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Directed Graphs and k-graphs: Topology of the Path Space and How It Manifests in the Associated C^*-algebra

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by

Samuel B.G. Webster, B.Math (Hons)

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Declaration

I, Samuel B.G. Webster, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Samuel B.G. Webster

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Contents

Abstract

Directed graphs and their higher-rank analogues provide an intuitive framework for the analysis of a broad class of C^*-algebras which we call graph algebras. Kumjian, Pask, Raeburn and Renault built a groupoid $\mathcal{G}_E$ from the infinite-path space of a locally finite directed graph E, and used the theory of groupoid C^*-algebras to define the graph C^*-algebra. Local finiteness was required so that $\mathcal{G}_E$ was locally compact and r-discrete, with unit space homeomorphic to the infinite path space of E. Similarly, the higher-rank graphs of Kumjian and Pask were initially studied with similar restrictive hypotheses in order to use groupoid based analysis of their higher-rank C^*-algebras. In particular, the topology on the path space of a directed graph or higher-rank graph is crucial in the analysis of graph C^*-algebras.

Drinen and Tomforde defined a process called desingularisation which can be used to extend many results about the C^*-algebras of locally finite directed graphs to those of arbitrary directed graphs. Drinen and Tomforde construct from an arbitrary directed graph E a row-finite directed graph $\widehat{E}$ with no sources such that $C^*(E)$ embeds in $C^*(\widehat{E})$ as a full corner. Subsequently, Farthing developed a partial desingularisation for higher-rank graphs, which constructs from a row-finite higher-rank graph Λ with sources a row-finite higher-rank graph $\widetilde{\Lambda}$ with no sources such that $C^*(\Lambda)$ embeds in $C^*(\widetilde{\Lambda})$ as a full corner.

In Chapter 2, we construct a topology on the path space of an arbitrary directed graph E and prove that it is locally compact and Hausdorff. We show that there is a homeomorphism ϕ_∞ from a subspace of the infinite-path space of the Drinen-Tomforde desingularisation $\widehat{E}$ onto the boundary-path space ∂E of E. We then show that there is a commutative C^*-subalgebra D_E of $C^*(E)$ which is homeomorphic to the continuous functions on ∂E. Concluding our results on directed graphs, we show that the embedding of $C^*(E)$ in $C^*(\widehat{E})$ restricts to an embedding of D_E in $D_{\widehat{E}}$ which implements ϕ_∞. In Chapter 3, we develop a modification of Farthing's desingularisation procedure for row-finite higher-rank graphs which contains cleaner proofs of her results. We use this modification to prove analogues for higher-rank graphs of the results from Chapter 2.

CHAPTER 1

Introduction

Cuntz and Krieger introduced and studied C^*-algebras associated to finite $(0,1)$-matrices in [**3**]. Within a year, Enomoto and Watatani showed in [**6**] how to interpret the Cuntz-Krieger relations and the hypotheses of Cuntz and Krieger's main theorems very naturally in terms of directed graphs. This opened many doors to operator algebraists: graph C^*-algebras have provided a rich supply of very tractable examples. In particular, the combinatorial properties of a graph are strongly tied to the algebraic properties of its C^*-algebra. Graph C^*-algebras include (up to Morita equivalence) all AF algebras [**4**] and all Kirchberg algebras with free abelian K_1 [**30**], as well many non-simple examples of purely infinite nuclear C^*-algebras. In [**12**], Kumjian and Pask introduced higher-rank analogues of directed graphs and associated to them C^*-algebras which broaden the class of graph C^*-algebras to a class including all tensor products of graph C^*-algebras (and thus many Kirchberg algebras whose K_1 contains torsion elements [**12**]), as well as (up to Morita equivalence) the irrational rotation algebras and many other examples of simple A$\mathbb{T}$-algebras with real rank zero [**15**]. See [**19**] for an excellent survey of the field.

There are several standard approaches to studying graph C^*-algebras. The original method for studying them uses groupoids in order to tap into the powerful theory of groupoid C^*-algebras [**24**] to study graph C^*-algebras. A groupoid is an object similar to a group but with multiplication only defined on some pairs of elements. In [**14**], Kumjian, Pask, Raeburn and Renault built a groupoid $\mathcal{G}_E$ from each directed graph E, then using Renault's theory of groupoid C^*-algebras, they defined the graph C^*-algebra to be the groupoid C^*-algebra $C^*(\mathcal{G}_E)$. By interpreting Renault's hypotheses in terms of the graph E from which $\mathcal{G}_E$ was built, Kumjian et al. were able to link properties of E to those of $C^*(\mathcal{G}_E)$. The analysis of [**14**] establishes among other things that $C^*(\mathcal{G}_E)$ is the universal C^*-algebra generated by a collection of partial isometries satisfying relations now known as the Cuntz-Krieger relations (see Section 2.3).

The results of [**14**] were proved only for graphs which are *locally finite*, meaning that each vertex emits and receives only finitely many edges. This is not to be confused with *row-finiteness*, which only requires each vertex to receive finitely many edges. A significantly different way to construct a groupoid $\mathcal{G}_E$ from a graph E

was introduced by Paterson in [**16**]. Paterson's construction proceeds via inverse semigroups, and provides a framework for a groupoid-based analysis of the graph algebras of non-row-finite directed graphs. Common to both groupoid models is that the unit space $\mathcal{G}_E^0$ of the groupoid, which must be locally compact and Hausdorff, is a collection of paths in the graph: for a row-finite graph with no sources, $\mathcal{G}_E^0$ is the collection of right-infinite paths in E; but for more complicated graphs, the infinite paths are replaced with *boundary paths* (for the definition see the prelude to Chapter 2). Hence the path space of a graph as a topological space is of great importance in the context of graph C^*-algebras. The path spaces of graphs are the central focus of this thesis.

Another popular method of studying graph C^*-algebras is to forgo the groupoid machinery used in earlier approaches, and analyse graph C^*-algebras with "bare hands" (for example [**1, 2, 21, 22**]). Such a direct analysis of graph C^*-algebras generally uses techniques similar (albeit much-refined) to those developed by Cuntz and Krieger in [**3**]. It provides cleaner proofs, and in particular finesses some of the technical hypothesis arising in a groupoid approach. Bates, Pask, Raeburn and Szymański in [**2**] used direct analysis to lift the no-sources limitation that had been present in all preceding studies.

Many results for row-finite directed graphs with no sources can be extended to arbitrary graphs via a process called *desingularisation*. Given an arbitrary directed graph E, Drinen and Tomforde show in [**5**] how to construct a row-finite directed graph F with no sources by adding vertices and edges to E in such a way that the C^*-algebra associated to F contains the C^*-algebra associated to E as a full corner. The modified graph F is now called a *Drinen-Tomforde desingularisation of* E. Although Drinen and Tomforde's process can be used to extend many results for row-finite directed graphs to arbitrary directed graphs, there are still open problems: for example, it is not yet known how to retrieve one of the major theorems — the gauge invariant uniqueness theorem — for arbitrary graphs via desingularisation. In this thesis we show how desingularisation affects the boundary path space of the graph; or more precisely, how it does not.

In [**28**], Robertson and Steger introduced and analysed higher-rank analogues of Cuntz-Krieger algebras associated to commuting families of $(0, 1)$-matrices. Kumjian and Pask in [**12**] introduced *higher-rank graphs* (or *k-graphs*) as analogues of directed graphs in order to study Robertson and Steger's higher-rank Cuntz-Krieger algebras using the techniques previously developed for directed graphs. Although the definition of a k-graph isn't quite as straightforward as that of a directed graph, k-graphs are a natural generalisation of directed graphs, and it is shown in [**12**, Example

1.3] that 1-graphs are precisely the path-categories of directed graphs. Like graph C^*-algebras, higher-rank graph C^*-algebras were first studied using groupoid techniques. In [**12**], Kumjian and Pask defined the k-graph C^*-algebra $C^*(\Lambda)$ to be the universal C^*-algebra for a set of Cuntz-Krieger relations among partial isometries associated to paths of the k-graph Λ. Using direct analysis, they proved a version of the gauge-invariant uniqueness theorem for k-graph algebras. They then constructed a path groupoid $\mathcal{G}_\Lambda$ from each k-graph Λ, and used the gauge invariant uniqueness theorem to prove that the groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$ is isomorphic to $C^*(\Lambda)$. This allowed them to plug into Renault's theory of groupoid C^*-algebras to analyse higher-rank graph C^*-algebras.

In [**21**], Raeburn, Sims and Yeend developed a "bare-hands" analysis of k-graph C^*-algebras. They found a slightly weaker alternative to the no-sources hypothesis from Kumjian and Pask's theorems called *local convexity* (Definition 3.0.13). The same authors later introduced *finitely aligned* k-graphs in [**22**], and gave a direct analysis of their C^*-algebras. This remains the most general class of k-graphs to which a C^*-algebra has been associated and studied in detail. Although no analogue of a Drinen-Tomforde desingularisation is currently available for higher-rank graphs, Farthing provided a construction in [**7**] analogous to that in [**2**] for removing the sources in a locally convex, row-finite higher-rank graph. The statement of the results of [**7**] do not contain the local convexity hypothesis, but Farthing alerted us to an issue in the proof of [**7**, Theorem 2.28] (see Remark 3.4.2), which arises when the graph is not locally convex.

Before we state the goals and results of this thesis in detail, we will review the definitions and properties of directed graphs and their higher-rank analogues.

Directed graphs and their C^*-algebras. A directed graph E consists of countable sets E^0 and E^1, and maps $r, s : E^1 \to E^0$. We think of elements of E^0 as vertices, and the elements of E^1 as edges. We call r, s *range* and *source* maps, and think of them as assigning a direction to each edge. We say E is *row-finite* if $|r^{-1}(v)| < \infty$ for all $v \in E^0$. For a row-finite graph E, a *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that

(CK1) $s_e^* s_e = p_{s(e)}$ for every $e \in E^1$; and
(CK2) $p_v = \sum_{\{e \in E^1 : r(e) = v\}} s_e s_e^*$ whenever $r^{-1}(v) \neq \emptyset$.

The *graph algebra, $C^*(E)$* is the universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s_e, p_v : v \in E^0, e \in E^1\}$. That is, if $\{t_e, q_v : v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, then there exists a $*$-homomorphism

$\pi_{t,q} : C^*(E) \to B$ such that $\pi_{t,q}(s_e) = t_e$ for every $e \in E^1$ and $\pi_{t,q}(p_v) = q_v$ for every $v \in E^0$.

In the groupoid-based approach to analysing graph C^*-algebras (for example, in [**14, 13, 16**]), the groupoid used is a locally compact Hausdorff groupoid $\mathcal{G}$ whose unit space contains as a dense subset the collection of infinite paths in E. The groupoid $\mathcal{G}$ is also r-discrete, meaning that the set of units $\mathcal{G}^0 = \{xx^{-1} : x \in \mathcal{G}\}$ is an open subset of $\mathcal{G}$.

In [**14**], the authors considered locally finite graphs E to ensure that $\mathcal{G}$ is a locally compact r-discrete groupoid. Since $\mathcal{G}$ as a set is built from the infinite-path space E^∞ of E, and we require that $\mathcal{G}^0$ is homeomorphic to E^∞, it is crucial that the infinite-path space of E is endowed with a locally compact Hausdorff topology. In [**16**], Paterson lifted the row-finiteness condition by first building an inverse semigroup S from E, then using the universal groupoid associated to S to build the graph C^*-algebra.

In [**2**], Bates et al. restricted attention to row-finite directed graphs E which may have sources. The construction in [**2**] adds to each source $v \in E^0$ an infinite path called a 'head', by which we mean a graph of the form

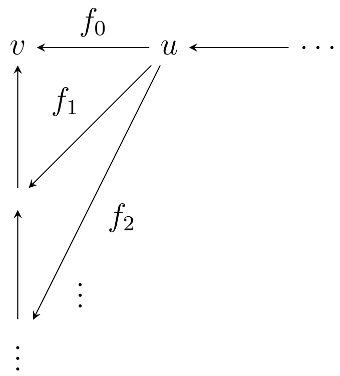
$$v \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots$$

Adding a head to each source in a row-finite directed graph E produces a row-finite graph $\widehat{E}$ with no sources. The authors of [**2**] showed that $C^*(E)$ embeds in $C^*(\widehat{E})$ as a full corner. They used this embedding to deduce theorems about $C^*(E)$ from existing theorems about $C^*(\widehat{E})$.

Drinen and Tomforde took this a step further in [**5**] by adding more complicated heads to infinite receivers. Under Drinen and Tomforde's procedure, an infinite receiver v such as

$$\begin{tikzcd} v & u \arrow[l, shift right, "f_i"'] \arrow[l, shift left] \arrow[l, phantom, "\vdots"] & \cdots \arrow[l] \end{tikzcd}$$

has a head added to it, and the infinite family of edges with range v is distributed down the head:



As in [**2**], the C^*-algebra of the modified graph contains the C^*-algebra of the original graph as a full corner. This allows many results to be extended from C^*-algebras of row-finite graphs with no-sources to arbitrary graph C^*-algebras. For example, Drinen and Tomforde proved that $C^*(E)$ is simple if and only if every cycle in E has an entry, E is cofinal, and every vertex in E can be reached from every infinite receiver [**5**, Corollary 2.15]. They also recovered the characterisation of purely infinite graph algebras of [**9**, Theorem 4]. They extended the characterisation of which graph algebras are AF [**5**, Corollary 2.13], and showed that a simple graph algebra is always either purely infinite or AF [**5**, Remark 2.16], extending the dichotomy of [**13**].

Motivated by Drinen and Tomforde's desingularisation, Raeburn showed in [**19**, §5] how to identify paths in a row-finite directed graph with no sources which can be 'collapsed' to a single vertex (see Section 2.2 for a precise definition). He then defined a desingularisation of an arbitrary graph E to be any pair (F, M), where F is a row-finite graph with no sources and M is a set of collapsible paths such that when all the paths in M are collapsed, the resulting graph F_M is isomorphic to E. In this thesis, we follow Raeburn's approach to desingularisations of graphs.

Higher-rank graphs and their C^*-algebras. In [**12**], Kumjian and Pask developed an analogue of directed graphs called higher-rank graphs as tool to study the higher-rank Cuntz-Krieger algebras of Robertson and Steger [**28**]. Given $k \in \mathbb{N}$, a *graph of rank* k (or k-graph) is a pair (Λ, d) consisting of a category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, which satisfies the *factorisation property*: for every $\lambda \in \mathrm{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. The degree map d is the higher-rank analogue of length. Although k-graphs are defined in terms of categories, no serious category theory is required to work with k-graphs. By the usual abuse of notation, we write $\lambda \in \Lambda$ to mean

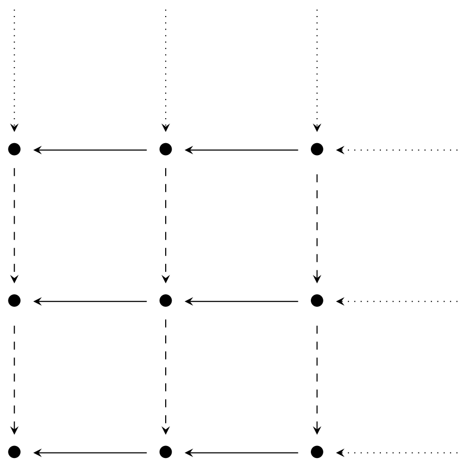


FIGURE 1. The 2-graph Ω_2.

$\lambda \in \operatorname{Mor}(\Lambda)$. We call elements of $\operatorname{Mor}(\Lambda)$ *paths* and elements of $\operatorname{Obj}(\Lambda)$ *vertices*. We identify Λ^0 with $\operatorname{Obj}(\Lambda)$ (for a justification see Remark 3.0.7).

To visualise a k-graph we draw its 1-*skeleton*: a directed graph with vertices Λ^0, and edges $\bigcup_{i=1}^k \Lambda^{e_i}$. To each edge we assign a colour determined by the edge's degree. In this thesis we tend to use 2-graphs for examples, and we draw edges of degree $(1,0)$ as solid lines, and edges of degree $(0,1)$ as dashed lines. In the literature these are often referred to as "blue" and "red" edges.

A particularly important class of examples of k-graphs are the k-graphs $\Omega_{k,m}$ defined as follows. Fix $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$. Let $\operatorname{Obj}(\Omega_{k,m}) = \{p \in \mathbb{N}^k : p_i \leq m_i \text{ for all } i \leq k\}$,

$$\operatorname{Mor}(\Omega_{k,m}) = \{(p,q) : p,q \in \operatorname{Obj}(\Omega_{k,m}) \text{ and } p_i \leq q_i \text{ for all } i \leq k\},$$

$r(p,q) = p$, $s(p,q) = q$ and $d(p,q) = q - p$, with composition given by $(p,q)(q,t) = (p,t)$. If $m = (\infty)^k$, we drop m from the subscript and just write Ω_k. The 1-skeleton of Ω_2 is depicted in Figure 1. The k-graphs $\Omega_{k,m}$ provide an intuitive model for paths in k-graphs: every path λ of degree m in a k-graph Λ determines a degree-preserving functor (i.e. a graph morphism) $x_\lambda : \Omega_{k,m} \to \Lambda$ by $x_\lambda(p,q) = \lambda''$, where $\lambda = \lambda'\lambda''\lambda'''$ and $d(\lambda') = p$, $d(\lambda'') = q - p$ and $d(\lambda''') = m - q$. A path in Λ is often identified with the associated graph morphism. In keeping with this model, we write $\lambda(p,q)$ to refer to the segment λ'' of λ of degree $q - p$ as factorized above. For example if λ is the path $anmh$ of degree $(2,2)$ in Figure 2, then $\lambda((0,1),(2,1))$ is the path eg of degree $(2,0)$. Infinite paths in a k-graph are defined to be k-graph morphisms $x : \Omega_k \to \Lambda$.

There are two major technical issues that arise in generalising results about directed graphs to higher-rank graphs. The first is that two paths $\mu, \nu \in \Lambda$ can be

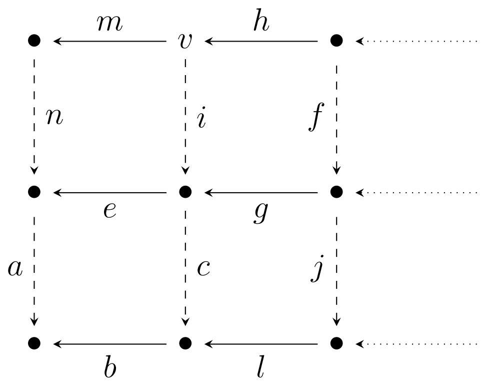


FIGURE 2. The 2-graph Λ.

subpaths of another larger path $\lambda \in \Lambda$ (so $\lambda = \mu\mu' = \nu\nu'$) with $d(\mu) \not\leq d(\nu)$ and $d(\nu) \not\leq d(\mu)$. For example, consider the 2-graph Λ in Figure 2. Here, the left hand rectangular path anm and the rectangular path blj that makes up the bottom part of the graph are both subpaths of the path $anmh = bljf$ of degree $(2, 2)$. This cannot happen in a directed graph: if two paths μ, ν are both initial segments of some longer path, then either μ is an initial segment of ν or vice versa.

The other technical issue is also an implication of the factorisation property. If a directed graph has a source, we can simply add an infinite path onto that source as in [**2**]. In k-graphs, there are many types of sources because a vertex may receive edges of some degrees but not of others. The 2-graph Λ whose skeleton appears in Figure 2 is infinite in the horizontal direction, but not in the vertical direction. So the vertex v is considered a source (as would any other vertex along that top row). Once we add an infinite path to v in the vertical direction, we must add more edges to ensure the factorisation property is satisfied. Thus the process of removing sources in a k-graph is significantly more complicated than in a directed graph. Farthing's construction [**7**] applied to the example of Figure 3.6.5 would extend the graph vertically, yielding a 2-graph isomorphic to the 2-graph Ω_2 depicted in Figure 1.

1.1. Overview of the Thesis

The overall goal of this thesis is to understand the path spaces of directed graphs and higher-rank graphs and investigate how these path spaces interact with desingularisation procedures such as those of Drinen-Tomforde and Farthing.

We begin in Chapter 2 by recalling the standard definitions and notation for directed graphs.

In Section 2.1 we construct a topology on the path space of an arbitrary directed graph E, and show that it is a locally compact Hausdorff topology. Although such results can already be found in the literature, detailed arguments are not usually provided. Our construction follows the approach of Paterson and Welch [**17**], and we fix a minor oversight in their work.

In Section 2.2, we introduce the notion of a desingularisation of a directed graph, and we construct the homeomorphism ϕ_∞, which identifies a subset of the infinite-path space of a desingularisation with the boundary-path space in the original graph.

Our results about desingularisations of directed graphs provide the foundation for our C^*-algebraic results. In Section 2.3, we recall some definitions and results pertaining to the C^*-algebras of directed graphs. First we show how the Cuntz-Krieger relations can be written in terms of paths instead of edges. We then recall that the C^*-algebra of a graph and that of its desingularisation are Morita equivalent. Lastly, we define the diagonal C^*-subalgebra of a graph C^*-algebra.

Section 2.4 contains the main results for directed graphs. First we build the homeomorphism h_E between the boundary-path space ∂E of an arbitrary graph E and the spectrum of its diagonal. We then show that for a desingularisation F of E, the isomorphism which embeds $C^*(E)$ as a full corner in $C^*(F)$ implements the homeomorphism ϕ_∞ constructed in Section 2.2 via the homeomorphisms h_E and h_F.

In Chapter 3, we turn our attention to k-graphs. We begin by recalling the definitions and standard notation for higher-rank graphs.

In Section 3.1 we build a topology for the path space of a higher-rank graph, and show that the path space is locally compact and Hausdorff under this topology. As in the directed graph setting, we follow the approach of [**17**].

Proving one of our main results (Theorem 3.3.1) posed problems using Farthings construction, motivating us to develop an improved version. In Section 3.2, given a k-graph Λ with sources, we construct a k-graph $\widetilde{\Lambda}$ with no sources such that Λ embeds in $\widetilde{\Lambda}$, and we describe some examples of the process. We prove that for row-finite k-graphs, our construction agrees with Farthing's [**7**], and that for 1-graphs, it coincides with the 'adding a head' construction of [**2**]. We also describe how our construction relates to the sets of paths appearing in the Cuntz-Krieger relations, and deduce that it preserves finite alignedness and row-finiteness of Λ.

In Section 3.3, we prove that given a row-finite k-graph Λ, there is a natural homeomorphism from the boundary-path space of Λ onto the space of infinite paths in $\widetilde{\Lambda}$ with range in the embedded copy of Λ. We provide examples and discussion showing that the topological basis constructed in Section 3.1 is the one we want.

In Section 3.4 we recall the definition of the Cuntz-Krieger algebra $C^*(\Lambda)$ of a higher-rank graph Λ. We show that if Λ is a row-finite k-graph and $\widetilde{\Lambda}$ is the graph with no sources obtained by applying the construction of Section 3.2 to Λ, then the embedding of Λ in $\widetilde{\Lambda}$ induces an isomorphism π of $C^*(\Lambda)$ onto a full corner of $C^*(\widetilde{\Lambda})$.

Section 3.5 contains results about the diagonal of a k-graph algebra which are analogous to those proved in Section 2.4 for 1-graphs. We identify the boundary-path space of a finitely aligned higher-rank graph with the spectrum of its diagonal C^*-algebra. We then show that the isomorphism π of Section 3.4 restricts to an isomorphism of diagonals which implements the homeomorphism of Section 3.3.

In Section 3.6 we investigate how the construction of a k-graph Λ from its k-coloured skeleton E relates to the topologies on their path spaces. In particular, we show that if E is row-finite, then the quotient topology on Λ inherited from its skeleton is precisely the topology described in Section 3.1. We provide an example to show that this doesn't necessarily happen when E is not row-finite.

CHAPTER 2

Directed Graphs

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each edge e, we call $s(e)$ the *source* of e and $r(e)$ the *range* of e; if $s(e) = v$ and $r(e) = w$, we say that v *emits* e and that w *receives* e, or that e is an edge from v to w. Since all graphs in this thesis are directed, we often just call a directed graph E a graph.

A *path of length* n in a directed graph E is a sequence $\mu = \mu_1 \cdots \mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n-1$.

$$\bullet \xleftarrow[\mu_1]{} \bullet \xleftarrow[\mu_2]{} \bullet \quad \cdots \quad \bullet \xleftarrow[\mu_n]{} \bullet$$

This convention, where the edges in a directed graph read from right to left, is a recent one adopted for reasons which become clear when we talk about graph C^*-algebras in §2.3. We write $|\mu| = n$ for the length of μ, and regard vertices as paths of length 0; we denote by E^n the set of paths of length n, and define $E^* := \bigcup_{n\in\mathbb{N}} E^n$. We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$. If μ and ν are paths with $s(\mu) = r(\nu)$, we write $\mu\nu$ for the path $\mu_1 \ldots \mu_{|\mu|}\nu_1 \cdots \nu_{|\nu|}$. For a set of vertices $V \subset E^0$ and a set of paths $F \subset E^*$, we define $VF := \{\mu \in F : r(\mu) \in V\}$ and $FV := \{\mu \in F : s(\mu) \in V\}$. If $V = \{v\}$, then we drop the braces and write vF to mean $\{v\}F$ and Fv to mean $F\{v\}$. We define the infinite paths E^∞ of E to be infinite strings $\mu_1 \ldots \mu_n \ldots$ such that $s(\mu_i) = r(\mu_{i+1})$ for all $i \geq 1$, we extend the range map to E^∞ by setting $r(\mu) = r(\mu_1)$, and for a set of vertices $V \subset E^0$, we define $VE^\infty := \{x \in E^\infty : r(x) \in V\}$.

If $r^{-1}(v)$ is finite for every $v \in E^0$, that is, every vertex in a graph E receives at most finitely many edges, we say that E is *row-finite*.

A vertex v is *singular* if either $|r^{-1}(v)| = \infty$, or $|r^{-1}(v)| = 0$. The *boundary paths* of E are defined by $\partial E := E^\infty \cup \{\alpha \in E^* : s(\alpha) \text{ is singular}\}$.

2.1. Topology

Our first aim is to construct a locally compact Hausdorff topology on $E^* \cup E^\infty$. For $\mu \in E^*$, we define the *cylinder set of* μ by

$$\mathcal{Z}(\mu) := \{\nu \in E^* \cup E^\infty : \nu = \mu\nu'\}.$$

Using Paterson and Welch's approach in [**17**], we identify elements of E^* with functions on E^*, an then use the topology of pointwise convergence. The open sets are defined to be the inverse images of open sets in $\{0,1\}^{E^*} = \prod_{\mu \in E^*}\{0,1\}$ (equipped with the product topology) under the map $\alpha : E^* \cup E^\infty \to \{0,1\}^{E^*}$ defined by

$$\alpha(w)(y) = \begin{cases} 1 & \text{if } w \in \mathcal{Z}(y), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.1}$$

The basis we end up with is slightly different to that in [**17**, Corollary 2.4], revealing a minor oversight of the authors.

Before stating our goal for this section, we recall the following definition. For a set X, a family of topological spaces $\{Y_i : i \in I\}$ and a family of functions $f_i : X \to Y_i$, there is a weakest topology on X that makes all of the f_i continuous (see [**18**, 1.4.5]). We call it the *initial topology* induced by the family $\{f_i : i \in I\}$. We give $\{0,1\}^{E^*}$ the topology of pointwise convergence, and then the topology we want on $E^* \cup E^\infty$ is the initial topology induced by $\{\alpha\}$.

