of this argument we need yet more notation. Therefore we give the core idea now. Consider two solutions $u_1, u_2$ of the problem (16) with identical initial data $u_0$. Assuming that $u_1 \ne u_2$, we may consider the difference in the spatial derivative operators $A_1 = A(u_1)$ and $A_2 = A(u_2)$:

$$\|A_1 - A_2\|_\xi.$$

Now Theorem 2.2 gives us some excellent control of this quantity. By virtue of the two solutions possessing identical initial data and by choosing time to be very small, we can force this norm to be arbitrarily small. This becomes crucial, as we can use this fact to construct an appropriate contraction mapping (from the linearised equation) and infer the existence of a unique solution.

We now give a summary of the proof, but as we mentioned earlier some notation is required. More details may be found in [**41**] and the references contained therein. For Banach spaces $E_0, E_1$ we denote by $E_\theta$ the complex interpolation space $[\overline{E}]_\theta$ and $\|\cdot\|_\theta$ the norm on $E_\theta$. For the basic facts of interpolation spaces the interested reader may refer to [**3, 63**]. The space $\mathcal{L}(E_1, E_0)$ is denotes the Banach space of all bounded linear operators from $E_1$ to $E_0$ and $\|\cdot\|_{\mathcal{L}(E_1,E_0)}$ is the corresponding norm. By $\mathcal{B}$ we denote the category of Banach spaces, whose elements and morphisms are the Banach spaces and bounded linear operators respectively. $\mathcal{B}_2$ denotes the category of densely injected Banach couples, that is, the elements of $\mathcal{B}_2$ are the spaces $\overline{E} := (E_0, E_1)$ with $E_1$ densely injected into $E_0$ and the morphisms $T : \overline{E} \to \overline{F}$ are the maps $T \in \mathcal{L}(E_0, F_0)$ satisfying $T \in \mathcal{L}(E_1, F_1)$.

The space $\mathcal{H}(\overline{E})$ is the set of all operators $A \in \mathcal{L}(E_1, E_0)$ such that $-A$ is the infinitesimal generator of an analytic semigroup on $E_0$. Let $\Sigma_\omega := \{\lambda \in C : \mathrm{Re}\,\lambda \geq \omega\}$, where $\omega \in \mathbb{R}$. A subset $\mathcal{A}$ of $\mathcal{H}(\overline{E})$ is said to be *regularly bounded* if it is bounded in $\mathcal{L}(E_1, E_0)$, there exist constants $M$ and $\omega$ such that

$$\Sigma_\omega \subset \rho(-A) \qquad \text{and} \qquad \|(\lambda + A)^{-1}\|_{\mathcal{L}(E_0)} \leq \frac{M}{1 + |\lambda|}$$

for all $\lambda \in \Sigma_\omega$ and all $A \in \mathcal{A}$, and if $\{(\omega + A)^{-1} : A \in \mathcal{A}\}$ is bounded in $\mathcal{L}(\overline{E})$. For $T > 0$, $\rho \in (0, 1)$ and a nonempty set $S$ of some Banach space $F$ we introduce the notation $C^\rho_T(S) := C^\rho([0,T],S)$. A subset $\mathcal{A}$ of $C^\rho_T(\mathcal{H}(\overline{E}))$ is said to be *regularly bounded* if $\{A : t \in [0,T], A \in \mathcal{A}\}$ is regularly bounded in $\mathcal{H}(\overline{E})$ and there exists a constant $L$ such that

$$\|A(s) - A(t)\|_{\mathcal{L}(\overline{E})} \leq L|s-t|^\rho,$$

for each $s,t \in [0,T]$ and $A \in \mathcal{A}$.

We must finally assume an additional property of our operator $A$.

(18)

For each $\beta \in (0,1)$ there is an open set $V \subset E_\beta$ and $A$ is locally Lipschitz on $V$.

REMARK. To make sense of this notation in the special case of surface diffusion flow in a local coordinate system, which is our focus regardless, one should refer to Escher, Mayer, Simonett [**16**]. There it is proved (among other things) that the surface diffusion flow generates a strongly continuous analytic semigroup. While there is no explicit proof of uniqueness or existence (the reference quoted is due to Amann, and the corresponding theorem in [**1**] is presented without proof), the proof that surface diffusion flow generates an analytic semigroup is valuable. As they also use similar methods to show that uniqueness holds for surface diffusion flow, the proof that (18) holds is also contained in [**16**].

THEOREM 2.9. *Suppose that* (18) *holds,* $0 < \beta < \alpha < 1$ *and* $u_0 \in V_\alpha := E_\alpha \cap V$. *Then there exists* $\tau > 0$ *such that* (16) *has a unique solution on* $[0,\tau]$.

PROOF. We assert that there exists a neighbourhood $W$ of $u_0$ in $V$ and a constant $L > 0$ such that $\{A(u) : u \in W\}$ is regularly bounded in $\mathcal{H}(\overline{E})$, and

$$\|A(u_1) - A(u_2)\|_{\mathcal{L}(E_1,E_0)} \leq L\|u_1 - u_2\|_\beta \text{ for every } u_1, u_2 \in W. \tag{19}$$

Since $E_\alpha \subset E_\beta$, the natural injection is continuous and $W \subset E_\beta$, there exist balls $B_\alpha(u_0, \epsilon) \subset E_\alpha$, $B_\beta(u_0, \delta)$, $B_\beta(u_0, 2\delta) \subset E_\beta$ with $\epsilon, \delta > 0$ such that

$$\overline{B}_\alpha(u_0, \epsilon) \subset E_\alpha$$

and

$$\overline{B}_\beta(u_0, \delta) \subset \overline{B}_\beta(u_0, 2\delta) \subset W.$$

From now on, we fix $\rho = \alpha - \beta \in (0, 1)$. Let $\tau \in (0, T)$ and

$$W_\tau := \{w \in C_\tau(W) : \|w(t) - w(t')\|_\beta \leq L|t - t'|^\rho, \forall t, t' \in [0, \tau]\}.$$

We set $A_w(\cdot) = A(w(\cdot))$ for $w \in W_\tau$. By (19) we have

$$\|A_w(t) - A_w(t')\|_{\mathcal{L}(E_1, E_0)} \leq L|t - t'|^\rho$$

for all $t, t' \in [0, \tau]$. Hence $\{A_w(\cdot) : w \in W_\tau\}$ is regularly bounded in $C_\tau^\rho(\mathcal{H}(\overline{E}))$. It follows from Theorem 2.5 that there exists a unique solution $u(\cdot, w)$ of the linear problem on $[0, \tau]$ and $u(\cdot, w) \in C_\tau(E_\alpha) \cap C_\tau^{\alpha-\beta}(E_\beta)$. Thus there exists $C > 0$ such that

$$\|u(t, w) - u(t', w)\|_\beta \leq C|t - t'|^\rho$$

for all $t, t' \in [0, \tau]$. This implies that for every $t \in [0, \tau]$ we have

$$\|u(t, w) - u_0\|_\beta \leq C\tau^\rho < \delta$$

for $\delta$ sufficiently small. That is, for some $\tau \in (0, T]$ we have $u(\cdot, w) \in W_\tau$ for every $w \in W_\tau$. On the other hand, by Theorem 2.2

$$\|u(t, w_1) - u(t, w_2)\|_\beta \leq C\tau^\rho \|u_0\|_\alpha \|A_{w1} - A_{w2}\|_{C_\tau(\mathcal{L}(E_1, E_0))} < \frac{1}{2}$$

if $\tau$ is sufficiently small. Therefore

$$[w \to u(\cdot, w)] : W_\tau \to W_\tau$$

is a contraction mapping with constant $\frac{1}{2}$ for some $\tau \in (0, T]$. Therefore there exists a unique $w = u(\cdot, w)$, which turns out to be a solution to (16) on $[0, \tau]$. It then follows from Theorem 2.5 that $u \in C([0, \tau], V_\alpha)$. This completes the proof. $\square$

### 4. Application to constrained surface diffusion flows

Let $f : M \times [0, T) \to \mathbb{R}^{n+1}$ be a constrained surface diffusion flow with velocity

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu. \tag{CSD}$$

We assume that the constraint function $h : [0, T) \to \mathbb{R}$ satisfies

$$\frac{d^2}{dt^2} h(0) \leq c(f_0), \tag{20}$$

where $f_0 : M \to \mathbb{R}^{n+1}$ is the initial data for the flow. The solvability of (CSD) with respect to the constraint function will depend upon how smooth the initial data is. For example, if we take

$$h = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M H d\mu}, \tag{21}$$

which is motivated in Chapter 3, we may compute the evolution of $h$ using Lemma 3.9 as

$$\begin{aligned}\frac{d}{dt} h = &\frac{\int_M \|\nabla H\|^2 H(\Delta H) - 2\left\langle \nabla H, \nabla \Delta^2 H\right\rangle + 2(\Delta H)\left\langle \nabla H, \nabla \|A\|^2\right\rangle d\mu}{\int_M H d\mu} \\ &- \frac{\int_M \|\nabla H\|^2 d\mu \int_M H^2(\Delta H + h) - \Delta^2 H + (\Delta H)\|A\|^2 d\mu}{\left(\int_M H d\mu\right)^2} \\ &+ \frac{\int_M \|\nabla H\|^2 d\mu \int_M H\|\nabla H\|^2 d\mu + \left\langle \nabla H, \nabla \|A\|^2\right\rangle d\mu}{\left(\int_M H d\mu\right)^2} + \frac{\int_M \|\nabla H\|^2 d\mu \int_M \|A\|^2 d\mu}{\left(\int_M H d\mu\right)^3}.\end{aligned}$$

Taking another time derivative results in a large mess, however the highest order term is easily seen to be

$$\int_M |\Delta^3 H|^2 d\mu$$

Therefore we can see that (20) is satisfied for this constraint function if $f_0 \in C^8$. The assumption (20) ensures that with $f_0$ smooth enough, $h'$ will remain bounded for a short time. (Otherwise, $h''(0)$ would be unbounded.)

We now show that (CSD) may be written locally as the evolution of a graph function. We will see that the resulting quasilinear fourth order equation is *strictly parabolic* in the sense of Petrovski. In a neighbourhood $U \subset M$, write

$$f(x,0) = \big(x, u(x,0)\big),$$

where $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ is called the *graph function* in $U$ corresponding to $f$. Note that the maximal time for the existence of the graph function will in general be smaller than the maximal time of existence for the immersion $f$. To obtain the true maximal time of existence, we simply take our current hypersurface $f(x,\tilde{T})$ and attempt to write $f$ once again locally as a graph. We continue in this manner until $f$ has lost regularity, and then in this case it may either not be possible to write $f$ locally as a graph or not be possible to satisfy the short time existence theorems for the graph functions covering $f$. In either case, this final time will be maximal.

Recall that the normal to the graph of $f$ in $U$ is given by

$$\nu = \frac{1}{\sqrt{1 + \|\nabla u\|^2}}\Big( - \nabla u, 1\Big),$$

where the derivative is the regular Euclidean derivative. The quantity in the square root is prolific in the coming equations so we make the notation

$$|v| = \frac{1}{\sqrt{1 + \|\nabla u\|^2}}.$$

The mean curvature is

$$H = \mathrm{div}(\nu) = -|v|\Delta u + |v|^3 \nabla_{ij} u \nabla_i u \nabla_j u.$$

We note that from this formula we can immediately prove parabolicity for the mean curvature flow equation $\frac{\partial}{\partial t}f = -H\nu$ (note the opposite sign), as the evolution for the graph function in this case would be

$$\frac{\partial}{\partial t}u = \Delta u - |v|^2\nabla_{ij}u\nabla_i u\nabla_j u.$$

The matrix of coefficients for the second order derivatives is

$$K_{ij} = \delta_{ij} - |v|^2\nabla_i u\nabla_j u,$$

and so

$$K(\xi,\xi) = |\xi|^2 - \frac{\Big(\xi_i\nabla_i u\Big)\Big(\xi_j\nabla_j u\Big)}{1+\|\nabla u\|^2} \geq 1 - \frac{\|\nabla u\|^2}{1+\|\nabla u\|^2} > 0,$$

where $\xi$ is a vector of unit length. Therefore the mean curvature flow equation is quasilinear parabolic, and we can infer short time existence using the method mentioned in the opening remarks.

The case for constrained surface diffusion flow is similar, but the situation is confused by a mess of extra lower order terms. We must compute

$$\Delta H = \Delta\Big(-|v|\Delta u\Big) + \Delta\Big(|v|^3\nabla_{ij}u\nabla_i u\nabla_j u\Big).$$

The details of this calculation are many but the procedure is simply differentiation, and the only difficulty is keeping track of all the indices. We instead state the evolution of the graph function by constrained surface diffusion flow below:

$$\begin{aligned}
\frac{\partial}{\partial t}u = -\Delta^2 u + |v|^2(\Delta\nabla_{ij}u)(\nabla_i u)(\nabla_j u),\ &\text{fourth order terms}\\
+|v|\Big[\ 2(\nabla_i\Delta u)(\nabla_{ij}u)(\nabla_j u) &+ (\Delta\nabla_i u)(\nabla_i u)(\Delta u)\\
+ 2(\Delta\nabla_i u)(\nabla_{ij}u)(\nabla_j u) &+ 2(\nabla_{ijk}u)(\nabla_{jk}u)(\nabla_i u)\\
+ 2(\nabla_{ijk}u)(\nabla_{ik}u)(\nabla_j u)\Big]&
\end{aligned}$$

$$
\begin{aligned}
&- 3|v|^3\Big[2(\nabla_{ijk}u)(\nabla_i u)(\nabla_j u)(\nabla_{lk}u)(\nabla_l u)\\
&\quad + (\Delta\nabla_l u)(\nabla_l u)(\nabla_{ij}u)(\nabla_i u)(\nabla_j u)\Big],\ \text{third order terms}\\
&+ |v|\Big[(\Delta u)\|\nabla_{(2)}u\|^2 + 2(\nabla_{jk}u)(\nabla_{ik}u)(\nabla_{ij}u)\Big]\\
&- 3|v|^3\Big[(\Delta u)(\nabla_{ij}u)(\nabla_j u)(\nabla_{kj}u)(\nabla_k u) + 2(\nabla_{jk}u)(\nabla_i u)(\nabla_{ij}u)(\nabla_{lk}u)(\nabla_l u)\\
&\quad + 2(\nabla_{ij}u)(\nabla_{ik}u)(\nabla_j u)(\nabla_{lk}u)(\nabla_l u) + (\nabla_{ij}u)(\nabla_i u)(\nabla_j u)\|\nabla_{(2)}u\|^2\Big]\\
&+ 15|v|^5(\nabla_{ij}u)(\nabla_{lk}u)(\nabla_{mk}u)(\nabla_i u)(\nabla_j u)(\nabla_l u)(\nabla_m u),\ \text{second order terms}\\
&+ |v|^{-1}h,\ \text{and the constraint function.}
\end{aligned}
$$

Note that we can simplify this expression by recognising terms as derivatives of norms, for example

$$(\nabla_{ij}u)(\nabla_i u)(\nabla_j u) = (\nabla_i u)(\nabla_i\|\nabla u\|^2).$$

Several other terms appear equal under the commutation of derivatives; however we do not wish to dwell on simplifying the evolution of the graph function. Important for us is that the purely nonlinear part is composed of first, second and third order derivatives of $u$ only, and the coefficient of the principal part consists only of first derivatives of $u$. Therefore the evolution of $u$ qualifies as quasilinear, and we must check that parabolicity is satisfied. For a fourth order equation to be strictly parabolic we must have that the coefficient matrix of the principal part is negative definite. Note the change of sign from the mean curvature case. The verification of this fact is almost identical to before, where

$$K_{ijkl} = -\delta_{ij}\delta_{kl} + \delta_{ij}|v|^2\nabla_k u\nabla_l u,$$

and so

$$K(\xi,\xi,\xi,\xi) = -|\xi|^4 + |\xi|^2 \frac{\Big(\xi_k \nabla_k u\Big)\Big(\xi_l \nabla_l u\Big)}{1 + \|\nabla u\|^2} \leq -1 + \frac{\|\nabla u\|^2}{1 + \|\nabla u\|^2} < 0,$$

where $\xi$ is a vector of unit length. Therefore the surface diffusion flow equation written locally as a graph is quasilinear parabolic. The constraint function presents no extra difficulty due to our assumption (20). The flow also generates a semi-group (see [**16**]) and the spatial operator is Locally Lipschitz, as required for the uniqueness theorem. Of course, we cannot use only one graph function to describe the whole of the evolution of $f$. Therefore, we use the inherent structure of $M$ to describe the evolution of $f$. First we reparametrise $M$ so that the domain of every parametrisation is a ball of radius 1. We then have

$$f_0(M) = \bigcup \varphi_i\Big(B_1(x_i)\Big)$$

where $\varphi_i : B_1(x_i) \to \mathbb{R}^{n+1}$ are the aforementioned parametrisations and $f_0 : M \to \mathbb{R}^{n+1}$ is the initial data for the flow we are interested in. Now we consider graphs $u_i : \varphi_i(B_1(x_i)) \to \mathbb{R}$ with the view of applying the theory of the previous sections. In each image $\varphi_i(B_1(x_i))$ we require the graphs $u_i$ to satisfy the evolution

$$\frac{\partial}{\partial t} u_i = (\Delta_i H_i + h)\nu_i,$$

where the subscript $i$ denotes the geometric data associated with that parametrisation. Note that the constraint function $h : I \to \mathbb{R}$ is global. Importantly, note also that the theory of the previous sections gives existence and uniqueness for each $u_i$, when $h$ is a known function of time. However, one may be concerned that this does not coincide with our original problem,

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu,$$

with $f_0 : M \to \mathbb{R}^{n+1}$ as initial data. Fortunately this is not the case, although it is not immediately obvious. The reason for this is that the flow is invariant under tangential diffeomorphisms. We will prove the following standard results, which may be found for example in Ecker [**12**].

Lemma 2.10. *The above formulation of surface diffusion flow is equivalent to* (SD)*, modulo a tangential diffeomorphism.*

Lemma 2.11. *Surface diffusion flow is invariant under tangential diffeomorphisms.*

Combining these with the previous remarks gives the following theorem.

Theorem 2.12. *Let $f_0 : M^n \to \mathbb{R}^{n+1}$ be a $C^4$ immersion with associated constraint function $h : I \to \mathbb{R}$, $h \in C^1$. Then there exists a maximal $0 < T \leq \infty$ and a unique constrained surface diffusion flow $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying*

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu.$$

What remains is to prove Lemma 2.10 and Lemma 2.11. We comment on the case of constraint functions which are not known functions of time at the end of this section. Using the notation above, the immersion $f$ may be written as

$$f(x, t) = \Big(\varphi(x, t), u(\varphi(x, t), t)\Big).$$

We compute

$$\frac{\partial}{\partial t} f = \Big(\frac{\partial \varphi}{\partial t}, Du \cdot \frac{\partial \varphi}{\partial t} + \frac{\partial u}{\partial t}\Big),$$

where we have suppressed the arguments of the functions. We have already shown that the graph $u$ satisfies a quasilinear fourth order evolution. However, in computing this evolution, we only used that the normal part of the speed is equal to $\Delta H + h$. That is, this formulation of constrained surface diffusion flow is

$$\left(\frac{\partial}{\partial t}f\right)^{\perp} = \Delta H + h,$$

where $(\cdot)^{\perp}$ denotes normal projection. This differs from our desired evolution by a tangential diffeomorphism $\phi$ satisfying

$$D_{\varphi(x,t)}f\left(\frac{\partial \phi}{\partial t}\right) = -\left(\frac{\partial f}{\partial t}\right)^{\top},$$

where $(\cdot)^{\top}$ denotes tangential projection. This proves the first lemma.

For the second, consider now an evolution

$$\left(\frac{\partial}{\partial t}f\right)^{\perp} = \Delta H + h.$$

Let $\phi(\cdot, t) : M \to M$ be a family of diffeomorphisms of $M$ satisfying

$$Df\Big(\phi(x,t),t\Big)\left(\frac{\partial \phi}{\partial t}(x,t)\right) = -\left(\frac{\partial f}{\partial t}\Big(\phi(x,t),t\Big)\right)^{\top}.$$

Now if we set $\tilde{f}_t(x) = \tilde{f}(x,t) = f\Big(\phi(x,t),t\Big)$ then $M_t = \tilde{f}_t(M) = f_t(M)$ and

$$\frac{\partial}{\partial t}\tilde{f} = \Delta H + h.$$

This shows the second lemma.

To finish this section we comment upon the case where $h$ is not a known function of time, but is given in terms of integrals of curvature as in for example (21). The preceding arguments give existence of a smooth solution to (CSD) for all constraint functions which are bounded for a short time. Now the assumption (20) ensures that this is the case for a ratio of integrals such as (21). The only issue remaining is to determine if one of the bounded functions of time (for which we have existence) coincides with the constraint function given as integrals of curvature. For this we use a fixed point argument.

THEOREM 2.13. *Let $f_0 : M^n \to \mathbb{R}^{n+1}$ be an immersion with associated constraint function $h$ both satisfying the assumption* (20)*. Then there exists a maximal* $0 < T \leq \infty$ *and a unique constrained surface diffusion flow $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfying*

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu.$$

PROOF. We consider the family of initial value problems

$$\frac{\partial}{\partial t} f_{\tilde{h}} = (\Delta H + \tilde{h})\nu$$

where $\tilde{h} \in C^1([0,T))$ is a known function of time and $f_0 = f(\cdot, 0) \in C^4(M^n)$. The preceding arguments give short time existence for each $f_{\tilde{h}}$. We will now show that at least one of the functions $\tilde{h}$ coincide with our given constraint function $h$, which is normally a ratio of integrals of curvature and not an a priori known function of time. We assume that $h$ satisfies the initial condition (20), which we note forces some measure of regularity on the immersion $f_0$.

Let $S = C^1([0,\delta])$ for some $\delta > 0$ which will be chosen. The theorem will be proved if we can apply Theorem 2.7 with the mapping $P : S \to S$ defined by

$$P\tilde{h} = h.$$

Noting that $C^1([0,\delta])$ is a compact, convex subset of the Banach space $C^1([0,T))$, we need to demonstrate that $P$ maps $S$ into itself and is continuous. Both of these follow from the assumption (20). In particular, we have that $h''(0) < c(f_0)$ and so $h'$ is continuous on $[0, \delta]$ for some $\delta > 0$, and so

$$h' = \frac{d}{dt} P\tilde{h} \in C^1([0, \delta]).$$

This also shows that $P'$ is bounded in the operator norm on $C^1([0, \delta])$ and so $P$ is continuous. Therefore we may apply Theorem 2.7 and deduce that at least one of the functions $\tilde{h}$ coincides with the given constraint function $h$ on an interval $[0, \delta] \subset [0, T)$. Observe that the uniqueness theory in Section 3 continues to apply unchanged. Repeating this argument by translating time $\tilde{t} = t - \delta$ and checking again (20) will give a sequence $\delta_i$ for which this argument is possible, and then the maximal time for the original problem (CSD) is $T = \lim_{i\to\infty} \delta_i$. $\square$

CHAPTER 3

# Lifespan theorem for constrained surface diffusion flows

## 1. Introduction

Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a family of compact immersed hypersurfaces $f(\cdot,t) = f_t : M \to f_t(M)$ with associated Laplace-Beltrami operator $\Delta$, unit normal vector field $\nu$, and mean curvature function $H$. The surface diffusion flow

$$\text{(SD)} \qquad \frac{\partial}{\partial t} f = (\Delta H)\nu,$$

and the more general constrained surface diffusion flows

$$\text{(CSD)} \qquad \frac{\partial}{\partial t} f = (\Delta H + h)\nu,$$

where $h : I \to \mathbb{R}$ and $I \supset [0,T)$, are the chief objects of interest for this chapter. We are motivated by the examples

$$h \equiv 0,\ \ h_H = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M H d\mu},\ \ \overline{h}_H = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M |H| d\mu},\ \text{and}\ h_K = \frac{-\int_M (\Delta H) K d\mu}{\int_M K d\mu}.$$

The first is simply surface diffusion flow (SD). Under this motion we compute

$$\frac{d}{dt}\text{Vol } M = \int_M \Delta H d\mu = 0, \text{ and}$$
$$\frac{d}{dt}\int_M d\mu = \int_M H \Delta H d\mu = -\int_M \|\nabla H\|^2 d\mu \le 0;$$

so that a manifold evolving by (SD) will exhibit conservation of enclosed volume and monotonic decreasing surface area. Further, surface area is preserved exactly when the mean curvature of $M_t$ is constant. It is these geometric characteristics of the surface diffusion flow which motivate the generalisation to constrained surface diffusion flows. For example, with $h = h_H$ we have

$$\begin{aligned}\frac{d}{dt}\int_M d\mu &= \int_M H\Delta H d\mu + h_H \int_M H d\mu \\ &= -\int_M \|\nabla H\|^2 d\mu + \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M H d\mu}\int_M H d\mu = 0,\end{aligned}$$

and now surface area is conserved. Volume is monotonic increasing or decreasing depending on the sign of $\int_M H d\mu$, and preserved only when $H$ is constant. Unfortunately, quantities which are expected to be preserved under second order flows (such as the mean curvature flow, or Ricci flow of metrics) are not in general preserved under fourth order flows. This is due to the absence of a maximum principle. In particular, $\int_M H d\mu$ could approach zero under (CSD) with $h = h_H$, which would cause the flow to be undefined, and most likely *without* a curvature singularity. This motivates the use of $\overline{h}_H$, where

$$\begin{aligned}\frac{d}{dt}\text{Vol } M &= \int_M (\Delta H + \overline{h}_H) d\mu = |M| \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M |H| d\mu} \geq 0, \quad \text{and} \\ \frac{d}{dt}\int_M d\mu &= \int_M H(\Delta H + \overline{h}_H) d\mu \\ &= -\int_M \|\nabla H\|^2 d\mu + \int_M \|\nabla H\|^2 d\mu \frac{\int_M H d\mu}{\int_M |H| d\mu} \leq 0.\end{aligned}$$

Here enclosed volume and surface area are monotonic increasing and decreasing respectively. Therefore we expect that the convergence of the (CSD) flow with $h = \overline{h}_H$ is faster than that of the surface diffusion flow (SD). We also have not only that surface area is stationary (constant in time) if and only if $H$ is constant, but volume also. Further, the flow speed itself is non-zero for surfaces of linear mean curvature. This leads us to believe that singularity development under (CSD) flow with $h = \overline{h}_H$ will be easier to understand compared with (SD) flow. (Consider for example a clothoid-type manifold.) Finally, we use an inequality of Burago-Zalgaller

[5] to infer

$$\int_M |H| d\mu \geq c_{BZ} |M_t|^{\frac{n}{n-1}} \geq c_{BZ}\Big(\text{Vol } M_0\Big) > 0,$$

where we also used the isoperimetric inequality and the fact that volume is monotonic increasing under this flow.

Following a similar line of reasoning gives rise to several other 'conservation' type flows. For example, with $h = h_K$ we calculate

$$\begin{aligned}\frac{d}{dt}\int_M H d\mu &= \int_M \Big[(H^2 - \|A\|^2)(\Delta H + h_K) - \Delta^2 H\Big] d\mu \\ &= \int_M K(\Delta H) d\mu + h_K \int_M K d\mu = 0,\end{aligned}$$

and so the mixed volume $\int_M H d\mu$ is always preserved under (CSD) flow with $h = h_K$. In this case $\int_M K d\mu$ is on the denominator of $h_K$, which is constant under the flow, and so similarly to $\overline{h}_H$ this is always defined. One expects that global analysis of flows such as this, which preserve a geometrically interesting quantity or keep it monotone in time, would lead to new geometric inequalities, or at least to new proofs of classical geometric inequalities. Due to the nature of our analysis, in particular the absence of a maximum principle, these inequalities would be regarding surfaces which satisfy certain curvature integral conditions.

The first step in any program of analysis is to establish a short time existence theorem. This is the subject of Chapter 2, and we extract the relevant result below.

THEOREM 3.1 (Short time existence). *For any smooth enough initial immersion $f_0 : M^n \to \mathbb{R}^{n+1}$ and constraint function $h : I \to \mathbb{R}$ with $I$ an interval containing $0$ and $h \in C^1(I)$, there exists a unique nonextendable smooth solution $f : M \times [0,T) \to \mathbb{R}^3$ to (CSD) with $f(\cdot, 0) = f_0$, where $0 < T \leq \infty$.*

Note that we sometimes interchange the term 'nonextendable' above with '$T$ is maximal', or equivalently 'the maximal time interval $[0,T)$'. Each of these mean the same thing: that the flow exists up to $T$ (possibly not including $T$), and that this is the greatest such $T$.

Motivated by the observation that (SD) flow can also be derived by considering the $H^{-1}$-gradient flow for the area functional (see Fife [**19**]), and the recent work of Kuwert and Schätzle [**36, 37**] on the gradient flow for the Willmore functional, we present the following theorem.

THEOREM 3.2 (Lifespan Theorem). *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact immersion with $C^\infty$ initial data evolving by*

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu. \tag{CSD}$$

*Further suppose that for some $j,k,l \in \mathbb{N}_0$ the constraint function $h : I \supset [0,T) \to \mathbb{R}$ obeys an estimate*

$$h \le \int_M P_j^2(A) + P_k^1(A) + P_l^0(A)d\mu. \tag{GC}$$

*Then there are constants $\rho > 0$, $\epsilon_0 > 0$, and $c < \infty$ such that if $\rho$ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu\Big|_{t=0} = \epsilon(x) \le \epsilon_0 \qquad \text{for any } x \in \mathbb{R}^{n+1} \tag{22}$$

*where $m = \max\{2k-2, 2j-k, l, n^2+n-2\}$; and there exists an absolute constant $C_{AB} \in (0,\infty)$ such that*

$$|M_t| \le C_{AB}, \qquad \text{for} \qquad 0 \le t \le \frac{1}{c}\rho^4; \tag{AB}$$

*then the maximal time $T$ of smooth existence for the flow* (CSD) *with initial data $f_0 = f(\cdot, 0)$ satisfies*

$$T \geq \frac{1}{c}\rho^4, \tag{23}$$

*and we have the estimate*

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu \leq c\epsilon_0 \qquad \text{for} \qquad 0 \leq t \leq \frac{1}{c}\rho^4. \tag{24}$$

Remark. Note that for any $\epsilon_0 > 0$, there is always a $\rho_0 > 0$ such that (22) holds for each $\rho \in (0, \rho_0)$. The radius $\rho > 0$ given by the theorem is certainly not unique, and there will be a $\rho_1 > 0$, $\rho_1 \in (0, \rho_0]$, such that the theorem holds for every $\rho \in (0, \rho_1)$. It is in this sense that we are allowed to choose $\rho$ sufficiently small, which will be useful in the coming arguments.

Remark. There is an inconvenient relationship between the three classes of flows (SD), (CSD) with $h$ an a priori known function of time, and (CSD) with $h$ a function of integrals of curvature quantities. The first and third are obviously included in the statement of Theorem 3.2 above, while the second is not. This is not satisfying since then for trivial constraint functions (the case of (SD) flow) one must assume that the initial manifold possesses local smallness of curvature in the $L^{(n^2+n-2)}$-norm. One would instead desire that this smallness be in the $L^n$ norm. It would appear intuitively obvious that if one was performing analysis of a constrained surface diffusion flow where the constraint function is known a priori then one may be able to obtain a statement stronger than Theorem 3.2 above. Indeed, for such simple functions as $h(t) = \frac{1}{1+t}$, $h(t) = \sin t$ and $h(t) = t$ both (AB) and (GC) are violated. Fortunately the intuition holds, and one may in fact obtain an analogous stronger version of Theorem 3.2 above restricted to the class of *simple* constraint functions, which are those that satisfy a bound $\|h\|_{\infty,J} < c(J)$ for each interval $J \subset [0,T)$. This is the subject of Appendix C and the statement is Theorem C.1. In particular, for these functions the smallness condition is in $L^n$, as we do not have the nasty interplay with global integrals in $h$ needing to be bounded by a local assumption on $f$ interfering with our integral estimates. We also do not need to assume the area bound (AB) in the case $n = 2$ for these more simple constraint functions. While the proof is relatively straightforward compared to that of Theorem 3.2 above, the result is stronger and the class of constraint functions includes those which do not satisfy (GC) for any $j, k, l$. Notably, this includes the surface diffusion flow, which is itself a new result and well motivated. Therefore we have devoted Appendix C to this alternative Lifespan Theorem. For the full picture one must really take both Theorem 3.2 and Theorem C.1 into account. As the argument is more straightforward in Appendix C, we recommend that the reader first look there for the flavour of the more complicated argument here. This will also serve to highlight the difficulties caused by the introduction of the constraint function.

Remark. The smallness assumption (22) above is not scale invariant. However, we can instead consider

$$\mu(M)^{\frac{m-n}{m}}\left(\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu\right)^{\frac{n}{m}}$$

and this is scale invariant. In light of (AB), one can see why the formally simpler (22) is sufficient.

Remark. The constant in the lower bound on maximal time, $\frac{1}{c}$, can be computed a priori as a function of $n$. For $n = 2$,

$$\frac{1}{c} = \frac{1}{4^3 c_0},$$

and for $n = 3$,

$$\frac{1}{c} = \frac{C_{AB}^{1/3}}{4^4 c_0},$$

where $c_0$ is the constant from Proposition 3.23.

The restriction on the dimension of the evolving immersion is due to both the exponent in the Michael-Simon Sobolev inequality, and the scale invariance of the total squared curvature functional in two dimensions. For flows where the evolution of the surface area is bounded (such as (SD) and (CSD) with $h = \overline{h}_H$) we have removed the latter restriction by considering (22), which is a natural generalisation of (1.4) in [**37**]. The size of $\epsilon_0$ is determined indirectly by the bound on surface area for the flow in question. As to the exponent in the Michael-Simon Sobolev inequality, the interplay between the evolution equations and our techniques using integral estimates forces $n < 4$; see Section 5 for a discussion of this issue. To our knowledge this cannot be improved.

At first glance, the choice in (22) may appear somewhat restrictive, since $\epsilon_0$ (the size of which is dictated by estimates to come) may be very small. However, it is clear that if the initial surface $M_0$ is of finite total curvature (that is, $\int_M \|A\|^n d\mu\Big|_{t=0} < \infty$), then there will exist a positive $\rho = \rho(\epsilon_0, M_0)$ such that (22) is satisfied. Therefore, in terms of allowable initial surfaces $M_0$, we are only excluding those for which the total curvature is infinite. Since short time existence for the flow (CSD) is itself not even valid on these 'singular' manifolds (they will not be smooth enough), this is a natural and quite general class of initial data.

Our proof relies on showing that (GC) and (AB) allows one to prove the conditional bound

$$\text{(CB)} \qquad \sup_{x\in\mathbb{R}^{n+1}}\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu < \epsilon_0 \quad \Longrightarrow \quad \|h\|_{\infty,I} < \infty.$$

In practice, the domain of the integral on the left hand side will be the support of a cutoff function. This is the key to treating the nontrivial constraint functions such as $h = \overline{h}_H, h_K$, and this is the subject of Section 3. Unfortunately, since $h_K$ does not permit the global area bound (AB), it remains just beyond our current techniques. It is in this sense which the two examples serve to differentiate between those constraint functions which are relatively easy to handle, and those which still present difficulty. The inequality

$$\sup_{x,y\in f(M)}|x-y| = \text{extrinsic diameter} = d_{ext} \le c_T(n)\int_M |H|^{n-1}d\mu$$

due to Topping [**62**] will play a major role, allowing us to prescribe a class of constraint functions which admit a 'localisation' procedure. The extra assumptions required are a growth condition, and a geometric condition: either bounded surface area or bounded total mean curvature. For $n = 3$, one requires (AB) regardless, and so we have concentrated on this condition.

We make one additional remark regarding this last point. Supposing that $n = 2$ and one enjoys the bound

$$\int_M |H|d\mu < c$$

uniformly, then the area bound (AB) is no longer required. One may recover Theorem 3.2 for these flows and furthermore the smallness condition is relaxed to be in $L^m$ where $m = \max\{2k-2, 2j-k, l, n\}$. Note that this implies for certain constraint functions the smallness may indeed be in $L^2$, which is far more desirable. Now recall the (CSD) flow with $h = h_K$, where we have by the structure of our flow:

$$\int_M H d\mu = \int_M H d\mu \Big|_{t=0} < c.$$

This is unfortunately still not enough to proceed, even by this alternative argument. Thus, yet again, the constraint function $h = h_K$ critically does not satisfy the required assumption.

In a more global sense, we present the Lifespan Theorem with a perspective toward further analysis of the (CSD) flows. In particular, as the statement depends on the concentration of the curvature of the initial surface, the result is particularly relevant to the analysis of asymptotic behaviour in the following respect. When considering a blowup of a singularity formed at some time $T < \infty$ of the (CSD) flow, we wish to have that some amount of the curvature concentrates in space. From the theorem, if $\rho(t)$ denotes the largest radius such that (22) holds at time $t$, then $\rho(t) \le \sqrt[4]{c(T-t)}$ and so at least $\epsilon_0$ of the curvature concentrates in a ball $f^{-1}(B_{\rho(T)}(x))$. That is,

$$\lim_{t\to T} \int_{f^{-1}(B_{\rho(t)}(x))} \|A\|^n d\mu \ge \epsilon_0,$$

where $x = x(t)$ is understood to be the centre of a ball where the integral above is maximised. This is a fundamental property of the blowups considered in Chapter 6.

Our motivation for the extension of (SD) to the more general class of flows (CSD) is essentially mathematical. Indeed, there does already exist a large body of work on (SD) flow itself, and study of (SD) alone is well motivated. First proposed by the physicist Mullins [**48**] in 1957 (two years before he proposed the mean curvature flow), it was originally designed to model the formation of tiny thermal grooves in phase interfaces where the contribution due to evaporation-condensation was insignificant. Some time later, Davi, Gurtin, Cahn and Taylor [**7, 10**] proposed many other physical models which give rise to the surface diffusion flow. These all exhibit a reduction of free surface energy and conservation of volume; an essential characteristic of (SD) flow. There are also other motivations for the study of (SD). For example, two years later Cahn, Elliot and Novick-Cohen [**6**] proved that (SD) is the singular limit of the Cahn-Hilliard equation with a concentration dependent mobility. Among other applications, this arises in the modeling of isothermal separation of compound materials.

Analysis of the surface diffusion flow began slowly, with the first works appearing in the early 80s. Baras, Duchon and Robert [**2**] showed the global existence of weak solutions for two dimensional strip-like domains in 1984. Later, in 1997 Elliot and Garcke [**15**] analysed (SD) flow of curves, and obtained local existence and regularity for $C^4$-initial curves, and global existence for small perturbations of circles. Significantly, Ito [**30**] showed in 1998 that convexity will not be preserved under (SD), even for smooth, rotationally symmetric, closed, compact, strictly convex initial hypersurfaces. Escher, Mayer and Simonett [**16**] gave several numerical schemes for modeling (SD) flow, and have also given the only two known numerical examples

[**45**] of the development of a singularity: a tubular spiral and thin-necked dumbbell. They also provide an example of an immersion which will self-intersect under the flow, a figure eight knot. In 2001, Simonett [**55**] used centre manifold techniques to show that for initial data $C^{2,\alpha}$-close to a sphere, both the surface diffusion and Willmore flows converge to a sphere in long time.

There have been many other important works on fourth order flows of a slightly different character, from Willmore flow (which includes some extra zero order terms in the speed of the flow) to Calabi flow (which is a fourth order flow of metrics). Significant contributions to the analysis of these flows by the authors Kuwert, Schätzle, Polden, Huisken, Mantegazza and Chrusciel [**8, 36, 37, 44, 52**] are particularly relevant, as the methods employed there are similar to ours here.

In our proof, we exploit the fact that for an $n$-dimensional immersion the integral

$$\int_M \|A\|^n d\mu$$

is scale invariant. The technique used by Struwe [**59**] in his paper on harmonic mappings of Riemannian surfaces, of using smallness of initial energy in a local sense, is then relevant, although as with all higher order flows the major difficulty is in overcoming the lack of powerful techniques unique to the second order case. In particular, we are without the maximum principle, and this implies that the geometry of the surface could devolve. Therefore we are forced to use integral estimates to derive derivative curvature bounds under a condition similar to (22), and in calculating these estimates it is crucial to only use inequalities which involve universal constants. Interpolation inequalities similar in nature to those used by

Ladyzhenskaya, Ural'tseva and Solonnikov [**39**] and Hamilton [**27**], and the Sobolev inequality of Michael-Simon [**47**], are invaluable in this regard.

The structure of this chapter is as follows. To apply the argument used by Struwe, we must prove two key local integral estimates. In Section 2 we collect various fundamental formulae from differential geometry, set our notation, and state some basic results. The goal of Section 3 is to show that the a priori bound (CB) is satisfied by a class of constraint functions, and to detail the localisation procedure required to use the essentially global constraint function in local integral estimates. Section 4 is concerned with estimating the evolution of local integrals of derivatives of curvature, and Section 5 combines these estimates with Sobolev inequalities, interpolation inequalities, and the results of Section 3 to conclude the two required key integral estimates. With these in hand, we adapt the argument of Struwe in Section 6 to prove the Lifespan Theorem. Section 7 contains some remarks on lifespan theorems for flows similar to (CSD).

We note that there is a similar theorem in Liu [**43**], applying only to the flow (SD). However, in that paper there are errors in the proof related to the rescaling, and to the usage of the interpolation inequalities. For example, if the integral quantity used is not scaling invariant one may not be able to choose a small enough $\epsilon_0$ in the hypothesis (22) without driving $\rho$ to zero. To our knowledge a corrected version has yet to appear.

Our proof of the Lifespan Theorem and the overall structure of this paper is inspired by the work of Kuwert and Schätzle [**37**] for the Willmore flow.

## 2. Notation and preliminary results

In this section we augment and summarise some of the background material from chapter one. In particular we define and derive some basic properties of the localisation functions which we will use. We have as our principal object of study a smooth immersion $f : M^n \to \mathbb{R}^{n+1}$ of an orientable compact hypersurface $M$ with induced metric $g_{ij}$ so that the pair $(M, g)$ is a Riemannian manifold. We denote by $A_{ij}$ the second fundamental form and the trace by the metric $g^{ij}A_{ij} = H$ is the mean curvature. Repeated indices are always summed from 1 to $n$ and we do not normalise the mean curvature. We use $\Gamma^k_{ij}$ for the Christoffel symbols, determined by the metric, and $\nabla$ for the covariant derivative on $M$.

The fundamental relations between components of the Riemann curvature tensor, the Ricci tensor and scalar curvature are given by Gauss' equation

$$R_{ijkl} = A_{ik}A_{jl} - A_{il}A_{jk},$$

with contractions

$$g^{jl}R_{ijkl} = R_{ik} = HA_{ik} - A_i^j A_j^k, \text{ and}$$
$$g^{ik}R_{ik} = R = H^2 - \|A\|^2.$$

We will need to interchange covariant derivatives; for vectors $X$ and covectors $Y$ we obtain

$$\nabla_{ij}X^h - \nabla_{ji}X^h = R^h_{ijk}X^k = (A_{lj}A_{ik} - A_{lk}A_{ij})g^{hl}X^k,$$

$$\nabla_{ij}Y_k - \nabla_{ji}Y_k = R_{ijkl}g^{lm}Y_m = (A_{lj}A_{ik} - A_{il}A_{jk})g^{lm}Y_m,$$

where $\nabla_{i_1 \dots i_n} = \nabla_{i_1} \cdots \nabla_{i_n}$. Further, recall the definition of the $P$-style terms from Chapter 1. Recall that we abuse the arbitrary absolute constant appearing in the $P$-style terms to encompass the norms of tensors (and other specialised contractions) under the induced metric. For example

$$\begin{aligned}
\|\nabla_{(2)} A\|^2 &= \left\langle \nabla_{(2)} A, \nabla_{(2)} A \right\rangle \\
&= 1 \cdot \left( \nabla_{(2)} A * \nabla_{(2)} A \right) + 0 \cdot \left( \nabla_{(1)} A * \nabla_{(3)} A \right) + 0 \cdot \left( A * \nabla_{(4)} A \right) \\
&= P_2^4(A).
\end{aligned}$$

This will occur throughout the chapter without further comment. In the coming sections we will be concerned with calculating the evolution of the iterated covariant derivatives of curvature quantities. For a tensor $T$ on $M$, the following less precise interchange of covariant derivatives formula will be useful to keep in mind:

$$\nabla_{ij} T = \nabla_{ji} T + P_2^0(A) * T.$$

In most of our integral estimates (especially those in sections 4 and 5), we will be including a function $\gamma : M \to \mathbb{R}$ in the integrand. Eventually, this will be specialised to a smooth cutoff function between concentric geodesic balls on $M$. For now however let us only assume that $\gamma = \tilde{\gamma} \circ f$, where

$$0 \leq \tilde{\gamma} \leq 1, \qquad \text{and} \qquad \|\tilde{\gamma}\|_{C^2(\mathbb{R}^{n+1})} \leq c_{\tilde{\gamma}} < \infty.$$

Using the chain rule, this implies $D\gamma = (D\tilde{\gamma} \circ f) Df$ and then $D^2\gamma = (D^2\tilde{\gamma} \circ f)(Df, Df) + (D\tilde{\gamma} \circ f) D^2 f(\cdot, \cdot)$. Using the expression (1) for the Christoffel symbols to convert the computations above to covariant derivatives, and the Weingarten relations to convert the derivatives of $\nu$ to factors of the second fundamental form with the basis vectors $\partial_i f$, we obtain the estimates

$$\|\nabla\gamma\| \le c_{\gamma 1}, \qquad \text{and} \qquad \|\nabla_{(2)}\gamma\| \le c_{\gamma 2}(1+\|A\|). \tag{25}$$

For a given $\rho > 0$, we also define the functions $\epsilon, \delta^{(p)} : \mathbb{R}^{n+1} \times [0, T^*] \to \mathbb{R}$ as

$$\epsilon(x) = \int_{f^{-1}(B_\rho(x))} \|A\|^2 d\mu, \quad \text{and} \quad \delta^{(p)}(x) = \int_{f^{-1}(B_\rho(x))} \|A\|^p d\mu.$$

We use the convention that

$$\sup_{x\in\mathbb{R}^{n+1}} \epsilon(x) \le \epsilon_0 \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^{n+1}} \delta^{(p)}(x) \le \delta_0^{(p)}.$$

At times we will instead consider the set $[\gamma > c] = \{p \in M : \gamma(p) > c\}$ or the set $[\gamma = c] = \{p \in M : \gamma(p) = c\}$ as the domain of the integrals in $\epsilon(x)$ and $\delta^{(p)}(x)$. The meaning will be clear by the context.

## 3. A priori estimates for the constraint function

Our constraint functions are by their nature global notions (being functions of time only). This is a distinct advantage in some areas of the analysis: evolution equations first order in time and of any order in space involve at most a linear factor of $h$.

When one wishes to prove local integral estimates however, the global nature of $h$ becomes an issue. We are faced with situations such as

$$\begin{aligned} \frac{d}{dt}\int_{f^{-1}(B_\rho(x))} \|A\|^2 d\mu + \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)}A\|^2 d\mu \\ \le h\int_{f^{-1}(B_{2\rho}(x))} \left(\|A\|^3 + \|A\|^2\right) d\mu + \text{“good terms”}, \end{aligned} \tag{26}$$

armed with a local smallness of curvature assumption

$$\sup_{\substack{x\in\mathbb{R}^3\\ t\in[0,T^*]}} \epsilon(x) \le \epsilon_0, \qquad \text{or} \qquad \sup_{\substack{x\in\mathbb{R}^3\\ t\in[0,T^*]}} \delta^{(p)}(x) \le \delta_0,$$

and tasked with absorbing the term involving $h$ into $\int_{f^{-1}(B_\rho(x))}\|\nabla_{(2)}A\|^2d\mu$, a local integral. Assume for the sake of example that $h = \int_M k(\kappa_1,\kappa_2)d\mu$ and obeys an estimate

$$h \le C_{ABS}\int_M \|A\|^2d\mu \int_M \|\nabla_{(2)}A\|^2d\mu,$$

where $C_{ABS}$ is an absolute constant. Then as a first attempt to 'localise' the integrals on the right one might estimate them by

$$\begin{aligned}
\int_M &\|A\|^2d\mu \int_M \|\nabla_{(2)}A\|^2d\mu \\
&\le c_\rho^2(t)\left[\sup_{x\in\mathbb{R}^3}\int_{f^{-1}(B_\rho(x))}\|A\|^2d\mu\right]\cdot\left[\sup_{x\in\mathbb{R}^3}\int_{f^{-1}(B_\rho(x))}\|\nabla_{(2)}A\|^2d\mu\right] \\
&\le c_\rho^2(t)\epsilon_0\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2d\mu,
\end{aligned}$$

where $c_\rho(t)$ is the number of extrinsic balls of radius $\rho$ required to cover $f(M_t)$ and $x_1 \in \mathbb{R}^3$ is a point where the second supremum is attained. The goal of course is to now bound $c_\rho^2(t)\epsilon_0$ by $\frac{1}{2C_{ABS}}$ (for example), and absorb the entire term on the left in (26). Unfortunately, this will in general be impossible. To attain a smaller $\epsilon_0$, one must drive $\rho$ to zero, but this will in turn drive $c_\rho$ to $\infty$. Further, the scaling is unfavourable, making it impossible to know a priori if any admissible $\rho > 0$ exists. Finally, $c_\rho$ is a function of time, and without a uniform bound we have little hope of absorbing the constraint function into a local integral.

With some minor modifications to the above idea, and assumptions on the flow, these problems can be overcome and the argument carries through. Our main result for this section is the following.

THEOREM 3.3. *For some $T^* < T$ let $f : M^n \times [0, T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with constraint function $h$ satisfying for some $j, k, l \in \mathbb{N}_0$*

$$h \le \int_M P_j^2(A) + P_k^1(A) + P_l^0(A) d\mu$$

*where for $m = \max\{2k-2, 2j-k, l, n^2+n-2\}$*

$$\sup_{x\in\mathbb{R}^3} \delta^m(x) \le \delta_0^m < \infty,$$

*and for an absolute constant $C_{AB}$*

$$|M_t| \le C_{AB}; \tag{AB}$$

*on $[0, T^*]$.*

*Then for any $\rho > 0$, $x \in \mathbb{R}^{n+1}$, $t \in [0, T^*]$ there exists an $x_1 \in \mathbb{R}^{n+1}$ such that for any $\theta > 0$,*

$$h \int_{f^{-1}(B_{2\rho}(x))} \Big(\|A\|^4 + \|A\|^2\Big) d\mu \le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + \frac{1}{\theta} C_{UGLY},$$

*if $j, k \ne 0$, and otherwise*

$$h \int_{f^{-1}(B_{2\rho}(x))} \Big(\|A\|^4 + \|A\|^2\Big) d\mu \le C_{UGLY},$$

*where $C_{UGLY} = C_{UGLY}(\delta_0^m, C_{AB}, \rho, j, k, l, n)$.*

Before we begin the proof we would like to show that $\overline{h}_H$ satisfies the assumptions of the theorem. By viewing mean curvature as the variation of area, one is led to (see Burago-Zalgaller [**5**]) the estimate

$$|M| \le c\Big(\int_M |H| d\mu\Big)^{\frac{n}{n-1}} \tag{27}$$

for a constant $c$ depending only on $n$. Using now the isoperimetric inequality we conclude

$$\frac{1}{\int_M |H| d\mu} \le c|M|^{\frac{1-n}{n}} \le c(\text{Vol } M)^{-1} \le c\text{Vol } M_0.$$

Therefore we may estimate

$$\overline{h}_H(t) = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M |H| d\mu} \le c(M_0) \int_M P_2^2(A) d\mu.$$

Thus for any dimension $n$ we take $m = (n-1)(n+2)$. Also, (AB) is satisfied with

$$C_{AB} = |M_0|.$$

Driving this estimate is the following result due to Topping [**62**].

THEOREM 3.4 (Topping). *Let $M^n$ be a compact connected $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Then its extrinsic diameter and its mean curvature $H$ are related by*

$$d_{ext} \le c_T(n) \int_M |H|^{n-1} d\mu.$$

Topping shows that in particular we may take $c_T(2) = \frac{32}{\pi}$. Please refer to the references in [**62**] for a history of this inequality and others similar to it. We note in particular that for our purposes, the earlier version of this inequality in Simon [**54**] is almost sufficient.

We first obtain an estimate for $c_\rho(t)$.

LEMMA 3.5. *Let $f : M^n \times [0, T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow satisfying* (AB). *Then for any $\rho$ such that $0 < \rho \le \frac{d_{ext}\sqrt{n+1}}{2}$ there exists an $x_2 \in \mathbb{R}^{n+1}$ where the following estimate holds:*

$$c_\rho(t) \le c(C_{AB}, \rho, n) \left( \int_{f^{-1}(B_\rho(x_2))} \|A\|^{(n-1)(n+2)} d\mu \right)^{n+1}.$$

REMARK. If $\rho > \frac{d_{ext}\sqrt{n+1}}{2}$ then $c_\rho(t) = 1$. We will always assume from now on that $0 < \rho \le \frac{d_{ext}\sqrt{n+1}}{2}$.

PROOF. We simply apply a covering argument, Topping's inequality, and then the Hölder inequality. Since we can cover $M_t$ by an $(n+1)$-cube with side length $d_{ext}$ and a ball of radius $\rho$ encloses an $(n+1)$-cube with side length $\frac{2\rho}{\sqrt{n+1}}$,

$$\begin{aligned}
c_\rho(t) &\le \left(\frac{d_{ext}\sqrt{n+1}}{2\rho}\right)^{n+1} \\
&\le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{n+1}\left(\int_M |H|^{n-1}d\mu\right)^{n+1} \\
&\le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{n+1}|M_t|^{\frac{(n+1)^2}{n+2}}\left(\int_M |H|^{(n-1)(n+2)}d\mu\right)^{\frac{n+1}{n+2}} \\
&\le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{n+1}|M_t|^{\frac{(n+1)^2}{n+2}}\left(\sup_{x\in\mathbb{R}^{n+1}} c_\rho(t)\int_{f^{-1}(B_\rho(x))}|H|^{(n-1)(n+2)}d\mu\right)^{\frac{n+1}{n+2}},
\end{aligned}$$

so

$$c_\rho(t) \le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{(n+1)(n+2)} C_{AB}^{(n+1)^2}\left(\int_{f^{-1}(B_\rho(x_2))}\|A\|^{(n-1)(n+2)}d\mu\right)^{n+1},$$

where $x_2$ is a point in $\mathbb{R}^{n+1}$ such that

$$\int_{f^{-1}(B_\rho(x_2))}\|A\|^{(n-1)(n+2)}d\mu = \sup_{x\in\mathbb{R}^{n+1}}\int_{f^{-1}(B_\rho(x))}\|A\|^{(n-1)(n+2)}d\mu.$$

□

REMARK. Since we can take $c_T(2) = \frac{32}{\pi}$, the conclusion in the theorem above for a (CSD) flow with $h = \overline{h}_H$ and $n = 2$ is

$$c_\rho(t) \le \left(\frac{32\sqrt{3}}{2\pi\rho}\right)^{12}|M_0|^9\left(\int_{f^{-1}(B_\rho(x_2))}\|A\|^4 d\mu\right)^3.$$

We now use the above to estimate $h$.

LEMMA 3.6. *Let $\theta > 0$ be a fixed positive number and $f : M \times [0, T^*] \to \mathbb{R}^{n+1}$ a* (CSD) *flow satisfying the assumptions of Theorem 3.3. Then for any $\rho > 0$ there exists a point $x_1 \in \mathbb{R}^{n+1}$ such that the constraint function $h$ satisfies the following* *estimate:*

$$h \le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c(\theta,\rho,n,j,k,l,C_{AB},\delta_0^m)\delta_0^m$$

*for $j,k,l$ not all equal to zero, and*

$$h \le c(\rho,n,C_{AB})\Big(\delta_0^m\Big)^{n+1}$$

*for $j=k=l=0$.*

Proof. Recall that

$$\sup_{x\in\mathbb{R}^{n+1}} \delta^m(x) \le \delta_0^m < \infty,$$

where $m = \max\{2j-2, 2k-j, l, n^2+n-2, 4\}$.

We will first prove the estimate assuming that $j \ge \max\{2, 2k+1\}$:

$$\begin{aligned}
h &\le \int_M P_j^2(A) + P_k^1(A) + P_l^0(A)d\mu \\
&\le c \sup_{x\in\mathbb{R}^{n+1}} c_\rho \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)}A\|\ \|A\|^{j-1}d\mu + \int_M \|\nabla A\|^2\|A\|^{j-2}d\mu \\
&\quad + c\int_M \|\nabla A\|\ \|A\|^{k-1}d\mu + c\int_M \|A\|^l d\mu \\
&\le \frac{\theta}{2}\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2\frac{c}{2\theta}\int_{f^{-1}(B_\rho(x_1))}\|A\|^{2j-2}d\mu \\
&\quad + c\int_M \|\nabla A\|^2\|A\|^{j-2}d\mu + c\int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\le \frac{\theta}{2}\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c(j)\int_M \langle A, \Delta A\rangle\ \|A\|^{j-2}d\mu \\
&\quad + c_\rho^2\frac{c}{2\theta}\int_{f^{-1}(B_\rho(x_1))}\|A\|^{2j-2}d\mu + c(\theta,j,k,l)\int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\le \theta\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2 c(\theta,j,k)\int_{f^{-1}(B_\rho(x_1))}\|A\|^{2j-2}d\mu \\
&\quad + c(\theta,j,k,l)\int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\le \theta\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2 c(\theta,j,k,C_{AB})\bigg(\int_{f^{-1}(B_\rho(x_1))}\|A\|^m d\mu\bigg)^{\frac{2j-2}{m}} \\
&\quad + c(\theta,j,k,l,C_{AB})\bigg(\sup_{x\in\mathbb{R}^{n+1}} c_\rho\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu\bigg)^{\frac{2k-j+l}{m}}
\end{aligned}$$

$$\leq \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^{\frac{2m+2k-j+l}{m}} c(\theta, j, k, l, C_{AB}) \left(\delta_0^m\right)^{\frac{j-2+2k+l}{m}}$$
$$\leq \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + c(\theta, \rho, n, j, k, l, C_{AB}) \left(\delta_0^m\right)^{\frac{(n+1)(2m+2k-j+l)+j-2+2k+l}{m}}.$$

The estimate is easier to prove in the subcases excluded above. When $j = 1$ we instead split the first integral by

$$\int_M P_j^2(A) d\mu \leq c \sup_{x\in\mathbb{R}^{n+1}} c_\rho \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)}A\| d\mu$$
$$\leq \frac{\theta}{2} \int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2 \frac{2}{\theta} \int_{f^{-1}(B_\rho(x_3))} 1 \; d\mu$$
$$\leq \frac{\theta}{2} \int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c(\theta, \rho, n, C_{AB}) \left(\delta_0^m\right)^{2n+2}.$$

When $j < 2k + 1$ we instead estimate the second integral by

$$\int_M P_k^2(A) d\mu \leq c \int_M \|\nabla A\| \; \|A\|^{k-1} d\mu$$
$$\leq c \int_M \|\nabla A\|^2 d\mu + c \int_M \|A\|^{2k-2} d\mu$$
$$\leq c \int_M \|\nabla_{(2)}A\| \; \|A\| d\mu + c \int_M \|A\|^{2k-2} d\mu$$
$$\leq \frac{\theta}{2} \int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2 \frac{2}{\theta} \int_{f^{-1}(B_\rho(x_3))} \|A\|^2 d\mu + c \int_M \|A\|^{2k-2} d\mu$$
$$\leq \frac{\theta}{2} \int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c(\theta, \rho, n, C_{AB}) \left[ \left(\delta_0^m\right)^{2n+2+\frac{2}{m}} + \left(\delta_0^m\right)^{\frac{2k-2}{m}} \right].$$

Note that in any case, the exponent of $\delta_0^m$ is greater than 1 due to the conditions on $m$. This gives the first part of the lemma.

