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Guanghua Lian
University of Wollongong

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Pricing Volatility Derivatives with Stochastic Volatility

*A thesis submitted in fulfillment of the
requirements for the award of the degree of*

Doctor of Philosophy

from

University of Wollongong

by

Guanghua Lian, B.Sc. (Sichuan University)

M.A. (Huazhong University of Science and Technology)

School of Mathematics and Applied Statistics

2010

CERTIFICATION

I, Guanghua Lian, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Guanghua Lian

March, 2010

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Abstract

Volatility derivatives are products where the volatility is the main underlying notion. These products are particularly important for market investors as they use them to have insight into the level of volatility to efficiently manage the market volatility risk. This thesis makes a contribution to literature by presenting a set of closed-form exact solutions for the pricing of volatility derivatives.

The first issue is the pricing of variance swaps, which is discussed in Chapter 2, 3, and 4. We first present an approach to solve the partial differential equation (PDE), based on the Heston (1993) two-factor stochastic volatility, to obtain closed-form exact solutions to price variance swaps with discrete sampling times. We then extend our approach to price forward-start variance swaps to obtain closed-form exact solutions. Finally, our approach is extended to price discretely-sampled variance by further including random jumps in the return and volatility processes. We show that our solutions can substantially improve the pricing accuracy in comparison with those approximations in literature. Our approach is also very versatile in terms of treating the pricing problem of variance swaps with different definitions of discretely-sampled realized variance in a highly unified way.

The second issue, which is covered in Chapter 5, and 6, is the pricing method for volatility swaps. Papers focusing on analytically pricing discretely-sampled volatility swaps are rare in literature, mainly due to the inherent difficulty associated with the nonlinearity in the pay-off function. We present a closed-form exact solution for the pricing of discretely-sampled volatility swaps, under the framework of Heston (1993) stochastic volatility model, based on the definition of the so-called average of realized volatility. Our closed-form exact solution for discretely-sampled volatility swaps can significantly reduce the computational time in obtaining numerical values for the discretely-sampled volatility swaps, and substantially improve the computational accuracy of discretely-sampled volatility swaps, comparing with the continuous sampling approximation. We also investigate the accuracy of the well-known convexity correction approximation in pricing volatility swaps. Through both theoretical analysis and numerical examples,

we show that the convexity correction approximation would result in significantly large errors on some specific parameters. The validity condition of the convexity correction approximation and a new improved approximation are also presented.

The last issue, which is covered in Chapter 7 and 8, is the pricing of VIX futures and options. We derive closed-form exact solutions for the fair value of VIX futures and VIX options, under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. As for the pricing of VIX futures, we show that our exact solution can substantially improve the pricing accuracy in comparison with the approximation in literature. We then demonstrate how to estimate model parameters, using the Markov Chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data. We also conduct empirical studies to examine the performance of the four different stochastic volatility models with or without jumps. Our empirical studies show that the Heston stochastic volatility model can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can indeed further improve the pricing the VIX futures. However, jumps added in the volatility process appear to add little improvement for pricing VIX futures. As for the pricing of VIX options, we point out the solution procedure of Lin & Chang (2009)'s pricing formula for VIX options is wrong, and alert the research community that this formula should not be further used. More importantly, we present a new closed-form pricing formula for VIX options and demonstrate its high efficiency in computing the numerical values of the price of a VIX option. The numerical examples show that results obtained from our formula consistently match up with those obtained from Monte Carlo simulation perfectly, verifying the correctness of our formula; while the results obtained from Lin & Chang (2009)'s pricing formula significantly differ from those from Monte Carlo simulation. Some other important and distinct properties of the VIX options (e.g., put-call parity, the hedging ratios) have also been discussed.

Contents

1	Introduction and Background	1
1.1	Volatility Derivatives	1
1.1.1	Variance Swaps	3
1.1.2	Volatility Swaps	6
1.1.3	VIX Futures and Options	8
1.2	Mathematical Background	10
1.2.1	Fundamental Pricing Theorems	10
1.2.2	Stochastic Calculus	12
1.2.3	Connections Between PDE and SDE	13
1.2.4	Transformations	14
1.2.5	Characteristic Function	16
1.3	Mathematical Models	17
1.3.1	Black-Scholes Model	17
1.3.2	Local Volatility Model	21
1.3.3	Stochastic Volatility Models	22
1.4	Literature Review	26
1.4.1	Variance Swaps and Volatility Swaps	26
1.4.2	VIX Futures and Options	30
1.5	Structure of Thesis	32
2	Pricing Variance Swaps with Discrete Sampling	37
2.1	Introduction	37
2.2	Pricing Variance Swaps	38
2.2.1	The Heston Stochastic Volatility Model	39