PROPOSITION 2.1.1. *Let E be a directed graph. For $\mu \in E^*$ and a finite subset $G \subset s(\mu)E^1$, define $\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{e\in G} \mathcal{Z}(\mu e)$. Then the collection*

$$\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)E^1 \textit{ is finite}\}$$

is a basis for the initial topology induced by $\{\alpha\}$. Moreover, it is a locally compact Hausdorff topology on $E^* \cup E^\infty$.

The motivation for Proposition 2.1.1 stemmed from the fact that no detailed construction of such a topology appears to have been published, even in the row-finite setting[1]. It is considered a folklore result, but a detailed proof is given on page 14. We begin by describing the topology on F^∞ when F is row-finite — in this situation, the basis for the topology is a little simpler.

PROPOSITION 2.1.2. *Let F be a row-finite graph. Then $\{\mathcal{Z}(\mu) \cap F^\infty : \mu \in F^*\}$ is a basis for the subspace topology on F^∞ inherited from $\prod_{\mathbb{N}} F^1$. Moreover, endowed with this topology, F^∞ is a locally compact Hausdorff space.*

[1]At least, I couldn't find one. The result is stated as [**14**, Corollary 2.2] without proof.

Proof. For a finite sequence $G = (g_1, g_2, \dots, g_N)$ of elements of F^1, define

$$\mathcal{Z}(G) := \Big\{(f_m)_{m=1}^{\infty} \in \prod_{\mathbb{N}} F^1 : f_n = g_n \text{ for } 1 \leq n \leq N\Big\}.$$

Since F^1 carries the discrete topology, the family

$$\{\mathcal{Z}(G) : G \text{ is a finite sequence in } F^1\}$$

is a basis for the product topology on $\prod_{\mathbb{N}} F^1$. Since $\mathcal{Z}(G) \cap F^\infty \neq \emptyset$ if and only if $g_1 \cdots g_N \in F^*$, the sets $\{\mathcal{Z}(\mu) \cap F^\infty : \mu \in F^*\}$ form a basis for the subspace topology on F^∞. We plan to show these sets are compact. To do so, we use the following result.

Claim 2.1.2.1. *For each $n \in \mathbb{N}$, let $F_n \subset F^1$ be finite. Then the product topology on $\prod_{n\in\mathbb{N}} F_n$ agrees with the relative topology on $\prod_{n\in\mathbb{N}} F_n$ inherited from $\prod_{\mathbb{N}} F^1$.*

Proof. Denote by X the set $\prod_{n\in\mathbb{N}} F_n$. Let τ_1 be the product topology on X, let τ_2 be the relative topology on X inherited from $\prod_{\mathbb{N}} F^1$, and let ϕ be the identity map on X. We aim to show that $\phi : (X, \tau_1) \to (X, \tau_2)$ is a homeomorphism. Since the F_n are finite, Tychonoff's theorem implies that τ_1 is a compact topology. Since F^1 is a Hausdorff space, and since products and subspaces of Hausdorff spaces are also Hausdorff, τ_2 is a Hausdorff topology. So ϕ is a bijection from a compact space onto a Hausdorff space, and hence it suffices to show that ϕ is continuous.

To see that ϕ is continuous, let $V = \mathcal{Z}(G)$ be a basic open set in $\prod_{\mathbb{N}} F^1$. If $V \cap X = \emptyset$, then $\phi^{-1}(V \cap X) = \emptyset$ is open in (X, τ_1). Suppose that $V \cap X \neq \emptyset$. Then

$$\phi^{-1}(V \cap X) = \{(f_i)_{i=1}^{\infty} \in X : f_i = g_i \text{ for } i \leq N\}$$

is a basic open set in (X, τ_1). $\square_{\text{Claim}}$

To see that F^∞ is locally compact we show that the basic open sets $\mathcal{Z}(\mu) \cap F^\infty$ are compact. First, we construct a set X_μ for each μ and show that it is compact in $\prod_{\mathbb{N}} F^1$. We then show that $\mathcal{Z}(\mu) \cap F^\infty$ is closed in X_μ. Fix $\mu \in F^*$, and for each $n \in \mathbb{N}$ define

$$F_n := \begin{cases} \{\mu_n\} & \text{for } 1 \leq n \leq |\mu| \\ \{e \in F^1 : s(\mu)F^{n-|\mu|-1}r(e) \neq \emptyset\} & \text{for } n > |\mu|. \end{cases}$$

Row-finiteness of F implies that F_n is finite for each $n \in \mathbb{N}$. Thus $\prod_{n\in\mathbb{N}} F_n$ is compact. By Claim 2.1.2.1, $X_\mu := \prod_{n\in\mathbb{N}} F_n$ with relative topology inherited from $\prod_{\mathbb{N}} F^1$ is also compact. Since $\mathcal{Z}(\mu) \cap F^\infty \subset X_\mu$, it suffices to show that $\mathcal{Z}(\mu) \cap F^\infty$ is closed. Since F^∞ satisfies the first axiom of countability (i.e. every neighborhood filter has a countable basis), it suffices to work with sequences. Let $(\lambda^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{Z}(\mu) \cap F^\infty$ converging to $\lambda \in X_\mu$; meaning that $\lambda^n_i \to \lambda_i$ for all $i \in \mathbb{N}$.

We show that $\lambda \in \mathcal{Z}(\mu) \cap F^\infty$. For each $j \in \mathbb{N}$, we have $\lambda_j^n \to \lambda_j$, so there exists M_j such that $n \geq M_j \implies \lambda_j^n = \lambda_j$. Fix $j \in \mathbb{N}$. Let $P_j = \max\{M_j, M_{j+1}\}$. Then $n \geq P_j \implies \lambda_j^n = \lambda_j$ and $\lambda_{j+1}^n = \lambda_{j+1}$. This implies that $s(\lambda_j) = s(\lambda_j^n) = r(\lambda_{j+1}^n) = r(\lambda_{j+1})$. Since this is true for all $j \in \mathbb{N}$, λ is a path in F and thus an element of $\mathcal{Z}(\mu) \cap F^\infty$. Now $\{\mathcal{Z}(\mu) \cap F^\infty : \mu \in F^*\}$ is a compact basis for F^∞, and thus F^∞ is locally compact. □

To prove Proposition 2.1.1, we use the following result.

PROPOSITION 2.1.3. *The map* $\alpha : E^* \cup E^\infty \to \{0,1\}^{E^*}$ *defined in* (2.1.1) *is continuous, open and injective.*

PROOF. That α is continuous is clear: the topology on $E^* \cup E^\infty$ is given by the inverse images $\alpha^{-1}(U)$ of open sets $U \subset \{0,1\}^{E^*}$. Since $\alpha(\alpha^{-1}(U)) = U \cap \alpha(E^* \cup E^\infty)$ is an open set, α is open.

To see that α is injective, suppose $\alpha(\mu) = \alpha(\nu)$. Then $\alpha(\mu)(\nu) = \alpha(\nu)(\nu) = 1$, and thus $\mu \in \mathcal{Z}(\nu)$. Similarly, $\nu \in \mathcal{Z}(\mu)$. Hence $\mu = \nu$. □

PROOF OF PROPOSITION 2.1.1. First we consider the topology on $\{0,1\}^{E^*}$. Given disjoint finite subsets $F, G \subset E^*$, define

$$U_\mu^{F,G} = \begin{cases} \{1\} & \text{if } \mu \in F, \\ \{0\} & \text{if } \mu \in G, \\ \{0,1\} & \text{otherwise.} \end{cases}$$

Then the sets $N(F,G) := \prod_{\mu \in E^*} U_\mu^{F,G}$, where F, G range over all finite, disjoint pairs of subsets of E^*, form a basis for the topology on $\{0,1\}^{E^*}$. Proposition 2.1.3 says that α is a homeomorphism onto its range, hence the sets $\alpha^{-1}(N(F,G))$ form a basis for a topology on $E^* \cup E^\infty$. These sets can be described as follows.

$$\begin{aligned} \lambda \in \alpha^{-1}(N(F,G)) &\iff \alpha(\lambda) \in N(F,G) \\ &\iff \alpha(\lambda)(\mu) = \begin{cases} 1 & \text{for } \mu \in F \\ 0 & \text{for } \mu \in G \end{cases} \\ &\iff \begin{cases} \lambda & \in \mathcal{Z}(\mu) \text{ for } \mu \in F, \\ \lambda & \notin \mathcal{Z}(\nu) \text{ for } \nu \in G. \end{cases} \\ &\iff \lambda \in \left(\bigcap_{\mu \in F} \mathcal{Z}(\mu) \right) \setminus \left(\bigcup_{\nu \in G} \mathcal{Z}(\nu) \right). \end{aligned}$$

We simplify these sets further. Fix finite $F, G \subset E^*$. If $\alpha^{-1}(N(F,G))$ is non empty, then $\bigcap_{\mu \in F} \mathcal{Z}(\mu) \neq \emptyset$. This implies that for $\mu, \nu \in F$, we have:

- $\mu \in \mathcal{Z}(\nu)$ if $|\mu| \geq |\nu|$, or
- $\nu \in \mathcal{Z}(\mu)$ if $|\nu| > |\mu|$.

Choosing μ such that $|\mu| = \max\{|\nu| : \nu \in F\}$, we have $\bigcap_{\nu \in F} \mathcal{Z}(\nu) = \mathcal{Z}(\mu)$. Let $G' = G \cap \mathcal{Z}(\mu)$. Then

$$\mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\nu) = \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G'} \mathcal{Z}(\nu).$$

Now let $G'' = \{\nu : \mu\nu \in G\}$. Then

$$\alpha^{-1}(N(F,G)) = \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G''} \mathcal{Z}(\mu\nu).$$

For $\mu \in E^*$ and a finite subset $G \subset s(\mu)E^*$, we define $\mathcal{Z}(\mu \setminus G) = \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu)$. By the above, each $\alpha^{-1}(N(F,G))$ has the form $\mathcal{Z}(\mu \setminus G)$ for some $\mu \in E^*$ and finite $G \subset s(\mu)E^*$.

Claim 2.1.1.1. *$\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)E^1$ is finite$\}$ and $\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)E^*$ is finite$\}$ are bases for the same topology.*

Proof. Fix $\mu \in E^*$, and a finite subset $G \subset s(\mu)E^*$. Let $\lambda \in \mathcal{Z}(\mu \setminus G)$. We seek $\alpha \in E^*$ and a finite set $F \subset s(\alpha)E^1$ such that

$$\lambda \in \mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G).$$

We consider two cases: λ is finite or λ is infinite. First suppose that $\lambda \in E^\infty$. Set $N = \max\{|\mu\nu| : \nu \in G\}$, $\alpha = \lambda_1 \cdots \lambda_N$, and $F = \emptyset$. Then $\mathcal{Z}(\alpha \setminus F) = \mathcal{Z}(\alpha)$ clearly contains λ. Since $|\alpha| \geq |\mu\nu|$ for all $\nu \in G$, we have $\mathcal{Z}(\alpha) \subset \mathcal{Z}(\mu \setminus G)$ as required.

Now suppose that $\lambda \in E^*$. Set $\alpha = \lambda$ and

$$F = \{(\mu\nu)_{|\lambda|+1} : \nu \in G \text{ satisfies } |\mu\nu| > |\lambda|\}.$$

Then $\mathcal{Z}(\alpha \setminus F) = \mathcal{Z}(\lambda \setminus F)$ clearly contains λ. To see that $\mathcal{Z}(\lambda \setminus F) \subset \mathcal{Z}(\mu \setminus G)$, fix $\beta \in \mathcal{Z}(\lambda \setminus F)$. Factor $\lambda = \mu\lambda'$, then we have $\beta = \lambda\beta' = \mu\lambda'\beta' \in \mathcal{Z}(\mu)$. We now show that $\lambda'\beta' \notin \bigcup_{\nu \in G} \mathcal{Z}(\nu)$. Fix $\nu \in G$. If $|\mu\nu| \leq |\lambda|$, then $|\nu| \leq |\lambda'|$. Since $\lambda' \notin \bigcup_{\nu \in G} \mathcal{Z}(\nu)$, we have $\lambda'\beta' \notin \bigcup_{\nu \in G} \mathcal{Z}(\nu)$. If $|\mu\nu| > |\lambda|$, then since $\beta'_1 \notin F$, we have $(\mu\lambda'\beta')_{|\lambda|+1} = \beta'_1 \neq (\mu\nu)_{|\lambda|+1}$. So $(\lambda'\beta')_{|\lambda|-|\mu|+1} \neq \nu_{|\lambda|-|\mu|+1}$. $\square_{\text{Claim}}$

So the collection

$$\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)^1 \text{ is finite}\}$$

form a basis for our topology on $E^* \cup E^\infty$.

To see $E^* \cup E^\infty$ is a locally compact Hausdorff space, we follow the strategy of [**17**] to show that $\mathcal{Z}(v)$ is compact for each $v \in E^0$. Proposition 2.1.3 implies that α is a homeomorphism onto its range, so it suffices to prove that $\alpha(\mathcal{Z}(v))$ is compact. Since $\{0,1\}^{E^*}$ is compact, we need only show that $\alpha(\mathcal{Z}(v))$ is closed. Let $\{\omega^{(n)} \in \mathcal{Z}(v) : n \in \mathbb{N}\}$ be such that $\alpha(\omega^{(n)}) \to f \in \{0,1\}^{E^*}$. We seek $\omega \in \mathcal{Z}(v)$ such that $f = \alpha(\omega)$. Let $A := \{\mu \in E^* : \alpha(\omega^{(n)})(\mu) \to 1\}$. Then for each $\mu, \nu \in A$ there exist N_μ, N_ν such that $n \geq \max\{N_\mu, N_\nu\}$ implies that $w^{(n)} \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$. In particular, $\mathcal{Z}(\mu) \cap \mathcal{Z}(\nu) \neq \emptyset$, and hence either $\nu = \mu\mu'$ or $\mu = \nu\nu'$; denote the longer path by $\beta_{\mu,\nu}$. Then $n \geq \max\{N_\mu, N_\nu\}$ implies that $\omega^{(n)} \in \mathcal{Z}(\beta_{\mu,\nu})$, so $\beta_{\mu,\nu} \in A$. Since A is countable, we can list it:

$$A = \{\nu^1, \nu^2, \ldots, \nu^m, \ldots\}.$$

Let $y^1 := \nu^1$, and iteratively define $y^n := \beta_{y^{n-1},\nu^n}$. Then $\{y^n : n \in \mathbb{N}\}$ satisfy $y_1^n y_2^n \cdots y_{|y^{n-1}|}^n = y^{n-1}$, and hence determines a unique path $\omega \in E^* \cup E^\infty$. We claim that $\nu \in A$ if and only if $\omega \in \mathcal{Z}(\nu)$. Firstly, for each $\nu^m \in A$, we have $y^m \in \mathcal{Z}(\nu^m)$. Then $\omega \in \mathcal{Z}(y^m) \subset \mathcal{Z}(\nu^m)$. Conversely, let $\omega \in \mathcal{Z}(\nu^m)$. Then $y^m \in \mathcal{Z}(\nu^m) \cap A$ implies that for large enough n we have $\omega^{(n)} \in \mathcal{Z}(y^m) \subset \mathcal{Z}(\nu^m)$, so $\nu^m \in A$.

We claim that $\alpha(\omega^{(n)}) \to \alpha(\omega)$. Fix $\nu \in E^*$. We will show that $\alpha(\omega^{(n)})(\nu) \to \alpha(\omega)(\nu)$. If $\alpha(\omega)(\nu) = 1$, then $\omega \in \mathcal{Z}(\nu)$. So $\nu \in A$, and hence $\omega^{(n)}(\nu) \to 1$. Now suppose $\alpha(\omega)(\nu) = 0$. So $\omega \notin \mathcal{Z}(\nu)$, and thus $\nu \notin A$. Since $\alpha(\omega^{(n)}) \to f \in \{0,1\}^{E^*}$, and $\alpha(\omega^{(n)})(\nu) \nrightarrow 1$, we must have $\alpha(\omega^{(n)})(\nu) \to 0$. So $\alpha(\omega^{(n)}) \to \alpha(\omega)$. Hence $\alpha(\mathcal{Z}(v))$ is closed, and thus compact. $\square$

2.2. Desingularisation

In this section we discuss Drinen and Tomforde's construction from [**5**] which modifies an arbitrary directed graph E to obtain a row-finite graph F in such a way that $C^*(F)$ contains $C^*(E)$ as a full corner. Originally, analysis of graph algebras was performed only for graphs with no sources and with at most finitely many edges attached to each vertex. Bates et al. in [**2**] overcame the no sources restriction by adding a 'head' onto each source in a graph E to form a new graph F with no sources, and showing that $C^*(F)$ contains $C^*(E)$ as a full corner. Recall that the graph C^*-algebra of a graph E is the universal C^*-algebra generated by partial isometries associated to paths in the graph subject to a set of Cuntz-Krieger relations (see page 3). For row-finite graphs, one of these relations says that for each $v \in E^0$ such that $vE^1 \neq \emptyset$, the associated partial isometry p_v is a projection equal to the sum $\sum_{\mu \in vE^1} s_e s_e^*$ of range projections associated to the edges incident on v. This poses immediate problems once you allow graphs to have *infinite receivers*: vertices v such that $|vE^1| = \infty$. It turns out that the right thing to do is to specify that the range projections $s_e s_e^*$ are all mutually orthogonal and satisfy $s_e s_e^* \leq p_{r(e)}$, and to insist that the equality $p_v = \sum_{\mu \in vE^1} s_e s_e^*$ holds only for vertices v such that $0 < |vE^1| < \infty$. That these modifications to the Cuntz-Krieger relations are the right thing to do was discovered independently using several different approaches to graph algebras, for example [**9, 16, 25, 29**].

Drinen and Tomforde's construction [**5**] starts with an arbitrary graph E and adds on a head wherever there is a singular vertex. When the singular vertex is an infinite receiver, the incoming edges are distributed along the head. The resulting graph F is now known as a Drinen-Tomforde desingularisation of E. Notice that at an infinite receiver, there is a choice in the way which edges are distributed along the appended head, and hence a Drinen-Tomforde desingularisation of E is not unique. Motivated by [**5**], Raeburn developed a 'collapsing' construction in [**19**, §5]. He defined a desingularisation by identifying paths in a row-finite graph F with no sources which we call *collapsible paths*, then 'collapsed' these paths to yield a graph E such that by applying Drinen and Tomforde's construction (and making the right choices along the way), we can recover F. The discussion [**19**, p44] shows that every graph permits a Drinen-Tomforde desingularisation in the sense of Definition 2.2.7. We are interested in how such constructions affect the path space. The following theorem is the goal for this section.

Theorem 2.2.1. *Let E be a directed graph and F be a Drinen-Tomforde desingularisation of E. Then E^0F^∞ is homeomorphic to ∂E.*

To prove Theorem 2.2.1, we define a map ϕ (Equation (2.2.1)) on finite paths in F with range and source in E. Using Lemma 2.2.9, we then use ϕ to define a map $\phi_\infty : E^0F^\infty \to \partial E$ as in (2.2.2), which we prove is a homeomorphism.

Let $\mu \in F^\infty$ and $e \in E^1$. We say that e *exits* μ if there exists $i \geq 1$ such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$; note that edges with source $r(\mu)$ are not considered exits of μ. We say that e *enters* μ if there exists $i \geq 1$ such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$.

Definition 2.2.2. Let F be a directed graph. We say that an infinite path $\mu \in F^\infty$ is *collapsible* if

(C1) μ has no exits,
(C2) $r^{-1}(r(\mu_i))$ is finite for every i,
(C3) $r^{-1}(r(\mu)) = \{\mu_1\}$,
(C4) $\mu_i \neq \mu_j$ for all $i \neq j$, and
(C5) μ has either zero or infinitely many entries.

Examples 2.2.3. In [**19**, p42] only (C1)–(C3) are present. Condition (C4) was added after we realized that a cycle with no entrance could be collapsible under the original definition, and (C5) was added to ensure that we only collapse paths (a process described in Remark 2.2.4) which yield singular vertices - thus avoiding

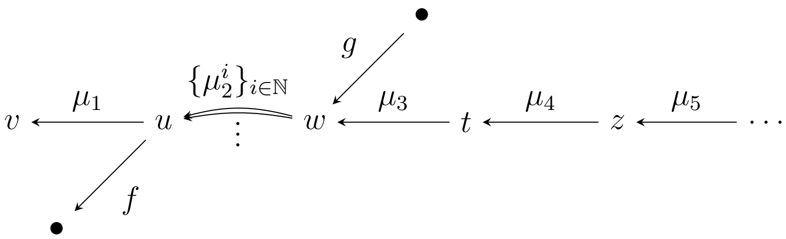


FIGURE 1. Examples of collapsible paths.

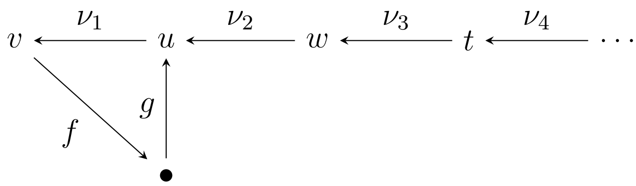


FIGURE 2. Further examples of collapsible paths.

a complication in the proof of [**19**, Proposition 5.2][2], the key result for this theory. These conditions are not all necessary to carry out the process of collapsing, but they ensure the simplest formulae, and also that we collapse as few paths as possible.

For example, consider Figure 1. The only collapsible path is $\mu_4\mu_5\ldots$. The paths $\mu_1\mu_2^i\ldots$ are not collapsible they have an exit at $r(\mu_2^i)$, thus failing (C1). Neither are $\mu_2^i\mu_3\ldots$, as $r^{-1}(r(\mu_2^i))$ is infinite, so they fail condition (C2). The path $\mu_3\mu_4\ldots$ fails (C3), since $r^{-1}(r(\mu_3)) = \{\mu_3, g\}$. In Figure 2, the only collapsible path is $\nu_3\nu_4\ldots$. The path $(\nu_1 g f)^\infty := \nu_1 g f \nu_1 g f \ldots$ is not collapsible as it fails (C4), and $\nu_1\nu_2\ldots$ is not collapsible either as it has exactly one entry, failing (C5).

REMARK 2.2.4. As the name suggests, we will collapse these paths to form a new graph. We then show that the boundary-path space of the new graph is homeomorphic to a subset of that of the original graph. Suppose that μ is a collapsible path in a row-finite graph F. Define $s_\infty(\mu) := \{s(\mu_i) : i \geq 1\}$ and

$$F^*(\mu) := \{\nu \in F^* : |\,\nu| > 1, \nu = \mu_1\mu_2\ \cdots \mu_{|\,\nu-1|}e \text{ for some } e \neq \mu_{|\nu|}\}.$$

Set $F^0_\mu := F^0 \setminus s_\infty(\mu)$ and $F^1_\mu := \big(F^1 \setminus (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\})\big) \cup \{e_\nu : \nu \in F^*(\mu)\}$, and extend the range and source maps to F^1_μ by setting $r(e_\nu) := r(\nu) = r(\mu)$ and $s(e_\nu) := s(\nu)$. Then F_μ is the graph obtained by collapsing the path μ in F. Notice that for $\alpha \in F^*_\mu$, $s(\alpha)$ is singular if and only if $s(\alpha) = r(\mu)$.

REMARK 2.2.5. Given a collection M of collapsible paths such that no two paths in M have any edge or vertex in common, we call the paths in M *disjoint*. We can

[2]The proof of [**19**, Proposition 5.2] contained an error when proving that the Cuntz-Krieger relation holds in F_μ at the vertex resulting from collapsing a path μ in with finitely many entries.

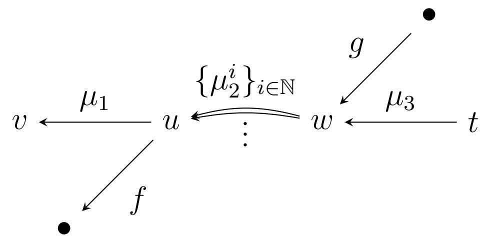


FIGURE 3. The graph obtained by collapsing the path in Figure 1

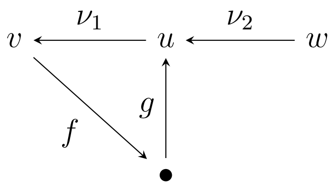


FIGURE 4. The graph obtained by collapsing the path in Figure 2

carry out the process described in Remark 2.2.4 on all the paths in M simultaneously, yielding a graph F_M which may no longer be row-finite.

EXAMPLES 2.2.6. Collapsing the path $\mu_4\mu_5\dots$ in Figure 1 yields the graph in Figure 3. Collapsing the path $\nu_3\nu_4\dots$ in Figure 2 yields the graph in Figure 4.

DEFINITION 2.2.7. Let E be a directed graph. A *Drinen-Tomforde desingularisation* of E is a pair (F, M) consisting of a row-finite graph F with no sources, and a collection M of disjoint collapsible paths such that $F_M \cong E$.

EXAMPLE 2.2.8. Consider the directed graphs E and F in Figures 5 and 6. The collapsible paths in F are $\mu := \mu_1\mu_2\dots$, $\lambda' := \lambda_2\dots$ and $\nu' := \nu_3\dots$. Let $M = \{\mu, \lambda', \nu'\}$. Then M is a set of disjoint collapsible paths. We have

$$F^*(\mu) = \{\mu_1\mu_2 h\} \cup \{\mu_1 \dots \mu_i g^i : i \geq 1\}$$

and $F^*(\lambda') = \emptyset = F^*(\nu')$. We have $F_M^0 = \{v, u, w, t, z\}$. For clarity of notation, we index the elements in $\{e_\nu : \nu \in F^*(\mu)\}$ by the edge closest to the source of the path, so

$$F_M^1 = \{\nu_1, \nu_2, \lambda_1, e_h\} \cup \{e_{g^i} : i \geq 1\}.$$

Then $F_M \cong E$, and thus (F, M) is a Drinen-Tomforde desingularisation of E.

Suppose E is a directed graph, and (F, M) is a Drinen-Tomforde desingularisation of E. Define $F^*(M) := \bigcup_{\mu\in M} F^*(\mu)$. Define $\phi' : (F^1 \cap E^1) \cup F^*(M) \to E^1$ by $\phi'|_{F^1\cap E^1} := \mathrm{id}_{F^1\cap E^1}$ and $\phi'|_{F^*(M)} : \nu \mapsto e_\nu$. So ϕ' acts as the identity on unchanged edges, and takes collapsible paths in F to the associated edges in E.

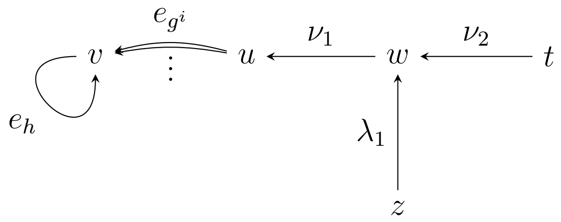


FIGURE 5. A directed graph E.