If $j = k = l = 0$ then obviously

$$h \leq c(\rho, n, C_{AB}) \left(\delta_0^m\right)^{n+1}.$$

This finishes the proof. □

Remark. In the special case where $h = \overline{h}_H$ and $n = 2$, the estimate reads

$$\overline{h}_H \leq \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + \frac{c_{BZ}}{4\theta\sqrt{\mathrm{Vol}\, M_0}} \left(\frac{16\sqrt{3}}{\rho\pi}\right)^2 4|M_0|^{\frac{37}{2}} \left(\delta_0^4\right)^{\frac{13}{2}},$$

where $c_{BZ}$ is the constant from the inequality (27) due to Burago-Zalgaller [**5**].

We are now ready to prove Theorem 3.3 as essentially a corollary to Lemma 3.6 above.

Proof of Theorem 3.3. First note that

$$\begin{aligned}\int_{f^{-1}(B_{2\rho}(x))}\Big(\|A\|^4+\|A\|^2\Big)d\mu &\le \sup_{x^*\in B_{2\rho}(x)} 4^{n+1}\int_{f^{-1}(B_{\rho}(x^*))}\Big(\|A\|^4+\|A\|^2\Big)d\mu\\ &\le 4^{n+1}C_{AB}^{1-\frac{4}{m}}\Big(\delta_0^m\Big)^{\frac{4}{m}}.\end{aligned}$$

By Lemma 3.6 we are now finished, choosing

$$\theta=\frac{\theta^*}{4^{n+1}C_{AB}^{1-\frac{4}{m}}\Big(\delta_0^m\Big)^{\frac{4}{m}}}.$$

□

Remark. In each of the previous inequalities we have been primarily concerned with integrals localised to a ball $f^{-1}(B_\rho(x))$. In the following sections where we derive the basic integral estimates, the domain of integration will instead be the set $[\gamma>0]=\{p\in M:\gamma(p)>0\}$, where $\gamma$ is as in equation (25). This is necessary to not only obtain the local integral estimates, but also to allow us enough freedom to choose various appropriate $\gamma$ functions, depending upon the situation. To bridge the gap between the two domains of integration we may choose $\gamma=\tilde\gamma\circ f$ to be such that

$$\chi_{B_\rho(x)}\le\tilde\gamma\le\chi_{B_{2\rho}(x)}$$

and $\gamma\in C^2(M)$. Then for a strictly positive integrand we crudely estimate

$$\int_{f^{-1}(B_\rho(x))}[\cdots]d\mu\le\int_{[\gamma>0]}[\cdots]d\mu\le\int_{f^{-1}(B_{2\rho}(x))}[\cdots]d\mu.$$

This is why in Theorem 3.3 we see integrals with balls of radii $2\rho$ on the left.

Theorem 4 gives us the opportunity to obtain the derivative of curvature estimates in the ball $B_\rho(x_1)$, but nowhere else. This is not enough to prove the Lifespan Theorem. However, we may still proceed by using the estimates in the ball $B_\rho(x_1)$ to bound the constraint function over all of $M_t$, and then once this is accomplished we can go back and prove the required derivative of curvature estimates everywhere else on $M_t$.

Corollary 3.7 (The curvature estimates on a special ball). *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with $h$ satisfying the assumptions of Theorem 3.3. Then there is a $\delta_0^m = \delta_0^m(n, M_0)$ such that if*

$$\sup_{t\in[0,T^*],x\in\mathbb{R}^{n+1}} \int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu \le \delta_0^m,$$

*there is an $x_1 \in \mathbb{R}^{n+1}$ such that*

$$\|\nabla_{(2)}A\|^2_{\infty,f^{-1}(B_\rho(x_1))} \le c\Big(\delta_0^m, T^*, C_{AB}, \rho, j, k, l, m, \alpha_0(2)\Big),$$

*where* $\alpha_0(2) = \displaystyle\sum_{j=0}^{2} \sup_{x\in\mathbb{R}^{n+1}} \|\nabla_{(j)}A\|_{2,f^{-1}(B_\rho(x))}\Big|_{t=0}.$

Proof. Observe that the smallness assumption and (AB) implies that

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu \le C_{AB}^{\frac{m-n}{m}} \left(\int_{f^{-1}(B_\rho(x)} \|A\|^m d\mu\right)^{\frac{n}{m}} \qquad \le C_{AB}^{\frac{m-n}{m}} \left(\delta_0^m\right)^{\frac{n}{m}} < \epsilon_0,$$

for

$$\delta_0^m < (\epsilon_0)^{\frac{m}{n}} C_{AB}^{\frac{n-m}{m}}.$$

Let $\gamma$ be a cutoff function on $M$ between a ball of radius $\rho$ and a ball of radius $2\rho$, as in the remark above. Then the smallness assumption (60) of Proposition 3.23 is satisfied for $\delta_0^m$ as above, that is

$$\sup_{[0,T^*]} \int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu \le \epsilon_0.$$

Recall the version (48) of Proposition 3.17 which does not require $h$ bounded. We restate this here with our choice of $\gamma$ as:

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_\rho(x))}\|A\|^2 d\mu + (2-\theta)\int_{f^{-1}(B_\rho(x))}\|\nabla_{(2)}A\|^2 d\mu \\ \le ch\int_{f^{-1}(B_{2\rho}(x))}([A*A]*A)\,d\mu + ch\int_{f^{-1}(B_{2\rho}(x))}\|A\|^2 d\mu \\ + c\int_{f^{-1}(B_{2\rho}(x))}\|A\|^2 d\mu + c\int_{f^{-1}(B_{2\rho}(x))}\Big([P_3^2(A)+P_5^0(A)]*A\Big)\,d\mu.\end{aligned}$$

Using Theorem 3.3 we obtain

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_\rho(x_1))}\|A\|^2 d\mu + (2-\theta)\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2 d\mu \\ \le c\delta_0^m + c\int_{f^{-1}(B_{2\rho}(x_1))}\Big([P_3^2(A)+P_5^0(A)]*A\Big)\,d\mu.\end{aligned}$$

Proceeding now exactly as in Proposition 3.23, we recover (22) at the point $x_1$. Note that the constant no longer depends on $\|h\|_\infty$. Moving on, we use the inequality above to conclude (64) in the case where there are no derivatives of curvature, with no additional factors of the constraint function on the right hand side. That is,

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_\rho(x_1))}\|A\|^2 d\mu + \frac{1}{2}\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2 d\mu \\ \le c\|A\|^2_{2,f^{-1}(B_{2\rho}(x_1))}(1+\|A\|^4_{\infty,f^{-1}(B_{2\rho}(x_1))}).\end{aligned}$$

Using this in the proof of Proposition 3.26 in place of Proposition 3.25 gives the required derivative of curvature bounds. □

REMARK. Allowable choices of $x_1$ depend upon the splitting of integrals in Lemma 3.6, and this depends upon $j,k$ and $l$. The proof of the next result will depend upon which class of allowable points is associated with the given constraint function.

We note that the assumption required is global, disguised as a local assumption. This is different to the case where we have no constraint function (such as for the surface diffusion or Willmore flows). However, even there, in the final argument used to prove the Lifespan Theorem one still requires this 'global disguised as local' assumption. We are merely introducing this concept earlier in the analysis.

Corollary 3.8 (The uniform bound for $h$.). *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with $h$ satisfying the assumptions of Theorem 3.3. Then there is a $\delta_0^m = \delta_0^m(n, M_0)$ such that if*

$$\sup_{[0,T^*],x\in\mathbb{R}^{n+1}} \int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu \le \delta_0^m, \tag{28}$$

*the constraint function satisfies the estimate*

$$\|h\|_{[0,T^*],\infty} \le c_h < \infty,$$

*where $c_h = c_h(\delta_0^m, C_{AB}, \rho, j, k, l, n)$.*

Proof. Using Corollary 3.7 above, we can directly estimate $h$ by localising as in the proof of Lemma 3.6. This is however contingent upon us retrieving integrals around one of the allowable points $x_1 \in \mathbb{R}^{n+1}$ from the conclusion of Corollary 3.7. So we must be somewhat careful with our estimates below.

Firstly, for the case where $j \ge \max\{2, 2k+1\}$,

$$\begin{aligned}
h &\le \int_M P_j^2(A) + P_k^1(A) + P_l^0(A) d\mu \\
&\le c \int_M \|\nabla_{(2)} A\| \ \|A\|^{j-1} + \|\nabla A\|^2 \|A\|^{j-2} + \|\nabla A\| \ \|A\|^{k-1} + \|A\|^l d\mu \\
&\le c_\rho c \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\| \ \|A\|^{j-1} d\mu + \int_M \|\nabla A\|^2 \|A\|^{j-2} d\mu \\
&\quad + c \int_M \|\nabla A\| \ \|A\|^{k-1} d\mu + c \int_M \|A\|^l d\mu \\
&\le \frac{1}{2} \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c(j) \int_M \langle A, \Delta A \rangle \ \|A\|^{j-2} d\mu \\
&\quad + c_\rho^2 \frac{c}{2} \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu + c(j,k,l) \int_M \|A\|^{2k-j} + \|A\|^l d\mu
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2d\mu + c(j)\int_M\|\nabla_{(2)}A\|\ \|A\|^{j-1}d\mu \\
&\qquad + c_\rho^2\frac{c}{2}\int_{f^{-1}(B_\rho(x_1))}\|A\|^{2j-2}d\mu + c(j,k,l)\int_M\|A\|^{2k-j}+\|A\|^l d\mu \\
&\le \int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2d\mu + c_\rho^2 c(j)\int_{f^{-1}(B_\rho(x_1))}\|A\|^{2j-2}d\mu \\
&\qquad + c(j,k,l)\int_M\|A\|^{2k-j}+\|A\|^l d\mu \\
&\le \int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2d\mu + c_\rho^2 c(j,C_{AB})\Big(\int_{f^{-1}(B_\rho(x_1))}\|A\|^m d\mu\Big)^{\frac{2j-2}{m}} \\
&\qquad + c(j,k,l,C_{AB})\sup_{x\in\mathbb{R}^{n+1}} c_\rho\Bigg[\Big(\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu\Big)^{\frac{2k-j}{m}} \\
&\qquad\qquad\qquad + \Big(\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu\Big)^{\frac{l}{m}}\Bigg] \\
&\le c_h(\delta_0^m, C_{AB}, \rho, j, k, l, n) < \infty.
\end{aligned}
$$

The other cases are simpler, and estimated as in Lemma 3.6, finished off using Corollary 3.7 as above. □

Remark. As remarked upon in the introduction, there is an alternative approach to this section. Based also on Topping's inequality, it works without the assumption (AB). However this requires monotonicity of $\int |H|$ on a ball around $x_1$, and does not give higher dimensional results. It is relevant to $h_K$ flow, where we have monotonicity of $\int H$ on the entire manifold, for all time. However the essential problem is that there is no known condition which rules out the case where mean curvature is becoming more negative in one part of the manifold and more positive in another part, such that the integral over the entire manifold is non-increasing, but for any small ball the integral $\int |H|$ is increasing. Also, even if such a case is ruled out, we have no way of ensuring that the special points $x_1$ are in the regions of $M$ where $\int |H|$ is monotone. What we really lack is a non-trivial condition we can impose on $M_0$ such that monotonicity of $\int H$ implies monotonicity of $\int |H|$, however without the maximum principle we have not been able to achieve this. Thus $h_K$ still presents difficulty.

We have thus shown that for the class of constraint functions satisfying the conditions of Theorem 3.3, the a priori conditional bound (CB) holds.

## 4. Evolution equations

We begin with the following evolution equations.

LEMMA 3.9. *For $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ evolving by $\partial_t f = F\nu$ the following equations hold:*

$$\begin{aligned}
&\frac{\partial}{\partial t} g_{ij} = 2FA_{ij}, \quad \frac{\partial}{\partial t} g^{ij} = -2FA^{ij}, \quad \frac{\partial}{\partial t} d\mu = (HF)d\mu,\\
&\frac{\partial}{\partial t}\nu = -\nabla F, \quad \frac{\partial}{\partial t} A_{ij} = -\nabla_{ij}F + FA_i^p A_{pj},\\
&\frac{\partial}{\partial t} H = -\Delta F - F\|A\|^2, \text{ and}\\
&\frac{\partial}{\partial t} A^o_{ij} = -S^o(\nabla_{(2)}F) + F\Big(A_i^p A_{pj} + \frac{1}{n} g_{ij}|A|^2 - \frac{2}{n} HA_{ij}\Big),
\end{aligned}$$

*where $S^o(T)$ denotes the tracefree part of a symmetric bilinear form $T$. If $F = \Delta H + h$ then the following evolution equation additionally holds:*

$$\frac{\partial}{\partial t} A_{ij} = -\Delta^2 A_{ij} + \|A\|^2 A_{ij} + (\Delta H - H + h)A_{ik}A_j^k.$$

PROOF. We begin by proving that the evolution of the unit normal $\nu$ is given by

$$\frac{\partial \nu}{\partial t} = -g^{ij}\frac{\partial F}{\partial x^i}\frac{\partial f}{\partial x^j} = -\nabla F.$$

Since $\nu$ is a unit vector, $(\nu|\nu) = 1$. Differentiating,

$$\left(\frac{\partial \nu}{\partial \xi} \;\middle|\; \nu\right) = 0, \tag{29}$$

and so any derivative of the normal is again normal to $\nu$, hence tangential to M.

Since $\{\partial_i f\}$ form a basis for $TM$, we may express the time derivative of $\nu$ as

$$\frac{\partial \nu}{\partial t} = g^{ij}\left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right)\frac{\partial f}{\partial x^i}. \tag{30}$$

As $\nu$ is normal to vectors in $TM$, $\left(\nu|\frac{\partial f}{\partial x^i}\right) = 0$. Differentiating this equation,

$$\begin{aligned}
0 &= \left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) + \left(\nu \;\middle|\; \frac{\partial}{\partial t}\frac{\partial f}{\partial x^i}\right)\\
&= \left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) + \left(\nu \;\middle|\; \frac{\partial}{\partial x^i}\frac{\partial f}{\partial t}\right)\\
&= \left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) + \left(\nu \;\middle|\; \frac{\partial (F\nu)}{\partial x^i}\right)\\
&= \left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) + \left(\nu \;\middle|\; \nu\frac{\partial F}{\partial x^i}\right) + F\left(\nu, \frac{\partial \nu}{\partial x^i}\right)\\
&= \left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) + \frac{\partial F}{\partial x^i}\left(\nu \;\middle|\; \nu\right)\\
&= \left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) + \frac{\partial F}{\partial x^i},
\end{aligned}$$

so

$$\left(\frac{\partial \nu}{\partial t} \;\middle|\; \frac{\partial f}{\partial x^i}\right) = -\frac{\partial F}{\partial x^i}. \tag{31}$$

Substituting (31) into (30), we have

$$\frac{\partial \nu}{\partial t} = -g^{ij}\frac{\partial F}{\partial x^i}\frac{\partial f}{\partial x^j},$$

as required.

We now move on to proving that the induced metric $(g_{ij})$ and inverse $(g^{ij})$ evolves by

$$\frac{\partial}{\partial t}g_{ij} = -2FA_{ij}, \text{ and } \frac{\partial}{\partial t}g^{ij} = 2FA^{ij}. \tag{32}$$

The induced metric is naturally specified by

$$g_{ij} = \left( \frac{\partial f}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right).$$

Note first that

$$\begin{aligned}
\left( \frac{\partial}{\partial t} \frac{\partial f}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right) &= \left( \frac{\partial (F\nu)}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right) \\
&= \left( \nu \frac{\partial F}{\partial x^i} + F \frac{\partial \nu}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right) \\
&= \frac{\partial F}{\partial x^i} \left( \nu \;\middle|\; \frac{\partial f}{\partial x^j} \right) + \left( F \frac{\partial \nu}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right) \\
&= 2F \left( \frac{\partial \nu}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right) = 2FA_{ij},
\end{aligned}$$

where we used the definition of $A_{ij}$ in the last step. Since the second fundamental form is symmetric, this gives

$$\frac{\partial}{\partial t} g_{ij} = \left( \frac{\partial}{\partial t} \frac{\partial f}{\partial x^i} \;\middle|\; \frac{\partial f}{\partial x^j} \right) + \left( \frac{\partial}{\partial t} \frac{\partial f}{\partial x^j} \;\middle|\; \frac{\partial f}{\partial x^i} \right) = 2FA_{ij}.$$

For the inverse, we first differentiate $g^{ik} g_{jk} = \delta^i_j$:

$$\begin{aligned}
0 &= \frac{\partial (g^{ik} g_{jk})}{\partial t} \\
&= \frac{\partial g^{ik}}{\partial t} g_{jk} + \frac{\partial g_{jk}}{\partial t} g^{ik} \\
&= \frac{\partial g^{ik}}{\partial t} g_{jk} + g^{ik} (2F h_{jk}) \\
&= \frac{\partial g^{ik}}{\partial t} g_{jk} + 2F A^i_j,
\end{aligned}$$

so

$$g^{kl} g_{jk} \frac{\partial g^{ik}}{\partial t} = \frac{\partial g^{il}}{\partial t} = -g^{kl} 2F A^i_j - = 2F A^{il},$$

and with the substitution $l \leftrightarrow j$, this finishes the proof of (32).

We will now need to make use of the rule for differentiating determinants:

$$\frac{\partial}{\partial x} \det A = \det A \; A^{ij} \frac{\partial}{\partial x} A_{ij}.$$

This is proved by considering the cofactor of $\frac{\partial A}{\partial A_{ij}}$, which is $\det\ A(A^{-1})_{ij} = \det\ A\ A^{ij}$. The formula then follows by summing over each entry.

We claim that the measure on $M$ evolves according to

$$\frac{\partial}{\partial t} d\mu = HFd\mu. \tag{33}$$

The induced surface measure $d\mu$ on $M$ is given by

$$d\mu = \sqrt{\det\ (g_{ij})} dx.$$

Differentiating,

$$\frac{\partial}{\partial t} d\mu = \frac{\partial}{\partial t}\left(\sqrt{\det\ (g_{ij})} dx\right) = \frac{1}{2}\sqrt{\det\ g}\ g^{ij}\frac{\partial}{\partial t} g_{ij} dx = Fg^{ij}A_{ij}d\mu = HFd\mu,$$

where we used the evolution of $g_{ij}$ in the last equality. This shows (33).

We shall now consider the evolution of the components of the second fundamental form, $A_{ij}$. These evolve by

$$\frac{\partial}{\partial t} A_{ij} = -\nabla_{ij}F + FA_{ik}A^k_j. \tag{34}$$

Recall that the components $A_{ij}$ are given by

$$A_{ij} = -\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f \;\middle|\; \nu\right).$$

Differentiating,

$$\begin{aligned}
\frac{\partial}{\partial t} A_{ij} &= -\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial f}{\partial t} \;\middle|\; \nu\right) - \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f \;\middle|\; \frac{\partial \nu}{\partial t}\right) \\
&= -\left(\frac{\partial}{\partial x^i}\left(\frac{\partial F}{\partial x^j}\nu + F\frac{\partial}{\partial x^j}\nu\right) \;\middle|\; \nu\right) - \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f \;\middle|\; \nabla F\right) \\
&= -\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}F - F\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\nu \;\middle|\; \nu\right) - \left(\Gamma^k_{ij}\frac{\partial f}{\partial x^k} - A_{ij}\nu \;\middle|\; \nabla F\right) \\
&= -\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}F - F\left(\frac{\partial}{\partial x^i}\left(A_{jk}g^{kl}\frac{\partial f}{\partial x^l}\right) \;\middle|\; \nu\right) - \Gamma^k_{ij}\left(\frac{\partial f}{\partial x^k} \;\middle|\; \nabla F\right) \\
&= -\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}F + \Gamma^k_{ij}\frac{\partial F}{\partial x^k}\right) - FA_{jk}g^{kl}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^l} f \;\middle|\; \nu\right)
\end{aligned}$$

$$= -\nabla_i\nabla_j F + FA_{jk}g^{kl}A_{li}$$

$$= -\nabla_i\nabla_j F + FA_{ik}A_j^k.$$

In the above we used the Gauss-Weingarten equations

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}f = \Gamma_{ij}^k\frac{\partial f}{\partial x^k} - A_{ij}\nu, \quad \frac{\partial}{\partial x^j}\nu = A_{jl}g^{lm}\frac{\partial f}{\partial x^m},$$

and the definition of the covariant derivative. Using this evolution and the evolution of the induced metric we may compute the evolution of the mean curvature:

$$\frac{\partial}{\partial t}H = g^{ij}\frac{\partial}{\partial t}A_{ij} + A_{ij}\frac{\partial}{\partial t}g^{ij} = -\Delta F - F\|A\|^2$$

and the tracefree second fundamental form:

$$\begin{aligned}
\frac{\partial}{\partial t}A^o_{ij} &= \frac{\partial}{\partial t}A_{ij} - \frac{1}{n}H\frac{\partial}{\partial t}g_{ij} - \frac{1}{n}g_{ij}\frac{\partial}{\partial t}H \\
&= -\nabla_i\nabla_j F + FA_{ik}A_j^k - \frac{2}{n}FHA_{ij} + \frac{1}{n}g_{ij}(\Delta F + F\|A\|^2) \\
&= -\Big(\nabla_i\nabla_j F - \frac{1}{n}g_{ij}\Delta F\Big) + FA_{ik}A_j^k - F\frac{2}{n}HA_{ij} + F\frac{1}{n}g_{ij}\|A\|^2 \\
&= -S^o(\nabla_{(2)}F) + F\Big(A_{ik}A_j^k - \frac{2}{n}HA_{ij} + \frac{1}{n}g_{ij}\|A\|^2\Big),
\end{aligned}$$

where $S^o(T)$ denotes the tracefree part of a symmetric bilinear form $T$. Specialising to the case where $F = \Delta H + h$, one may use Simons' identity (SI) to obtain

$$\frac{\partial}{\partial t}A_{ij} = -\Delta^2 A_{ij} + \|A\|^2A_{ij} + (\Delta H - H + h)A_{ik}A_j^k.$$

$\square$

We will also need to know the general structure of the evolution of the Christoffel symbols. Note that any derivative of the Christoffel symbols is a tensor.

LEMMA 3.10. *For $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ evolving by $\partial_t f = F\nu$ the Christoffel symbols evolve by*

$$\frac{\partial}{\partial t}\Gamma = FP_1^1(A) + A * \nabla F. \tag{35}$$

PROOF. Recall the following formula for the Christoffel symbols in a local coordinate system which is torsion free:

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}g_{il} - \frac{\partial}{\partial x^l}g_{ij}\right).$$

Differentiating the above and using the evolution of the induced metric (32),

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}\frac{\partial g^{kl}}{\partial t}&\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}g_{il} - \frac{\partial}{\partial x^l}g_{ij}\right)\\
&+ \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial t}g_{jl} + \frac{\partial}{\partial x^j}g_{il} - \frac{\partial}{\partial x^l}g_{ij}\right)\\
&+ \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}\frac{\partial}{\partial t}g_{il} - \frac{\partial}{\partial x^l}g_{ij}\right)\\
&+ \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}g_{il} - \frac{\partial}{\partial x^l}\frac{\partial}{\partial t}g_{ij}\right)\\
= (-FA^{kl})&\left(\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{jl} - \frac{\partial}{\partial x_l}g_{ij}\right) + \Gamma_{ij}^k + FP_1^1(A) + A * \nabla F\\
= \Gamma * (1&+ F * A) + FP_1^1(A) + A * \nabla F.
\end{aligned}$$

Evaluating the above at a point where $\Gamma \equiv 0$ gives (35), and so finishes the proof of the Lemma. □

The rest of this section is in two parts. The first aims to prove in detail the estimates needed to present a proof of the first result needed to drive our overall argument, Proposition 3.23. Since the proof of the second estimate, Proposition 3.26, requires similar results but with more generality, we present the estimates needed for this after the simpler versions, in a slightly more succinct fashion.

Using the curvature evolution equations in Lemma 3.9 we calculate the evolution of the total squared curvature.

LEMMA 3.11. *Let* $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ *be a* (CSD) *flow, and* $\gamma$ *as in* (25). *Then for any* $s \geq 0$,

$$\begin{aligned}\frac{d}{dt}\int_M \|A\|^2\gamma^s d\mu = &-2\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + \int_M \|A\|^2\partial_t\gamma^s d\mu \\ &+ 2\int_M \langle(\nabla\gamma^s)A, \Delta\nabla A\rangle - \left\langle(\nabla\gamma^s)(\nabla A), \nabla_{(2)}A\right\rangle d\mu \\ &+ \int_M [(P_3^2(A) + hA * A) * A]\gamma^s d\mu.\end{aligned}$$

PROOF. This is simply differentiation followed by two applications of integration by parts. First we differentiate,

$$\begin{aligned}\frac{d}{dt}\int_M \|A\|^2\gamma^s d\mu &= \int_M (\partial_t\|A\|^2)\gamma^s + \|A\|^2(\partial_t\gamma^s) + \|A\|^2\gamma^s(\partial_t d\mu)\ \ d\mu \\ &= \int_M 2\langle\partial_t A, A\rangle\gamma^s d\mu + \int_M 2(\partial_t g^{ik})g^{jl}h_{ij}h_{kl}\gamma^s d\mu \\ &\quad + \int_M \|A\|^2\partial_t\gamma^s d\mu + \int_M \|A\|^2(H\Delta H + Hh)\gamma^s d\mu.\end{aligned}$$

We leave the third integral for this proof. The second and fourth integral are both of the form

$$\int_M [(P_3^2(A) + hA * A) * A]\gamma^s d\mu.$$

For our purposes in this chapter we are not concerned with the precise algebraic nature of the nonlinearities. We now deal with the first integral. Note that integration by parts does not give a boundary term as all our manifolds are compact and without boundary. Using interchange of covariant derivative we calculate,

$$\begin{aligned}\int_M \langle\partial_t A, A\rangle\gamma^s d\mu = &-\int_M \left\langle\Delta^2 A, A\right\rangle\gamma^s d\mu + \int_M P_3^2(A) * A\gamma^s d\mu \\ &+ \int_M hA * A * A\gamma^s d\mu\end{aligned}$$

$$
\begin{aligned}
&= -\int_M \langle \nabla^{qp}\nabla_{qp}A, A\rangle \gamma^s d\mu \\
&\quad + \int_M \Big(\nabla\big[(A*A - A*A)*\nabla A\big]\Big) * A\gamma^s d\mu \\
&\quad + \int_M P_3^2(A) * A\gamma^s d\mu + \int_M hA*A*A\gamma^s d\mu \\
&= -\int_M \langle \nabla^{pq}\nabla_{qp}A, A\rangle \gamma^s d\mu \\
&\quad + \int_M \Big(\nabla_{(2)}\big[(A*A - A*A)*A\big]\Big) * A\gamma^s d\mu \\
&\quad + \int_M P_3^2(A) * A\gamma^s d\mu + \int_M hA*A*A\gamma^s d\mu \\
&= \int_M \langle \Delta\nabla A, \nabla A\rangle \gamma^s d\mu + \int_M \langle \Delta\nabla A, (\nabla\gamma^s)A\rangle d\mu \\
&\quad + \int_M P_3^2(A) * A\gamma^s d\mu + \int_M hA*A*A\gamma^s d\mu \\
&= -\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu \\
&\quad + \int_M \langle \Delta\nabla A, (\nabla\gamma^s)A\rangle d\mu - \int_M \Big\langle \nabla_{(2)}A, (\nabla\gamma^s)\nabla A\Big\rangle d\mu \\
&\quad + \int_M P_3^2(A) * A\gamma^s d\mu + \int_M hA*A*A\gamma^s d\mu.
\end{aligned}
$$

Combining the evaluation of each of the integrals above gives the lemma. $\square$

Routine estimates refine the previous lemma into the following form.

LEMMA 3.12. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow, and $\gamma$ as in* (25). *Fix $\delta > 0$. Then for any $s \geq 4$,*

$$
\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^s d\mu + (2-\delta)\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu \\
&\qquad \leq c\int_M \|A\|^2\partial_t\gamma^s d\mu + ch\int_M P_3^0(A)\gamma^s d\mu + c\int_M \|A\|^2\gamma^{s-4} d\mu \\
&\qquad\qquad + c\int_M [(P_3^2(A) + P_5^0(A)) * A]\gamma^s d\mu,
\end{aligned}
$$

*where $c = c(c_{\gamma 1}, c_{\gamma 2}, s)$.*

Proof. We wish to deal with the leftover terms from integrating by parts in the previous lemma. In the following proof and in fact throughout we use the following two inequalities extensively:

$$\text{For any } \epsilon > 0,\ a, b \in \mathbb{R} \qquad ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \qquad \text{and}$$

$$\text{For any tensors } A, B, \qquad |\langle A, B\rangle| \leq \|A\|\ \|B\|;$$

which are of course the Cauchy inequality and the Cauchy-Schwartz inequality respectively.

Estimate the undesirable integrals from before by

$$\begin{aligned}
&\int_M \langle (\nabla\gamma^s)A, \Delta\nabla A\rangle\, d\mu - \int_M \Big\langle (\nabla\gamma^s)(\nabla A), \nabla_{(2)}A\Big\rangle\, d\mu \\
&\qquad\qquad = -\int_M \Big\langle (\nabla_{(2)}\gamma^s)A, \nabla_{(2)}A\Big\rangle\, d\mu - \int_M \Big\langle (\nabla\gamma^s)(\nabla A), \nabla_{(2)}A\Big\rangle\, d\mu \\
&\qquad\qquad\quad - \int_M \Big\langle (\nabla\gamma^s)(\nabla A), \nabla_{(2)}A\Big\rangle\, d\mu \\
&\qquad\qquad \leq \int_M \|A\|\ \|\nabla_{(2)}A\| \Big[s(s-1)c_{\gamma 1}^2\gamma^{s-2} + sc_{\gamma 2}\gamma^{s-1}(1+\|A\|)\Big]\, d\mu \\
&\qquad\qquad\quad + 2sc_{\gamma 1}\int_M \|\nabla A\|\ \|\nabla_{(2)}A\|\gamma^{s-1} d\mu \\
&\qquad\qquad \leq (\delta_1+\delta_2+\delta_3+\delta_4)\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu \\
&\qquad\qquad\quad + \left(\frac{[s(s-1)c_{\gamma 1}^2]^2}{4\delta_1} + \frac{[sc_{\gamma 2}]^2}{4\delta_2}\right)\int_M \|A\|^2\gamma^{s-4} d\mu \\
&\qquad\qquad\quad + \frac{[sc_{\gamma 2}]^2}{4\delta_3}\int_M \|A\|^4\gamma^{s-2} d\mu + \frac{[sc_{\gamma 1}]^2}{\delta_4}\int_M \|\nabla A\|^2\gamma^{s-2} d\mu.
\end{aligned}$$

Therefore, choosing $\sum_i \delta_i = \delta$ (where $\delta$ is in the statement of the lemma) and combining with the previous lemma will finish the proof, if we can estimate the remaining integrals.

The first is simple, using

$$\|A\|^4\gamma^{s-2} \leq \frac{1}{2}\|A\|^6\gamma^s + \frac{1}{2}\|A\|^2\gamma^{s-4} = \frac{1}{2}P_5^0(A) * A\gamma^s + \frac{1}{2}\|A\|^2\gamma^{s-4}.$$

For the second we need a baby interpolation inequality, which we derive as follows:

$$\begin{aligned}\int_M \|\nabla A\|^2\gamma^{s-2}d\mu &= -\int_M \left\langle(\nabla\gamma^{s-2})A, \nabla A\right\rangle d\mu - \int_M \left\langle \Delta A, A\right\rangle \gamma^{s-2}d\mu\\ &\le (s-2)c_{\gamma 1}\int_M \|\nabla A\|\ \|A\|\gamma^{s-3}d\mu + \int_M \|\nabla_{(2)}A\|\ \|A\|\gamma^{s-2}d\mu\\ &\le \delta_5\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + \beta\int_M \|\nabla A\|^2\gamma^{s-2}d\mu\\ &\quad + \left(\frac{1}{4\delta_5} + \frac{[(s-2)c_{\gamma 1}]^2}{4\beta}\right)\int_M \|A\|^2\gamma^{s-4}d\mu.\end{aligned}$$

Therefore, for any $\beta > 0$ we have

$$\begin{aligned}(1-\beta)\int_M \|\nabla A\|^2\gamma^{s-2}d\mu \le \delta_5 &\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu\\ &+ \left(\frac{1}{4\delta_5} + \frac{[(s-2)c_{\gamma 1}]^2}{4\beta}\right)\int_M \|A\|^2\gamma^{s-4}d\mu.\end{aligned}$$

Inserting this into our previous estimate we have

$$\begin{aligned}&\int_M \left\langle(\nabla\gamma^s)A, \Delta\nabla A\right\rangle d\mu - \int_M \left\langle(\nabla\gamma^s)(\nabla A), \nabla_{(2)}A\right\rangle d\mu\\ &\qquad\le \left(\delta_1+\delta_2+\delta_3+\delta_4+\delta_5\frac{[sc_{\gamma 1}]^2}{(1-\beta)\delta_4}\right)\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu\\ &\qquad\quad + \Bigg(\frac{[s(s-1)c_{\gamma 1}^2]^2}{4\delta_1} + \frac{[sc_{\gamma 2}]^2}{4\delta_2} + \frac{[sc_{\gamma 2}]^2}{8\delta_3}\\ &\qquad\qquad + \frac{[sc_{\gamma 1}]^2}{4(1-\beta)\delta_4\delta_5} + \frac{[s(s-2)c_{\gamma 1}^2]^2}{4\delta_4(1-\beta)\beta}\Bigg)\int_M \|A\|^2\gamma^{s-4}d\mu\\ &\qquad\quad + \frac{[sc_{\gamma 2}]^2}{8\delta_3}\int_M P_5^0(A)*A\gamma^{s-2}d\mu.\end{aligned}$$

We are now finished, by any reasonable choice for our constants $\delta_i$ and $\beta$. For example, let $\delta > 0$ be any fixed positive number (as in the statement of the lemma), and choose

$$\beta = \frac{1}{2},\quad \delta_1+\delta_2+\delta_3 = \frac{\delta}{4},\quad \delta_4 = \frac{\delta}{2},\quad \delta_5 = \frac{\delta^2}{16[sc_{\gamma 1}]^2}.$$

□

We conclude this series of estimates by proving the following.

PROPOSITION 3.13. *Let $f : M \times [0,T) \to \mathbb{R}^3$ be a* (CSD) *flow, and $\gamma$ as in* (25). *Fix $\delta > 0$. Then for any $s \geq 4$,*

$$\frac{d}{dt}\int_M \|A\|^2\gamma^s d\mu + (2-\delta)\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu$$
$$\leq c(1+|h|)\int_M \|A\|^2\gamma^{s-4}d\mu + c(1+h)\int_M \|A\|^6\gamma^s d\mu + c\int_M \|A\|P_3^2(A)\gamma^s d\mu,$$

*where $c = c(c_{\gamma 1}, c_{\gamma 2}, s)$.*

PROOF. This proposition is essentially an evaluation of the time derivative $\partial_t\gamma^s$ in the previous lemma. The other terms are easily estimated by

$$P_5^0(A) * A \leq c\|A\|^6, \quad \text{and} \quad P_3^0(A)\gamma^s \leq c\|A\|^2\gamma^{s-4} + c\|A\|^6\gamma^s.$$

Differentiating,

$$\begin{aligned}\partial_t\gamma^s &= s\gamma^{s-1}\partial_t(\tilde{\gamma}\circ f)\\ &\leq sc_{\tilde{\gamma}}\gamma^{s-1}|\Delta H + h|,\end{aligned}$$

so

$$\begin{aligned}\int_M \|A\|^2\partial_t\gamma^s d\mu &\leq sc_{\tilde{\gamma}}\int_M \|A\|^2\gamma^{s-1}|\Delta H + h|d\mu\\ &\leq sc_{\tilde{\gamma}}|h|\int_M \|A\|^2\gamma^{s-4}d\mu + \delta_1\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu\\ &\qquad + \frac{[sc_{\tilde{\gamma}}]^2}{4\delta_1}\int_M \|A\|^4\gamma^{s-2}d\mu\\ &\leq sc_{\tilde{\gamma}}|h|\int_M \|A\|^2\gamma^{s-4}d\mu + \delta_1\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu\\ &\qquad + \frac{[sc_{\tilde{\gamma}}]^2}{8\delta_1}\left(\int_M \|A\|^6\gamma^s d\mu + \int_M \|A\|^2\gamma^{s-4}d\mu\right).\end{aligned}$$

Substituting into the previous lemma and absorbing $\delta_1\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu$ on the left gives the result. $\square$

We are now, modulo a multiplicative Sobolev inequality, ready to give the proof of our first major integral estimate, Proposition 3.23, used in Section 5 to prove the Lifespan Theorem. However, for the second important estimate we need to consider the more general case where we have $k$ derivatives of curvature. The evolution equation for the iterated covariant derivative of the second fundamental form is given below.

LEMMA 3.14. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a (CSD) flow. Then the following equation holds:*

$$\frac{\partial}{\partial t}\nabla_{(k)}A = -\Delta^2\nabla_{(k)}A + hP_2^k(A) + P_3^{k+2}(A).$$

PROOF. We will use the evolution equation for $A$ and interchange of covariant derivative. From Lemma 3.9 we have

$$\partial_t\nabla_{(k)}A = \nabla_{(k)}\partial_t A = -\nabla_{(k)}\Delta^2 A + P_3^{k+2}(A) + hP_2^k(A).$$

Now to obtain the result we interchange covariant derivatives $4k$ times:

$$\begin{aligned}
\partial_t\nabla_{(k)}A &= -\nabla_{(k)}\nabla^p\nabla_p\Delta A + P_3^{k+2}(A) + hP_2^k(A)\\
&= -\nabla_{(k-1)}\nabla^p\nabla_{(1)}\nabla_p\Delta A - \nabla_{(k-1)}[P_2^0(A) * \nabla\Delta A] + P_3^{k+2}(A) + hP_2^k(A)\\
&= -\nabla^p\nabla_{(k)}\nabla_p\Delta A\\
&\qquad + P_3^{k+2}(A) + hP_2^k(A) - \sum_{j=1}^{k}\nabla_{(k-j)}[(j+2)P_2^0(A) * \nabla_{(j)}\Delta A]\\
&= -\nabla^p\nabla_{(k)}\nabla_p\Delta A + P_3^{k+2}(A) + hP_2^k(A)\\
&= -\Delta\nabla_{(k)}\Delta A\\
&\qquad + P_3^{k+2}(A) + hP_2^k(A) - \sum_{j=1}^{k}\nabla_{(k-j+1)}[(j+1)P_2^0(A) * \nabla_{(j-1)}\Delta A]\\
&= -\Delta\nabla_{(k)}\Delta A + P_3^{k+2}(A) + hP_2^k(A)
\end{aligned}$$

$$
\begin{aligned}
&= -\Delta\nabla^p\nabla_{(k)}\nabla_p A \\
&\qquad + P_3^{k+2}(A) + hP_2^k(A) - \sum_{j=1}^{k} \nabla_{(k-j+2)}[(j+2)P_2^0(A) * \nabla_{(j-1)}\nabla A] \\
&= -\Delta\nabla^p\nabla_{(k)}\nabla_p A + P_3^{k+2}(A) + hP_2^k(A) \\
&= -\Delta^2\nabla_{(k)}A \\
&\qquad + P_3^{k+2}(A) + hP_2^k(A) - \sum_{j=1}^{k} \nabla_{(k-j+3)}[(j+1)P_2^0(A) * \nabla_{(j-1)}A] \\
&= -\Delta^2\nabla_{(k)}A + P_3^{k+2}(A) + hP_2^k(A);
\end{aligned}
$$

note that here we allow several exotic constants to appear in the collection of terms $P_3^{k+2}(A)$. These are all universal however and are collected in the $P$-term as mentioned in the definition of the $P$-style terms earlier. □

The following is an easy consequence of the above lemma.

Corollary 3.15. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow. Then the following equation holds:*

$$\frac{\partial}{\partial t}\|\nabla_{(k)}A\|^2 = -2\left\langle \nabla_{(k)}A, \nabla^p\Delta\nabla_p\nabla_{(k)}A \right\rangle + [hP_2^k(A) + P_3^{k+2}(A)] * \nabla_{(k)}A.$$

Proof. We simply use the previous lemma as follows:

$$
\begin{aligned}
\partial_t\|\nabla_{(k)}A\|^2 &= 2\left\langle \nabla_{(k)}A, \partial_t\nabla_{(k)}A \right\rangle + (k+2)\partial_t g * \nabla_{(k)}A * \nabla_{(k)}A \\
&= -2\left\langle \nabla_{(k)}A, \Delta^2\nabla_{(k)}A + P_3^{k+2}(A) + hP_2^k(A) \right\rangle \\
&\qquad + 2(k+2)[(\Delta H)A + hA] * \nabla_{(k)}A * \nabla_{(k)}A \\
&= -2\left\langle \nabla_{(k)}A, \Delta^2\nabla_{(k)}A \right\rangle + [hP_2^k(A) + P_3^{k+2}(A)] * \nabla_{(k)}A
\end{aligned}
$$

$$
\begin{aligned}
&= -2\left\langle \nabla_{(k)}A, \nabla^p \Delta \nabla_p \nabla_{(k)}A \right\rangle \\
&\qquad + \sum_{j=1}^{2} \left(\nabla_{(2-j)}[P_2^0(A) * \nabla_{(k+j)}A]\right) * \nabla_{(k)}A \\
&\qquad + [hP_2^k(A) + P_3^{k+2}(A)] * \nabla_{(k)}A \\
&= -2\left\langle \nabla_{(k)}A, \nabla^p \Delta \nabla_p \nabla_{(k)}A \right\rangle + [hP_2^k(A) + P_3^{k+2}(A)] * \nabla_{(k)}A.
\end{aligned}
$$

□

Using Corollary 3.15, we derive the following integral identity.

Corollary 3.16. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow, and $\gamma$ as in* (25). *Then for any $s \geq 0$,*

$$
\begin{aligned}
\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2 \gamma^s d\mu + 2\int_M \|\nabla_{(k+2)}A\|^2 \gamma^s d\mu &= \int_M \|\nabla_{(k)}A\|^2 (\partial_t \gamma^s) d\mu \\
+ 2\int_M \left\langle (\nabla\gamma^s)(\nabla_{(k)}A), \Delta\nabla_{(k+1)}A \right\rangle d\mu &- 2\int_M \left\langle (\nabla\gamma^s)(\nabla_{(k+1)}A), \nabla_{(k+2)}A \right\rangle d\mu \\
+ \int_M \gamma^s [(P_3^{k+2}(A) + hP_2^k(A)) * \nabla_{(k)}A] d\mu.&
\end{aligned}
$$

Proof. First we differentiate,

$$
\begin{aligned}
\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2 \gamma^s d\mu &= \int_M (\partial_t \|\nabla_{(k)}A\|^2)\gamma^s + \|\nabla_{(k)}A\|^2 \Big[(\partial_t \gamma^s) + \gamma^s(\partial_t d\mu)\Big] \ d\mu \\
&= -2\int_M \Big[\left\langle \nabla_{(k)}A, \nabla^p \Delta \nabla_p \nabla_{(k)}A \right\rangle + [hP_2^k(A) + P_3^{k+2}(A)] * \nabla_{(k)}A\Big]\gamma^s d\mu \\
&\quad + \int_M (k+2)(\partial_t g^{i_1 j_1}) g^{i_2 j_2} \cdots g^{i_k j_k} \nabla_{i_1 \cdots i_k} A_{i_{k+1} i_{k+2}} \nabla_{j_1 \cdots j_k} A_{j_{k+1} j_{k+2}} \gamma^s d\mu \\
&\quad + \int_M \|\nabla_{(k)}A\|^2 \partial_t \gamma^s d\mu + \int_M \|\nabla_{(k)}A\|^2 (H\Delta H + Hh)\gamma^s d\mu.
\end{aligned}
$$

We leave the third integral for this proof. The second and fourth integral are both of the form

$$
\int_M \Big[(P_3^{k+2}(A) + hA * \nabla_{(k)}A) * \nabla_{(k)}A\Big]\gamma^s d\mu.
$$

We now deal with the first integral. Integration by parts gives

$$\begin{aligned}
-2\int_M\Big[\Big\langle\nabla_{(k)}A,\nabla^p\Delta\nabla_p\nabla_{(k)}A\Big\rangle+[hP_2^k(A)+P_3^{k+2}(A)]*\nabla_{(k)}A\Big]\gamma^s d\mu&\\
=-2\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu+2\int_M\Big\langle\Delta\nabla_{(k+1)}A,(\nabla\gamma^s)\nabla_{(k)}A\Big\rangle\,d\mu&\\
-2\int_M\Big\langle\nabla_{(k+2)}A,(\nabla\gamma^s)\nabla_{(k+1)}A\Big\rangle\,d\mu&\\
+\int_M\Big([hP_2^k(A)+P_3^{k+2}(A)]*\nabla_{(k)}A\Big)\gamma^s d\mu.&
\end{aligned}$$

Combining the evaluation of each of the integrals above gives the statement of the lemma. □

Remark. It is easy to compute an 'allowable' explicit expression for the large constant $c$. For example, if $N_p^q(A)$ denotes a $P_p^q(A)$ term with all non-zero constants set to one, we have

$$\begin{aligned}
\int_M\gamma^s\frac{\partial}{\partial t}\|\nabla_{(k)}A\|^2d\mu+2\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu&=2\int_M\Big\langle(\nabla\gamma^s)(\nabla_{(k)}A),\Delta\nabla_{(k+1)}A\Big\rangle\,d\mu\\
&\quad-2\int_M\Big\langle(\nabla\gamma^s)(\nabla_{(k+1)}A),\nabla_{(k+2)}A\Big\rangle\,d\mu\\
&\quad+(4+10k+2\cdot 7\cdot 3^k)\int_M\gamma^s[(N_3^{k+2}(A)+hN_2^k(A))*\nabla_{(k)}A]d\mu,
\end{aligned}$$

and note that here any non-zero constants in the $*$ operator terms have been similarly set to one.

We now wish to use interpolation to estimate the extraneous terms from integration by parts. For $k=1$, the required inequality follows easily (for $\theta,\beta>0$):

(36)

$$(1-\beta)\int_M\|\nabla A\|^2\gamma^{s-2}d\mu\le\theta\int_M\|\nabla_{(2)}A\|^2\gamma^s d\mu+\frac{\beta+\theta[(s-2)c_{\gamma 1}]^2}{4\beta\theta}\int_M\|A\|^2\gamma^{s-4}d\mu.$$

For $k > 1$ however we need a more powerful version of the above. Let $2 \le p < \infty$, $k \in \mathbb{N}$, $s \ge kp$, and $\theta > 0$. Then we have

(37)
$$\left(\int_M \|\nabla_{(k)}A\|^p\gamma^s d\mu\right)^{\frac{1}{p}} \le \theta\left(\int_M \|\nabla_{(k+1)}A\|^p\gamma^{s+p}d\mu\right)^{\frac{1}{p}} + c\left(\int_{[\gamma>0]} \|A\|^p\gamma^{s-kp}d\mu\right)^{\frac{1}{p}},$$

where $c = c(\theta, c_{\gamma 1}, s, p)$. This is proved by induction on the inequality (36). Details can be found in [**37**], or alternately Appendix A. We now estimate the equality in Corollary 3.16.

Proposition 3.17. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow and $\gamma$ a cutoff function as in* (25). *Then if $h$ satisfies* (CB) *on the support of $\gamma$ and $\|A\|^2_{2,[\gamma>0]} < \epsilon_0$, we have for a fixed $\theta > 0$ and $s \ge 2k+4$,*

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu &+ (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\ &\le (c+ch)\int_M \|A\|^2\gamma^{s-4-2k}d\mu + ch\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu \\ &\qquad + c\int_M \left([P_3^{k+2}(A) + P_5^k(A)]*\nabla_{(k)}A\right)\gamma^s d\mu,\end{aligned}$$

*where $c = c(c_{\gamma 1}, c_{\gamma 2}, s, k, \|h\|_{\infty,[0,T)}, \theta)$.*

Proof. We will use Corollary 3.16 and equation (37) to deal with the derivatives (both spatial and temporal) of $\gamma$.

Corollary 3.16 implies

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu &+ 2\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\ &= \int_M (\partial_t\gamma^s)\|\nabla_{(k)}A\|^2 d\mu + \int_M [(P_3^{k+2}(A) + hP_2^k(A))*\nabla_{(k)}A]\gamma^s d\mu \\ &\qquad + 2\int_M \left\langle(\nabla\gamma^s)(\nabla_{(k)}A), \Delta\nabla_{(k+1)}A\right\rangle d\mu \\ &\qquad - 2\int_M \left\langle(\nabla\gamma^s)(\nabla_{(k+1)}A), \nabla_{(k+2)}A\right\rangle d\mu. \qquad (38)\end{aligned}$$

Since $\partial_t \gamma^s = s\gamma^{s-1}(D\tilde{\gamma}\circ f)[(\Delta H + h)\nu]$,

$$
\begin{aligned}
\int_M \|\nabla_{(k)}A\|^2 \partial_t \gamma^s d\mu &= s\int_M \|\nabla_{(k)}A\|^2 \gamma^{s-1}(D\tilde{\gamma}\circ f)[(\Delta H + h)\nu] d\mu \\
&= s\int_M \|\nabla_{(k)}A\|^2 \gamma^{s-1}(D\tilde{\gamma}\circ f)[(\Delta H)\nu] d\mu \\
&\quad + s\int_M \|\nabla_{(k)}A\|^2 \gamma^{s-1}(D\tilde{\gamma}\circ f)[h\nu] d\mu. \qquad (39)
\end{aligned}
$$

For the first integral in (39) we begin with integration by parts:

$$
\begin{aligned}
& s\int_M \|\nabla_{(k)}A\|^2 \gamma^{s-1}(D\tilde{\gamma}\circ f)[(\Delta H)\nu] d\mu \\
&\quad = -s\int_M \nabla^p \left(\|\nabla_{(k)}A\|^2\gamma^{s-1}\right)(D_\nu\tilde{\gamma}\circ f)(\nabla_p H) d\mu \\
&\qquad - s\int_M \left(\|\nabla_{(k)}A\|^2\gamma^{s-1}\right)(\nabla^p D_\nu\tilde{\gamma}\circ f)(\nabla_p H) d\mu \\
&\quad = -2s\int_M \left\langle \nabla^p\nabla_{(k)}A, \nabla_{(k)}A\right\rangle \gamma^{s-1}(D_\nu\tilde{\gamma}\circ f)\nabla_p H d\mu \\
&\qquad - s(s-1)\int_M \|\nabla_{(k)}A\|^2\gamma^{s-2}\left\langle \nabla\gamma, \nabla H\right\rangle (D_\nu\tilde{\gamma}\circ f) d\mu \\
&\qquad - s\int_M \left(\|\nabla_{(k)}A\|^2\gamma^{s-1}\right)\Big((D^2\tilde{\gamma}\circ f)(\nu, e_p)\Big)(\nabla_p H) d\mu \\
&\quad \le 4sc_{\tilde{\gamma}}\int_M \left(\|\nabla_{(k+1)}A\|\gamma^{\frac{s}{2}-1}\right)\left(\|\nabla_{(k)}A\| \; \|\nabla H\|\gamma^{\frac{s}{2}}\right) d\mu \\
&\qquad + s(s-1)c_{\tilde{\gamma}}c_{\gamma 1}\int_M \|\nabla_{(k)}A\|^2\|\nabla H\|\gamma^{s-2} d\mu \\
&\qquad + sc_{\gamma 2}\int_M \|\nabla_{(k)}A\|^2\|\nabla H\|\Big(1 + \|A\|\Big)\gamma^{s-1} d\mu \\
&\quad \le 4sc_{\tilde{\gamma}}\int_M \left(\|\nabla_{(k+1)}A\|\gamma^{\frac{s}{2}-1}\right)\left(\|\nabla_{(k)}A\| \; \|\nabla H\|\gamma^{\frac{s}{2}}\right) d\mu \\
&\qquad + s\Big[(s-1)c_{\tilde{\gamma}}c_{\gamma 1} + c_{\gamma 2}\Big]\int_M \|\nabla_{(k)}A\|^2\|\nabla H\|\gamma^{s-2} d\mu \\
&\qquad + sc_{\gamma 2}\int_M \|\nabla_{(k)}A\|^2\|\nabla H\| \; \|A\|\gamma^{s-1} d\mu.
\end{aligned}
$$

Note that we used Kato's inequality (for some tensor $T$)

$$\| \nabla \|T\| \| \le \|\nabla T\|$$

in the last step.

The first integral obviously splits (using Cauchy's inequality) by

$$\begin{aligned}
&4sc_{\tilde\gamma}\int_M \left(\|\nabla_{(k+1)}A\|\gamma^{\frac{s}{2}-1}\right)\left(\|\nabla_{(k)}A\|\ \|\nabla H\|\gamma^{\frac{s}{2}}\right)d\mu\\
&\qquad\le 2sc_{\tilde\gamma}\int_M\|\nabla_{(k+1)}A\|^2\gamma^{s-2}d\mu+2sc_{\tilde\gamma}\int_M[P_3^{k+2}(A)*\nabla_{(k)}A]\gamma^s d\mu.
\end{aligned}\tag{40}$$

It is considerably less obvious that the other two integrals can be estimated as

$$\begin{aligned}
\int_M\|\nabla_{(k)}A\|^2\|\nabla H\|\gamma^{s-2}d\mu &\le \frac{1}{2}\int_M\left(P_3^{k+2}(A)*\nabla_{(k)}A\right)\gamma^s d\mu\\
&\quad+\frac{\theta}{2}\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu+\frac{1}{2}\Big(c_I+c_\theta\Big)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu
\end{aligned}$$

and

$$\begin{aligned}
\int_M\|\nabla_{(k)}A\|^2\|\nabla H\|\ \|A\|\gamma^{s-1}d\mu &\le \frac{1}{2}\int_M\left(P_3^{k+2}(A)*\nabla_{(k)}A\right)\gamma^s d\mu\\
&\quad+\frac{\theta}{4}\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu+\frac{1}{4}\Big(c_I+c_\theta\Big)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu\\
&\quad+\frac{1}{4}\int_M\left(P_5^k(A)*\nabla_{(k)}A\right)\gamma^s d\mu.
\end{aligned}\tag{41}$$

Here the constant $c_\theta$ is the constant in (37) and $c_I = c(1, c_{\gamma 1}, s, p)$ is the constant in (37) with $\theta = 1$.

We now describe how to obtain estimate (41). This is where the interpolation inequality (37) becomes useful. Two separate applications and Jensen's inequality gives the inequalities

$$\int_M\|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu\le\int_M\|\nabla_{(k+1)}A\|^2\gamma^{s-2}d\mu+c_I\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu,\tag{42}$$

and

$$\int_M\|\nabla_{(k+1)}A\|^2\gamma^{s-2}d\mu\le\theta\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu+c_\theta\int_{[\gamma>0]}\|A\|^2\gamma^{s-2-2k}d\mu.\tag{43}$$

Note that using equation (43) we can finish the estimate (40). Combining (42) and (43) we obtain

$$\int_M \|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu \le \theta\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + (c_I + c_\theta)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu. \tag{44}$$

Using (44) we can now derive the first estimate in (41):

$$\begin{aligned}
&\int_M \|\nabla_{(k)}A\|^2\|\nabla H\|\gamma^{s-2}d\mu \\
&\quad\le \frac{1}{2}\int_M \Big(\|\nabla_{(k)}A\|\ \|\nabla H\|^2\Big)\|\nabla_{(k)}A\|\gamma^s d\mu + \frac{1}{2}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu \\
&\quad\le \frac{1}{2}\int_M \Big(P_3^{k+2}(A) * \nabla_{(k)}A\Big)\gamma^s d\mu \\
&\qquad + \frac{\theta}{2}\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + \frac{1}{2}\Big(c_I + c_\theta\Big)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu.
\end{aligned}$$

For the second integral, we first estimate

$$\begin{aligned}
&\int_M \|\nabla_{(k)}A\|^2\|A\|^2\gamma^{s-2}d\mu \\
&\quad\le \frac{1}{2}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu + \frac{1}{2}\int_M \|\nabla_{(k)}A\|^2\|A\|^4\gamma^s d\mu \\
&\quad\le \frac{\theta}{2}\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + \frac{1}{2}\int_M \Big(P_5^k(A) * \nabla_{(k)}A\Big)\gamma^s d\mu \\
&\qquad + \frac{1}{2}\Big(c_I + c_\theta\Big)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu,
\end{aligned} \tag{45}$$

where we used (44) again. Using (45) above we can derive the second estimate in (41):

$$\begin{aligned}
&\int_M \|\nabla_{(k)}A\|^2\|\nabla H\|\ \|A\|\gamma^{s-1}d\mu \\
&\quad\le \frac{1}{2}\int_M \Big(\|\nabla_{(k)}A\|\ \|\nabla H\|^2\Big)\|\nabla_{(k)}A\|\gamma^s d\mu + \frac{1}{2}\int_M \|\nabla_{(k)}A\|^2\|A\|^2\gamma^{s-2}d\mu \\
&\quad\le \frac{1}{2}\int_M \Big(P_3^{k+2}(A) * \nabla_{(k)}A\Big)\gamma^s d\mu + \frac{\theta}{4}\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\qquad + \frac{1}{4}\Big(c_I + c_\theta\Big)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu + \frac{1}{4}\int_M \Big(P_5^k(A) * \nabla_{(k)}A\Big)\gamma^s d\mu.
\end{aligned}$$

This proves (41), and combining (41) with (40) estimates the first integral in (39).