2.2.2	Variance Swaps	40
2.2.3	Our Approach to Price Variance Swaps	43
2.3	Numerical Examples and Discussions	57
2.3.1	Monte Carlo Simulations	58
2.3.2	The Validity of the Continuous Approximation	61
2.3.3	Comparison with Other Solutions	65
2.4	Conclusion	70
3	Pricing Forward-Start Variance Swaps	71
3.1	Introduction	71
3.2	Our Solution Approach	73
3.2.1	Forward-Start Variance Swaps	73
3.2.2	Forward Characteristic Function	76
3.2.3	Pricing Forward-Start Variance Swaps	78
3.3	Numerical Results and Discussions	82
3.3.1	Continuous Sampling Approximation	82
3.3.2	Monte Carlo Simulations	83
3.3.3	The Effect of Forward Start	85
3.3.4	The Effect of Mean-reverting Speed	87
3.3.5	The Effect of Realized-Variance Definitions	89
3.3.6	The Effect of Sampling Frequencies	90
3.4	Conclusion	92
4	Pricing Variance Swaps with Stochastic Volatility and Random Jumps	94
4.1	Introduction	94
4.2	Our Solution Approach	96
4.2.1	Affine Model Specification	97
4.2.2	Pricing Variance Swaps	98
4.3	Numerical Results and Discussions	103
4.3.1	Continuous Sampling Approximation	104
4.3.2	Monte Carlo Simulations	108

4.3.3	The Effect of Realized-Variance Definitions	110
4.3.4	The Effect of Jump Diffusion	111
4.3.5	The Effect of Sampling Frequencies	116
4.4	Conclusion	118
5	Pricing Volatility Swaps with Discrete Sampling	120
5.1	Introduction	120
5.2	Our Solution Approach	123
5.2.1	Volatility Swaps	123
5.2.2	Pricing Volatility Swaps	125
5.3	Numerical Results and Discussions	129
5.3.1	Monte Carlo Simulations	130
5.3.2	Other Definition of Realized Volatility	132
5.3.3	Continuous Sampling Approximation	134
5.3.4	The Effect of Realized-Variance Definitions	136
5.4	Conclusion	138
6	Examining the Accuracy of the Convexity Correction Approximation	140
6.1	Introduction	140
6.2	Convexity Correction and Convergence Analysis	143
6.3	Illustrations and Discussions	149
6.3.1	Volatility Swaps in Heston Model	149
6.3.2	Volatility Swaps in GARCH Model	154
6.3.3	VIX Futures in SVJJ Model	157
6.4	Conclusion	162
7	Pricing VIX Futures	164
7.1	Introduction	164
7.2	VIX Futures Models	169
7.2.1	Volatility Index	170
7.2.2	Affine Model Specification	171
7.2.3	Pricing VIX Futures	175

7.2.4	Numerical Examples	184
7.3	Empirical Studies	191
7.3.1	The Econometric Methodology	193
7.3.2	Data Description	196
7.3.3	Empirical Results	199
7.3.4	Comparative Studies of Pricing Performance	201
7.4	Conclusion	206
8	Pricing VIX Options	208
8.1	Introduction	208
8.2	VIX Options	211
8.2.1	Our Formula	214
8.3	Numerical Results and Discussions	218
8.3.1	Lin & Chang (2009)'s Formula	218
8.3.2	Monte Carlo Simulations	220
8.3.3	Numerical Results	220
8.3.4	Properties of VIX Options	223
8.4	Conclusion	227
9	Concluding Remarks	228
A	A Sample Term Sheet of A Variance Swap	231
B	Proofs for Chapter 2	232
B.1	Proof of Proposition 1	232
B.2	The Derivation of Eq. (2.32)	234
B.3	The Derivation of Eq. (2.55)	235
C	Proof for Chapter 3 and 4	236
D	The Laplace Transform of the Realized Variance in Chapter 6	239
E	Proof for Chapter 7	240
	Bibliography	242

List of Figures

1.1	The cash flow of a variance swap at maturity	5
1.2	The payoffs of variance and volatility swaps for long position with strike=20 volatility points and notional amount $L=2,000,000$. . .	8
1.3	The implied volatility of ASX SPI 200 index call options	20
2.1	A comparison of fair strike values of actual-return variance swaps obtained from our closed-form solution, the continuous approxi- mation and the Monte Carlo simulations, based on the Heston stochastic volatility model	59
2.2	A comparison of fair strike values of log-return variance swaps obtained from our closed-form solution, the continuous approxi- mation and the Monte Carlo simulations, based on the Heston stochastic volatility model	60
2.3	Calculated fair strike values of actual-return and log-return vari- ance swaps as a function of sampling frequency	63
2.4	Calculated fair strike values of actual-return and log-return vari- ance swaps as a function of tenor	65
2.5	The comparison of our results with those of Broadie & Jain (2008) for log-return variance swaps	67
2.6	The effect of alternative measures of realized variance	69
3.1	Calculated fair strike values as a function of sampling frequency .	84
3.2	Calculated fair strike values as a function of the starting time of sampling while the total sampling period is held as a constant, $T_e - T_s = 1$	85