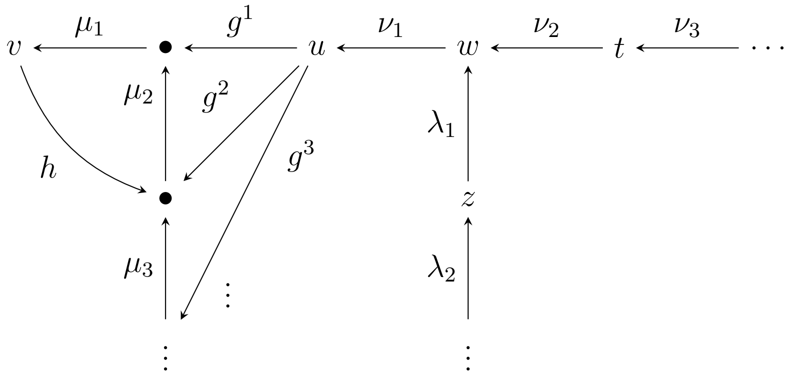


FIGURE 6. A directed graph F.

If $\beta \in F^*$ with $r(\beta), s(\beta) \in E^0$, then β has the form $\beta = b^1 b^2 \cdots b^n$ where each $b^k \in (F^1 \cap E^1) \cup F^*(M)$. Define $E^0 F^* E^0 := \{\beta \in F^* : r(\beta), s(\beta) \in E^0\}$. We extend the map ϕ' above to a map ϕ on finite paths: define $\phi : E^0 F^* E^0 \to E^*$ by

$$\phi(\beta) := \phi(b^1 b^2 \cdots b^n) = \phi'(b^1) \cdots \phi'(b^n). \tag{2.2.1}$$

We will extend this map to $E^0 F^\infty$, and ultimately show that it is a homeomorphism from $E^0 F^\infty$ to ∂E. To do so precisely we use the following results.

LEMMA 2.2.9. *Let E be a directed graph, and (F, M) be a desingularisation of E. If $\lambda \in E^0 F^\infty$, then either*

- *$\lambda = l^1 \cdots l^k \mu$ for some $\mu \in M$ and $l^i \in (F^1 \cap E^1) \cup F^*(M)$, or*
- *$\lambda = l^1 l^2 \cdots l^n \ldots$ where $l^i \in (F^1 \cap E^1) \cup F^*(M)$.*

PROOF. Fix $\lambda \in E^0 F^\infty$. We construct the l^i inductively. Either $\lambda_1 \in F^1 \cap E^1$, or $\lambda_1 = \mu_1$ for some $\mu \in M$. If $\lambda_1 \in F^1 \cap E^1$, then let $l^1 = \lambda_1$. If $\lambda_1 = \mu_1$, there are two cases to consider:

(i) $\lambda_i = \mu_i$ for all $i \in \mathbb{N}$, in which case $\lambda = \mu$; or
(ii) there exists k such that $\lambda_i = \mu_i$ for all $i < k$ and $\lambda_k \neq \mu_k$, in which case we set $l^1 = \mu_1 \cdots \mu_{k-1} \lambda_k$. Since paths in M have no edges in common, we have $l^1 \in F^*(\mu)$.

In case (i). $\lambda = \mu$, in which case we are done. In case (ii), $\lambda = l^1\lambda'$ for some $\lambda' \in F^\infty$. Since $s(l^1)$ is an element of E^0, we have $r(\lambda') \in E^0$ and we can repeat the process, applying it to λ' to get l^2. Iterating will either terminate with $\lambda = l^1 \cdots l^n\mu$ where $\mu \in M$, or continue ad infinitum, in which case $\lambda = l^1 \cdots l^n \cdots$. □

We now define the map $\phi_\infty : E^0F^\infty \to \partial E$:

$$\phi_\infty(\lambda) := \begin{cases} \phi(\lambda') & \text{if } \lambda = \lambda'\mu \text{ for some } \mu \in M, \\ \phi'(\lambda^1)\cdots\phi'(\lambda^n)\cdots & \text{if } \lambda = l^1 \cdots l^n \cdots. \end{cases} \tag{2.2.2}$$

PROPOSITION 2.2.10 (Adapted from [**5**, 2.6]). *Let E be a directed graph, and (F, M) be a desingularisation of E. Then ϕ and ϕ_∞, defined as in* (2.2.1) *and* (2.2.2) *respectively, are bijections and preserve range and source.*

PROOF. Since ϕ' is a bijection, it follows that ϕ is a bijection. Similarly, since ϕ' preserves range and source, so does ϕ. Since the paths in M are disjoint, it follows that ϕ_∞ is injective. To see that ϕ_∞ is surjective, fix $\alpha \in \partial E$. Either $\alpha \in E^\infty$, or $\alpha \in E^*$ and $s(\alpha)$ is singular. Suppose first that $\alpha \in E^\infty$. Since (F, M) is a desingularisation of E, each α_i is either an element of $E^1 \cap F^1$, or of the form e_ν for some $\nu \in F^*(M)$. For each i, define

$$g_i = \begin{cases} \alpha_i & \text{if } \alpha_i \in E^1 \cap F^1, \\ \nu & \text{if } \alpha_i = e_\nu \text{ for some } \nu \in F^*(M). \end{cases}$$

Since $r(e_\nu) = r(\nu)$ and $s(e_\nu) = s(\nu)$ for each $\nu \in F^*(M)$, we have $g^1 \cdots g^n \cdots \in F^\infty$, and hence $\phi_\infty(g^1 \cdots g^n \cdots) = \alpha$.

Now suppose that $\alpha \in E^*$ and $s(\alpha)$ is singular. Then there exists $\mu \in M$ such that $s(\alpha) = r(\mu)$ in F. Since ϕ preserves r and s, we have $\phi^{-1}(\alpha)\mu \in F^\infty$ and $r(\phi^{-1}(\alpha)\mu) = r(\alpha) \in E^0$. So $\phi_\infty(\phi^{-1}(\alpha)\mu) = \phi(\phi^{-1}(\alpha)) = \alpha$. Thus ϕ_∞ is surjective. □

We now have the tools to prove Theorem 2.2.1.

PROOF OF THEOREM 2.2.1. We will show that the map ϕ_∞ defined in (2.2.2) is a homeomorphism. Proposition 2.2.10 says that ϕ_∞ is a bijection, so it suffices to show that ϕ_∞ and ϕ_∞^{-1} are continuous.

To see that ϕ_∞ is continuous, fix $\alpha \in E^*$ and a finite subset $G \subset s(\alpha)E^1$, so $\mathcal{Z}(\alpha \setminus G) \cap \partial E$ is a basic open set in ∂E. We show that $\phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E)$ is open. If $\mathcal{Z}(\alpha \setminus G) \cap \partial E = \emptyset$ then $\phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E) = \emptyset$ is open. So suppose that $\mathcal{Z}(\alpha \setminus G) \cap \partial E \neq \emptyset$, and fix $\lambda \in \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E)$. We seek $\gamma \in F^*$ such that

$$\lambda \in \mathcal{Z}(\gamma) \cap E^0F^\infty \subset \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E).$$

We consider two cases:

(i) λ is either equal to $l^1 l^2 \cdots$, or $\lambda = l^1 \cdots l^k \mu$ with $k > |\alpha|$; or

(ii) $\lambda = l^1 \cdots l^{|\alpha|}\mu$.

where $\mu \in M$, and $l^i \in (F^1 \cap E^1) \cup F^*(M)$ for each i.

In case (i), let $\gamma = l^1 \cdots l^{|\alpha|+1}$. Clearly $\lambda \in \mathcal{Z}(\gamma) \cap E^0F^\infty$. Now suppose that $y \in \mathcal{Z}(\gamma) \cap E^0F^\infty$. We have $\phi_\infty(\lambda) = \phi_\infty(l^1 \cdots l^{|\alpha|} \cdots) \in \mathcal{Z}(\alpha \setminus G) \cap \partial E$, so $\phi'(l^1) \cdots \phi'(l^{|\alpha|}) = \alpha$ and $\phi'(l^{|\alpha|+1}) \notin G$. Hence

$$\phi_\infty(y) = \phi_\infty(\gamma y') = \phi_\infty(l^1 \cdots l^{|\alpha|+1} y') \in \mathcal{Z}(\alpha \setminus G) \cap \partial E.$$

So $\mathcal{Z}(\gamma) \cap E^0F^\infty \subset \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E)$.

In case (ii), we have $\phi_\infty(\lambda) = \phi(l^1 \cdots l^{|\alpha|}) \in \mathcal{Z}(\alpha \setminus G) \cap \partial E$, so $\phi(l^1 \cdots l^{|\alpha|}) = \alpha$. Since ϕ preserves r and s, we have $s(\alpha) = r(\mu)$ in F, and thus $s(\alpha)$ is singular in E. Since $G \subset s(\alpha)E^1$, it follows from condition (C3) that $G \subset \phi(F^*(M))$. Let $N = \max_{\nu \in \phi^{-1}(G)} |\nu|$. Each $\nu \in G \cap E^N$ has the form $\mu_1 \cdots \mu_{N-1} e$, where $e \neq \mu_N$. Set $\gamma = \phi^{-1}(\alpha)\mu_1 \cdots \mu_N$. Then $\lambda = \phi^{-1}(\alpha)\mu \in \mathcal{Z}(\gamma) \cap E^0F^\infty$. To see that

$$E^0F^\infty \subset \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E,$$

fix $y \in \mathcal{Z}(\gamma) \cap E^0F^\infty$. Then

$$\phi_\infty(y) = \phi_\infty(\gamma y') = \phi_\infty(\phi^{-1}(\alpha)\mu_1 \cdots \mu_N y') = \alpha\phi_\infty(\mu_1 \cdots \mu_N y').$$

Now either y' is the rest of μ, or $y' = \mu_{N+1} \cdots \mu_{N+K} e y''$ for some $e \neq \mu_{N+K+1}$ and $y'' \in F^\infty$. If y' is the rest of μ (so $y' = \mu_{N+1} \cdots \mu_{N+K} \cdots$) then $y = \lambda \in \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E)$ by assumption. In the other case, we have $y = \mu_1 \cdots \mu_N y' = \nu y''$ for some $\nu \in F^*(M)$, with $|\nu| = N + K + 1$. Then by choice of N, we have $\phi'(\nu) = e_\nu \notin G$. Hence

$$\phi_\infty(y) = \alpha\phi_\infty(\nu y'') = \alpha e_\nu \phi_\infty(y'') \in \mathcal{Z}(\alpha \setminus G) \cap \partial E.$$

So $y \in \mathcal{Z}(\gamma) \cap E^0F^\infty \subset \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E)$.

Now, to prove ϕ_∞^{-1} is continuous, fix $\gamma \in F^*$. Then $\mathcal{Z}(\gamma) \cap E^0F^\infty$ is a basic open set in E^0F^∞. If $\mathcal{Z}(\gamma) \cap E^0F^\infty = \emptyset$ then $\phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty) = \emptyset$ is open, so suppose that $\mathcal{Z}(\gamma) \cap E^0F^\infty \neq \emptyset$. Let $x \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty)$. We seek $\alpha \in E^*$ and a finite subset $G \subset s(\alpha)E^1$ such that

$$x \in \mathcal{Z}(\alpha \setminus G) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty).$$

Let $\lambda \in \mathcal{Z}(\gamma) \cap E^0F^\infty$ be the unique element such that $x = \phi_\infty(\lambda)$. Write $\lambda = \gamma\lambda'$ where $\lambda' \in F^\infty$. We consider two cases:

(i) $x \in E^\infty$, or

(ii) $x \in E^*$ and $s(x)$ is singular.

In case (ii), we have $x = \phi_\infty(\gamma\lambda') \in E^\infty$. This implies that λ does not have the form $\nu\mu$ for $\nu \in E^*$ and $\mu \in M$. That is, λ does not 'start' with a collapsible path. Hence by Lemma 2.2.9 we can write $\lambda = l^1 l^2 \dots$ for some $l^i \in (E^1 \cap F^1) \cup F^*(M)$. Let $j = \min\{i \in \mathbb{N} : |l^1 \dots l^i| \geq |\gamma|\}$. Set $\alpha = \phi(l^1 \dots l^j)$ and $G = \emptyset$. We claim that $x \in \mathcal{Z}(\alpha) \cap \partial E$, and that $\mathcal{Z}(\alpha) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0 F^\infty)$. We have

$$x = \phi_\infty(l^1 \dots l^j l^{j+1} \dots) = \phi(l^1 \dots l^j)\phi_\infty(l^{j+1} \dots) = \alpha\phi_\infty(l^{j+1} \dots).$$

So $x \in \mathcal{Z}(\alpha) \cap \partial E$. To see that $\mathcal{Z}(\alpha) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0 F^\infty)$ fix $y \in \mathcal{Z}(\alpha) \cap \partial E$. So $y = \alpha y'$ for some $y' \in \partial E$. Then

$$\begin{aligned}
\phi_\infty^{-1}(y) &= \phi_\infty^{-1}(\alpha y') \\
&= \phi_\infty^{-1}(\phi(l^1 \dots l^j) y') \\
&= l^1 \dots l^j \phi_\infty^{-1}(y') \\
&= \gamma\gamma' \phi_\infty^{-1}(y') \qquad \text{for some } \gamma' \in F^*.
\end{aligned}$$

So $\phi_\infty^{-1}(y) \in \mathcal{Z}(\gamma) \cap E^0 F^\infty$, and hence $y \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0 F^\infty)$.

In case (ii), we have $\lambda = \gamma\lambda' = \omega\mu$ for some $\omega \in F^*$ and $\mu \in M$. Let $\alpha := x$. Our choice of G depends on $|\gamma|$, so we argue in cases:

(1) If $|\gamma| \leq |\omega|$, let $G = \emptyset$.
(2) If $|\gamma| > |\omega|$, then $\gamma = \omega\mu_1 \dots \mu_j$ for some $j \in \mathbb{N}$; let

$$G = \{e_\nu : \nu = \mu_1 \dots \mu_k \nu_{k+1} \in F^*(\mu), \text{ and } k < j\}.$$

Since $x \in \mathcal{Z}(\alpha \setminus G) \cap \partial E$ by definition, we just need to show that

$$\mathcal{Z}(x \setminus G) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0 F^\infty).$$

Fix $y \in \mathcal{Z}(x \setminus G) \cap \partial E$, so $y = xy'$ for some $y' \in \partial E$. Since $x = \phi_\infty(\lambda) = \phi_\infty(\omega\mu) = \phi(\omega)$, we have $\phi_\infty^{-1}(y) = \phi_\infty^{-1}(xy') = \omega\phi_\infty^{-1}(y')$.

In case (1), $|\gamma| \leq |\omega|$ implies that $\omega = \gamma\omega'$ for some $\omega' \in F^*$, so

$$\phi_\infty^{-1}(y) = \gamma\omega'\phi_\infty^{-1}(y') \in \mathcal{Z}(\gamma) \cap E^0 F^\infty.$$

Hence $y \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0 F^\infty)$.

For case (2), observe that if $y' \in E^0$, then $y = x \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0 F^\infty)$ by assumption. So suppose $|y'| \geq 1$. Then $s(x)$ is an infinite receiver, and thus $y'_1 = e_\nu$ for some $\nu \in F^*(\mu)$. Since $y \in \mathcal{Z}(x \setminus G)$, $y'_1 \notin G$, so $\nu = \mu_1 \dots \mu_k \nu_{k+1}$ for some

$k \geq j$, and thus

$$\begin{aligned}\phi_\infty^{-1}(y) = \phi_\infty^{-1}(xy') &= \omega\phi_\infty^{-1}(y_1'y_2'\dots) \\ &= \omega\phi_\infty^{-1}(e_\nu y_2'\dots) \\ &= \omega\nu\phi_\infty^{-1}(y_2'\dots) \\ &= \omega\mu_1\dots\mu_j\dots\mu_k\nu_{k+1}\phi_\infty^{-1}(y_2'\dots) \\ &= \gamma\mu_{j+1}\dots\mu_k\nu_{k+1}\phi_\infty^{-1}(y_2'\dots)\end{aligned}$$

is an element of $\mathcal{Z}(\gamma) \cap E^0F^\infty$. So $y \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty)$, and hence $\phi_\infty : E^0F^\infty \to \partial E$ is a homeomorphism. □

2.3. Graph C^*-algebras

Let E be a directed graph. Define

$$E^{\leq n} := \{\mu \in E^* : |\mu| = n, \text{ or } |\mu| < n \text{ and } s(\mu)E^1 = \emptyset\}.$$

A *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{s_v : v \in E^0\}$ and partial isometries $\{s_\mu : \mu \in E^*\}$ such that $\{s_\mu : \mu \in E^{\leq n}\}$ have mutually orthogonal ranges for each $n \in \mathbb{N}$, and such that

(CK1) $s_\mu^* s_\mu = s_{s(\mu)}$ for every $\mu \in E^*$;
(CK2) $s_\mu s_\mu^* \leq s_{r(\mu)}$ for every $\mu \in E^*$; and
(CK3) $s_v = \sum_{\nu \in vE^{\leq n}} s_\nu s_\nu^*$ for every $v \in E^0$ and $n \in \mathbb{N}$ such that $|vE^{\leq n}| < \infty$.

The C^*-algebra of E is the universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_\mu : \mu \in E^*\}$. The existence of such a C^*-algebra follows from an argument like that of [**19**, Proposition 1.21].

These relations are slightly different to the standard Cuntz-Krieger relations appearing elsewhere (for example in [**1, 5, 19**]). In these papers, a Cuntz-Krieger E-family is defined to be a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges such that

(G1) $s_e^* s_e = p_{s(e)}$ for every $e \in E^1$;
(G2) $s_e s_e^* \leq P_{r(e)}$ for every $e \in E^1$;
(G3) $p_v = \sum_{e \in vE^1} s_e s_e^*$ for every $v \in E^0$ such that $0 < |vE^1| < \infty$.

When we are working with higher-rank graphs later in this thesis, there is also a set of Cuntz-Krieger relations which, as a consequence of the structure of a higher-rank graph, are easier to work with when stated in terms of paths. Since the majority of this thesis deals with higher-rank graphs, we will use the relations (CK1)–(CK3) for the sake of consistency. First we prove that our definition of a Cuntz-Krieger E-family is equivalent to the one that is usually stated.

LEMMA 2.3.1. *Let E be a directed graph. Let $\{p_v : v \in E^0\}$ be mutually orthogonal projections, and $\{s_e : e \in E^1\}$ be partial isometries. For $v \in E^0$, let $s_v = p_v$ and for $\mu \in E^*$ let $s_\mu = s_{\mu_1} \dots s_{\mu_{|\mu|}}$. Then $\{s_e : e \in E^1\}$ have mutually orthogonal ranges and $\{p_v, s_e : v \in E^0, e \in E^1\}$ satisfy* (G1)–(G3) *if and only if $\{s_\mu : \mu \in E^{\leq n}\}$ have mutually orthogonal ranges for each $n \in \mathbb{N}$ and $\{s_\mu : \mu \in E^*\}$ satisfy* (CK1)–(CK3).

PROOF. If $\{s_\mu : \mu \in E^{\leq n}\}$ have mutually orthogonal ranges for each $n \in \mathbb{N}$ and $\{s_\mu : \mu \in E^*\}$ satisfy (CK1)–(CK3), then clearly $\{s_e : e \in E^1\}$ have mutually orthogonal ranges and $\{p_v, s_e : v \in E^0, e \in E^1\}$ satisfy (G1)–(G3). So suppose that $\{s_e : e \in E^1\}$ have mutually orthogonal ranges and that $\{p_v, s_e : v \in E^0, e \in E^1\}$ satisfy (G1)–(G3). To see (CK1) holds, calculate

$$\begin{aligned} s_\mu^* s_\mu &= s_{\mu_2 \dots \mu_n}^* s_{\mu_1}^* s_{\mu_1} s_{\mu_2 \dots \mu_n} \\ &= s_{\mu_2 \dots \mu_n}^* s_{s(\mu_1)} s_{\mu_2 \dots \mu_n} && \text{by (G1)} \\ &= s_{\mu_2 \dots \mu_n}^* s_{r(\mu_2)} s_{\mu_2 \dots \mu_n} \\ &= s_{\mu_1 \dots \mu_{n-1}}^* s_{\mu_1 \dots \mu_{n-1}}. \end{aligned}$$

So (CK1) follows from an induction on $|\mu|$.

For (CK2), we have

$$\begin{aligned} s_\mu s_\mu^* &= s_{\mu_1 \dots \mu_{n-1}} s_{\mu_n} s_{\mu_n}^* s_{\mu_1 \dots \mu_{n-1}}^* \\ &\leq s_{\mu_1 \dots \mu_{n-1}} s_{r(\mu_n)} s_{\mu_1 \dots \mu_{n-1}}^* && \text{by (G2)} \\ &= s_{\mu_1 \dots \mu_{n-1}} s_{s(\mu_{n-1})} s_{\mu_1 \dots \mu_{n-1}}^* \\ &= s_{\mu_1 \dots \mu_{n-1}} s_{\mu_1 \dots \mu_{n-1}}^*. \end{aligned}$$

So another induction on $|\mu|$ gives the result.

For (CK3), fix $\mu \in E^*$ such that $|s(\mu)E^1| < \infty$. If $s(\mu)E^1 = \emptyset$, we have $\mu \in E^{\leq |\mu|+1}$, so $\mu E^* \cap E^{\leq |\mu|+1} = \{\mu\}$. Then

$$s_\mu s_\mu^* = \sum_{\nu \in \mu E^* \cap E^{\leq |\mu|+1}} s_\nu s_\nu^*.$$

If $s(\mu)E^1 \neq \emptyset$, then

$$\begin{aligned} s_\mu s_\mu^* &= s_\mu s_{s(\mu)} s_\mu^* \\ &= s_\mu \Big(\sum_{e \in s(\mu)E^1} s_e s_e^* \Big) s_\mu^* && \text{by (G3)} \\ &= \sum_{e \in s(\mu)E^1} s_{\mu e} s_{\mu e}^* \\ &= \sum_{\nu \in \mu E^* \cap E^{\leq |\mu|+1}} s_\nu s_\nu^*. \end{aligned}$$

Hence

$$\sum_{\mu\in vE^{\leq n}} s_\mu s_\mu^* = \sum_{\mu\in vE^{\leq n}} \Big(\sum_{\nu\in\mu E^*\cap E^{\leq|\mu|+1}} s_\nu s_\nu^* \Big) = \sum_{\nu\in vE^{\leq n+1}} s_\nu s_\nu^*,$$

and an induction on n gives the result.

To see $\{s_\mu s_\mu^* : \mu \in E^{\leq n}\}$ are mutually orthogonal, let $\mu,\nu \in E^{\leq n}$ such that $\mu \neq \nu$. Since $\mu,\nu \in E^{\leq n}$, we know that neither $\mu \neq \nu\nu'$ nor $\nu \neq \mu\mu'$ (otherwise the shorter path would not be an element of $E^{\leq n}$). Then there exists $j \leq \min\{|\mu|,|\nu|\}$ such that $\mu_i = \nu_i$ for $i < j$ and $\mu_j \neq \nu_j$. Since $\{s_e : e \in E^1\}$ have mutually orthogonal ranges, we have $s_{\mu_j}^* s_{\nu_j} = 0$, so

$$\begin{aligned}(s_\mu s_\mu^*)(s_\nu s_\nu^*) &= s_\mu s_{\mu_{j+1}\dots\mu_{|\mu|}} s_{\mu_j}^* (s_{\mu_1\dots\mu_{j-1}}^* s_{\nu_1\dots\nu_{j-1}}) s_{\nu_j} s_{\nu_{j+1}\dots\nu_{|\nu|}} s_\nu^* \\ &= s_\mu s_{\mu_{j+1}\dots\mu_{|\mu|}} s_{\mu_j}^* s_{\nu_j} s_{\nu_{j+1}\dots\nu_{|\nu|}} s_\nu^* \qquad \text{by (CK1)} \\ &= 0\end{aligned}$$

□

REMARK 2.3.2. When E^0 is infinite, $C^*(E)$ does not have an identity. To see why, let $\{s_\mu : \mu \in E^*\}$ be the universal Cuntz-Krieger E-family generating $C^*(E)$. Then $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu,\nu \in E\}$ (see [**19**, Proposition 1.21]). If $1 \in C^*(E)$, then there exists a finite sum $a := \sum_{\mu,\nu\in F} a_{\mu,\nu} s_\mu s_\nu^*$ such that $\|1-a\| < 1/2$. Fix $v \in E^0$ such that $r(\mu) \neq v$ for any $\mu \in F$. Then

$$1/2 \geq \|1-a\|\|s_v\| \geq \|(1-a)s_v\| = \|s_v - 0\| = \|s_v\| = 1.$$

If A is a C^*-algebra without identity, it is often useful to embed it in a larger C^*-algebra with an identity like the multiplier algebra $M(A)$ of A. We use multiplier algebras to make sense of certain infinite sums in $C^*(E)$. An infinite sum of mutually orthogonal projections in a C^*-algebra cannot converge in norm: each difference $\sum_{n=M+1}^N p_n$ between partial sums is also a projection and hence has norm 1. The following lemma [**19**, Lemma 2.10] nevertheless allows us to safely talk about infinite sums of vertex projections in $C^*(E)$. The proof supplied in [**19**] does not depend on the row-finiteness of E, so it has been left out of the statement here.

LEMMA 2.3.3 ([**19**, Lemma 2.10]). *Let E be a directed graph, and fix $V \subset E^0$. Then there is a projection p_V in $M(C^*(E))$ such that*

$$p_V s_\mu s_\nu^* = \begin{cases} s_\mu s_\mu^* \ \textit{if}\ r(\mu) \in V \\ 0 \ \textit{if}\ r(\mu) \notin V. \end{cases} \tag{2.3.1}$$

We can now state the key result, which relates the C^*-algebra of E to that of its desingularisation.

Proposition 2.3.4 ([**19**, Proposition 5.2]). *Suppose that μ is a collapsible path in a row-finite graph F. Define $s_\infty(\mu) := \{s(\mu_i) : i \geq 1\}$ and*

$$F^*(\mu) := \{\nu \in F^* : |\nu| > 1, \nu = \mu_1\mu_2 \ \cdots \mu_{|\nu-1|}e \text{ for some } e \neq \mu_{|\mu|}\}.$$

Let $F^0_\mu := F^0 \backslash s_\infty(\mu)$ and $F^1_\mu := \big(F^1 \backslash (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\})\big) \cup \{e_\nu : \nu \in F^*(\mu)\}$, and extend the range and source maps to F^1_μ by $r(e_\nu) := r(\nu) = r(\mu)$ and $s(e_\nu) := s(\nu)$. Let $p_{F^0_\mu}$ be the projection from Lemma 2.3.3. Let $\{s_e, p_v : e \in F^1, v \in F^0\}$ and $\{t_e, q_v : e \in F^1_\mu, v \in F^0_\mu\}$ be the generators of $C^*(F)$ and $C^*(F_\mu)$. Then $p_{F^0_\mu}C^*(F)p_{F^0_\mu}$ is a full corner in $C^*(F)$, and there is an isomorphism π of $C^*(F_\mu)$ onto $p_{F^0_\mu}C^*(F)p_{F^0_\mu}$ such that $\pi(q_v) = p_v$ for $v \in F^0_\mu$, $\pi(t_e) = s_e$ for $e \in F^1 \setminus (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\})$, and $\pi(t_{e_\nu}) = s_\nu$ for $\nu \in F^*(\mu)$.

For a directed graph E, we call $C^*(\{s_\mu s_\mu^* : \mu \in E\}) \subset C^*(E)$ the diagonal C^*-algebra of E and denote it D_E, dropping the subscript when confusion is unlikely. We denote the spectrum of a commutative C^*-algebra B by $\Delta(B)$. Given a homomorphism $\pi : A \to B$ of commutative C^*-algebras, we denote by π^* the induced map from $\Delta(B)$ to $\Delta(A)$ such that $\pi^*(f)(y) = f(\pi(y))$ for all $f \in \Delta(B)$ and $y \in A$.