For the second integral in (39), we note

$$s\int_M \|\nabla_{(k)}A\|^2\gamma^{s-1}(D\tilde{\gamma}\circ f)[h\nu]d\mu \le hsc_{\tilde{\gamma}}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-1}d\mu$$
$$\le hsc_{\tilde{\gamma}}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu.$$

Applying equation (44), we estimate this term as

$$hsc_{\tilde{\gamma}}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu$$
$$\le \theta hsc_{\tilde{\gamma}}\int_M \|\nabla_{(k+2)}A\|^2\gamma^{s}d\mu + hsc_{\tilde{\gamma}}(c_I+c_\theta)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu.$$

Summarising, we have shown

$$\int_M \|\nabla_{(k)}A\|^2\partial_t\gamma^s d\mu \le (\theta_1+\theta_2 h)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu$$
$$+ (c+ch)\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu$$
$$+ c\int_M [(P_3^{k+2}(A)+P_5^k(A))*\nabla_{(k)}A]\gamma^s d\mu, \tag{46}$$

for any fixed $\theta_1,\theta_2>0$, where $c=c(c_{\gamma 1},c_{\gamma 2},s,k,\theta_1,\theta_2)$. The leading order term on the right is absorbed into the same term on the left in equation (38). Note that we need to use (CB) to ensure that $\theta_2\not\to 0$, since then $c\to\infty$. We also note that this choice of $\theta_2$ introduces a dependence on $\|h\|_{\infty,[0,T)}$ into the constant $c$. This gives half the proposition.

Recalling (38), we will therefore be finished if we can deal with the integrals

$$2\int_M \Big\langle(\nabla\gamma^s)(\nabla_{(k)}A),\Delta\nabla_{(k+1)}A\Big\rangle d\mu - 2\int_M \Big\langle(\nabla\gamma^s)(\nabla_{(k+1)}A),\nabla_{(k+2)}A\Big\rangle d\mu.$$

Given our earlier work, these are not so bad. The second integral is estimated as

$$-2\int_M \Big\langle(\nabla\gamma^s)(\nabla_{(k+1)}A),\nabla_{(k+2)}A\Big\rangle d\mu$$
$$\le \theta_3\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + \frac{[sc_{\gamma 1}]^2}{\theta_3}\int_M \|\nabla_{(k+1)}A\|^2\gamma^{s-2}d\mu$$

(47)

$$\leq \left(\theta_3 + \frac{\theta_4[sc_{\gamma 1}]^2}{\theta_3}\right)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + \frac{[sc_{\gamma 1}]^2}{\theta_3}c_{\theta_4}\int_{[\gamma>0]}\|A\|^2\gamma^{s-4-2k}d\mu,$$

where we used (43) again. We need to integrate by parts on the first integral:

$$\begin{aligned}
&\int_M \left\langle (\nabla\gamma^s)(\nabla_{(k)}A), \Delta\nabla_{(k+1)}A\right\rangle d\mu \\
&\quad = -\int_M \left\langle (\nabla_{(2)}\gamma^s)(\nabla_{(k)}A), \nabla_{(k+2)}A\right\rangle d\mu \\
&\qquad - \int_M \left\langle (\nabla\gamma^s)(\nabla_{(k+1)}A), \nabla_{(k+2)}A\right\rangle d\mu \\
&\quad \leq s(s-1)c_{\gamma 1}^2\int_M \|\nabla_{(k)}A\|\;\|\nabla_{(k+2)}A\|\gamma^{s-2}d\mu \\
&\qquad + sc_{\gamma 2}\int_M \|\nabla_{(k)}A\|\;\|\nabla_{(k+2)}A\|\gamma^{s-1}d\mu \\
&\qquad + sc_{\gamma 2}\int_M \|\nabla_{(k)}A\|\;\|\nabla_{(k+2)}A\|\;\|A\|\gamma^{s-1}d\mu \\
&\qquad - \int_M \left\langle (\nabla\gamma^s)(\nabla_{(k+1)}A), \nabla_{(k+2)}A\right\rangle d\mu.
\end{aligned}$$

The last integral is estimated exactly as in (47). We have also seen the other three integrals before. To make this explicit, we split them with

$$\begin{aligned}
&\int_M \|\nabla_{(k)}A\|\;\|\nabla_{(k+2)}A\|\gamma^{s-2}d\mu + \int_M \|\nabla_{(k)}A\|\;\|\nabla_{(k+2)}A\|\gamma^{s-1}d\mu \\
&\qquad + \int_M \|\nabla_{(k)}A\|\;\|\nabla_{(k+2)}A\|\;\|A\|\gamma^{s-1}d\mu \\
&\quad \leq \theta\int_M \|\nabla_{(k+2)}A\|^2\gamma^{s-2}d\mu + \frac{1}{2\theta}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-4}d\mu \\
&\qquad + \frac{1}{2\theta}\int_M \|\nabla_{(k)}A\|^2\|A\|^2\gamma^{s-2}d\mu.
\end{aligned}$$

The first integral is absorbed, and the other two are dealt with exactly as in the proof of inequality (41). This finishes the proof. $\square$

To prove Corollary 3.7 we also need a version of the above estimate where we do not assume (CB). For this purpose, we state the following version of Proposition 3.17. The proof differs only in that the integrals with $h$ are not estimated.

Proposition 3.18. *Let $f : M \times [0,T) \to \mathbb{R}^3$ be a* (CSD) *flow and $\gamma$ a cutoff function as in* (25)*. Then for a fixed $\theta > 0$ and $s \geq 2k+4$,*

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\qquad \leq ch\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu + ch\int_M \|\nabla_{(k)}A\|^2\gamma^{s-1}d\mu \\
&\qquad\qquad + c\int_M \|A\|^2\gamma^{s-4-2k}d\mu + c\int_M \left([P_3^{k+2}(A)+P_5^k(A)]*\nabla_{(k)}A\right)\gamma^s d\mu,
\end{aligned} \tag{48}$$

*where $c = c(c_{\gamma 1}, c_{\gamma 2}, s, k)$.*

Remark. As our main result is a lower bound on the maximal time of existence for a (CSD) flow, one may be interested in a more explicit expression for the constants involved in Proposition 3.17. As we will see, these constants play an important role in determining the numerical value of the lower bound. Also, the reader unfamiliar with so many nested inequalities may be suspicious that our claim of absorbing the high derivatives on the left is in fact valid; indeed, several of the constants depend on each other and a choice of one small will make another larger. Therefore, we will now present explicitly the constants from Proposition 3.17. As the computation is long and tedious, yet relatively simple, we present only two steps. Carefully summarising and factorising, we claim

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + 2\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\quad \leq \Bigg[2sc_{\tilde{\gamma}}\theta_1 + \frac{s[(s-1)c_{\tilde{\gamma}}c_{\gamma 1}+c_{\gamma 2}]}{2}\theta_2 + \frac{sc_{\gamma 2}}{4}\theta_3 + 8\theta_4 + \frac{8(sc_{\gamma 1})^2}{\theta_4}\theta_5 \\
&\qquad + 2s[(s-1)c_{\gamma 1}^2 + c_{\gamma 2}]\left(\theta_6 + \frac{1}{4\theta_6}\theta_7\right) + 2sc_{\gamma 2}\left(\theta_8 + \frac{1}{8\theta_8}\theta_9\right)\Bigg]\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\quad + \Bigg[2sc_{\tilde{\gamma}}c_{\theta_1} + \frac{s[(s-1)c_{\tilde{\gamma}}c_{\gamma 1}+c_{\gamma 2}]}{2}(c_I + c_{\theta_2}) + \frac{sc_{\gamma 2}}{4}(c_I + c_{\theta_3}) + \frac{4(sc_{\gamma 1})^2}{\theta_4}c_{\theta_5} \\
&\qquad + \frac{s[(s-1)c_{\gamma 1}^2+c_{\gamma 2}]}{2\theta_6}(c_I + c_{\theta_7}) + \frac{sc_{\gamma 2}}{4\theta_8}(c_I + c_{\theta_9})\Bigg]\int_{[\gamma>0]} \|A\|^2\gamma^{s-4-2k}d\mu
\end{aligned}$$

$$+\left[c_{3.(k+2)} + 2sc_{\tilde{\gamma}} + \frac{s[(s-1)c_{\tilde{\gamma}}c_{\gamma 1} + c_{\gamma 2}] + sc_{\gamma 2}}{2}\right]\int_M \Big(M_3^{k+2}(A) * \nabla_{(k)}A\Big)\gamma^s d\mu$$
$$+\left[\frac{sc_{\gamma 2}}{4}\Big(1 + \frac{1}{\theta_8}\Big)\right]\int_M \Big(M_5^k(A) * \nabla_{(k)}A\Big)\gamma^s d\mu$$
$$+\,h\left[c_{2.(k)h}\int_M \Big(M_2^k(A) * \nabla_{(k)}A\Big)\gamma^s d\mu + sc_{\tilde{\gamma}}\int_M \|\nabla_{(k)}A\|^2\gamma^{s-1}d\mu\right],$$

(49)

for any $\theta_i > 0$.

Some notation here needs explaining. As the motivation for the use of the letter $P$ is that $P_i^j(T)$ is *"a polynomial with terms containing $i$ derivatives of $j$ copies of $T$"*, we have used $M_i^j(T)$ to denote the corresponding *monomial*. However the situation is a little more complicated than that of a standard monic polynomial of real variables. The constants present in each of the $P$ terms vary from term to term in the summation, and of course we do not have any positivity of these terms to take a maximum of all the constants. What we have done is this: each of the $M$ terms may be estimated by

$$M_i^j(T) \le \sum_{k_1+\ldots+k_j=i} \|\nabla_{(k_1)}T\| \cdot \cdots \cdot \|\nabla_{(k_j)}T\|.$$

Then the constants $c_{3.(k+2)}$ and $c_{2.(k)h}$ are the maximum of the absolute value of all the constants in each the the $P$ terms

$$P_3^{k+2}(A) * \nabla_{(k)}A, \text{ and } hP_2^k(A) * \nabla_{(k)}A$$

multiplied by $n^{2k+8}$ and $n^{2k+6}$ respectively (since the dimension of our immersed manifold is $n$, and $A$ is a (0,2)-tensor).

Let $\theta > 0$ be any real number. We wish to choose our constants $\theta_i$ such that the entire coefficient of $\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu$ on the left becomes equal to $\theta$. There are nine terms and so we simply choose each $\theta_i$ iteratively to force each term to equal $\frac{1}{9}\theta$. These choices are:

$$\theta_1 = \frac{1}{18sc_{\tilde\gamma}}\theta, \qquad \theta_2 = \frac{2}{9s[(s-1)c_{\tilde\gamma}c_{\gamma 1} + c_{\gamma 2}]}\theta, \qquad \theta_3 = \frac{4}{9sc_{\gamma 2}}\theta, \qquad \theta_4 = \frac{1}{36}\theta,$$
$$\theta_5 = \frac{1}{2592(sc_{\gamma 1})^2}\theta^2, \qquad \theta_6 = \frac{1}{18s[(s-1)c_{\gamma 1}^2 + c_{\gamma 2}]}\theta,$$
$$\theta_7 = \frac{1}{1296\Big(s[(s-1)c_{\gamma 1}^2 + c_{\gamma 2}]\Big)^2}\theta^2, \qquad \theta_8 = \frac{1}{18sc_{\gamma 2}}\theta, \qquad \theta_9 = \frac{1}{2592(sc_{\gamma 2})^2}\theta^2.$$

Note that these choices also set the coefficients of the other terms, and in particular determines each $c_{\theta_i}$. This means that our above equation (49) becomes

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\quad\le c_i \int_{[\gamma>0]} \|A\|^2 \gamma^{s-4-2k} d\mu \\
&\qquad + c_{ii}\int_M \Big(M_3^{k+2}(A) * \nabla_{(k)}A\Big)\gamma^s d\mu + c_{iii}\int_M \Big(M_5^k(A) * \nabla_{(k)}A\Big)\gamma^s d\mu \\
&\qquad + hc_{iv}\int_M \Big(M_2^k(A) * \nabla_{(k)}A\Big)\gamma^s d\mu + hc_v \int_M \|\nabla_{(k)}A\|^2\gamma^{s-1} d\mu,
\end{aligned}$$

where $c_i, \ldots, c_v$ are the constants from (49) with the choices for the $\theta_i$ indicated above.

## 5. Integral estimates with small concentration of curvature

We will first need a few Sobolev and interpolation inequalities. The argument for $n = 3$ is by necessity different to that for $n = 2$. This is due to the important role played by the Michael-Simon Sobolev inequality.

THEOREM 3.19 (Michael-Simon Sobolev inequality [**47**]). *Let $f : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion. Then for any $u \in C_c^1(M)$ we have*

$$\Big(\int_M u^{n/(n-1)} d\mu\Big)^{(n-1)/n} \le \frac{4^{n+1}}{\omega_n^{1/n}} \int_M \|\nabla u\| + u|H| d\mu,$$

*where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.*

Notice the exponent on the left. Our eventual goal for this section is to prove local $L^\infty$ estimates for all derivatives of curvature. Our main tool to convert $L^p$ bounds to $L^\infty$ bounds is the following theorem, which is an $n$-dimensional analogue of Theorem 5.6 from [**37**]. The proof is contained in Appendix A.

THEOREM 3.20. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a smooth immersed hypersurface. For $u \in C^1_c(M)$, $n < p \le \infty$, $0 \le \beta \le \infty$ and $0 < \alpha \le 1$ where $\frac{1}{\alpha} = \left(\frac{1}{n} - \frac{1}{p}\right)\beta + 1$ we have*

$$\|u\|_\infty \le c\|u\|_\beta^{1-\alpha}(\|\nabla u\|_p + \|Hu\|_p)^\alpha, \tag{50}$$

*where $c = c(n, p, \beta)$.*

The proof follows ideas from [**39**] and [**37**]. Due to the exponent in the Michael-Simon Sobolev inequality (which is itself an isoperimetric obstruction), it is not possible to decrease the lower bound on $p$, even at the expense of other parameters in the inequality.

The basic problem which this lower bound on $p$ causes is that the evolution equations from section 4 will only give a nice relationship between $\|\nabla_{(k)}A\|^2_{2,[\gamma>0]}$ and $\|\nabla_{(k+2)}A\|_{2,[\gamma>0]}$, and to exploit this relationship we must take $p = 4$ in Theorem 3.20 above.

One can see this by the following. Consider the quantity $\|\nabla T\|_p^p$ where $T$ is a tensor on $f$. Then

$$\|\nabla T\|_p^p \le c\int_M T * \nabla_{(2)}T * [\nabla T]^{p-2} d\mu.$$

We must estimate the integral on the right such that we recover *both* a term $\|\nabla_{(2)}T\|_2$ and $\|\nabla T\|_p$. That is,

$$\|\nabla T\|_p^p \le c\int_M T * \nabla_{(2)}T * [\nabla T]^{p-2}d\mu \le c\|T\|_\infty\|\nabla_{(2)}T\|_2\|\nabla T\|_{2p-4}^{p-2}.$$

Only if $p = 2p-4$ may we conclude that

$$\|\nabla T\|_p \le c\|T\|_\infty^{\frac{1}{2}}\|\nabla_{(2)}T\|_2^{\frac{1}{2}}.$$

Therefore if we are to use an estimate such as the above we are forced to consider only $p = 4$, and thus only $n = 2$ and $n = 3$. This is highlighted in the following local refinement to Theorem 3.20.

Proposition 3.21. *Let $n \in \{2,3\}$. Then for any tensor $T$ on $f : M^n \to \mathbb{R}^{n+1}$ and $\gamma$ as in* (25),

$$\|T\|_{\infty,[\gamma=1]}^4 \le c\|T\|_{2,[\gamma>0]}^{4-n}\Big(\|\nabla_{(2)}T\|_{2,[\gamma>0]}^n + \|TA^2\|_{2,[\gamma>0]}^n + \|T\|_{2,[\gamma>0]}^n\Big), \tag{51}$$

*where $c = c(c_{\gamma 1}, n)$. Assume $T = A$, and if $n = 3$ also assume* (AB). *Then there exists an $\epsilon_0 = \epsilon_0(c_{\gamma 1}, c_{\gamma 2}, n)$ such that if*

$$\|A\|_{n,[\gamma>0]}^n \le \epsilon_0$$

*we have*

$$\|A\|_{\infty,[\gamma=1]}^{8n-12} \le c\epsilon_0\Big(\|\nabla_{(2)}A\|_{2,[\gamma>0]}^{2n^2-3n} + \epsilon_0\Big), \tag{52}$$

*with $c = c(c_{\gamma 1}, c_{\gamma 2}, n, \epsilon_0)$ for $n = 2$ and $c = c(c_{\gamma 1}, c_{\gamma 2}, n, \epsilon_0, C_{AB})$ for $n = 3$.*

Proof. We wish to obtain an $L^\infty$ norm estimate for the tensor $T$ in terms of the concentration of $T$ in a small region of $M$, specified by $\gamma$. The proof proceeds in two parts: first we will estimate an arbitrary tensor $S$, and then we will localise the estimate for $S$ by using a $\gamma$ function. Precisely, we specialise the estimate for $S$ to

$S = T\gamma^2$, taking care to factor out the quantity $\|T\|^2_{2,[\gamma>0]}$ to conclude our desired inequality.

Take $p = 4$, $\beta = 2$ in Theorem 3.20 to obtain

$$\|S\|_\infty \le c\|S\|_2^{\frac{4-n}{n+4}}\Big(\|\nabla S\|_4 + \|S\ H\|_4\Big)^{\frac{2n}{n+4}}. \tag{53}$$

We now use integration by parts and the Hölder inequality to derive

$$\begin{aligned}
\|\nabla S\|_4^4 &\le \int_M S * (\nabla_{(2)}S\|\nabla S\|^2 + 2\nabla S * \nabla S * \nabla_{(2)}S)d\mu \\
&\le c\|S\|_\infty\|\nabla S\|_4^2\|\nabla_{(2)}S\|_2, \text{ so}
\end{aligned}$$

$$\|\nabla S\|_4 \le c\|S\|_\infty^{\frac{1}{2}}\|\nabla_{(2)}S\|_2^{\frac{1}{2}}. \tag{54}$$

Combine equation (54) above with (53) and use Jensen's inequality to obtain

$$\|S\|_\infty \le c\|S\|_2^{\frac{4-n}{n+4}}\Big[(\|S\|_\infty^{\frac{1}{2}}\|\nabla_{(2)}S\|_2^{\frac{1}{2}})^{\frac{2n}{n+4}} + \|S\ H\|_4^{\frac{2n}{n+4}}\Big]. \tag{55}$$

Using Hölder's inequality we estimate

$$\|S\ H\|_4^{\frac{2n}{n+4}} \le \Big(\|S^2\|_\infty^{\frac{1}{4}}\|S^{\frac{1}{2}}H\|_4\Big)^{\frac{2n}{n+4}} \le \|S\|_\infty^{\frac{n}{n+4}}\|S^{\frac{1}{2}}H\|_4^{\frac{2n}{n+4}},$$

and combining this with (55) above we conclude

$$\begin{aligned}
\|S\|_\infty^4 &= \Big(\|S\|_\infty^{1-\frac{n}{n+4}}\Big)^{n+4} \\
&\le \Big(c\|S\|_2^{\frac{4-n}{n+4}}\Big(\|\nabla_{(2)}S\|_2^{\frac{n}{n+4}} + \|S^{\frac{1}{2}}H\|_4^{\frac{2n}{n+4}}\Big)\Big)^{n+4} \\
&\le c\|S\|_2^{4-n}\Big(\|\nabla_{(2)}S\|_2^n + \|S\ H^2\|_2^n\Big).
\end{aligned} \tag{56}$$

We now turn our attention to localising the estimate for $S$. As mentioned earlier, for this purpose we set $S = T\gamma^2$. We first evaluate and estimate the second derivative term $\|\nabla_{(2)}S\|_2^2$:

$$\|\nabla_{(2)}S\|_2^2 = \int_M \|\nabla_{(2)}(T\gamma^2)\|^2 d\mu$$

$$
\begin{aligned}
&\le \int_M \|\nabla_{(2)}T\|^2\gamma^4 d\mu + 4\int_M \|\nabla T\|^2 \|\nabla\gamma^2\|^2 d\mu + \int_M \|T\|^2\|\nabla_{(2)}\gamma^2\|^2 d\mu \\
&\le \int_M \|\nabla_{(2)}T\|^2\gamma^4 d\mu + 8\int_M \|\nabla T\|^2\|\nabla\gamma\|^2\gamma^2 d\mu \\
&\qquad + 2\int_M \|T\|^2 \left[\|\nabla_{(2)}\gamma\|\gamma + \|\nabla\gamma\|^2\right]^2 d\mu \\
&\le \int_M \|\nabla_{(2)}T\|^2\gamma^4 d\mu + c\|\nabla\gamma\|_\infty^2 \int_M \|\nabla T\|^2\gamma^2 d\mu \\
&\qquad + c\int_M \|T\|^2\|\nabla_{(2)}\gamma\|^2\gamma^2 d\mu + c\|\nabla\gamma\|_\infty^4 \int_{[\gamma>0]} \|T\|^2 d\mu. \qquad (57)
\end{aligned}
$$

We interpolate the first derivative term:

$$
\begin{aligned}
\int_M \|\nabla T\|^2\gamma^2 d\mu &\le \int_M \|T\|\ \|\nabla_{(2)}T\|\gamma^2 d\mu + c\|\nabla\gamma\|_\infty \int_M \|T\|\ \|\nabla T\|\gamma d\mu \\
&\le c\int_M \left(\|T\|^2 + \|\nabla_{(2)}T\|^2\right)\gamma^2 d\mu + c(1+\|\nabla\gamma\|_\infty^2)\int_{[\gamma>0]} \|T\|^2 d\mu \\
&\qquad + \frac{1}{2}\int_M \|\nabla T\|^2\gamma^2 d\mu,
\end{aligned}
$$

and thus

$$
\int_M \|\nabla T\|^2\gamma^2 d\mu \le c\int_M \|\nabla_{(2)}T\|^2\gamma^2 d\mu + c(1+\|\nabla\gamma\|_\infty^2)\int_{[\gamma>0]} \|T\|^2 d\mu.
$$

Inserting this result into (57), and estimating

$$
\int_M \|A\|^2\|T\|^2\gamma^4 d\mu \le \frac{1}{2}\int_{[\gamma>0]} \|T\|^2 d\mu + \frac{1}{2}\int_M \|T\|^2\|A\|^4\gamma^4 d\mu,
$$

we obtain

$$
\|\nabla_{(2)}S\|_2^2 \le c\int_{[\gamma>0]} \|\nabla_{(2)}T\|^2 + \|T\|^2 d\mu + c\int_M \|T\|^2\|A\|^4\gamma^4 d\mu. \qquad (58)
$$

Combining this with our estimate for $\|S\|_\infty$ earlier, inequality (56), gives

$$
\begin{aligned}
\|S\|_\infty^4 &\le c\|S\|_2^{4-n}\Big(\|\nabla_{(2)}T\|_{2,[\gamma>0]}^n + \|T\|_{2,[\gamma>0]}^n + \|S\ H^2\|_2^n + \|TA^2\gamma^4\|_2^n\Big) \\
&\le c\|T\|_{2,[\gamma>0]}^{4-n}\Big(\|\nabla_{(2)}T\|_{2,[\gamma>0]}^n + \|T\|_{2,[\gamma>0]}^n + \|TA^2\|_{2,[\gamma>0]}^n\Big). \qquad (59)
\end{aligned}
$$

Estimating $\|T\|_{\infty,[\gamma=1]}^4 \le \|S\|_\infty^4$ proves (51).

Now set $T = A$. For $n = 2$, Lemma 3.22 implies

$$\int_M \|A\|^6\gamma^4 d\mu \le c_{S1}\|A\|^2_{2,[\gamma>0]}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \|A\gamma^{\frac{2}{3}}\|_6^6\Big) + c\|A\|^4_{2,[\gamma>0]},$$

and absorbing on the left we obtain

$$(1-\epsilon_0 c_{S1})\int_M \|A\|^6\gamma^4 d\mu \le c\|A\|^2_{2,[\gamma>0]}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \|A\|^2_{2,[\gamma>0]}\Big), \text{ so}$$
$$\int_M \|A\|^6\gamma^4 d\mu \le \frac{c}{1-\epsilon_0 c_{S1}}\|A\|^2_{2,[\gamma>0]}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \|A\|^2_{2,[\gamma>0]}\Big),$$

where $c_{S1}$ is the constant from Lemma 3.22. Inserting this into (59) gives the second statement for $n = 2$. For $n = 3$, Lemma 3.22 gives

$$\int_M \|A\|^6\gamma^4 d\mu \le c_{S1}\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \|A\gamma^{\frac{2}{3}}\|_6^6\Big) + \theta\|\nabla_{(2)}A\|^2_{2,[\gamma>0]}$$
$$+ c\Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).$$

Choosing $\theta < \sqrt{\epsilon_0}$ this becomes

$$\int_M \|A\|^6\gamma^4 d\mu \le 2c_{S1}\sqrt{\epsilon_0}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \|A\gamma^{\frac{2}{3}}\|_6^6\Big)$$
$$+ c\Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big),$$

and again absorbing on the left we obtain

$$(1-2c_{S1}\sqrt{\epsilon_0})\int_M \|A\|^6\gamma^4 d\mu \le c\sqrt{\epsilon_0}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \sqrt{\epsilon_0}\Big)$$
$$\Longrightarrow \int_M \|A\|^6\gamma^4 d\mu \le c\sqrt{\epsilon_0}\Big(\|\nabla_{(2)}A\|^2_{2,[\gamma>0]} + \sqrt{\epsilon_0}\Big).$$

We combine this with (59) and estimate to obtain

$$\|A\|^4_{\infty,[\gamma=1]} \le c(C_{AB})^{\frac{1}{6}}\|A\|_{3,[\gamma>0]}\left(\|\nabla_{(2)}A\|^3_{2,[\gamma>0]} + (C_{AB})^{\frac{1}{2}}\|A\|^3_{3,[\gamma>0]} + \Big(\int_M \|A\|^6\gamma^4 d\mu\Big)^{\frac{3}{2}}\right)$$
$$\le c\epsilon_0^{1/3}\left(\|\nabla_{(2)}A\|^3_{2,[\gamma>0]} + \epsilon_0 + \epsilon_0^{3/4}\Big(\|\nabla_{(2)}A\|^3_{2,[\gamma>0]} + \epsilon_0^{3/4}\Big)\right),$$

and upon cubing both sides we recover the second statement for $n = 3$ and so we are done. □

The following multiplicative Sobolev inequality, which we already used above, is a combination of the Michael-Simon Sobolev inequality and standard integral estimates.

LEMMA 3.22. *Let $\gamma$ be as in (25). Then for an immersed surface $f : M^2 \to \mathbb{R}^3$ we have*

$$\int_M \|A\|^6 \gamma^s d\mu + \int_M \|A\|^2 \|\nabla A\|^2 \gamma^s d\mu \le c \int_{[\gamma>0]} \|A\|^2 d\mu \int_M (\|\nabla_{(2)} A\|^2 + \|A\|^6) \gamma^s d\mu + c(c_{\gamma 1})^4 \Big( \int_{[\gamma>0]} \|A\|^2 d\mu \Big)^2,$$

*and for an immersion $f : M^3 \to \mathbb{R}^4$,*

$$\int_M \|A\|^6 \gamma^s d\mu + \int_M \|A\|^2 \|\nabla A\|^2 \gamma^s d\mu \le \theta \int_M \|\nabla_{(2)} A\|^2 \gamma^s d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]} \int_M \Big( \|\nabla_{(2)} A\|^2 + \|A\|^6 \Big) \gamma^s d\mu + c(c_{\gamma 1})^3 \Big( \|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]} \Big),$$

*where $\theta \in (0, \infty)$ and $c = c(s, \theta)$ is an absolute constant.*

PROOF. The first statement is Lemma 4.2 in [**37**]. For the second, first observe that

$$\begin{aligned}
\int \|\nabla A\|^3 \gamma^s d\mu &\le \int_M \Big( \langle A, \Delta A \rangle * \nabla A + A * \nabla A * \nabla \|\nabla A\| \Big) \gamma^s d\mu \\
&\quad + s \int_M \Big( A * \nabla A * \nabla A * \nabla \gamma \Big) \gamma^{s-1} d\mu \\
&\le 2 \int_M \|A\| \; \|\nabla A\| \; \|\nabla_{(2)} A\| \gamma^s d\mu + s c_{\gamma 1} \int_M \Big( \|\nabla A\|^2 \|A\| \Big) \gamma^{s-1} d\mu \\
&\le \frac{1}{4\theta} \int_M \|\nabla_{(2)} A\|^2 \gamma^s d\mu + \theta \int_M \|A\|^2 \|\nabla A\|^2 \gamma^s d\mu \\
&\quad + \frac{(s c_{\gamma 1})^3 4^2}{3} \int_M \|A\|^3 \gamma^{2s-3} d\mu + \frac{1}{6} \int_M \|\nabla A\|^3 \gamma^s d\mu \\
&\le \frac{1}{4\theta} \int_M \|\nabla_{(2)} A\|^2 \gamma^s d\mu + \frac{\theta^3}{3} \int_M \|A\|^6 \gamma^s d\mu + \frac{(s c_{\gamma 1})^3 4^2}{3} \int_M \|A\|^3 \gamma^{2s-3} d\mu \\
&\quad + \frac{5}{6} \int_M \|\nabla A\|^3 \gamma^s d\mu,
\end{aligned}$$

so

$$\int\|\nabla A\|^3 d\mu \le \frac{3}{2\theta}\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + 2\theta^3\int_M \|A\|^6\gamma^s d\mu + 2(sc_{\gamma 1})^3 4^2\int_{[\gamma>0]}\|A\|^3 d\mu,$$

for any $\theta \in (0,\infty)$.

Now we use the Michael-Simon Sobolev inequality with $u = \|A\|^4\gamma^{2s/3}$ to estimate

$$\begin{aligned}
\Big(\int_M \|A\|^6\gamma^s d\mu\Big)^{\frac{2}{3}} &\le c\int_M \Big\|\nabla\Big(\|A\|^4\gamma^s\Big)\Big\| d\mu + c\int_M H\|A\|^4\gamma^s d\mu \\
&\le c\int_M \|A\|^3\|\nabla A\|\gamma^s d\mu + c\int_M \|A\|^4\|\nabla\gamma\|\gamma^{s-1}d\mu + c\int_M \|A\|^5\gamma^s d\mu \\
&\le c\int_M \|A\|^3\|\nabla A\|\gamma^s d\mu + c\int_M \|A\|^5\gamma^s d\mu + c(c_{\gamma 1})^2\|A\|^3_{3,[\gamma>0]} \\
&\le c\int_M \|\nabla A\|^2\|A\|\gamma^s d\mu + c\int_M \|A\|^5\gamma^s d\mu + c(c_{\gamma 1})^2\|A\|^3_{3,[\gamma>0]} \\
&\le c\int_M \|\nabla A\|^2\|A\|\gamma^s d\mu + \Big(\int_M \|A\|^6\gamma^s d\mu\Big)^{\frac{2}{3}}\Big(\int_{[\gamma>0]}\|A\|^3\Big)^{\frac{1}{3}} \\
&\quad + c(c_{\gamma 1})^2\|A\|^3_{3,[\gamma>0]},
\end{aligned}$$

so

$$\begin{aligned}
\int_M \|A\|^6\gamma^s d\mu &\le c\Big(\int_M \|\nabla A\|^2\|A\|\gamma^s d\mu\Big)^{\frac{3}{2}} + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|A\|^6\gamma^s d\mu \\
&\quad + c(c_{\gamma 1})^3\|A\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\le c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|\nabla A\|^3\gamma^s d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|A\|^6\gamma^s d\mu \\
&\quad + c(c_{\gamma 1})^3\|A\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\le c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^6\Big)\gamma^s d\mu + c(c_{\gamma 1})^3\|A\|^{\frac{9}{2}}_{3,[\gamma>0]}.
\end{aligned}$$

This estimates the first term. For the second, we can employ a more direct technique using our estimates above,

$$\int_M \|A\|^2\|\nabla A\|^2\gamma^s d\mu \le c\int_M \|A\|^6\gamma^s d\mu + c\int_M \|\nabla A\|^3\gamma^s d\mu$$

$$\leq \theta \int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + c_\theta \|A\|^{\frac{3}{2}}_{3,[\gamma>0]} \int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^6\Big)\gamma^s d\mu$$
$$+ c_\theta (c_{\gamma 1})^3 \Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).$$

This estimates the second term, and combining the two estimates above finishes the proof. □

The proposition used for the constructive part of the argument used to prove the Lifespan Theorem can be proved now.

Proposition 3.23. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a* (CSD) *flow with $h$ satisfying* (CB) *and $\gamma$ a cutoff function as in* (25)*. Additionally, if $n = 3$ assume* (AB)*. Then there is an $\epsilon_0 = \epsilon_0\Big(c_{\gamma 1}, c_{\gamma 2}, \|h\|_{\infty,[0,T^*]}\Big)$ such that if*

$$\epsilon = \sup_{[0,T^*]} \int_{[\gamma>0]} \|A\|^n d\mu \leq \epsilon_0 \tag{60}$$

*then for any $t \in [0,T^*]$ we have*

$$\begin{aligned} &\int_{[\gamma=1]} \|A\|^2 d\mu + \int_0^t \int_{[\gamma=1]} (\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6) d\mu d\tau \\ &\qquad \leq \int_{[\gamma>0]} \|A\|^2 d\mu \Big|_{t=0} + c\epsilon^{\frac{2}{n}} t, \end{aligned} \tag{61}$$

*where $c = c\Big(c_{\gamma 1}, c_{\gamma 2}, \|h\|_{\infty,[0,T^*]}, C_{AB}\Big)$.*

Proof. The idea of the proof is to integrate Proposition 3.17, and then use the multiplicative Sobolev inequality Lemma 3.22. This will introduce a multiplicative factor of $\|A\|_{n,[\gamma>0]}$ in front of several integrals, which we can then absorb on the left.

Setting $k = 0$ and $s = 4$ in Proposition 3.17 we have

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu$$
$$\leq (c + ch)\int_{[\gamma>0]} \|A\|^2 d\mu + ch\int_M ([A*A]*A)\,\gamma^4 d\mu$$

$$+ c\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big)\,\gamma^4 d\mu.$$

First we estimate the $P$-style terms:

$$\begin{aligned}
&\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big)\,\gamma^4 d\mu \\
&\quad\le c\int_M \Big(\Big[\|A\|^2\cdot\|\nabla_{(2)}A\| + \|\nabla A\|^2\|A\| + \|A\|^5\Big]\|A\|\Big)\gamma^4 d\mu \\
&\quad\le c\int_M \Big[\|A\|^3\|\nabla_{(2)}A\| + \|\nabla A\|^2\|A\|^2 + \|A\|^6\Big]\gamma^4 d\mu \\
&\quad\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\int_M (\|A\|^6 + \|\nabla A\|^2\|A\|^2)\gamma^4 d\mu.
\end{aligned}$$

We use Lemma 3.22 to estimate the second integral and obtain for $n = 2$

$$\begin{aligned}
&\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big)\,\gamma^4 d\mu \\
&\quad\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\int_{[\gamma>0]}\|A\|^2 d\mu \int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu \\
&\qquad + c\Big(\int_{[\gamma>0]}\|A\|^2 d\mu\Big)^2,
\end{aligned} \tag{62}$$

and for $n = 3$

$$\begin{aligned}
&\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big)\,\gamma^4 d\mu \\
&\quad\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu \\
&\qquad + c\Big(c_{\gamma 1}\Big)^3\Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).
\end{aligned} \tag{63}$$

We add the integrals $\int_M \|A\|^6\gamma^4 d\mu$ and $\int_M \|\nabla A\|^2\|A\|^2\gamma^4 d\mu$ to the estimate of Proposition 3.17 (with $k = 0$, $s = 4$) and obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu \\
&\quad\le (c + ch)\int_{[\gamma>0]}\|A\|^2 d\mu + ch\int_M ([A * A] * A)\,\gamma^4 d\mu \\
&\qquad + c\int_M \Big(\|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu + c\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big)\,\gamma^4 d\mu \\
&\quad\le c(1 + h^2)\int_{[\gamma>0]}\|A\|^2 d\mu + c\int_M \Big(\|A\|^3\|\nabla_{(2)}A\| + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu.
\end{aligned}$$

For $n=2$, we use the estimate (62) above and obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu\\
&\qquad \le c(1+h^2)\int_{[\gamma>0]}\|A\|^2 d\mu + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu\\
&\qquad\qquad + c\int_{[\gamma>0]}\|A\|^2 d\mu \int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu + c\Big(\int_{[\gamma>0]}\|A\|^2 d\mu\Big)^2.
\end{aligned}$$

For $n=3$, we use instead (63) to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu\\
&\qquad \le c(1+h^2)\int_{[\gamma>0]}\|A\|^2 d\mu + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu\\
&\qquad\qquad + c\Big(c_{\gamma 1}\Big)^3\Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).\\
&\qquad \le c(1+h^2)C_{AB}^{\frac{1}{3}}\|A\|^2_{3,[\gamma>0]} + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu\\
&\qquad\qquad + c\Big(c_{\gamma 1}\Big)^3\Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).
\end{aligned}$$

Absorbing, we obtain for $n=2$

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta-\epsilon_0)\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu\\
&\qquad \le c(1+\epsilon_0 + \|h\|^2_{\infty,[0,T^*]})\epsilon\\
&\qquad \le c\epsilon,
\end{aligned}$$

and for $n=3$

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta-\sqrt{\epsilon_0})\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu\\
&\qquad \le c\Big(1 + C_{AB}^{\frac{1}{3}} + C_{AB}^{\frac{1}{3}}\|h\|^2_{\infty,[0,T^*]} + \epsilon_0^{\frac{23}{6}} + \epsilon_0^{\frac{4}{3}}\Big)\epsilon^{\frac{2}{3}}\\
&\qquad \le c\epsilon^{\frac{2}{3}}.
\end{aligned}$$

For $\theta$, $\epsilon_0$ small enough we have

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + \int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu \le c\epsilon^{\frac{2}{n}},$$

where for $n=3$, $c$ depends additionally on $C_{AB}$. Integrating, we have

$$\begin{aligned}\int_{[\gamma=1]} \|A\|^2\gamma^4 d\mu + \int_0^t \int_{[\gamma=1]} (\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6) d\mu d\tau \\ \le \int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c\epsilon^{\frac{2}{n}},\end{aligned}$$

where we used the fact $[\gamma = 1] \subset [\gamma > 0]$ and $0 \le \gamma \le 1$, with

$$c = c(\epsilon_0, \|h\|_{\infty,[0,t^*]}, c_{\gamma 1}, c_{\gamma 2}, C_{AB}).$$

□

Remark. The assumption (AB) is required for the three dimensional case due to the fact that $L^2$ norms naturally arise when computing the evolution equations of various integral quantities, see the proof of Corollary 3.16 and Proposition 3.17. Forcing $L^3$ norms in these inequalities for the purpose of the above proof introduces changes in the exponents of the $P$-terms, and to deal with this one would need to prove an altered form of Lemma 3.22. This altered form will still require (AB) to handle the different exponents in the integrals. So it seems to us that for the three dimensional case it is not possible to avoid assuming (AB), which is required to obtain results for non-simple constraint functions regardless (see Theorem 3.3 and Appendix C).

It remains only to prove the estimate used in the contradiction branch of the argument used to prove the Lifespan Theorem. For this, we need an interpolation inequality, and a preliminary proposition. We will only state the required interpolation inequality; the proof can be found in Appendix A or [**37**].

Proposition 3.24. *Let $0 \le i_1, \dots, i_r \le k$, $i_1 + \dots + i_r = 2k$ and $s \ge 2k$. Then for any tensor $T$ defined on an immersed hypersurface $f$ we have*

$$\int_M \nabla_{(i_1)} T * \cdots * \nabla_{(i_r)} T \gamma^s d\mu \le c \|T\|^{r-2}_{\infty,[\gamma>0]} \left( \int_M \|\nabla_{(k)} T\|^2 \gamma^s d\mu + \|T\|^2_{2,[\gamma>0]} \right).$$

We now use this to derive the required proposition.

Proposition 3.25. *Suppose $f : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is a* (CSD) *flow and $\gamma : M \to \mathbb{R}$ a cutoff function as in* (25)*. Then, for $s \ge 2k+4$ the following estimate holds:*

$$\begin{aligned} &\frac{d}{dt}\int_M \|\nabla_{(k)} A\|^2 \gamma^s d\mu + \int_M \|\nabla_{(k+2)} A\|^2 \gamma^s d\mu \\ &\qquad \le c\|A\|^4_{\infty,[\gamma>0]} \int_M \|\nabla_{(k)} A\|^2 \gamma^s d\mu + c\|A\|^2_{2,[\gamma>0]}(1 + \|A\|^4_{\infty,[\gamma>0]}) \\ &\qquad\quad + ch\left( h^{\frac{1}{3}} \int_M \|\nabla_{(k)} A\|^2 \gamma^s d\mu + (1 + h^{\frac{1}{3}})\|A\|^2_{2,[\gamma>0]} \right). \end{aligned} \tag{64}$$

Proof. The proposition will be proved using Proposition 3.17 if we can establish the following inequality:

$$\begin{aligned} &(c + ch)\int_M \|A\|^2 \gamma^{s-4-2k} d\mu + ch\int_M \left( \nabla_{(k)}[A * A] * \nabla_{(k)} A \right) \gamma^s d\mu \\ &\qquad + c\int_M \left( [P_3^{k+2}(A) + P_5^k(A)] * \nabla_{(k)} A \right) \gamma^s d\mu \\ &\le \frac{1}{2}\int_M \|\nabla_{(k+2)} A\|^2 \gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]} \int_M \|\nabla_{(k)} A\|^2 \gamma^s d\mu \\ &\qquad + c\|A\|^2_{2,[\gamma>0]}(1 + \|A\|^4_{\infty,[\gamma>0]}) \\ &\qquad + ch\left( h^{\frac{1}{3}} \int_M \|\nabla_{(k)} A\|^2 \gamma^s d\mu + (1 + h^{\frac{1}{3}})\|A\|^2_{2,[\gamma>0]} \right). \end{aligned} \tag{65}$$

We estimate each of the four terms on the left hand side of (65) in turn. First note that

$$(1+h)\int_M \|A\|^2\gamma^{s-4-2k}d\mu \le \|A\|^2_{2,[\gamma>0]} + h\|A\|^2_{2,[\gamma>0]}. \tag{66}$$

For the second term, let $r=3$ and $i_1+i_2=k$, $i_3=k$ in Corollary 3.24 to obtain

$$\begin{aligned}
h\int_M \Big(\nabla_{(k)}[A*A]*\nabla_{(k)}A\Big)\gamma^s d\mu &\le ch\sum_{\substack{i_1+i_2=k\\0\le i_j\le k}}\int_M \nabla_{(i_1)}A*\nabla_{(i_2)}A*\nabla_{(i_3)}A\gamma^s d\mu\\
&\le ch\|A\|_\infty\left(\int_M\|\nabla_{(k)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right)\\
&\le c\|A\|^4_\infty\left(\int_M\|\nabla_{(k)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right)\\
&\quad + h^{\frac{4}{3}}\left(\int_M\|\nabla_{(k)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right), \qquad (67)
\end{aligned}$$

using Young's inequality. This estimates the second term in (65).

The fourth term is also straightforward. Let $r=6$ in Corollary 3.24 to obtain

$$\int_M P_5^k(A)*\nabla_{(k)}A\gamma^2 d\mu \le c\|A\|^4_{\infty,[\gamma>0]}\left(\int_M\|\nabla_{(k)}A\|\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right). \tag{68}$$

The third term takes a little more effort to estimate. First note

$$\begin{aligned}
\int_M P_3^{k+2}(A)*\nabla_{(k)}A\gamma^s d\mu &= \int_M\Big(\nabla_{(k+2)}A*A*A\Big)*\nabla_{(k)}A\gamma^s d\mu\\
&\quad + \sum_{\substack{i_1+i_2+i_3=k+2\\0\le i_j\le k+1}}\int_M \nabla_{(i_1)}A*\nabla_{(i_2)}A*\nabla_{(i_3)}A*\nabla_{(i_4)}A\gamma^s d\mu.
\end{aligned}$$

Since $i_1+i_2+i_3=k+2$ and $i_4=k$, and in particular each $i_j\le k+1$, we can use Corollary 3.24 with $k+1$ derivatives and $r=4$ to estimate

$$\begin{aligned}
\int_M P_3^{k+2}(A)*\nabla_{(k)}A\gamma^s d\mu &\le \theta_1\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\int_M\|A\|^4\|\nabla_{(k)}A\|^2\gamma^s d\mu\\
&\quad + c\|A\|^2_{\infty,[\gamma>0]}\left(\int_M\|\nabla_{(k+1)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right)\\
&\le \theta_1\int_M\|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]}\int_M\|\nabla_{(k)}A\|^2\gamma^s d\mu\\
&\quad + c\|A\|^2_{\infty,[\gamma>0]}\left(\int_M\|\nabla_{(k+1)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right).
\end{aligned}$$

We could now proceed by interpolating the $\int_M \|\nabla_{(k+1)}A\|^2\gamma^s d\mu$ term using integration by parts. A quicker (although equivalent) method however is to simply invoke Lemma 5.1 from [**37**] (or Lemma A.3 in Appendix A) with $p = q = 2r$, $\alpha = 0$, $\beta = 1$, $t = 0$, and obtain

$$\begin{aligned}
&\|A\|^2_{\infty,[\gamma>0]}\int_M \|\nabla_{(k+1)}A\|\gamma^s d\mu \\
&\qquad \le c\|A\|^2_{\infty,[\gamma>0]}\left(\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu\right)^{\frac12}\left(\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu\right)^{\frac12} \\
&\qquad\qquad + c\|A\|^2_{\infty,[\gamma>0]}\left(\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu\right)^{\frac12}\left(\int_M \|\nabla_{(k+1)}A\|^2\gamma^{s-2} d\mu\right)^{\frac12} \\
&\qquad \le \theta_2\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu \\
&\qquad\qquad + \int_M \|\nabla_{(k+1)}A\|^2\gamma^{s-2} d\mu.
\end{aligned}$$

Since $s \ge 2k+4$, and $s-2 \ge 2(k+1)$ we can use Lemma 5.2 from [**37**] (or Lemma A.4 from Appendix A) to obtain

$$\int_M \|\nabla_{(k+1)}A\|^2\gamma^{s-2} d\mu \le \theta_3\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^2_{2,[\gamma>0]}.$$

Therefore we can finally estimate the third term by

$$\begin{aligned}
&\int_M P_3^{k+2}(A) * \nabla_{(k)}A\gamma^s d\mu \\
&\quad \le \theta_1\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu \\
&\qquad\quad + c\|A\|^2_{\infty,[\gamma>0]}\|A\|^2_{2,[\gamma>0]} + c\|A\|^2_{\infty,[\gamma>0]}\int_M \|\nabla_{(k+1)}A\|^2\gamma^s d\mu \\
&\quad \le \theta_1\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu \\
&\qquad\quad + c\|A\|^2_{2,[\gamma>0]}(1+\|A\|^4_{\infty,[\gamma>0]}) + c\|A\|^2_{\infty,[\gamma>0]}\int_M \|\nabla_{(k+1)}A\|^2\gamma^s d\mu \\
&\quad \le (\theta_1+\theta_2+\theta_3)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu \\
&\qquad\quad + c\|A\|^2_{2,[\gamma>0]}(1+\|A\|^4_{\infty,[\gamma>0]}).
\end{aligned} \tag{69}$$

Combining the inequalities (66), (67), (68) and (69) we have

$$
\begin{aligned}
(c+ch)\int_M \|A\|^2\gamma^{s-4-2k}d\mu &+ ch\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu \\
&+ c\int_M \left([P_3^{k+2}(A)+P_5^k(A)]*\nabla_{(k)}A\right)\gamma^s d\mu \\
\le (\theta_1+\theta_2+\theta_3)&\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu + c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu \\
&+ c\|A\|^2_{2,[\gamma>0]}(1+\|A\|^4_{\infty,[\gamma>0]}) \\
&+ ch\left(h^{\frac{1}{3}}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + (h^{\frac{1}{3}}+1)\|A\|^2_{2,[\gamma>0]}\right).
\end{aligned}
$$

Choosing $\theta_1+\theta_2+\theta_3 = \frac{1}{2}$ completes the proof of (65), and so the proposition is proved. □

We now finish this section with a proof of the higher derivatives of curvature estimate, which will allow us to both bound the constraint function in balls other than the 'special ball' (see Corollary 8) and perform the contradiction part of our overall argument used to prove the Lifespan Theorem.

PROPOSITION 3.26. *Let $n\in\{2,3\}$. Suppose $f: M^n\times[0,T^*]\to\mathbb{R}^{n+1}$ is a* (CSD) *flow with $h$ satisfying* (CB) *and $\gamma$ as in* (25)*. If $n=3$ assume in addition* (AB)*. Then there is an $\epsilon_0$ depending on the constants in* (25) *and $\|h\|_{\infty,[0,T^*]}$ such that if*

$$\sup_{[0,T^*]}\int_{[\gamma>0]}\|A\|^n d\mu \le \epsilon_0, \tag{70}$$

*we can conclude*

$$\|\nabla_{(k)}A\|^2_{\infty,[\gamma=1]} \le c\Big(k,T^*,c_{\gamma1},c_{\gamma2},\|h\|_{\infty,[0,T^*]},\alpha_0(k+2),C_{AB}\Big), \tag{71}$$

*where* $\alpha_0(k) = \displaystyle\sum_{j=0}^{k}\|\nabla_{(j)}A\|_{2,[\gamma>0]}\Big|_{t=0}$.

PROOF. The idea is to use our previous estimates and then integrate. The $\epsilon_0$ which we will use is exactly the same as that in Proposition 3.23.

We fix $\gamma$ and consider cutoff functions $\gamma_{\sigma,\tau}$ which will allow us to combine our previous estimates. Define for $0 \le \sigma < \tau \le 1$ functions $\gamma_{\sigma,\tau} = \psi_{\sigma,\tau} \circ \gamma$ satisfying $\gamma_{\sigma,\tau} = 0$ for $\gamma \le \sigma$ and $\gamma_{\sigma,\tau} = 1$ for $\gamma \ge \tau$. The function $\psi_{\sigma,\tau}$ is chosen such that $\gamma_{\sigma,\tau}$ satisfies equation (25), although with different constants. Acceptable choices are

$$c_{\gamma_{\sigma,\tau}1} = \|\nabla\psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 1}, \text{ and } c_{\gamma_{\sigma,\tau}2} = \max\{\|\nabla_{(2)}\psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 1}^2, \|\nabla\psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 2}\}.$$

Using the cutoff function $\gamma_{0,\frac{1}{2}}$ instead of $\gamma$ in Proposition 3.23 gives

$$\int_0^{T^*}\int_{[\gamma_{0,\frac{1}{2}}=1]} \|\nabla_{(2)}A\|^2 + \|A\|^6 d\mu d\tau \le c\epsilon_0^{\frac{2}{n}} T^* + \|A\|^2_{2,[\gamma>0]}\Big|_{t=0}, \quad \text{so}$$

$$\int_0^{T^*}\int_{[\gamma\ge\frac{1}{2}]} \|\nabla_{(2)}A\|^2 + \|A\|^6 d\mu d\tau \le c\epsilon_0(1+T^*) \tag{72}$$

for $n = 2$ and

$$\int_0^{T^*}\int_{[\gamma\ge\frac{1}{2}]} \|\nabla_{(2)}A\|^2 + \|A\|^6 d\mu d\tau \le c\epsilon_0^{\frac{2}{3}}(C_{AB}^{\frac{1}{3}} + T^*)$$

for $n = 3$.

Next, using $\gamma_{\frac{1}{2},\frac{3}{4}}$ in (51) and inequality (72) above we obtain for $n = 2$

$$\begin{aligned}\int_0^T \|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]} d\tau &\le c\epsilon_0(c\epsilon_0(1+T^*) + \epsilon_0 T^*) \\ &\le c\epsilon_0. \end{aligned} \tag{73}$$

For $n = 3$ we have

$$\begin{aligned}\int_0^T \|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]} d\tau &\le c(C_{AB})^{\frac{1}{3}}\epsilon_0^{\frac{2}{3}}\Big(2[c\epsilon_0^{\frac{2}{3}}(C_{AB}^{\frac{1}{3}} + T^*)]^{\frac{3}{2}} + c\epsilon_0(C_{AB})^{\frac{1}{2}}(T^*)^{\frac{3}{2}}\Big) \\ &\le c\epsilon_0, \end{aligned} \tag{74}$$

where $c = c\Big(\|h\|_\infty, c_{\gamma 1}, c_{\gamma 2}, T^*, n, \epsilon_0\Big)$ for $n = 2$ and $c = c\Big(\|h\|_\infty, c_{\gamma 1}, c_{\gamma 2}, T^*, n, \epsilon_0, C_{AB}\Big)$ for $n = 3$. We use the convention that for the remainder of this proof all constants $c$ will depend on these quantities for $n = 2$ and $n = 3$ respectively.

Note that by (CB) we trivially have $\|h\|_{\infty,[0,T^*]} \le c$.

We now use (64) with $\gamma_{\frac{3}{4},\frac{7}{8}}$. Factorising, we have

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu &\le c\|A\|^4_{\infty,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu \\ &\quad + c\|A\|^2_{2,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\Big(1+\|A\|^4_{\infty,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\Big) \\ &\quad + ch\Big(h^{\frac{1}{3}}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu + (1+h^{\frac{1}{3}})\|A\|^2_{2,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\Big) \\ &\le c\Big(\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h^{\frac{4}{3}}\Big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu \\ &\quad + c\|A\|^2_{2,[\gamma\ge\frac{3}{4}]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h+h^{\frac{4}{3}}\Big).\end{aligned}$$

We wish to solve this differential inequality using Gronwall's inequality, Lemma A.1. We will use the integral version (considering the integrals in the above expressions as functions of time), since we can bound the integrals of relevant quantities, as we have shown above.

Integrating,

$$\begin{aligned}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu - \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\Big|_{t=0} \\ \le c\int_0^t \Big[\Big(\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h^{\frac{4}{3}}\Big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\Big]d\tau \\ + c\int_0^t \Big[\|A\|^2_{2,[\gamma\ge\frac{3}{4}]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h+h^{\frac{4}{3}}\Big)\Big]d\tau.\end{aligned} \tag{75}$$

Now from our earlier calculation (73) and assumption (CB) we have

$$\int_0^t \Big(\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h^{\frac{4}{3}}\Big)d\tau \le c,$$

and, using our assumption (70)

$$c\int_0^t \Big[\|A\|^2_{2,[\gamma\ge\frac{3}{4}]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h+h^{\frac{4}{3}}\Big)\Big]d\tau \le c.$$

Also, we have

$$\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\Big|_{t=0} \le c\alpha_0(k),$$

where $\alpha_0$ is as in the statement of the proposition.

Therefore, inequality (75) is of the form

$$\alpha(t) \le \beta(t) + \int_c^t \lambda(\tau)\alpha(\tau)d\tau,$$

where

$$\alpha(t) = \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu,$$
$$\beta(t) = \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\Big|_{t=0} + c\int_0^t \Big[\|A\|^2_{2,[\gamma\ge\frac{3}{4}]}\Big(1 + \|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]} + h + h^{\frac{4}{3}}\Big)\Big]\,d\tau,$$

and

$$\lambda(t) = \|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]} + h^{\frac{4}{3}}.$$

Noting that $\beta$ and $\int \lambda d\tau$ are bounded by the constants shown above, we can invoke Gronwall's inequality and conclude

$$\int_{[\gamma\ge\frac{7}{8}]}\|\nabla_{(k)}A\|^2d\mu \le \beta(t) + \int_0^t \beta(\tau)\lambda(\tau)e^{\int_\tau^t\lambda(\nu)d\nu}d\tau \le c\Big(k,\alpha_0(k)\Big).$$

Trivially, we also have

$$\int_{[\gamma\ge\frac{7}{8}]}\|\nabla_{(k+2)}A\|^2d\mu \le c\Big(k+2,\alpha_0(k+2)\Big).$$

Therefore using (52) with $\gamma_{\frac{7}{8},\frac{15}{16}}$ we can in fact bound $\|A\|_\infty$ on a smaller ball:

$$\|A\|^{8n-12}_{\infty,[\gamma\ge\frac{15}{16}]} \le c\epsilon_0\Big(\Big[c(2,\alpha_0(2))\Big]^{\frac{2n^2-3n}{2}} + \epsilon_0\Big).$$

Finally, using (51) with $T = \nabla_{(k)}A$ and $\gamma = \gamma_{\frac{15}{16},1}$ we obtain

$$\begin{aligned}\|\nabla_{(k)}A\|^4_{\infty,[\gamma=1]} \le c\|\nabla_{(k)}A\|^{4-n}_{2,[\gamma>\frac{15}{16}]}\Big(&\|\nabla_{(k+2)}A\|^n_{2,[\gamma>\frac{15}{16}]}\\ &+ (\|A\|^{2n}_{\infty,[\gamma>\frac{15}{16}]}+1)\|\nabla_{(k)}A\|^n_{2,[\gamma>\frac{15}{16}]}\Big)\\ \le c\Big(k,\alpha_0(k+2)\Big).&\end{aligned}$$

This completes the proof of the proposition. $\square$

Remark. This proposition is essential in the overall argument, however we note that to obtain bounds for all derivatives of curvature using this result we must assume that the initial data $f_0$ is not only 'smooth enough' (see short time existence theorem), but in fact $C^\infty$. This is why we have smooth initial data in the statement of the Lifespan Theorem. It is probably possible to use a different argument for the very high derivatives of curvature, more similar to classical theory, and then we would only require $C^4$ initial data in the case of surface diffusion flow. However since we are not overly concerned with the regularity of our initial data we have not pursued that here.

## 6. Proof of the Lifespan Theorem

We begin by reducing the problem to the case where $\rho = 1$ in (22). Observe that if $\rho \neq 1$ we may rescale our surface $f$ to $\tilde{f}(x,t) = \frac{1}{\rho} f(x, t\rho^4)$ in order to return to the case $\rho = 1$. This preserves our key integral estimates (multiplying some terms by a constant) and most crucially the integral quantity

$$\int_{f^{-1}(B_\rho)} \|A\|^p d\mu$$

scales to

$$\int_{\tilde{f}^{-1}(B_1)} \rho^{-p+n} \|\tilde{A}\|^p d\tilde{\mu},$$

where $n$ is the dimension of the manifold $M$. In our cases, $n = 2$ or $n = 3$ and the integral is scale invariant if $p = 2$ or $p = 3$ respectively. For the details of this scaling, please see Appendix B. We will show that in the $\rho = 1$ setting

$$\tilde{T} \geq \frac{1}{c},$$

and scale back to the case of $f^{-1}(B_\rho(x))$. Due to the contribution of the fourth order term $\Delta H$ in our governing equation (CSD), to maintain our earlier integral estimates we scale time by a factor of $\rho^4$. Note that $h$ may scale in a non-invariant fashion but this introduces a single change in the constant $c$ only, and certainly a scaled $h$ (we only perform this rescaling once) continues to satisfy (CB). Therefore we will conclude

$$\frac{T}{\rho^4} \geq \frac{1}{c} \qquad \Longrightarrow \qquad T \geq \frac{1}{c}\rho^4,$$

which is equation (23). The estimate (24) valid during this time comes along 'for free' in a sense, due to the structure of our argument.