3.3	Calculated fair strike values as a function of the starting time of sampling while the terminating time of sampling is held as a constant, $T_e = 1$	87
3.4	Calculated fair strike values as a function of the starting time of sampling while the total sampling period is held as a constant, $T_e - T_s = 1$	88
4.1	Calculated fair strike values in the SVJJ model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)	109
4.2	Calculated fair strike values in the SV model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)	111
4.3	Calculated fair strike values in the SVJ model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)	112
4.4	Calculated fair strike values in the SVVJ model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)	114
5.1	A comparison of fair strike prices of volatility swaps based on our explicit pricing formula and the Monte Carlo simulations	130
5.2	A comparison of fair strike prices of volatility swaps based on the two definitions of realized volatility obtained from our explicit pricing formula, the Monte Carlo simulations, and the corresponding continuous sampling approximations.	136
6.1	A comparison of the exact volatility strike and the approximations based on the Heston model	151
6.2	Relative pricing errors of the second order approximation as a function of SCV ratio in Heston model	153
6.3	A comparison of the volatility strikes from the finite difference and those from approximations in the GARCH model	156

6.4	Relative pricing errors of the second order approximation as a function of SCV ratio in GARCH model	157
6.5	A comparison of the VIX futures strikes from the exact formula and those from the convexity correction approximation in the SVJJ model	159
6.6	Relative pricing errors of the second order approximation in pricing VIX futures as a function of SCV ratio in SVJJ model	160
6.7	A comparison of VIX futures strikes obtained from the exact formula and the second-order and the third-order approximations in the Heston model	161
7.1	A comparison of VIX futures strikes obtained from our exact formula, the MC simulations and Lin (2007)'s approximation, as a function of tenor, based on the SVJJ model	186
7.2	A comparison of VIX futures strikes obtained from our exact formula, the MC simulations and Lin (2007)'s approximation, as a function of "vol of vol", based on the SVJJ model	187
7.3	A comparison of VIX futures strikes obtained from our exact formula and the approximations in literature, as a function of tenor, based on the Heston model	190
7.4	A comparison of VIX futures strikes obtained from our exact formula and the approximations in literature, as a function of "vol of vol", based on the Heston model	192
7.5	The historical data of VIX index and S&P500 index from Jun. 1990 to Aug. 2008	198
7.6	A comparison of the term structures of average VIX futures prices obtained from empirical market data and the four models	204
7.7	A comparison of the steady-rate VIX density functions obtained from empirical market data and the four models	205
8.1	A Comparison of the Prices of VIX Options Obtained from Our Exact Formula and the Formula in Lin & Chang (2009), as A Function of Tenor, based on the Heston Model ($K = 13$)	221

8.2	A Comparison of VIX Futures Strikes Obtained from Our Exact Formula and the Formulae in Literature, as A Function of Tenor, based on the Heston Model	223
8.3	The Delta of VIX Options with different maturities: $\tau = 5, 20, 40$ and 128 days, based on the SVJJ Model.	225
8.4	The Prices of VIX Options, as A Function of the Time to Maturity, based on the SVJJ Model.	226
A.1	A sample term sheet of a variance swap written on the variance of S&P500.	231

List of Tables

2.1	The strike prices of discretely-sampled actual-return variance swaps obtained from our closed-form solution Eq. (2.36), the continuous approximation and MC simulations	60
2.2	Relative errors and computational time of MC simulations in calculating the strike prices of actual-Return variance swaps	61
2.3	The sensitivity of strike price of variance swap (daily sampling) .	70
3.1	The numerical results of discrete model, continuous model and MC simulations	85
3.2	The sensitivity of strike price of variance swap (daily sampling) .	91
4.1	The numerical results of discrete model, continuous model and MC simulations	109
4.2	The sensitivity of the strike price of a variance swap (weekly sampling)	118
5.1	The numerical results of volatility-average swaps obtained from our analytical pricing formula, MC simulations and continuous sampling approximation	131
5.2	Relative errors and computational time of MC simulations	131
5.3	The sensitivity of the strike price of a volatility swap (daily sampling)	138
6.1	Strikes of one-year maturity volatility swaps obtained from the exact pricing formula and the approximations in the Heston model	152
6.2	The relative errors of the three approximations in the three intervals	153
7.1	Parameters for SV, SVJ and SVJJ models	185

7.2	Descriptive statistics of VIX and daily settlement prices of the VIX futures across maturities	199
7.3	The parameters of the SV, SVJ, SVCJ, and SVSCJ models esti- mated from the MCMC method	200
7.4	The test of pricing performance of the four models	203