2.4. The Diagonal and the Spectrum

The goal for this section is the following theorem.

Theorem 2.4.1. *Let E be a directed graph and (F, M) be a Drinen-Tomforde desingularisation of E. Let $\phi_\infty : E^0F^\infty \to \partial E$ be the homeomorphism from Theorem 2.2.1, let $p_{E^0} \in M(C^*(F))$ be the projection obtained in Lemma 2.3.3, and let $\pi : C^*(F_\mu) \to p_{E^0}C^*(F)p_{E^0}$ be the isomorphism from Proposition 2.3.4. Then $\pi(D_E) = p_{E^0}D_Fp_{E^0}$, and there exist homeomorphisms $h_E : \partial E \to \Delta(D_E)$ and $h : E^0F^\infty \to \Delta(p_{E^0}D_Fp_{E^0})$ such that the following diagram commutes.*

$$\begin{tikzcd} E^0F^\infty \arrow[r, "\phi_\infty"] \arrow[d, "h"'] & \partial E \arrow[d, "h_E"] \\ \Delta(p_{E^0}D_Fp_{E^0}) \arrow[r, "\pi^*"'] & \Delta(D_E) \end{tikzcd}$$

We prove Theorem 2.4.1 on page 33. First, we establish some technical results.

Lemma 2.4.2. *Let E be a directed graph, and let $F \subset E^*$ be finite. For $\mu \in F$, define*

$$q^F_\mu := s_\mu s_\mu^* \prod_{\mu\mu' \in F\backslash\{\mu\}} (s_\mu s_\mu^* - s_{\mu\mu'}s_{\mu\mu'}^*).$$

Then the q^F_μ *are mutually orthogonal projections in* $\operatorname{span}\{s_\mu s_\mu^* : \mu \in F\}$*, and for each* $\nu \in F$*, we have*

$$s_\nu s_\nu^* = \sum_{\nu\nu' \in F} q^F_{\nu\nu'}. \tag{2.4.1}$$

Proof. We prove (2.4.1) by induction on $|F|$. If $|F| = 1$, then (2.4.1) is trivially satisfied. Now suppose that (2.4.1) holds for all F with $|F| < n$, and fix F with $|F| = n$. Let $\lambda \in F$ be of maximal length, and define $G = F \setminus \{\lambda\}$. Then $q^F_\lambda = s_\lambda s_\lambda^*$, and for each $\mu \in G$ we have

$$q^F_\mu = \begin{cases} q^G_\mu & \text{if } \lambda \notin \mathcal{Z}(\mu) \\ q^G_\mu(s_\mu s_\mu^* - s_\lambda s_\lambda^*) = q^G_\mu - q^G_\mu s_\lambda s_\lambda^* & \text{if } \lambda \in \mathcal{Z}(\mu). \end{cases}$$

Fix $\mu \in G$. If $\lambda \notin \mathcal{Z}(\mu)$, the inductive hypothesis implies that

$$\sum_{\mu\mu' \in F} q^F_{\mu\mu'} = \sum_{\mu\mu' \in G} q^G_{\mu\mu'} = s_\mu s_\mu^*.$$

If $\lambda \in \mathcal{Z}(\mu)$, then

$$\begin{aligned} \sum_{\mu\mu' \in F} q^F_{\mu\mu'} &= \sum_{\mu\mu' \in G} (q^G_\mu - q^G_\mu s_\lambda s_\lambda^*) + q^F_\lambda \\ &= \sum_{\mu\mu' \in G} q^G_\mu - \sum_{\mu\mu' \in G} q^G_\mu s_\lambda s_\lambda^* + s_\lambda s_\lambda^* \\ &= s_\mu s_\mu^* - s_\mu s_\mu^* s_\lambda s_\lambda^* + s_\lambda s_\lambda^* \qquad \text{by the inductive hypothesis} \\ &= s_\mu s_\mu^* - s_\lambda s_\lambda^* + s_\lambda s_\lambda^* \qquad \text{since } \lambda \in \mathcal{Z}(\mu) \\ &= s_\mu s_\mu^*, \end{aligned}$$

establishing (2.4.1).

That the q^F_μ are projections follows from Lemma A.0.7. That they are mutually orthogonal follows from (2.4.1). □

Remark 2.4.3. Let E be a directed graph, and let $F \subset E^*$ be finite. For $\mu \in F$, let $F_\mu = \{\mu' \in s(\mu)E \setminus \{s(\mu)\} : \mu\mu' \in F\}$. We claim that

$$q^F_\mu = s_\mu\Big(\prod_{\mu' \in F_\mu} (s_{s(\mu)} - s_{\mu'} s_{\mu'}^*)\Big) s_\mu^*.$$

To see this, fix $\nu \in F_\mu$. We have

$$
\begin{aligned}
s_\mu\Big(\prod_{\mu'\in F_\mu}&(s_{s(\mu)} - s_{\mu'}s_{\mu'}^*)\Big)s_\mu^* \\
&= s_\mu\Big(\prod_{\mu'\in F_\mu\setminus\{\nu\}}(s_{s(\mu)} - s_{\mu'}s_{\mu'}^*)\Big)(s_{s(\mu)} - s_\nu s_\nu^*)s_\mu^* \\
&= s_\mu\Big(\prod_{\mu'\in F_\mu\setminus\{\nu\}}(s_{s(\mu)} - s_{\mu'}s_{\mu'}^*)\Big)s_\mu^* s_\mu(s_{s(\mu)} - s_\nu s_\nu^*)s_\mu^* \\
&= \Big(s_\mu\Big(\prod_{\mu'\in F_\mu\setminus\{\nu\}}(s_{s(\mu)} - s_{\mu'}s_{\mu'}^*)\Big)s_\mu^*\Big)(s_\mu s_\mu^* - s_{\mu\nu}s_{\mu\nu}^*).
\end{aligned}
$$

Now an induction on $|F_\mu|$ gives

$$
s_\mu\Big(\prod_{\mu'\in F_\mu}(s_{s(\mu)} - s_{\mu'}s_{\mu'}^*)\Big)s_\mu^* = s_\mu s_\mu^* \prod_{\mu'\in F_\mu}(s_\mu s_\mu^* - s_{\mu\mu'}s_{\mu\mu'}^*) = q_\mu^F.
$$

To go further we need some more definitions. We say that $\mu, \nu \in E^*$ have *common extension* if either $\mu = \nu\nu'$ or $\nu = \mu\mu'$, and call the longer path the *minimal common extension* of μ and ν. A set $F \subset E^*$ is *exhaustive* if for every $\mu \in E^*$ there exists $\nu \in F$ such that μ and ν have common extension. We denote the set of finite exhaustive sets by $\mathcal{FE}(E)$, and for a vertex v we define $v\mathcal{FE}(E) := \{F \in \mathcal{FE}(E) : F \subset vE^*\}$.

The following lemma is stated for row-finite directed graphs as [**19**, Corollary 1.14(b)]. The proof is marginally different for arbitrary directed graphs, and is supplied here.

Lemma 2.4.4. *Let E be a directed graph, and let $\mu, \nu \in E^*$. Then*

$$
s_\mu^* s_\nu = \begin{cases} s_{\nu'}^* & \text{if } \mu = \nu\nu' \\ s_{\mu'} & \text{if } \nu = \mu\mu' \\ 0 & \text{otherwise.} \end{cases}
$$

Furthermore,

$$
(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \begin{cases} s_\mu s_\mu^* & \text{if } \mu = \nu\nu' \\ s_\nu s_\nu^* & \text{if } \nu = \mu\mu' \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.2}
$$

Proof. If $\mu = \nu\nu'$, then

$$
s_\mu^* s_\nu = s_{\nu\nu'}^* s_\nu = s_{\nu'}^* s_\nu^* s_\nu = s_{\nu'}^* s_{s(\nu)} = s_{\nu'}^*,
$$

hence

$$
(s_\mu s_\mu^*)(s_\nu s_\nu^*) = s_\mu s_{\nu\nu'}^* s_\nu s_\nu^* = s_\mu s_{\nu'}^* s_\nu^* s_\nu s_\nu^* = s_\mu s_{\nu'}^* s_\nu^* = s_\mu s_{\nu\nu'}^* = s_\mu s_\mu^*.
$$

Similar calculations show that if $\nu = \mu\mu'$, then we have $s_\mu^* s_\nu = s_{\mu'}$ and $(s_\mu s_\mu^*)(s_\nu s_\nu^*) = s_\nu s_\nu^*$.

Otherwise, we have $\mu_1 \dots \mu_n \neq \nu_1 \dots \nu_n$ where $n = \min\{|\mu|, |\nu|\}$. Without loss of generality, suppose that $|\mu| > |\nu|$. Let $\lambda = \mu_1 \dots \mu_n$, and let μ' be such that $\mu = \lambda\mu'$. Then

$$s_\mu^* s_\nu = s_{\mu'}^* s_\lambda^* s_\nu = s_{\mu'}^* s_{s(\lambda)} s_\lambda^* s_\nu s_{s(\nu)} = s_{\mu'}^* s_\lambda^* (s_\lambda s_\lambda^* s_\nu s_\nu^*) s_\nu = 0,$$

hence

$$(s_\mu s_\mu^*)(s_\nu s_\nu^*) = s_\mu (s_\mu^* s_\nu) s_\nu^* = 0. \qquad \square$$

THEOREM 2.4.5. *Let E be a directed graph. Then $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E\}$, and for each $x \in \partial E$ there exists a unique $h_E(x) \in \Delta(D)$ such that*

$$h_E(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $x \mapsto h_E(x)$ is a homeomorphism of ∂E onto $\Delta(D)$.

PROOF. We will first show that $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\}$. Equation (2.4.2) implies that $\operatorname{span}\{s_\mu s_\mu^* : \mu \in E^*\}$ is closed under multiplication and thus is a $*$-subalgebra of $C^*(E)$. Hence the closed span is a C^*-algebra. Since D is the smallest C^*-subalgebra of $C^*(E)$ containing the generators $\{s_\mu s_\mu^* : \mu \in E^*\}$, we have $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\}$.

Fix $x \in \partial E$ and $\sum_{\mu \in F} b_\mu s_\mu s_\mu^* \in \operatorname{span}\{s_\mu s_\mu^* : \mu \in E^*\}$. Let $n = \max\{p \in \mathbb{N} : x_1 \dots x_p \in F\}$, and define $F_x := \{\mu' \in x(n)E \setminus \{x(n)\} : x(0,n)\mu' \in F\}$.

CLAIM 2.4.5.1. *The projection $q^F_{x_1 \dots x_n} \neq 0$.*

PROOF. First suppose that $s(x_n)E^* = \emptyset$. Then $F_x = \emptyset$, hence

$$q^F_{x_1 \dots x_n} = s_{x_1 \dots x_n} s^*_{x_1 \dots x_n} \neq 0.$$

Now suppose that $s(x_n)E^* \neq \emptyset$. We first show that there exists $\nu \in s(x_n)E^*$ such that for each $\mu' \in F_x$, ν and μ' have no common extension. We argue in cases; we know $s(x_n)$ is not a source in E, so our cases are

(i) $s(x)$ is a source in E, and $|x| > n$;
(ii) $s(x)$ is an infinite receiver;
(iii) $x \in E^\infty$.

In case (i), let $\nu = x_{n+1} \dots x_{|x|}$. Then by choice of n, ν has no common extension with any μ' in F_x. In case (ii), such a ν exists since $|F_x| \leq |F| < |s(x)E^*| = \infty$. In case (iii), let $k = \max\{|\mu'| : \mu' \in F_x\}$. Then it follows from our choice of n that $\nu = x_{n+1} \dots x_{n+k}$ is not a common extension of any μ' in F_x.

So we have $\nu \in s(x_n)E^*$ such that ν and μ' have no common extension for all $\mu' \in F_x$. Thus by Lemma 2.4.4, we have $s_\nu s_\nu^* s_{\mu'} s_{\mu'}^* = 0$ for all $\mu' \in F_x$. Applying Lemma A.0.7 with $p = s_{s(x_n)}$, $q_0 = s_\nu s_\nu^*$, $Q = F_x$, we have $\prod_{\mu' \in F_x}(s_{s(x_n)} - s_{\mu'} s_{\mu'}^*) \neq 0$. So

$$q^F_{x_1 \dots x_n} = s_{x_1 \dots x_n} \prod_{\mu' \in F_x} (s_{s(x_n)} - s_{\mu'} s_{\mu'}^*) s_{x_1 \dots x_n}^* \neq 0. \qquad \square_{\text{Claim}}$$

We now calculate

$$\begin{aligned}
\Big\| \sum_{\nu \in F} b_\mu s_\mu s_\mu^* \Big\| &= \Big\| \sum_{\nu \in F} \Big(\sum_{\substack{\mu \in F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big) q_\nu^F \Big\| \\
&= \max_{\substack{\nu \in F \\ q_\nu^F \neq 0}} \Big\{ \Big| \sum_{\substack{\mu \in F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big| \Big\} \qquad \text{by Lemma A.0.6} \\
&\geq \Big| \sum_{\substack{\mu \in F \\ x_1 \dots x_n \in \mathcal{Z}(\mu)}} b_\mu \Big| \qquad \text{by Claim 2.4.5.1.}
\end{aligned}$$

Hence the formula

$$h_E(x)\Big(\sum_{\mu \in F} b_\mu s_\mu s_\mu^*\Big) = \sum_{\substack{\mu \in F \\ x \in \mathcal{Z}(\mu)}} b_\mu \tag{2.4.3}$$

determines a well-defined, norm-decreasing linear map $h_E(x)$ on $\operatorname{span}\{s_\mu s_\mu^* : \mu \in E\}$.

We now show that $h_E(x)$ is a homomorphism. Since $h_E(x)$ is linear and norm-decreasing, it suffices to calculate

$$\begin{aligned}
h_E(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) &= \begin{cases} 1 & \text{if } \alpha \in \mathcal{Z}(\mu) \text{ and } x \in \mathcal{Z}(\alpha) \\ & \text{or } \mu \in \mathcal{Z}(\alpha) \text{ and } x \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\alpha) \subset \mathcal{Z}(\mu) \\ & \text{or } x \in \mathcal{Z}(\mu) \subset \mathcal{Z}(\alpha), \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\alpha) \cap \mathcal{Z}(\mu) \\ 0 & \text{otherwise.} \end{cases} \\
&= h_E(x)(s_\mu s_\mu^*) h_E(x)(s_\alpha s_\alpha^*).
\end{aligned}$$

Since $h(x)$ is a nonzero bounded linear map on a dense subspace of D, and since multiplication is continuous, $h(x)$ extends uniquely to a nonzero homomorphism $h(x) : D \to \mathbb{C}$.

It remains to show that $h_E : \partial E \to \Delta(D)$ is a homeomorphism. The trickiest part is to show that h_E is onto.

CLAIM 2.4.5.2. *The map h_E is surjective.*

PROOF. Fix $\phi \in \Delta(D)$. We seek $x \in \partial E$ such that $h_E(x) = \phi$. We have that $\phi(p) \in \{0, 1\}$ for any projection $p \in D$, and that for each $n \in \mathbb{N}$, $\{s_\mu s_\mu^* : |\mu| = n\}$ are mutually orthogonal projections. It then follows that for each n there exists at most one $\nu^n \in E^n$ such that $\phi(s_{\nu^n} s_{\nu^n}^*) = 1$. Let

$$S := \{n \in \mathbb{N} : \text{ there exists } \nu^n \in E^n \text{ such that } \phi(s_{\nu^n} s_{\nu^n}) = 1\}.$$

Since ϕ is nonzero, S is nonempty. If $\nu = \mu\nu'$ and $\phi(s_\nu s_\nu^*) = 1$, then

$$1 = \phi(s_\nu s_\nu^*) = \phi(s_\nu s_\nu^* s_\mu s_\mu^*) = \phi(s_\nu s_\nu^*)\phi(s_\mu s_\mu^*),$$

so $\phi(s_\mu s_\mu^*) = 1$. This implies that if $n \in S$ and $m \leq n$, then $m \in S$, and $\nu^n(0, m) = \nu^m$. It follows that S is equal either to $\mathbb{N}$, or to $\{1, \ldots, N\}$ for some N.

If $S = \mathbb{N}$, define $x \in E^\infty$ by $x(0, n) = \nu^n$ for all n. If $S = \{1, \ldots, N\}$, define $x := \nu^N$. We will show that $x \in \partial E$ and $h_E(x) = \phi$. We first show that $x \in \partial E$. This is trivial if $S = \mathbb{N}$, so suppose that $S = \{1, \ldots, N\}$. We must show that $|s(x)E^1| \in \{0, \infty\}$. To see this, we suppose that $s(x)E^1$ is finite and nonempty and seek a contradiction. By (CK3), we have $\phi(s_x s_x^*) = \sum_{e \in s(x)E^1} \phi(s_{xe} s_{xe}^*)$, so there exists $e \in s(x)E^1$ such that $\phi(s_{xe} s_{xe}^*) = 1$, giving $N + 1 \in S$, a contradiction.

Now we show that $h_E(x) = \phi$. For each $\mu \in E^*$ we have

$$\begin{aligned} \phi(s_\mu s_\mu^*) = 1 &\iff |\mu| \in S \text{ and } \nu^{|\mu|} = \mu \\ &\iff x(0, |\mu|) = \mu \\ &\iff h_E(x)(s_\mu s_\mu^*) = 1. \end{aligned}$$

Since both $\phi(s_\mu s_\mu^*)$ and $h_E(x)(s_\mu s_\mu^*)$ only take values in $\{0, 1\}$, it follows that $h_E(x) = \phi$. $\square_{\text{Claim}}$

To see h is injective, suppose that $h_E(x) = h_E(y)$. Then for each $n \in \mathbb{N}$, let $n_x = \min\{n, |x|\}$. Then we have

$$h_E(y)(s_{x(0,n_x)} s_{x(0,n_x)}^*) = h_E(x)(s_{x(0,n_x)} s_{x(0,n_x)}^*) = 1$$

Hence $y(0, n \wedge |x|) = x(0, n \wedge |x|)$ for all $n \in \mathbb{N}$. By symmetry, we also have that $y(0, n \wedge |y|) = x(0, n \wedge |y|)$ for all n. In particular, $|x| = |y|$ and $y(0, n) = x(0, n)$ for all $n \leq |x|$. Thus $x = y$.

We now show that h_E is continuous. Suppose that $x^n \to x$. Since the topology on $\Delta(D)$ is that of pointwise convergence, we must show that $h_E(x^n)(a) \to h_E(x)(a)$ for each $a \in D$. We will first show that for each $\mu \in E^*$, there exists N such that

$n \geq N$ implies that $h_E(x^n)(s_\mu s_\mu^*) = h_E(x)(s_\mu s_\mu^*)$. Since $x^n \to x$, there exists N_0 such that $n \geq N_0$ implies that $x^n(0, |\mu| \wedge |x|) = x(0, |\mu| \wedge |x|)$. Fix $n \geq N_0$. Suppose that $h_E(x)(s_\mu s_\mu^*) = 1$. Then $x(0, |\mu|) = \mu$. In particular, $x^n(0, |\mu|) = x(0, |\mu|) = \mu$, so $h_E(x^n)(s_\mu s_\mu^*) = 1$. Now suppose that $h_E(x)(s_\mu s_\mu^*) = 0$. Then $x(0, |\mu| \wedge |x|) \neq \mu$, so $x^n(0, |\mu| \wedge |x|) \neq \mu$, and thus $h_E(x^n)(s_\mu s_\mu^*) = 0$. Since $h_E(x)$ and the $h_E(x^n)$ are linear, it follows that $h_E(x^n)$ converges to $h_E(x)$ for $x \in \operatorname{span}\{s_\mu s_\mu^* : \mu \in E\}$. We now argue that h_E us continuous on D. Fix $a \in D$ and $\varepsilon > 0$. We seek $N \in \mathbb{N}$ such that $n \geq N$ implies that $|h_E(x^n)(a) - h_E(x)(a)| < \varepsilon$. Fix a sequence $\{a_m : n \in \mathbb{N}\} \subset \operatorname{span}\{s_\mu s_\mu^* : \mu \in E\}$ such that $a_m \to a$. So there exists N_1 such that $n \geq N_0$ implies that $\|a_n - a\| < \varepsilon/3$. Fix $m \in \mathbb{N}$. Since h_E is continuous on $\operatorname{span}\{s_\mu s_\mu^* : \mu \in E\}$, there exists N_2 such that $n \geq N_2$ implies that $|h_E(x^n)(a_m) - h_E(x)(a_m)| < \varepsilon/3$. Let $N = \max\{N_1, N_2\}$, and fix $n \geq N$. Then since $h_E(x)$ is norm decreasing for every $x \in \partial E$, we have

$$\begin{aligned}
|h_E(x^n)(a) - h_E(x)(a)| &= \big|h_E(x^n)(a) - h_E(x^n)(a_n) \\
&\qquad + h_E(x^n)(a_n) - h_E(x)(a_n) \\
&\qquad\qquad + h_E(x)(a_n) - h_E(x)(a)\big| \\
&\leq |h_E(x^n)(a) - h_E(x^n)(a_n)| \\
&\qquad + |h_E(x^n)(a_n) - h_E(x)(a_n)| \\
&\qquad\qquad + |h_E(x)(a_n) - h_E(x)(a)| \\
&< |h_E(x^n)(a - a_n)| \\
&\qquad + \frac{\varepsilon}{3} + |h_E(x)(a_n - a)| \\
&\leq \|a - a_n\| + \frac{\varepsilon}{3} + \|a - a_n\| \\
&= \varepsilon.
\end{aligned}$$

So h_E is continuous on D.

Finally, we show that h_E is open. Since h_E is a bijection, it suffices to show that h_E^{-1} is continuous. Suppose that $h_E(x^n) \to h_E(x)$. We will show that $x^n \to x$. Fix $\mu \in E^*$ such that $x \in \mathcal{Z}(\mu)$, so $h_E(x)(s_\mu s_\mu^*) = 1$. Since $h_E(x^n) \to h_E(x)$ in $\Delta(D)$ and $h_E(x^n)(s_\mu s_\mu^*) \in \{0, 1\}$ for all n, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $h_E(x^n)(s_\mu s_\mu^*) = 1$. So $x^n \in \mathcal{Z}(\mu)$ for $n \geq N$. Since $\mathcal{Z}(\mu)$ are a basis for the topology on ∂E, it follows that $x^n \to x$ in ∂E. $\square$

We can now prove our main result for this chapter.

Proof of Theorem 2.4.1. Let E be a directed graph with Drinen-Tomforde desingularisation (F, M). Then $\pi(s_\mu) = t_{\phi^{-1}(\mu)}$ for each $\mu \in E^*$.

Let $\phi : E^0F^*E^0 \to E^*$ be the bijection from Proposition 2.2.10. It follows from Lemma 2.3.3 that there exists a projection $p_{E^0} \in M(C^*(F))$ such that

$$p_{E^0} t_\mu t_\mu^* p_{E^0} = \begin{cases} t_\mu t_\mu^* & \text{if } r(\mu) \in E^0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.4}$$

Denote by π the isomorphism $\pi : C^*(E) \cong p_{E^0} C^*(F) p_{E^0}$ of Proposition 2.3.4.

We will show that π maps D_E onto $p_{E^0} D_F p_{E^0}$. Since

$$p_{E^0} t_\mu t_\mu^* p_{E^0} = \begin{cases} t_\mu t_\mu^* & \text{if } r(\mu) \in E^0, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\pi(D_E) \subset p_{E^0} D_F p_{E^0}$. To see the reverse inclusion, fix $\mu \in F^*$. We must show that $p_{E^0} t_\mu t_\mu^* p_{E^0} \in \pi(D_E)$. If $r(\mu) \notin E^0$ then $p_{E^0} t_\mu t_\mu^* p_{E^0} = 0 \in \pi(D_E)$, so suppose that $r(\mu) \in E^0$. If $s(\mu) \in E^0$, then

$$p_{E^0} t_\mu t_\mu^* p_{E^0} = t_\mu t_\mu^* = \pi(s_{\phi(\mu)} s_{\phi(\mu)}^*) \in \pi(D_E).$$

Now suppose that $s(\mu) \notin E^0$, then $s(\mu) = s(\nu_n)$ for some collapsible path $\nu \in F^\infty$ and $n \in \mathbb{N}$. By definition of a collapsible path, ν has no exits except at $r(\nu)$. Thus $\mu = \mu'\nu_n$, where $\mu' = \mu(0, |\mu|-1)$. Furthermore, $s(\mu')F^1$ is finite, thus (CK3) implies that

$$s_{\nu_n} s_{\nu_n}^* = s_{s(\mu')} - \sum_{f \in s(\mu')F^1 \setminus \{\nu_n\}} s_f s_f^*.$$

Then

$$\begin{aligned} p_{E^0} s_\mu s_\mu^* p_{E^0} &= p_{E^0} s_{\mu'} s_{\nu_n} s_{\nu_n}^* s_{\mu'}^* p_{E^0} \\ &= p_{E^0} s_{\mu'} \big(s_{s(\mu')} - \sum_{f \in s(\mu')F^1 \setminus \{\nu_n\}} s_f s_f^* \big) s_{\mu'}^* p_{E^0} \\ &= p_{E^0} s_{\mu'} s_{\mu'}^* p_{E^0} - \sum_{f \in s(\mu')F^1 \setminus \{\nu_n\}} p_{E^0} s_{\mu'} s_f s_f^* s_{\mu'}^* p_{E^0}. \end{aligned} \tag{2.4.5}$$

We proceed by induction on n. If $n = 1$, then $\mu' \in E^*$, and since $s(f) \in E^0$ for all $f \in s(\mu')F^1 \setminus \{\nu_1\}$, it follows that $p_{E^0} s_\mu s_\mu^* p_{E^0} \in \pi(D_E)$. Suppose, as an inductive hypothesis, that for every $\lambda \in F^*$ such that there exists a collapsible path $\nu \in F^\infty$ with $s(\lambda) = s(\nu_{n-1})$, we have $p_{E^0} s_\lambda s_\lambda^* p_{E^0} \in \pi(D_E)$. Since $s(\mu) = s(\nu_n)$ for some collapsible path ν, the inductive hypothesis implies that $p_{E^0} s_{\mu'} s_{\mu'}^* p_{E^0} \in \pi(D_E)$, and since $s(f) \in E^0$ for all $f \in s(\mu')F^1 \setminus \{\nu_n\}$, we have

$$\sum_{f \in s(\mu')F^1 \setminus \{\nu_n\}} p_{E^0} s_{\mu' f} s_{\mu' f}^* p_{E^0} \in \pi(D_E).$$

It then follows from (2.4.5) that $p_{E^0} s_\mu s_\mu^* p_{E^0} \in \pi(D_E)$, as required. So $\pi(D_E) = p_{E^0} D_F p_{E^0}$.