We make the definition

$$\eta(t) = \sup_{x\in\mathbb{R}^3} \int_{f^{-1}(B_1(x))} \|A\|^n d\mu. \tag{76}$$

By covering $B_1$ with several translated copies of $B_{\frac{1}{2}}$ there is a constant $c_\eta$ such that

$$\eta(t) \leq c_\eta \sup_{x\in\mathbb{R}^3} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^n d\mu. \tag{77}$$

Note that $c_\eta = 4^{n+1}$ is sufficient.

By short time existence we have that $f(M \times [0,t])$ is compact for $t < T$ and so the function $\eta : [0,T) \to \mathbb{R}$ is continuous. We now define

(78)

$$t_0^{(n)} = \begin{cases} \sup\{0 \leq t \leq \min(T,\lambda_2) : \eta(\tau) \leq 3c_\eta\epsilon_0 \ \text{ for } \ 0 \leq \tau \leq t\}, & n = 2, \\ \sup\{0 \leq t \leq \min(T,\lambda_3) : \eta(\tau) \leq 3c_{P24}c_\eta C_{AB}^{1/3}\epsilon_0^{2/3} \ \text{ for } \ 0 \leq \tau \leq t\}, & n = 3, \end{cases}$$

where $\lambda_n$ is a parameter to be specified later. The constant $c_{P24}$ is the maximum of 1 and the constant from Proposition 3.26 with $k = 0$. Recall that we assume (AB) in the case where $n = 3$. Note that the $\epsilon_0$ on the right hand side of the inequality is from equation (22).

The proof continues in three steps. First, we show that it must be the case that $t_0^{(n)} = \min(T, \lambda_n)$. Second, we show that if $t_0^{(n)} = \lambda_n$, then we can conclude the Lifespan Theorem. Finally, we prove by contradiction that if $T \neq \infty$, then $t_0^{(n)} \neq T$. We label these steps as

$$t_0^{(n)} = \min(T, \lambda_n), \tag{79}$$

$$t_0^{(n)} = \lambda_n \quad \Longrightarrow \quad \text{Lifespan Theorem}, \tag{80}$$

$$T \neq \infty \quad \Longrightarrow \quad t_0^{(n)} \neq T. \tag{81}$$

The three statements (79), (80), (81) together imply the Lifespan Theorem. We expand the sketch of the argument given above as follows: first notice that by (79) $t_0^{(n)} = \lambda_n$ or $t_0^{(n)} = T$, and if $t_0^{(n)} = \lambda_n$ then by (80) we have the Lifespan Theorem. Also notice that if $t_0^{(n)} = \infty$ then $T = \infty$ and the Lifespan Theorem follows from estimate (84) below (used to prove statement (80)). Therefore the only remaining case where the Lifespan Theorem may fail to be true is when $t_0^{(n)} = T < \infty$. But this is impossible by statement (81), so we are finished.

We now give the proof of the first step, statement (79). From the assumption (22),

$$\eta(0) \le \epsilon_0 < \begin{cases} 3c_\eta \epsilon_0, & \text{for } n = 2 \\ 3c_{P24} c_\eta C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3, \end{cases}$$

and therefore (78) implies $t_0^{(n)} > 0$. Assume for the sake of contradiction that $t_0^{(n)} < \min(T, \lambda_n)$. Then from the definition (78) of $t_0^{(n)}$ and the continuity of $\eta$ we have

$$\eta\left(t_0^{(n)}\right) = \begin{cases} 3c_\eta \epsilon_0, & \text{for } n = 2 \\ 3c_{P24} c_\eta C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3, \end{cases} \tag{82}$$

so long as $\epsilon_0 \le 1$ and $C_{AB}, c_{P24} \ge 1$. Recall Proposition 3.23. We will now set $\gamma$ to be a cutoff function as in (25) such that

$$\chi_{B_{\frac{1}{2}}(x)} \le \tilde{\gamma} \le \chi_{B_1(x)},$$

for any $x \in M_t$. Choosing a small enough $\epsilon_0$ (by varying $\rho$ in (22)), (78) implies that the smallness condition (60) is satisfied on $[0, t_0^{(n)})$. Due to (CB), we also have that $\|h\|_{\infty,[0,t_0^{(n)})} < \infty$. Therefore we have satisfied all the requirements of Proposition 3.23, and so we conclude

$$\begin{aligned} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^2 d\mu &\le \int_{f^{-1}(B_1(x))} \|A\|^2 d\mu \bigg|_{t=0} + c_0 c_\eta \epsilon_0^{\frac{2}{n}} t \\ &\le \begin{cases} 2\epsilon_0, & \text{for } n = 2 \text{ and } \lambda_2 = \frac{1}{c_0 c_\eta}, \\ 2c_{P24} C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3 \text{ and } \lambda_3 = c_{P24} \frac{C_{AB}^{1/3}}{c_0 c_\eta}, \end{cases} \end{aligned} \tag{83}$$

for all $t \in [0, t^*]$, where $t^* < t_0^{(n)}$ and $c_0$ is the constant from Proposition 3.23. That is, equation (83) above is true for all $t \in \left[0, t_0^{(n)}\right)$. We combine this with (77) and Proposition 3.26 to conclude

$$\eta(t) \le c_{P24}^{n-2} c_\eta \sup_{x \in \mathbb{R}^{n+1}} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^2 d\mu \le \begin{cases} 2c_\eta \epsilon_0, & \text{for } n = 2 \\ 2c_{P24} c_\eta C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3, \end{cases} \tag{84}$$

where $0 \le t < t_0^{(n)}$.

Since $\eta$ is continuous, we can let $t \to t_0^{(n)}$ and obtain a contradiction with (82). Therefore, with the choice of $\lambda_n$ in equation (83), the assumption that $t_0^{(n)} < \min(T, \lambda_n)$ is incorrect. Thus we have shown (79), the first of our three steps.

We in fact have also proved the second step (80). Observe that if $t_0^{(n)} = \lambda_n$ then by the definition (78) of $t_0^{(n)}$,

$$T \ge \lambda_n,$$

which is (23). Also, (84) implies (24). That is, we have proved if $t_0^{(n)} = \lambda_n$, then the lifespan theorem holds, which is the second step (80). It only remains to prove equation (81).

We assume

$$t_0^{(n)} = T \neq \infty;$$

since if $T = \infty$ then (23) holds automatically and again (84) implies (24). Note also that we can safely assume $T < \lambda_n$, since otherwise we can apply step two to conclude the Lifespan Theorem.

Our strategy is to show that in this case the flow exists smoothly up to and including time $T$, allowing us to extend the flow, thus contradicting the finite maximality of $T$ from short time existence.

To begin short time existence again at time $T$, we need to prove that the limiting object $M_T$ has at least *some* regularity. Since the constraint function $h$ satisfies the estimate (22), if we can establish regularity for $f(\cdot, T)$ then we will have taken case of the constraint function also. We will show that $M_T$ is in fact smooth, by obtaining a uniform bound on all the derivatives of $f$ on the interval $[0, T)$. This will then allow us to assert that the convergence $M_t \to M_T$ is uniform, that the limiting object $M_T$ is unique and that $M_T$ is smooth. This will be enough to not only start short time existence again at $T$, but our entire argument.

Our main tool is Proposition 3.26. Since $T = t_0^{(n)}$, (84) implies the smallness condition (70) and we have

$$\|\nabla_{(m)} A\|^2_{\infty, B_{\frac{1}{2}}(x)} \leq c\Big(m, T, c_{\gamma 1}, c_{\gamma 2}, \|h\|_{\infty,[0,T)}, \alpha_0(m+2)\Big), \tag{85}$$

for any $t \in [0,T)$ and by the definition of $\eta$ (see equations (22) and (76)) for any $x \in M_t$, where $\alpha_0$ is defined as in Proposition 3.26. That is, we have a pointwise bound

$$\|\nabla_{(m)}A\|_\infty \le c, \tag{86}$$

on all of $M_t$, where the constant $c$ is as in inequality (85) above. We take the convention that all constants $c$ in the estimates below also depend on the quantities in (85).

We will now work towards converting the bound (86) to bounds on the coordinate derivatives of $f$. First however, we must show that there does exist a limiting Riemannian manifold $M_T$, and that the topology of the evolving manifolds (which is determined by the metric) is equivalent to the topology of the limiting object. If we cannot show this, then extending the flow beyond the maximal time $T$ might not be a contradiction, since we may have a different flow. We will use a result from Hamilton [**27**] which proves that all the evolving metrics are equivalent to the metric of $M_T$. (A stronger statement.) This is a standard argument, and although Hamilton used this with Ricci flow, it is important to many other flows. The first appearance of this argument in the context of a hypersurface flow is Huisken [**28**] on mean curvature flow.

Recall the evolution of the metric:

$$\frac{\partial}{\partial t}g = -2\left\langle A, \Delta H + h\right\rangle \le A * \nabla_{(2)}A + A\|h\|_\infty.$$

Therefore by (86),

$$\|\nabla_{(m)}\partial_t g\| \le c. \tag{87}$$

Similarly, the evolution of the Christoffel symbols is bounded as

$$\frac{\partial}{\partial t}\Gamma_{ij}^{k} \leq \nabla_{(3)}A * A + \nabla_{(2)}A * \nabla A + \nabla A \|h\|_{\infty},$$

so again by (86) we have

$$\|\nabla_{(m)}\partial_t \Gamma_{ij}^{k}\|_{\infty} \leq c. \tag{88}$$

LEMMA (Hamilton [**27**], Lemma 14.2). *Let $g_{ij}$ be a time dependent metric on a compact manifold $M$ for $0 \leq t < T \leq \infty$. Suppose that*

$$\int_0^T \max_{M_t} \left| \frac{\partial g_{ij}}{\partial t} \right| dt \leq C.$$

*Then the metrics $g_{ij}(t)$ are all equivalent, and they converge as $t \to T$ uniformly to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.*

By (87) the hypothesis of the lemma is satisfied and we have that the metrics $g(t)$, for $0 \leq t < T$ are equivalent. Choose a local chart with $\frac{1}{C} \leq g_{ij}(t) \leq C$ on a neighbourhood $U \subset M$, $t \in [0, T)$. Let $\Gamma$ be the Christoffel symbols associated with this chart and denote $m$ iterated coordinate derivatives by $\partial_{(m)}$. For any tensor $T$ we have the formula

$$\nabla_{(m)}T = \partial_{(m)}T + \sum_{l=1}^{m} \sum_{k+k_1+\cdots+k_l=m-l} \partial_{(k)}T \cdot \partial_{(k_1)}\Gamma \cdots \partial_{(k_l)}\Gamma. \tag{89}$$

This is immediate for $m = 1$ and then follows by induction. The base case is the definition of the covariant derivative:

$$\nabla T = \partial T + T * \Gamma.$$

It is instructive to see how we move from $m = 1$ to $m = 2$ before handling the inductive step. The derivation is

$$\nabla_{(2)}T = \nabla(\partial T + T * \Gamma)$$

$$
\begin{aligned}
&= \partial(\partial T + T * \Gamma) + \Gamma * (\partial T + T * \Gamma) \\
&= \partial_{(2)} T + \partial(T * \Gamma) + \Gamma * \partial T + T * \Gamma * \Gamma \\
&= \partial_{(2)} T + \Big[ T * \partial\Gamma + T * \Gamma * \Gamma + \partial T * \Gamma \Big] \\
&= \partial_{(2)} T + \sum_{k+k_1=1} \partial_{(k)} T * \partial_{(k_1)} \Gamma + \sum_{k+k_1+k_2=0} \partial_{(k)} T * \partial_{(k_1)} \Gamma * \partial_{(k_2)} \Gamma \\
&= \partial_{(2)} T + \sum_{l=1}^{2} \sum_{k+k_1+\dots+k_l=m-l} \partial_{(k)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma .
\end{aligned}
$$

Using the inductive hypothesis, the following derivation finishes the induction proof of (89).

$$
\begin{aligned}
\nabla_{(m)} T &= \nabla\left( \partial_{(m-1)} T + \sum_{l=1}^{m-1} \sum_{k+k_1+\dots+k_l=m-l-1} \partial_{(k)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma \right) \\
&= \partial_{(m)} T + \Gamma * \partial_{(m-1)} T \\
&\quad + \Gamma * \left( \sum_{l=1}^{m-1} \sum_{k+k_1+\dots+k_l=m-l-1} \partial_{(k)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma \right) \\
&\quad + \partial\left( \sum_{l=1}^{m-1} \sum_{k+k_1+\dots+k_l=m-l-1} \partial_{(k)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma \right) \\
&= \partial_{(m)} T + \Gamma * \partial_{(m-1)} T \\
&\quad + \sum_{l=1}^{m-1} \sum_{k+k_1+\dots+k_l=m-l-1} \partial_{(k)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma * \partial_{(0)} \Gamma \\
&\quad + \sum_{l=1}^{m-1} \sum_{k+k_1+\dots+k_l=m-l-1} \partial_{(k+1)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma \\
&\quad + \sum_{l=1}^{m-1} l \sum_{k+k_1+\dots+k_l=m-l-1} \partial_{(k)} T \cdot \partial_{(k_1+1)} \Gamma \cdots \partial_{(k_l)} \Gamma \\
&= \partial_{(m)} T + \sum_{l=1}^{m} \sum_{k+k_1+\dots+k_l=m-l} \partial_{(k)} T \cdot \partial_{(k_1)} \Gamma \cdots \partial_{(k_l)} \Gamma .
\end{aligned}
$$

Set $\sigma_m = \|\Gamma\| + \ldots + \|\partial_{(m)}\Gamma\|$. Then

$$
\|\partial_{(m)} T\| \le c(m, \sigma_{m-1}) (\|\nabla_{(m)} T\| + \|\partial_{(m-1)} T\| + \ldots + \|T\|). \tag{90}
$$

We wish to refine the expression on the right hand side of (89) to include only covariant derivatives. For clarity we state this as a lemma.

LEMMA 3.27. *For any tensor $T$ we have*

$$\|\partial_{(m)}T\| \le c(m, \sigma_{m-1})(\|\nabla_{(m)}T\| + \|\nabla_{(m-1)}T\| + \ldots + \|T\|). \tag{91}$$

PROOF. The proof is straightforward and again by induction: for the base case we have

$$\|\partial T\| \le c(1, \sigma_0)(\|\nabla T\| + \|T\|),$$

by (90), and then

$$\begin{aligned}
\|\partial_{(m)}T\| &\le c(m, \sigma_{m-1})(\|\nabla_{(m)}T\| + \|\partial_{(m-1)}T\| + \ldots + \|T\|) \\
&\le c(m, \sigma_{m-1})\Big(\|\nabla_{(m)}T\| \\
&\qquad\qquad + c(m-1, \sigma_{m-2})(\|\nabla_{(m-1)}T\| + \ldots + \|T\|) \\
&\qquad\qquad + c(m-2, \sigma_{m-3})(\|\nabla_{(m-2)}T\| + \ldots + \|T\|) \\
&\qquad\qquad \vdots \\
&\qquad\qquad + c(1, \sigma_0)(\|\nabla T\| + \|T\|)\Big) \\
&= c(m, \sigma_{m-1}) \sum_{i=0}^{m} \left( \left( \sum_{j=1}^{m-i} c(m-j, \sigma_{m-j-1}) \right) \|\nabla_{(i)}T\| \right),
\end{aligned}$$

using the induction hypothesis and taking the convention that $c(0, \sigma_{-1}) = 0$. □

Noting that coordinate and time derivatives of the Christoffel symbols $\Gamma$ are tensors, we apply (91) to $\frac{\partial}{\partial t}\Gamma$ and $\partial_{(m)}\Gamma$. Using (86) and (88) with this we obtain

$$\|\partial_{(m)} \frac{\partial}{\partial t}\Gamma\|_\infty \le c(m, T, f_0, \|h\|_\infty),$$

and this implies

$$\|\partial_{(m)}\Gamma\|_\infty \le c(m, T, f_0, \|h\|_\infty). \tag{92}$$

We claim now that

$$\|\partial_{(k)}\nabla_{(l)}A\|_\infty,\ \|\partial_{(m+1)}f\|_\infty \le c(m,T,f_0,\|h\|_\infty),\ \text{for } k+l=m\ge 0. \tag{93}$$

Using (86) and $\|\partial f\|^2 = n$, this clearly holds for $m=0$. For the induction step, let $k+l=m+1$ and then

$$\begin{aligned}
\partial_{(k)}\nabla_{(l)}A - \nabla_{(m+1)}A &= \partial_{(k)}\nabla_{(l)}A - \nabla_{(k)}\nabla_{(l)}A \\
&= \sum_{j=1}^{k}(\partial_{(j)}\nabla_{(k+l-j)} - \partial_{(j-1)}\nabla_{(k+l+1-j)})A \\
&= \sum_{j=1}^{k}\partial_{(j-1)}(\partial-\nabla)\nabla_{(k+l-j)}A \\
&= \sum_{j=1}^{k}\partial_{(j-1)}(\nabla_{(k+l-j)}A * A * \partial f), \qquad (94)
\end{aligned}$$

where we used the identity

$$\partial T - \nabla T = T * A * \partial f,$$

for any tangential tensor $T$ in the last step. This is easily seen by differentiating $(T\,|\,\nu)$ as follows. Let $\{e_i,\nu\}_{1\le i\le n}$ be a choice of Gaussian coordinates centered at a point $p\in M_t$ and then compute in a neighbourhood of $p$:

$$\begin{aligned}
&D_{e_j}(T\,|\,\nu) = 0 \\
\Longrightarrow\ &\left(D_{e_j}T\,\middle|\,\nu\right) = -\left(T\,|\,D_{e_j}\nu\right) \\
\Longrightarrow\ &\left(D_{e_j}T\,\middle|\,\nu\right)e_j = -\left(T\,|\,D_{e_j}\nu\right)e_j \\
\Longrightarrow\ &\partial T - \nabla T = T * A * \partial f.
\end{aligned}$$

Now from the induction hypothesis we have that for any $i$ where $0\le i\le j-1\le k-1\le m$, the quantities $\partial_{(i)}\nabla_{(k+l-j)}A$, $\partial_{(i)}A$ and $\partial_{(i+1)}f$ are bounded. Therefore, the above derivation (94) shows the first part of claim (93). The second part of (93) is an easy consequence of the Gauss-Weingarten relations

$$\partial_{(2)} f = A + \partial f * \Gamma$$

and (86), (92).

Considering the governing equation (CSD) and equations (86), (93), we have that

$$\|\partial_{(k)} \frac{\partial}{\partial t} f\|_\infty, \|\partial_{(k)} f\|_\infty \le c(m, T, f_0, \|h\|_{\infty,[0,T)}). \tag{95}$$

Hence the convergence $f(\cdot, t) \to f(\cdot, T)$ is in the $C^\infty$ topology and $M_T$ is smooth. We have that $f(\cdot, T)$ is a smooth immersion as the metrics at each time $t$ are uniformly equivalent and $g(t) \to g(T)$. Finally, by short time existence, we can extend the solution to an interval $[0, T + \delta]$, contradicting the maximality of $T$.

This establishes (81) and the theorem is proved. □

## 7. Concluding remarks

As mentioned earlier, Kuwert and Schätzle [**37**] proved a Lifespan Theorem for the Willmore flow,

$$\frac{\partial}{\partial t} f = \big(\Delta H + Q(A)\big)\nu,$$

where they considered surfaces immersed in $\mathbb{R}^n$ via $f$, i.e. $f : M^2 \to \mathbb{R}^n$. Note that in one codimension $Q(A) = \|A^o\|^2 H$. We first remark that one may use their setup of the evolution equation (using the induced Laplacian along the normal bundle) to obtain the Lifespan Theorem we proved here in arbitrary codimension. While the core argument remains identical, there is additional notation to introduce and the blowup analysis with associated long time existence and exponential convergence to spheres (see chapters 6 and 7) will not be valid in arbitrary codimension. We have therefore omitted this analysis.

We also remark that one may consider the evolution equation

$$\frac{\partial}{\partial t}f = \Big(\Delta H + \tilde{Q}(A)\Big)\nu,$$

where $f : M^2 \to \mathbb{R}^3$, with $\tilde{Q}(A)$ a term which may be estimated as

$$\tilde{Q} \le P_3^0(A) \tag{96}$$

and recover a Lifespan Theorem. One may employ the techniques which we presented in Sections 4 and 5, or of course an adaptation of those in [**37**], to obtain this result. This is essentially due to the integral estimates not depending on the precise form of the $P$-style terms. It may be possible to improve the growth condition (96) above to include some derivatives and more copies of $A$, however we have not pursued this. Of course combining this remark with the analysis we present in this chapter for constrained flows will give a lifespan theorem for flows of the form

$$\frac{\partial}{\partial t} = \Big(\Delta H + P_3^0(A) + h\Big)\nu.$$

Apart from constrained Willmore flows (for which one may compute constraint functions which give monotone area, volume, etc) we are not aware of any interesting examples of such flows. For immersions of dimension greater than 3, one will still be frustrated by the Sobolev inequality Theorem 3.20, and the local version Proposition 3.21. We are not aware of any technique which may be used to completely remove this restriction.

CHAPTER 4

# Gap lemma for constrained surface diffusion flows

## 1. Introduction

We begin our discussion by first recalling the classical gap lemma, typically seen in an introductory course on real analysis.

THEOREM (Classical Gap Lemma (CGL)). *Let $A \subset \mathbb{R}$ and assume there is an $a \in \mathbb{R}$ such that $a = \sup A$. Then for every $\epsilon > 0$, there is a $b \in A$ such that $|a - b| < \epsilon$.*

The (CGL) says that, although $a$ may not be in $A$ (that is, we cannot choose $\epsilon = 0$), we can become as close to $a$ as we wish.

At first glance this appears far removed from our work here in evolution equations, deep in the context of differential geometry; at least to the point where such a result is far entrenched in the required background, and not even worthy of comment, let alone be called a theorem.

But as with many similar results in introductory calculus, they stem from an abstract, hazy body of properties that mathematicians are interested in and when new, unknown objects of higher complexity appear, mathematicians tend to run tests on the new objects to be sure that they make sense and classify which of these properties remain true or false, and under which conditions.

That said however, our gap lemma is not a direct analog of (CGL). Indeed, a direct analog would be just as easy to prove as the classical gap lemma. The relationship our gap lemma has with (CGL) is more like that of a second cousin, rather than a direct ancestor or descendent. We will make this precise. Consider a smoothly varying one parameter family of manifolds $M_t$, where $t \in [0, T)$, equipped with any metric $|\cdot| : M_t \mapsto \mathbb{R}$ so that $(|\cdot|, \{M_t\})$ is a metric space. Then an interpretation of (CGL) in this context could be:

Theorem (Manifold Gap Lemma (MGL)). *Assume $M_T$ exists. Then for any $\epsilon > 0$, there is a $t_1 \in [0, T)$ such that $|M_{t_1} - M_T| < \epsilon$.*

Let us consider families of manifolds (and Riemannian metrics) which evolve by some law, such as mean curvature flow, Ricci flow, and in particular surface diffusion and the constrained surface diffusion flows. Unfortunately, if $T$ is maximal then we cannot in general expect (MGL) to remain useful. Indeed, for the flows mentioned many interesting things can happen at the final time $T$, and the manifold $M_T$ may possess any number of singularities. This is in contrast with the fact that the limiting object $M_T$ will essentially always exist (at least in the weak varifold sense, for example). It is clear that if we wish to recover a relevant statement, we must impose some measure of regularity on $M_T$, and then question not how 'close' we can become to $M_T$, but attempt to obtain information regarding the *geometry* of $M_T$.

With this in mind, we state our main theorem for this chapter.

Theorem 4.1 (Gap Lemma). *Suppose $n \in \{2, 3\}$ and let $f : M^n \to \mathbb{R}^{n+1}$ be a compact immersion with $(\Delta H + h) \equiv 0$. Then if assumptions* (GLA1) *and* (GLA2) *are satisfied*

$$\text{(GL)} \qquad M^n = S^n$$

*where $S^n$ is an embedded $n$-dimensional sphere in $\mathbb{R}^{n+1}$. If $f : M^n \to \mathbb{R}^{n+1}$ is instead a proper immersion then we must assume in addition* (GLA3) *to recover* (GL)*, and allow $S^n$ to denote a union of embedded spheres and planes.*

Before detailing the conditions (GLA1), (GLA2) and (GLA3) we make some brief comments relating this statement to our previous observations. First, the hypothesis of the theorem includes that $f(\cdot)$ is a stationary surface under (CSD) flow, and so in the 'one parameter family of manifolds' context we may assume that $T = \infty$. Second, although this statement includes only the $\epsilon = 0$ case from (MGL) in any rigor, certainly choosing a suitable metric for comparison of manifolds would give a statement similar to (MGL) for other values of $\epsilon$. It is also worth mentioning that one may easily prove that in any dimension stationary surfaces possess constant mean curvature under (SD) flow:

$$0 = \int_M H\Delta H d\mu = -\int_M \|\nabla H\|^2 d\mu.$$

In the case of (CSD) flow one may perform a similar computation:

$$0 = \int_M H\Delta H d\mu + h\int_M H d\mu = -\int_M \|\nabla H\|^2 d\mu + h\int_M H d\mu,$$

and so if

$$h\int_M H d\mu < \int_M \|\nabla H\|^2 d\mu$$

one obtains the same conclusion, that stationary solutions possess constant mean curvature. This simple computation is of course not sufficient to obtain Theorem 4.1, although it does highlight the kind of conditions which we must impose upon

$h$ to obtain our desired result. This is natural, since of the possible choices for $h$ any number of them may be poorly behaved on spheres (non-zero for example). We must rule out these constraint functions with the conditions for our theorem.

We make one final remark on the above computation. It is independent of $n$. This is remarkable since throughout this work the intrinsic dimension of our manifolds is tightly restricted to $n = 2$ or $n = 3$, with the latter even causing difficulty at times. This may only imply that the elementary computation above is too simple to see the full complexities of the problem at hand, although it may also indicate that Theorem 4.1 remains valid in higher dimensions, and that our techniques here are suffering from a technical disability.

The conditions for the theorem to be true are global smallness of the total trace-free curvature, a growth condition on $h$ and, in the case where $M_t$ is not compact in the limit, a bound on the growth of the curvature at infinity. Although they are somewhat restrictive, the result gives an important feeling of the stationary manifolds for a well-behaved class of constrained surface diffusion flows.

We say $h$ satisfies (GLA1) if for some $k \in (1, \infty]$,

$$\text{(GLA1)} \quad h^2|M| \le \frac{1}{kc_1}\int_M \|\nabla_{(2)}A^o\|^2 + \|A\|^2\|\nabla A^o\|^2 + H^2\|\nabla H\|^2 + \|A\|^4\|A^o\|^2 d\mu.$$

where $c = c(n)$ is the constant in the leading term on the right in Proposition 4.7. Note that certainly (GLA1) can be checked and satisfied a priori. Also note that this growth condition is *global*, and so none of the problems related to localised estimates come into the analysis for the Gap Lemma, unlike the Lifespan Theorem and Interior Estimates in chapters 3 and 5 respectively.

The other assumptions are as follows. We say that $f : M^n \rightarrow \mathbb{R}^{n+1}$ satisfies (GLA2) if for a given $\epsilon_0 > 0$,

$$\int_M \|A^o\|^n d\mu < \epsilon_0. \tag{GLA2}$$

One should think of this as the averaged distance from $M$ to a sphere in $L^n$. The constant $\epsilon_0$ is again known a priori and is the chief restriction. In the proof of Proposition 4.7 we determine exactly how small $\epsilon_0$ must be. Finally, we say that $f : M^n \rightarrow \mathbb{R}^{n+1}$ satisfies (GLA3) if

$$\liminf_{\rho\rightarrow\infty} \frac{1}{\rho^4} \int_{f^{-1}(B_\rho(0))} \|A\|^2 d\mu = 0. \tag{GLA3}$$

Let us say that we are interested in a given (CSD) flow, and wish to determine if the Gap Lemma holds for our flow. Then we must check (GLA1), and typically while doing so we will wish to use the other assumption, (GLA2). The way we do this is by observing that while both $\delta = \frac{1}{ck}$ and $\epsilon_0$ are fixed a priori, this gives not a fixed pair of values for which the theorem is true but rather a whole range of values. Therefore, we are 'free' to further tighten (GLA2) so that we can at least prove (GLA1). Depending on the flow and the constraint function, this may be no further restriction at all (in the case where $\epsilon_0$ is already required to be smaller than needed to prove (GLA1)), or a significant restriction (in the case where the $\epsilon_0$ required to prove (GLA1) is much smaller than that required from (GLA2)).

To strengthen the Gap Lemma to a more complete stability of spheres in $L^2$ result, we would need to weaken the assumptions on the curvature to conditions at initial time, and of course consider the flow equation instead of only stationary solutions. This is far from trivial, and such a procedure was carried out successfully by Kuwert & Schätzle [**36**], who we credit for the inspiration and structure of our argument here. In this overall sense, the Gap Lemma we prove is (similarly to the Lifespan Theorem) in preparation for global analysis of the (CSD) flows. We will be continuing our analysis toward obtaining our own stability of spheres result, which is the subject of Chapter 7.

A version of the Lifespan Theorem with the smallness assumption on the trace-free curvature instead of the full curvature would already be enough to attain this goal, although this is a departure from the method of Kuwert and Schätzle, whose technique (using the fact that Willmore flow is a gradient flow of total curvature) is superior in the sense that they immediately rule out development of type 2 singularities in the limit. For us, it will be a delicate interplay between the Lifespan Theorem and the Gap Lemma. This second version of the Lifespan Theorem is yet another avenue for further research in this area.

## 2. Preparation

Unlike the Lifespan Theorem, the Gap Lemma does not make use of derivative integral estimates. Instead, we deal directly with integral estimates where the right hand side involves the speed of the flow. Throughout this chapter we will employ the notation $F = \Delta H + h$, for some specified constraint function $h$.

We begin with some elementary computations. For our (CSD) flows, we have

$$\begin{aligned} \|\partial_t f\| &= \|(F)\nu\| = |F| \\ &= |(\Delta H) + h| \leq |\Delta H| + |h|. \end{aligned} \tag{97}$$

This implies

$$\int_M |\Delta H|\gamma^s d\mu + \int_M |h|\gamma^s d\mu \geq \int_M \|\partial_t f\|\gamma^s d\mu.$$

We can also obtain an inequality in the reverse direction by

$$
\begin{aligned}
F &= \Delta H + h \\
\implies \Delta H &= F - h \\
\implies |\Delta H| &= |F - h| \leq |F| + |h|, \qquad (98)
\end{aligned}
$$

so

$$\int_M |\Delta H|\gamma^s d\mu \leq \int_M |F|\gamma^s d\mu + \int_M |h|\gamma^s d\mu.$$

The overall idea of the argument is to prove estimates like

$$
\begin{aligned}
\int_M \|\nabla_{(2)}A\|^2 + \|A\|^4\|A^o\|^2 d\mu \leq \int_M \|\Delta A\|^2 d\mu \leq \int_M \|\nabla_{(2)}H\|^2 d\mu \leq \int_M |\Delta H|^2 d\mu \\
\leq \int_M |F|^2 d\mu.
\end{aligned}
$$

Then the proof is, in essence, that if $f(\cdot)$ is a (CSD)-surface, $|F(\cdot)| = 0$, and then we obtain that $\|A\| = 0$, $\|A^o\| = 0$, or both are zero at final time. This implies $f(\cdot, T)$ is an embedded plane or sphere. Of course we cannot prove estimates exactly as above; there are some error terms and the constraint function $h$ forces us to work a little harder.

But even before we have these troubles, the first problem is how to exploit the symmetries and fundamental theorems of differential geometry to obtain relationships like

$$
\begin{aligned}
\|\nabla_{(2)}A\| &\leq c\|\Delta A\| + \text{`error'} \\
&\leq c\|\nabla_{(2)}H\| + \text{`error'} \\
&\leq c|\Delta H| + \text{`error'}.
\end{aligned}
$$

It is not easy to obtain these relationships as stated. However, with the introduction of the tracefree second fundamental form

$$A^o = A - \tfrac{1}{n}Hg,$$

such relationships become much easier to prove. One of the nice symmetries that this definition makes apparent is

$$\tfrac{n}{n-1}\nabla_i (A^o)^i_j = \nabla_j H. \tag{99}$$

This is especially useful when combined with the adjoint $\nabla^*$ of $\nabla$, using which the previous expression becomes

$$-\tfrac{n}{n-1}\nabla^* A^o = \nabla H.$$

To prove (99), simply note

$$g^{ij}\nabla_i (A^o)_{jk} = g^{ij}\nabla_i (A_{jk} - \tfrac{1}{n}g_{jk}H) = \nabla_i A^i_k - \tfrac{1}{n}\nabla^k H = (1 - \tfrac{1}{n})\nabla_k H,$$

by Codazzi.

We will also need a Simons' identity for $A^o$. Using (2),

$$\begin{aligned}
\Delta A^o_{ij} &= \Delta A_{ij} - \tfrac{1}{n}g_{ij}\Delta H \\
&= \nabla_{ij} H - \tfrac{1}{n}g_{ij}\Delta H - \|A\|^2 A_{ij} + H A_{ik} A^k_j \\
&= S^o\Big(\nabla_{(2)} H\Big) + \tfrac{1}{n}H^2 A^o_{ij} - \|A^o\|^2 A^o_{ij} + H S^o\Big(A^o_{i(\cdot)} A^o_{j(\cdot)}\Big) \\
&= S^o\Big(\nabla_{(2)} H\Big) + \tfrac{1}{n}H^2 A^o_{ij} + A^o * A^o * A^o + A * A^o * A^o, \qquad (100)
\end{aligned}$$

where $S^o(T)$ is the symmetric tracefree part of a bilinear form $T$.

## 3. Estimating $\|\nabla_{(2)} A^o\|$ in terms of $|\Delta H|$

We begin this process with the following lemma.

LEMMA 4.2. *For an immersion $f : M^n \to \mathbb{R}^{n+1}$ and $\gamma$ as in* (25),

$$\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \int_M H^2\|A^o\|^2\gamma^2 d\mu \le c\frac{1}{(c_{\gamma 1})^2}\int_M |F|^2\gamma^4 d\mu + c|h|^2\frac{1}{(c_{\gamma 1})^2}\int_M \gamma^4 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu, \tag{101}$$

*where $c = c(n)$.*

PROOF. Multiplying (100) by $A^o\gamma^2$ and integrating by parts,

$$\begin{aligned}\int_M \langle A^o, \Delta A^o\rangle \gamma^2 d\mu &= -\int_M \|\nabla A^o\|^2\gamma^2 d\mu - 2\int_M \langle (\nabla\gamma)A^o, \nabla A^o\rangle \gamma d\mu \\ &= \int_M \Big\langle A^o, S^o(\nabla_{(2)}H)\Big\rangle \gamma^2 d\mu + \frac{1}{n}\int_M H^2 \langle A^o, A^o\rangle \gamma^2 d\mu \\ &\quad + \int_M \Big(A^o * A^o * A^o * A^o\Big)\gamma^2 d\mu.\end{aligned} \tag{102}$$

Note that the trace-free part of $\nabla_{(2)}H$ is given by $\nabla_{(2)}H - \frac{1}{2}\Delta H g = S^o(\nabla_{(2)}H)$. Using this we obtain

$$\begin{aligned}\int_M \Big\langle A^o, S^o(\nabla_{(2)}H)\Big\rangle \gamma^2 d\mu &= \int_M \Big\langle A^o, \nabla_{(2)}H - \frac{1}{2}g_{ij}\Delta H\Big\rangle \gamma^2 d\mu \\ &= \int_M \Big\langle A^o, \nabla_{(2)}H\Big\rangle \gamma^2 d\mu - \frac{1}{2}\int_M \Big(\text{trace } A^o\Big)\Delta H\gamma^2 d\mu \\ &= \int_M \Big\langle A^o, \nabla_{(2)}H\Big\rangle \gamma^2 d\mu \\ &= -\int_M \Big(\nabla_i (A^o)^i_j\Big)(\nabla^j H)\gamma^2 d\mu - 2\int_M (\nabla_i\gamma)(A^o)^i_j(\nabla^j H)\gamma d\mu.\end{aligned} \tag{103}$$

Combining estimates (102) and (103) gives

$$\begin{aligned}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{1}{n}\int_M H^2\|A^o\|^2\gamma^2 d\mu &= 2\int_M \Big[(\nabla_i\gamma)(A^o)^i_j(\nabla^j H) - \langle \nabla\gamma A^o, \nabla A^o\rangle\Big]\gamma d\mu \\ &\quad + \int_M \Big(\nabla_i (A^o)^i_j\Big)(\nabla^j H)\gamma^2 d\mu - \int_M \Big(A^o * A^o * A^o * A^o\Big)\gamma^2 d\mu.\end{aligned}$$

Estimating,

$$\begin{aligned}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{1}{n}\int_M H^2\|A^o\|^2\gamma^2 d\mu &\le \frac{n-1}{n}\int_M \|\nabla H\|^2\gamma^2 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu \\ &\quad + \epsilon_1\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{c}{4\epsilon_1}(c_{\gamma 1})^2\int_{[\gamma>0]}\|A^o\|^2 d\mu\end{aligned}$$

$$+\int_M \|\nabla H\|^2\gamma^2 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A^o\|^2 d\mu,$$

choosing $\epsilon_1 = \frac{3n-5}{3}$ we have

$$\frac{5}{3n}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{1}{n}\int_M H^2\|A^o\|^2\gamma^2 d\mu \le \frac{n-1}{n}\int_M \|\nabla H\|^2\gamma^2 d\mu$$
$$+ c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A^o\|^2 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu.$$

We estimate the first term on the right by

$$\begin{aligned}\int_M \|\nabla H\|^2\gamma^2 d\mu &= -\int_M \langle H, \Delta H\rangle \gamma^2 d\mu - 2\int_M \langle \nabla\gamma, \nabla H\rangle H\gamma d\mu\\ &= -\int_M \langle H, \Delta H\rangle \gamma^2 d\mu - \frac{2n}{n-1}\int_M (\nabla_j\gamma)\Big(\nabla^i (A^o)_i^j\Big) H\gamma d\mu\\ &\le \frac{1}{(c_{\gamma 1})^2}\int_M |\Delta H|^2\gamma^4 d\mu + 4(c_{\gamma 1})^2\int_{[\gamma>0]} H^2 d\mu\\ &\quad + c\int_M \Big(\nabla\gamma * A * \nabla A^o\Big)\gamma d\mu\\ &\le \frac{1}{(c_{\gamma 1})^2}\int_M |\Delta H|^2\gamma^4 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu\\ &\quad + \frac{n}{n-1}\frac{1}{3n}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu,\end{aligned}\tag{104}$$

where we used (99). Combining the inequalities gives

$$\begin{aligned}&\frac{5}{3n}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{1}{n}\int_M H^2\|A^o\|^2\gamma^2 d\mu\\ &\quad\le \frac{1}{3n}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{c}{(c_{\gamma 1})^2}\int_M |\Delta H|^2\gamma^4 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu\\ &\qquad + \int_M \|A^o\|^4\gamma^2 d\mu.\end{aligned}$$

Absorbing $\int_M \|\nabla A^o\|^2\gamma^2 d\mu$ on the left, this becomes

$$\begin{aligned}&\frac{4}{3n}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \frac{1}{n}\int_M H^2\|A^o\|^2\gamma^2 d\mu\\ &\quad\le \frac{c}{(c_{\gamma 1})^2}\int_M |\Delta H|^2\gamma^4 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu\\ &\quad= \frac{c}{(c_{\gamma 1})^2}\int_M |F|^2\gamma^4 d\mu - |h|^2\frac{c}{(c_{\gamma 1})^2}\int_M \gamma^4 d\mu - \frac{c}{(c_{\gamma 1})^2}h\int_M (\Delta H)\gamma^4 d\mu\\ &\qquad + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu,\end{aligned}\tag{105}$$

since

$$|F|^2 = (\Delta H + h)^2 = |\Delta H|^2 + |h|^2 - 2h(\Delta H).$$

Note that

$$\begin{aligned} -ch\int_M (\Delta H)\gamma^4 d\mu &= ch\int_M \langle \nabla H, \nabla\gamma\rangle \gamma^3 d\mu \\ &\le \frac{1}{3n}\int_M \|\nabla A^o\|^2\gamma^2 d\mu + c(c_{\gamma 1})^2 h^2 \int_M \gamma^4 d\mu. \end{aligned}$$

Combining this with (105) above and absorbing, we finally obtain the result. □

A slight variation to the above proof also gives the following estimate.

COROLLARY 4.3. *For an immersion $f : M^n \to \mathbb{R}^{n+1}$ and $\gamma$ as in* (25),

$$\begin{aligned} &\int_M \|\nabla A^o\|^2\gamma^2 d\mu + \int_M H^2\|A^o\|^2\gamma^2 d\mu \\ &\quad\le c\frac{1}{(c_{\gamma 1})^2}\int_M |F|^2\gamma^4 d\mu + ch\int_M H\gamma^2 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0]} \|A\|^2 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu, \end{aligned}$$

*where $c = c(n)$.*

PROOF. Instead of (104) use

$$\begin{aligned} -\int_M \langle H, \Delta H\rangle \gamma^2 &= -\int_M \langle H, F\rangle \gamma^2 d\mu + h\int_M H\gamma^2 d\mu \\ &\le \frac{1}{(c_{\gamma 1})^2}\int_M |F|^2\gamma^4 d\mu + h\int_M H\gamma^2 d\mu + (c_{\gamma 1})^2\int_{[\gamma>0]} H^2 d\mu. \end{aligned}$$

The remaining terms are estimated identically to before. □

Using Codazzi and interchange of covariant derivatives, we improve the left hand side of the previous estimate.

LEMMA 4.4. *For an immersion $f : M^n \to \mathbb{R}^{n+1}$ we have*

$$\int_M \|\nabla_{(2)} H\|^2\gamma^4 d\mu + \int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu + \frac{n-1}{n}\int_M H^2\|\nabla H\|^2\gamma^4 d\mu$$

$$+\frac{1}{n(n-1)}\int_M H^4\|A^o\|^2\gamma^4 d\mu \le c\int_M |F|^2\gamma^4 d\mu + c\int_M |h|^2\gamma^4 d\mu + c(c_{\gamma 1})^4\int_{[\gamma>0]}\|A\|^2 d\mu$$
$$+ c\int_M (\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^6)\gamma^4 d\mu,$$

*where* $c = c(n)$.

Proof. Gauss-Bonnet, interchange of covariant derivative and Codazzi yield the following identity

$$\nabla^*(\nabla_{(2)}H) = \nabla(\nabla^*\nabla H) - \frac{n-1}{n^2}H^2\nabla H + A * A^o * \nabla A^o,$$

where $\nabla^*$ is the formal adjoint of $\nabla$. Taking an inner product on both sides of the above equation with $(\nabla H)\gamma^4$ and then integrating by parts gives

$$\int_M \left\langle \nabla H, \nabla^*(\nabla_{(2)}H)\right\rangle \gamma^4 d\mu = \int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + 4\int_M \gamma^3(\nabla\gamma * \nabla H * \nabla_{(2)}H)d\mu$$
$$= -\frac{n-1}{n^2}\int_M H^2\left\langle \nabla H, \nabla H\right\rangle \gamma^4 d\mu + \int_M \left\langle \nabla H, A * A^o * \nabla A^o\right\rangle \gamma^4 d\mu$$
$$- \int_M \left\langle \nabla H, \nabla\Delta H\right\rangle \gamma^4 d\mu,$$

and so, integrating by parts once more,

$$\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + \frac{n-1}{n^2}\int_M H^2\|\nabla H\|^2\gamma^4 d\mu \le c\int_M (A * A^o * \nabla A^o * \nabla A^o)\gamma^4 d\mu$$
$$+ c\int_M \gamma^3(\nabla\gamma * \nabla H * \nabla_{(2)}H)d\mu + c\int_M |\Delta H|^2\gamma^4 d\mu. \tag{106}$$

Note that we used $\int_M \left\langle \nabla H, \nabla^*(\nabla_{(2)}H)\right\rangle \gamma^4 d\mu = -\int_M \left\langle \nabla H, \Delta\nabla H\right\rangle \gamma^4 d\mu$.

Recall that from our earlier calculations

$$|\Delta H|^2 = |F|^2 + |h|^2 - 2Fh.$$

Inserting this into (106) we obtain

$$\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + \frac{n-1}{n^2}\int_M H^2\|\nabla H\|^2\gamma^4 d\mu \le \int_M [|F|^2 + |h|^2 - 2Fh]\gamma^4 d\mu$$

$$+\frac{n^2-n+1}{n^2}\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + c(c_{\gamma 1})^2\int_M \|\nabla H\|^2\gamma^2 d\mu$$

$$+\delta_1\int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu + c\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^4 d\mu, \tag{107}$$

where we also estimated $\int_M(\nabla\gamma * \nabla H * \nabla_{(2)}H)\gamma^3 d\mu$ and $\int_M(A * A^o * \nabla A^o * \nabla A^o)\gamma^4 d\mu$, and used $A = A^o - \frac{1}{n}Hg$.

We now estimate the $\|\nabla H\|^2$ term. Using (99) and then Corollary 4.3,

$$\begin{aligned}
\int_M \|\nabla H\|^2\gamma^2 d\mu &\le c\int_M \|\nabla A^o\|\gamma^2 d\mu \\
&\le c(c_{\gamma 1})^{-2}\int_M |F|^2\gamma^4 d\mu + c\int_M \|A^o\|^4\gamma^2 d\mu \\
&\quad + c(c_{\gamma 1})^2\int_{[\gamma>0]}\|A\|^2 d\mu + ch\int_M H\gamma^2 d\mu \\
&\le c(c_{\gamma 1})^{-2}\int_M |F|^2\gamma^4 d\mu + ch\int_M H\gamma^2 d\mu \\
&\quad + c(c_{\gamma 1})^{-2}\int_M \|A^o\|^6\gamma^4 d\mu + c(c_{\gamma 1})^2\int_{[\gamma>0}\|A\|^2 d\mu.
\end{aligned}$$

Inserting into (107) yields

$$\begin{aligned}
\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + \int_M H^2\|\nabla H\|^2\gamma^4 d\mu &\le c\int_M \Big(|F|^2 - 2Fh + |h|^2\Big)\gamma^4 d\mu \\
&\quad + c(c_{\gamma 1})^2 h\int_M H\gamma^2 d\mu + c\int_M \|A^o\|^6\gamma^4 d\mu + c(c_{\gamma 1})^4\int_{[\gamma>0]}\|A\|^2 d\mu \\
&\quad + c\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^4 d\mu + \delta_1\int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu.
\end{aligned} \tag{108}$$

The last integral is critical. For the Gap Lemma to work we need some positive non-derivative term on the left. We obtain it from the last integral as follows.

$$\begin{aligned}
&\int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu \\
&\quad = -\int_M H^2\left\langle A^o, \Delta A^o\right\rangle\gamma^4 d\mu - 2\int_M H\left\langle(\nabla H)A^o, \nabla A^o\right\rangle\gamma^4 d\mu \\
&\qquad - 4\int_M H^2\left\langle(\nabla\gamma)A^o, \nabla A^o\right\rangle\gamma^3 d\mu \\
&\quad = -\int_M H^2\Big\langle A^o, \nabla_{(2)}H + \tfrac{1}{n}H^2A^o + A^o * A^o * A^o + A * A^o * A^o\Big\rangle\gamma^4 d\mu
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\int_M H(\nabla H * \nabla A^o * A^o)\gamma^4 d\mu - 4\int_M H^2(\nabla A^o * A^o * \nabla\gamma)\gamma^3 d\mu \\
&= -\int_M H^2 \left\langle A^o, \nabla_{(2)} H\right\rangle \gamma^4 d\mu - \frac{1}{n}\int_M H^2 \left\langle A^o, H^2 A^o\right\rangle \gamma^4 d\mu \\
&\quad - \int_M H^2 \left\langle A^o, A^o * A^o * A^o\right\rangle \gamma^4 d\mu - 2\int_M H(\nabla H * \nabla A^o * A^o)\gamma^4 d\mu \\
&\quad - 4\int_M H^2(\nabla A^o * A^o * \nabla\gamma)\gamma^3 d\mu \\
&= -\int_M H^2 \left\langle \nabla^* A^o, \nabla H\right\rangle \gamma^4 d\mu + 2\int_M H(\nabla_p H)(A^o)^p_q(\nabla^q H)\gamma^4 d\mu \\
&\quad + 4\int_M H^2 \left\langle A^o, \nabla H \nabla\gamma\right\rangle \gamma^3 d\mu - \frac{1}{n}\int_M H^4\|A^o\|^2\gamma^4 d\mu - c\int_M H^2\|A^o\|^4\gamma^4 d\mu \\
&\quad - 2\int_M H(\nabla H * \nabla A^o * A^o)\gamma^4 d\mu - 4\int_M H^2(\nabla A^o * A^o * \nabla\gamma)\gamma^3 d\mu.
\end{aligned}
$$

Using (99) we estimate the equality by

$$
\begin{aligned}
&\int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu \\
&\quad \le -\frac{1}{n}\int_M H^4\|A^o\|^2\gamma^4 d\mu + \frac{n-1}{n}\int_M H^2\|\nabla H\|^2\gamma^4 d\mu \\
&\qquad + c\int_M H^2\|A^o\|^4\gamma^4 d\mu + c\int_M H\|\nabla H\|\cdot\|\nabla A^o\|\cdot\|A^o\|\gamma^4 d\mu \\
&\qquad + c(c_{\gamma 1})\int_M H^2\|\nabla A^o\|\cdot\|A^o\|\gamma^3 d\mu \\
&\quad \le -\frac{1}{n}\int_M H^4\|A^o\|^2\gamma^4 d\mu + \frac{n-1}{n}\int_M H^2\|\nabla H\|^2\gamma^4 d\mu + c\int_M H^2\|A^o\|^4\gamma^4 d\mu \\
&\qquad + (\delta_2 + \delta_3)\int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu + c_{\delta_2}\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^4 d\mu \\
&\qquad + c_{\delta_3}(c_{\gamma 1})^2\int_M H^2\|A^o\|^2\gamma^2 d\mu.
\end{aligned}
$$

Let $\delta_2 + \delta_3 = \frac{n-1}{n}$. Then

$$
\begin{aligned}
&\int_M H^2\|\nabla A^o\|^2\gamma^4 d\mu + \int_M H^4\|A^o\|^2\gamma^4 d\mu - (n-1)\int_M H^2\|\nabla H\|^2\gamma^4 d\mu \\
&\quad \le c\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^4 d\mu + c\int_M \|A^o\|^6\gamma^4 d\mu + c(c_{\gamma 1})^4\int_{[\gamma>0]}\|A^o\|^2 d\mu,
\end{aligned} \tag{109}
$$

where we estimated

$$
(c_{\gamma 1})^2\int_M H^2\|A^o\|^2\gamma^2 d\mu \le \frac{1}{4}\int_M H^4\|A^o\|^2\gamma^4 d\mu + (c_{\gamma 1})^4\int_{[\gamma>0]}\|A^o\|^2 d\mu.
$$

Combining (108) and (109), we conclude

$$\begin{aligned}
\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + \int_M H^2\|\nabla H\|^2\gamma^4 d\mu + \delta_1 \int_M H^4\|A^o\|^2\gamma^4 d\mu& \\
\le c\int_M \Big(|F|^2 - 2Fh + |h|^2\Big)\gamma^4 d\mu + c(c_{\gamma 1})^2 h \int_M H\gamma^2 d\mu& \\
+ c\int_M \|A^o\|^6\gamma^4 d\mu + c(c_{\gamma 1})^4 \int_{[\gamma>0]}\|A\|^2 d\mu& \\
+ c\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^4 d\mu + \delta_1(n-1)\int_M H^2\|\nabla H\|^2\gamma^4 d\mu& \\
\le c\int_M \Big(|F|^2 + |h|^2\Big)\gamma^4 d\mu + c\int_M \|A^o\|^6\gamma^4 d\mu + c(c_{\gamma 1})^4 \int_{[\gamma>0]}\|A\|^2 d\mu& \\
+ c\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^4 d\mu + \delta_1(n-1)\int_M H^2\|\nabla H\|^2\gamma^4 d\mu.&
\end{aligned}$$

Choosing $\delta_1 = \frac{1}{n(n-1)}$ gives the result. $\square$

We further exploit the symmetry relations to convert the left hand side to an expression involving $\|\nabla_{(2)}A^o\|$.

LEMMA 4.5. *For an immersion $f : M^n \to \mathbb{R}^{n+1}$ we have*

$$\begin{aligned}
\int_M \|\nabla_{(2)}A^o\|^2\gamma^4 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2\gamma^4 d\mu + \int_M H^2\|\nabla H\|^2\gamma^4 d\mu& \\
+ \int_M H^4\|A^o\|^2\gamma^4 d\mu + \int_M \|A\|^4\|A^o\|^2\gamma^4 d\mu& \\
\le c\int_M |F|^2\gamma^4 d\mu + c\int_M |h|^2\gamma^4 d\mu + c(c_{\gamma 1})^4 \int_{[\gamma>0]}\|A\|^2 d\mu& \\
+ c_1\int_M (\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^6)\gamma^4 d\mu,&
\end{aligned}$$

*where $c = c(n)$.*

PROOF. We will again use a consequence of interchange and Codazzi,

$$\nabla^*(\nabla_{(2)}A^o) = \nabla(\nabla^*\nabla A^o) + A * A * \nabla A^o. \tag{110}$$

Multiplying (110) by $\gamma^4\nabla A^o$ and integrating by parts,

$$\int_M \Big\langle \nabla A^o, \nabla^*(\nabla_{(2)}A^o)\Big\rangle \gamma^4 d\mu = -\int_M \langle \nabla A^o, \Delta\nabla A^o\rangle \gamma^4 d\mu$$

$$\begin{aligned}
\implies &\int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu + 4\int_M \Big\langle (\nabla\gamma)\nabla A^o, \nabla_{(2)} A^o \Big\rangle \gamma^3 d\mu \\
&\le \int_M \langle \nabla A^o, \nabla(\nabla^* \nabla A^o)\rangle \gamma^4 d\mu + \int_M \langle \nabla A^o, A * A * \nabla A^o\rangle \gamma^4 d\mu \\
&= \int_M \|\Delta A^o\|^2 \gamma^4 d\mu + \int_M \langle \nabla A^o, A * A * \nabla A^o\rangle \gamma^4 d\mu \\
&\qquad + 4\int_M (\nabla_p \gamma)(\nabla^p (A^o)_{qr})(\Delta (A^o)^{qr})\gamma^3 d\mu \\
\implies &\int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu \\
&\le \int_M \|\Delta A^o\|^2 \gamma^4 d\mu + c\int_M \|\nabla A^o\|^2 \|A\|^2 \gamma^4 d\mu + c\int_M (\nabla\gamma * \nabla A^o * \nabla_{(2)} A^o)\gamma^3 d\mu \\
&\le \int_M \|\Delta A^o\|^2 \gamma^4 d\mu + \delta_1 \int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu + c_{\delta 1}(c_{\gamma 1})^2 \int_M \|\nabla A^o\|^2 \gamma^4 d\mu \\
&\qquad + c\int_M \|\nabla A^o\|^2 \|A\|^2 \gamma^4 d\mu.
\end{aligned}$$

Choosing $\delta_1 = \frac{1}{2}$, absorbing $\int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu$ on the left and multiplying by 2 we have

$$\begin{aligned}
&\int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu \\
&\qquad \le 2\int_M \|\Delta A^o\|^2 \gamma^4 d\mu + c\int_M \|\nabla A^o\|^2 \|A\|^2 \gamma^4 d\mu + c(c_{\gamma 1})^2 \int_M \|\nabla A^o\|^2 \gamma^4 d\mu.
\end{aligned} \tag{111}$$

Now use (100) to compute

$$\begin{aligned}
\int_M \|\Delta A^o\|^2 \gamma^4 d\mu &= \int_M \Big\langle S^o(\nabla_{(2)} H), \Delta A^o \Big\rangle \gamma^4 d\mu + \frac{1}{n}\int_M H^2 \langle A^o, \Delta A^o\rangle \gamma^4 d\mu \\
&\qquad + \int_M \langle \Delta A^o, A^o * A^o * A^o + HS^o(A^o * A^o)\rangle \gamma^4 d\mu \\
&\le \int_M \|\nabla_{(2)} H\| \cdot \|\Delta A^o\| \gamma^4 d\mu + \frac{1}{n}\int_M H^2 \|A^o\| \cdot \|\Delta A^o\| \gamma^4 d\mu \\
&\qquad + \int_M \|\Delta A^o\| \cdot \|A^o\|^3 \gamma^4 d\mu + \int_M \|\Delta A^o\| \cdot H \cdot \|A^o\|^2 \gamma^4 d\mu \\
&\le \sum_{i=1}^5 \delta_i \int_M \|\Delta A^o\|^2 \gamma^4 d\mu + c_{\delta_2} \int_M \|\nabla_{(2)} H\|^2 \gamma^4 d\mu \\
&+ c_{\delta_3} \int_M \|A^o\|^6 \gamma^4 d\mu + c_{\delta_4} \int_M H^4 \|A^o\|^2 \gamma^4 d\mu + c_{\delta_5} \int_M H^2 \|A^o\|^4 \gamma^4 d\mu.
\end{aligned}$$

Choose $\sum_{i=1}^5 \delta_i = \frac{1}{2}$ and absorb $\int_M \|\Delta A^o\|^2\gamma^4 d\mu$ on the left to obtain

$$\int_M \|\Delta A^o\|^2\gamma^4 d\mu \le c\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + c\int_M H^4\|A^o\|^2\gamma^4 d\mu + c\int_M \|A^o\|^6\gamma^4 d\mu,$$

where we also estimated

$$H^2\|A^o\|^2 \le \frac{1}{2}H^4\|A^o\|^2 + \frac{1}{2}\|A^o\|^6.$$

We combine this with (111) and conclude

$$\begin{aligned}\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu \le{}& c\int_M \|\nabla_{(2)}H\|^2\gamma^4 d\mu + c\int_M H^4\|A^o\|^2\gamma^4 d\mu + c\int_M \|A^o\|^6\gamma^4 d\mu \\ &+ c\int_M \|\nabla A^o\|^2\|A\|^2\gamma^4 d\mu + c(c_{\gamma 1})^2\int_M \|\nabla A^o\|^2\gamma^4 d\mu. \qquad (112)\end{aligned}$$

Combining (112) with corollary 4.3 and lemma 4.4 gives the result. □

Recall Lemma 3.22. A similar proof, where we consider $A^o$ instead of $A$, yields the following multiplicative Sobolev inequalities.