We now construct the homeomorphism h. Since p_{E^0} commutes with D_F, the space $p_{E^0}D_Fp_{E^0}$ is an ideal of D_F. Then [**23**, Propositions A26(a) and A27(b)] imply that the map $k : \phi \mapsto \phi|_{p_{E^0}D_Fp_{E^0}}$ is a homeomorphism of $\{\phi \in \Delta(D_F) : \phi|_{p_{E^0}D_Fp_{E^0}} \neq 0\}$ onto $\Delta(p_{E^0}D_Fp_{E^0})$. Since F has no singular vertices, $\partial F = F^\infty$. Let $h_F : F^\infty \to \Delta(D_F)$ be the homeomorphism obtained from Theorem 2.4.5. Fix $x \in E^0F^\infty$. Then there exists $\lambda \in F^*$ such that $h_F(x)(t_\lambda t_\lambda^*) \neq 0$, so $h_F(x) \in \mathrm{dom}(k)$ for all $x \in E^0F^\infty$. We define $h := k \circ h_F|_{E^0F^\infty} : E^0F^\infty \to \Delta(p_{E^0}D_Fp_{E^0})$.

Let $h_E : \partial E \to \Delta(D)$ be the homeomorphism obtained from Theorem 2.4.5, let $\phi_\infty : E^0F^\infty \to \partial E$ be the homeomorphism from Theorem 2.2.1, and let $\pi^* : \Delta(p_{E^0}D_Fp_{E^0}) \to \Delta(D_E)$ be the map $\phi \mapsto \phi \circ \pi$ induced by π. We show that the diagram on page 27 commutes by showing that $h_E \circ \phi_\infty = \pi^* \circ h$. Let $x \in E^0F^\infty$, and fix $\mu \in E^*$. Since $(h_E \circ \phi_\infty)(x)$ and $h(x)$ are homomorphisms, and since π is an isomorphism, it suffices to show that

$$(h_E \circ \phi_\infty)(x)(s_\mu s_\mu^*) = (\pi^* \circ h)(x)(s_\mu s_\mu^*). \tag{2.4.6}$$

Since $\mu \in E^*$, we have $t_{\phi^{-1}(\mu)}t^*_{\phi^{-1}(\mu)} \in p_{E^0}D_Fp_{E^0}$. Then the right-hand side of (2.4.6) becomes

$$\begin{aligned}\pi^*(h(x))(s_\mu s_\mu^*) &= (h(x) \circ \pi)(s_\mu s_\mu^*)\\ &= h(x)(t_{\phi^{-1}(\mu)}t^*_{\phi^{-1}(\mu)})\\ &= h_F(x)|_{pD_Fp}(t_{\phi^{-1}(\mu)}t^*_{\phi^{-1}(\mu)}) \qquad\qquad \text{since } r(x) \in E^0\\ &= \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\phi^{-1}(\mu)) \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We break the left-hand side of (2.4.6) into cases:

(i) $\phi_\infty(x) \in E^\infty$, or
(ii) $\phi_\infty(x) \in E^*$.

In case (i), the left-hand side of (2.4.6) becomes

$$h_E(\phi_\infty(x))(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \phi_\infty(x) \in \mathcal{Z}(\mu) \\ 0 & \text{otherwise.}\end{cases}$$

Since

$$\begin{aligned}\phi_\infty(x) \in \mathcal{Z}(\mu) &\iff \phi_\infty(x) = \mu\mu' \qquad \text{for some } \mu' \in E^\infty\\ &\iff x = \phi_\infty^{-1}(\mu\mu') = \phi^{-1}(\mu)\phi_\infty^{-1}(\mu'),\end{aligned}$$

we have

$$h_E(\phi_\infty(x))(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\phi^{-1}(\mu)) \\ 0 & \text{otherwise} \end{cases}$$

as required.

In case (ii), $\phi_\infty(x) = \phi(x')$, where $x = x'\nu$ for some collapsible path $\nu \in M$. The left hand side of (2.4.6) then becomes

$$h_E(\phi(x'))(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \phi(x') \in \mathcal{Z}(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

Since ϕ is a bijection, $x' = \phi^{-1}(\mu)x'' \iff \phi(x') = \mu\phi(x'')$, so equation (2.4.6) is satisfied, and thus $h_E \circ \phi_\infty(x) = \pi^* \circ h(x)$. $\square$

CHAPTER 3

Higher-Rank Graphs

Higher-rank graphs are a higher-dimensional analogue of directed graphs. They were introduced by Kumjian and Pask in [**12**] as a visual model for the higher-rank Cuntz-Krieger algebras of Robertson and Steger [**28**]. A higher-rank graph (or k-graph) is a countable category Λ with a degree functor d satisfying the factorisation property (Definition 3.0.6). We think of $d : \Lambda \to \mathbb{N}^k$ as a generalisation of the notion of length. In this section we construct a locally compact Hausdorff topology on the path space of a higher-rank graph. Since our motivation comes from higher-rank graph C^*-algebras, and since C^*-algebras have only been associated to *finitely aligned* (Definition 3.0.13) k-graphs to date, we consider only finitely aligned k-graphs here.

As yet there is no analogue of Drinen and Tomforde's desingularisation procedure for finitely aligned higher-rank graphs. Farthing detailed a partial desingularisation of high-rank graphs [**7**] which is analogous to the 'adding a head' construction in [**14**]. However, the complexity of higher-rank graphs makes even this a comparatively complicated task. In this section, we develop a modification of Farthing's desourcification technique, and use it extend the results about directed graphs from Chapter 2 to the higher-rank graph setting. As a side benefit, our construction seems to have led to simplified proofs of Farthing's results for arbitrary *row-finite* (Definition 3.0.13) higher-rank graphs. The difference between Farthing's construction and ours is that our construction is based on a set of paths $\partial\Lambda$ in the higher-rank graph which we call the *boundary paths* (Definition 3.0.15), whereas Farthing's is based on the set of paths $\Lambda^{\leq\infty}$ (Definition 3.0.15). Various notions of boundary paths have appeared in the literature, and as a part of our analysis, we discuss the relationship between them.

We view $\mathbb{N}^k$ as a category with $\mathrm{Obj}(\mathbb{N}^k) = \{\star\}$, $\mathrm{Mor}(\mathbb{N}^k) = \mathbb{N}^k$ and with composition defined by addition.

Definition 3.0.6. Given $k \in \mathbb{N}$, a *graph of rank k* (or k-graph) is a pair (Λ, d) consisting of a countable category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, which satisfies the *factorisation property*: for every $\lambda \in \mathrm{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. Elements $\lambda \in \mathrm{Mor}(\Lambda)$ are called *paths*. We follow the usual abuse of notation, and write $\lambda \in \Lambda$ to mean $\lambda \in \mathrm{Mor}(\Lambda)$. For $m \in \mathbb{N}^k$ we define $\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m\}$. For a subset $F \subset \Lambda$, and $V \subset \mathrm{Obj}(\Lambda)$, we write $VF := \{\lambda \in F : r(\lambda) \in V\}$ and $FV := \{\lambda \in F : s(\lambda) \in V\}$. If $V = \{v\}$, we drop the braces and write vF and Fv. A morphism between two k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ which respects the degree maps.

Remark 3.0.7. Fix $v \in \mathrm{Obj}(\Lambda)$. Since $(\mathrm{id}_v)^2 = \mathrm{id}_v$, that d is a functor forces $d(\mathrm{id}_v) = 0$. Now suppose that $\lambda \in v\Lambda^0$. Then $\mathrm{id}_{r(\lambda)} \lambda = \lambda = \lambda \, \mathrm{id}_{s(\lambda)}$. Since $d(\lambda) = 0 + 0$, and since $d(\mathrm{id}_{r(\lambda)}) = 0 = d(\mathrm{id}_{s(\lambda)})$, the factorisation property implies that $\mathrm{id}_v = \mathrm{id}_{r(\lambda)} = \lambda$. So $v\Lambda^0 = \{\mathrm{id}_v\}$, and we henceforth identify $\mathrm{Obj}(\Lambda)$ with Λ^0. We refer to elements of Λ^0 as *vertices*.

Remark 3.0.8. Recall from the introduction that to visualise a k-graph we draw its 1-*skeleton*: a directed graph with vertices Λ^0 and edges $\bigcup_{i=1}^k \Lambda^{e_i}$. To each edge we assign a colour determined by the edge's degree. In this thesis we tend to use 2-graphs for examples, and we draw edges of degree $(1, 0)$ as solid lines, and edges of degree $(0, 1)$ as dashed lines. In the literature these are often referred to as "blue" and "red" edges.

Example 3.0.9. For $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, we define k-graphs $\Omega_{k,m}$ as follows. Set $\mathrm{Obj}(\Omega_{k,m}) = \{p \in \mathbb{N}^k : p_i \leq m_i \text{ for all } i \leq k\}$,

$$\mathrm{Mor}(\Omega_{k,m}) = \{(p, q) : p, q \in \mathrm{Obj}(\Omega_{k,m}) \text{ and } p_i \leq q_i \text{ for all } i \leq k\},$$

$r(p, q) = p$, $s(p, q) = q$ and $d(p, q) = q - p$, with composition given by $(p, q)(q, t) = (p, t)$. If $m = (\infty)^k$, we drop m from the subscript and write Ω_k. The 1-skeleton of Ω_2 is depicted in Figure 1.

Remark 3.0.10. The graphs $\Omega_{k,m}$ provide an intuitive model for paths. Every path λ of degree m in a k-graph Λ determines a k-graph morphism $x_\lambda : \Omega_{k,m} \to \Lambda$. To see this, let $p, q \in \mathbb{N}^k$ be such that $p \leq q \leq m$. Define $x_\lambda(p, q) = \lambda''$, where $\lambda = \lambda'\lambda''\lambda'''$; and $d(\lambda') = p$, $d(\lambda'') = q - p$ and $d(\lambda''') = m - q$. In this way, paths in Λ are often identified with the graph morphisms $x_\lambda : \Omega_{k,m} \to \Lambda$. This provides convenient notation for referring to segments of paths. For example, we refer to the segment λ'' of λ (as factorized above) as $\lambda(p, q)$, and for $n \leq m$, we refer to the vertex $r(\lambda(n, m)) = s(\lambda(0, n))$ as $\lambda(n)$. By analogy, for $m \in (\mathbb{N} \cup \{\infty\})^k$ we define $\Lambda^m := \{x : \Omega_{k,m} \to \Lambda : x \text{ is a graph morphism.}\}$. For clarity of notation, if $m = (\infty)^k$ we write Λ^∞.

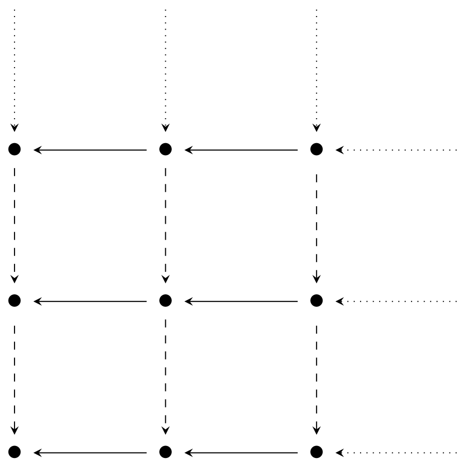


FIGURE 1. The 2-graph Ω_2.

Define

$$W_\Lambda := \bigcup_{n \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^n.$$

We call W_Λ the *path space* of Λ. We drop the subscript when confusion is unlikely.

Since finite and infinite paths are fundamentally different objects, that one can compose finite paths with a infinite paths isn't immediately obvious. The following proposition shows how to do so.

For $m, n \in \mathbb{N}^k$, we denote by $m \wedge n$ the coordinate-wise minimum, and by $m \vee n$ the coordinate-wise maximum. For example, $(4, 3, 2) \vee (0, 4, 1) = (4, 4, 2)$, and $(4, 3, 2) \wedge (0, 4, 1) = (0, 3, 1)$. With no parentheses, $\vee$ and $\wedge$ take priority over the group operation: $a - b \wedge c$ means $a - (b \wedge c)$.

PROPOSITION 3.0.11. *Let Λ be a k-graph. Suppose $\lambda \in \Lambda$ and suppose that $x \in W_\Lambda$ satisfies $r(x) = s(\lambda)$. Then there exists a unique k-graph morphism $\lambda x : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $(\lambda x)(d(\lambda), n + d(\lambda)) = x(0, n)$ for all $n \leq d(x)$.*

PROOF. Fix $p, q \in \mathbb{N}^k$ with $p \leq q \leq d(\lambda) + d(x)$. Since $q, d(\lambda) \leq d(\lambda) + d(x)$, we have $q \vee d(\lambda) \leq d(\lambda) + d(x)$. So $0 \leq q \vee d(\lambda) - d(\lambda) \leq d(x)$. Since $\lambda x(0, q \vee d(\lambda) - d(\lambda)) \in \Lambda$, it can be viewed as a k-graph morphism from $\Omega_{k,q \vee d(\lambda)}$ into Λ. We then define $\lambda x : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ by

$$(\lambda x)(p, q) := \big(\lambda x(0, q \vee d(\lambda) - d(\lambda))\big)(p, q).$$

Clearly, λx is a k-graph morphism.

To see that $(\lambda x)(0, d(\lambda)) = \lambda$, we calculate

$$\begin{aligned}(\lambda x)(0, d(\lambda)) &= \big(\lambda x(0, d(\lambda) \vee d(\lambda) - d(\lambda))\big)(0, d(\lambda)) \\ &= (\lambda r(x))(0, d(\lambda)) \\ &= \lambda.\end{aligned}$$

To see that $(\lambda x)(d(\lambda), n+d(\lambda)) = x(0, n)$ for all $n \leq d(x)$, fix $n \leq d(x)$ and calculate

$$\begin{aligned}&(\lambda x)(d(\lambda), n + d(\lambda)) \\ &\qquad = \big(\lambda x(0, (n + d(\lambda)) \vee d(\lambda) - d(\lambda))\big)(d(\lambda), n + d(\lambda)) \\ &\qquad = (\lambda x(0, n))(d(\lambda), n + d(\lambda)) \\ &\qquad = x(0, n).\end{aligned}$$

For uniqueness, suppose that $\phi : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ is a k-graph morphism satisfying $\phi(0, d(\lambda)) = \lambda$ and $\phi(d(\lambda), n + d(\lambda)) = x(0, n)$ for all $n \leq d(x)$. Then for $p \leq q \leq d(\lambda) + d(x)$, we have

$$\begin{aligned}\phi(p, q) &= \big(\phi(0, q \vee d(\lambda))\big)(p, q) \\ &= \big(\lambda\phi(d(\lambda), q \vee d(\lambda))\big)(p, q) \\ &= \big(\lambda x(0, q \vee d(\lambda) - d(\lambda))\big)(p, q) \\ &= (\lambda x)(p, q).\end{aligned}$$

□

DEFINITION 3.0.12. For $\lambda, \mu \in \Lambda$, we write

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$$

for the collection of pairs which give *minimal common extensions* of λ and μ, and denote the set of minimal common extensions by

$$\mathrm{MCE}(\lambda, \mu) := \{\lambda\alpha : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)\} = \{\mu\beta : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)\}.$$

DEFINITION 3.0.13. A k-graph Λ is *row-finite* if for each $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, the set $v\Lambda^m$ is finite; Λ has *no sources* if $v\Lambda^m \neq \emptyset$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

We say that Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

As in [**21**, Definition 3.1], a k-graph Λ is *locally convex* if for all $v \in \Lambda^0$, all $i, j \in \{1, \ldots k\}$ with $i \neq j$, all $\lambda \in v\Lambda^{e_i}$ and all $\mu \in v\Lambda^{e_j}$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty. Roughly speaking, local convexity stipulates that Λ contains no subgraph resembling:

$$\begin{tikzcd} u \arrow[d, dashed, "\mu"'] & \\ v & w \arrow[l, "\lambda"] \end{tikzcd}$$

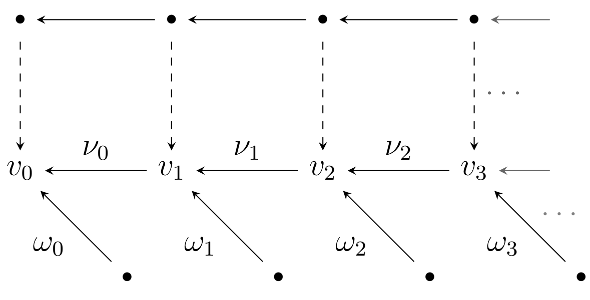
Definition 3.0.14. For $v \in \Lambda^0$, a subset $E \subset v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$ (or equivalently $\mathrm{MCE}(\lambda, \mu) \neq \emptyset$). We denote the set of all *finite exhaustive subsets* of Λ by $\mathcal{FE}(\Lambda)$, and for $v \in \Lambda^0$, we write $v\mathcal{FE}(\Lambda)$ for the set $\{E \in \mathcal{FE}(\Lambda) : E \subset v\Lambda\}$.

Definition 3.0.15. An element $x \in W$ is a *boundary path* if for all $n \in \mathbb{N}^k$ with $n \leq d(x)$ and for all $E \in x(n)\mathcal{FE}(\Lambda)$ there exists $m \in \mathbb{N}^k$ such that $x(n, n+m) \in E$. We write $\partial\Lambda$ for the set of all boundary paths, and for $v \in \Lambda^0$, write $v\partial\Lambda$ for $\{x \in \partial\Lambda : r(x) = v\}$.

We define the set $\Lambda^{\leq\infty}$ as follows. A k-graph morphism $x : \Omega_{k,m} \to \Lambda$ is an element of $\Lambda^{\leq\infty}$ if there exists $n_x \leq d(x)$ such that for $n \in \mathbb{N}^k$ satisfying $n_x \leq n \leq d(x)$ and $n_i = d(x)_i$, we have $x(n)\Lambda^{e_i} = \emptyset$.

The set $\Lambda^{\leq\infty}$ of Definition 3.0.15 was introduced by Raeburn, Sims and Yeend in [**22**], and its elements were referred to there as "boundary paths". These "boundary paths" were used in [**22**] to construct a nonzero Cuntz-Krieger Λ-family [**22**, Proposition 2.12]. Farthing, Muhly and Yeend introduced the set $\partial\Lambda$ of Definition 3.0.15 in [**8**]; in order to construct a groupoid to which Renault's theory of groupoid C^*-algebras [**24**] applied, they required a path space which was locally compact and Hausdorff in an appropriate topology, and $\Lambda^{\leq\infty}$ did not suffice. The differences between $\partial\Lambda$ and $\Lambda^{\leq\infty}$ can be easily seen if Λ contains any infinite receivers (e.g. any path in a 1-graph Λ with source an infinite receiver is an element of $\partial\Lambda \setminus \Lambda^{\leq\infty}$), but can even show itself in the row-finite case if Λ is not locally convex.

Example 3.0.16. Suppose Λ is the 2-graph with the skeleton pictured below.



Consider the paths $\nu = \nu_0\nu_1\ldots$, and $\omega^n = \nu_0\nu_1\ldots\nu_{n-1}\omega_n$, $n = 0, 1, 2, \ldots$. Observe that $\nu \notin \Lambda^{\leq\infty}$: for each $n \in \mathbb{N}$, we have $d(\nu)_2 = 0 = (n, 0)_2$, and $\nu((n, 0))\Lambda^{e_2} = v_n\Lambda^{e_2} \neq \emptyset$.

We claim that $\nu \in \partial\Lambda$. We have $d(\nu) = (\infty, 0)$, so to prove that $\nu \in \partial\Lambda$ we must show that for each $m \in \mathbb{N}$, and each $E \in \nu((m,0))\mathcal{FE}(\Lambda) = v_m\mathcal{FE}(\Lambda)$, there exists $p \in \mathbb{N}$ such that $\nu((m,0),(m+p,0)) \in E$. Fix $m \in \mathbb{N}$ and $E \in v_m\mathcal{FE}(\Lambda)$. Since E is exhaustive, for each $n \geq m$, there exists $\lambda^n \in E$ such that $\mathrm{MCE}(\lambda^n, \nu_m \dots \nu_{n-1}\omega_n) \neq \emptyset$. Since each $s(\omega_n)$ receives no edges, each λ^n must be an initial segment of $\nu_m \dots \nu_{n-1}\omega_n$: either $\lambda^n = \nu_m \dots \nu_{n-1}\omega_n$, or λ^n takes the form $\nu_m \dots \nu_p$ for $m \leq p < n$. Since E is finite, it cannot contain $\nu_m \dots \nu_{n-1}\omega_n$ for every $n \geq m$, so it must contain $\nu_m \dots \nu_p$ for some $p \in \mathbb{N}$. So $\nu((m,0),(m+p)) = \nu_m \dots \nu_p$ belongs to E.

Remark 3.0.17. The 2-graph of example 3.0.16 first appeared in Robertson's honours thesis [**26**] to illustrate a subtlety arising in Farthing's procedure [**7**] for removing sources in k-graphs when the k-graphs in question are not locally convex. It was for this reason that Robertson restricted his attention to locally convex k-graphs in the main results of [**26**]. As it turns out, the issue arises precisely because Farthing's construction notionally extends each element of $\Lambda^{\leq\infty}$ to an infinite path, but not elements of $\partial\Lambda$. It was this which motivated us to develop the construction we use in Section 3.2 where $\partial\Lambda$ replaces $\Lambda^{\leq\infty}$ in Farthing's scheme.

Proposition 3.0.18. *Suppose Λ is a finitely aligned k-graph. Then $\Lambda^{\leq\infty} \subset \partial\Lambda$. If Λ is row-finite and locally convex, then $\Lambda^{\leq\infty} = \partial\Lambda$.*

To prove this we use the following lemma.

Lemma 3.0.19. *Let Λ be a row-finite, locally convex k-graph, and suppose that $v \in \Lambda^0$ satisfies $v\Lambda^{e_i} \neq \emptyset$ for some $i \leq k$. Then $v\Lambda^{e_i} \in v\mathcal{FE}(\Lambda)$.*

Proof. Since Λ is row-finite, $v\Lambda^{e_i}$ is finite. To see that it is exhaustive, let $\mu \in v\Lambda$. If $d(\mu)_i > 0$, then $g = \mu(0, e_i) \in v\Lambda^{e_i}$ implies that $\Lambda^{\min}(\mu, g) \neq \emptyset$. Suppose $d(\mu)_i = 0$. Let $\mu = \mu_1 \dots \mu_n$ be a factorisation of μ such that $|d(\mu_j)| = 1$ for each $j \leq n$. Since Λ is locally convex, $r(\mu_1)\Lambda^{e_i} = v\Lambda^{e_i} \neq \emptyset$ implies that $s(\mu_1)\Lambda^{e_i} \neq \emptyset$. Then $s(\mu_1)\Lambda^{e_i} \neq \emptyset$ implies that $s(\mu_2)\Lambda^{e_i} \neq \emptyset$. Continuing in this way, we see that $s(\mu)\Lambda^{e_i} = s(\mu_n)\Lambda^{e_i} \neq \emptyset$. Fix $g \in s(\mu)\Lambda^{e_i}$. Let $f := (\mu g)(0, e_i)$. Then $f \in v\Lambda^{e_i}$. Since $d(\mu_i) = 0$, we have $d(\mu g) = d(\mu) \vee d(f)$. Hence $(g, (\mu g)(e_i, d(\mu g))) \in \Lambda^{\min}(\mu, f)$ as required. $\square$

Proof of Proposition 3.0.18. Fix $x \in \Lambda^{\leq\infty}$. Then there exists $n_x \in \mathbb{N}^k$ such that $n_x \leq d(x)$, and whenever $n \in \mathbb{N}^k$ satisfies $n_x \leq n \leq d(x)$,

$$n_i = d(x)_i \Longrightarrow x(n)\Lambda^{e_i} = \emptyset. \tag{3.0.7}$$

To see $x \in \partial\Lambda$, we show that for all $m \leq d(x)$ and all $E \in x(m)\mathcal{FE}(\Lambda)$, there exists $\lambda \in E$ such that $m + d(\lambda) \leq d(x)$ and $x(m, m + d(\lambda)) = \lambda$.

Fix $m \le d(x)$ and $E \in x(m)\mathcal{FE}(\Lambda)$. Define $t \in \mathbb{N}^k$ by

$$t_i := \begin{cases} d(x)_i & \text{if } d(x)_i < \infty, \\ \max_{\lambda \in E} \big(n_x \vee (m + d(\lambda))\big)_i & \text{if } d(x)_i = \infty. \end{cases}$$

Then $x(m,t) \in x(m)\Lambda$, so there exists $\lambda \in E$ such that $\Lambda^{\min}(x(m,t),\lambda)$ is nonempty. Let $\alpha, \beta \in \Lambda^{\min}(x(m,t),\lambda)$. We first show that $d(\alpha) = 0$. Since $x \in \Lambda^{\le\infty}$ and $n_x \le t \le d(x)$, we have $d(x)_i < \infty$ implies that $x(t)\Lambda^{e_i} = \emptyset$. So for each i such that $d(x)_i < \infty$, we have $d(\alpha)_i = 0$. Now suppose that $d(x)_i = \infty$. Then $d(x(m,t))_i = t_i - m_i \ge d(\lambda)_i$. So $d(x(m,t)\alpha)_i = \max\{d(x(m,t))_i, d(\lambda)_i\} = d(x(m,t))_i$, giving $d(\alpha)_i = 0$. Then we have $x(m,t) = \lambda\beta$, so $x(m, m + d(\lambda)) = \lambda$.

Now suppose Λ is row-finite and locally convex. We want to show $\partial\Lambda \subset \Lambda^{\le\infty}$. Fix $x \in \partial\Lambda$, and $n \in \mathbb{N}^k$ such that $n \le d(x)$ and $n_i = d(x)_i$. It suffices to show that $x(n)\Lambda^{e_i} = \emptyset$, for then $n_x = 0$ satisfies (3.0.7). Since $n_i = d(x)_i$, we have $x(n)\Lambda^{e_i} \notin x(n)\mathcal{FE}(\Lambda)$. Lemma 3.0.19 then implies that $x(n)\Lambda^{e_i} = \emptyset$. □

3.1. Topology

Our first aim is to construct a locally compact Hausdorff topology on the path space W of a finitely aligned k-graph Λ. As we did for directed graphs, we follow the approach of Paterson and Welch in [**17**]. We show that the sets in Theorem 3.1.1 are precisely the inverse images of basic open sets in $\{0,1\}^\Lambda$ (equipped with the product topology) under the map $\alpha : W \to \{0,1\}^\Lambda$ defined by

$$\alpha(w)(y) = \begin{cases} 1 & \text{if } w = yw' \text{ for some } w' \in W, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.1}$$

For $\mu \in \Lambda$ the *cylinder set of* μ is

$$\mathcal{Z}(\mu) := \{\nu \in W : \nu(0, d(\mu)) = \mu\}.$$

For a finite subset $G \subset s(\mu)\Lambda$, we define

$$\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu). \tag{3.1.2}$$

Our goals for this section are the following two theorems.

THEOREM 3.1.1. *Let Λ be a finitely aligned k-graph. Then the collection*

$$\Big\{\mathcal{Z}(\mu \setminus G) : \mu \in \Lambda \text{ and } G \subset \bigcup_{i=1}^{k} (s(\mu)\Lambda^{e_i}) \text{ is finite}\Big\}$$

form a basis for the initial topology on W induced by $\{\alpha\}$.

THEOREM 3.1.2. *Let Λ be a finitely-aligned higher-rank graph. The topology on W generated by the basic open sets given by Theorem 3.1.1 is a locally compact Hausdorff topology.*

We first require some definitions and a lemma. Let F be a set of paths in a k-graph Λ. A path $\beta \in W$ is a *common extension of the paths in F* if for each $\mu \in F$, we can write $\beta = \mu\beta_\mu$ for some $\beta_\mu \in W$. If in addition $d(\beta) = \bigvee_{\mu \in F} d(\mu)$, then β is a *minimal common extension of the paths in F*. We denote the set of all minimal common extensions of the paths in F by $\mathrm{MCE}(F)$. Since $\mathrm{MCE}(\{\mu,\nu\}) = \mathrm{MCE}(\mu,\nu)$, this definition is consistent with Definition 3.0.12.