LEMMA 4.6. *Suppose $\gamma$ is as in (25) and $s \ge 4$. Then for an immersion $f : M^2 \to \mathbb{R}^3$*

$$\begin{aligned}&\int_M \|A^o\|^6\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\ &\quad\le c_2\int_{[\gamma>0]}\|A^o\|^2 d\mu \int_M \left(\|\nabla_{(2)}A^o\|^2 + \|A\|^2\|\nabla A^o\|^2 + \|A\|^2\|A^o\|^4\right)\gamma^s d\mu \\ &\qquad + c(c_{\gamma 1})^4\left(\int_{[\gamma>0]}\|A^o\|^2 d\mu\right)^2,\end{aligned}$$

*and for an immersion $f : M^3 \to \mathbb{R}^4$*

$$\begin{aligned}&\int_M \|A^o\|^6\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\ &\le \delta\int_M \|\nabla_{(2)}A^o\|^2\gamma^s d\mu + c_2\|A^o\|_{3,[\gamma>0]}^{\frac{3}{2}}\int_M \left(\|\nabla_{(2)}A^o\|^2 + \|A\|^2\|A^o\|^4 + \|A^o\|^6\right)\gamma^s d\mu \\ &\quad + c(c_{\gamma 1})^3\|A^o\|_{3,[\gamma>0]}^{\frac{9}{2}},\end{aligned}$$

*where $\delta \in (0,\infty)$ and $c = c(s,n)$.*

Proof. We first consider the $n = 2$ case. Keep in mind that since we obtain many of the below estimates through the use of the Michael-Simon Sobolev inequality, one can only safely use them if $n = 2$. We will provide the proof of the $n = 3$ case separately.

Our overall strategy is to apply the Michael-Simon Sobolev inequality with $u = \|A^o\|^3\gamma^{\frac{s}{2}}$ and $u = \|\nabla A^o\| \cdot \|A\|\gamma^{\frac{s}{2}}$, and then use Hölder and other standard integral estimates.

Beginning with $u = \|A^o\|^3\gamma^{\frac{s}{2}}$:

$$\begin{aligned}
\int_M \|A^o\|^6\gamma^s d\mu \le c\bigg[ &\int_M \|A^o\|^2\|\nabla A^o\|\gamma^{\frac{s}{2}} d\mu + \int_M \|A^o\|^3\|\nabla\gamma\|\gamma^{\frac{s}{2}-1} d\mu \\
&+ \int_M \|A^o\|^3|H|\gamma^{\frac{s}{2}} d\mu\bigg]^2 \\
\le\, &c\bigg(\int_M \|A^o\|^2\|\nabla A^o\|\gamma^{\frac{s}{2}} d\mu\bigg)^2 + c\bigg(\int_M \|A^o\|^3\|\nabla\gamma\|\gamma^{\frac{s}{2}-1} d\mu\bigg)^2 \\
&+ c\bigg(\int_M \|A^o\|^3|H|\gamma^{\frac{s}{2}} d\mu\bigg)^2 \\
\le\, &c\|A^o\|^2_{2,[\gamma>0]}\int_M \Big(\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^4 H^2\Big)\gamma^s d\mu \\
&+ c(c_{\gamma 1})^2\bigg(\int_{[\gamma>0]} \|A^o\|^2 d\mu\bigg)\bigg(\int_M \|A^o\|^4\gamma^{s-2} d\mu\bigg) \\
\le\, &c\|A^o\|^2_{2,[\gamma>0]}\int_M \Big(\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^4 H^2 + \|A^o\|^6\Big)\gamma^s d\mu \\
&+ c(c_{\gamma 1})^4\bigg(\int_{[\gamma>0]} \|A^o\|^2 d\mu\bigg)^2.
\end{aligned}$$

This estimates the first term. For the second we need to work a tiny bit harder. First we derive the formula

$$\bigg(\int_M \|\nabla A^o\|^2\gamma^{\frac{s}{2}} d\mu\bigg)^2 \le c\bigg(\int_M \|A^o\| \cdot \|\nabla_{(2)} A^o\|\gamma^{\frac{s}{2}} d\mu\bigg)^2 + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]}. \tag{113}$$

To show (113), we use integration by parts, Kato, Cauchy and then Jensen's inequality:

$$
\begin{aligned}
\int_M \|\nabla A^o\|^2 \gamma^{\frac{s}{2}} d\mu &\leq c \int_M \|A^o\| \cdot \|\nabla_{(2)} A^o\| \gamma^{\frac{s}{2}} d\mu \\
&\qquad + c \int_M \|\nabla A^o\| \cdot \|A^o\| \cdot \|\nabla\gamma\| \gamma^{\frac{s}{2}-1} d\mu \\
&\leq c \int_M \|A^o\| \cdot \|\nabla_{(2)} A^o\| \gamma^{\frac{s}{2}} d\mu \\
&\qquad + \frac{1}{\sqrt{2}} \int_M \|\nabla A^o\|^2 \gamma^{\frac{s}{2}} d\mu + c(c_{\gamma 1})^2 \|A^o\|^2_{2,[\gamma>0]} \\
\Longrightarrow \quad \left( \int_M \|\nabla A^o\|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 &\leq c \left( \int_M \|A^o\| \cdot \|\nabla_{(2)} A^o\| \gamma^{\frac{s}{2}} d\mu \right)^2 \\
&\qquad + \frac{1}{2} \left( \int_M \|\nabla A^o\|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 + c(c_{\gamma 1})^4 \|A^o\|^4_{2,[\gamma>0]},
\end{aligned}
$$

and absorbing on the left gives (113).

Note that we obtain as a corollary to the proof above

$$
\left( \int_M \|\nabla A^o\| \cdot \|A^o\| \cdot \|\nabla\gamma\| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \leq c \left( \int_M \|A^o\| \cdot \|\nabla_{(2)} A^o\| \gamma^{\frac{s}{2}} d\mu \right)^2 + c(c_{\gamma 1})^4 \|A^o\|^4_{2,[\gamma>0]}.
$$

Now we use the Michael-Simon Sobolev inequality with $u = \|\nabla A^o\| \cdot \|A\| \gamma^{\frac{s}{2}}$:

$$
\begin{aligned}
\int_M \|\nabla A^o\|^2 \|A^o\|^2 \gamma^s d\mu &\leq c \bigg[ \int_M \|\nabla_{(2)} A^o\| \cdot \|A^o\| \gamma^{\frac{s}{2}} d\mu + \int_M \|\nabla A^o\|^2 \gamma^{\frac{s}{2}} d\mu \\
&\qquad + \int_M \|\nabla A^o\| \cdot \|A^o\| \cdot \|\nabla\gamma\| \gamma^{\frac{s}{2}-1} d\mu \\
&\qquad + \int_M \|\nabla A^o\| \cdot \|A^o\| |H| \gamma^{\frac{s}{2}} d\mu \bigg]^2 \\
&\leq c \left( \int_M \|\nabla_{(2)} A^o\| \cdot \|A^o\| \gamma^{\frac{s}{2}} d\mu \right)^2 + c \left( \int_M \|\nabla A^o\|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
&\qquad + c \left( \int_M \|\nabla A^o\| \cdot \|A^o\| \|\nabla\gamma\| \gamma^{\frac{s}{2}-1} d\mu \right)^2 \\
&\qquad + c \left( \int_M \|\nabla A^o\| \cdot \|A^o\| \cdot |H| \gamma^{\frac{s}{2}} d\mu \right)^2 \\
&\leq c \left( \int_M \|\nabla_{(2)} A^o\| \cdot \|A^o\| \gamma^{\frac{s}{2}} d\mu \right)^2
\end{aligned}
$$

$$+ c\Big( \int_M \|\nabla A^o\| \cdot \|A^o\| \cdot \|\nabla\gamma\|\gamma^{\frac{s}{2}-1} d\mu\Big)^2$$

$$+ c\|A^o\|^2_{2,[\gamma>0]} \int_M \Big(\|\nabla A^o\|^2\|A\|^2\Big)\gamma^s d\mu + c\|A^o\|^4_{2,[\gamma>0]}$$

$$\le c\|A^o\|^2_{2,[\gamma>0]} \int_M \Big(\|\nabla_{(2)} A^o\|^2 + \|\nabla A^o\|^2\|A\|^2\Big)\gamma^s d\mu$$

$$+ c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]},$$

Where we used (113) and the corollary to (113) in the last two lines above.

Combining the two main inequalities proved above gives the first statement of the lemma.

We now turn to the $n = 3$ case. First observe that

$$\begin{aligned}
\int \|\nabla A^o\|^3\gamma^s d\mu &\le \int_M \Big(\langle A^o, \Delta A^o\rangle * \nabla A^o + A^o * \nabla A^o * \nabla\|\nabla A^o\|\Big)\gamma^s d\mu \\
&\quad + s\int_M \Big(A^o * \nabla A^o * \nabla A^o * \nabla\gamma\Big)\gamma^{s-1} d\mu \\
&\le 2\int_M \|A^o\|\cdot\|\nabla A^o\|\cdot\|\nabla_{(2)} A^o\|\gamma^s d\mu + sc_{\gamma 1}\int_M \Big(\|\nabla A^o\|^2\|A^o\|\Big)\gamma^{s-1} d\mu \\
&\le \frac{1}{4\delta}\int_M \|\nabla_{(2)} A^o\|^2\gamma^s d\mu + \delta\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\
&\quad + \frac{(sc_{\gamma 1})^3 4^2}{3}\int_M \|A^o\|^3\gamma^{2s-3} d\mu + \frac{1}{6}\int_M \|\nabla A^o\|^3\gamma^s d\mu \\
&\le \frac{1}{4\delta}\int_M \|\nabla_{(2)} A^o\|^2\gamma^s d\mu + \frac{\delta^3}{3}\int_M \|A^o\|^6\gamma^s d\mu + \frac{(sc_{\gamma 1})^3 4^2}{3}\int_M \|A^o\|^3\gamma^{2s-3} d\mu \\
&\quad + \frac{5}{6}\int_M \|\nabla A^o\|^3\gamma^s d\mu,
\end{aligned}$$

so

$$\int \|\nabla A^o\|^3\gamma^s d\mu \le \frac{3}{2\delta}\int_M \|\nabla_{(2)} A^o\|^2\gamma^s d\mu + 2\delta^3\int_M \|A^o\|^6\gamma^s d\mu + 2(sc_{\gamma 1})^3 4^2\int_{[\gamma>0]} \|A^o\|^3 d\mu,$$

for any $\delta \in (0, \infty)$.

Now we use the Michael-Simon Sobolev inequality with $u = \|A^o\|^4\gamma^{2s/3}$ to estimate

$$
\begin{aligned}
\Big(\int_M \|A^o\|^6\gamma^s d\mu\Big)^{\frac{2}{3}} &\le c\int_M \Big\|\nabla\Big(\|A^o\|^4\gamma^{2s/3}\Big)\Big\| d\mu + c\int_M |H|\cdot\|A^o\|^4\gamma^{2s/3}d\mu \\
&\le c\int_M \|A^o\|^3\|\nabla A^o\|\gamma^{2s/3}d\mu + c\int_M \|A^o\|^4\|\nabla\gamma\|\gamma^{2s/3-1}d\mu \\
&\quad + c\int_M \|A\|^{4/3}\|A^o\|^{11/3}\gamma^{2s/3}d\mu \\
&\le c\int_M \|A^o\|^3\|\nabla A^o\|\gamma^{2s/3}d\mu + c\int_M \|A^o\|^5\gamma^{4s/3-2}d\mu + c(c_{\gamma 1})^2\|A^o\|^3_{3,[\gamma>0]} \\
&\quad + c\Big(\int_{[\gamma>0]}\|A^o\|^3 d\mu\Big)^{\frac{1}{3}}\Big(\int_M \|A\|^2\|A^o\|^4\gamma^s d\mu\Big)^{\frac{2}{3}} \\
&\le c\int_M \|\nabla A^o\|^2\|A^o\|\gamma^s d\mu + c\int_M \|A^o\|^5\gamma^s d\mu + c(c_{\gamma 1})^2\|A^o\|^3_{3,[\gamma>0]} \\
&\quad + c\Big(\int_{[\gamma>0]}\|A^o\|^3 d\mu\Big)^{\frac{1}{3}}\Big(\int_M \|A\|^2\|A^o\|^4\gamma^s d\mu\Big)^{\frac{2}{3}} \\
&\le c\int_M \|\nabla A^o\|^2\|A^o\|\gamma^s d\mu + \Big(\int_M \|A^o\|^6\gamma^s d\mu\Big)^{\frac{2}{3}}\Big(\int_{[\gamma>0]}\|A^o\|^3\Big)^{\frac{1}{3}} \\
&\quad + c\Big(\int_{[\gamma>0]}\|A^o\|^3 d\mu\Big)^{\frac{1}{3}}\Big(\int_M \|A\|^2\|A^o\|^4\gamma^s d\mu\Big)^{\frac{2}{3}} + c(c_{\gamma 1})^2\|A^o\|^3_{3,[\gamma>0]},
\end{aligned}
$$

so

$$
\begin{aligned}
\int_M \|A^o\|^6\gamma^s d\mu &\le c\Big(\int_M \|\nabla A^o\|^2\|A^o\|\gamma^s d\mu\Big)^{\frac{3}{2}} + c\|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|A^o\|^6\gamma^s d\mu \\
&\quad + c\|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|A\|^2\|A^o\|^4\gamma^s d\mu + c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\le c\|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|\nabla A^o\|^3\gamma^s d\mu + c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\quad + c\|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \Big(\|A^o\|^6 + \|A\|^2\|A^o\|^4\Big)\gamma^s d\mu \\
&\le c\|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \Big(\|\nabla_{(2)}A^o\|^2 + \|A\|^2\|A^o\|^4 + \|A^o\|^6\Big)\gamma^s d\mu \\
&\quad + c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}.
\end{aligned}
$$

This estimates the first term. For the second, we can employ a more direct technique using our estimates above,

$$\begin{aligned}\int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu &\le c\int_M \|A^o\|^6\gamma^s d\mu + c\int_M \|\nabla A^o\|^3\gamma^s d\mu \\ &\le \delta\int_M \|\nabla_{(2)}A^o\|^2\gamma^s d\mu \\ &\quad + c_\delta\|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \Big(\|\nabla_{(2)}A^o\|^2 + \|A^o\|^6\Big)\gamma^s d\mu \\ &\quad + c_\delta(c_{\gamma 1})^3\Big(\|A^o\|^3_{3,[\gamma>0]} + \|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).\end{aligned}$$

This estimates the second term, and combining the two estimates above finishes the proof of the second statement. □

Remark. We have only included the above multiplicative Sobolev inequality for the $n = 2, 3$ cases due to the corresponding limitation on the Lifespan Theorem from the outset (see Chapter 3), which will also make itself known when we prove curvature and interior estimates (see Chapter 5). In fact, whenever one requires $L^\infty$ estimates, this limitation will impose itself. One should keep in mind however that for the sole purpose of the Gap Lemma, it is possible that the restriction on the dimension of $M$ is not required.

The following proposition is the final estimate, alluded to in our introduction, which will allow us to proceed with the proof of the gap lemma. The smallness assumption here is not mysterious: we will see that $\epsilon_0 = \frac{1}{c_1c_2}$ is good enough for $n = 2$, and $\epsilon_0 < \frac{1-\delta}{c_1c_2}$ for some $1 > \delta > 0$ when $n = 3$, where $c_1$ and $c_2$ are the same constants as before, from Lemma 4.5 and Lemma 4.6 respectively.

**Proposition 4.7.** *Suppose $n \in \{2,3\}$ and $\gamma$ is as in (25). Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersion with*

$$\int_{[\gamma>0]} \|A^o\|^n d\mu < \epsilon_0. \tag{114}$$

*Then we have*

$$\begin{aligned}
&\int_M \|\nabla_{(2)}A^o\|^2\gamma^4 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2\gamma^4 d\mu + \int_M \|A\|^4\|A^o\|^2\gamma^4 d\mu \\
&+ \int_M H^2\|\nabla H\|^2\gamma^4 d\mu \le c\int_M \Big(|F|^2 + |h|^2\Big)\gamma^4 d\mu + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]} \\
&\qquad + c(c_{\gamma 1})^4\|A\|^2_{2,[\gamma>0]} + (n-2)c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]},
\end{aligned}$$

*where $c = c(n)$.*

**Proof.** Recall Lemma 4.5 and Lemma 4.6. We combine these to obtain in the $n = 2$ case

$$\begin{aligned}
&\int_M \|\nabla_{(2)}A^o\|^2\gamma^4 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2\gamma^4 d\mu + \int_M \|A\|^4\|A^o\|^2\gamma^4 d\mu \\
&+ \int_M H^2\|\nabla H\|^2\gamma^4 d\mu \\
&\le c\int_M \Big(|F|^2 + |h|^2\Big)\gamma^4 d\mu + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]} + c(c_{\gamma 1})^4\|A\|^2_{2,[\gamma>0]} \\
&\qquad + c_1c_2\|A^o\|^2_{2,[\gamma>0]}\int_M \Big(\|\nabla_{(2)}A^o\|^2 + \|\nabla A^o\|^2\|A\|^2 + \|A\|^2\|A^o\|^4\Big)\gamma^4 d\mu \\
\Longrightarrow &\Big(1 - c_1c_2\|A^o\|^2_{2,[\gamma>0]}\Big)\Bigg[\int_M \|\nabla_{(2)}A^o\|^2\gamma^4 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2\gamma^4 d\mu \\
&\qquad + \int_M \|A\|^4\|A^o\|^2\gamma^4 d\mu\Bigg] + \int_M H^2\|\nabla H\|^2\gamma^4 d\mu \\
&\le c\int_M \Big(|F|^2 + |h|^2\Big)\gamma^4 d\mu + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]} + c(c_{\gamma 1})^4\|A\|^2_{2,[\gamma>0]}.
\end{aligned}$$

Therefore, with $\|A^o\|^2_{2,[\gamma>0]} < \epsilon_0 < (c_1c_2)^{-1}$ we can divide by $(1-\epsilon_0)$ to conclude the result.

For the $n = 3$ case we must proceed slightly differently. We have

$$
\begin{aligned}
&\int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu + \int_M \|A\|^2 \|\nabla A^o\|^2 \gamma^4 d\mu + \int_M \|A\|^4 \|A^o\|^2 \gamma^4 d\mu + \int_M H^2 \|\nabla H\|^2 \gamma^4 d\mu \\
&\quad \le c \int_M \Big( |F|^2 + |h|^2 \Big) \gamma^4 d\mu + c(c_{\gamma 1})^4 \|A^o\|^4_{2,[\gamma>0]} + c(c_{\gamma 1})^4 \|A\|^2_{2,[\gamma>0]} \\
&\qquad + \delta \int_M \|\nabla_{(2)} A^o\|^2 \gamma^s d\mu + c(c_{\gamma 1})^3 \|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\qquad + c_1 c_2 \|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]} \int_M \Big( \|\nabla_{(2)} A^o\|^2 + \|\nabla A^o\|^2 \|A\|^2 + \|A\|^2 \|A^o\|^4 \Big) \gamma^4 d\mu \\
&\Longrightarrow \Big(1 - \delta - c_1 c_2 \|A^o\|^{\frac{3}{2}}_{3,[\gamma>0]}\Big) \bigg[ \int_M \|\nabla_{(2)} A^o\|^2 \gamma^4 d\mu + \int_M \|A\|^2 \|\nabla A^o\|^2 \gamma^4 d\mu \\
&\qquad\qquad + \int_M \|A\|^4 \|A^o\|^2 \gamma^4 d\mu \bigg] + \int_M H^2 \|\nabla H\|^2 \gamma^4 d\mu \\
&\quad \le c \int_M \Big( |F|^2 + |h|^2 \Big) \gamma^4 d\mu + c(c_{\gamma 1})^4 \|A^o\|^4_{2,[\gamma>0]} + c(c_{\gamma 1})^3 \|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\qquad + c(c_{\gamma 1})^4 \|A\|^2_{2,[\gamma>0]},
\end{aligned}
$$

and the second statement follows choosing $\delta < 1$ and requiring

$$
\epsilon_0 < \frac{1 - \delta}{c_1 c_2}.
$$

$\square$

Remark. The extra term appearing in the $n = 3$ case,

$$
(c_{\gamma 1})^3 \|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]},
$$

will have no effect on our result. This is because we will assume $\|A^o\|$ is *globally* small, and then $(c_{\gamma 1})$ will dominate the integral.

We can now give our main argument for this chapter.

## 4. Proof of the gap lemma

The first point to note is that if for some compact hypersurface the principal curvatures are all equal, then the hypersurface must be an immersed plane (where they are all zero) or an immersed sphere (where they are all positive). We will show that this must be the case if the hypothesis of the gap lemma holds: that $f : M^n \longrightarrow \mathbb{R}^{n+1}$ is an immersion which is stationary under (CSD) flow, possesses small total tracefree curvature and the constraint function $h$ obeys a growth condition.

Let $p \in M$. We set the cutoff function $\gamma$ to be such that

$$\gamma(p) = \varphi\left(\frac{1}{\rho}|f(p)|\right),$$

where $\varphi \in C^1(\mathbb{R})$ and

$$\begin{aligned}
&\varphi(s) = 1 \text{ for } s < \tfrac{1}{2},\\
&\varphi(s) = 0 \text{ for } s \geq 1, \text{ and}\\
&\varphi(s) \geq 0 \text{ for any } s.
\end{aligned}$$

Then $c_{\gamma 1} = \frac{c}{\rho}$. Recall that in our estimates we never use the second derivative of $\gamma$. Taking $\rho \nearrow \infty$ in Proposition 4.7, we have

$$\begin{aligned}
\int_M &\|\nabla_{(2)}A^o\|^2 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2 d\mu + \int_M \|A\|^4\|A^o\|^2 d\mu + \int_M H^2\|\nabla H\|^2 d\mu\\
&\leq c \liminf_{\rho\to\infty} \int_M h^2\gamma^4 d\mu + c\liminf_{\rho\to\infty}(c_{\gamma 1})^4\|A\|^4_{2,[\gamma>0]}\\
&\quad + c\liminf_{\rho\to\infty}(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]} + (n-2)\liminf_{\rho\to\infty}(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]},\\
&\leq c\liminf_{\rho\to\infty}\int_{f^{-1}(B_\rho(0))} h^2 d\mu + c\liminf_{\rho\to\infty}\frac{1}{\rho^4}\|A\|^4_{2,f^{-1}(B_\rho(0))}\\
&\leq ch^2|M| + c\liminf_{\rho\to\infty}\frac{1}{\rho^4}\|A\|^4_{2,f^{-1}(B_\rho(0))}.
\end{aligned}$$

Note that the terms involving $A^o$ vanished due to the global bounded tracefree curvature assumption. We now use the remaining assumptions. Recall

$$\liminf_{\rho\to\infty}\frac{1}{\rho^4}\int_{f^{-1}(B_\rho(0))}\|A\|^2 d\mu = 0, \text{ and}$$

$$h^2|M| \leq \frac{1}{kc}\int_M \|\nabla_{(2)}A^o\|^2 + \|A\|^2\|\nabla A^o\|^2 + H^2\|\nabla H\|^2 + \|A\|^4\|A^o\|^2 d\mu.$$

Note that the first equation above is automatic if $M$ is compact, and assumed to be true otherwise. Absorbing $h^2|M|$ on the left,

$$\int_M \|\nabla_{(2)}A^o\|^2 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2 d\mu + \int_M \|A\|^4\|A^o\|^2 d\mu + \int_M H^2\|\nabla H\|^2 d\mu$$
$$= 0.$$

This gives us a lot of information about the kind of stationary surface we have. In particular, we note that

$$\|A\| = 0,\ \|A^o\| = 0, \text{ or both.}$$

In the first and third instance, we have an immersed plane, as this implies all the curvatures are zero. In the second instance, this implies all the curvatures are equal and so $M$ could also be an immersed sphere. For the compact case, we can of course exclude immersed planes. Therefore for the compact case, $f : M \to S^n$, where $S^n$ is an immersed sphere, and in the proper immersion case $S^n$ is a union of immersed planes and spheres. Note also that in either case, $\Delta H \equiv 0$ on $M$ and also $h = 0$. This shows (GL) holds.

We finish by strengthening the statement from immersed to embedded. Let $S^n$ be the union of immersed planes and spheres from above. Then, since $M$ is geodesically complete, $f : (M, g) \to S^n$ is a global isometry, and so $f(\cdot)$ is in fact an embedding.

CHAPTER 5

# Curvature and interior estimates for constrained surface diffusion flows

## 1. Introduction.

We have two main results for this chapter. First, we prove pointwise curvature estimates where the speed of the flow appears on the right hand side.

THEOREM (Partial curvature estimates). *Suppose $n \in \{2,3\}$, $\rho > 0$, and let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow where for any $x \in \mathbb{R}^{n+1}$,*

$$\|A^o\|^n_{n,f^{-1}(B_\rho(x))} \le \|A^o\|^n_{n,f^{-1}(B_\rho(x_0))} < \epsilon_0,$$

*where $\epsilon_0 > 0$ is as in Proposition 4.7. Further, assume that the constraint function $h$ satisfies* (A2)*, and in the case where $n = 3$ that* (AB) *is satisfied. Then*

$$\|A^o\|^2_{\infty,f^{-1}(B_{\rho/2}(x_0))}$$
$$\le c\|A^o\|^{\frac{4-n}{2}}_{2,f^{-1}(B_\rho(x_0))}\Big[\|F\|^{\frac{n}{2}}_{2,f^{-1}(B_\rho(x_0))} + \frac{1}{\rho^n}\|A\|^{\frac{n}{2}}_{2,f^{-1}(B_\rho(x_0))} + (c_h)^{\frac{n}{4}} + (n-2)\epsilon_0\Big],$$

*where $c = c(n, \epsilon_0, C_{AB})$.*

We also obtain an analogous statement for the full curvature tensor, Corollary 5.9. Our method of proof here is a variation on our fundamental mode of argument from Chapter 3 on the Lifespan Theorem. Since our overall focus is on showing that for certain initial manifolds a class of constrained surface diffusion flows both exist for all time and converge to a sphere, and the tracefree second fundamental form measures in some sense the pointwise difference from $M$ to a sphere, we will be enhancing the role of the tracefree second fundamental form in our analysis. This altered focus leads to the $L^\infty$ estimates for the tracefree second fundamental form above.

One result obtained 'along the way' is Proposition 5.10, and there one can see the crucial role played by the tracefree second fundamental form. This will be useful in the future when we turn to analysis of the asymptotic behaviour of constrained surface diffusion flows.

An interesting feature of the proof of the curvature estimates above is the usage of a different growth condition, (A2). This appears more restrictive than the previous growth condition, (GC). However since we only assume smallness of the tracefree second fundamental form (as opposed to the full second fundamental form), we cannot use (GC). This is detailed in Theorem 5.6.

Our second main result in this chapter is the following theorem.

Theorem (Interior estimates). *Suppose $n \in \{2,3\}$ and $f : M^n \times (0,T^*] \to \mathbb{R}^{n+1}$ is a* (CSD) *flow with $h$ satisfying the conditions of the Lifespan Theorem. Further assume that*

$$\sup_{t\in(0,T^*]}\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu \le \epsilon(x),$$

*where $T^* \le c(n)\rho^4$ and $m = m(h)$ is as in the Lifespan Theorem. Then for any $k \in \mathbb{N}_0$ we have at time $t \in (0,T^*]$ the estimates*

$$\|\nabla_{(k)}A\|_{2,f^{-1}(B_{\rho/2}(x))} \le c(k)\sqrt{\epsilon(x)}t^{-\frac{k}{4}}$$

$$\|\nabla_{(k)}A\|_{\infty,f^{-1}(B_{\rho/2}(x))} \le c(k)\sqrt{\epsilon(x)}t^{-\frac{k+1}{4}},$$

*where* $c_k = c_k\Big(k, n, \rho, T^*, \|\nabla_{(k)}A\|_{2,f^{-1}(B_\rho(x_0))}|_{t=0}\Big)$.

This is in essence a sharpening (but not a sharp version) of Proposition 3.26 from Chapter 3 on the Lifespan Theorem. The proof involves the use of a time-based localisation function and is given in Section 4.

As with each of our previous chapters, the interior estimates are also in preparation for asymptotic analysis. In this case, we will use them on parabolic cylinders to ensure the existence of a blow up immersion with certain properties. Apart from this, they are also of independent interest. It would be particularly interesting to determine *sharp* constants in the interior estimates, however to this author's knowledge the distinct lack of example evolutions for even surface diffusion flow makes this very difficult.

This chapter is organised as follows. Section 2 is devoted to using elementary evolution equations to prove integral estimates. These are similar to those in Chapter 4, where the speed of the flow is involved in the resulting estimates. The difference here however is that the speed of the flow is kept on the left hand side, as a 'good' term. Section 3 incorporates small curvature assumptions into the integral estimates from Section 2. We also prove some Sobolev inequalities and conclude the pointwise curvature estimates in this section. Section 4 is devoted to proving the interior estimates.

## 2. Energy based integral estimates.

We begin by proving elementary evolution equations. Unlike Chapter 3 however, our focus here is on deriving estimates where the speed of the flow $F = \Delta H + h$ is on the left hand side.

LEMMA 5.1. *Let $\gamma$ be a cut off function as in* (25)*. Then the following equalities hold for a* (CSD) *flow $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$.*

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + \int_M F^2\gamma^s d\mu = h^2|M_t|_{[\gamma>0]} &+ h\int_M (\Delta H)\gamma^s d\mu \\ &+ \frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu - \int_M FH\|A^o\|^2\gamma^s d\mu \\ &+ \int_M H\left\langle \nabla F, \nabla\gamma^s\right\rangle - F\left\langle \nabla H, \nabla\gamma^s\right\rangle d\mu,\end{aligned}$$

*and*

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu + \int_M F^2\gamma^s d\mu = h^2|M_t|_{[\gamma>0]} &+ h\int_M (\Delta H)\gamma^s d\mu \\ &+ \int_M \|A^o\|^2(\partial_t\gamma^s)d\mu + \int_M FH\|A^o\|^2\gamma^s d\mu \\ &- 2\int_M F(A^o)^i_j(A^o)^j_k(A^o)^k_i\gamma^s d\mu \\ &+ 2\int_M \left\langle A^o, \nabla F\nabla\gamma^s\right\rangle H + F\left\langle \nabla^* A^o, \nabla\gamma^s\right\rangle d\mu.\end{aligned}$$

PROOF. This follows from computing the evolution of the integral of squared mean curvature, and squared tracefree curvature respectively. We then integrate by parts twice to obtain the good term, an integral of the speed $F$ squared, on the left.

Recall the evolution equations from Lemma 3.9. Using these,

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu &= \int_M H(-\Delta F - \|A\|^2 F)\gamma^s d\mu + \int_M \frac{1}{2}H^2(\partial_t\gamma^s)d\mu \\ &\quad + \int_M \frac{1}{2}H^3 F\gamma^s d\mu \\ &= -\int_M H(\Delta F)\gamma^s d\mu - \int_M FH\|A^o\|^2\gamma^s d\mu\end{aligned}$$

$$+\frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu,$$

now using $(\Delta H)(\Delta H + h) = F^2 - h\Delta H - h^2$ and integration by parts twice,

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + \int_M F^2\gamma^s d\mu \le h\int_M(\Delta H)\gamma^s d\mu + h^2|M_t|_{[\gamma>0]} - \int_M FH\|A^o\|^2\gamma^s d\mu\\ + \frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu + \int_M H\left\langle\nabla F,\nabla\gamma^s\right\rangle - F\left\langle\nabla H,\nabla\gamma^s\right\rangle d\mu.\end{aligned}$$

This proves the first statement. The second is similar:

$$\begin{aligned}\frac{d}{dt}\int_M\|A^o\|^2\gamma^s d\mu &= 2\int_M\Big\langle A^o, -S^o(\nabla_{(2)}F) + F\Big[A_i^pA_{pj} + \tfrac{1}{2}g_{ij}\|A\|^2 - HA_{ij}\Big]\Big\rangle\gamma^s d\mu\\ &\quad + \int_M\|A^o\|^2HF\gamma^s d\mu + \int_M\|A^o\|^2(\partial_t\gamma^s)d\mu\\ &= -2\int_M\Big\langle A^o,\nabla_{(2)}F\Big\rangle\gamma^s d\mu - 2\int_M F(A^o)_j^i(A^o)_k^j(A^o)_i^k\gamma^s d\mu\\ &\quad + \int_M\|A^o\|^2HF\gamma^s d\mu + \int_M\|A^o\|^2(\partial_t\gamma^s)d\mu,\end{aligned}$$

since $\langle A^o, A_i^pA_{pj} - HA_{ij}\rangle = (A^o)_j^i(A^o)_k^j(A^o)_i^k$. Using $\nabla_{(2)}^*A = \frac{1}{2}\Delta H$, $(\Delta H)(\Delta H + h) = F^2 - h\Delta H - h^2$ and integration by parts twice we obtain

$$\begin{aligned}\frac{d}{dt}\int_M\|A^o\|^2\gamma^s d\mu + \int_M F^2\gamma^s d\mu \le h^2|M_t|_{[\gamma>0]} + h\int_M(\Delta H)\gamma^s d\mu\\ + \int_M\|A^o\|^2(\partial_t\gamma^s)d\mu + \int_M FH\|A^o\|^2\gamma^s d\mu\\ - 2\int_M F(A^o)_j^i(A^o)_k^j(A^o)_i^k\gamma^s d\mu\\ + 2\int_M(A^o)_{ij}(\nabla^iF)(\nabla^j\gamma^s)H + F\left\langle\nabla^*A^o,\nabla\gamma^s\right\rangle d\mu.\end{aligned}$$

This proves the second statement, and so we are finished. $\square$

For our second lemma, we estimate some potentially troubling terms from Lemma 5.1 above. Our motivation here is maximising the utility of the estimate in Proposition 4.7 and the multiplicative Sobolev inequality Lemma 4.6.

LEMMA 5.2. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow, $\gamma$ as in* (25) *and $\delta_1, \delta_2, \delta_3$ be fixed positive numbers. Then the following inequalities hold:*

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + \int_M F^2\gamma^s d\mu &\le \frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu + 2h\int_M \left\langle A^o, \nabla_{(2)}\gamma^s\right\rangle d\mu \\ &+ c_{\delta_2}h^2|M_t|_{[\gamma>0]} + \delta_3\int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu \\ &+ c_{\delta_3}\int_M \|\nabla A^o\|^2\|A^o\|^2\gamma^s d\mu + \delta_2\int_M \|A\|^4\|A^o\|^2\gamma^s d\mu \\ &+ c_{\delta_2}(c_{\gamma 1})^4\int_M \|A^o\|^2\gamma^{s-4}d\mu \\ &+ \int_M H\left\langle\nabla F, \nabla\gamma^s\right\rangle - F\left\langle\nabla H, \nabla\gamma^s\right\rangle d\mu,\end{aligned}$$

*and*

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu + (1-\delta_1)\int_M F^2\gamma^s d\mu &\le \int_M \|A^o\|^2(\partial_t\gamma^s)d\mu + 2h\int_M \left\langle A^o, \nabla_{(2)}\gamma^s\right\rangle d\mu \\ &+ c_{\delta_2}h^2|M_t|_{[\gamma>0]} + \delta_3\int_M \|\nabla A^o\|^2H^2\gamma^s d\mu + c_{\delta_1}\int_M \|A^o\|^6\gamma^s d\mu \\ &+ c_{\delta_3}\int_M \|\nabla A^o\|^2\|A^o\|^2\gamma^s d\mu + \delta_2\int_M \|A\|^4\|A^o\|^2\gamma^s d\mu \\ &+ c_{\delta_2}(c_{\gamma 1})^4\int_M \|A^o\|^2\gamma^{s-4}d\mu \\ &+ 2\int_M (A^o)_{ij}(\nabla^i F)(\nabla^j\gamma^s)H + F\left\langle\nabla^* A^o, \nabla\gamma^s\right\rangle d\mu,\end{aligned}$$

*where $c_{\delta_i} = c_{\delta_i}(s, n, \delta_i)$.*

PROOF. We must estimate the terms $\int_M FH\|A^o\|^2\gamma^s d\mu$ and $\int_M F(A^o)^i_j(A^o)^j_k(A^o)^k_i\gamma^s d\mu$ from Lemma 5.1. Using integration by parts and the identity $\nabla H = \frac{n}{n-1}\nabla^* A^o$,

$$\begin{aligned}\int_M (\Delta H)H\|A^o\|^2\gamma^s d\mu &= -\int_M \left\langle\nabla H, \nabla(H\|A^o\|^2\gamma^s)\right\rangle d\mu \\ &= -\int_M \|\nabla H\|^2\|A^o\|^2\gamma^s d\mu - \int_M H\left\langle\nabla H, \nabla(\|A^o\|^2\gamma^s)\right\rangle d\mu \\ &= -\int_M \|\nabla H\|^2\|A^o\|^2\gamma^s d\mu - s\int_M H\|A^o\|^2\left\langle\nabla H, \nabla\gamma\right\rangle\gamma^{s-1}d\mu \\ &\quad - 2\int_M H\|A^o\|\left\langle\nabla H, \nabla\|A^o\|\right\rangle\gamma^s d\mu\end{aligned}$$

$$\leq \frac{4n^2(\delta_3^{-1}-1)}{(n-1)^2}\int_M \|\nabla^* A^o\|^2\|A^o\|^2\gamma^s d\mu + \delta_3\int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu$$
$$+ s^2(c_{\gamma 1})^2\int_M H^2\|A^o\|^2\gamma^{s-2}d\mu + \frac{1}{4}\int_M \|\nabla H\|^2\|A^o\|^2\gamma^s d\mu,$$

so

$$\int_M FH\|A^o\|^2\gamma^s d\mu \leq c_{\delta_3}\int_M \|\nabla A^o\|^2\|A^o\|^2\gamma^s d\mu + \delta_3\int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu$$
$$+ c_{\delta_2}(c_{\gamma 1})^4\int_M \|A^o\|^2\gamma^{s-4}d\mu + \delta_2\int_M \|A\|^4\|A^o\|^2\gamma^s d\mu$$
$$+ h\int_M H\|A^o\|^2\gamma^s d\mu.$$

Also,

$$h\int_M H\|A^o\|^2\gamma^s d\mu \leq c_{\delta_3}h^2|M_t|_{[\gamma>0]} + \delta_3\int_M \|A\|^4\|A^o\|^2\gamma^s d\mu.$$

This estimates the first integral. The second is easily estimated by

$$\int_M F(A^o * A^o * A^o)\gamma^s d\mu \leq \delta_1\int_M F^2\gamma^s d\mu + c_{\delta_1}\int_M \|A^o\|^6\gamma^s d\mu.$$

Finally, note that

$$h\int_M (\Delta H)\gamma^s d\mu = 2h\int_M (\nabla^*_{(2)}A^o)\gamma^s d\mu = 2h\int_M \left\langle A^o, \nabla_{(2)}\gamma^s\right\rangle d\mu.$$

Combining these estimates with Lemma 5.1 earlier finishes the proof. □

The third lemma below is an estimate which deals with the extraneous terms from the derivatives of the cutoff function $\gamma$, resulting from our extensive usage of integration by parts. These are by nature 'good' terms, expected in any localised integral estimates. We will also group the terms and perform some other minor alterations to those in Lemma 5.2 above. We then will finally be able to apply the estimate in Proposition 4.7 and the multiplicative Sobolev inequality Lemma 4.6 to conclude our first 'small energy' result for this chapter.

LEMMA 5.3. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow, $\gamma$ as in* (25) *and $\delta_1, \delta_2, \delta_3$ be fixed positive numbers. Then the following inequalities hold:*

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu + (1-\delta_1)\int_M F^2\gamma^s d\mu &\le ch^2|M_t|_{[\gamma>0]} + \delta_2\int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu \\ &+ \delta_3\int_M \|A\|^4\|A^o\|^2\gamma^s d\mu + c\int_M (\|A^o\|^6 + \|\nabla A^o\|^2\|A^o\|^2)\gamma^s d\mu \\ &+ c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\int_M \|A^o\|^2\gamma^{s-4}d\mu,\end{aligned}$$

*and*

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + (1-\delta_1)\int_M F^2\gamma^s d\mu &\le ch^2|M_t|_{[\gamma>0]} + c\int_M (\|A\|^2\|\nabla A\|^2 + \|A\|^6)\gamma^s d\mu \\ &+ c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\int_M \|A\|^2\gamma^{s-4}d\mu,\end{aligned}$$

*where $c = c(s, n, \delta_i)$.*

PROOF. We first compute, using integration by parts and the definition of $\gamma$,

$$\begin{aligned}2h\int_M \Big\langle A^o, \nabla_{(2)}\gamma^s\Big\rangle d\mu &+ 2\int_M (A^o)_{ij}(\nabla^i\gamma^s)(\nabla^j F)d\mu + 2\int_M F\left\langle \nabla^* A^o, \nabla\gamma^s\right\rangle d\mu \\ &= 2h\int_M \Big\langle A^o, \nabla_{(2)}\gamma^s\Big\rangle d\mu - 2\int_M (\Delta H + h)\Big\langle A^o, \nabla_{(2)}\gamma^s\Big\rangle d\mu \\ &\quad (-2+2)\int_M F\left\langle \nabla^* A^o, \nabla\gamma^s\right\rangle d\mu \\ &= -2\int_M (\Delta H)\Big\langle A^o, \nabla_{(2)}\gamma^s\Big\rangle d\mu \\ &= -2s\int_M (\Delta H)\Big\langle A^o, (D^2\tilde\gamma\circ f)g + (D\tilde\gamma\circ f)A\Big\rangle\gamma^{s-1}d\mu \\ &\quad + 2s(s-1)\int_M (\Delta H)\left\langle A^o, (\nabla\gamma)(\nabla\gamma)\right\rangle\gamma^{s-2}d\mu.\end{aligned}$$

Note that $\langle A^o, (D^2\tilde\gamma\circ f)g\rangle = \sum_{i,j=1}^n A^o_{ij}(D_{ij}\tilde\gamma)g_{ij}$, and this is not in general zero, as each term in the sum is scaled by the second derivatives of $\tilde\gamma$. Continuing,

$$\begin{aligned}2h\int_M \Big\langle A^o, \nabla_{(2)}\gamma^s\Big\rangle d\mu &+ 2\int_M (A^o)_{ij}(\nabla^i\gamma^s)(\nabla^j F)d\mu + 2\int_M F\left\langle \nabla^* A^o, \nabla\gamma^s\right\rangle d\mu \\ &\le 2s\int_M |\Delta H|\Big(\|A^o\|(c_{\tilde\gamma 2}) + \|A^o\|^2(c_{\tilde\gamma 1})\Big)\gamma^{s-1}d\mu \\ &\quad + 2s(s-1)\int_M |\Delta H|\ \|A^o\|(c_{\gamma 1})^2\gamma^{s-2}d\mu\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\delta}{4}\int_M |\Delta H|^2\gamma^s d\mu + \frac{8s^2}{\delta}(c_{\tilde\gamma 2})^2\int_M \|A^o\|^2\gamma^{s-2}d\mu \\
&\quad + \frac{8s^2}{\delta}(c_{\tilde\gamma 1})^2\int_M \|A^o\|^4\gamma^{s-2}d\mu \\
&\quad + 2s(s-1)\int_M |\Delta H|\ \|A^o\|(c_{\gamma 1})^2\gamma^{s-2}d\mu \\
&\le \frac{\delta}{2}\int_M |\Delta H|^2\gamma^s d\mu + \frac{8s^2}{\delta}(c_{\tilde\gamma 1})^2\int_M \|A^o\|^4\gamma^{s-2}d\mu \\
&\quad + \delta^{-1}\Big(8s^2 + 8s^2(s-1)^2\Big)\Big[(c_{\tilde\gamma 2})^2 + (c_{\gamma 1})^4\Big]\int_M \|A^o\|^2\gamma^{s-4}d\mu \\
&\le \frac{\delta}{2}\int_M |\Delta H|^2\gamma^s d\mu + \int_M \|A^o\|^6\gamma^s d\mu \\
&\quad + \delta^{-2}\Big(8s^2 + 8s^2(s-1)^2 + 8s^4\Big)\Big[(c_{\tilde\gamma 2})^2 + (c_{\gamma 1})^4 + (c_{\tilde\gamma 1})^4\Big]\int_M \|A^o\|^2\gamma^{s-4}d\mu.
\end{aligned}
$$

Of course, the constant in front of the $\|A^o\|^2_{2,[\gamma>0]}$ term is far from optimal. The representation above is just personal preference. The point is that $\delta$ is a fixed positive number (smaller than 1), and so the coefficient is a function of $s$ and the constants in (25) only, and can be written in the form indicated in the statement of the lemma.

Finally we estimate the time derivative of $\gamma$ as follows.

$$
\begin{aligned}
\int_M \|A^o\|^2(\partial_t\gamma^s)d\mu &= s\int_M \|A^o\|^2(\gamma^{s-1}\left\langle D\tilde\gamma, \nu\right\rangle F)d\mu \\
&\le \frac{\delta}{4}\int_M F^2\gamma^s d\mu + \delta^{-1}s^2(c_{\tilde\gamma 1})^2\int_M \|A^o\|^4\gamma^{s-2}d\mu \\
&\le \frac{\delta}{4}\int_M F^2\gamma^s d\mu + c_\delta(c_{\tilde\gamma 1})^4\int_M \|A^o\|^2\gamma^{s-4}d\mu + \int_M \|A^o\|^6 d\mu.
\end{aligned}
$$

Combining these inequalities with lemma 5.2 and absorbing finishes the proof for $\int_M \|A^o\|^2\gamma^s d\mu$. For $\int_M H^2\gamma^s d\mu$, the proof is identical, except for estimating $H$ and $A^o$ by $A$ where appropriate. □

We finish this section by proving an estimate which will be useful in the following chapter.

LEMMA 5.4. *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is a* (CSD) *flow, $\gamma$ as in* (25) *and $s \ge 4$. Then there exist absolute constants $\epsilon_0, c_e > 0$ such that if*

$$\int_{[\gamma>0]} \|A^o\|^2 d\mu < \epsilon_0$$

*and if*

$$-h\int_M H\|A^o\|^2\gamma^s d\mu \le \frac{c_e}{2}\int_M \|\nabla_{(2)}H\|^2 + \|\nabla A^o\|^2 H^2 + H^4\|A^o\|^2 d\mu$$

*then we have*

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu &+ c_e\int_M \Big(\|\nabla_{(2)}H\|^2 + \|\nabla A^o\|^2 H^2 + H^4\|A^o\|^2\Big)\gamma^s d\mu\\ &- \frac{c_e}{2}\int_M \|\nabla_{(2)}H\|^2 + \|\nabla A^o\|^2 H^2 + H^4\|A^o\|^2 d\mu\\ &\le c\Big[(c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]},\end{aligned} \tag{115}$$

*for an absolute constant $\infty > c > 0$ depending only on $s$.*

PROOF. We proceed similarly to the proof of Lemma 5.1. Differentiating,

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu &= \int_M H(-\Delta F - \|A\|^2 F)\gamma^s d\mu + \int_M \frac{1}{2}H^2(\partial_t\gamma^s)d\mu + \int_M \frac{1}{2}H^3 F\gamma^s d\mu\\ &= -\int_M H(\Delta F)\gamma^s d\mu - \int_M FH\|A^o\|^2\gamma^s d\mu + \frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu\\ &= -\int_M H(\Delta^2 H)\gamma^s d\mu - \int_M (\Delta H)H\|A^o\|^2\gamma^s d\mu - h\int_M H\|A^o\|^2\gamma^s d\mu\\ &\quad + \frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu.\end{aligned}$$

Integrating by parts twice, we have

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu &+ \int_M |\Delta H|^2\gamma^s d\mu\\ &= s\int_M H\left\langle \nabla\Delta H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu - s\int_M (\Delta H)\left\langle \nabla H, \nabla\gamma\right\rangle\gamma^{s-1}d\mu\\ &\quad - \int_M (\Delta H)H\|A^o\|^2\gamma^s d\mu - h\int_M H\|A^o\|^2\gamma^s d\mu + \frac{1}{2}\int_M H^2(\partial_t\gamma^s)d\mu.\end{aligned} \tag{116}$$

We will estimate each term in turn. First we deal with the spatial derivatives of $\gamma$.

$$
\begin{aligned}
s\int_M & H\left\langle \nabla\Delta H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu - s\int_M (\Delta H)\left\langle \nabla H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu \\
&\le s\int_M H(\Delta H)(\Delta\gamma)\gamma^{s-1}d\mu + s(s-1)\int_M H(\Delta H)\|\nabla\gamma\|^2\gamma^{s-2}d\mu \\
&\quad + s\int_M (\Delta H)\left\langle \nabla H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu \\
&\quad + \delta_1\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2}{4\delta_1}(c_{\gamma 1})^2\int_M \|\nabla H\|^2\gamma^{s-2}d\mu \\
&\le s\int_M H(\Delta H)(\Delta\gamma)\gamma^{s-1}d\mu + s(s-1)\int_M H(\Delta H)\|\nabla\gamma\|^2\gamma^{s-2}d\mu \\
&\quad + 2\delta_1\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2}{2\delta_1}(c_{\gamma 1})^2\int_M \|\nabla H\|^2\gamma^{s-2}d\mu \\
&\le s\int_M H(\Delta H)(\Delta\gamma)\gamma^{s-1}d\mu \\
&\quad + \delta_2\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2(s-1)^2}{4\delta_2}(c_{\gamma 1})^4\|H\|^2_{2,[\gamma>0]} \\
&\quad + 2\delta_1\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2}{2\delta_1}(c_{\gamma 1})^2\int_M \|\nabla H\|^2\gamma^{s-2}d\mu \\
&\le s(c_{\gamma 2})\int_M H(\Delta H)(1+|H|)\gamma^{s-1}d\mu \\
&\quad + \delta_2\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2(s-1)^2}{4\delta_2}(c_{\gamma 1})^4\|H\|^2_{2,[\gamma>0]} \\
&\quad + 2\delta_1\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2}{2\delta_1}(c_{\gamma 1})^2\int_M \|\nabla H\|^2\gamma^{s-2}d\mu \\
&\le \delta_3\int_M |\Delta H|\gamma^s d\mu + \frac{s^2}{4\delta_3}(c_{\gamma 2})^2\int_M H^2\gamma^{s-2}d\mu + s(c_{\gamma 2})\int_M H^2(\Delta H)\gamma^{s-1}d\mu \\
&\quad + \delta_2\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2(s-1)^2}{4\delta_2}(c_{\gamma 1})^4\|H\|^2_{2,[\gamma>0]} \\
&\quad + 2\delta_1\int_M |\Delta H|^2\gamma^s d\mu + \frac{s^2}{2\delta_1}(c_{\gamma 1})^2\int_M \|\nabla H\|^2\gamma^{s-2}d\mu.
\end{aligned}
$$

For the third integral we integrate by parts to obtain

$$
\begin{aligned}
(c_{\gamma 2})\int_M H^2(\Delta H)\gamma^{s-1}d\mu &\le \delta_4\int_M H^2\|\nabla H\|^2\gamma^s d\mu + \frac{1}{2\delta_4}(c_{\gamma 1}c_{\gamma 2})^2(s-1)^2\|H\|^2_{2,[\gamma>0]} \\
&\quad + \frac{1}{2\delta_4}(c_{\gamma 2})^2\int_M \|\nabla H\|^2\gamma^{s-2}d\mu.
\end{aligned}
$$

Combining this with our previous estimate we have

$$
\begin{aligned}
s&\int_M H\left\langle \nabla\Delta H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu - s\int_M (\Delta H)\left\langle \nabla H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu\\
&\le (2\delta_1+\delta_2+\delta_3)\int_M |\Delta H|\gamma^s d\mu + \delta_4\int_M H^2\|\nabla H\|^2\gamma^s d\mu\\
&\quad+\left(\frac{s^2(s-1)^2}{4\delta_2}(c_{\gamma 1})^4+\frac{s^2}{8\delta_3}(c_{\gamma 2})^2+\frac{1}{2\delta_4}(c_{\gamma 1}c_{\gamma 2})^2(s-1)^2\right)\|H\|^2_{2,[\gamma>0]}\\
&\quad+\left(\frac{s^2}{2\delta_1}(c_{\gamma 1})^2+\frac{1}{2\delta_4}(c_{\gamma 2})^2\right)\int_M\|\nabla H\|^2\gamma^{s-2}d\mu.
\end{aligned}
$$

Now using the inequality

$$
\int_M\|\nabla H\|^2\gamma^{s-2}d\mu \le 2\int_M |H|\;|\Delta H|\gamma^{s-2}d\mu + (s-2)^2(c_{\gamma 1})^2\int_M H^2\gamma^{s-4}d\mu
$$

we obtain

$$
\begin{aligned}
s&\int_M H\left\langle \nabla\Delta H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu - s\int_M (\Delta H)\left\langle \nabla H, \nabla\gamma\right\rangle \gamma^{s-1}d\mu\\
&\le (2\delta_1+\delta_2+\delta_3)\int_M |\Delta H|\gamma^s d\mu + \delta_4\int_M H^2\|\nabla H\|^2\gamma^s d\mu\\
&\quad+\left(\frac{s^2(s-1)^2}{4\delta_2}(c_{\gamma 1})^4+\frac{s^2}{8\delta_3}(c_{\gamma 2})^2+\frac{1}{2\delta_4}(c_{\gamma 1}c_{\gamma 2})^2(s-1)^2\right)\|H\|^2_{2,[\gamma>0]}\\
&\quad+\left(\frac{(s-2)^2}{2\delta_4}(c_{\gamma 1}c_{\gamma 2})^2+\frac{s^2(s-2)^2}{2\delta_1}(c_{\gamma 1})^4\right)\|H\|^2_{2,[\gamma>0]}\\
&\quad+\left(\frac{s^2}{\delta_1}(c_{\gamma 1})^2+\frac{1}{\delta_4}(c_{\gamma 2})^2\right)\int_M |H|\;|\Delta H|\gamma^{s-2}d\mu\\
&\le (3\delta_1+\delta_2+\delta_3+\delta_4)\int_M |\Delta H|\gamma^s d\mu + \delta_4\int_M H^2\|\nabla H\|^2\gamma^s d\mu\\
&\quad+\Bigg(\frac{s^2(s-1)^2}{4\delta_2}(c_{\gamma 1})^4+\frac{(s-1)^2}{2\delta_4}(c_{\gamma 1}c_{\gamma 2})^2+\frac{s^4}{4\delta_1^3}(c_{\gamma 1})^4+\frac{s^2}{8\delta_3}(c_{\gamma 2})^2\\
&\qquad+\frac{(s-2)^2}{2\delta_4}(c_{\gamma 1}c_{\gamma 2})^2+\frac{s^2(s-2)^2}{2\delta_1}(c_{\gamma 1})^4+\frac{1}{\delta_4^3}(c_{\gamma 2})^4\Bigg)\|H\|^2_{2,[\gamma>0]}.
\end{aligned}
$$

For brevity, we combine this estimate with (116) now. Choose $\delta_1,\delta_2,\delta_3$ such that $3\delta_1+\delta_2+\delta_3=\delta_4$. Then

$$
\frac{d}{dt}\int_M\frac{1}{2}H^2\gamma^s d\mu + (1-2\delta_4)\int_M|\Delta H|^2\gamma^s d\mu
$$

$$
\begin{aligned}
&\le \delta_4 \int_M \|\nabla H\|^2 H^2 \gamma^s d\mu + c\Big((c_{\gamma 1})^4 + (c_{\gamma 1}c_{\gamma 2})^2 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big)\|A\|^2_{2,[\gamma>0]} \\
&\quad - \int_M (\Delta H) H \|A^o\|^2 \gamma^s d\mu - h\int_M H\|A^o\|^2\gamma^s d\mu + \frac{1}{2}\int_M H^2(\partial_t \gamma^s) d\mu. \qquad (117)
\end{aligned}
$$

We now estimate the time derivative of $\gamma$.

$$
\begin{aligned}
\int_M H^2(\partial_t\gamma^s)d\mu &= \int_M H^2(\Delta H)(\nabla_\nu \tilde{\gamma})\gamma^{s-1}d\mu \\
&\le (c_{\tilde{\gamma}1})\int_M H^2(\Delta H)\gamma^{s-1}d\mu \\
&\le 2(c_{\tilde{\gamma}1})\int_M \|\nabla H\|^2|H|\gamma^{s-1}d\mu + (c_{\tilde{\gamma}1}c_{\gamma 1})\int_M \|\nabla H\|\ |H|^2\gamma^{s-2}d\mu \\
&\le (2\delta_5 + 2\delta_6)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu + \frac{1}{8\delta_5}(c_{\gamma 1}c_{\tilde{\gamma}1})^2\int_M |H|^2\gamma^{s-2}d\mu \\
&\qquad + \frac{1}{8\delta_6}(c_{\tilde{\gamma}1})^2\int_M |H|\ |\Delta H|\gamma^{s-2}d\mu + \frac{(s-2)^2}{16\delta_6}(c_{\tilde{\gamma}1}c_{\gamma 1})^2\int_M H^2\gamma^{s-4}d\mu \\
&\le (2\delta_5 + 2\delta_6)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu + 2\delta_4\int_M |\Delta H|^2\gamma^s d\mu \\
&\qquad + \left(\frac{(s-2)^2}{16\delta_6}(c_{\tilde{\gamma}1}c_{\gamma 1})^2 + \frac{1}{8\delta_5}(c_{\gamma 1}c_{\tilde{\gamma}1})^2 + \frac{1}{128\delta_6^2\delta_4}(c_{\tilde{\gamma}1})^4\right)\|H\|^2_{2,[\gamma>0]}.
\end{aligned}
$$

Combining this estimate with (117) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + (1-3\delta_4)\int_M |\Delta H|^2\gamma^s d\mu \\
&\quad\le (\delta_4+\delta_5+\delta_6)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu - \int_M (\Delta H)H\|A^o\|^2\gamma^s d\mu - h\int_M H\|A^o\|^2\gamma^s d\mu \\
&\qquad + c\Big((c_{\gamma 1})^4 + (c_{\gamma 1}c_{\gamma 2})^2 + (c_{\tilde{\gamma}1}c_{\gamma 2})^2 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big)\|A\|^2_{2,[\gamma>0]}. \qquad (118)
\end{aligned}
$$

We begin to estimate the second term with

$$
\begin{aligned}
&\int_M (\Delta H)H\|A^o\|^2\gamma^s d\mu + \int_M \|\nabla H\|^2\|A^o\|^2\gamma^s d\mu \\
&\quad\le 2\int_M \|\nabla H\|\ \|\nabla A^o\|\ \|A^o\|\ H\gamma^s d\mu + s(c_{\gamma 1})\int_M \|\nabla H\|\ \|A^o\|^2 H\gamma^{s-1}d\mu \\
&\quad\le \delta_7\int_M \|\nabla H\|^2 H^2\gamma^s d\mu + \frac{1}{\delta_7}\int_M \|\nabla A^o\|^2\|A^o\|^2\gamma^s d\mu \\
&\qquad + \delta_8\int_M H^2\|A^o\|^4\gamma^s d\mu + \frac{s^2(c_{\gamma 1})^2}{4\delta_8}\int_M \|\nabla H\|^2\gamma^{s-2}d\mu.
\end{aligned}
$$

Estimating the $\int_M \|\nabla H\|^2\gamma^{s-2}d\mu$ term as earlier, we combine this with (118) above to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + (1-4\delta_4)\int_M |\Delta H|^2\gamma^s d\mu \\
&\qquad \le (\delta_4+\delta_5+\delta_6+\delta_7)\int_M \|\nabla H\|^2H^2\gamma^s d\mu + \delta_8\int_M H^2\|A^o\|^4\gamma^s d\mu \\
&\qquad\quad + \frac{1}{\delta_7}\int_M \|\nabla A^o\|^2\|A^o\|^2\gamma^s d\mu - h\int_M H\|A^o\|^2\gamma^s d\mu \\
&\qquad\quad + c\Big((c_{\gamma 1})^4 + (c_{\gamma 1}c_{\gamma 2})^2 + (c_{\tilde{\gamma} 1}c_{\gamma 2})^2 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big)\|A\|^2_{2,[\gamma>0]}.
\end{aligned} \tag{119}
$$

To work with (119) above we must invoke a multiplicative Sobolev inequality and a consequence of the fundamental relations of differential geometry. The latter is:

$$
\begin{aligned}
&c_3\int_M \Big(\|\nabla_{(2)}H\|^2 + H^2\|\nabla H\|^2 + H^4\|A^o\|^2\Big)\gamma^s d\mu \\
&\qquad \le \int_M |\Delta H|^2\gamma^s d\mu + c\int_M \Big(\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^6\Big)\gamma^s d\mu + c(c_{\gamma 1})^4\int_{[\gamma>0]}\|A\|^2 d\mu.
\end{aligned}
$$

This is a corollary to estimates (108) and (109). Unfortunately, it is not quite strong enough. We improve the left hand side with the inequality

$$
\begin{aligned}
&\frac{c_3}{2}\int_M H^2\|\nabla A^o\|^2\gamma^s d\mu + \frac{c_3}{32}\int_M H^4\|A^o\|^2\gamma^s d\mu - \frac{c_3}{2}\int_M \|\nabla_{(2)}H\|^2\gamma^s d\mu \\
&\qquad - c(c_{\gamma 1})^4\|A^o\|^2_{2,[\gamma>0]} - c\int_M \Big(\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^6\Big)\gamma^s d\mu \\
&\qquad \le \frac{c_3}{2}\int_M H^2\|\nabla H\|^2\gamma^s d\mu.
\end{aligned} \tag{120}
$$

This gives us the very useful inequality

$$
\begin{aligned}
&c_3\int_M \Big(\|\nabla_{(2)}H\|^2 + H^2\|\nabla A^o\|^2 + H^4\|A^o\|^2\Big)\gamma^s d\mu \\
&\le \int_M |\Delta H|^2\gamma^s d\mu + c(c_{\gamma 1})^4\|A\|^2_{2,[\gamma>0]} + c\int_M \Big(\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^6\Big)\gamma^s d\mu.
\end{aligned} \tag{121}
$$

We must prove (120). The technique is similar to the estimates from Chapter 4 and so we will proceed quickly. Integrating by parts and estimating we have

$$\int_M H^2\|\nabla A^o\|^2\gamma^s d\mu$$

$$
\begin{aligned}
&\le \int_M H^2\|\nabla H\|^2\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\
&\quad + s(c_{\gamma 1})\int_M H^2\|A^o\|\ \|\nabla A^o\|\gamma^{s-1}d\mu - \int_M H^2\left\langle A^o, \Delta A^o\right\rangle\gamma^s d\mu \\
&\le \int_M H^2\|\nabla H\|^2\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\
&\quad + \theta_1\int_M H^4\|A^o\|^2\gamma^s d\mu + \frac{s^2(c_{\gamma 1})^2}{4\theta_1}\int_M \|\nabla A^o\|^2\gamma^{s-2}d\mu - \int_M H^2\left\langle A^o, \Delta A^o\right\rangle\gamma^s d\mu \\
&\le \int_M H^2\|\nabla H\|^2\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\
&\quad - \int_M H^2\left\langle A^o, \Delta A^o\right\rangle\gamma^s d\mu - \frac{s^2(c_{\gamma 1})^2}{2\theta_1}\int_M \left\langle A^o, \Delta A^o\right\rangle\gamma^{s-2}d\mu \\
&\quad + \theta_1\int_M H^4\|A^o\|^2\gamma^s d\mu + \frac{s^2(s-2)^2(c_{\gamma 1})^4}{4\theta_1}\int_M \|A^o\|^2\gamma^{s-4}d\mu,
\end{aligned}
$$

where we used the inequality

$$
\frac{(c_{\gamma 1})^2}{2}\int_M \|\nabla A^o\|^2\gamma^{s-2}d\mu \le -(c_{\gamma 1})^2\int_M \left\langle A^o, \Delta A^o\right\rangle\gamma^{s-2}d\mu + \frac{(s-2)^2(c_{\gamma 1})^4}{2}\int_M \|A^o\|^2\gamma^{s-4}d\mu.
$$

Simons' identity for $\Delta A^o$ implies that

$$
\left\langle A^o, \Delta A^o\right\rangle = \left\langle A^o, \nabla_{(2)}H\right\rangle - \|A^o\|^4 + \frac{1}{2}\|A^o\|^2H^2 + H(A^o)_{pq}(A^o)^p_r(A^o)^{qr}.
$$

Using this we obtain

$$
\begin{aligned}
&\int_M H^2\|\nabla A^o\|^2\gamma^s d\mu + \frac{1}{2}\int_M H^4\|A^o\|^2\gamma^s d\mu \\
&\quad\le \int_M H^2\|\nabla H\|^2\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\
&\quad\quad - \int_M H^2\Big(\left\langle A^o, \nabla_{(2)}H\right\rangle - \|A^o\|^4 + H(A^o)_{pq}(A^o)^p_r(A^o)^{qr}\Big)\gamma^s d\mu \\
&\quad\quad - \frac{s^2(c_{\gamma 1})^2}{2\theta_1}\int_M \Big(\left\langle A^o, \nabla_{(2)}H\right\rangle - \|A^o\|^4 + \frac{1}{2}\|A^o\|^2H^2 + H(A^o)_{pq}(A^o)^p_r(A^o)^{qr}\Big)\gamma^{s-2}d\mu \\
&\quad\quad + \theta_1\int_M H^4\|A^o\|^2\gamma^s d\mu + \frac{s^2(s-2)^2(c_{\gamma 1})^4}{4\theta_1}\int_M \|A^o\|^2\gamma^{s-4}d\mu \\
&\quad\le \int_M H^2\|\nabla H\|^2\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu + \Big(\frac{1}{4\theta_3} + \frac{1}{4\theta_4}\Big)\int_M \|A^o\|^6\gamma^s d\mu \\
&\quad\quad + (\theta_2+\theta_3)\int_M \|\nabla_{(2)}H\|^2\gamma^s d\mu + (\theta_1 + \frac{1}{4\theta_2} + \theta_4 + \theta_5)\int_M H^4\|A^o\|^2\gamma^s d\mu \\
&\quad\quad - \int_M H^3(A^o)_{pq}(A^o)^p_r(A^o)^{qr}\gamma^s d\mu - \frac{s^2(c_{\gamma 1})^2}{2\theta_1}\int_M H(A^o)_{pq}(A^o)^p_r(A^o)^{qr}\gamma^{s-2}d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{s^2(s-2)^2}{4\theta_1}+\frac{s^4}{2\theta_1^2\theta_3}+\frac{s^4}{8\theta_1^2\theta_5}\Big)(c_{\gamma 1})^4\int_M\|A^o\|^2\gamma^{s-4}d\mu\\
\le&\int_M H^2\|\nabla H\|^2\gamma^s d\mu+\int_M\|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu+\Big(\frac{1}{4\theta_3}+\frac{1}{4\theta_4}\Big)\int_M\|A^o\|^6\gamma^s d\mu\\
&+(\theta_2+\theta_3)\int_M\|\nabla_{(2)}H\|^2\gamma^s d\mu+(\theta_1+\frac{1}{4\theta_2}+\theta_4+\theta_5+\theta_6)\int_M H^4\|A^o\|^2\gamma^s d\mu\\
&+\frac{1}{4\theta_6}\int_M H^2\|A^o\|^4\gamma^s d\mu+\int_M H^2\|A^o\|^4\gamma^s d\mu\\
&+\Big(\frac{s^2(s-2)^2}{4\theta_1}+\frac{s^4}{8\theta_1^2}+\frac{s^4}{2\theta_1^2\theta_3}+\frac{s^4}{8\theta_1^2\theta_5}\Big)(c_{\gamma 1})^4\int_M\|A^o\|^2\gamma^{s-4}d\mu\\
\le&\int_M H^2\|\nabla H\|^2\gamma^s d\mu+(\theta_2+\theta_3)\int_M\|\nabla_{(2)}H\|^2\gamma^s d\mu\\
&+\int_M\|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu+\Big(\frac{1}{4\theta_3}+\frac{1}{4\theta_4}+\frac{1}{4\theta_7}+\frac{1}{4\theta_8}\Big)\int_M\|A^o\|^6\gamma^s d\mu\\
&+\Big(\theta_1+\frac{1}{4\theta_2}+\theta_4+\theta_5+\theta_6+\frac{\theta_7}{16\theta_6^2}+\theta_8\Big)\int_M H^4\|A^o\|^2\gamma^s d\mu\\
&+\Big(\frac{s^2(s-2)^2}{4\theta_1}+\frac{s^4}{8\theta_1^2}+\frac{s^4}{2\theta_1^2\theta_3}+\frac{s^4}{8\theta_1^2\theta_5}\Big)(c_{\gamma 1})^4\int_M\|A^o\|^2\gamma^{s-4}d\mu.
\end{aligned}
$$

Choosing appropriately small $\theta_i > 0$ and multiplying both sides by $\frac{c_1}{2}$ in the above allows us to conclude (120). For completeness we give one possible set of choices here:

$$
\theta_1=\theta_4=\theta_5=\theta_8=\frac{1}{48},\quad \theta_2=\frac{3}{4},\quad \theta_3=\frac{1}{4},\quad \theta_6=\frac{1}{92},\ \text{and}\ \theta_7=\frac{1}{55296}.
$$

This establishes (121).

The multiplicative Sobolev inequality we will use is a variant on Lemma 4.6. The statement is

$$
\begin{aligned}
&\int_M\|A^o\|^6\gamma^s d\mu+\int_M\|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu\\
&\qquad\le c_2\int_{[\gamma>0]}\|A^o\|^2 d\mu\int_M\Big(\|\nabla_{(2)}H\|^2+H^2\|\nabla A^o\|^2+H^2\|A^o\|^4\Big)\gamma^s d\mu\\
&\qquad\qquad+c(c_{\gamma 1})^4\left(\int_{[\gamma>0]}\|A^o\|^2 d\mu\right)^2. \qquad (122)
\end{aligned}
$$

Combining both (121) and (122) with our running estimate (119), we have

$$
\begin{aligned}
\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + c_3(1-4\delta_4)\Big(\int_M \|\nabla_{(2)}H\|^2\gamma^s d\mu + \int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu + \int_M H^4\|A^o\|^2\gamma^s d\mu\Big) \\
\le (\delta_4+\delta_5+\delta_6+\delta_7)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu + \delta_8\int_M H^2\|A^o\|^4\gamma^s d\mu \\
+ \frac{1+c_3\delta_7}{\delta_7}\int_M \big(\|A^o\|^2\|\nabla A^o\|^2 + \|A^o\|^6\big)\gamma^s d\mu - h\int_M H\|A^o\|^2\gamma^s d\mu \\
+ c\bigg((c_{\gamma 1})^4 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\bigg)\|A\|^2_{2,[\gamma>0]} \\
\le (\delta_4+\delta_5+\delta_6+\delta_7)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu + \delta_8\int_M H^2\|A^o\|^4\gamma^s d\mu \\
+ c_2\frac{1+c_3\delta_7}{\delta_7}\bigg(\|A^o\|^2_{2,[\gamma>0]}\int_M \big(\|\nabla_{(2)}H\|^2 + H^2\|\nabla A^o\|^2 + H^2\|A^o\|^4\big)\gamma^s d\mu \\
+ c(c_{\gamma 1})^4\left(\int_{[\gamma>0]}\|A^o\|^2 d\mu\right)^2\bigg) \\
- h\int_M H\|A^o\|^2\gamma^s d\mu + c\bigg((c_{\gamma 1})^4 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\bigg)\|A\|^2_{2,[\gamma>0]}.
\end{aligned}
$$

Using the inequality

$$
\delta_8\int_M H^2\|A^o\|^4\gamma^s d\mu \le \delta_8\int_M H^4\|A^o\|^2\gamma^s d\mu + \frac{\delta_8}{4}\int_M \|A^o\|^6\gamma^s d\mu
$$

the above becomes

$$
\begin{aligned}
\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + c_3(1-4\delta_4)\Big(\int_M \|\nabla_{(2)}H\|^2\gamma^s d\mu + \int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu + \int_M H^4\|A^o\|^2\gamma^s d\mu\Big) \\
\le (\delta_4+\delta_5+\delta_6+\delta_7)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu + \delta_8\int_M H^4\|A^o\|^2\gamma^s d\mu \\
+ c_2\frac{4+4c_3\delta_7+\delta_7\delta_8}{4\delta_7}\bigg(\|A^o\|^2_{2,[\gamma>0]}\int_M \big(\|\nabla_{(2)}H\|^2 + H^2\|\nabla A^o\|^2 + H^2\|A^o\|^4\big)\gamma^s d\mu \\
+ c(c_{\gamma 1})^4\left(\int_{[\gamma>0]}\|A^o\|^2 d\mu\right)^2\bigg) - h\int_M H\|A^o\|^2\gamma^s d\mu \\
+ c\bigg((c_{\gamma 1})^4 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\bigg)\|A\|^2_{2,[\gamma>0]}.
\end{aligned}
$$

We have one final refinement. Observe that by estimating

$$
c\epsilon_0\int_M H^2\|A^o\|^4\gamma^s d\mu \le c^2\epsilon_0^2\int_M H^4\|A^o\|^2\gamma^s d\mu + \frac{1}{4}\int_M \|A^o\|^6\gamma^s d\mu
$$

one may strengthen our estimate to the following:

$$
\begin{aligned}
\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + c_3(1-4\delta_4)\Big(\int_M \|\nabla_{(2)}H\|^2\gamma^s d\mu + \int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu + \int_M H^4\|A^o\|^2\gamma^s d\mu\Big)& \\
\le (\delta_4+\delta_5+\delta_6+\delta_7)\int_M \|\nabla H\|^2 H^2\gamma^s d\mu& \\
+\left(\delta_8 + \Big(\frac{\delta_8}{3} + \frac{4(1+c_3\delta_7)}{3\delta_7}\Big)(\epsilon_0 c_2)^2\right)\int_M H^4\|A^o\|^2\gamma^s d\mu& \\
+ c_2\frac{4+4c_3\delta_7+\delta_7\delta_8}{8\delta_7}\Bigg(\|A^o\|^2_{2,[\gamma>0]}\int_M \Big(\|\nabla_{(2)}H\|^2 + \|\nabla A^o\|^2H^2\Big)\gamma^s d\mu& \\
+ c(c_{\gamma 1})^4\left(\int_{[\gamma>0]}\|A^o\|^2 d\mu\right)^2\Bigg)& \\
- h\int_M H\|A^o\|^2\gamma^s d\mu + c\Big((c_{\gamma 1})^4 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big)\|A\|^2_{2,[\gamma>0]}.&
\end{aligned}
$$

Recall that $\|\nabla H\|^2 = 4\|\nabla^* A^o\|^2 \le 4\|\nabla A^o\|^2$. Choosing sufficiently small $\delta_i > 0$ we absorb to obtain

$$
\begin{aligned}
\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu + c_e\int_M \Big(\|\nabla_{(2)}H\|^2 + \|\nabla H\|^2H^2 + H^4\|A^o\|^2\Big)\gamma^s d\mu& \\
\le -h\int_M H\|A^o\|^2\gamma^s d\mu + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]}& \\
+ c\Big((c_{\gamma 1})^4 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big)\|A\|^2_{2,[\gamma>0]}.&
\end{aligned}
$$

In this step we also enforce a condition upon the magnitude of $\epsilon_0$. For the sake of definiteness we give specific choices. Note first that we may assume $c_3 \le 1$. Then let

$$
\delta_i = \frac{1}{16}c_3, \qquad \text{and assume} \qquad \epsilon_0 \le \frac{(c_3)^2}{144c_2}.
$$

In this case we have

$$
c_3(1-4\delta_4) - \sum_{i=4}^{8}\delta_i - (\epsilon_0 c_2)^2\Big(\frac{\delta_8}{3} + \frac{4+4c_3\delta_7}{3\delta_7}\Big) - \epsilon_0\frac{c_2}{4\delta_7}(4+4c_3\delta_7+\delta_7\delta_8) \ge \frac{1}{4}c_3,
$$

and so with these choices

$$
c_e = \frac{1}{4}c_3.
$$

Finally, our very restrictive assumption on the constraint function:

$$-h\int_M H\|A^o\|^2\gamma^s d\mu \le \frac{c_e}{2}\int_M \|\nabla_{(2)}H\|^2 + \|\nabla A^o\|^2H^2 + H^4\|A^o\|^2 d\mu$$

implies the result. □

Remark. The hypothesis of the lemma contains the particular cutoff function $\gamma$ chosen. It is desirable to state a condition on the constraint function $h$ which does not involve $\gamma$. Unfortunately, there are several ways to do this and none appear any better than the others. Since all follow from the condition given in the lemma, we have not changed this. Perhaps the most direct way is to estimate:

$$-h\int_M H\|A^o\|^2 \le \frac{1}{4}\Big(\int_M H\|A^o\|^2\gamma^s d\mu\Big)^4 + \frac{3}{4}h^{\frac{4}{3}} \le \epsilon_0^3\int_M H^4\|A^o\|^2\gamma^s d\mu + \frac{3}{4}h^{\frac{4}{3}}.$$

Then a sufficient condition for the lemma to hold is to require

$$\frac{3}{4}h^{\frac{4}{3}} \le \frac{c_e}{2}\int_M \|\nabla_{(2)}H\|^2 + \|\nabla A^o\|^2H^2 + H^4\|A^o\|^2 d\mu.$$

This condition is more readily satisfied than the previous, although it is strictly stronger in that fewer constraint functions satisfy this condition compared with the one given in the statement of the lemma. For example one may choose

$$h = \int_M \|A^o\|^{1/2}\|\nabla H\|^{3/2}H^{3/2}d\mu$$

among many others.

Remark. For surface diffusion flow, the left hand side of the estimate (115) does not include the negative integrals.

## 3. Integral estimates with small curvature

We now present the first smallness result of this chapter.

Proposition 5.5. *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a (CSD) flow and $\gamma$ be as in (25). Then there exists an $\epsilon_0 > 0$ such that while*

$$\int_{[\gamma>0]} \|A^o\|^n d\mu < \epsilon_0, \tag{123}$$

*we have*

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu &+ c_1\int_M (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu \\ &\le ch^2|M_t|_{[\gamma>0]} + c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^{(6-n)}\epsilon_0^{\frac{6-n}{2}},\end{aligned}$$

*and*

$$\begin{aligned}\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu &+ c_1\int_M (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu \\ &\le ch^2|M_t|_{[\gamma>0]} + c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^{(6-n)}\epsilon_0^{\frac{6-n}{2}},\end{aligned}$$

*where $c_1 = c_1(s,n,\epsilon_0) > 0$ and $c = c(s,n)$.*

Proof. The two inequalities which drive this proof are the third multiplicative Sobolev inequality Lemma 4.6, which we invoke in the weaker form:

$$\begin{aligned}\int_M \|A^o\|^6\gamma^s d\mu &+ \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu \\ &\le c\Big(\int_{[\gamma>0]}\|A^o\|^n d\mu\Big)^{\frac{4-n}{2}}\int_M \Big(\|\nabla_{(2)}A^o\|^2 + \|\nabla A^o\|^2\|A\|^2 + \|A\|^2\|A^o\|^4\Big)\gamma^s d\mu \\ &\quad + c(c_{\gamma 1})^{(6-n)}\Big(\int_{[\gamma>0]}\|A^o\|^n d\mu\Big)^{\frac{6-n}{2}} + \delta(n-2)\int_M \|\nabla_{(2)}A^o\|^2\gamma^s d\mu,\end{aligned}$$

and the key smallness estimate in Proposition 4.7,

$$\begin{aligned}\int_M (\|\nabla_{(2)}A\|^2 &+ \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu \\ &\le c\int_M \Big(|F|^2 + |h|^2\Big)\gamma^4 d\mu + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]} + c(c_{\gamma 1})^4\|A\|^2_{2,[\gamma>0]} \\ &\quad + (n-2)c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]},\end{aligned}$$

Note that only the second inequality needs the smallness assumption. Rearranging the second inequality gives

$$\int_M \Big(|F|^2 + |h|^2\Big)\gamma^s d\mu + c(c_{\gamma 1})^4\Big[\|A^o\|^4_{2,[\gamma>0]} + \|A\|^2_{2,[\gamma>0]}\Big] + (n-2)c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}$$
$$\geq c\int_M (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu$$

Combining this estimate with that of Lemma 5.3 earlier, we obtain

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu + (1-\delta_1)\Big[c\int_M (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu \\ - h^2|M_t|_{[\gamma>0]} - c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]} - c(c_{\gamma 1})^4\|A\|^2_{2,[\gamma>0]} \\ - (n-2)c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big] \\ \leq ch^2|M_t|_{[\gamma>0]} + \delta_2\int_M \|\nabla A^o\|^2 H^2\gamma^s d\mu \\ + \delta_3\int_M \|A\|^4\|A^o\|^2\gamma^s d\mu + c\int_M (\|A^o\|^6 + \|\nabla A^o\|^2\|A^o\|^2)\gamma^s d\mu \\ + c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\int_M \|A^o\|^2\gamma^{s-4}d\mu.\end{aligned}$$

Rearranging,

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu + \Big(1 - \textstyle\sum_i\delta_i\Big)\int_M (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu \\ \leq ch^2|M_t|_{[\gamma>0]} + c\int_M (\|A^o\|^6 + \|\nabla A^o\|^2\|A^o\|^2)\gamma^s d\mu \\ + c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\Big(\|A\|^2_{2,[\gamma>0]} + \|A^o\|^4_{2,[\gamma>0]}\Big) \\ + (n-2)c(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}.\end{aligned}$$

We now use Lemma 4.6 and the smallness assumption (123) to absorb the second integral on the right to the left hand side. That is,

$$\int_M \|A^o\|^6\gamma^s d\mu + \int_M \|A^o\|^2\|\nabla A^o\|^2\gamma^s d\mu$$
$$\leq c\Big(\int_{[\gamma>0]}\|A^o\|^n d\mu\Big)^{\frac{4-n}{2}}\int_M \Big(\|\nabla_{(2)}A^o\|^2 + \|\nabla A^o\|^2\|A\|^2 + \|A\|^2\|A^o\|^4\Big)\gamma^s d\mu$$

$$
\begin{aligned}
&\quad + c(c_{\gamma 1})^{(6-n)}\Big(\int_{[\gamma>0]}\|A^o\|^n d\mu\Big)^{\frac{6-n}{2}} + \delta(n-2)\int_M \|\nabla_{(2)}A^o\|^2\gamma^s d\mu,\\
&\le \Big[c\epsilon_0^{\frac{4-n}{2}} + \delta(n-2)\Big]\int_M \Big(\|\nabla_{(2)}A^o\|^2 + \|\nabla A^o\|^2\|A\|^2 + \|A\|^2\|A^o\|^4\Big)\gamma^s d\mu\\
&\quad + c(c_{\gamma 1})^{(6-n)}\epsilon_0^{\frac{6-n}{2}}.
\end{aligned}
$$

Absorbing, we have therefore shown

$$
\begin{aligned}
&\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu\\
&\quad + \Big(1 - \textstyle\sum_i \delta_i - c^*\epsilon_0 - \delta(n-2)\Big)\displaystyle\int_M \Big(\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2\Big)\gamma^s d\mu\\
&\le ch^2|M_t|_{[\gamma>0]} + c\Big[(c_{\tilde\gamma 2})^2 + (c_{\tilde\gamma 1})^4 + (c_{\gamma 1})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^{(6-n)}\epsilon_0^{\frac{6-n}{2}}.
\end{aligned}
$$

Choosing $\delta_i$,$\delta$ and requiring $\epsilon_0$ to be such that

$$1 - \sum_i \delta_i - c^*\epsilon_0 - \delta(n-2) = c_1 > 0$$

finishes the proof of the first statement. The second follows similarly. □

The result above is good, however it is not good enough for us to continue. We need to absorb the error term $h^2|M_t|_{[\gamma>0]}$ or otherwise deal with it to proceed toward the interior estimates. The most natural method to overcome this difficulty is to impose a growth condition on our constraint function, and this is what we do next. We will need some of the work already done for the estimates involved in the proof of the Lifespan Theorem, see Chapter 3 Section 3 for the details.

The main difference here is the focus moving away from *small total curvature* to a combination of *bounded total curvature* and *small tracefree curvature*. In this respect, the uniform bound for $h$ obtained earlier in Corollary 3.8 is useless. However Theorem 3.3 from that section is still useful, in the following modified form.

THEOREM 5.6. *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with constraint function $h$ satisfying for some $j,k,l,p \in \mathbb{N}_0$*

$$h^2 \le c\Big(1 + \int_M \|A^o\|^n d\mu\Big)^p \int_M P_j^2(A) + P_k^1(A) + P_l^0(A)\ d\mu \tag{A2}$$

*where for $m = \max\{2k-2, 2j-k, l, n^2+n-2\}$*

$$\sup_{x\in\mathbb{R}^{n+1}} \delta^m(x) \le \delta_0^m < \infty$$

*and for an absolute constant $C_{AB}$*

$$|M_t| \le C_{AB}; \tag{AB}$$

*on $[0,T^*]$.*

*Then for any $\rho > 0$, $x \in \mathbb{R}^{n+1}$, $t \in [0,T^*]$ there exists an $x_1 \in \mathbb{R}^{n+1}$ such that if*

$$\int_{f^{-1}(B_\rho(x_1))} \|A^o\|^n d\mu \le \epsilon_0 < \infty, \tag{124}$$

*we have*

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_{\rho/2}(x_1))} \|A^o\|^2 d\mu + \frac{c_1}{2}\int_{f^{-1}(B_{\rho/2}(x_1))} (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2) d\mu \\ \le c_h + \frac{c}{\rho^4}\|A\|^2_{2,f^{-1}(B_{2\rho}(x_1))} + \frac{c}{\rho^{6-n}}\epsilon_0^{\frac{6-n}{2}},\end{aligned}$$

*and*

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_{\rho/2}(x_1))} \frac{1}{2}H^2 d\mu + \frac{c_1}{2}\int_{f^{-1}(B_{\rho/2}(x_1))} (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2) d\mu \\ \le c_h + \frac{c}{\rho^4}\|A\|^2_{2,f^{-1}(B_{2\rho}(x_1))} + \frac{c}{\rho^{6-n}}\epsilon_0^{\frac{6-n}{2}},\end{aligned}$$

*if $j,k \ne 0$, and otherwise*

$$h \le c_h,$$

*where $c_h = c_h(\delta_0^m, C_{AB}, \rho, j, k, l, n) < \infty$ and $\epsilon_0$ is as in Proposition 5.5.*

Proof. If $j, k = 0$ the theorem follows from Theorem 3.3. Otherwise, we use Proposition 5.5 above to compute:

$$\begin{aligned}
\frac{d}{dt}\int_M \|A^o\|^2\gamma^s d\mu &+ c_1\int_M (\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2)\gamma^s d\mu \\
&\le ch^2|M_t|_{[\gamma>0]} + c\Big[(c_{\tilde{\gamma}2})^2 + (c_{\tilde{\gamma}1})^4 + (c_{\gamma 1})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^{(6-n)}\epsilon_0^{\frac{6-n}{2}} \\
&\le C_{AB}\left[\left(1+\int_M \|A^o\|^n d\mu\right)^p \int_M P_j^2(A) + P_k^1(A) + P_l^0(A)\ d\mu\right] \\
&\quad + c\Big[(c_{\tilde{\gamma}2})^2 + (c_{\tilde{\gamma}1})^4 + (c_{\gamma 1})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^{(6-n)}\epsilon_0^{\frac{6-n}{2}}.
\end{aligned}$$

Now using Lemma 3.6 we have for any $\theta^* > 0$

$$\begin{aligned}
\left(1+\int_M \|A^o\|^n d\mu\right)^p &\int_M P_j^2(A) + P_k^1(A) + P_l^0(A)\ d\mu \\
&\le \left(1+\int_M \|A^o\|^n d\mu\right)^p \Bigg[\theta\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2 d\mu \\
&\qquad + c(\theta,\rho,n,j,k,l,C_{AB})\Big(\delta_0^m\Big)^{\frac{(n+1)(2m+2k-j+l)+j-2+2k+l}{m}}\Bigg] \\
&\le \theta^*\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2 d\mu \\
&\qquad + c(\epsilon_0,\rho,n,j,k,l,p,C_{AB})\Big(\delta_0^m\Big)^{\frac{(n+1)(2m+2k-j+l)+j-2+2k+l}{m}},
\end{aligned}$$

choosing

$$\theta = \theta^*\Big(C_{AB}^{1/p}[1 + c_\rho\epsilon_0]\Big)^{-p}.$$

Note that this is allowed by the bound on $c_\rho(t)$, Lemma 3.5, and the boundedness of $\epsilon_0$. Therefore, we can use the above estimate for $\theta^* \le \frac{1}{2C_{AB}c_1}$ to absorb the term with $h$ and conclude the theorem. □

Remark. Because we cannot assume small curvature, we will not recover the uniform bound for $h$. This is the reason why we must assume the much more restrictive growth condition (A2) instead of that allowed in the Lifespan Theorem.

Our next step is another Sobolev-type inequality, which we will use presently to obtain $L^\infty$ estimates for the curvature. Note that here the dimension of $M$ plays a key role, for exactly the same reasons as outlined in Chapter 3, Section 5.

LEMMA 5.7. *Suppose $n \in \{2,3\}$ and let $\gamma$ be as in (25). Then for any $C^1$ tensor $T$ on $M^n$ and $s \geq 2$,*

$$\|T\gamma^s\|_\infty^4 \leq c\|T\gamma^s\|_2^{4-n}\Bigg[\Big(\int_M \|\nabla_{(2)}T\|^2\gamma^{2s}d\mu\Big)^{\frac{n}{2}} + \Big(\int_M H^4\|T\|^2\gamma^{2s}d\mu\Big)^{\frac{n}{2}} + (c_{\gamma 1})^{2n}\Big(\int_{[\gamma>0]}\|T\|^2 d\mu\Big)^{\frac{n}{2}}\Bigg],$$

*where $c = c(n,s)$.*

PROOF. We use the interpolation inequality: for any $C^2$ tensor $S$,

$$\|S\|_\infty^{4+n} \leq c\|S\|_2^{4-n}(\|\nabla S\|_4 + \|H\cdot S\|_4)^{2n},$$

from Theorem A.2. We invoke this with $S = T\gamma^s$ to infer

$$\|T\gamma^s\|_\infty^{4+n} \leq c\|T\gamma^s\|_2^{4-n}\Big(\|\nabla(T\gamma^s)\|_4^{2n} + \|H\cdot T\gamma^s\|_4^{2n}\Big). \tag{125}$$

Now

$$\|H\cdot T\gamma^s\|_4^{2n} \leq \|T\gamma^s\|_\infty^n\Big(\int_M H^4\|T\|^2\gamma^{2s}d\mu\Big)^{\frac{n}{2}},$$

which gives the second term in the statement of the lemma, and

$$\begin{aligned}\|\nabla(T\gamma^s)\|_4^4 &= \int_M \Big[(\nabla T)\gamma^s + sT(\nabla\gamma)\gamma^{s-1}\Big]^4 d\mu\\ &\leq c\int_M \|\nabla T\|^4\gamma^{4s}d\mu + c(c_{\gamma 1})^4\int_M \|T\|^4\gamma^{4s-4}d\mu,\end{aligned}$$

so

$$\|\nabla(T\gamma^s)\|_4^{2n} \leq c\Big(\int_M \|\nabla T\|^4\gamma^{4s}d\mu\Big)^{\frac{n}{2}} + c(c_{\gamma 1})^{2n}\Big(\int_M \|T\|^4\gamma^{4s-4}d\mu\Big)^{\frac{n}{2}}. \tag{126}$$

The second term in (126) above is easy to estimate

$$(c_{\gamma 1})^{2n}\Big(\int_M \|T\|^4\gamma^{4s-4}d\mu\Big)^{\frac{n}{2}} \le (c_{\gamma 1})^{2n}\|T\gamma^s\|_\infty^n\Big(\int_{[\gamma>0]}\|T\|^2 d\mu\Big)^{\frac{n}{2}},$$

where we needed $s \ge 2$. This gives the last term in the statement of the lemma.

We will use integration by parts and Young's inequality to estimate the first term in (126):

$$\begin{aligned}
\int_M \|\nabla T\|^4\gamma^{4s}d\mu &\le 3\int_M \|T\|\ \|\nabla_{(2)}T\|\ \|\nabla T\|^2\gamma^{4s}d\mu \\
&\quad + 4s\int_M \|T\|\ \|\nabla T\|^3\|\nabla\gamma\|\gamma^{4s-1}d\mu \\
&\le \Big[\frac{1}{4}\int_M \|\nabla T\|^4\gamma^{4s}d\mu + 9\int_M \|T\|^2\|\nabla_{(2)}T\|^2\gamma^{4s}d\mu\Big] \\
&\quad + \Big[\frac{1}{4}\int_M \|\nabla T\|^4\gamma^{4s}d\mu + 1728s^4\int_M \|T\|^4\|\nabla\gamma\|^4\gamma^{4s-4}d\mu\Big] \\
&\le \frac{1}{2}\int_M \|\nabla T\|^4\gamma^{4s}d\mu \\
&\quad + \|T\gamma^s\|_\infty^2\Big[c\int_M \|\nabla_{(2)}T\|^2\gamma^{2s}d\mu + c(c_{\gamma 1})^4\int_M \|T\|^2\gamma^{2s-4}d\mu\Big].
\end{aligned}$$

Therefore

$$\int_M \|\nabla T\|^4\gamma^{4s}d\mu \le c\|T\gamma^s\|_\infty^2\Big[\int_M \|\nabla_{(2)}T\|^2\gamma^{2s}d\mu + (c_{\gamma 1})^4\int_{[\gamma>0]}\|T\|^2 d\mu\Big],$$

and so

(127)
$$\Big(\int_M \|\nabla T\|^4\gamma^{4s}d\mu\Big)^{\frac{n}{2}} \le c\|T\gamma^s\|_\infty^n\Big[\Big(\int_M \|\nabla_{(2)}T\|^2\gamma^{2s}d\mu\Big)^{\frac{n}{2}} + (c_{\gamma 1})^{2n}\Big(\int_{[\gamma>0]}\|T\|^2 d\mu\Big)^{\frac{n}{2}}\Big],$$

where we again needed $s \ge 2$. Combining (125), (126) and (127) gives

$$\begin{aligned}
\|T\gamma^s\|_\infty^{4+n} \le c\|T\gamma^s\|_2^{4-n}\|T\gamma^s\|_\infty^n\Big[&\Big(\int_M \|\nabla_{(2)}T\|^2\gamma^{2s}d\mu\Big)^{\frac{n}{2}} + \Big(\int_M H^4\|T\|^2\gamma^{2s}d\mu\Big)^{\frac{n}{2}} \\
&+ (c_{\gamma 1})^{2n}\Big(\int_{[\gamma>0]}\|T\|^2 d\mu\Big)^{\frac{n}{2}}\Big].
\end{aligned} \tag{128}$$

Since Lemma 5.7 is trivial if either $\gamma \equiv 0$ or $T \equiv 0$, assume otherwise and then divide by $\|\gamma^s T\|_\infty^n$ in (128) to finish the proof. $\square$

Lemma 5.7 combined with some earlier estimates leads to partial curvature estimates. We show that the supremum of the tracefree curvature is bounded by an expression incorporating the speed of the flow as the principal driving factor. As expected, the constraint function causes some trouble, by adding a large constant on the right hand side. The following result, analogous to Theorem 3.3, is one of our major theorems for this chapter.

THEOREM 5.8 (Partial curvature estimates). *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow, $\gamma$ a cutoff function as in* (25) *with $\chi_{B_{\rho/2}} < \tilde{\gamma} < \chi_{B_\rho}$ and suppose that for any $x \in \mathbb{R}^{n+1}$, $\|A^o\|^n_{n,f^{-1}(B_\rho(x))} \le \|A^o\|^n_{n,f^{-1}(B_\rho(x_0))} < \epsilon_0$ where $\epsilon_0 > 0$ is as in Proposition 4.7. Further, assume that the constraint function $h$ satisfies* (A2)*, and in the case where $n = 3$ that* (AB) *is satisfied. Then*

$$
\begin{aligned}
&\|A^o\|^2_{\infty,f^{-1}(B_{\rho/2}(x_0))}\\
&\qquad\le c\|A^o\|^{\frac{4-n}{2}}_{2,f^{-1}(B_\rho(x_0))}\Big[\|F\|^{\frac{n}{2}}_{2,f^{-1}(B_\rho(x_0))} + \frac{1}{\rho^n}\|A\|^{\frac{n}{2}}_{2,f^{-1}(B_\rho(x_0))} + (c_h)^{\frac{n}{4}} + (n-2)\epsilon_0\Big],
\end{aligned}
$$

*where $c = c(n, \epsilon_0, C_{AB})$.*

PROOF. We use Proposition 4.7 to obtain

$$
\begin{aligned}
\int_{f^{-1}(B_{\rho/2}(x_0))} &\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2 d\mu\\
&\le c\int_{f^{-1}(B_\rho(x_0))} F^2 d\mu + ch^2|M_t|_{f^{-1}(B_\rho(x_0))}\\
&\quad + \frac{c}{\rho^4}\Big[\|A^o\|^4_{2,f^{-1}(B_\rho(x_0))} + \|A\|^2_{2,f^{-1}(B_\rho(x_0))}\Big]\\
&\quad + c(n-2)(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}.
\end{aligned}
$$

Therefore, using Theorem 5.6 we have

$$\begin{aligned}
\int_{f^{-1}(B_{\rho/2}(x_0))} &\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2 d\mu \\
&\le c_h + c\int_{f^{-1}(B_\rho(x_0))} F^2 d\mu + c(n-2)(c_{\gamma 1})^3\|A^o\|^{\frac{9}{2}}_{3,[\gamma>0]}, \\
&\quad + \frac{c}{\rho^4}\Big[\|A^o\|^4_{2,f^{-1}(B_\rho(x_0))} + \|A\|^2_{2,f^{-1}(B_\rho(x_0))}\Big]
\end{aligned}$$

where $c_h$ is as in Theorem 5.6. Lemma 5.7 above with $T = A^o$ gives

$$\begin{aligned}
\|A^o\|^4_{\infty,f^{-1}(B_{\rho/2}(x_0))} \le c\|A^o\|^{4-n}_{2,f^{-1}(B_\rho(x_0))}\bigg[&\Big(\int_{f^{-1}(B_\rho(x_0))}\|\nabla_{(2)}A^o\|^2\mu\Big)^{\frac{n}{2}} \\
&+ \Big(\int_{f^{-1}(B_\rho(x_0))} H^4\|A^o\|^2 d\mu\Big)^{\frac{n}{2}} \\
&+ \frac{1}{\rho^{2n}}\Big(\int_{f^{-1}(B_\rho(x_0))}\|A^o\|^2 d\mu\Big)^{\frac{n}{2}}\bigg].
\end{aligned}$$

Combining these inequalities, we have

$$\begin{aligned}
&\|A^o\|^4_{\infty,f^{-1}(B_{\rho/2}(x_0))} \\
&\quad \le c\|A^o\|^{4-n}_{2,f^{-1}(B_\rho(x_0))}\bigg[\Big(\int_{f^{-1}(B_\rho(x_0))} F^2 d\mu\Big)^{\frac{n}{2}} \\
&\qquad + \frac{1}{\rho^{2n}}\Big[\|A^o\|^{2n}_{2,f^{-1}(B_\rho(x_0))} + \|A\|^n_{2,f^{-1}(B_\rho(x_0))} + \|A^o\|^n_{2,f^{-1}(B_\rho(x_0))}\Big] \\
&\qquad + (c_h)^{\frac{n}{2}} + (n-2)(c_{\gamma 1})^{\frac{9}{2}}\|A^o\|^{\frac{27}{4}}_{3,[\gamma>0]}\bigg].
\end{aligned}$$

Taking square roots and using the smallness assumption with, in the $n = 3$ case, (AB) and the Hölder inequality, gives the result. $\square$

An identical proof gives an analogous estimate for the full curvature tensor.

Corollary 5.9. *Suppose $n \in \{2, 3\}$ and let $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow, $\gamma$ a cutoff function as in* (25) *with $\chi_{B_{\rho/2}} < \tilde{\gamma} < \chi_{B_\rho}$ and suppose that for any $x \in \mathbb{R}^3$, $\|A\|^n_{n,f^{-1}(B_\rho(x))} \le \|A\|^n_{n,f^{-1}(B_\rho(x_0))} < \epsilon_0$ where $\epsilon_0 > 0$ is as in Proposition 4.7. Further, assume that the constraint function $h$ satisfies* (A2)*, and in the case where*

*$n = 3$ that* (AB) *is satisfied. Then*

$$\|A\|^2_{\infty,f^{-1}(B_{\rho/2}(x_0))}$$
$$\leq c\|A\|^{\frac{4-n}{2}}_{2,f^{-1}(B_\rho(x_0))}\Big[\|F\|^{\frac{n}{2}}_{2,f^{-1}(B_\rho(x_0))} + \frac{1}{\rho^n}\|A\|^{\frac{n}{2}}_{2,f^{-1}(B_\rho(x_0))} + (c_h)^{\frac{n}{4}} + (n-2)\epsilon_0\Big],$$

*where $c = c(n, \epsilon_0, C_{AB})$.*

For later application we prove the following estimate.

Proposition 5.10. *Suppose $f : M^2 \times [0, T^*] \to \mathbb{R}^3$ is a* (CSD) *flow where there exists a $\sigma < \infty$ such that*

$$\int_M \|A\|^2 d\mu \leq \sigma \qquad on \qquad [0, T^*].$$

*Then there exist constants $\epsilon_1 = \epsilon_1(M_0)$, $c_2 = c_2(\epsilon_0, \sigma) > 0$ and $x_1 \in \mathbb{R}^3$ such that if $\rho > 0$ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} \|A^o\|^2 d\mu\Big|_{t=0} \leq \int_{f^{-1}(B_\rho(x_0))} \|A^o\|^2 d\mu\Big|_{t=0} < \epsilon_1 \text{ for every } x \in \mathbb{R}^3, \tag{129}$$

*and the constraint function $h$ satisfies* (A2)*, then at any time $0 \leq t < t_1 = \min\{c_2\rho^4, T^*\}$ we have*

$$\int_{f^{-1}(B_\rho(x_1))} \|A^o\|^2 d\mu + \int_0^t \int_{f^{-1}(B_\rho(x_1))} \Big(\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2\Big) d\mu d\tau$$
$$\leq c[\epsilon_1 + (c_h + \sigma\rho^{-4})t], \tag{130}$$

*and*

$$\int_0^t \|A^o\|^4_{\infty,f^{-1}(B_\rho(x_1))} d\tau \leq c[\epsilon_1 + (c_h + \sigma\rho^{-4})t]. \tag{131}$$

*Also, for $0 < \rho' \leq \rho$ and $\tau \leq \min\{c_2(\rho')^4, T^*\}$ we have*

$$\int_{f^{-1}(B_{\rho'/2}(x_1))} \|A\|^2 d\mu\Big|_{t=\tau} \leq \int_{f^{-1}(B_{\rho'}(x_1))} \|A\|^2 d\mu\Big|_{t=0} + c(c_h + \sigma(\rho')^{-4})\tau.$$

PROOF. Motivated by the fact that $2^4$ balls $B_{\rho/2}$ can be used to cover a ball $B_\rho$, we set $\epsilon_1 \le \frac{\epsilon_0}{4\cdot 2^4}$, where $\epsilon_0$ is as in Proposition 5.5. Note, importantly, that this implies $\epsilon_1 < \epsilon_0$. Assume (129) is satisfied on $[0,t]$ and integrate the estimate in Theorem 5.6 to obtain

(132)

$$\int_{f^{-1}(B_{\rho/2}(x_1))} \|A^o\|^2 d\mu + \frac{c_1}{2}\int_0^t \int_{f^{-1}(B_{\rho/2}(x_1))} \|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2 d\mu d\tau$$
$$\le \epsilon_1 + c(c_h + \sigma\rho^{-4})t.$$

Since

$$0 < t \le \frac{\epsilon_0}{c4\cdot 2^4}\left(\frac{1}{\sigma + c_h\rho^4}\right)\rho^4 = c_2\rho^4,$$

we use (132) to derive

$$\begin{aligned}\int_{f^{-1}(B_\rho(x_1))} \|A^o\|^2 d\mu + c_1\int_0^t \int_{f^{-1}(B_\rho(x_1))} &\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2\|A\|^2 + \|A\|^4\|A^o\|^2 d\mu d\tau \\ &\le 2^4\Big(\epsilon_1 + c(c_h + \sigma\rho^{-4})t\Big) \\ &\le 2^4\left(\epsilon_0\frac{1}{4\cdot 2^4} + \epsilon_0\frac{1}{4\cdot 2^4}\right) \\ &\le \frac{\epsilon_0}{2}.\end{aligned}$$

Therefore (124) holds up to time $t = t_1$ and (130) follows. Using a covering argument and combining the estimate of Lemma 5.7 with (130) above gives

$$\begin{aligned}&\int_0^t \|A^o\|^4_{\infty, f^{-1}(B_{\rho/2}(x_1))} d\tau \\ &\quad\le c\int_0^t \|A^o\|^2_{2,f^{-1}(B_\rho(x_1))}\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 + \|A\|^4\|A^o\|^2 + \rho^{-4}\|A^o\|^2 d\mu d\tau \\ &\quad\le c\epsilon_0\int_0^t\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 + \|A\|^4\|A^o\|^2 d\mu d\tau + c\epsilon_0^2\rho^{-4} \\ &\quad\le c\epsilon_0\Big(\epsilon_1 + c(c_h + \sigma\rho^{-4})t + c\epsilon_0\rho^{-4}\Big) \\ &\quad\le c\Big[\epsilon_1 + (c_h + \sigma\rho^{-4})t\Big].\end{aligned}$$

This shows (131). Integrating both conclusions of Theorem (5.6) and combining the result completes the proof. □

## 4. Proof of the interior estimates

We return to our old growth condition (GC) and use the a priori estimates for the constraint function proved in Chapter 3. In particular, note that this maximises the 'overlap' in the class of constraint functions to which both the Lifespan Theorem and Interior Estimates apply. In the statement below we use the convention that $0 < T^* < T' < T$.

THEOREM 5.11 (Interior estimates). *Suppose $n \in \{2,3\}$ and let $f : M^n \times (0,T'] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with the constraint function $h$ satisfying the conditions of the Lifespan Theorem. Further assume* (AB) *and that*

$$\sup_{0<t\le T^*}\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu \le \epsilon_0,\ \textit{for } T^* \le c\rho^4, \tag{133}$$

*where $m = m(h)$ is as in the Lifespan Theorem. Then for any $k \in \mathbb{N}_0$ we have at time $t \in (0,T^*]$ the estimates*

$$\|\nabla_{(k)}A\|_{2,f^{-1}(B_{\rho/2}(x))} \le c_k\sqrt{\epsilon_0}\,t^{-\frac{k}{4}} \tag{134}$$

$$\|\nabla_{(k)}A\|_{\infty,f^{-1}(B_{\rho/2}(x))} \le c_k\sqrt{\epsilon_0}\,t^{-\frac{k+1}{4}}, \tag{135}$$

*for some $x \in \mathbb{R}^{n+1}$, where $c_k = c_k\Big(k,n,\rho,T^*,\|\nabla_{(k)}A\|_{2,f^{-1}(B_\rho(x_0))}\big|_{t=0}\Big)$.*

PROOF. We may assume $\rho = 1$, since if otherwise we instead consider the scaled immersions $f_\rho(p,t) = \frac{1}{\rho}f(p,\rho^4 t)$. Recall now the estimates from Proposition 3.23 and Proposition 3.26:

$$\int_0^{T^*}\int_{f^{-1}(B_{\frac{3}{4}}(x_0))}\|\nabla_{(2)}A\|^2 + \|A\|^6 d\mu dt \le c\epsilon_0,\ \text{and} \tag{136}$$

$$\int_0^{T^*} \|A\|^4_{\infty,f^{-1}(B_{\frac{3}{4}}(x_0))}dt \le c\epsilon. \tag{137}$$

Note that the constant above, and all further constants in this proof, depend additionally on $C_{AB}$ for $n=3$. We now fix a cutoff function $\gamma = \tilde\gamma \circ f$ by choosing $\tilde\gamma \in C^2_c(\mathbb{R}^3)$ such that

$$\chi_{B_{\frac{1}{2}(x_0)}} \le \tilde\gamma \le \chi_{B_1(x_0)}, \qquad \text{and} \qquad \|D\tilde\gamma\|_\infty + \|D^2\tilde\gamma\|_\infty \le c.$$

We also define a family of cutoff functions in time $\eta_j$ by

$$\eta_j(t) = \begin{cases} 0, & t \le (j-1)\frac{T^*}{m} \\ \frac{m}{T^*}\Big(t-(j-1)\frac{T^*}{m}\Big), & (j-1)T^* < t < j\frac{T^*}{m} \\ 1, & t \ge j\frac{T^*}{m}, \end{cases}$$

where $j \in [0,m]$ and $m \in \mathbb{N}_0$. Note that $\eta_0 \equiv 1$ on $[0,T^*]$, $\eta_m(T^*) = 1$ and the derivative satisfies

$$\eta_j'(t) = \frac{m}{T^*} \quad \text{for} \quad t \in \Big((j-1)\frac{T^*}{m}, j\frac{T^*}{m}\Big),$$

zero elsewhere. This is succinctly written as

$$0 \le \eta_j' \le \frac{m}{T^*}\eta_{j-1}, \quad t \in \Big[0, j\frac{T^*}{m}\Big], \tag{138}$$

although of course remains valid for $t > j\frac{T^*}{m}$ and $t < 0$. Recall the following inequality from the proof of Proposition 3.26:

$$\begin{aligned} \frac{d}{dt}\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu + \frac{1}{2}\int_M \|\nabla_{(2j+2)}A\|^2\gamma^{4j+8}d\mu \\ \le c\Big(\|A\|^4_{\infty,f^{-1}(B_{\frac{3}{4}}(x_0))} + h^{\frac{4}{3}}\Big)\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu \\ + c\|A\|^2_{2,f^{-1}(B_{\frac{3}{4}}(x_0))}\Big(1 + \|A\|^4_{\infty,f^{-1}(B_{\frac{3}{4}}(x_0))} + h + h^{\frac{4}{3}}\Big). \end{aligned}$$

We compute

$$
\begin{aligned}
&\frac{d}{dt}\Big(\eta_j\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu\Big)\\
&\qquad= \eta_j'(t)\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu + \eta_j(t)\frac{d}{dt}\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu\\
&\qquad\le c\Big(\|A\|^4_{\infty,f^{-1}(B_{\frac34}(x_0))} + h^{\frac43}\Big)\Big[\eta_j(t)\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu\Big]\\
&\qquad\quad + c\epsilon_0\Big(1+\|A\|^4_{\infty,f^{-1}(B_{\frac34}(x_0))} + h + h^{\frac43}\Big)\eta_j(t)\\
&\qquad\quad - \eta_j(t)\frac12\int_M \|\nabla_{(2j+2)}A\|^2\gamma^{4j+8}d\mu\\
&\qquad\quad + \eta_{j-1}(t)\frac{m}{T^*}\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu.
\end{aligned}
\tag{139}
$$

We claim, for $0\le j\le m$, $t\in(0,T^*]$,

$$
\eta_j\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu + \frac12\int_0^t \eta_j(\tau)\Big[\int_M \|\nabla_{(2j+2)}A\|^2\gamma^{4j+8}d\mu\Big]d\tau \le \frac{c(m)\epsilon_0}{(T^*)^j}. \tag{140}
$$

The proof is by induction on (139). For $j=0$, (140) follows by combining (133) with (136). That is,

$$
\int_M \|A\|^2\gamma^4 d\mu + \frac12\int_0^t\int_M \|\nabla_{(2)}A\|^2\gamma^8 d\mu d\tau \le c\epsilon_0.
$$

Integrating (139) on $[0,T^*]$ gives, for $j\ge1$,

$$
\begin{aligned}
&\eta_j\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu + \frac12\int_0^t \eta_j(\tau)\Big[\int_M \|\nabla_{(2j+2)}A\|^2\gamma^{4j+8}d\mu\Big]d\tau\\
&\qquad\le c\int_0^t \Big(1+\|A\|^4_{\infty,f^{-1}(B_{\frac34}(x_0))}\Big)\Big[\eta_j(t)\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu\Big]d\tau\\
&\qquad\quad + c\epsilon_0\int_0^t\Big(1+\|A\|^4_{\infty,f^{-1}(B_{\frac34}(x_0))}\Big)d\tau\\
&\qquad\quad + \frac{m}{T^*}\int_0^t \eta_{j-1}(\tau)\Big[\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu\Big]d\tau,
\end{aligned}
$$

where we estimated $0\le\eta_j\le1$ and used $\eta_j(0)=0$ if $j\ge1$. We also used (CB). Invoking (137) and Gronwall's inequality (c.f. the proof of Proposition 3.26):

$$
\eta_j\int_M \|\nabla_{(2j)}A\|^2\gamma^{4j+4}d\mu + \frac12\int_0^t \eta_j(\tau)\Big[\int_M \|\nabla_{(2j+2)}A\|^2\gamma^{4j+8}d\mu\Big]d\tau
$$

$$\leq c\epsilon_0 + c\Big(\frac{m}{T^*}\Big)\Big(\frac{c(m)}{(T^*)^{j-1}}\Big)\epsilon_0$$

$$\leq \frac{c(m)\epsilon_0}{(T^*)^j}. \tag{141}$$

With (141) we have shown (140). Therefore, at $t = T^*$,

$$\int_M \|\nabla_{(2m)}A\|^2\gamma^{4m+4}d\mu \leq \epsilon_0\frac{c(m)}{(T^*)^m}.$$

We interpolate with one of our interpolation inequalities from Appendix A to obtain the analogous statement for the odd derivatives; Lemma A.4 for example. Renaming $T^*$ to $t$, we have shown (135). The $L^\infty$ estimate (134) follows from Proposition 3.21 and (137) exactly as in the proof of Proposition 3.26. $\square$

CHAPTER 6

# Blow up analysis for constrained surface diffusion flows

## 1. Introduction

In this chapter we will detail the blowup construction and the properties of such which we shall use in our asymptotic analysis. This is instrumental in concluding our final major result, long time existence and convergence to spheres for certain initial manifolds and certain (CSD) flows. Briefly, the idea is as follows. From the Lifespan Theorem, we know that the only way for our flow to halt and lose regularity is if the curvature concentrates. To contradict this, we will assume we have the tracefree curvature initially small in $L^2$ and also that we are still plagued by a finite time curvature singularity. The strategy from here is to study intensely the properties of the finite time singularity. We construct a sequence of rescalings around the final time, a so-called 'blowup', and study the properties of the limit. In this chapter we will show that the blowup is a non-umbilic stationary surface (with $\Delta H + h \equiv 0$) with small tracefree curvature and controlled growth of curvature at infinity. This is in direct contradiction with the Gap Lemma, and so we must not have had a finite time singularity at all. In this chapter we will be concerned with the construction and properties of the blowup. We leave the asymptotic analysis to Chapter 7.

This technique, of analysing rescaled solutions to attack problems of singularity development, has appeared in the literature for quite some time. However it is only recently that it has been applied to problems in geometric analysis. For our purposes, the key references are again Kuwert and Schätzle [**36, 38**]. Our technique however differs from that of [**36**] in several ways, as it must. There, Willmore flow is studied, and being an $L^2$ gradient flow of curvature lends itself naturally to this analysis. This can be seen in many ways, although the most telling is surely the stationary nature of the blowup. This is related to two key facts: the scale invariance of the $L^2$ norm of curvature and the flow, and the monotonic decrease of the $L^2$ norm of curvature in time. Both of these facts combine to give one that the blowup at *any* finite time singularity is stationary. We do not even recover this statement, and we must work a lot harder to obtain a useful analogue. To begin with, we do not have any $L^2$ norm of curvature monotone in time. Thus we first need to show that the $L^2$ norm of curvature, while not monotone, is at least well-controlled. This is Almost Preservation, Theorem 6.3. With this in hand, one can obtain that the blowup is stationary by the use of one of our earlier localised integral estimates; however this is of course only valid if the $L^2$ norm of tracefree curvature at initial time is small. Therefore, we do not obtain that *all* blowups of finite time singularities are stationary. We only obtain that blowups with small tracefree curvature are stationary, which is a strictly weaker result than that in [**36**]. Given that the natural energy for surface diffusion flow is surface area, the same as for mean curvature flow, one would expect self-similar or translating solutions to be a common result of rescaling. Our analysis here only excludes this possibility—in fact excludes the possibility of any singularity at all—in the case where one already has small tracefree curvature, which is more in line with what one would expect.

Note that we *do not* show the reverse implication: that a stationary blowup satisfies a small tracefree curvature condition.

For the constrained flows, the above difficulties are only part of the story. In the case where we have a non-trivial constraint function, the surface area becomes difficult to control from below. While excluding planes is no challenge (the blowup we construct will have some curvature), our main method of contradicting the possible development of a sphere in the limit is to prove a uniform lower bound on area. This results in more conditions placed upon the constraint function, and is the subject of Proposition 6.7.

Throughout this chapter, and indeed for the rest of the thesis, we will only be considering flows of surfaces, i.e. families of immersions $f : M^2 \times [0, T) \to \mathbb{R}^3$, and only in one codimension. There is good reason for this. The asymptotics of our previous estimates and the covering argument we have used to obtain crucial properties of our rescaled solutions requires it. If we had codimension greater than one, then we would need too many ambient balls to cover key regions of $M$. This problem does not arise in the case of the Willmore flow [**36**], as there it is easy to use the gradient flow structure of the equation instead to obtain the valuable properties of the blowup.