LEMMA 3.1.3. *Let F be a finite set of paths in a k-graph Λ. Then*

$$\bigcap_{\mu \in F} \mathcal{Z}(\mu) = \bigcup_{\beta \in \mathrm{MCE}(F)} \mathcal{Z}(\beta).$$

PROOF. Let $\alpha \in \bigcup_{\beta \in \mathrm{MCE}(F)} \mathcal{Z}(\beta)$. Then there exists $\beta \in \mathrm{MCE}(F)$ such that $\alpha = \beta\alpha'$. So for all $\mu \in F$ we have $\alpha = \mu\beta_\mu\alpha' \in \mathcal{Z}(\mu)$. Thus $\alpha \in \bigcap_{\mu \in F} \mathcal{Z}(\mu)$.

Now suppose $\alpha \in \bigcap_{\mu \in F} \mathcal{Z}(\mu)$. So $\alpha \in \mathcal{Z}(\mu)$ for all $\mu \in F$. This implies that α is a common extension of paths in F. Let $\beta = \alpha\left(0, \bigvee_{\mu \in F} d(\mu)\right)$. Then $\beta \in \mathrm{MCE}(F)$ and $\alpha \in \mathcal{Z}(\beta)$. $\square$

PROOF OF THEOREM 3.1.1. We first describe the topology on $\{0,1\}^\Lambda$. Given disjoint finite subsets $F, G \subset \Lambda$ and $\mu \in \Lambda$, define

$$U_\mu^{F,G} = \begin{cases} \{1\} & \text{if } \mu \in F, \\ \{0\} & \text{if } \mu \in G, \\ \{0,1\} & \text{otherwise.} \end{cases}$$

Then the sets

$$N(F,G) := \prod_{\mu \in \Lambda} U_\mu^{F,G}$$

where F, G range over all finite disjoint pairs of subsets of Λ form a basis for the topology on $\{0,1\}^\Lambda$.

An identical argument to that used in Proposition 2.1.3 shows that α is a homeomorphism onto its range, and hence the sets $\alpha^{-1}(N(F,G))$ are a basis for a topology on W. These sets can be described as follows.

$$\begin{aligned}
\lambda \in \alpha^{-1}(N(F,G)) &\iff \alpha(\lambda) \in N(F,G) \\
&\iff \alpha(\lambda)(\mu) = \begin{cases} 1 & \text{for } \mu \in F \\ 0 & \text{for } \mu \in G \end{cases} \\
&\iff \begin{cases} \lambda & \in \mathcal{Z}(\mu) \text{ for } \mu \in F, \\ \lambda & \notin \mathcal{Z}(\nu) \text{ for } \nu \in G. \end{cases} \\
&\iff \lambda \in \left(\bigcap_{\mu \in F} \mathcal{Z}(\mu) \right) \setminus \left(\bigcup_{\nu \in G} \mathcal{Z}(\nu) \right) \\
&\iff \lambda \in \left(\bigcup_{\mu \in \mathrm{MCE}(F)} \mathcal{Z}(\mu) \right) \setminus \left(\bigcup_{\nu \in G} \mathcal{Z}(\nu) \right).
\end{aligned}$$

Since each $\alpha^{-1}(N,(F,G)) = \bigcup_{\lambda \in F} \alpha^{-1}(N(\{\lambda\},G))$, we only need the sets F containing a single element. If we set $G' = G \cap \mathcal{Z}(\mu)$ then

$$\mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\nu) = \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G'} \mathcal{Z}(\nu),$$

so the sets $\alpha^{-1}(N(\{\lambda\},G))$, where $G \subset \mathcal{Z}(\mu)$, are a basis for the same topology. Furthermore, by appropriately adjusting G (i.e. taking $G' = \{\nu : \mu\nu \in G\}$ then relabeling $G = G'$), the basis sets $\mathcal{Z}(\mu) \setminus \left(\bigcup_{\nu \in G} \mathcal{Z}(\nu) \right)$ become

$$\mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu) = \mathcal{Z}(\mu \setminus G),$$

as defined in equation (3.1.2). To finish the proof, it suffices to show that for $\mu \in \Lambda$, a finite subset $G \subset s(\mu)\Lambda$ and $\lambda \in \mathcal{Z}(\mu \setminus G)$, there exist $\alpha \in \Lambda$ and a finite $F \subset \bigcup_{i=1}^{k} \left(s(\alpha)\Lambda^{e_i} \right)$ such that

$$\lambda \in \mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G).$$

Let $N := \left(\bigvee_{\nu \in G} d(\mu\nu) \right) \wedge d(\lambda)$ and $\alpha = \lambda(0, N)$. To define F, we first define a set F_ν associated to each $\nu \in G$, then take $F = \bigcup_{\nu \in G} F_\nu$. Fix $\nu \in G$. We consider the following cases:

(1) If $N \geq d(\mu\nu)$, let $F_\nu = \emptyset$.
(2) If $N \not\geq d(\mu\nu)$, then either
  (a) $\mathrm{MCE}(\alpha, \mu\nu) = \emptyset$, in which case let $F_\nu = \emptyset$, or
  (b) $\mathrm{MCE}(\alpha, \mu\nu) \neq \emptyset$, which requires a little more work:

Since $N \not\geq d(\mu\nu)$, there exists $j_\nu \leq k$ such that $N_{j_\nu} < d(\mu\nu)_{j_\nu}$. Hence each $\gamma \in \mathrm{MCE}(\alpha, \mu\nu)$ satisfies $d(\gamma)_{j_\nu} = (N \vee d(\mu\nu))_{j_\nu} > N_{j_\nu}$. Define $F_\nu = \{\gamma(N, N + e_{j_\nu}) : \gamma \in \mathrm{MCE}(\alpha, \mu\nu)\}$. Since Λ is finitely aligned, F_ν is finite.

We now show that $\lambda \in \mathcal{Z}(\alpha \setminus F)$. We have $\lambda \in \mathcal{Z}(\alpha)$ by choice of α. If $F = \emptyset$ we are done. If not, then fix $\nu \in G$ such that $F_\nu \neq \emptyset$, and fix $e \in F_\nu$. We will show that $\lambda \notin \mathcal{Z}(\alpha e)$. We have $e = \gamma(N, N + e_{j_\nu})$ for some $\gamma \in \mathrm{MCE}(\alpha, \mu\nu)$. Since $N_{j_\nu} < d(\mu\nu)_{j_\nu} \leq \big(\bigvee_{\nu \in G} d(\mu\nu)\big)_{j_\nu}$, we have $d(\lambda)_{j_\nu} = N_{j_\nu} < (N + e_{j_\nu})_{j_\nu} = d(\alpha e)_{j_\nu}$. This implies that $\lambda \notin \mathcal{Z}(\alpha e)$. Since this holds for all $\nu \in G$ and $e \in F_\nu$, we have $\lambda \in \mathcal{Z}(\alpha \setminus F)$.

We now show that $\mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G)$. Fix $\beta \in \mathcal{Z}(\alpha \setminus F)$. Since $\alpha \in \mathcal{Z}(\mu)$, we have $\beta \in \mathcal{Z}(\mu)$. Fix $\nu \in G$. We will show that $\beta \notin \mathcal{Z}(\mu\nu)$. We argue the following cases separately:

(1) Suppose that $N \geq d(\mu\nu)$. Since $\beta \in \mathcal{Z}(\alpha) = \mathcal{Z}(\lambda(0, N))$ and $\lambda \notin \mathcal{Z}(\mu\nu)$, it follows that $\beta \notin \mathcal{Z}(\mu\nu)$.
(2) If $N \not\geq d(\mu\nu)$, then either
   (a) $\mathrm{MCE}(\alpha, \mu\nu) = \emptyset$, in which case $\beta \in \mathcal{Z}(\alpha)$ implies that $\beta \notin \mathcal{Z}(\mu\nu)$; or
   (b) $\mathrm{MCE}(\alpha, \mu\nu) \neq \emptyset$, which requires a little more work:

Suppose, for a contradiction, that $\beta \in \mathcal{Z}(\mu\nu)$. Then $\beta = \gamma\beta'$ for some $\gamma \in \mathrm{MCE}(\alpha, \mu\nu)$. By our choice of F, we have $\beta(N, N + e_{j_\nu}) \neq \gamma(N, N + e_{j_\nu})$. So $\beta(0, d(\gamma)) \neq \gamma$, a contradiction. Hence $\beta \notin \mathcal{Z}(\mu\nu)$.

Since this holds for all $\nu \in G$, we have $\beta \in \mathcal{Z}(\mu \setminus G)$. □

For the proof of Theorem 3.1.2, we use the following technical results.

LEMMA 3.1.4. *Let $\{\nu^{(n)}\}$ be a sequence of paths in Λ such that*

(i) *$d(\nu^{(n+1)}) \geq d(\nu^{(n)})$ for all $n \in \mathbb{N}$, and*
(ii) *$\nu^{(n+1)}\big(0, d(\nu^{(n)})\big) = \nu^{(n)}$ for all $n \in \mathbb{N}$.*

Then there exists a unique $\omega \in W$ such that $d(\omega) = \bigvee_{n \in \mathbb{N}} d(\nu^{(n)})$ and $\omega\big(0, d(\nu^{(n)})\big) = \nu^{(n)}$ for all $n \in \mathbb{N}$.

PROOF. Let $m = \bigvee_{n \in \mathbb{N}} d(\nu^{(n)}) \in (\mathbb{N} \cup \{\infty\})^k$.

CLAIM 3.1.4.1. *For $a \in \mathbb{N}^k$ with $a \leq m$, there exists $N_a \in \mathbb{N}$ such that $d(\nu^{(N_a)}) \geq a$.*

PROOF. Let $q \in \{1, \dots, k\}$. Since $a \leq m$, there exists $t_q \in \mathbb{N}$ such that $d(\nu^{(t_q)}) \geq a_q$. Let $N_a = \max\{t_1, \dots, t_q\}$. Then $d(\nu^{(n+1)}) \geq d(\nu^{(n)})$ implies that $d(\nu^{(N_a)}) \geq a$. $\square_{\text{Claim}}$

We now define ω and show that it has the required properties. For each $(p, q) \in \Omega_{k,m}$ apply Claim 3.1.4.1 with $a = q$ and define $\omega(p, q) = \nu^{(N_q)}(p, q)$.

CLAIM 3.1.4.2. *$\omega : \Omega_{k,m} \to \Lambda$ is a well-defined graph morphism.*

Proof. We first check that ω is well defined. To do this we show that it does not depend on the choice of N_q from Claim 3.1.4.1. Suppose that $(p, q) \in \mathrm{Mor}(\Omega_{k,m})$, and that $M, N \in \mathbb{N}$ satisfy $d(\nu^M), d(\nu^N) \geq q$. Without loss of generality we can assume that $M \geq N$. Condition (ii) from the Lemma hypothesis implies that $\nu^M(0, d(\nu^N)) = \nu^N$. Since $d(\nu^M), d(\nu^N) \geq q$, we have $\nu^M(p, q) = \nu^N(p, q)$. So ω is well-defined.

To see that ω is a functor, we use that $\nu^{(n)}$ is a functor: for each $(p, q) \in \mathrm{Mor}(\Omega_{k,m})$, we have

$$r(\omega(p, q)) = r(\nu^{(N_q)}(p, q)) = \nu^{(N_q)}(p), \text{ and}$$

$$\omega(r(p, q)) = \omega(p) = \nu^{(N_p)}(p).$$

Since $d(\nu^{(N_q)}), d(\nu^{(N_p)}) \geq p$, we have $\nu^{(N_q)}(p) = \nu^{(N_p)}(p)$. So $r(\omega(p, q)) = \omega(r(p, q))$. Similar calculations show that ω respects the source, identity morphism, and composition maps:

$$s(\omega(p, q)) = s(\nu^{(N_q)}(p, q)) = \nu^{(N_q)}(q) = \omega(q) = \omega(s(p, q)),$$

$$\omega(\mathrm{id}_p) = \omega(p, p) = \nu^{(N_p)}(p, p) = \nu^{(N_p)}(\mathrm{id}_p) = \mathrm{id}_{\nu^{(N_p)}(p)} = \mathrm{id}_{\omega(p)},$$

and

$$\begin{aligned} \omega\left((p, q) \circ (q, v)\right) &= \omega(p, v) \\ &= \nu^{(N_v)}(p, v) \\ &= \nu^{(N_v)}((p, q) \circ (q, v)) \\ &= \nu^{(N_v)}(p, q) \circ \nu^{(N_v)}(q, v) \\ &= \nu^{(N_q)}(p, q) \circ \nu^{(N_v)}(q, v) \\ &= \omega(p, q) \circ \omega(q, v). \end{aligned}$$

So ω is a functor.

To see ω is a graph morphism, we check that it preserves the degree map. This follows because the $\nu^{(n)}$ are graph morphisms:

$$d(\omega(p, q)) = d(\nu^{(N_q)}(p, q)) = q - p = d(p, q). \qquad \square_{\text{Claim}}$$

Since $d(\omega) = m = \bigvee_{n \in \mathbb{N}} d(\nu^{(n)})$ by definition, it remains only to show that $\omega(0, d(\nu^{(n)})) = \nu^{(n)}$ for all n, and that ω is the unique morphism with these properties. To see that $\omega\left(0, d(\nu^{(n)})\right) = \nu^{(n)}$ for all $n \in \mathbb{N}$, take $a = d(\nu^{(n)})$ in Claim 3.1.4.1.

This gives $M_n := N_{d(\nu^{(n)})}$ such that $d(\nu^{(M_n)}) \geq d(\nu^{(n)})$. Then hypothesis (ii) of the Lemma implies that

$$\omega\left(0, d(\nu^{(n)})\right) = \nu^{(M_n)}\left(0, d(\nu^{(n)})\right) = \nu^{(n)}.$$

To see that ω is unique, suppose that $\omega' \in \Lambda^m$ is such that $\omega'(0, d(\nu^{(n)})) = \nu^{(n)}$ for all $n \in \mathbb{N}$. Fix $p \leq m$. Claim 3.1.4.1 gives $N_p \in \mathbb{N}$ such that $d(\nu^{(N_p)}) \geq p$. Then $\omega'(0,p) = \nu^{(N_p)}(0,p) = \omega(0,p)$. Since this is true for all $p \leq m = d(\omega) = d(\omega')$, we have $\omega' = \omega$. $\square$

Proof of Theorem 3.1.2. Fix $v \in \Lambda^0$. We follow the strategy of [**17**, Theorem 2.2] to show $\mathcal{Z}(v)$ is compact. Proposition 2.1.3 implies that α is a homeomorphism onto its range, so it suffices to prove that $\alpha(\mathcal{Z}(v))$ is compact. Since $\{0,1\}^\Lambda$ is compact, we need only show that $\alpha(\mathcal{Z}(v))$ is closed in $\{0,1\}^\Lambda$. Suppose that $(\omega^{(n)})_{n\in\mathbb{N}}$ is a sequence in $\mathcal{Z}(v)$ such that $\alpha(\omega^{(n)}) \to f \in \{0,1\}^\Lambda$. We seek $\omega \in \mathcal{Z}(v)$ such that $f = \alpha(\omega)$.

Define $A = \{\nu \in \Lambda : \alpha(\omega^{(n)})(\nu) \to 1 \text{ as } n \to \infty\}$. Then $A \neq \emptyset$ since $v \in A$. Let $d(A) := \bigvee_{\nu\in A} d(\nu)$.

Claim 3.1.2.1. *There exists $\omega \in v\Lambda^{d(A)}$ such that:*

- *$d(\omega) \geq d(\mu)$ for all $\mu \in A$, and*
- *$\omega(0,n) \in A$ for all $n \in \mathbb{N}^k$ with $n \leq d(A)$.*

Proof. To define ω we construct a sequence of paths to which we will apply Lemma 3.1.4. We first show that for each pair $\mu, \nu \in A$, there exists a unique path $\beta_{\mu,\nu} \in \mathrm{MCE}(\mu,\nu) \cap A$. Fix $\mu, \nu \in A$. Then $\alpha(\omega^n)(\mu) \to 1$ and $\alpha(\omega^n)(\nu) \to 1$. So there exists N such that $n \geq N$ implies that $\omega^n = \mu\mu' = \nu\nu'$. So for each $n \geq N$, there exist $\beta^n \in \mathrm{MCE}(\mu,\nu)$ such that $\omega^n = \beta^n(\omega^n)'$. Since $\mathrm{MCE}(\mu,\nu)$ is finite, there exists M such that $\omega^n = \beta^M(\omega^n)'$ for infinitely many n. Denote these n by $\{n_k : k \in \mathbb{N}\}$, and define $\beta_{\mu,\nu} := \beta^M$. So $\alpha(\omega^{(n_k)})(\beta_{\mu,\nu}) = 1$ for all k. Since $\alpha(\omega^n)$ converges, it follows that for large enough n we have $\alpha(\omega^n)(\beta_{\mu,\nu}) = 1$. So $\beta_{\mu,\nu} \in A$. For uniqueness, suppose that $\phi \in \mathrm{MCE}(\mu,\nu) \cap A$. Then since $d(\phi) = d(\mu) \vee d(\nu)$ and $\phi \in A$, it follows that for large n we have

$$\beta_{\mu,\nu} = \omega^n(0, d(\mu) \vee d(\nu)) = \phi.$$

We now construct our sequence of paths. Since A is countable, we can list the elements of A as

$$A = \{\nu^1, \nu^2, \ldots, \nu^m, \ldots\}.$$

Let $y^1 := \nu^1$, then iteratively define $y^n = \beta_{y^{n-1},\nu^n}$. Then

$$d(y^n) = d(y^{n-1}) \vee d(\nu^n) \geq d(y^{n-1}),$$

and $y^n(0, y^{n-1}) = y^{n-1}$. By Lemma 3.1.4, there exists a unique $\omega \in W$ satisfying $d(\omega) = d(A)$ and $\omega(0, d(y^n)) = y^n$ for all n.

To see that $\omega(0, n) \in A$ for each $n \in \mathbb{N}^k$ with $n \leq d(A)$, fix such an n. Claim 3.1.4.1 implies that there exists $N_n \in \mathbb{N}$ such that $d(y^{N_n}) \geq n$. Then since each $y^m \in A$ by definition, for large enough m we have

$$\omega^m = y^{N_n}(\omega^m)' = \omega(0, n)(y^{N_n})'(\omega^m)'.$$

That is, $\omega(0, n) \in A$. $\square_{\text{Claim}}$

To see $\alpha(\mathcal{Z}(v))$ is closed, we show that $\alpha(\omega^{(n)}) \to \alpha(\omega)$ as $n \to \infty$. Fix $\lambda \in \Lambda$. We aim to show that $\alpha(\omega^{(n)})(\lambda) \to \alpha(\omega)(\lambda)$. Suppose that $\alpha(\omega)(\lambda) = 1$. Then $\lambda = \omega(0, d(\lambda)) \in A$ by Claim 3.1.2.1, and thus $\alpha(\omega^{(n)})(\lambda) \to 1$ as $n \to \infty$.

Now suppose that $\alpha(\omega)(\lambda) = 0$. We argue that $\alpha(\omega^{(n)})(\lambda) \to 0$ in two cases: either $d(\lambda) \leq d(\omega)$, or not. Suppose that $d(\lambda) \not\leq d(\omega)$. Then by Claim 3.1.2.1, $\lambda \notin A$. This means that $\alpha(\omega^{(n)})(\lambda) \not\to 1$. Since $\alpha(\omega^{(n)})$ converges to either 0 or 1, we must have $\alpha(\omega^{(n)})(\lambda) \to 0$.

Suppose that $d(\lambda) \leq d(\omega)$. Then since $\omega(0, d(\lambda)) \in A$, there exists N such that $n \geq N$ implies that $\omega^{(n)} = \omega(0, d(\lambda))\tau^{(n)}$. Furthermore, $\alpha(\omega)(\lambda) = 0$ implies that $\omega(0, d(\lambda)) \neq \lambda$. So for all $n \geq N$ we have $\omega^{(n)} \neq \lambda\tau^{(n)}$. This implies that $\alpha(\omega^{(n)})(\lambda) \not\to 1$. Since $\alpha(\omega^{(n)})$ converges, we must have $\alpha(\omega^{(n)})(\lambda) \to 0$. $\square$

3.2. Removing Sources

In this section, given a finitely aligned k-graph Λ, we construct a k-graph $\widetilde{\Lambda}$ with no sources which contains a subgraph isomorphic to Λ. In the sections following, we investigate how the boundary-path space of Λ relates to that of $\widetilde{\Lambda}$. We then show that if Λ is row-finite, then $C^*(\Lambda)$ is isomorphic to a full corner of $C^*(\widetilde{\Lambda})$. Our construction is modelled on Farthing's construction in [**7**], and thus most of the proofs are inspired by hers. The crucial difference is that our construction involves extending paths in $\partial\Lambda$, whereas Farthing's extends paths from $\Lambda^{\leq\infty}$. Interestingly, although $\partial\Lambda$ and $\Lambda^{\leq\infty}$ are potentially different when Λ is row-finite and not locally convex, our construction and Farthing's yield isomorphic k-graphs except in the non-row-finite case (see Examples 3.2.1 and Proposition 3.2.12).

We follow Robertson and Sims' notational refinement [**27**] of Farthing's desourcification: we construct a new k-graph in which the original k-graph is embedded, whereas Farthing's construction adds bits onto the existing k-graph. This simplifies many arguments involving $\widetilde{\Lambda}$.

One of our key goals was to show that there is a homeomorphism π from the space of infinite paths of $\widetilde{\Lambda}$ with range in the embedded copy of Λ to the boundary path space of Λ. Showing that this map is surjective proved difficult using Farthing's construction, and this was the primary motivation for developing a new one.

DEFINITION 3.2.1. Define a relation $\approx$ on $V_\Lambda := \{(x;m) : x \in \partial\Lambda, m \in \mathbb{N}^k\}$ by: $(x;m) \approx (y;p)$ if and only if

(V1) $x(m \wedge d(x)) = y(p \wedge d(y))$; and
(V2) $m - m \wedge d(x) = p - p \wedge d(y)$.

DEFINITION 3.2.2. Define a relation $\sim$ on $P_\Lambda := \{(x;(m,n)) : x \in \partial\Lambda, m \leq n \in \mathbb{N}^k\}$ by: $(x;(m,n)) \sim (y;(p,q))$ if and only if

(P1) $x(m \wedge d(x), n \wedge d(x)) = y(p \wedge d(y), q \wedge d(y))$;
(P2) $m - m \wedge d(x) = p - p \wedge d(y)$; and
(P3) $n - m = q - p$.

REMARK 3.2.3. It is clear from their definitions that both $\approx$ and $\sim$ are equivalence relations.

LEMMA 3.2.4. *Suppose that* $(x;(m,n)) \sim (y;(p,q))$. *Then* $n - n \wedge d(x) = q - q \wedge d(y)$.

PROOF. (P1) implies that $n \wedge d(x) - m \wedge d(x) = q \wedge d(y) - p \wedge d(y)$. Then (P3) implies that

$$n - m - (n \wedge d(x) - m \wedge d(x)) = p - q - (q \wedge d(y) - p \wedge d(y)).$$

Reordering the terms, we have

$$n - n \wedge d(x) - (m - m \wedge d(x)) = q - q \wedge d(y) - (p - p \wedge d(y)).$$

It then follows from (P2) that

$$n - n \wedge d(x) = q - q \wedge d(y).$$

□

Let $\widetilde{P_\Lambda} := P_\Lambda/\sim$ and $\widetilde{V_\Lambda} := V_\Lambda/\approx$. The class in $\widetilde{P_\Lambda}$ of $(x;(m,n)) \in P_\Lambda$ is denoted $[x;(m,n)]$, and similarly the class in $\widetilde{V_\Lambda}$ of $(x;m) \in V_\Lambda$ is denoted $[x;m]$.

To define range and source maps, observe that if $(x;(m,n)) \sim (y;(p,q))$, then $(x;m) \approx (y;p)$ by definition, and $(x;n) \approx (y;q)$ by Lemma 3.2.4. We define range and source maps as follows.

DEFINITION 3.2.5. Define $\widetilde{r}, \widetilde{s} : \widetilde{P_\Lambda} \to \widetilde{V_\Lambda}$ by:

$$\widetilde{r}([x;(m,n)]) = [x;m] \qquad \text{and}$$
$$\widetilde{s}([x;(m,n)]) = [x,n].$$

We now need to define composition. For each $m \in \mathbb{N}^k$, we define the *shift map* $\sigma^m : \bigcup_{n \geq m} \Lambda^n \to \Lambda$ by $\sigma^m(\lambda)(p,q) = \lambda(p+m, q+m)$. So σ^m essentially 'chops off' an initial segment of degree m.

PROPOSITION 3.2.6. *Suppose that Λ is a k-graph and let $[x;(m,n)]$ and $[y;(p,q)]$ be elements of $\widetilde{P_\Lambda}$ satisfying $[x;n] = [y;p]$. Let $z := x(0, n \wedge d(x))\sigma^{p \wedge d(y)}y$. Then*

(1) *$z \in \partial\Lambda$;*
(2) *$m \wedge d(x) = m \wedge d(z)$ and $n \wedge d(x) = n \wedge d(z)$;*
(3) *$x(m \wedge d(x), n \wedge d(x)) = z(m \wedge d(z), n \wedge d(z))$ and $y(p \wedge d(y), q \wedge d(y)) = z(n \wedge d(z), (n+q-p) \wedge d(z))$.*

PROOF. Part (1): By [**8**, Lemma 5.13(1)] we have $\sigma^{p \wedge d(y)}y \in \partial\Lambda$. Since $[x;n] = [y;p]$, (V1) says that $x(n \wedge d(x)) = y(p \wedge d(y))$. Then [**8**, Lemma 5.13(2)] implies that $z = x(0, n \wedge d(x))\sigma^{p \wedge d(y)}y \in \partial\Lambda$.

For part (2), we show that the equations hold coordinate-wise. Fix $i \leq k$. We will show that

$$\min\{m_i, d(x)_i\} = \min\{m_i, d(z)_i\} \text{ and} \tag{3.2.1}$$

$$\min\{n_i, d(x)_i\} = \min\{n_i, d(z)_i\}. \tag{3.2.2}$$

Since $[x;n] = [y;p]$, we have $n - n \wedge d(x) = p - p \wedge d(y)$. This implies that

$$p_i \leq d(y)_i \iff n_i \leq d(x)_i. \tag{3.2.3}$$

We argue in cases:

(i) $p_i > d(y)_i$, or
(ii) $p_i \leq d(y)_i$.

In case (i), equation (3.2.3) implies that $n_i > d(x)_i$. Then

$$\begin{aligned} d(z)_i &= \min\{n_i, d(x)_i\} + d(y)_i - \min\{p_i, d(y)_i\} \\ &= d(x)_i + d(y)_i - d(y)_i \\ &= d(x)_i. \end{aligned}$$

This gives both (3.2.1) and (3.2.2).

In case (ii), equation (3.2.3) implies that $n_i \leq d(x)_i$. This forces

$$d(z)_i = (n \wedge d(x))_i + d(y)_i - (p \wedge d(y))_i = d(y)_i + n_i - p_i \geq n_i \geq m_i,$$

and $m_i \leq n_i \leq d(x)_i$. Hence (3.2.1) and (3.2.2) are satisfied.

Part (3): Since $z(0, n \wedge d(x)) = x(0, n \wedge d(x))$, part (2) implies that

$$z(m \wedge d(z), n \wedge d(z)) = z(m \wedge d(x), n \wedge d(x)) = x(m \wedge d(x), n \wedge d(x)),$$

giving the first equality. We now prove that the second equality holds. Part (2) implies that

$$\sigma^{n\wedge d(z)}z = \sigma^{n\wedge d(x)}z = \sigma^{p\wedge d(y)}y,$$

so it suffices to show that

$$(n+q-p)\wedge d(z) - n\wedge d(z) = q\wedge d(y) - p\wedge d(y). \tag{3.2.4}$$

Since $d(z) = d(y)+n-p$, we have $(n+q-p)\wedge d(z) = q\wedge d(y)+n-p$. Furthermore, by part (2) we have $n\wedge d(z) = n\wedge d(x)$. So the left-hand side of (3.2.4) becomes

$$(n+q-p)\wedge d(z) - n\wedge d(z) = q\wedge d(y) - p + (n - n\wedge d(x)).$$

Since $[x;n] = [y;p]$, we have $n - n\wedge d(x) = p - p\wedge d(y)$, and thus

$$q\wedge d(y) - p + (p - p\wedge d(y)) = q\wedge d(y) - p\wedge d(y),$$

giving (3.2.4). □

Fix $[x;(m,n)], [y;(p,q)] \in \widetilde{P_\Lambda}$ such that $[x;n] = [y;p]$, and let $z = x(0, n\wedge d(x))\sigma^{p\wedge d(y)}y$. The purpose of the next lemma is to show that the formula

$$[x;(m,n)] \circ [y;(p,q)] = [z;(m,n+q-p)] \tag{3.2.5}$$

determines a well-defined composition.

Lemma 3.2.7. *Let $[x;(m,n)], [y;(m,n)] \in \widetilde{P_\Lambda}$ be such that $[x;n] = [y;p]$. Let $z = x(0,n\wedge d(x))\sigma^{p\wedge d(y)}y$. If $(x;(m,n)) \sim (x';(m',n'))$ and $(y;(p,q)) \sim (y';(p',q'))$, then $z' := x'(0,n'\wedge d(x'))\sigma^{p'\wedge d(y')}y'$ satisfies*

$$(z';(m',n'+q'-p') \sim (z;(m,n+q-p)).$$

Proof. We must show that

(P1) $z'(m'\wedge d(z'), (n'+q'-p')\wedge d(z')) = z(m\wedge d(z); (m+q-p)\wedge d(z))$
(P2) $m' - m'\wedge d(z) = m - m\wedge d(z)$
(P3) $n'+q'-p'-m' = n+q-p-m$.

Since $(x;(m,n)) \sim (x';(m',n'))$ and $(y;(p,q)) \sim (y';(p',q'))$, the relation $\sim$ gives us

(a)
$$x'(m'\wedge d(x'), n'\wedge d(x')) = x(m\wedge d(x), n\wedge d(x))$$
$$y'(p'\wedge d(y'), q'\wedge d(y')) = y(p\wedge d(y), q\wedge d(y)),$$

(b)
$$m' - m'\wedge d(x') = m - m\wedge d(x)$$
$$p' - p'\wedge d(y') = p - p\wedge d(y)$$

(c)
$$n' - m' = n - m$$
$$q' - p' = q - p.$$

Then (P3) follows from (c). For (P2), notice that by (b) we have $m' - m' \wedge d(x') = m - m \wedge d(x)$. Then Proposition 3.2.6(2) gives (P2). It remains to show that (P1) holds. By Proposition 3.2.6(3), we have

$$\begin{aligned} z(m \wedge d(z), &(n + q - p) \wedge d(z)) \\ &= z(m \wedge d(z), n \wedge d(z))\ z(n \wedge d(z), (n + q - p) \wedge d(z)) \\ &= x(m \wedge d(x), n \wedge d(x))\ y(p \wedge d(y), q \wedge d(y)). \end{aligned}$$

Then (P1) follows from (a), completing the proof. □

Define $\mathrm{id} : \widetilde{V_\Lambda} \to \widetilde{P_\Lambda}$ by $\mathrm{id}_{[x;m]} = [x; (m, m)]$.

PROPOSITION 3.2.8. $\widetilde{\Lambda} := (\widetilde{V_\Lambda}, \widetilde{P_\Lambda}, \widetilde{r}, \widetilde{s}, \circ, \mathrm{id})$ *is a category.*

PROOF. We must show

(i) $\widetilde{r}(\mathrm{id}_{[x;m]}) = [x; m] = \widetilde{s}(\mathrm{id}_{[x;m]})$ for all $[x; m] \in \widetilde{V_\Lambda}$.
(ii) $\widetilde{s}([x; (m, n)] \circ [y; (p, q)]) = \widetilde{s}([y; (p, q)])$ and
$\widetilde{r}([x; (m, n)] \circ [y; (p, q)]) = \widetilde{r}([x; (m, n)])$
for all composable pairs $[x; (m, n)], [y; (p, q)] \in \widetilde{P_\Lambda}$
(iii) $([x; (m, n)] \circ [y; (p, q)]) \circ [w; (a, b)] = [x; (m, n)] \circ ([y; (p, q)] \circ [w; (a, b)])$ for all $[x; (m, n)], [y; (p, q)], [w; (a, b)] \in \widetilde{P_\Lambda}$ satisfying $[x; n] = [y; p]$ and $[y; q] = [w; a]$.
(iv) $[x; (m, n)] \circ \mathrm{id}_{\widetilde{s}([x;(m,n)])} = [x; (m, n)]$ and
$\mathrm{id}_{\widetilde{r}([x;(m,n)])} \circ [x; (m, n)] = [x; (m, n)]$ for all $[x; (m, n)] \in \widetilde{P_\Lambda}$.

Parts (i), (ii) and (iv) follow directly from the definitions of $\widetilde{r}, \widetilde{s}, \circ$ and $\mathrm{id}_{[x;m]}$.

Part (iii), though the notation is unavoidably complicated, is a fairly straightforward argument. Let $[x; (m, n)], [y; (p, q)], [w; (a, b)] \in \widetilde{P_\Lambda}$ such that $[x; n] = [y; p]$ and $[y; q] = [w; a]$. Define $z_{x,y} := x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. Then

$$[x; (m, n)] \circ [y; (p, q)] = [z_{x,y}; (m, n + p - q)].$$

Now define $z_{xy,w} := z_{x,y}(0, (n + p - q) \wedge d(z_{x,y}))\sigma^{a \wedge d(w)} w$, so

$$\begin{aligned} ([x; (m, n)] \circ [y; (p, q)]) \circ [w; (a, b)] &= [z_{x,y}; (m, n + p - q)] \circ [w; (a, b)] \\ &= [z_{xy,w}; (m, n + p - q + b - a)]. \end{aligned}$$

Similarly, we can define $z_{y,w}$ and $z_{x,yw}$ such that

$$\begin{aligned}[x;(m,n)]\circ([y;(p,q)]\circ[w;(a,b)]) &= [x;(m,n)]\circ[z_{y,w};(p,q+b-a)]\\ &= [z_{x,yw};(m,n+q+b-a-p)].\end{aligned}$$

We must show that $(z_{xy,w};(m,n+p-q+b-a)) \sim (z_{x,yw};(m,n+q+b-a-p))$, so we will verify that conditions (P1)–(P3) are satisfied.

We first verify that (P1) holds. Two applications of Proposition 3.2.6(3) give

$$\begin{aligned}&z_{xy,w}(m\wedge d(z_{xy,w}),(n+p-q+b-a)\wedge d(z_{xy,w}))\\ &\quad= z_{x,y}(m\wedge d(z_{x,y}),(n+p-q)\wedge d(z_{x,y}))w(a\wedge d(w),b\wedge d(w))\\ &\quad= x(m\wedge d(x),n\wedge d(x))y(p\wedge d(y),q\wedge d(y))w(a\wedge d(w),b\wedge d(w)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&z_{x,yw}(m\wedge d(z_{x,yw}),(n+q+b-a-p)\wedge d(z_{x,yw}))\\ &\quad= x(m\wedge d(x),n\wedge d(x))z(y,w)(p\wedge d(z_{y,w}),(q+b-a)\wedge d(z_{y,w})\\ &\quad= x(m\wedge d(x),n\wedge d(x)y(p\wedge d(y),q\wedge d(y))w(a\wedge d(w),b\wedge d(w)).\end{aligned}$$

Thus (P1) holds. To verify that (P2) holds, we apply Proposition 3.2.6(2) to obtain

$$m-m\wedge d(z_{xy,w}) = m-m\wedge d(z_{x,y}) = m-m\wedge d(x).$$

Similarly, $m-m\wedge d(z_{x,yw}) = m-m\wedge d(x)$ giving (P2). (P3) follows directly since $(n+p-q+b-a)-m = (n+q+b-a-p)-m$. So we have $(z_{xy,w};(m,n+p-q+b-a)) \sim (z_{x,yw};(m,n+q+b-a-p))$, proving (iii). □

DEFINITION 3.2.9. Define $\widetilde{d}:\widetilde{\Lambda}\to\mathbb{N}^k$ by $\widetilde{d}(v)=\star$ for all $v\in\widetilde{V_\Lambda}$, and $\widetilde{d}([x;(m,n)])=n-m$ for all $[x;(m,n)]\in\widetilde{P_\Lambda}$.

PROPOSITION 3.2.10. *The map $\widetilde{d}$ defined above satisfies the factorisation property. Hence with $\widetilde{\Lambda}$ as in Proposition 3.2.8, $(\widetilde{\Lambda},\widetilde{d})$ is a k-graph with no sources.*

PROOF. Fix $[w;(a,b)]\in\widetilde{P_\Lambda}$. Let $t,u\in\mathbb{N}^k$ be such that $b-a=t+u$. Then we have

$$\begin{aligned}[w;(a,a+t)]\circ[w;(a+t,a+t+u)] &= [w;(a,a+t)]\circ[w;(a+t,b)]\\ &= [w;(a,b)].\end{aligned}$$

To see that this factorisation is unique, suppose that $[x;(m,n)],[y;(p,q)]$ are such that $n-m=t$, $q-p=u$ and $[x;(m,n)]\circ[y;(p,q)]=[w;(a,b)]$. Then $[w;a]=[x;m]$, $[w;b]=[y;q]$, and $[x;n]=[y;p]$. We aim to show that $(w;(a,a+t))\sim(x;(m,n))$, and that $(w;(a+t,a+t+u))\sim(y;(p,q))$.

We first show that $(w;(a,a+t))\sim(x;(m,n))$. So we need to verify that

(P1) $w(a \wedge d(\omega), (a+t) \wedge d(w)) = x(m \wedge d(x), n \wedge d(x))$.
(P2) $a - a \wedge d(w) = m - m \wedge d(x)$.
(P3) $a + t - a = n - m$.

Since $[w; a] = [x; m]$, (V2) gives (P2). We chose n and m such that $t = n - m$, thus (P3) holds. It remains to verify that (P1) holds. Let $z := x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. Then $[w; (a, b)] = [z; (m, n + q - p)]$. By Proposition 3.2.6(3), we have

$$\begin{aligned} w(a \wedge d(w), b \wedge d(w)) &= z(m \wedge d(z), (n+q-p) \wedge d(z)) \\ &= z(m \wedge d(z), n \wedge d(z)) z(n \wedge d(z), (n+q-p) \wedge d(z)) \\ &= x(m \wedge d(x), n \wedge d(x)) y(p \wedge d(y), q \wedge d(y)). \end{aligned} \tag{3.2.6}$$

Hence

$$b \wedge d(w) - a \wedge d(w) = n \wedge d(x) - m \wedge d(x) + q \wedge d(y)) - p \wedge d(y), \tag{3.2.7}$$

and

$$w(a \wedge d(w), a \wedge d(w) + n \wedge d(x) - m \wedge d(x)) = x(m \wedge d(x), n \wedge d(x)).$$

So it suffices to show that

$$(a+t) \wedge d(w) = a \wedge d(w) + n \wedge d(x) - m \wedge d(x).$$

Rearranging and substituting $t = n - m$, we see that it suffices to show that

$$(a + n - m) \wedge d(w) - n \wedge d(x) = a - m.$$

Fix $i \leq k$. We will prove that

$$\min\{a_i + n_i - m_i, d(w)_i\} - \min\{n_i, d(x)_i\} = a_i - m_i. \tag{3.2.8}$$

We argue in two cases:

(i) $n_i \leq d(x)_i$, or
(ii) $n_i > d(x)_i$.

Case (i). Equation (3.2.8) holds if and only if $a_i + n_i - m_i \leq d(w)_i$. To prove this, we suppose that $a_i + n_i - m_i > d(w)_i$ and seek a contradiction. Since $m_i \leq n_i$, we have $m_i \leq d(x)_i$. Then (P2) implies that $a_i \leq d(w)_i$. Since $b - a = t + u$, we have $b \geq a + n - m$. Then

$$a_i \leq d(w)_i < a_i + n_i - m_i \leq b_i.$$

Since $[w; b] = [y; q]$ and $[x; n] = [y; p]$, it follows from (V2) that $b_i \leq d(w)_i \iff q_i \leq d(y)_i$ and $n_i \leq d(x)_i \iff p_i \leq d(y)_i$. So $q_i > d(y)_i$ and $p_i \leq d(y)_i$. Now (3.2.7) implies that

$$d(w)_i - a_i = n_i - m_i + d(y)_i - p_i.$$

Since $d(y)_i - p_i \geq 0$, we have

$$d(w)_i = a_i + n_i - m_i + d(y)_i - p_i \geq a_i + n_i - m_i,$$

which contradicts that $a_i + n_i - m_i > d(w)_i$. So $a_i + n_i - m_i \leq d(w)_i$, and hence

$$\min\{a_i + n_i - m_i, d(w)_i\} - \min\{n_i, d(x)_i\} = a_i + n_i - m_i - n_i = a_i - m_i.$$

Case (ii). We suppose that $a_i + n_i - m_i \leq d(w)_i$, and seek a contradiction. Since $n_i \geq m_i$, we have $a_i \leq d(w)_i$. Hence $m_i \leq d(x)_i$. Since $n_i > d(x)_i$, we have $p_i > d(y)_i$. Furthermore, $p_i \leq q_i$ implies that $q_i > d(y)_i$, so $b_i > d(w)_i$. Then (3.2.7) becomes

$$d(w)_i - a_i = d(x)_i - m_i + d(y)_i - d(y)_i = d(x)_i - m_i.$$

This implies that

$$0 \leq d(w)_i - a_i - (n_i - m_i) = d(x)_i - m_i - (n_i - m_i) = d(x)_i - n_i < 0,$$

contradicting our supposition. So $a_i + n_i - m_i > d(w)_i$. Hence (3.2.8) holds if $d(w)_i - d(x)_i = a_i - m_i$. If $m_i > d(x)_i$, then $a_i > d(w)_i$ and the result follows directly from (P2). Otherwise, $m_i \leq d(x)_i$ and $a_i \leq d(w)_i < a_i + n_i - m_i \leq b_i$. Hence $q_i, p_i \geq d(y)_i$. Equation (3.2.7) then implies that

$$d(w)_i - a_i = d(x)_i - m_i,$$

as required. Thus we have $(w; (a, a + t)) \sim (x; (n, m))$.

Now we will show that $(w; (a + t, a + t + u)) \sim (y; (p, q))$. So we need to verify that

(P1) $w(a + t \wedge d(\omega), (a + t + u) \wedge d(w)) = y(p \wedge d(y), q \wedge d(y))$.
(P2) $a + t - (a + t) \wedge d(w) = p - p \wedge d(y)$.
(P3) $a + t + u - (a + t) = q - p$.

Equation (P3) is true by assumption. Substituting (P1) into (3.2.6), we have

$$w(a \wedge d(w), b \wedge d(w)) = w(a \wedge d(\omega), (a + t) \wedge d(w))y(p \wedge d(y), q \wedge d(y)).$$

Then (P1) follows by the factorisation property in Λ. It remains to verify (P2). Since (3.2.8) holds for all $i \leq k$, we can rearrange it to obtain

$$a + n - m - (a + n - m) \wedge d(w) = n - n \wedge d(x).$$

Since $[x; n] = [y; p]$, we have $n - n \wedge d(x) = p - p \wedge d(y)$. Then

$$\begin{aligned} a + t - (a + t) \wedge d(w) &= a + n - m - (a + n - m) \wedge d(w) \\ &= n - n \wedge d(x) \\ &= p - p \wedge d(y), \end{aligned}$$

as required.

This shows $(\widetilde{\Lambda}, \widetilde{d})$ is a k-graph. Suppose $v \in \widetilde{\Lambda}^0$. Then $v = [x; m]$ for some $x \in \partial\Lambda$ and $m \in \mathbb{N}^k$. Then $[x; (m, m + e_i)] \in \widetilde{\Lambda}^{e_i}$ for all $i \leq k$. Thus $\widetilde{\Lambda}$ has no sources. $\square$

3.2.1. Some Examples. If we allow infinite receivers, our construction yields a different k-graph to Farthing's construction in [**7**, §2]. To see how this might happen, we work through an elementary example. Consider the 1-graph E with vertices $E^0 = \{u, v\}$, and edges $E^1 = \{f_i : i \in \mathbb{N}\}$ with $r(f_i) = v$ and $s(f_i) = u$ for all $i \in \mathbb{N}$. The graph E looks like

$$\begin{tikzcd} v & u \arrow[l, Rightarrow, bend right, "f_i"'] \end{tikzcd} \quad \vdots$$

We first use Farthing's construction to desourcify E. Using the notation established in [**27**], our vertices take the form $[x; m]$, where $x \in \Lambda^{\leq\infty} = E^1 \cup \{u\}$ and $m \in \mathbb{N}$. Fix a vertex $[f_j; p]$ for some $p \in \mathbb{N}^k$. Fix $i \in \mathbb{N}$. Then since $d(f_j) = d(f_i)$, we have

$$p - p \wedge d(f_i) = p - p \wedge d(f_j),$$

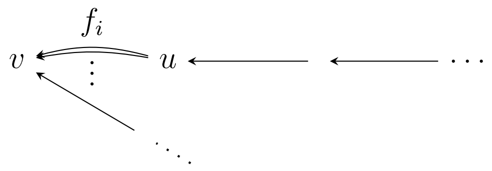
and

$$f_i(p \wedge d(f_i)) = s(f_i) = s(f_j) = f_j(p \wedge d(f_j)).$$

This implies that $[f_j; p] = [f_i; p]$ for all $i, j \in \mathbb{N}$ and $p \in \mathbb{N}^k$. Similarly, $[u; p - 1] = [f_j; p]$ for all $p > 1$. So any of the "new" vertices appended to E can be written as $[u; m]$ for some $m \geq 1$. Furthermore, any path between two of these appended vertices has the form $[u; (m, n)]$, and simple calculations show that for all $i \in \mathbb{N}$ and $1 \leq m \leq n$, we have $[f_j; (m, n)] = [f_i; (m, n)] = [u; (m - 1, n - 1)]$. This implies that the construction adds an infinite path with range u. Hence the Farthing desourcification $\overline{E}$ below is the same as the "adding heads" construction by Bates, et al in [**2**].

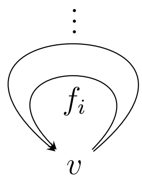
$$\begin{tikzcd} v & u \arrow[l, Rightarrow, bend right, "f_i"'] & \arrow[l] & \arrow[l] \cdots \end{tikzcd} \quad \vdots$$

Applying the construction detailed in this thesis, the same argument as in the previous paragraph shows that there is a head added at the vertex u, just as in Farthing's construction. However, we also have to consider the vertex $v \in \partial\Lambda \setminus \Lambda^{\leq\infty}$. Fix $j, n, m \in \mathbb{N}$. By (V1), we have $[v; m] = [f_j, n]$ if and only if $m = n = 0$. For $m \geq 1$, the vertices $\{[v; m] : m \in \mathbb{N}\}$ are distinct from those defined using the f_i. Thus, in our new desourcification, we append a head to the boundary path v:



It is intriguing that following Drinen and Tomforde's desingularisation, a head is also added at infinite receivers like this, and then the ranges of the edges f_i are distributed along this head — we cannot help but wonder whether this might suggest an approach to a Drinen-Tomforde desingularisation for k-graphs.

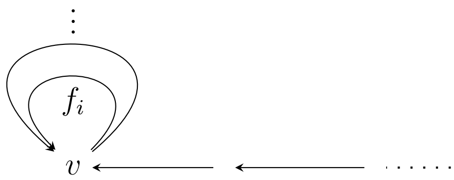
Another interesting example is the graph E with $E^0 = \{v\}$ and $E^1 = \{f_i : i \in \mathbb{N}\}$ where $s(f_i) = r(f_i) = v$ for all $i \in \mathbb{N}$. That is, an infinite number of loops on a single vertex v:



Here we have $E^{\leq\infty} = \emptyset$, so Farthing's construction yields a 1-graph $\overline{E} \cong E$. Since v belongs to every finite exhaustive set in E, we have $\partial E = E^*$. Furthermore $[f_j; p] = [f_i; p] = [v; p]$ for all $i, j, p \in \mathbb{N}$, and

$$[f_j; (p, q)] = [f_i; (p, q)] = [v; (p-1, q-1)]$$

for all i, j, p, q such that $1 < p \leq q$. Thus there is exactly one path between any two of the added vertices, resulting an infinite tail coming into v, yielding the graph illustrated below



3.2.2. Row-finite 1-graphs. While one expects this style of desourcification (both Farthing's and the one contained in this thesis) to agree with adding heads to a row-finite 1-graph as in [**2**], this appears not to have been checked anywhere.

PROPOSITION 3.2.11. *Let E be a row-finite directed graph and F be the graph obtained by adding heads to sources, as in* [**2**, p4]. *Let Λ be the 1-graph associated to E. Then $\widetilde{\Lambda} \cong F^*$, where F^* is a the path-category of F.*

PROOF. We will construct 1-graph morphisms $\eta : \widetilde{\Lambda} \to F^*$ and $\xi : F^* \to \widetilde{\Lambda}$, and show that $\xi = \eta^{-1}$. We first define $\eta' : P_\Lambda \to F^*$ as follows. Fix $x \in \partial E$ and

$m, n \in \mathbb{N}$. Then either $x \in E^\infty$, or $x \in E^*$ and $s(x)$ is a source in E. For $x \in E^\infty$, define $\eta'((x;(m,n))) = x(m,n)$. For $x \in E^*$, let μ_x be the head added to $s(x)$, and define $\eta'((x;(m,n))) = (x\mu_x)(m,n)$. We now check that η' respects the equivalence relation $\sim$ on P_Λ. Let $y \in \partial\Lambda$ and $p, q \in \mathbb{N}$ be such that $(y;(p,q)) \sim (x;(m,n))$. Then

$$m \le d(x) \iff p \le d(y) \qquad \text{by (P2)} \tag{3.2.9}$$

$$n \le d(x) \iff q \le d(y) \qquad \text{by Lemma 3.2.4} \tag{3.2.10}$$

We consider the following three cases:

(i) $m \le n \le d(x)$;
(ii) $m \le d(x) \le n$;
(iii) $d(x) \le m \le n$.

In case (i), (3.2.9) and (3.2.10) tell us that $p \le q \le d(y)$. Then (P1) implies that $x(m,n) = y(p,q)$. Thus $\eta'((x;(m,n))) = x(m,n) = y(p,q) = \eta'((y;(p,q)))$.

In case (ii), condition (P1) says that $x(m,d(x)) = y(p,d(y))$. So $s(x) = s(y)$, and $\mu_x = \mu_y$. Lemma 3.2.4 implies that $n - d(x) = q - d(y)$. Then

$$\begin{aligned}\eta'((x;(m,n))) &= (x\mu_x)(m,n)\\ &= x(m,d(x))\mu_x(0,n-d(x))\\ &= y(p,d(y))\mu_y(0,q-d(y))\\ &= (y\mu_y)(p,q)\\ &= \eta'((y;(p,q))).\end{aligned}$$

In case (iii), (P1) implies that

$$x(m \wedge d(x), n \wedge d(x)) = s(x) = y(p \wedge d(y), q \wedge d(y)) = y(p \wedge d(y)).$$

Since $s(x)$ is a source in E, so is $y(p \wedge d(y))$. This implies that $d(y) \le p$. So $s(x) = s(y)$, and hence $\mu_x = \mu_y$. We then have

$$\begin{aligned}\eta'((x;(m,n))) &= (x\mu_x)(m,n)\\ &= \mu_x(m-d(x),n-d(x))\\ &= \mu_y(p-d(y),q-d(y)) && \text{by (P2) and Lemma 3.2.4}\\ &= (y\mu_y)(p,q)\\ &= \eta'((y;(p,q))).\end{aligned}$$

So η' respects $\sim$. We now define $\eta : \widetilde{\Lambda} \to F^*$ by

$$\eta([x;(m,n)]) = \eta'((x;(m,n))).$$

We check that η is a graph morphism. Straightforward calculations show that the range, source and degree maps are preserved. We must work to that show composition is preserved. Fix $[x;(m,n)],[y;(p,q)] \in \widetilde{\Lambda}$ such that $[x;n]=[y;p]$. Let $z = x(0, n\wedge d(x))\sigma^{p\wedge d(y)}(y)$. Then $\eta([x;(m,n)][y;(p,q)]) = \eta([z;(m,n+q-p)])$. We argue the following cases:

(i) $z \in E^*$ and $p \le d(y)$,
(ii) $z \in E^*$ and $p > d(y)$,
(iii) $z \in E^\infty$.

For cases (i) and (ii), first observe that if $z \in E^*$, then $s(z) = s(y)$ is a source. So $y \in E^*$ and hence $\mu_z = \mu_y$. Then

$$\begin{aligned}
\eta([z;(m,n+q-p)]) &= (z\mu_z)(m,n+p-q) \\
&= (x(0,n\wedge d(x))\sigma^{p\wedge d(y)}(y)\mu_y)(m,n+q-p) \\
&= (x(0,n\wedge d(x))y(p\wedge d(y),d(y))\mu_y)(m,n+q-p) \qquad (3.2.11)
\end{aligned}$$

Case (i). Suppose that $z \in E^*$ and $p \le d(y)$. Since $[x;n]=[y;p]$, condition (V2) implies that $n \le d(x)$. Then (3.2.11) becomes

$$\begin{aligned}
&\eta([z;(m,n+q-p)]) \\
&\quad = \big(x(0,n\wedge d(x))y(p\wedge d(y),d(y))\mu_y\big)(m,n+q-p) \\
&\quad = \big(x(0,n)y(p,d(y))\mu_y\big)(m,n+q-p) \\
&\quad = x(m,n)(y(p,d(y))\mu_y)(0,q-p) \\
&\quad = x(m,n)(y\mu_y)(p,q) \\
&\quad = \eta([x;(m,n)])\eta([y;(p,q)]).
\end{aligned}$$

Case (ii). Suppose that $p > d(y)$. Condition (V2) implies that $n > d(x)$ and $n - d(x) = p - d(y)$. Then (V1) gives $s(x) = s(y)$. Hence $\mu_y = \mu_x$, and (3.2.11) becomes

$$\begin{aligned}
&\eta([z;(m,n+q-p)]) \\
&\quad = (x(0,n\wedge d(x))y(p\wedge d(y),d(y))\mu_y)(m,n+q-p) \\
&\quad = (x\mu_x)(m,n+q-p) \\
&\quad = \big((x\mu_x)(m,n)\big)\big((x\mu_x)(n,n+q-p)\big) \\
&\quad = \big((x\mu_x)(m,n)\big)\big(\mu_x(n-d(x),n-d(x)+q-p)\big) \\
&\quad = \big((x\mu_x)(m,n)\big)\big(\mu_y(p-d(y),p-d(y)+q-p)\big) \\
&\quad = \big((x\mu_x)(m,n)\big)\big((y\mu_y)(p,q)\big) \\
&\quad = \eta([x;(m,n)])\eta([y;(p,q)]).
\end{aligned}$$

Case (iii). Suppose that $z \in E^\infty$. Then $y \in E^\infty$, so $p \le d(y)$. By (V2), $n \le d(x)$. So $z = x(0,n)\sigma^p(y)$, and

$$\begin{aligned}
\eta([z;(m,n+q-p)]) &= z(m,n+q-p) \\
&= x(m,n)\sigma^p(y)(0,q-p) \\
&= x(m,n)y(p,q) \\
&= \eta([x;(m,n)])\eta([y;(p,q)]).
\end{aligned}$$

So $\eta : \widetilde{\lambda} \to F$ is a graph morphism.