## 2. Compactness theorem and construction of blowup

We will be constructing a sequence of immersions and wish to study the geometric properties of a 'limit' immersion. The following theorem, a localisation of a result due to Langer [**40**] by Kuwert and Schätzle [**36**], defines precisely what is meant by 'limit' immersion, and gives sufficient conditions for its existence.

THEOREM 6.1. *Let $f_j : M_j \to \mathbb{R}^3$ be a sequence of proper immersions, where $M_j$ is a surface without boundary. Let*

$$M_j(R) = \{p \in M_j : |f_j(p)| < R\}$$

*and assume the bounds*

$$\mu_j(M_j(R)) \le c(R) \quad \text{for any } R > 0,$$
$$\|\nabla_{(k)} A\|_{\infty, M_j} \le c(k) \quad \text{for any } k \in \mathbb{N}_0.$$

*Then there exists a proper immersion $\tilde{f} : \tilde{M} \to \mathbb{R}^3$, where $\tilde{M}$ is again a surface without boundary, such that after passing to a subsequence we have a representation*

$$f_j \circ \phi_j = \tilde{f} + u_j \quad \text{on} \quad \tilde{M}(j) = \{p \in \tilde{M} : |\tilde{f}(p)| < j\}$$

*with the following properties:*

$$\phi_j : \tilde{M}(j) \to U_j \subset M_j \text{ is diffeomorphic,}$$
$$M_j(R) \subset U_j \text{ if } j \ge j(R),$$
$$u_j \in C^\infty(\tilde{M}(j), \mathbb{R}^3) \text{ is normal along } \tilde{f},$$
$$\|\tilde{\nabla}_{(k)} u_j\|_{\infty, \tilde{M}(j)} \to 0 \text{ as } j \to \infty \text{ for any } k \in \mathbb{N}_0. \tag{142}$$

The theorem says that on any ball $B_R(0)$ the immersion $f_j$ can be written as a normal graph with small norm for $j$ large over a limit immersion $\tilde{f}$, after suitably reparametrising with $\phi_j$.

Let $f : M^2 \times [0, T) \to \mathbb{R}^3$ be a smooth (CSD) flow defined on a closed surface $M^2$, where $0 < T \le \infty$. Define

$$\eta(r, t) = \sup_{x \in \mathbb{R}^3} \int_{f^{-1}(B_r(x))} \|A\|^2 d\mu.$$

Let $r_j$ be an arbitrary decreasing sequence with $r_j \searrow 0$ and assume that

$$t_j = \inf\{t \geq 0 : \eta(r_j, t) > \epsilon_1\} < T,$$

where $\epsilon_1 = \epsilon_0 c_0$ and $\epsilon_0 > 0, c_0 = \frac{1}{c}$ are as in the Lifespan Theorem.

LEMMA 6.2. *With the definitions above, we have*

$$\int_{f^{-1}(B_{r_j}(x))} \|A\|^2 d\mu \bigg|_{t=t_j} \leq \epsilon_1 \text{ for any } x \in \mathbb{R}^3,$$

*and*

$$\int_{f^{-1}\overline{(B_{r_j}(x_j))}} \|A\|^2 d\mu \bigg|_{t=t_j} \geq \epsilon_1 \text{ for some } x_j \in \mathbb{R}^3.$$

PROOF. The first statement is a direct consequence of the definition of $t_j$. For the second, fix $j$ and consider a sequence $\nu_i \to \infty$, $\nu_0 < r_j^2$. Consider times $t_j + \nu_i^{-1} \searrow t_j$ and radii $r_j - \nu_i^{-2} \nearrow r_j$. Then for each $i$ there exists an $\big(x_j\big)_i$ such that

$$\int_{f^{-1}\left(B_{r_j - \nu^{-2}}((x_j)_i)\right)} \|A\|^2 d\mu \bigg|_{t=t_j+\nu^{-1}} \geq \epsilon_1.$$

Taking $\nu_i \to \infty$ in the above equation gives the second statement. ☐

We now rescale $f$. Define immersions

$$f_j : M^2 \times \Big[-r_j^{-4} t_j, r_j^{-4}(T - t_j)\Big) \to \mathbb{R}^3, \qquad f_j(p,t) = \frac{1}{r_j}\Big(f(p, t_j + r_j^4 t) - x_j\Big).$$

The sequence of immersions $f_j$ can be thought of as 'zooming in' on the assumed curvature singularity at time $T$. Let $\eta_j(r,t)$ be $\eta$ with respect to the immersion $f_j$. Then from the lemma above we have $\eta_j(1,t) \leq \epsilon_1$ for $t \leq 0$ and

$$\int_{f_j^{-1}\overline{(B_1(0))}} \|A\|^2 d\mu \bigg|_{t=0} \geq \epsilon_1.$$

The Lifespan Theorem implies $r_j^{-4}(T - t_j) \geq c_0$ and also that

$$\eta_j(1,t) \leq \epsilon_0 \quad \text{for} \quad -r_j^{-4} t_j < 0 < t \leq c_0. \tag{143}$$

Thus we may apply the Interior Estimates of the previous chapter on parabolic cylinders $B_1(x) \times (t-1, t]$ to obtain

$$\|\nabla_{(k)} A\|_{\infty, f_j} \le c(k) \quad \text{for} \quad -r_j^{-4} t_j + 1 \le t \le c_0, \tag{144}$$

for every $k \in \mathbb{N}_0$. We also need a local area bound (c.f. Proposition 6.7). Since we know the Willmore energy is bounded from Theorem 6.3, it is enough to use a lemma due to Simon [**54**] to conclude

$$\frac{|f_j^{-1}(B_R(0))|}{R^2} \le c\bigg( \int_{f_j^{-1}(B_{2R}(0))} \|A\|^2 d\mu + 4\pi\chi(M) \bigg) < \infty.$$

That is, we do not need to assume (AB). Using Theorem 6.1 with the sequence $f_j = f_j(\cdot, 0) : M^2 \to \mathbb{R}^3$ we obtain a limit immersion $\tilde{f}_0 : \tilde{M} \to \mathbb{R}^3$. Let $\phi_j : \tilde{M}(j) \to U_j \subset M^2$ be as in (142). Then the reparametrisation

$$f_j(\phi_j, \cdot) : \tilde{M}(j) \times [0, c_0] \to \mathbb{R}^3 \tag{145}$$

is a (CSD) flow with initial data

$$f_j(\phi_j, 0) = \tilde{f}_0 + u_j : \tilde{M}(j) \to \mathbb{R}^3. \tag{146}$$

The flows (145) satisfy the curvature bounds (144) and have initial data converging locally in $C^k$, for every $k \in \mathbb{N}_0$, to the immersion $\tilde{f}_0 : M^2 \to \mathbb{R}^3$. By converting the curvature bounds to partial derivative bounds in parabolic cylinders (as in the proof of the Lifespan Theorem, final step) we obtain the locally smooth convergence

$$f_j(\phi_j, \cdot) \to \tilde{f}, \tag{147}$$

where $\tilde{f} : \tilde{M} \times [0, c_0] \to \mathbb{R}^3$ is a (CSD) flow with initial data $f_0$.

## 3. Blow up with small initial tracefree curvature

We wish to show that the blowup $\tilde{f}$ is stationary. Unfortunately, we do not have a guarantee that our (CSD) flows are gradient flows in any practical sense, and so can not expect to obtain a result analogous to [**36**]. Indeed, our argument differs in several fundamental ways and the conclusion is much weaker. However, we still obtain sufficient information to proceed along the same fundamental lines as [**36**] and obtain, in the end, long time existence. Our argument involves the previously proved estimate Lemma 5.4 and the following almost preservation of small tracefree curvature theorem.

THEOREM 6.3 (Almost Preservation). *Let $f : M^2 \times [0,T) \rightarrow \mathbb{R}^3$ be a* (CSD) *flow with $h$ satisfying the hypothesis of Proposition 5.10. Then for any $\epsilon_0 > 0$ there exists a constant $\epsilon_1 = \epsilon_1(\epsilon_0) > 0$ such that if*

$$\int_M \|A^o\|^2 d\mu\bigg|_{t=0} \le \epsilon_1 \tag{148}$$

*then for all $t \in [0,T]$*

$$\int_M \|A^o\|^2 d\mu \le \epsilon_0. \tag{149}$$

PROOF. This proof is a somewhat straightforward application of Proposition 5.10. Although we cannot simply 'apply' Proposition 5.10 with '$\rho = \infty$', this is still the underlying idea.

Let $f_j$ be the sequence of rescaled immersions constructed above. Recall the sequence of times $t_j$, where $t_j \nearrow T$ and each time is chosen to correspond with a concentration of $\epsilon_0$ curvature at the scale $r_j$. We work in intervals $[0, t_j]$. From short time existence, we have the existence of $\sigma_j$ such that $\|A\|_2^2 \le \sigma_j$, although we acknowledge that (from for example the Lifespan Theorem) in the case where $T < \infty$, $\sigma_j$ necessarily approaches infinity. Let $\rho_j = \max\{\sigma_j^{-1}, r_j^{-1}\}$, so that we have $\rho_j \nearrow \infty$. Now, applying Proposition 5.10 in time intervals $[0, t_j]$ with $\rho = \rho_j$ we obtain

$$\int_{f^{-1}(B_{\rho_j}(0)} \|A^o\|^2 d\mu < \epsilon_0 \quad \text{for} \quad t \in \Big[0, \min\{t_j, c\sigma_j^{-1}\rho_j^4\}\Big]. \tag{150}$$

Taking $\rho_j \nearrow \infty$ we have

$$\int_M \|A^o\|^2 d\mu < \epsilon_0 \quad \text{for} \quad t \in [0, T],$$

as required. □

REMARK. It is easy to see that for any $\epsilon_0$ one may take $\epsilon_1 = 2^{-6}\epsilon_0$. However, this is far from optimal. Since (148) is satisfied over the whole manifold, the covering argument used in the proof of Proposition 5.10 may be improved by considering a sequence of radii $\rho_i \nearrow \rho$ instead of only $\rho/2$. Then as $i$ and $\rho$ increase, one obtains a sequence $(\epsilon_1)_i \nearrow \epsilon_0$. So the above condition (148) may be strengthened to $\|A^o\|_2^2\Big|_{t=0} < \epsilon_0$.

Note that if we had a statement to the effect of

$$\int_M \|A^o\|^2 d\mu < \epsilon_0 \quad \Longrightarrow \quad \text{guaranteed short time existence,}$$

then Theorem 6.3 above would immediately imply that for (CSD) flows with small initial tracefree curvature we have long time existence. Unfortunately the relevant theorem, the Lifespan Theorem, requires $\|A\|_2^2$ small. With a little thought one realises that such a condition can not be in general satisfied (consider $\epsilon_0 < 2\pi$).

This implies that such a direct approach is unrealistic. This is why we have chosen to employ a ‘round about’ method to demonstrate long time existence.

We now show that the blowup is stationary.

THEOREM 6.4. *Let $f : M^2 \times [0,T) \to \mathbb{R}^3$ be a* (CSD) *flow with the constraint function $h$ satisfying* (GC)*, the hypotheses of Lemma 5.4, Proposition 5.10 and*

$$\text{(AS)} \qquad \Delta H \equiv 0 \Longrightarrow h = 0.$$

*Then there exists an absolute constant $\epsilon_1 > 0$ such that if*

$$\int_M \|A^o\|^2 d\mu\bigg|_{t=0} \le \epsilon_1,$$

*the blowup $\tilde{f}$ as constructed above is stationary.*

PROOF. First, note that by the limiting construction Proposition 6.3, the hypothesis of Lemma 5.4 is satisfied for all $\gamma$ as in (25). Therefore we use Lemma 5.4 to imply

$$\begin{aligned}
\frac{d}{dt}\int_M \frac{1}{2}H^2\gamma^s d\mu &+ c_e\int_M \Big(\|\nabla_{(2)}H\|^2 + \|\nabla H\|^2H^2 + H^4\|A^o\|^2\Big)\gamma^s d\mu \\
&- \frac{c_e}{2}\int_M \|\nabla_{(2)}H\|^2 + \|\nabla H\|^2H^2 + H^4\|A^o\|^2 d\mu \\
&\le c\Big[(c_{\tilde{\gamma}1})^4 + (c_{\gamma 1})^4 + (c_{\gamma 2})^2 + (c_{\gamma 2})^4\Big]\|A\|^2_{2,[\gamma>0]} + c(c_{\gamma 1})^4\|A^o\|^4_{2,[\gamma>0]},
\end{aligned}$$

Set $\gamma$ to be a cutoff function for $f_j$ on $U_j$. Then rearranging the above we have

$$\begin{aligned}
&c_e\int_{U_j}\|\nabla_{(2)}H\|^2 + \|\nabla H\|^2H^2 + H^4\|A^o\|^2 d\mu \\
&-\frac{c_e}{2}\int_{M_j}\|\nabla_{(2)}H\|^2 + \|\nabla H\|^2H^2 + H^4\|A^o\|^2 d\mu \\
&\quad\le -\frac{d}{dt}\int_{U_j}\frac{1}{2}H^2 d\mu + c\Big[\frac{1}{j^4} + \frac{1}{j^8}\Big]\|A\|^2_{2,M_j(R+1)} + c\frac{1}{j^4}\|A^o\|^4_{2,M_j(R+1)},
\end{aligned}$$

where $R = R(j)$ is the largest integer such that $M_j(R) \subset U_j$. Denote by $I$ the integrand $\|\nabla_{(2)}H\|^2 + \|\nabla H\|^2H^2 + H^4\|A^o\|^2$. With a slight abuse of notation we now compute

$$
\begin{aligned}
&\frac{c_e}{2}\int_0^{c_0}\int_{\tilde M} I(\tilde f(\phi_\infty,\cdot))d\mu_{\tilde f(\phi_\infty,\cdot)}dt \\
&\quad = \lim_{j\to\infty}\left(c_e\int_0^{c_0}\left(\int_{U_j} I(f_j(\cdot,\cdot))d\mu_j - \frac{1}{2}\int_{M_j} I(f_j(\cdot,\cdot))d\mu_j\right)dt\right) \\
&\quad \le \lim_{j\to\infty}\Bigg(\left(\frac{1}{j^4}+\frac{1}{j^8}\right)\int_0^{c_0}\|A\|^2_{2,M_j(R+1)}dt + \frac{1}{j^4}\int_0^{c_0}\|A^o\|^2_{2,M_j(R+1)}dt \\
&\qquad - c\Big[\int_{U_j} H^2 d\mu\Big|_{t=c_0} - \int_{U_j} H^2 d\mu\Big|_{t=0}\Big]\Bigg) \\
&\quad \le \lim_{j\to\infty}\Bigg(\left(\frac{1}{j^4}+\frac{1}{j^8}\right)\int_0^{c_0}\|A\|^2_{2,M_j(R+1)}dt + \frac{1}{j^4}\int_0^{c_0}\|A^o\|^2_{2,M_j(R+1)}dt \\
&\qquad - c\Big[\int_{M(j)} H^2 d\mu\Big|_{t=t_j+r_j^4c_0} - \int_{M(j)} H^2 d\mu\Big|_{t=t_j}\Big]\Bigg) \\
&\quad \le \lim_{j\to\infty}\Bigg(\left(\frac{1}{j^4}+\frac{1}{j^8}\right)\int_0^{c_0} 3\sqrt{3}(cj)^3\eta_j(1,t)dt + \frac{1}{j^4}\int_0^{c_0}\epsilon_0 dt \\
&\qquad - c\Big[\int_{M(j)} H^2 d\mu\Big|_{t=t_j+r_j^4c_0} - \int_{M(j)} H^2 d\mu\Big|_{t=t_j}\Big]\Bigg) \\
&\quad \le \lim_{j\to\infty}\Bigg(\left(\frac{1}{j^4}+\frac{1}{j^8}\right)\int_0^{c_0} 3\sqrt{3}(cj)^3\epsilon_0 dt + \frac{1}{j^4}\int_0^{c_0}\epsilon_0 dt \\
&\qquad - c\Big[\int_{M(j)} H^2 d\mu\Big|_{t=t_j+r_j^4c_0} - \int_{M(j)} H^2 d\mu\Big|_{t=t_j}\Big]\Bigg) \\
&\quad \le \lim_{j\to\infty}\Bigg(\left(\frac{1}{j}+\frac{1}{j^5}\right)3\sqrt{3}c^3\epsilon_0 c_0 + \frac{1}{j^4}\epsilon_0 c_0 \\
&\qquad - c\Big[\int_{M(j)} H^2 d\mu\Big|_{t=t_j+r_j^4c_0} - \int_{M(j)} H^2 d\mu\Big|_{t=t_j}\Big]\Bigg).
\end{aligned}
$$

We will now bring in the limit. Note that we used a covering argument with (143) in the above computation. Therefore,

$$
\begin{aligned}
&\frac{c_e}{2}\int_0^{c_0}\int_{\tilde M} I(\tilde f(\phi_\infty,\cdot))d\mu_{\tilde f(\phi_\infty,\cdot)}dt \\
&\quad \le -c\lim_{j\to\infty}\Big[\int_{M(j)} H^2 d\mu\Big|_{t=t_j+r_j^4c_0} - \int_{M(j)} H^2 d\mu\Big|_{t=t_j}\Big]
\end{aligned}
$$

$$\le -c\Big[\int_{\tilde M} H^2 d\mu\Big|_{t=T} - \int_{\tilde M} H^2 d\mu\Big|_{t=T}\Big]$$
$$\le 0.$$

We used the fact that $\lim_{j\to\infty} t_j + r_j^4 c_0 = \lim_{j\to\infty} t_j = T$ above. Note that we also need to ensure that the limit

$$\lim_{t\to T}\int_M H^2 d\mu$$

exists. A lower bound is trivial ($M$ is closed) and an upper bound follows from Theorem 6.3, so we know

$$4\pi \le \int_M H^2 d\mu \le 4\pi + \epsilon_0.$$

It remains only to rule out the possibility that $\|H\|_2^2$ oscillates with infinite frequency approaching the final time. This is easy to do using an argument similar to that of Theorem 6.3 with the estimate of Theorem 5.6, which we briefly summarise. Let $\delta \in (0,T)$ and assume that for some $t^* \in [\delta, T)$ and $c > 0$ we have $\frac{\partial}{\partial t}\|H\|_2^2\Big|_{t=t^*} > c$. This contradicts Theorem 5.6 at $t = t^*$, after taking $\rho \to \infty$ as in the proof of Theorem 6.3.

This shows that $I(\tilde f) = 0$ and so (among other facts) $\Delta H(\tilde f) = 0$. Therefore, using (AS) we have that $h = 0$ and so

$$\frac{\partial}{\partial t}\tilde f = 0.$$

This finishes the proof. □

Remark. The condition (AS) above is natural, although necessarily restrictive. Since our overall goal is to prove that if one begins a (CSD) flow with small distance to a sphere then the flow exponentially converges to a sphere, we need spheres themselves to be well-behaved. Of course one can construct constraints $h$ which satisfy all of our previous conditions yet are not zero on a sphere. Now we know by using the Gap Lemma (for example) with surface diffusion flow that the only compact manifolds with $\Delta H \equiv 0$ and small tracefree curvature are spheres. Therefore it becomes natural that we demand $h = 0$ for spheres. Indeed, this is the essence of the growth condition placed on $h$ in Chapter 4 on the Gap Lemma. The condition (AS), viewed in light of the Gap Lemma, is thus equivalent to the growth condition in Chapter 4, when one considers manifolds which satisfy a small tracefree curvature condition and restricted growth of curvature at infinity. Since these are exactly the manifolds which interest us, one may safely consider (AS) to be no further restriction to the growth condition already required by the Gap Lemma, although of course outside this set the two conditions differ.

REMARK. One may bypass the separate statement of the Gap Lemma by noticing that if $I(\tilde{f}) = 0$ then $\tilde{f}$ must be umbilic, and combined with the nontriviality of the blowup (Theorem 6.8 below) obtain long time existence in a slightly more efficient manner. However, since the Gap Lemma is of independent interest and follows naturally during the course of proving the required estimates for this chapter, we have treated it independently.

LEMMA 6.5. *The blowup $\tilde{f}$ constructed above is not a union of planes.*

PROOF. Due to the smooth convergence in (147) and the second conclusion in Lemma 6.2 we have

$$\int_{\tilde{f}^{-1}\overline{(B_1(0))}} \|A\|^2 d\mu \geq \epsilon_1 > 0.$$

$\square$

LEMMA 6.6. *If the blowup $\tilde{f}$ constructed above contains a connected component $C$, then in fact $\tilde{M} = C$ and $M$ is diffeomorphic to $C$.*

PROOF. For $j$ sufficiently large, $\phi_j(C)$ is open and closed in $M$. By the connectedness of $M$ we have $M = \phi_j(C)$ and thus $\tilde{M} = C$. $\square$

The reason for the previous two lemmas is to prove that $\tilde{f}$ is nontrivial. The following area bound, which requires $\|A^o\|_2^2$ small, is crucial for treating nontrivial constraint functions. The main difficulty here is that the speed of our flow differs from $\Delta H$ by the constraint function, whose impact on the evolving area element is hard to control. The proof follows [**36**], where this also causes difficulty, in the form of the extra zero order term $H\|A^o\|^2$.

PROPOSITION 6.7. *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is a* (CSD) *flow with small tracefree curvature*

$$\int_M \|A^o\|^2 d\mu \le \epsilon < \epsilon_1,$$

*and with this assumption the constraint function satisfies both*

$$\begin{aligned} -\mu(M) \int_M \|\nabla_{(2)} H\|^2 d\mu - \int_M H^2 \|\nabla H\|^2 d\mu \\ \le h \int_M H d\mu \le 2 \int_M \|\nabla A^o\|^2 d\mu + \int_M H^2 \|A^o\|^2 d\mu + c \int_M \|A^o\|^4 d\mu \end{aligned}$$

*and* (A2). *Then*

$$(1 - c\epsilon)\mu(M)\Big|_{t=0} \le \mu(M) \le (1 + c\epsilon)\mu(M)\Big|_{t=0}. \tag{151}$$

PROOF. The evolution of the area is

$$\frac{d}{dt}\mu(M) = \frac{d}{dt}\int_M d\mu = -\int_M \|\nabla H\|^2 d\mu + h \int_M H d\mu.$$

Simons' identity for $A^o$ implies that

$$2\int_M \|\nabla A^o\|^2 d\mu + \int_M H^2\|A^o\|^2 d\mu = \int_M \|\nabla H\|^2 d\mu + \int_M A^o * A^o * A^o * A^o d\mu.$$

Combining these equalities we have

$$\begin{aligned}\frac{d}{dt}\mu(M) + 2\int_M \|\nabla A^o\|^2 d\mu + \int_M H^2\|A^o\|^2 d\mu \\ = h\int_M H d\mu + \int_M A^o * A^o * A^o * A^o d\mu.\end{aligned}$$

Using our hypothesis this becomes

$$\frac{d}{dt}\mu(M) \le \mu(M)\|A^o\|_\infty^4.$$

From Proposition 5.10 we have

$$\int_0^t \|A^o\|_\infty^4 d\tau \le c\epsilon.$$

So, using Gronwall's inequality, we obtain

$$\mu(M) \le (1 + c\epsilon)\mu(M)\Big|_{t=0}.$$

For the lower bound, observe by the Michael-Simon Sobolev inequality that

$$\begin{aligned}\int_M \|\nabla H\|^2 d\mu &\le c\Big(\int_M \|\nabla_{(2)} H\| + H\|\nabla H\| d\mu\Big)^2 \\ &\le c\mu(M)\int_M \|\nabla_{(2)} H\|^2 + H^2\|\nabla H\|^2 d\mu.\end{aligned}$$

Combining this again with the evolution of the area element we have

$$\begin{aligned}\frac{d}{dt}\mu(M) &\ge -c\mu(M)\int_M \|\nabla_{(2)} H\|^2 + H^2\|\nabla H\|^2 d\mu + h\int_M H d\mu \\ &\ge -c\mu(M)\int_M \|\nabla_{(2)} H\|^2 + H^2\|\nabla H\|^2 d\mu.\end{aligned}$$

Using Proposition 5.10 again gives the lower bound required and finishes the proof.

$\square$

THEOREM 6.8 (Nontriviality of the blowup). *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is a* (CSD) *flow with constraint function satisfying*

$$h\int_M H d\mu \geq -\mu(M)\int_M \|\nabla_{(2)} H\|^2 d\mu - \int_M H^2\|\nabla H\|^2 d\mu,$$

*the growth condition* (A2) *and let $\tilde{f}$ be the blowup constructed above. Then none of the components of $\tilde{f}$ parametrises a round sphere. In particular, the blowup has a component which is a compact or noncompact nonumbilic stationary* (CSD) *surface.*

PROOF. Assume that there is a component of $\tilde{f}$ which parametrises a round sphere. Then Lemma 6.6 implies that $\tilde{f} : \tilde{M} \to \mathbb{R}^3$ is an embedded round sphere, that is, has no further components. Therefore the surface area of the blowup does not explode. The measure behaves under scaling by

$$\mu(M)\Big|_{t=t_j} = r_j^2 \mu_j(M)\Big|_{t=0}$$

and so we have

$$\mu(M)\Big|_{t=T} = \lim_{j\to\infty} \mu(M)\Big|_{t=t_j} = \lim_{j\to\infty} r_j^2\mu_j(M)\Big|_{t=0} = 0.$$

Since the maps $f_j(\cdot, 0)$ are $C^k$-close to a round sphere (up to the diffeomorphism $\phi$) we have

$$\int_M \|A^o\|^2 d\mu\Big|_{t=t_j} = \int_{M_j} \|A^o\|^2 d\mu\Big|_{t=0} \to 0.$$

Therefore for sufficiently large $j$ we may apply Proposition 6.7 and obtain a contradiction with the lower area bound. $\square$

CHAPTER 7

# Long time existence and convergence to spheres for surface diffusion flow

## 1. Introduction

In this chapter we prove the following theorem.

THEOREM 7.1. *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is a* (SD) *flow. Then there exists a constant $\epsilon_1 > 0$ such that if*

$$\int_M \|A^o\|^2 d\mu\Big|_{t=0} \le \epsilon_1 \tag{152}$$

*then $T = \infty$ and $f$ converges exponentially to a round sphere.*

One way to view the condition (152) is that the deviation of $f$ from being round is small in an averaged sense. This result can then be viewed as a kind of stability of spheres theorem in the $L^2$ norm. Simonett [**55**] used centre manifold techniques to show that the statement of Theorem 7.1 holds under the stronger assumption that $f_0$ is $C^{2,\alpha}$-close to a round sphere. Our analysis here is completely different, as it must be, and as noted throughout the thesis we have drawn inspiration instead from the work of Kuwert and Schätzle [**36, 37**] on the Willmore flow of surfaces. There they prove Theorem 7.1 for Willmore flow. All of the additional difficulties we have encountered in earlier chapters are due to the lack of a very special zero order curvature term in the speed of the flow, and the addition of a difficult to control global term (the constraint function). The problems caused range from obtaining 'good' terms in our integral estimates, to the fact that we do not enjoy the structure of an $L^2$ gradient flow of curvature, and thus certain trivial facts become highly non-trivial. For example, in Willmore flow one has

$$\frac{d}{dt}\int_M |A^o|^2 d\mu \le 0,$$

in fact better than this, for "free". With surface diffusion flow this is no longer true. Instead, we must rely on Theorem 6.3, which is strictly weaker. Note that although this weakness carries through to the rest of the blow up analysis, where we obtain a weaker result than Kuwert and Schätzle, our final major result Theorem 7.1 is no weaker than the analogous Theorem 5.1 in [**36**]. This is due to the particular weakness of our blow up analysis: the results only hold in the case where (152) is satisfied, whereas for Willmore flow the blow up analysis does not require (152) to be satisfied. However, since our main theorem requires (152) regardless, one does not 'see' this shortcoming of the blow up analysis from the outset.

We briefly demonstrate an application of Theorem 7.1. Consider the quantity

$$I(t) = \frac{\int_M d\mu}{\text{Vol } M_t} = \frac{\mu(M_t)}{\text{Vol } M_t},$$

sometimes called the *isoperimetric ratio*. Let $f_0 : M \to \mathbb{R}^3$ be a surface satisfying (152), and let $f : M \times [0,T) \to \mathbb{R}^3$ be the surface diffusion flow with initial data $f_0$. Then

$$\frac{d}{dt}I(t) = \frac{-\int_M \|\nabla H\|^2 d\mu}{\text{Vol } M_0} \le 0. \tag{153}$$

By Theorem 7.1, $f$ approaches a round 2-sphere $S$ with the volume equal to the volume of $f_0$. This sphere has radius

$$r = \sqrt[3]{\frac{3\text{Vol } M_0}{4\pi}}.$$

Integrating (153) and taking limits we have

$$\begin{aligned}
\frac{\mu(M_0)}{\text{Vol } M_0} &\geq \frac{\mu(S)}{\text{Vol } S} \\
&= \frac{4\pi\Big(\sqrt[3]{\frac{3\text{Vol } M_0}{4\pi}}\Big)^2}{\text{Vol } M_0}, \text{ so} \\
\Big(\mu(M_0)\Big)^3 &\geq \frac{9(4\pi)^3}{(4\pi)^2}\Big(\text{Vol } M_0\Big)^2 \\
&= 36\pi\Big(\text{Vol } M_0\Big)^2,
\end{aligned}$$

the isoperimetric inequality (with optimal constant) for 2-surfaces in $\mathbb{R}^3$ satisfying (152).

One may wonder on a possible upper bound for $\epsilon_1$. Unfortunately, there is a dearth of analytic examples of surfaces flowing by surface diffusion in the literature, and so at this time we do not have any analog of the bound given in [**38**].

## 2. Long time existence

We begin by establishing that surface diffusion flows with small initial tracefree curvature exist for all time, that is for these flows $T = \infty$. For this, the only issue is to rule out out possible concentrations of curvature at finite times. Now that we are armed with the analysis from Chapters 3 through to 6 there are several possible approaches to establishing Theorem 7.1. Our approach is to here give some elementary arguments, which were outlined heuristically throughout the thesis, whereas the next section is devoted to the technicalities involved. In particular, for this section we establish $T = \infty$, while in the next section we utilise a different kind of 'blowup' (in fact there is no scaling) to establish exponential subconvergence to spheres.

PROPOSITION 7.2. *Suppose* $f : M^2 \times [0, T) \to \mathbb{R}^3$ *is a* (SD) *flow. Then there exists a constant* $\epsilon_1 > 0$ *such that if*

$$\int_M \|A^o\|^2 d\mu \bigg|_{t=0} < \epsilon_1$$

*then* $T = \infty$.

PROOF. Assume otherwise, and then by the Lifespan Theorem there exists a $T < \infty$ such that curvature concentrates at time $T$. Note that we may assume $T > 0$. Performing a blow up construction as in Chapter 6 at $T$ we recover a stationary blow up $\tilde{f}$ with small tracefree curvature due to Proposition 6.3. Now observe that the Gap Lemma implies $\tilde{f}$ must be a plane or sphere, which contradicts the nontriviality of the blow up, Theorem 6.8. Thus there does not exist a finite time when curvature concentrates, and so $T = \infty$. □

This establishes long time existence, however we know little of the asymptotic behaviour of our limits in the case where $T = \infty$. The following is straightforward in light of Almost Preservation and the Gap Lemma.

LEMMA 7.3. *For surface diffusion flows satisfying* (152)*, curvature cannot concentrate at final time.*

PROOF. Almost preservation implies that if indeed the curvature did concentrate in infinite time, we would have a compact surface $f(\cdot, \infty)$ with small tracefree curvature and a curvature singularity. This contradicts the Gap Lemma. □

This resolves the issue of possible exotic singularities developing asymptotically slowly. Note that in the case where (152) is not satisfied any one of a whole menagerie of singularities may develop in finite or infinite time, and excluding or providing examples of these are valuable contributions to the literature. At this time, the only ones known are due to Meyer [**45**].

For surface diffusion flow (for constrained flows, a similar argument with Proposition 6.7 applies) we have the existence of a non-zero positive finite limit $\mu(M)|_{t=T}$ due to the uniform bounds

$$0 < 2\sqrt{3\pi}\sqrt{\text{Vol } M}\Big|_{t=0} = 2\sqrt{3\pi}\sqrt{\text{Vol } M} \le \mu(M) \le \mu(M)\Big|_{t=0} < \infty. \tag{154}$$

We finish this section with another lemma, which states that at some scale the curvature will never concentrate. This is similar to Lemma 5.3 in [**36**]. The proof is contained in the proof of Proposition 7.2.

Corollary 7.4. *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is an* (SD) *flow. Then there exists a radius $r_0 > 0$ such that*

$$\int_{f^{-1}\left(B_{r_0}(x)\right)} \|A\|^2 d\mu \le \epsilon_1, \textit{ for every } x \in \mathbb{R}^3,$$

*where $\epsilon_1 > 0$ is as in the construction of the blow up.*

Note that the above gives an alternate proof of both Proposition 7.2 and Lemma 7.3 above, which is more similar in spirit to [**36**].

## 3. Exponential smooth convergence to round spheres.

We first prove, as in [**36**], that under the assumption (152) convergence to round spheres is smooth. This is similar to the analysis from Chapter 6.

Proposition 7.5. *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is an* (SD) *flow satisfying* (152). *Then for any sequence $t_j \nearrow \infty$ there exist $x_j \in \mathbb{R}^3$ and $\phi_j \in Diff(M)$ such* *that, after passing to a subsequence, the immersions $f(\phi_j, t) - x_j$ converge smoothly to an embedded round sphere.*

Proof. Let $p \in M$ be arbitrary and set $x_j = f(p, t_j)$. By Corollary 7.4 and the Interior Estimates, Theorem 5.11, we have for each $t_j \geq 1$

$$\|\nabla_{(k)} A\|_\infty \leq c_k. \tag{155}$$

We also have an area bound easily as $\frac{d}{dt}\mu(M) \leq 0$. Then by Theorem 6.1 we infer the existence of a properly immersed surface $\tilde{f} : \tilde{M} \to \mathbb{R}^3$ and diffeomorphisms $\phi_j : \tilde{M}(j) \to U_j \subset M$ such that, after selection of a subsequence,

$$f(\phi_j, t_j) - x_j \longrightarrow \tilde{f}$$

locally in $C^k$ on $\tilde{M}$. On $\tilde{M}(j)$ we consider the surface diffusion flows

$$g_j(p, t) = f(\phi_j(p), t_j + t) - x_j, \text{ for } t \geq -t_j.$$

These flows satisfy the interior estimates (155) and the initial data ($t = 0$) converges to $\tilde{f}$. Arguing as in (145) we obtain the local smooth convergence of each $g_j$ on $\tilde{M} \times [0, \infty)$ to a surface diffusion flow $g : \tilde{M} \times [0, \infty) \to \mathbb{R}^3$ with initial data $\tilde{f}$. But now, using the argument of Theorem 6.4, we obtain that

$$\int_{t_j}^{t_{j+1}} \int_M |\Delta H|^2 d\mu d\tau \searrow 0, \text{ as } j \nearrow \infty.$$

Therefore $\tilde{f}$ is a stationary (SD) surface. The Gap Lemma implies that $\tilde{f}$ must be a union of planes and spheres, however we can exclude several components using the proof of Lemma 6.6 and planes are impossible due to the upper area bound. Therefore $\tilde{f}$ must be an embedded round sphere, and subconvergence is smooth. $\square$

The above implies that Theorem 6.3 may in fact be strengthened to

$$\int_M \|A^o\|^2 d\mu \searrow 0 \text{ as } t \nearrow \infty,$$

and further that we must also enjoy almost preservation of the full second fundamental form, as

$$\int_M \|A\|^2 d\mu \to 4\pi \text{ as } t \nearrow \infty,$$

We need one more estimate before we can prove exponential decay of curvature. The proof is a simpler version of that given already in Lemma 5.4.

LEMMA 7.6. *Suppose $f : M^2 \times [0,T) \to \mathbb{R}^3$ is a* (SD) *flow. Then there exists an absolute constant $\epsilon_0 > 0$ such that if*

$$\int_M \|A^o\|^2 d\mu < \epsilon_0$$

*and*

$$\liminf_{\rho\to\infty} \frac{1}{\rho^4} \int_{f^{-1}(B_\rho(0))} \|A\|^2 d\mu = 0$$

*then we have*

$$\frac{d}{dt}\int_M \|A^o\|^2 d\mu + \frac{1}{100}\int_M \Big(\|\nabla_{(2)}A\|^2 + \|\nabla A\|^2 H^2 + \|A^o\|^2 H^4\Big) d\mu \le 0.$$

PROOF. The argument is similar to that of Lemma 5.4. Let $\gamma$ be a cutoff function as in (25) on a ball of radius $\rho$. We begin by combining Lemma 5.3 with Proposition 4.7:

$$\begin{aligned}\frac{d}{dt}\int_M \|A^o\|^2\gamma^4 d\mu + (1-\delta_1)\Big[\int_M \|\nabla_{(2)}A^o\|^2\gamma^4 d\mu + \int_M \|A\|^2\|\nabla A^o\|^2\gamma^4 d\mu \\ + \int_M \|A\|^4\|A^o\|^2\gamma^4 d\mu + \int_M H^2\|\nabla H\|^2\gamma^4 d\mu \\ - \frac{c}{\rho^4}\|A^o\|^4_{2,[\gamma>0]} - \frac{c}{\rho^4}\|A\|^2_{2,[\gamma>0]}\Big]\end{aligned}$$

$$\le \delta_2 \int_M \|\nabla A^o\|^2 H^2 \gamma^4 d\mu + \delta_3 \int_M \|A\|^4 \|A^o\|^2 \gamma^4 d\mu$$
$$+ c \int_M (\|A^o\|^6 + \|\nabla A^o\|^2 \|A^o\|^2) \gamma^4 d\mu + \frac{c}{\rho^4} \int_{[\gamma>0]} \|A^o\|^2 d\mu.$$

Simplifying, we choose $\delta_i = \frac{1}{6}$ and obtain

$$\frac{d}{dt} \int_M \|A^o\|^2 \gamma^4 d\mu + \frac{1}{50} \Big[ \int_M \|\nabla_{(2)} A\|^2 \gamma^4 d\mu + \int_M \|A\|^2 \|\nabla A\|^2 \gamma^4 d\mu + \int_M \|A\|^4 \|A^o\|^2 \gamma^4 d\mu \Big]$$
$$\le c \int_M (\|A^o\|^6 + \|\nabla A^o\|^2 \|A^o\|^2) \gamma^4 d\mu + \frac{c}{\rho^4} \|A^o\|^2_{2,[\gamma>0]} + \frac{c}{\rho^4} \|A\|^2_{2,[\gamma>0]}.$$

Note that we also added $\int_M \|A^o\|^2 \|\nabla A^o\|^2 \gamma^4 d\mu$ to both sides and used the inequalities

$$\|\nabla_{(2)} A\|^2 \le 25 \|\nabla_{(2)} A^o\|^2, \text{ and } \|\nabla A\|^2 \le 25 \|\nabla A^o\|^2.$$

These are easily proved by estimating

$$\nabla A = \nabla(A^o + \tfrac{1}{2} g H) = \nabla A^o + g_{ij} \nabla^j H = \nabla A^o + 2g \nabla^* A^o$$

and

$$\nabla_{(2)} A = \nabla(\nabla A^o + 2g \nabla^* A^o) = \nabla_{(2)} A^o + 2g \nabla \nabla^* A^o.$$

Invoking the Sobolev inequality (Lemma 3.22), as in the proof of Lemma 5.4, we absorb the first term on the right hand side to conclude

$$\frac{d}{dt} \int_M \|A^o\|^2 \gamma^4 d\mu + \frac{1}{100} \Big[ \int_M \|\nabla_{(2)} A\|^2 \gamma^4 d\mu + \int_M \|A\|^2 \|\nabla A\|^2 \gamma^4 d\mu + \int_M \|A\|^4 \|A^o\|^2 \gamma^4 d\mu \Big]$$
$$\le \frac{c}{\rho^4} \|A^o\|^2_{2,[\gamma>0]} + \frac{c}{\rho^4} \|A^o\|^4_{2,[\gamma>0]} + \frac{c}{\rho^4} \|A\|^2_{2,[\gamma>0]},$$

where we required $\epsilon_0 < \frac{1}{50c}$. Taking $\rho \nearrow \infty$ concludes the proof. $\square$

We finish this chapter by proving exponential decay of curvature. Our proof follows that in [**36**].

Proposition 7.7. *Suppose* $f : M^2 \times [0,T) \to \mathbb{R}^3$ *is an* (SD) *flow satisfying* (152). *Then there exists a* $\lambda > 0$ *such that as* $t \nearrow \infty$ *the following asymptotic* *statements hold:*

$$\|\nabla_{(k)}A\|_\infty \le c_k e^{-\lambda t},$$

$$\|A^o\|_\infty \le c_0 e^{-\lambda t},$$

*for* $k \ge 1$.

PROOF. Let $\mathcal{A} = \mu(M)\Big|_{t=T}$ be as in (154). Then Proposition 7.5 above implies that the sectional curvature and mean curvature satisfy

$$\|K\|_\infty \longrightarrow \frac{4\pi}{\mathcal{A}},$$

and

$$\|H^2\|_\infty \longrightarrow \frac{16\pi}{\mathcal{A}},$$

as $t \nearrow \infty$. Therefore there exists a $t_H < \infty$ such that

$$H^2 \ge c_H > 0, \text{ for all } t \ge t_H.$$

From now on we assume $t \ge t_H$. Now we invoke Lemma 7.6. Note that we may assume $c_H \le 1$. Using the above we have

$$\frac{d}{dt}\int_M \|A^o\|^2 d\mu + \frac{c_H^2}{100}\int_M \|\nabla_{(2)}A\|^2 + \|\nabla A\|^2 + \|A^o\|^2 d\mu \le 0.$$

Integrating gives

$$\int_M \|A^o\|^2 d\mu + \frac{c_H^2}{100}\int_t^\infty \int_M \|\nabla_{(2)}A\|^2 + \|\nabla A\|^2 d\mu d\tau \le e^{-2\lambda t}, \tag{156}$$

where

$$\lambda = \frac{c_H^2}{200}.$$

Note that we needed Proposition 7.5 for $\int_M \|A^o\|^2 d\mu\Big|_{t=T} = 0$ in the above. From this estimate and again Proposition 7.5 we can obtain exponential decay in a standard way. From Proposition 3.17, taking $\rho \nearrow \infty$ gives

$$\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2 d\mu + \int_M \|\nabla_{(k+2)}A\|^2 d\mu \le \int_M \left(P_3^{k+2}(A) + P_5^k(A)\right) * \nabla_{(k)}A d\mu.$$

Note that the term $c\|A\|^2_{2,[\gamma>0]}$ disappeared due to the dependence of the constant on $\rho$. From Proposition 7.5 we know that $A$ and all its derivatives remain bounded as $t \nearrow \infty$, so we estimate

$$\int_M P_2^0(A) * \nabla_{(k+2)}A * \nabla_{(k)}A d\mu \le \epsilon \int_M \|\nabla_{(k+2)}A\|^2 d\mu + c_\epsilon \int_M \|\nabla_{(k)}A\|^2 d\mu,$$

$$\int_M \left(\tilde{P}_3^{k+2}(A) + P_5^m(A)\right) * \nabla_{(k)}A d\mu \le c\sum_{j=1}^{k+1}\int_M \|\nabla_{(j)}A\|^2 d\mu,$$

where the constant is not universal (i.e. depends on derivatives of curvature). $\tilde{P}_3^{k+2}(A)$ denotes all terms of type $P_3^{k+2}(A)$ that do not contain the $(k+2)$-th derivative.

We thus obtain

$$\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2 d\mu + \frac{1}{2}\int_M \|\nabla_{(k+2)}A\|^2 d\mu \le c\sum_{j=1}^{k+1}\int_M \|\nabla_{(j)}A\|^2 d\mu.$$

A proof by induction using (156) then gives

$$\|\nabla_{(k)}A\|_2^2 + \frac{c_H^2}{100}\int_t^\infty \|\nabla_{(k+2)}A\|_2^2 d\tau \le e^{-2\lambda t}.$$

This gives us the estimates

$$\|A^o\|_2 \le ce^{-\lambda t}, \text{ and } \|\nabla_{(k)}A\|_2 \le ce^{-\lambda t}.$$

Using Proposition 51 as in the proof of the Interior Estimates finishes the proof. □

REMARK. Note that in the proof of the above we in fact showed that after a fixed time translation, the sign of the mean curvature is preserved. This is interesting in that a typically second order phenomenon remains true (after some time) in the fourth order setting.

## 4. On constrained surface diffusion flows

For a result analogous to Theorem 7.1 pertaining to the constrained surface diffusion flows, at present the conditions placed upon the constraint function are too restrictive. While there are contrived examples of constraint functions (apart from $h = 0$) which will satisfy these constraints, there are no motivating examples which we know of. This is an interesting question to address in future investigations. Despite this, we state the theorem relevant to (CSD) flows for the interested reader. The proof is identical to that of Theorem 7.1.

THEOREM 7.8. *Suppose* $f : M^2 \times [0,T) \to \mathbb{R}^3$ *is a* (CSD) *flow with constraint function satisfying* (A2), (AS), *the hypothesis of Lemma 5.4 and*

$$h\int_M H d\mu \geq -\mu(M)\int_M \|\nabla_{(2)}H\|^2 d\mu - \int_M H^2\|\nabla H\|^2 d\mu.$$

*Then there exist constants* $\epsilon_1 > 0$ $m \geq 2$ *such that if*

$$\mu(M_0)^{\frac{m-2}{m}}\Big(\int_M \|A^o\|^m d\mu\Big)^{\frac{2}{m}}\Big|_{t=0} \leq \epsilon_1$$

*then* $T = \infty$ *and* $f$ *converges exponentially to a round sphere.*

APPENDIX A

# Inequalities

We collect here several inequalities which we use throughout the thesis. This is certainly not an exhaustive account, but the more common or more useful of them are included, along with proofs.

LEMMA A.1 (Gronwall's Inequality). *Let $f, g, h : I \to \mathbb{R}$ where $I \subset \mathbb{R}$ is bounded and connected with $c = \inf I$, and $f, g, h$ are continuous and integrable on $I$. Then, if $g \geq 0$ and*

$$f(t) \leq g(t) + \int_c^t h(\tau)f(\tau)d\tau, \tag{157}$$

*we can conclude*

$$f(t) \leq g(t) + \int_c^t g(\tau)h(\tau)e^{\int_\tau^t h(\nu)d\nu}d\tau, \tag{158}$$

*for any $t \in I$.*

PROOF. The idea of the proof is to take a useful test function and then combine the derivative of such with our assumptions to conclude the lemma.

For $\nu, \tau \in I$, let

$$\varphi(\tau) = e^{-\int_c^\tau h(\nu)d\nu} \int_c^\tau h(\nu)f(\nu)d\nu.$$

Differentiating,

$$\varphi'(\tau) = h(\tau)e^{-\int_c^\tau h(\nu)d\nu} \left( f(\tau) - \int_c^\tau h(\nu)f(\nu)d\nu \right).$$

The term in brackets is estimated by our assumption (157), so we obtain

$$\varphi'(\tau) \leq g(\tau)h(\tau)e^{-\int_c^\tau h(\nu)d\nu}.$$

Integrating,

$$e^{-\int_c^t h(\nu)d\nu} \int_c^t h(\tau)f(\tau)d\tau = \int_c^t \varphi'(\tau)d\tau \leq \int_c^t g(\tau)h(\tau)e^{-\int_c^\tau h(\nu)d\nu}d\tau.$$

Using again the assumption (157),

$$\begin{aligned}
e^{-\int_c^t h(\nu)d\nu}\Big(f(t) - g(t)\Big) &\leq \int_c^t g(\tau)h(\tau)e^{-\int_c^\tau h(\nu)d\nu}d\tau \\
e^{-\int_c^t h(\nu)d\nu} f(t) &\leq e^{-\int_c^t h(\nu)d\nu} g(t) + \int_c^t g(\tau)h(\tau)e^{-\int_c^\tau h(\nu)d\nu}d\tau \\
f(t) &\leq g(t) + e^{\int_c^t h(\nu)d\nu} \int_c^t g(\tau)h(\tau)e^{-\int_c^\tau h(\nu)d\nu}d\tau \\
&\leq g(t) + \int_c^t g(\tau)h(\tau)e^{\int_c^t h(\nu)d\nu - \int_c^\tau h(\nu)d\nu}d\tau.
\end{aligned}$$

Since $\tau \leq t$, the integrals in the exponent combine and we can conclude (158). □

The following interpolation inequality is used to prove the second multiplicative Sobolev inequality (51). This interpolation inequality is proved using the Michael-Simon Sobolev inequality and an induction argument primarily due to Ladyzhenskaya [**39**]. We present here an $n$-dimensional version based upon the modern 2-dimensional version given in [**37**].

THEOREM A.2. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a smooth immersed hypersurface. For $u \in C_c^1(M)$, $n < p \leq \infty$, $0 \leq m \leq \infty$ and $0 < \alpha \leq 1$ where $\frac{1}{\alpha} = \left(\frac{1}{n} - \frac{1}{p}\right)m + 1$ we have*

(LZ) $$\|u\|_\infty \leq c\|u\|_m^{1-\alpha}(\|\nabla u\|_p + \|Hu\|_p)^\alpha,$$

*where $c = c(n, m, p)$.*

PROOF. Assume $u \geq 0$. Define

$$C_{MSS}(\|\nabla u\|_p + \||H|u\|_p) =: Q(u),$$

where the constant $C_{MSS}$ is the constant from the Michael-Simon Sobolev inequality above. Note that in particular $C_{MSS}$ is an absolute constant and does not depend on $u$. Now scale $u$ by

$$\tilde{u} = \frac{1}{Q(u)} u$$

for $Q(u) \neq 0$ and if $Q(u) = 0$ set $\tilde{u} = u$. Then $Q(\tilde{u}) = 1$, that is

$$C_{MSS}(\|\nabla \tilde{u}\|_p + \||H|\tilde{u}\|_p) = 1. \tag{159}$$

Note that since we scale the image of $u$, the derivative $\nabla u$ (in fact any derivative) scales the same as the original function $u$.

Let $q = \frac{p}{p-1} \in \left[1, \frac{n}{n-1}\right)$, and $\tau \geq 0$. Then by the Michael-Simon Sobolev inequality

$$\begin{aligned} \|\tilde{u}^{1+\tau}\|_{n/(n-1)} &\leq C_{MSS}\left(\int_M \|\nabla \tilde{u}^{1+\tau}\| d\mu + \int_M \tilde{u}^{1+\tau}|H| d\mu\right) \\ &= C_{MSS}\Big((1+\tau)\|(\nabla \tilde{u})\tilde{u}^\tau\|_1 + \|\tilde{u}^{1+\tau}|H|\|_1\Big). \end{aligned} \tag{160}$$

Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{p-1}{p} = 1$, we use the Hölder inequality and equation (159) to conclude

$$\begin{aligned} \|\tilde{u}^{1+\tau}\|_{n/(n-1)} &\leq C_{MSS}\|\tilde{u}^\tau\|_q\Big((1+\tau)\|\nabla \tilde{u}\|_p + \|\tilde{u}|H|\|_p\Big) \\ &\leq (1+\tau)\|\tilde{u}^\tau\|_q Q(\tilde{u}) \\ &\leq (1+\tau)\|\tilde{u}^\tau\|_q. \end{aligned} \tag{161}$$

We will now proceed with several induction arguments. Set $k = \frac{n}{q(n-1)} \in \left(1, \frac{n}{n-1}\right]$ and, taking the $(1+\tau)$-th root, rewrite (161) as

$$\|\tilde{u}\|_{kq(1+\tau)} \leq (1+\tau)^{\frac{1}{1+\tau}} \|\tilde{u}\|_{q\tau}^{\frac{\tau}{1+\tau}}. \tag{162}$$

Since

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty,$$

we look to taking a sequence $\tau_\nu$ in (162) and then letting $\nu \to \infty$. For this purpose we set the following constants:

$$\tau_0 = \frac{m}{q} \in \left(m - \frac{m}{n}, m\right] \qquad\qquad \tau_{\nu+1} = k(1+\tau_\nu)$$
$$\epsilon_\nu = \frac{\tau_\nu}{1+\tau_\nu} \in [0,1) \qquad\qquad c_\nu = (1+\tau_\nu)^{\frac{1}{1+\tau_\nu}},$$

for $\nu \in \mathbb{N}_0$. We now rewrite (162), replacing $\tau$ with $\tau_\nu$:

$$\|\tilde{u}\|_{\tau_{\nu+1}q} \leq c_\nu \|\tilde{u}\|_{q\tau_\nu}^{\epsilon_\nu}. \tag{163}$$

These definitions imply the following formula for $\tau_\nu$ by induction

$$\begin{aligned}\tau_\nu &= k(1+\tau_{\nu-1}) = k(1+k(1+\tau_{\nu-2})) = k + k^2 + k^2\tau_{\nu-2} \\ &= \ldots = k^\nu \tau_0 + \sum_{\mu=1}^{\nu} k^\mu\end{aligned}$$

where the base case is given by

$$\tau_1 = k(1+\tau_0) = k^1\tau_0 + \sum_{\mu=1}^{1} k^\mu$$

and the inductive step is easy to show as

$$\begin{aligned}\tau_{\nu+1} &= k(1+\tau_\nu) = k + k^{\nu+1}\tau_0 + \sum_{\mu=2}^{\nu+1} k^\mu \\ &= k^{\nu+1}\tau_0 + \sum_{\mu=1}^{\nu+1} k^\mu,\end{aligned}$$

where we used the inductive hypothesis in the second equality. Adding 1 to each side gives

$$1 + \tau_\nu = k^\nu \tau_0 + \sum_{\mu=0}^{\nu} k^\mu. \tag{164}$$

From (163) we obtain again by a similar induction

$$\begin{aligned}
\|\tilde{u}\|_{\tau_\nu q} &\leq c_{\nu-1}\|\tilde{u}\|_{q\tau_{\nu-1}}^{\epsilon_{\nu-1}} \\
&\leq c_{\nu-1}\left(c_{\nu-2}\|\tilde{u}\|_{q\tau_{\nu-2}}^{\epsilon_{\nu-2}}\right)^{\epsilon_{\nu-1}} \\
&= c_{\nu-1}c_{\nu-2}^{\epsilon_{\nu-1}}\|\tilde{u}\|_{q\tau_{\nu-2}}^{\epsilon_{\nu-2}\epsilon_{\nu-1}} \\
&= c_{\nu-1}c_{\nu-2}^{\epsilon_{\nu-1}}c_{\nu-3}^{\epsilon_{\nu-1}\epsilon_{\nu-2}}\|\tilde{u}\|_{q\tau_{\nu-3}}^{\epsilon_{\nu-3}\epsilon_{\nu-2}\epsilon_{\nu-1}} \\
&\vdots \\
&= \left(\prod_{i=1}^{\nu} c_{\nu-i}^{\prod_{j=1}^{i-1}\epsilon_{\nu-j}}\right)\|\tilde{u}\|_m^{\prod_{j=1}^{\nu}\epsilon_{\nu-j}}. \qquad (165)
\end{aligned}$$

For this induction the base case follows from (163) by

$$\begin{aligned}
\|\tilde{u}\|_{\tau_1 q} &\leq c_0\|\tilde{u}\|_{q\tau_0}^{\epsilon_0} \\
&\leq c_0\|\tilde{u}\|_m^{\epsilon_0} \\
&= \left(\prod_{i=1}^{1} c_{1-i}^{\prod_{j=1}^{i-1}\epsilon_{1-j}}\right)\|\tilde{u}\|_m^{\prod_{j=1}^{1}\epsilon_{1-j}}
\end{aligned}$$

and the inductive step also follows from (163) and the inductive hypothesis by

$$\begin{aligned}
\|\tilde{u}\|_{\tau_{\nu+1} q} &\leq c_\nu\|\tilde{u}\|_{q\tau_\nu}^{\epsilon_\nu} \\
&\leq c_\nu\left(\prod_{i=1}^{\nu+1} c_{\nu-i}^{\prod_{j=1}^{i-1}\epsilon_{\nu-j}}\right)^{\epsilon_\nu}\|\tilde{u}\|_m^{\epsilon_\nu\prod_{j=1}^{\nu}\epsilon_{\nu-j}}, \\
&= \left(\prod_{i=1}^{\nu+1} c_{\nu+1-i}^{\prod_{j=1}^{i-1}\epsilon_{\nu+1-j}}\right)\|\tilde{u}\|_m^{\prod_{j=1}^{\nu+1}\epsilon_{\nu+1-j}}.
\end{aligned}$$

Since $\epsilon_\nu < 1$ and $c_\nu > 1$ we estimate

$$\begin{aligned}
\ln\left(\prod_{i=1}^{\nu} c_{\nu-i}^{\prod_{j=1}^{i-1}\epsilon_{\nu-j}}\right) &\leq \ln\prod_{i=1}^{\nu} c_{\nu-i} \\
&= \sum_{i=1}^{\nu}\ln c_{\nu-i} \\
&= \sum_{i=1}^{\nu}\frac{1}{1+\tau_{\nu-i}}\ln(1+\tau_{\nu-i}). \qquad (166)
\end{aligned}$$

From (164) we can estimate by the arithmetic mean-geometric mean inequality

$$\frac{1}{c}k^\nu \leq 1+\tau_\nu \leq ck^\nu,$$

for $c = c(m,p)$. We use this with (166) to conclude

$$\begin{aligned}
\ln\left(\prod_{i=1}^{\nu} c_{\nu-i}^{\prod_{j=1}^{i-1}\epsilon_{\nu-j}}\right) &= \sum_{i=1}^{\nu} \frac{1}{\frac{1}{c}k^{\nu-i}} \ln(ck^{\nu-i}) \\
&= \sum_{i=1}^{\nu} ck^{i-\nu}(\ln c + (\nu-i)\ln k) \\
&\leq \sum_{i=0}^{\infty} ck^{-i}(\ln c + i\ln k) \\
&= c(m,p) < \infty. \qquad (167)
\end{aligned}$$

Note that we relabeled the terms in the series above so that we can take a limit in $\nu$ later. We now use $\tau_{\nu+1} = k(1+\tau_\nu)$ and (164) to obtain again by induction

$$\begin{aligned}
\prod_{j=1}^{\nu}\epsilon_{\nu-j} &= \prod_{j=0}^{\nu-1}\epsilon_j \\
&= \frac{\tau_0}{1+\tau_0}\frac{\tau_1}{1+\tau_1}\cdots\frac{\tau_{\nu-1}}{1+\tau_{\nu-1}} \\
&= \frac{\tau_0\tau_1\cdots\tau_{\nu-2}(k(1+\tau_{\nu-2}))}{(1+\tau_0)(1+\tau_1)\cdots(1+\tau_{\nu-2})(1+\tau_{\nu-1})} \\
&\ \ \vdots \\
&= k^{\nu-1}\frac{\tau_0}{1+\tau_{\nu-1}}.
\end{aligned}$$

For this induction argument we simply note that by the definition of $\epsilon_\nu$,

$$\begin{aligned}
\prod_{j=0}^{1-1}\epsilon_j &= \epsilon_0 \\
&= \frac{\tau_0}{1+\tau_0} \\
&= k^{1-1}\frac{\tau_0}{1+\tau_{1-1}},
\end{aligned}$$

which is the base case. The inductive step is again easy as

$$\prod_{j=0}^{\nu}\epsilon_j = \epsilon_\nu\prod_{j=0}^{\nu-1}\epsilon_j$$

$$
\begin{aligned}
&= \frac{\tau_\nu}{1+\tau_\nu}\left(k^{\nu-1}\frac{\tau_0}{1+\tau_{\nu-1}}\right)\\
&= k^{\nu-1}\frac{k(1+\tau_{\nu-1})}{1+\tau_\nu}\frac{\tau_0}{1+\tau_{\nu-1}}\\
&= k^{\nu}\frac{\tau_0}{1+\tau_\nu},
\end{aligned}
$$

where the second equality is from the inductive hypothesis and the third equality is from the definition of $\tau_\nu$.

Recall again equation (164). We first take limits as follows:

$$
\begin{aligned}
\lim_{\nu\to\infty}\frac{k^{\nu-1}}{1+\tau_{\nu-1}} &= \lim_{\nu\to\infty}\left(\tau_0+\sum_{i=0}^{-\nu}k^i\right)^{-1}\\
&= \frac{1}{\tau_0+\frac{k}{k-1}},
\end{aligned}
$$

and now unravelling the definitions we find

$$
\begin{aligned}
\prod_{j=0}^{\infty}\epsilon_j &= \frac{\tau_0}{\tau_0+\frac{k}{k-1}} = \frac{\frac{m}{q}}{\frac{m}{q}+\frac{n}{q(n-1)(\frac{n}{q(n-1)}-1)}} = \frac{m}{m+\frac{n}{\frac{n}{q}-(n-1)}} = \frac{m+\frac{nq}{n-q(n-1)}-\frac{nq}{n-q(n-1)}}{m+\frac{nq}{n-q(n-1)}}\\
&= 1-\frac{nq}{m(n-q(n-1))+nq} = 1-\frac{1}{1+m(\frac{1}{q}-\frac{n-1}{n})} = 1-\frac{1}{1+m(\frac{p-1}{p}-\frac{n-1}{n})}\\
&= 1-\frac{1}{1+m(\frac{1}{n}-\frac{1}{p})} = 1-\alpha,
\end{aligned}
$$

which explains our choice of the constant $\alpha$. Note that for the proof to work we require $m > 0$ and $\alpha \in (0, 1]$, which in turn requires $\frac{1}{n} - \frac{1}{p} > 0$. This is why it must be the case that $p > n$. If one of these conditions is violated, then the scaling will not 'close' at the end of the proof.

Finally we let $\nu \to \infty$ in (165) and conclude, using (159),

$$
\begin{aligned}
&\|\tilde{u}\|_\infty \le c\|\tilde{u}\|_m^{1-\alpha}\\
&\qquad = cC_{MSS}^{\alpha}\|\tilde{u}\|_m^{1-\alpha}(\|\nabla\tilde{u}\|_p+\||H|\tilde{u}\|_p)^\alpha\\
\implies &\|u\|_\infty \le Q(u)c(m,n,p)\frac{1}{Q(u)^{1-\alpha}}\frac{1}{Q(u)^{\alpha}}\|u\|_m^{1-\alpha}(\|\nabla u\|_p+\||H|u\|_p)^\alpha
\end{aligned}
$$

$$= c(m,n,p)\|u\|_m^{1-\alpha}(\|\nabla u\|_p + \||H|u\|_p)^\alpha$$

and so we can see that the scaling of the image of $u$ forces the exponent $\alpha$ on the right hand side.

It still remains to remove the positivity assumption on $u$. Trivially, we can split $u$ into

$$u = u^+ - u^-$$

where the operations $(\ )^+$ and $(\ )^-$ are the positive and negative part respectively. The problem is that these operations are not closed in $C^1$. To overcome this, we first approximate $u$ by a sequence of functions $u_\epsilon \to u$ where the convergence is in $W^{1,p}$. This weakens the regularity of $u$ to $W^{1,p}$ and in this space the positive and negative part operations are closed. We then split $u$ as previously indicated. This completes the proof. $\square$

We use a large number of interpolation inequalities in the proof of the Lifespan Theorem. A series of 5 form a logical progression to the very important interpolation inequality which allows us to estimate the $P$-style terms. We present these results beginning with the lemma below.

Lemma A.3. *Let* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 \le p,q,r \le \infty$ *and* $\alpha + \beta = 1$, $\alpha, \beta \ge 0$. *For* $s \ge \max\{\alpha q, \beta p\}$ *and* $-\frac{1}{p} \le t \le \frac{1}{q}$ *we have*

$$\left(\int_M \|\nabla T\|^{2r}\gamma^s d\mu\right)^{\frac{1}{r}} \le c\left(\int_M \|T\|^q \gamma^{s(1-tq)} d\mu\right)^{\frac{1}{q}} \left(\int_M \|\nabla_{(2)}T\|^p \gamma^{s(1+tp)} d\mu\right)^{\frac{1}{p}}$$
$$+ c\|\nabla\gamma\|_\infty \left(\int_M \|T\|^q \gamma^{s-\alpha q} d\mu\right)^{\frac{1}{q}} \left(\int_M \|\nabla T\|^p \gamma^{s-\beta p} d\mu\right)^{\frac{1}{p}},$$

*where* $c = c(r,s)$.

PROOF. This is a simple consequence of integration by parts and the Hölder inequality with $\frac{1}{q}+\frac{1}{p}+\frac{1}{\frac{r}{r-1}}=1$, as follows

$$
\begin{aligned}
&\int_M \langle \nabla T, \nabla T\rangle \|\nabla T\|^{2r-2}\gamma^s d\mu \\
&\quad\le \int_M \|T\|\ \|\nabla T\|(2r-2)\|\nabla T\|^{2r-3}\|\nabla_{(2)}T\|\gamma^s d\mu \\
&\qquad + \int_M \|T\|\ \|\nabla T\|\ \|\nabla T\|^{2r-2} s\|\nabla\gamma\|\gamma^{s-1}d\mu \\
&\quad\le (2r-2)\int_M \|T\|\ \|\nabla T\|^{2r-2}\|\nabla_{(2)}T\|\gamma^s d\mu \\
&\qquad + s\|\nabla\gamma\|_\infty \int_M \|T\|\ \|\nabla T\|^{2r-1}\gamma^{s-1}d\mu \\
&\quad\le \left(\int_M \|\nabla T\|^{2r}\gamma^s d\mu\right)^{\frac{r-1}{r}} \Bigg[(2r-2)\left(\int_M \|T\|^q\gamma^{s(1-tq)}d\mu\right)^{\frac{1}{q}}\left(\int_M \|\nabla_{(2)}T\|^p\gamma^{s(1+tp)}d\mu\right)^{\frac{1}{p}} \\
&\qquad\qquad + s\|\nabla\gamma\|_\infty\left(\int_M \|T\|^q\gamma^{s-\alpha q}d\mu\right)^{\frac{1}{q}}\left(\int_M \|\nabla T\|^p\gamma^{s-\beta p}d\mu\right)^{\frac{1}{p}}\Bigg].
\end{aligned}
$$

If $\nabla T\equiv 0$ then the result is trivial. Assuming otherwise, we obtain by division

$$
\begin{aligned}
\left(\int_M \|\nabla T\|^{2r}\gamma^s d\mu\right)^{1-\frac{r-1}{r}} &\le (2r-2)\left(\int_M \|T\|^q\gamma^{s(1-tq)}d\mu\right)^{\frac{1}{q}}\left(\int_M \|\nabla_{(2)}T\|^p\gamma^{s(1+tp)}d\mu\right)^{\frac{1}{p}} \\
&\quad + s\|\nabla\gamma\|_\infty\left(\int_M \|T\|^q\gamma^{s-\alpha q}d\mu\right)^{\frac{1}{q}}\left(\int_M \|\nabla T\|^p\gamma^{s-\beta p}d\mu\right)^{\frac{1}{p}},
\end{aligned}
$$

which is the statement of the lemma. □

LEMMA A.4. *For $2\le p<\infty$ and $s\ge p$ we have*

$$
\left(\int_M \|\nabla T\|^p\gamma^s d\mu\right)^{\frac{1}{p}} \le \epsilon\left(\int_M \|\nabla_{(2)}T\|^p\gamma^{s+p}d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|T\|^p\gamma^{s-p}d\mu\right)^{\frac{1}{p}},
$$

*where $c=c(\epsilon,p,s,\|\nabla\gamma\|_\infty)$ and $\epsilon>0$.*

PROOF. Let $p=q=2r$, $\alpha=1$, $\beta=0$ and $t=\frac{1}{s}$ in Lemma A.3 to obtain

$$
\begin{aligned}
\left(\int_M \|\nabla T\|^p\gamma^s d\mu\right)^{\frac{2}{p}} &\le c\left(\int_M \|T\|^p\gamma^{s-p}d\mu\int_M \|\nabla_{(2)}T\|\gamma^{s+p}d\mu\right)^{\frac{1}{p}} \\
&\quad + c\|\nabla\gamma\|_\infty\left(\int_M \|T\|^p\gamma^{s-p}d\mu\int_M \|\nabla T\|\gamma^s d\mu\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\leq \frac{1}{2}\left(\int_M \|\nabla T\|^p \gamma^s d\mu\right)^{\frac{2}{p}} + \epsilon\left(\int_M \|\nabla_{(2)} T\|^p \gamma^{s+p} d\mu\right)^{\frac{2}{p}}$$
$$+ c\left(\int_M \|T\|^p \gamma^{s-p} d\mu\right)^{\frac{2}{p}},$$

where $c = c(\epsilon, p, s, \|\nabla\gamma\|_\infty)$ and $\epsilon > 0$. Subtracting and taking square roots gives the statement of the lemma. □

LEMMA A.5. *For* $2 \leq p < \infty$, $k \in \mathbb{N}$, $s \geq kp$, $c = c(\epsilon, p, s, \|\nabla\gamma\|_\infty)$ *and* $\epsilon > 0$ *we have*

$$\left(\int_M \|\nabla_{(k)} T\|^p \gamma^s d\mu\right)^{\frac{1}{p}} \leq \epsilon\left(\int_M \|\nabla_{(k+1)} T\|^p \gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|T\|^p \gamma^{s-kp} d\mu\right)^{\frac{1}{p}}.$$

PROOF. As mentioned, for $k = 1$ the lemma holds by Lemma A.4. We now proceed to assume

$$\left(\int_M \|\nabla_{(k)} S\|^p \gamma^s d\mu\right)^{\frac{1}{p}} \leq \lambda\left(\int_M \|\nabla_{(k+1)} S\|^p \gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|S\|^p \gamma^{s-kp} d\mu\right)^{\frac{1}{p}}, \tag{168}$$

and attempt to prove for any $\epsilon > 0$

$$\left(\int_M \|\nabla_{(k+1)} S\|^p \gamma^s d\mu\right)^{\frac{1}{p}} \leq \epsilon\left(\int_M \|\nabla_{(k+2)} S\|^p \gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|S\|^p \gamma^{s-(k+1)p} d\mu\right)^{\frac{1}{p}}. \tag{169}$$

This will finish the induction. Let $T = \nabla_{(k)} S$ in Lemma A.4 to obtain

$$\left(\int_M \|\nabla_{(k+1)} S\|^p \gamma^s d\mu\right)^{\frac{1}{p}} \leq \lambda\left(\int_M \|\nabla_{(k+2)} S\|^p \gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|\nabla_{(k)} S\|^p \gamma^{s-p} d\mu\right)^{\frac{1}{p}}. \tag{170}$$

Since we assumed (168), we know $s \geq kp$. Therefore $s - p \geq (k-1)p$. Using (168) with $s - p$ instead of $s$, we obtain

$$\left(\int_M \|\nabla_{(k)} S\|^p \gamma^{s-p} d\mu\right)^{\frac{1}{p}} \leq \frac{\delta}{c}\left(\int_M \|\nabla_{(k+1)} S\|^p \gamma^s d\mu\right)^{\frac{1}{p}} + c_0\left(\int_M \|S\|^p \gamma^{s-(k+1)p} d\mu\right)^{\frac{1}{p}}. \tag{171}$$

Combining (170) and (171) we get

$$\begin{aligned}\left(\int_M \|\nabla_{(k+1)}S\|^p\gamma^s d\mu\right)^{\frac{1}{p}} &\le \lambda\left(\int_M \|\nabla_{(k+2)}S\|^p\gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|\nabla_{(k)}S\|^p\gamma^{s-p} d\mu\right)^{\frac{1}{p}}\\ &\le \lambda\left(\int_M \|\nabla_{(k+2)}S\|^p\gamma^{s+p} d\mu\right)^{\frac{1}{p}} + \delta\left(\int_M \|\nabla_{(k+1)}S\|^p\gamma^{s} d\mu\right)^{\frac{1}{p}}\\ &\quad + c\left(\int_M \|S\|^p\gamma^{s-(k+1)p} d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

Absorbing the second term on the right gives

$$\left(\int_M \|\nabla_{(k+1)}S\|^p\gamma^s d\mu\right)^{\frac{1}{p}} \le \frac{\lambda}{1-\delta}\left(\int_M \|\nabla_{(k+2)}S\|^p\gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c\left(\int_M \|S\|^p\gamma^{s-(k+1)p} d\mu\right)^{\frac{1}{p}},$$

where $c = c(\lambda, \delta, p, s, \|\nabla\gamma\|_\infty) < \infty$. The above formula is valid for any $\delta, \lambda > 0$. Choosing $\delta = \frac{1}{2}$ and $\lambda = \frac{\epsilon}{2}$ proves (169) and finishes the proof. □

The following lemma appears in Hamilton [**27**].

Lemma A.6 (Hamilton, Lemma 11.5). *Let $f(k)$ be a real valued function of the integer $k$ for $0 \le k \le n$. If $f(k)$ satisfies*

$$f(k) \le Cf(k-1)^{1/2} f(k+1)^{1/2}, \tag{172}$$

*then*

$$f(k) \le C^{k(n-k)} f(0)^{1-k/n} f(n)^{k/n}. \tag{173}$$

Theorem A.7. *For $k \in \mathbb{N}$, $1 \le i \le k$ and $s \ge 2k$ we have the inequality*

$$\left(\int_M \|\nabla_{(i)}T\|^{\frac{2k}{i}}\gamma^s d\mu\right)^{\frac{i}{2k}} \le c\|T\|_{\infty,[\gamma>0]}^{1-\frac{i}{k}}\left(\left(\int_M \|\nabla_{(k)}T\|\gamma^s d\mu\right)^{\frac{1}{2}} + \|T\|_2\right)^{\frac{1}{2}}$$

*where $c = c(k, d, s, \|\nabla\gamma\|_\infty)$.*

Proof. We proceed to set the following constants to use in our induction

$$a_i = \left(\int_M \|\nabla_{(i)}T\|^{\frac{2k}{i}}\gamma^s d\mu\right)^{\frac{i}{2k}}, \qquad a_0 = \|T\|_{\infty,[\gamma>0]}$$

$$b_i = \left(\int_M \|T\|^{\frac{2k}{i}} d\mu\right)^{\frac{i}{2k}}, \qquad b_0 = \|T\|_{\infty,[\gamma>0]}.$$

Let $r = \frac{k}{i}$,$p = \frac{2k}{i+1}$,$q = \frac{2k}{i-1}$, $t = \alpha = 0$ and $\beta = 1$ in (A.3) to obtain, for $s \geq k$

$$a_i^2 \leq c a_{i-1}\left(a_{i+1} + \left(\int_M \|\nabla_{(i)}T\|^{\frac{2k}{i+1}}\gamma^{s-\frac{2k}{i+1}} d\mu\right)^{\frac{i+1}{2k}}\right).$$

Let $s \geq 2k$ and then by (A.5) we have

$$\begin{aligned}\left(\int_M \|\nabla_{(i)}T\|^{\frac{2k}{i+1}}\gamma^{s-\frac{2k}{i+1}} d\mu\right)^{\frac{i+1}{2k}} &\leq c\left(\int_M \|\nabla_{(i+1)}T\|^{\frac{2k}{i+1}}\gamma^{s} d\mu\right)^{\frac{i+1}{2k}} \\ &\quad + c\left(\int_M \|T\|^{\frac{2k}{i+1}}\gamma^{s-\frac{2k}{i+1}-i\frac{2k}{i+1}} d\mu\right)^{\frac{i+1}{2k}} \\ &\leq c(a_{i+1} + b_{i+1}).\end{aligned}$$

Since $b_i^2 \leq b_{i-1}b_{i+1}$ by the Hölder inequality, we obtain

$$(a_i + b_i)^2 \leq c(a_{i-1} + b_{i-1})(a_{i+1} + b_{i+1}), \text{ for } 1 \leq i \leq k-1.$$

So, we have that $f(i) = a_i + b_i$ satisfied (172), and so by the convex functions result (173), we have

$$\begin{aligned}a_i \leq a_i + b_i &\leq c(a_0 + b_0)^{1-\frac{i}{k}}(a_k + b_k)^{\frac{i}{k}} \\ &\leq c\|T\|_{\infty,[\gamma>0]}^{1-\frac{i}{k}}\left(\left(\int_M \|\nabla_{(k)}T\|^2\gamma^s d\mu\right)^{\frac{1}{2}} + \|T\|_2\right)^{\frac{i}{k}},\end{aligned}$$

which is the desired statement. □

We can finally finish this series of estimates with a result suitable for application to the $P$-style terms.

COROLLARY A.8. *Let $0 \leq i_1, \ldots, i_r \leq k$, $i_1 + \ldots + i_r = 2k$ and $s \geq 2k$. Then we have*

$$\int_M \nabla_{(i_1)}T * \cdots * \nabla_{(i_r)}T\gamma^s d\mu \leq c\|T\|_{\infty,[\gamma>0]}^{r-2}\left(\int_M \|\nabla_{(k)}T\|^2\gamma^s d\mu + \|T\|_{2,[\gamma>0]}^2\right).$$

Proof. This follows from Lemma A.7 and the generalised Hölder inequality. For the purposes of notation, if at least one of the $i_j$ are zero, reindex and assume for some $0 \le l \le r$ we have $i_1, \dots, i_l \neq 0$ and $i_{l+1}, \dots, i_r = 0$.

We derive

$$\begin{aligned}
&\int_M \nabla_{(i_1)} T * \cdots * \nabla_{(i_r)} T \gamma^s d\mu \\
&\quad \le \|T\|_{\infty,[\gamma>0]}^{r-l} \prod_{j=1}^{l} \left( \int_M \|\nabla_{(i_j)} T\|^{\frac{2k}{i_j}} \gamma^s d\mu \right)^{\frac{i_j}{2k}} \\
&\quad \le \|T\|_{\infty,[\gamma>0]}^{r-l} \prod_{j=1}^{l} \left[ \|T\|_{\infty,[\gamma>0]}^{1-\frac{i_j}{k}} \left( \left( \int_M \|\nabla_{(k)} T\|^2 \gamma^s d\mu \right)^{\frac{1}{2}} \right)^{\frac{i_j}{k}} \right] \\
&\quad \le c \|T\|_{\infty,[\gamma>0]}^{r-2} \left( \int_M \|\nabla_{(k)} T\|^2 \gamma^s d\mu + \|T\|_{2,[\gamma>0]}^2 \right),
\end{aligned}$$

which is the desired statement. $\square$

APPENDIX B

# Scaling a hypersurface flow

One technique which has gained enormous popularity among analysts working on geometric evolution equations is that of 'scaling'. This is in some sense a loaded term, being that many associated methods fall under this umbrella, from singularity analysis, integral estimates, to covering arguments and simplifications. We will use scaling methods several times, where each application plays a critical role in the encompassing argument. These are in the proof of the Lifespan Theorem, the proof of the Gap Lemma, and perhaps most important of all in the construction of a blowup at an assumed finite time singularity.

The application of scaling used in the proof of the Lifespan Theorem of Chapter 3 is more classical in nature. In this appendix, we will explain how the quantity

$$\int_{f^{-1}(B_\rho(x))} \|A\|^p d\mu \tag{174}$$

transforms under a scaling of the independent variables in an immersion $f : M^n \rightarrow \mathbb{R}^{n+1}$. Our immediate aim is to prove that (174) is scaling invariant for $p = n$. Our motivation for choosing this particular kind of scaling is to make the transformation

$$\int_{B_1} |A|^p d\mu \qquad \longrightarrow \qquad \int_{B_\rho} |A|^p d\mu.$$

This is an absolutely crucial step in the proof of the Lifespan Theorem. If we did not fix the radius $\rho = \mathrm{diam}_{[\gamma>0]}$ of our cutoff function $\gamma$, then the constants in (25) would depend on the radius $\rho$ and this would introduce a circular dependency on

$\epsilon_0$ which may drive $\rho \to 0$. This would make the entire argument invalid. We will present the proof of the scale invariance of $\|A\|_n^n$ at the end of this appendix.

For the purpose of obtaining a scaling invariant governing equation, we will need to scale time by a factor different to that of space. One can see this (using results to come in this appendix), and differentiate a rescaled immersion $\tilde{f}(x, \tilde{t}(t)) = \alpha f(x,t)$ in time to obtain

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} = \alpha \frac{\partial f}{\partial t}\frac{\partial \tilde{t}}{\partial t} = \frac{\partial \tilde{t}}{\partial t}\Big(\alpha^4 (\tilde{\Delta}\tilde{H})\tilde{\nu}\Big).$$

So now if

$$\frac{\partial \tilde{t}}{\partial t} = \alpha^{-4},$$

our governing equation is invariant under scaling by $\alpha$ on the new time interval $0 \le \tilde{t} < \tilde{T}$, as desired. Thus we choose the following rescaling:

$$\tilde{x} = x, \qquad \text{and} \qquad \tilde{t} = \alpha^{-4} t.$$

There is no scaling in the domain of $f$ (this is a completely different issue altogether). We commonly write this particular rescaling in abbreviated form as

$$\tilde{f}(\tilde{x}, \tilde{t}) = \alpha f(x, \alpha^{-4} t).$$

Under this rescaling the new governing equation will be

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} = (\tilde{\Delta}\tilde{H})\tilde{\nu}.$$

We now prove the results used in the above computations, beginning by determining how the metric scales.

LEMMA B.1. *$\tilde{g}_{ij} = \alpha^2 g_{ij}$ and $\tilde{g}^{ij} = \alpha^{-2} g^{ij}$.*

PROOF. From the definition,

$$\begin{aligned}\tilde{g}_{ij} &= \left( \frac{\partial \tilde{f}}{\partial \tilde{x}^i} \,\middle|\, \frac{\partial \tilde{f}}{\partial \tilde{x}^j} \right)\\ &= \left( \alpha \frac{\partial f}{\partial x^i} \,\middle|\, \alpha \frac{\partial f}{\partial x^j} \right)\\ &= \alpha^2 \left( \frac{\partial f}{\partial x^i} \,\middle|\, \frac{\partial f}{\partial x^j} \right)\\ &= \alpha^2 g_{ij}.\end{aligned}$$

We also compute $\tilde{g}^{ij}$. In the following, $\tilde{g}$ (without the subscripts) refers to the matrix.

$$\begin{aligned}I = \tilde{g}\tilde{g}^{-1} &= \alpha^2 g \tilde{g}^{-1}\\ \Rightarrow \tilde{g}^{-1} &= \frac{1}{\alpha^2} g^{-1},\\ \Rightarrow \tilde{g}^{ij} &= \frac{1}{\alpha^2} g^{ij}.\end{aligned}$$

$\square$

Using the above we can easily prove the scale invariance of the covariant derivative.

LEMMA B.2. $\nabla = \tilde{\nabla}$.

PROOF. It is enough to determine how the Christoffel symbols scale, and thus using the fact that we have no torsion we can consider the scaling of the right hand side in the following equation:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right).$$

Therefore

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2} \alpha^2 \tilde{g}^{kl} \left( \alpha^{-2} \frac{\partial}{\partial x^i} \tilde{g}_{jl} + \alpha^{-2} \frac{\partial}{\partial x^j} \tilde{g}_{il} - \alpha^{-2} \frac{\partial}{\partial x^l} \tilde{g}_{ij} \right) = \Gamma_{ij}^k.$$

□

Second, we consider the induced measure on the surface $d\mu$. Denote the scaled measure on $\tilde{f}$ by $d\tilde{\mu}$.

Lemma B.3. $d\tilde{\mu} = \alpha^n d\mu$.

Proof. Note that $n$-dimensional Lebesgue (and Hausdorff) measure is invariant under scaling. We have

$$\begin{aligned} d\tilde{\mu} &= \sqrt{\det(\tilde{g})}d\mathcal{L}^n = \sqrt{\det(\alpha^2 g)}d\mathcal{L}^n \\ &= \sqrt{\alpha^{2n}\det(g)}d\mathcal{L}^n = \alpha^n\sqrt{\det(g)}d\mathcal{L}^n \\ &= \alpha^n d\mu. \end{aligned}$$

□

Finally, we compute $\tilde{h}_{ij}$.

Lemma B.4. $\tilde{h}_{ij} = \alpha h_{ij}$.

Proof. From the definition,

$$\tilde{h}_{ij} = \left( \frac{\partial \tilde{f}}{\partial \tilde{x}^i} \middle| \frac{\partial \tilde{\nu}}{\partial \tilde{x}^j} \right),$$

where $\tilde{\nu}$ is a local choice of unit normal for $\tilde{f}$. Noting that $\tilde{\nu} = \nu$ (the base point changes but nothing else), we compute

$$\tilde{h}_{ij} = \left( \frac{\partial \tilde{f}}{\partial \tilde{x}^i} \middle| \frac{\partial \tilde{\nu}}{\partial \tilde{x}^j} \right) = \left( \alpha\frac{\partial f}{\partial x^i} \middle| \frac{\partial \nu}{\partial x^j} \right) = \alpha h_{ij}.$$

□

We finish by determining how $\|A\|^p_{p,f^{-1}(B_\rho(x))}$ scales. Our balls in question are interpreted as preimages under $f$ and $\tilde{f}$; that is, assuming the balls are centred at the origin,

$$\begin{aligned}\tilde{f}^{-1}(B_1) &= \{\tilde{f}^{-1}(\tilde{y}) : \tilde{y} \in \mathbb{R}^{n+1}, |\tilde{y}| < 1\}\\ &= \{f^{-1}(y) : y \in \mathbb{R}^{n+1}, |\alpha y| < 1\}\\ &= \{x \in M : |f(x)| < \alpha^{-1}\}\\ &= f^{-1}(B_{\frac{1}{\alpha}}),\end{aligned}$$

after rescaling. Therefore we see that choosing $\alpha = \frac{1}{\rho}$ in the proof of the Lifespan Theorem will allow us to transform a ball of radius $\rho$ into a ball of radius 1. Recall that

$$\|A\|^2 = g^{ik}g^{jl}h_{ij}h_{kl}.$$

Thus, we must consider how each of $(g_{ij})$, $(h_{ij})$ and $d\mu$ change under scaling. Thus the scaled norm of curvature $\|\tilde{A}\|^p$ is

$$\|\tilde{A}\|^p = \left(\tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{ij}\tilde{h}_{kl}\right)^{\frac{p}{2}} = \left(\frac{1}{\alpha^2}\frac{1}{\alpha^2}\alpha\alpha g^{ik}g^{jl}h_{ij}h_{kl}\right)^{\frac{p}{2}} = \frac{1}{\alpha^p}\|A\|^p.$$

Therefore, the scaled integral (174) is

$$\int_{B_{\frac{1}{\alpha}}} \alpha^{p-n}|\tilde{A}|^2 d\tilde{\mu}.$$

And so we understand that the integrand is invariant under scaling if $p = n$.

APPENDIX C

# Lifespan theorem for simple constrained surface diffusion flows

## 1. Introduction

There is a natural class of constraint functions which give rise to a more elegant and less convoluted statement of the Lifespan Theorem than that given in Chapter 3. For these constraint functions, which we define momentarily, the smallness assumption is only required in $L^n$, and is automatically scale invariant.

A constraint function $h : [0, T) \subset I \to \mathbb{R}$ is *trivial* if $h = 0$, in which case we recover (SD) flow. We further deem that a constraint function $h$ which satisfies an estimate

$$\|h\|_{\infty,J} \le c_h < \infty$$

on any interval $J \subset [0, T)$ with $c_h = c_h(J)$ as *simple*. The corresponding simple constrained surface diffusion flows admit the following theorem.

THEOREM C.1 (Lifespan Theorem). *Suppose $n \in \{2, 3\}$ and let $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a simple constrained surface diffusion flow. Then there are constants $\rho > 0$, $\epsilon_0 > 0$, and $c < \infty$ such that if $\rho$ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu\Big|_{t=0} = \epsilon(x) \le \epsilon_0 \qquad \textit{for } m = 2, n, \textit{ any } x \in \mathbb{R}^{n+1}, \tag{175}$$

*and $h$ is simple on $\left[0, \frac{1}{c}\rho^4\right]$, then the maximal time $T$ satisfies*

$$T \ge \frac{1}{c}\rho^4, \tag{176}$$

*and we have the estimate*

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n + \|A\|^2 d\mu \le c\epsilon(x) \qquad \textit{for} \qquad 0 \le t \le \frac{1}{c}\rho^4. \tag{177}$$

While this seems to be strictly less general than the corresponding Lifespan Theorem from Chapter 3 on the nonsimple constrained flows, there does not appear to be an easy relationship between the two. For example, there are simple constraint functions such as those which grow in $t$ (even exponentially), as well as uniformly bounded functions such as the trigonometric functions, or decaying functions such as $h(t) = e^{-t}$, which do not fit into the argument of Chapter 3. Further, each of these do not satisfy the assumption (AB), the global bound on evolving area. Therefore, to obtain a full picture of the properties a given constrained surface diffusion flow exhibits, one must take into account both the Lifespan Theorem of Chapter 3 and that of this appendix.

Our strategy for presenting this proof is to refer to the estimates in Chapter 3 when appropriate and at other times present arguments which are substantially simpler in nature or give better results than those required by Chapter 3. It should be noted that one can construct the proof in this appendix from [**37**] or Chapter 3.This appendix references the interpolation and Sobolev inequalities of Appendix A and Chapter 3, and Propositions 3.17, 3.25 which carry over without change to the case of simple constrained surface diffusion flows. Otherwise, this appendix presents a complete proof of Theorem C.1.

## 2. Integral estimates

Recall the proof of Proposition 3.17. Using the same argument, we obtain the following analogous estimate. Note that we do not need to assume any smallness of curvature here.

PROPOSITION C.2. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a simple constrained surface diffusion flow with $\gamma$ a cutoff function as in* (25). *Then for a fixed $\theta > 0$ and $s \geq 2k+4$,*

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\qquad \leq (c+ch)\int_M \|A\|^2\gamma^{s-4-2k}d\mu + ch\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu \\
&\qquad\qquad + c\int_M \left([P_3^{k+2}(A) + P_5^k(A)]*\nabla_{(k)}A\right)\gamma^s d\mu,
\end{aligned}$$

*where $c = c(c_{\gamma1}, c_{\gamma2}, s, k, c_h, \theta)$.*

For the case of the simple constrained surface diffusion flows, we will exert significantly more effort in removing the restriction (AB) seen throughout Chapter 3. While there we needed it regardless to deal with non-simple constraint functions, here we will not have any control over the evolving surface area a priori, and so it does not make much sense.

We will again need various Sobolev and interpolation inequalities. As these are identical to those required in Chapter 3, we will simply refer to them when required. We now give a proof of the first key estimate we require to demonstrate Theorem C.1.

Proposition C.3. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a simple constrained surface diffusion flow and $\gamma$ a cutoff function as in* (25). *Then there is an $\epsilon_0 = \epsilon_0\Big(c_{\gamma 1}, c_{\gamma 2}, c_h([0,T^*])\Big)$ such that if*

$$\epsilon = \sup_{[0,T^*]} \int_{[\gamma>0]} \|A\|^n d\mu \le \epsilon_0 \tag{178}$$

*then for any $t \in [0,T^*]$ we have*

$$\begin{aligned} &\int_{[\gamma=1]} \|A\|^2 d\mu + \int_0^t \int_{[\gamma=1]} (\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6) d\mu d\tau \\ &\qquad\le \Big(1+(n-2)t\Big) \int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c\Big(t + (n-2)e^t\Big)\epsilon^{\frac{2}{n}}, \end{aligned} \tag{179}$$

*where $c = c\Big(c_{\gamma 1}, c_{\gamma 2}, c_h([0,T^*])\Big)$.*

Proof. The idea of the proof is to integrate Proposition C.2, and then use the multiplicative Sobolev inequality Lemma 3.22. This will introduce a multiplicative factor of $\|A\|_{n,[\gamma>0]}$ in front of several integrals, which we can then absorb on the left.

Setting $k = 0$ and $s = 4$ in Proposition C.2 we have

$$\begin{aligned} \frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu &\le (c+ch)\int_{[\gamma>0]} \|A\|^2 d\mu \\ + ch\int_M ([A*A]*A)\,\gamma^4 d\mu + c\int_M &\Big([P_3^2(A) + P_5^0(A)]*A\Big)\,\gamma^4 d\mu. \end{aligned}$$

First we estimate the $P$-style terms:

$$\begin{aligned} &\int_M \Big([P_3^2(A) + P_5^0(A)]*A\Big)\,\gamma^4 d\mu \\ &\quad\le c\int_M \Big(\Big[\|A\|^2\cdot\|\nabla_{(2)}A\| + \|\nabla A\|^2\cdot\|A\| + \|A\|^5\Big]\|A\|\Big)\gamma^4 d\mu \\ &\quad\le c\int_M \Big[\|A\|^3\cdot\|\nabla_{(2)}A\| + \|\nabla A\|^2\cdot\|A\|^2 + \|A\|^6\Big]\gamma^4 d\mu \\ &\quad\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\int_M (\|A\|^6 + \|\nabla A\|^2\|A\|^2)\gamma^4 d\mu. \end{aligned}$$

We use Lemma 3.22 to estimate the second integral and obtain for $n = 2$

$$\begin{aligned}
&\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big) \gamma^4 d\mu \\
&\quad\le \theta \int_M \|\nabla_{(2)} A\|^2 \gamma^4 d\mu + c \int_{[\gamma>0]} \|A\|^2 d\mu \int_M (\|\nabla_{(2)} A\|^2 + \|A\|^6) \gamma^4 d\mu \\
&\qquad + c\Big(\int_{[\gamma>0]} \|A\|^2 d\mu\Big)^2,
\end{aligned} \tag{180}$$

and for $n = 3$

$$\begin{aligned}
&\int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big) \gamma^4 d\mu \\
&\quad\le \theta \int_M \|\nabla_{(2)} A\|^2 \gamma^4 d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]} \int_M (\|\nabla_{(2)} A\|^2 + \|A\|^6) \gamma^4 d\mu \\
&\qquad + c\Big(c_{\gamma 1}\Big)^3 \Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big)
\end{aligned} \tag{181}$$

We add the integrals $\int_M \|A\|^6 \gamma^4 d\mu$ and $\int_M \|\nabla A\|^2 \|A\|^2 \gamma^4 d\mu$ to the estimate of Proposition C.2 (with $k = 0$, $s = 4$) and obtain

$$\begin{aligned}
&\frac{d}{dt} \int_M \|A\|^2 \gamma^4 d\mu + (2-\theta) \int_M \Big(\|\nabla_{(2)} A\|^2 + \|A\|^2 \|\nabla A\|^2 + \|A\|^6\Big) \gamma^4 d\mu \\
&\quad\le (c + ch) \int_{[\gamma>0]} \|A\|^2 d\mu + ch \int_M ([A * A] * A) \gamma^4 d\mu \\
&\qquad + c \int_M \Big(\|A\|^2 \|\nabla A\|^2 + \|A\|^6\Big) \gamma^4 d\mu + c \int_M \Big([P_3^2(A) + P_5^0(A)] * A\Big) \gamma^4 d\mu \\
&\quad\le c(1+h^2) \int_{[\gamma>0]} \|A\|^2 d\mu + c \int_M \Big(\|A\|^3 \|\nabla_{(2)} A\| + \|A\|^2 \|\nabla A\|^2 + \|A\|^6\Big) \gamma^4 d\mu.
\end{aligned}$$

For $n = 2$, we use the estimate (180) above and obtain

$$\begin{aligned}
&\frac{d}{dt} \int_M \|A\|^2 \gamma^4 d\mu + (2-\theta) \int_M \Big(\|\nabla_{(2)} A\|^2 + \|A\|^2 \|\nabla A\|^2 + \|A\|^6\Big) \gamma^4 d\mu \\
&\quad\le c(1+h^2) \int_{[\gamma>0]} \|A\|^2 d\mu + \theta \int_M \|\nabla_{(2)} A\|^2 \gamma^4 d\mu \\
&\qquad + c \int_{[\gamma>0]} \|A\|^2 d\mu \int_M (\|\nabla_{(2)} A\|^2 + \|A\|^6) \gamma^4 d\mu + c\Big(\int_{[\gamma>0]} \|A\|^2 d\mu\Big)^2.
\end{aligned}$$

For $n = 3$, we use instead (181) to obtain

$$\begin{aligned}
\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu &+ (2-\theta)\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu \\
&\le c(1+h^2)\int_{[\gamma>0]}\|A\|^2 d\mu + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu \\
&\qquad + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu \\
&\qquad + c\Big(c_{\gamma 1}\Big)^3\Big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\Big).
\end{aligned}$$

Absorbing, we obtain for $n = 2$

$$\begin{aligned}
\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu &+ (2-\theta-\epsilon_0)\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu \\
&\le c(1+\epsilon_0+\|h\|^2_{\infty,[0,T^*]})\epsilon \\
&\le c\epsilon,
\end{aligned}$$

and for $n = 3$

$$\begin{aligned}
\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu &+ (2-\theta-\sqrt{\epsilon_0})\int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu \\
&\le c\Big(1+\|h\|^2_{\infty,[0,T^*]}\Big)\int_{[\gamma>0]}\|A\|^2 d\mu + c\Big(\epsilon_0^{\frac{23}{6}} + \epsilon_0^{\frac{4}{3}}\Big)\epsilon^{\frac{2}{3}}.
\end{aligned}$$

For $\theta$, $\epsilon_0$ small enough we have

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + \int_M \Big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\Big)\gamma^4 d\mu \le c\epsilon^{\frac{2}{n}} + c(n-2)\int_{[\gamma>0]}\|A\|^2 d\mu,$$

with

$$c = c(\epsilon_0, c_h([0,t^*]), c_{\gamma 1}, c_{\gamma 2}).$$

Integrating, we have for $n = 2$

$$\begin{aligned}
\int_{[\gamma=1]}\|A\|^2\gamma^4 d\mu + \int_0^t\int_{[\gamma=1]}(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6)\,d\mu\, d\tau \\
\le \int_{[\gamma>0]}\|A\|^2 d\mu\Big|_{t=0} + c\epsilon t,
\end{aligned}$$

where we used the fact $[\gamma = 1] \subset [\gamma > 0]$ and $0 \le \gamma \le 1$. For $n = 3$ we use a covering argument and Gronwall's inequality after integrating to obtain

$$\begin{aligned}
\int_{[\gamma=1]} \|A\|^2\gamma^4 d\mu + \int_0^t \int_{[\gamma=1]} (\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6) d\mu d\tau \\
&\le \int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c\epsilon^{\frac{2}{3}} t + c\int_0^t \left( \int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c\epsilon^{\frac{2}{3}}\tau \right) e^{\int_\tau^t c d\nu} d\tau \\
&= (1+ct)\int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c\epsilon^{\frac{2}{3}} t + c\epsilon^{\frac{2}{3}} \int_0^t \tau e^{c(t-\tau)} d\tau \\
&\le (1+ct)\int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c(t+e^t)\epsilon^{\frac{2}{3}}.
\end{aligned}$$

This finishes the proof. □

We now move on to obtaining estimates for the higher derivatives of curvature in $L^\infty$. The first issue is in dealing with the $P$-style terms from Proposition C.2. Note that the proof of Proposition 3.25 in Chapter 3 did not require the constraint function bounded in any way. Therefore, we will use the same result. For the convenience of the reader we restate it here.

PROPOSITION C.4. *Suppose $f : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is a constrained surface diffusion flow and $\gamma$ a cutoff function as in* (25)*. Then, for $s \ge 2k+4$ the following estimate holds:*

$$\begin{aligned}
\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + \int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\
&\le c\|A\|^4_{\infty,[\gamma>0]} \int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + c\|A\|^2_{2,[\gamma>0]}(1+\|A\|^4_{\infty,[\gamma>0]}) \\
&\quad + ch\left( h^{\frac{1}{3}} \int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + (1+h^{\frac{1}{3}})\|A\|^2_{2,[\gamma>0]} \right).
\end{aligned} \tag{182}$$

We now prove the higher derivatives of curvature estimates. The proof here is completely analogous to that of Proposition 3.26. We present it here with the improvement of not requiring (AB).

Proposition C.5. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a simple constrained surface diffusion flow and $\gamma$ is as in (25). Then there is an $\epsilon_0$ depending on the constants in (25) and $c_h([0,T^*])$ such that if*

$$\sup_{[0,T^*]} \int_{[\gamma>0]} \|A\|^n d\mu \le \epsilon_0, \tag{183}$$

*we can conclude*

$$\|\nabla_{(k)} A\|^2_{\infty,[\gamma=1]} \le c\Big(k, T^*, c_{\gamma 1}, c_{\gamma 2}, c_h([0,T^*]), \alpha_0(k+2)\Big), \tag{184}$$

*where* $\alpha_0(k) = \displaystyle\sum_{j=0}^{k} \|\nabla_{(j)} A\|_{2,[\gamma>0]}\Big|_{t=0}$.

Proof. As before, the idea is to use our previous estimates and then integrate. The $\epsilon_0$ which we will use is exactly the same as that in Proposition C.3.

We fix $\gamma$ and consider cutoff functions $\gamma_{\sigma,\tau}$ which will allow us to combine our previous estimates. Define for $0 \le \sigma < \tau \le 1$ functions $\gamma_{\sigma,\tau} = \psi_{\sigma,\tau} \circ \gamma$ satisfying $\gamma_{\sigma,\tau} = 0$ for $\gamma \le \sigma$ and $\gamma_{\sigma,\tau} = 1$ for $\gamma \ge \tau$. The function $\psi_{\sigma,\tau}$ is chosen such that $\gamma_{\sigma,\tau}$ satisfies (25), although with different constants. Acceptable choices are

$$c_{\gamma_{\sigma,\tau}1} = \|\nabla \psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 1}, \text{ and } c_{\gamma_{\sigma,\tau}2} = \max\{c_{\gamma 1}^2 \|\nabla_{(2)} \psi_{\sigma,\tau}\|_\infty, c_{\gamma 2}\|\nabla \psi_{\sigma,\tau}\|_\infty\}.$$

Using the cutoff function $\gamma_{0,\frac12}$ instead of $\gamma$ in Proposition C.3 gives

$$\int_{[\gamma_{0,\frac12}=1]} \|A\|^2 d\mu + \int_0^{T^*} \int_{[\gamma_{0,\frac12}=1]} \|\nabla_{(2)} A\|^2 + \|A\|^6 d\mu d\tau \le c\epsilon_0^{\frac{2}{n}} T^* + \|A\|^2_{2,[\gamma>0]}\Big|_{t=0}$$

which is

$$\int_{[\gamma\ge\frac12]} \|A\|^2 d\mu + \int_0^{T^*} \int_{[\gamma\ge\frac12]} \|\nabla_{(2)} A\|^2 + \|A\|^6 d\mu d\tau \le c(1+T^*)\epsilon_0 \tag{185}$$

for $n=2$ and for $n=3$

$$\int_{[\gamma\ge\frac12]} \|A\|^2 d\mu + \int_0^{T^*} \int_{[\gamma\ge\frac12]} \|\nabla_{(2)} A\|^2 + \|A\|^6 d\mu d\tau \le c(1+T^*)\Big(\delta + \epsilon_0^{\frac23}\Big),$$

where $\delta = \|A\|^2_{2,[\gamma>0]}\Big|_{t=0}$. We do not need any smallness of $\delta$, this is simply notation.

Recall the multiplicative Sobolev inequality Proposition 3.21:

$$\|T\|^4_{\infty,[\gamma=1]} \le c\|T\|^{4-n}_{2,[\gamma>0]}\Big(\|\nabla_{(2)}T\|^n_{2,[\gamma>0]} + \|TA^2\|^n_{2,[\gamma>0]} + \|T\|^n_{2,[\gamma>0]}\Big). \tag{51}$$

Using this with $\gamma_{\frac{1}{2},\frac{3}{4}}$ and (185) above we obtain for $n=2$

$$\begin{aligned}\int_0^T \|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}d\tau &\le c\epsilon_0(c\epsilon_0(1+T^*)+\epsilon_0 T^*)\\ &\le c\epsilon_0.\end{aligned} \tag{186}$$

For $n=3$ we similarly obtain

$$\begin{aligned}\int_0^T \|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}d\tau &\le \sqrt{c(1+T^*)\Big(\delta+\epsilon_0^{\frac{2}{3}}\Big)}\Big[c(1+T^*)\Big(\delta+\epsilon_0^{\frac{2}{3}}\Big)\Big]^{\frac{3}{2}}\\ &\le c\Big(\sqrt{\delta}+\epsilon_0^{\frac{1}{3}}\Big),\end{aligned} \tag{187}$$

where

$$c = c\Big(c_h([0,T^*]), c_{\gamma 1}, c_{\gamma 2}, T^*, n, \epsilon_0\Big).$$

We now use (182) with $\gamma_{\frac{3}{4},\frac{7}{8}}$. Factorising, we have

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu &\le c\|A\|^4_{\infty,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\\ &\quad + c\|A\|^2_{2,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\Big(1+\|A\|^4_{\infty,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\Big)\\ &\quad + ch\Big(h^{\frac{1}{3}}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu + (1+h^{\frac{1}{3}})\|A\|^2_{2,[\gamma_{\frac{3}{4},\frac{7}{8}}\ge 0]}\Big)\\ &\le c\Big(\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h^{\frac{4}{3}}\Big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\\ &\quad + c\|A\|^2_{2,[\gamma\ge\frac{3}{4}]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac{3}{4}]}+h+h^{\frac{4}{3}}\Big).\end{aligned}$$

We wish to solve this differential equation using Gronwall's inequality. The constraint function is obviously bounded, and we can bound the integrals of the relevant curvature quantities, as we have shown above. Integrating,

$$\begin{aligned}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu - \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu\Big|_{t=0} \\ \le c\int_0^t \Big[\Big(\|A\|^4_{\infty,[\gamma\ge\frac34]} + h^{\frac43}\Big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu\Big]\,d\tau \\ + c\int_0^t \Big[\|A\|^2_{2,[\gamma\ge\frac34]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac34]} + h + h^{\frac43}\Big)\Big]\,d\tau. \end{aligned} \tag{188}$$

Now from our earlier calculation (186) we have

$$\int_0^t \Big(\|A\|^4_{\infty,[\gamma\ge\frac34]} + h^{\frac43}\Big)d\tau \le c,$$

and, using our assumption (183)

$$c\int_0^t \Big[\|A\|^2_{2,[\gamma\ge\frac34]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac34]} + h + h^{\frac43}\Big)\Big]\,d\tau \le c.$$

Also, we have

$$\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu\Big|_{t=0} \le c\alpha_0(k),$$

where $\alpha_0$ is as in the statement of the proposition.

Therefore, equation (188) is of the form

$$\alpha(t) \le \beta(t) + \int_c^t \lambda(\tau)\alpha(\tau)d\tau,$$

where

$$\begin{aligned}\alpha(t) &= \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu,\\ \beta(t) &= \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu\Big|_{t=0} + c\int_0^t \Big[\|A\|^2_{2,[\gamma\ge\frac34]}\Big(1+\|A\|^4_{\infty,[\gamma\ge\frac34]} + h + h^{\frac43}\Big)\Big]\,d\tau,\end{aligned}$$

and

$$\lambda(t) = \|A\|^4_{\infty,[\gamma\ge\frac34]} + h^{\frac43}.$$

Noting that $\beta$ and $\int \lambda d\tau$ are bounded by the constants shown above, we can invoke Gronwall's inequality and conclude

$$\int_{[\gamma\geq\frac{7}{8}]} \|\nabla_{(k)}A\|^2 d\mu \leq \beta(t) + \int_0^t \beta(\tau)\lambda(\tau)e^{\int_\tau^t \lambda(\nu)d\nu} d\tau \leq c\Big(k, \alpha_0(k)\Big).$$

Trivially, we also have

$$\int_{[\gamma\geq\frac{7}{8}]} \|\nabla_{(k+2)}A\|^2 d\mu \leq c\Big(k+2, \alpha_0(k+2)\Big).$$

Therefore using (51) with $\gamma_{\frac{7}{8},\frac{15}{16}}$, and taking into account the $n = 3$ statement of Lemma 3.22 we can bound $\|A\|_\infty$ on a smaller ball:

$$\|A\|^4_{\infty,[\gamma\geq\frac{15}{16}]} \leq c(0, \alpha_0(0))^{\frac{4-n}{2}} \Big( \big(c(2, \alpha_0(2))^{\frac{n}{2}} + \big(c(0, \alpha_0(0))^{\frac{n}{2}} \Big) \leq c.$$

Finally, using (51) with $T = \nabla_{(k)}A$ and $\gamma = \gamma_{\frac{15}{16},1}$ we obtain

$$\begin{aligned}\|\nabla_{(k)}A\|^4_{\infty,[\gamma=1]} \leq c\|\nabla_{(k)}A\|^{4-n}_{2,[\gamma>\frac{15}{16}]} \Big( &\|\nabla_{(k+2)}A\|^n_{2,[\gamma>\frac{15}{16}]} \\ &+ (\|A\|^{2n}_{\infty,[\gamma>\frac{15}{16}]} + 1)\|\nabla_{(k)}A\|^n_{2,[\gamma>\frac{15}{16}]}\Big) \\ \leq c\Big(k, \alpha_0(k+2)\Big).&\end{aligned}$$

This completes the proof. □

## 3. Proof of the Lifespan Theorem

The proof is analogous to that presented in Chapter 3, however as there are some simplifications required (related to the absence of a constraint function) we will still present it here for completeness. We begin by scaling $\tilde{f}(x,t) = \frac{1}{\rho}f(x, \rho^4 t)$. Note that $\|A\|^n_n$ is scale invariant, and so we may assume $\rho = 1$. Note that $h$ may scale in a non-invariant fashion but this introduces a single change in the constant $c_h$ only, and certainly a scaled simple $h$ (we only perform this rescaling once) remains simple.

We make the definition

$$\eta(t) = \sup_{x\in\mathbb{R}^3} \int_{f^{-1}(B_1(x))} \|A\|^n d\mu. \tag{189}$$

By covering $B_1$ with several translated copies of $B_{\frac{1}{2}}$ there is a constant $c_\eta$ such that

$$\eta(t) \le c_\eta \sup_{x\in\mathbb{R}^3} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^n d\mu. \tag{190}$$

Note that $c_\eta = 4^{n+1}$ is sufficient.

By short time existence we have that $f(M \times [0,t])$ is compact for $t < T$ and so the function $\eta : [0,T) \to \mathbb{R}$ is continuous. We now define

$$t_0^{(n)} = \begin{cases} \sup\{0 \le t \le \min(T,\lambda_2) : \eta(\tau) \le 3c_\eta\epsilon_0 \ \text{ for } \ 0 \le \tau \le t\}, & n = 2, \\ \sup\{0 \le t \le \min(T,\lambda_3) : \eta(\tau) \le 3c_{P5}c_\eta c^*(\delta + \epsilon_0^{\frac{2}{3}}) \ \text{ for } \ 0 \le \tau \le t\}, & n = 3, \end{cases} \tag{191}$$

where $\delta = \sup_{x\in\mathbb{R}^4} \|A\|^2_{2,f^{-1}(B_1(x))}\Big|_{t=0}$, $\lambda_n$ is a parameter to be specified later and

$$c^* = c_{P5} + c_0 c_\eta e^{c_{P}5/c_0 c_\eta}.$$

The constant $c_{P5}$ is the maximum of 1 and the constant from Proposition C.5, and $c_0$ is the maximum of all the constants on the right hand side of Proposition C.3. Note that the $\epsilon_0$ on the right hand side of the inequality is from equation (175). Unlike earlier in Proposition C.5, we require $\delta$ small as described in the statement of Theorem C.1.

The proof continues in three steps. First, we show that it must be the case that $t_0^{(n)} = \min(T,\lambda_n)$. Second, we show that if $t_0^{(n)} = \lambda_n$, then we can conclude the Lifespan Theorem. Finally, we prove by contradiction that if $T \ne \infty$, then $t_0^{(n)} \ne T$. We label these steps as

$$t_0^{(n)} = \min(T,\lambda_n), \tag{192}$$

$$t_0^{(n)} = \lambda_n \quad \Longrightarrow \quad \text{Lifespan Theorem}, \tag{193}$$

$$T \neq \infty \implies t_0^{(n)} \neq T. \tag{194}$$

The three statements (192), (193), (194) together imply the Lifespan Theorem. We expand the sketch of the argument given above as follows: first notice that by (192) $t_0^{(n)} = \lambda_n$ or $t_0^{(n)} = T$, and if $t_0^{(n)} = \lambda_n$ then by (193) we have the Lifespan Theorem. Also notice that if $t_0^{(n)} = \infty$ then $T = \infty$ and the Lifespan Theorem follows from estimate (197) below (used to prove statement (193)). Therefore the only remaining case where the Lifespan Theorem may fail to be true is when $t_0^{(n)} = T < \infty$. But this is impossible by statement (194), so we are finished.

We now give the proof of the first step, statement (192). From the assumption (175),

$$\eta(0) \le \epsilon_0 < \begin{cases} 3c_\eta \epsilon_0, & \text{for } n = 2 \\ 3c_{P5}c_\eta c^*(\delta + \epsilon_0^{\frac{2}{3}}), & \text{for } n = 3, \end{cases}$$

and therefore (191) implies $t_0^{(n)} > 0$. Assume for the sake of contradiction that $t_0^{(n)} < \min(T, \lambda_n)$. Then from the definition (191) of $t_0^{(n)}$ and the continuity of $\eta$ we have

$$\eta\Big(t_0^{(n)}\Big) = \begin{cases} 3c_\eta \epsilon_0, & \text{for } n = 2 \\ 3c_{P5}c_\eta c^*(\delta + \epsilon_0^{\frac{2}{3}}), & \text{for } n = 3, \end{cases} \tag{195}$$

so long as $\epsilon_0 \le 1$ and $c_{P5} \ge 1$. Recall Proposition C.3. We will now set $\gamma$ to be a cutoff function as in (25) such that

$$\chi_{B_{\frac{1}{2}}(x)} \le \tilde{\gamma} \le \chi_{B_1(x)},$$

for any $x \in M_t$. Choosing a small enough $\epsilon_0$ (by varying $\rho$ in (175)), definition (191) implies that the smallness condition (178) is satisfied on $[0, t_0^{(n)})$. Therefore we have satisfied all the requirements of Proposition C.3, and so we conclude

(196)
$$\begin{aligned}\int_{f^{-1}(B_{\frac{1}{2}}(x))}\|A\|^2 d\mu &\le \Big(1+(n-2)t\Big)\int_{f^{-1}(B_1(x))}\|A\|^2 d\mu\Big|_{t=0} + c\Big(t+(n-2)e^t\Big)c_\eta \epsilon^{\frac{2}{n}}\\ &\le \begin{cases} 2\epsilon_0, & \text{for } n=2 \text{ and } \lambda_2 = \frac{1}{c_0 c_\eta},\\ 2c_{P5}c^*(\delta+\epsilon_0^{\frac{2}{3}}) & \text{for } n=3 \text{ and } \lambda_3 = c_{P5}\frac{1}{c_0 c_\eta},\end{cases}\end{aligned}$$

for all $t \in [0, t^*]$, where $t^* < t_0^{(n)}$. Thus equation (196) above is true for all $t \in \Big[0, t_0^{(n)}\Big)$. We combine this with (190) to conclude

$$\eta(t) \le c_{P5}^{n-2} c_\eta \sup_{x\in\mathbb{R}^3}\int_{f^{-1}(B_{\frac{1}{2}}(x))}\|A\|^n d\mu \le \begin{cases} 2c_\eta\epsilon_0, & \text{for } n=2,\\ 2c_{P5}c_\eta c^*(\delta+\epsilon_0^{\frac{2}{3}}), & \text{for } n=3,\end{cases} \tag{197}$$

where $0 \le t < t_0^{(n)}$.

Since $\eta$ is continuous, we can let $t \to t_0^{(n)}$ and obtain a contradiction with (195). Therefore, with the choice of $\lambda_n$ in equation (196), the assumption that $t_0^{(n)} < \min(T, \lambda_n)$ is incorrect. Thus we have shown (192). We have also proved the second step (193). Observe that if $t_0^{(n)} = \lambda_n$ then by the definition (191) of $t_0^{(n)}$,

$$T \ge \lambda_n,$$

which is (176). Also, (197) implies (177). That is, we have proved if $t_0^{(n)} = \lambda_n$, then the Lifespan Theorem holds, which is the second step (193). It only remains to prove equation (194).

We assume

$$t_0^{(n)} = T \ne \infty;$$

since if $T = \infty$ then (176) holds automatically and again (197) implies (177). Note also that we can safely assume $T < \lambda_n$, since otherwise we can apply step two to conclude the Lifespan Theorem.

Our strategy is to show that in this case the flow exists smoothly up to and including time $T$, allowing us to extend the flow, thus contradicting the finite maximality of $T$ from short time existence. Since $h$ is simple, it presents no difficulty, and is always bounded. As in Chapter 3, Section 6, we use Proposition C.5 to convert the higher covariant derivatives of curvature bounds to partial derivatives of the immersion bounds. That is, we have

$$\|\partial_{(k)}\frac{\partial}{\partial t}f\|_\infty, \|\partial_{(k)}f\|_\infty \le c(m, T, f_0, \|h\|_{\infty,[0,T)}),$$

for any $k \in \mathbb{N}$. This is enough to show that the convergence $f(\cdot, t) \to f(\cdot, T)$ is in the $C^\infty$ topology and $M_T$ is smooth. We have that $f(\cdot, T)$ is a smooth immersion as the metrics at each time $t$ are uniformly equivalent and $g(t) \to g(T)$. Finally, by short time existence, we can extend the solution to an interval $[0, T + \delta]$, contradicting the maximality of $T$. This establishes (194) and the theorem is proved. $\square$

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