We now wish to construct another graph morphism $\xi : F^* \to \widetilde{\Lambda}$ and show that $\xi = \eta^{-1}$. Let $\nu \in F^*$. To define ξ we first need some preliminary notation. ξ will be defined casewise, broken up as follows:

(i) $\nu \in E^*$,

(ii) $r(\nu) \in E^*$ and $s(\nu) \in F^* \setminus E^*$, or

(iii) $r(\nu), s(\nu) \in F^* \setminus E^*$.

If $\nu \in E^*$, fix $\alpha_\nu \in s(\nu)\partial E$. If ν has $r(\nu) \in E^*$ and $s(\nu) \in F^* \setminus E^*$, let $p_\nu = \max\{p \in \mathbb{N} : \nu(0,p) \in E^*\}$. Then $\nu(p_\nu)$ is a source in E^*, and $\nu(0,p_\nu) \in \partial E$. If $\nu \in F^* \setminus E^*$, then ν is a segment of a head μ_ν added to a source in E^*, and we let q_ν be such that $\nu = \mu_\nu(q_\nu, q_\nu + d(\mu))$.

We then define ξ by

$$\xi(\nu) = \begin{cases} [\nu\alpha_\nu;(0,d(\nu))] & \text{if } \nu \in E^* \\ [\nu(0,p_\nu);(0,d(\nu))] & \text{if } r(\nu) \in E^* \text{ and } s(\nu) \notin E^* \\ [r(\mu_\nu);(q_\nu,q_\nu+d(\nu))] & \text{if } r(\nu), s(\nu) \in F^* \setminus E^*. \end{cases}$$

To see that ξ is well-defined, we must show that in case (i), $\xi(\nu)$ does not depend on the choice of α_ν. Suppose that $\nu \in E^*$ and $\alpha_\nu, \beta_\nu \in \partial E$. We claim that $(\nu\alpha_\nu; (0, d(\nu))) \sim (\nu\beta_\nu; (0, d(\nu)))$. Indeed, equations (P2) and (P3) are trivially satisfied, and (P1) is easy to see:

$$(\nu\alpha_\nu)(0, d(\nu)) = \nu = \nu\beta_\nu(0, d(\nu)).$$

Straightforward calculations show that ξ preserves the range, source and degree maps. We show that it preserves composition. Let $\lambda, \nu \in F^*$ be such that $s(\lambda) = r(\nu)$. Then

$$\xi(\lambda\nu) = \begin{cases} [\lambda\nu\alpha_{\lambda\nu}; (0, d(\lambda\nu))] & \text{if } \lambda\nu \in E^* \\ [(\lambda\nu)(0, p_{\lambda\nu}); (0, d(\lambda\nu))] & \text{if } r(\lambda\nu) \in E^* \text{ and } s(\lambda\nu) \in F^* \setminus E^* \\ [r(\mu_{\lambda\nu}); (q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda\nu))] & \text{if } r(\lambda\nu), s(\lambda\nu) \in F^* \setminus E^*. \end{cases}$$

If $\lambda\nu \in E^*$, then $\lambda, \nu \in E^*$.

$$\begin{aligned} \xi(\lambda)\xi(\nu) &= [\lambda\nu\alpha_{\lambda\nu}; (0, d(\lambda))][\nu\alpha_{\lambda\nu}; (0, d(\nu))] \\ &= [(\lambda\nu\alpha_{\lambda\nu})(0, d(\lambda))\sigma^0(\nu\alpha_{\lambda\nu}); (0, d(\lambda) + d(\nu))] \\ &= [\lambda\nu\alpha_{\lambda\nu}; (0, d(\lambda\nu))] \\ &= \xi(\lambda\nu). \end{aligned}$$

If $r(\lambda\nu) \in E^*$ and $s(\lambda\nu) \in F^* \setminus E^*$, we argue in two separate cases: (i) $p_{\lambda\nu} \leq d(\lambda)$ or (ii) $p_{\lambda\nu} > d(\lambda)$.

If $p_{\lambda\nu} \leq d(\lambda)$, then $\lambda(p_{\lambda\nu})$ is a source and ν is a segment of the head $\mu_{\lambda(p_{\lambda\nu})}$ added at $\lambda(p_{\lambda\nu})$; namely

$$\nu = \mu_{\lambda(p_{\lambda\nu})}(d(\lambda) - p_{\lambda\nu}, d(\lambda) - p_{\lambda\nu} + d(\nu)).$$

Then

$$\begin{aligned} \xi(\lambda)\xi(\nu) &= [\lambda(0, p_{\lambda\nu}); (0, d(\lambda))][\lambda(p_{\lambda\nu}); (d(\lambda) - p_{\lambda\nu}, d(\lambda) - p_{\lambda\nu} + d(\nu))] \\ &= [\lambda(0, p_{\lambda\nu})\sigma^0\lambda(p_{\lambda\nu}); (0, d(\lambda) + d(\lambda) - p_{\lambda\nu} + d(\nu) - d(\lambda) + p_{\lambda\nu})] \\ &= [(\lambda\nu)(0, p_{\lambda\nu}); (0, d(\lambda\nu))] \\ &= \xi(\lambda\nu). \end{aligned}$$

If $p_{\lambda\nu} > d(\lambda)$, then $\lambda \in E^*$ and $\nu(p_{\lambda\nu} - d(\lambda))$ is a source. So $p_\nu = p_{\lambda\nu} - d(\lambda)$, and $\nu(0, p_{\lambda\nu} - d(\lambda)) \in s(\lambda)\partial E$. Then

$$\begin{aligned} \xi(\lambda)\xi(\nu) &= [\lambda\nu(0, p_{\lambda\nu} - d(\lambda)); (0, d(\lambda))][\nu(0, p_{\lambda\nu} - d(\lambda)); (0, d(\nu))] \\ &= [(\lambda\nu(0, p_{\lambda\nu} - d(\lambda)))(0, d(\lambda))\sigma^0(\nu(0, p_{\lambda\nu} - d(\lambda))); (0, d(\lambda) + d(\nu))] \\ &= [(\lambda\nu)(0, p_{\lambda\nu}); (0, d(\lambda\nu))] \\ &= \xi(\lambda\nu). \end{aligned}$$

If $r(\lambda\nu), s(\lambda\nu) \in F^* \setminus E^*$, then

$$\begin{aligned}\lambda\nu &= \mu_{\lambda\nu}(q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda\nu)) \\ &= \mu_{\lambda\nu}(q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda))\mu_{\lambda\nu}(q_{\lambda\nu} + d(\lambda), q_{\lambda\nu} + d(\lambda\nu)).\end{aligned}$$

The factorisation property then implies that $\lambda = \mu_{\lambda\nu}(q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda))$ and $\nu = \mu_{\lambda\nu}(q_{\lambda\nu} + d(\lambda), q_{\lambda\nu} + d(\lambda\nu))$. So

$$\begin{aligned}\xi(\lambda)\xi(\nu) &= [r(\mu_{\lambda\nu}); (q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda))][r(\mu_{\lambda\nu}); (q_{\lambda\nu} + d(\lambda), q_{\lambda\nu} + d(\lambda\nu))] \\ &= [r(\mu_{\lambda\nu}); (q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda) + q_{\lambda\nu} + d(\lambda\nu) - q_{\lambda\nu} - d(\lambda))] \\ &= [r(\mu_{\lambda\nu}); (q_{\lambda\nu}, q_{\lambda\nu} + d(\lambda\nu))] \\ &= \xi(\lambda\nu).\end{aligned}$$

So $\xi : F^* \to \widetilde{\Lambda}$ is a graph morphism.

We now check that $\xi \circ \eta = 1_{\widetilde{\Lambda}}$ and $\eta \circ \xi = 1_{F^*}$. Fix $\nu \in F^*$. Then

$$\begin{aligned}\eta(\xi(\nu)) &= \begin{cases} \eta([\nu\alpha_\nu; (0, d(\nu))]) & \text{if } \nu \in E^* \\ \eta([\nu(0, p_\nu); (0, d(\nu))]) & \text{if } r(\nu) \in E^* \text{ and } s(\nu) \in F^* \setminus E^* \\ \eta([r(\mu_\nu); (q_\nu, q_\nu + d(\nu))]) & \text{if } r(\nu), s(\nu) \in F^* \setminus E^*. \end{cases} \\ &= \begin{cases} (\nu\alpha_\nu)(0, d(\nu)) & \text{if } \nu \in E^* \\ \nu(0, p_\nu)\mu_{\nu(0,p_\nu)}(0, d(\nu)) & \text{if } r(\nu) \in E^* \text{ and } s(\nu) \in F^* \setminus E^* \\ \mu_\nu(q_\nu, q_\nu + d(\nu)) & \text{if } r(\nu), s(\nu) \in F^* \setminus E^*. \end{cases} \\ &= \nu.\end{aligned}$$

Now fix $[x; (m, n)] \in \widetilde{\Lambda}$. We argue that $\xi \circ \eta([x; (m, n)]) = [x; (m, n)]$ in cases: (i) $x \in E^\infty$, and (ii) $x \in E^*$. In case (i), we have $x(m, n) \in E^*$ and

$$\xi(\eta([x; (m, n)])) = \xi(x(m, n)) = [x(m, n)\sigma^n(x); (0, n - m)] = [x; (m, n)].$$

For case (ii), let μ_x be the head at $s(x)$. Then we have

$$\xi(\eta([x; (m, n)])) = \xi((x\mu_x)(m, n)).$$

We argue that $\xi((x\mu_x)(m, n)) = [x; (m, n)]$ separately for each of the three cases in the definition of ξ: first suppose that $(x\mu_x)(m, n) \in E^*$. Then

$$\xi((x\mu_x)(m, n)) = \xi(x(m, n)) = [x(m, n)\sigma^n(x); (0, n - m)] = [x; (m, n)].$$

Now suppose that $r((x\mu_x)(m,n)) \in E^*$ and $s((x\mu_x)(m,n)) \in F^* \setminus E^*$. Let $p_x = \max\{p \in \mathbb{N} : (x\mu_x)(m,p) \in E^*\}$. Then $x\mu_x(0,p_x) = x$, and

$$\begin{aligned}\xi((x\mu_x)(m,n)) &= [(x\mu_x)(m,p_x);(0,n-m)]\\ &= [(x\mu_x)(0,p_x);(m,n)]\\ &= [x;m,n].\end{aligned}$$

Lastly, if $(x\mu_x)(m),(x\mu_x)(n) \in F^* \setminus E^*$, then $(x\mu_x)(m,n) = \mu_x(m-d(x),n-d(x))$ and

$$\xi((x\mu_x)(m,n)) = [r(\mu_x);(m-d(x),n-d(x))] = [x;(m,n)].$$

So $\xi = \eta^{-1}$, and $\eta : \widetilde{\Lambda} \to F^*$ is a graph isomorphism. □

When Λ is row-finite and locally convex, Proposition 3.0.18 implies that $\Lambda^{\leq\infty} = \partial\Lambda$. In this case our construction is essentially the same as that of Farthing [**7**, §2], with notation adopted as in [**27**]. If Λ is row-finite but not locally convex, then $\Lambda^{\leq\infty} \subset \partial\Lambda$ (Example 3.0.16 shows this may be a strict containment). Thus it is reasonable to suspect that our construction would result in a larger path space than Farthing's. Interestingly, this is not the case.

Proposition 3.2.12. *Let Λ be a row-finite k-graph. Suppose that $x \in \partial\Lambda \setminus \Lambda^{\leq\infty}$ and $m \leq n \in \mathbb{N}^k$. Then there exists $y \in \Lambda^{\leq\infty}$ such that $(x;(m,n)) \sim (y;(m,n))$.*

Proof. Since $x \notin \Lambda^{\leq\infty}$, there exists $q \geq n \wedge d(x)$ and $i \leq k$ such that $q \leq d(x)$, $q_i = d(x)_i$, and $x(q)\Lambda^{e_i} \neq \emptyset$. Let

$$J := \{i \leq k : q_i = d(x)_i \text{ and } x(q)\Lambda^{e_i} \neq \emptyset\}.$$

Since $x \in \partial\Lambda$, for each $E \in x(q)\mathcal{FE}(\Lambda)$ there exists $t \in \mathbb{N}^k$ such that $x(q,q+t) \in E$. Since $q_i = d(x)_i$ for all $i \in J$, the set $\bigcup_{i\in J} x(q)\Lambda^{e_i}$ contains no such segments of x, and thus cannot be finite exhaustive. Since Λ is row-finite, $\bigcup_{i\in J} x(q)\Lambda^{e_i}$ is finite, so $\bigcup_{i\in J} x(q)\Lambda^{e_i}$ is not exhaustive. Thus there exists $\mu \in x(q)\Lambda$ such that $\mathrm{MCE}(\mu,\nu) = \emptyset$ for all $\nu \in \bigcup_{i\in J} x(q)\Lambda^{e_i}$. By [**22**, Lemma 2.11], $s(\mu)\Lambda^{\leq\infty} \neq \emptyset$. Let $z \in s(\mu)\Lambda^{\leq\infty}$, and define $y := x(0,q)\mu z$. Then $y \in \Lambda^{\leq\infty}$ by [**22**, Lemma 2.10].

Now we show that $(x;(m,n)) \sim (y;(m,n))$. Condition (P3) is trivially satisfied. To see that (P1) and (P2) hold, we show that $n \wedge d(x) = n \wedge d(y)$. Firstly, let $i \in J$. If $d(\mu z)_i \neq 0$, then $(\mu z)(0,d(\mu)+e_i) \in \mathrm{MCE}(\mu,\nu)$ for $\nu = (\mu z)(0,e_i) \in r(\mu)\Lambda^{e_i} = x(q)\Lambda^{e_i}$, a contradiction. So $d(\mu z)_i = 0$ for each $i \in J$. This implies that $d(y)_i = d(x)_i$ for all $i \in J$. Now suppose that $i \notin J$. Then either $x(q)\Lambda^{e_i} = \emptyset$ or $q_i < d(x)_i$. If $x(q)\Lambda^{e_i} = \emptyset$ then $d(y)_i = d(x)_i$. So suppose that $q_i < d(x)_i$. Since $n \wedge d(x) \leq q$, it follows that $n_i < d(x)_i$ and $n_i \leq q_i$. Then since $q \leq d(y)$ we have $n_i \leq d(y)_i$, hence $(n \wedge d(x))_i = n_i = (n \wedge d(y))_i$.

So we have that for each $i \leq k$, either $d(y)_i = d(x)_i$, or $(n \wedge d(x))_i = n_i = (n \wedge d(y))_i$. Hence $m \leq n$ implies that $m \wedge d(x) = m \wedge d(y)$, whence it follows that (P2) holds. Furthermore, since $y(0,q) = x(0,q)$ and $n \wedge d(x) \leq q$, we have

$$x(m \wedge d(x), n \wedge d(x)) = y(m \wedge d(y), n \wedge d(y)),$$

verifying (P1). $\square$

We wish to be able to identify Λ with a subgraph of $\widetilde{\Lambda}$. The following results allow us to do so.

Proposition 3.2.13. *Suppose that Λ is a k-graph, and that $\lambda \in \Lambda$. Then $s(\lambda)\partial\Lambda \neq \emptyset$. If $x, y \in s(\lambda)\partial\Lambda$, then $\lambda x, \lambda y \in \partial\Lambda$ and $(\lambda x; (0, d(\lambda))) \sim (\lambda y; (0, d(\lambda)))$. Moreover, there is an injective k-graph morphism $\iota : \Lambda \to \widetilde{\Lambda}$ such that for $\lambda \in \Lambda$*

$$\iota(\lambda) = [\lambda x; (0, d(\lambda))] \textit{ for any } x \in s(\lambda)\partial\Lambda.$$

Proof. By [**8**, Lemma 5.15], we have $v\partial\Lambda \neq \emptyset$ for all $v \in \Lambda^0$. In particular, we have $s(\lambda)\partial\Lambda \neq \emptyset$. Let $x, y \in s(\lambda)\partial\Lambda$. Then [**8**, Lemma 5.13(ii)] says that $\lambda x, \lambda y \in \partial\Lambda$. To show that $(\lambda x; (0, d(\lambda))) \sim (\lambda y; (0, d(\lambda)))$, equations (P1)–(P3) are easily verified:

(P1) $(\lambda x)\big(0 \wedge d(\lambda x), d(\lambda) \wedge d(\lambda x)\big) = (\lambda x)\big(0, d(\lambda)\big) = \lambda$,
and similarly $(\lambda y)\big(0 \wedge d(\lambda y), d(\lambda) \wedge d(\lambda y)\big) = \lambda$.
(P2) $0 - 0 \wedge d(\lambda x) = 0 = 0 - 0 \wedge d(\lambda y)$, and
(P3) $d(\lambda) - 0 = d(\lambda) = d(\lambda) - 0$.

We now show that ι is an functor. We first show that it preserves range and source. Since $\lambda x \in \partial\Lambda$, we have

$$\iota(s(\lambda)) = [s(\lambda)x; 0] = [\lambda x; d(\lambda)] = s([\lambda x; (0, d(\lambda))]) = s(\iota(\lambda)), \text{ and}$$
$$\iota(r(\lambda)) = [r(\lambda)\lambda x; 0] = [\lambda x; 0] = r([\lambda x; (0, d(\lambda))]) = r(\iota(\lambda)).$$

To see that ι respects composition, suppose that $\mu \in \Lambda$, and that $\lambda \in s(\mu)\Lambda$. Fix $x \in s(\mu)\partial\Lambda$ and $y \in s(\lambda)\partial\Lambda$. Define

$$z = (\mu x)(0, d(\mu))\sigma^{0 \wedge d(\lambda y)}(\lambda y).$$

So

$$\iota(\mu)\iota(\lambda) = [\mu x; (0, d(\mu))][\lambda y; (0, d(\lambda))] = [z; (0, d(\mu) + d(\lambda))].$$

Then $z = (\mu x)(0, d(\mu))\lambda y = \mu\lambda y$. Hence

$$[z; (0, d(\mu) + d(\lambda))] = [\mu\lambda y; (0, d(\mu\lambda))] = \iota(\mu\lambda).$$

To see that ι is a k-graph morphism, we must show that it preserves degree:

$$d(\iota(\mu)) = d([\mu x; (0, d(\mu))]) = d(\mu).$$

Finally, we show that ι is injective. Suppose that $\lambda, \mu \in \Lambda$, and that $\iota(\lambda) = \iota(\mu)$. Then $[\lambda x; (0, d(\lambda))] = [\mu x; (0, d(\mu))]$, and (P1) implies that $\lambda = \lambda x(0, d(\lambda)) = \mu x(0, d(\mu)) = \mu$. $\square$

We want to extend ι to an injection of W_Λ into $W_{\widetilde{\Lambda}}$. The next proposition shows that any injective k-graph morphism defined on Λ can be extended to W_Λ.

Proposition 3.2.14. *Let Λ, Γ be k-graphs and $\phi : \Lambda \to \Gamma$ be a k-graph morphism. Let $x \in W_\Lambda \setminus \Lambda$, then $\phi(x) : \Omega_{k,d(x)} \to W_\Gamma$ defined by $\phi(x)(p, q) = \phi(x(p, q))$ belongs to W_Γ.*

Proof. We need to show that $\phi(x)$ respects the range, source, composition and degree of elements in $\Omega_{k,d(x)}$. Fix $p, q \in \mathbb{N}^k$ such that $p \leq q \leq d(x)$. Then

$$\begin{aligned} s(\phi(x)(p, q)) &= s(\phi(x(p, q))) \\ &= \phi(s(x(p, q))) && \text{since } \phi \text{ is a } k\text{-graph morphism} \\ &= \phi(x(q)) \\ &= \phi(x)(q) \\ &= \phi(x)(s(p, q)). \end{aligned}$$

Similarly, $\phi(x)$ preserves the range of (p, q). For composition, we calculate

$$\begin{aligned} \phi(x)(p, q)\phi(x)(q, m) &= \phi(x(p, q))\phi(x(q, m)) \\ &= \phi(x(p, q)x(q, m)) \\ &= \phi(x(p, m)) \\ &= \phi(x)(p, m) \\ &= \phi(x)((p, q)(q, m)). \end{aligned}$$

Lastly, we verify that $\phi(x)$ preserves degree. This again follows from ϕ being a k-graph morphism:

$$d(\phi(x)(p, q)) = d(\phi(x(p, q))) = d((p, q)). \qquad \square$$

In particular, we can extend ι to paths with non-finite degree. The next result says that composition works as expected for non-finite paths. It is a 'folklore' result, and we provide details for completeness.

Proposition 3.2.15. *Let Λ, Γ be k-graphs and $\phi : \Lambda \to \Gamma$ be a k-graph morphism. Let $\lambda \in \Lambda$, $x \in s(\lambda)W_\Lambda$, and suppose that $n \in \mathbb{N}^k$ satisfies $n \leq d(x)$. Then*

(1) *$\phi(\lambda)\phi(x) = \phi(\lambda x)$; and*

(2) $\sigma^n(\phi(x)) = \phi(\sigma^n(x))$.

PROOF. For part (1), fix $p, q \in \mathbb{N}^k$ with $p \le q \le d(\lambda x)$. We need to show that

$$(\phi(\lambda)\phi(x))(p,q) = \phi(\lambda x)(p,q).$$

Since ϕ is a k-graph morphism, we have

$$\begin{aligned}(\phi(\lambda)\phi(x))(p,q) &= \big(\phi(\lambda)\phi(x)(0, q \vee d(\lambda) - d(\lambda))\big)(p,q)\\ &= \big(\phi(\lambda)\phi(x(0, q \vee d(\lambda) - d(\lambda)))\big)(p,q)\\ &= \phi\big(\big(\lambda x(0, q \vee d(\lambda) - d(\lambda))\big)(p,q)\big)\\ &= \phi((\lambda x)(p,q))\\ &= \phi(\lambda x)(p,q).\end{aligned}$$

For part (2), fix $n \in \mathbb{N}^k$ such that $n \le d(x)$. Let $p, q \in \mathbb{N}^k$ such that $p \le q \le d(x) - n$, then

$$\begin{aligned}\sigma^n(\phi(x))(p,q) &= \phi(x)(p+n, q+n)\\ &= \phi(x(p+n, q+n))\\ &= \phi(\sigma^n(x)(p,q))\\ &= \phi(\sigma^n(x))(p,q).\end{aligned}$$ $\square$

REMARK 3.2.16. As one would expect, the extension of an injective k-graph morphism to W_Λ is also injective. In particular, the map $\iota : \Lambda \to \widetilde{\Lambda}$ has an injective extension $\iota : W_\Lambda \to W_{\widetilde{\Lambda}}$. To see this, suppose that $\phi : \Lambda \to \Gamma$ is an injective k-graph morphism, and fix $x, y \in W_\Lambda$. Then

$$\begin{aligned}\phi(x) = \phi(y) &\iff \phi(x)(p,q) = \phi(y)(p,q) \qquad \text{for all } p, q \le d(x)\\ &\iff \phi(x(p,q)) = \phi(y(p,q))\\ &\iff x(p,q) = y(p,q)\\ &\iff x = y.\end{aligned}$$

To prove a few more consistency results about $\widetilde{\Lambda}$ we need to be able to 'project' paths from $\widetilde{\Lambda}$ onto the embedding $\iota(\Lambda)$ of Λ. For $y \in \partial\Lambda$ define

$$\pi([y; (m,n)]) = [y; (m \wedge d(y), n \wedge d(y))]. \tag{3.2.12}$$

The next result is asserted in [**27**]. The proof is from Robertson's honours thesis [**26**], but is unpublished.

PROPOSITION 3.2.17. *Let Λ be a k-graph. Then $\pi : \widetilde{\Lambda} \to \iota(\Lambda)$ defined in* (3.2.12) *is a surjective functor, and is a projection in the sense that $\pi(\pi([y; (m,n)])) = \pi([y; (m,n)])$ for all $[y; (m,n)] \in \widetilde{\Lambda}$. In particular, $\pi|_{\iota(\Lambda)} = \mathrm{id}_{\iota(\Lambda)}$.*

Proof. To see that π is a functor, we must show that it preserves range, source, composition and the identity. Fix $[x;(m,n)], [y;(p,q)] \in \widetilde{\Lambda}$ and suppose that $[x;n] = [y;p]$. We first check that π preserves the source map:

$$\begin{aligned} r(\pi([x;(m,n)])) &= r([x; m \wedge d(x), n \wedge d(x)]) \\ &= [x; m \wedge d(x)] \\ &= \pi([x;m]) \\ &= \pi(r([x;(m,n)])). \end{aligned}$$

A similar argument shows that π preserves range. To see that π preserves composition, let $z := x(0, n \wedge d(x))\sigma^{p \wedge d(y)}(y)$. Since $m \le n$ and $p \le q$, we have $m \wedge d(z) \le n \wedge d(z)$ and $n \wedge d(z) \le (n+q-p) \wedge d(z)$. Hence

$$\begin{aligned} &\pi([x;(m,n)] \circ [y;(p,q)]) \\ &\qquad = \pi([z; m, n+q-p]) \\ &\qquad = [z; (m \wedge d(z), (n+q-p) \wedge d(z))] \\ &\qquad = [z; (m \wedge d(z), n \wedge d(z))] \circ [z; (n \wedge d(z), (n+q-p) \wedge d(z))]. \end{aligned}$$

By Proposition 3.2.6(3), this is equal to

$$\begin{aligned} &[x; (m \wedge d(x), n \wedge d(x))] \circ [y; (p \wedge d(y), q \wedge d(y))] \\ &\qquad\qquad = \pi([x;(m,n)]) \circ \pi([y;(p,q)]). \end{aligned}$$

Lastly, since $m \ge m \wedge d(x)$ and $n \ge n \wedge d(x)$, we have

$$\begin{aligned} \pi(\pi([x;(m,n)])) &= \pi([x;(m \wedge d(x), n \wedge d(x))]) \\ &= [x;(m \wedge d(x), n \wedge d(x))] \\ &= \pi([x;(m,n)]). \end{aligned}$$

□

The following lemmas are used to prove that $\widetilde{\Lambda}$ is finitely aligned whenever Λ is.

Lemma 3.2.18. *Let Λ be a k-graph. Suppose that $\lambda, \mu \in \widetilde{\Lambda}$ satisfy $\lambda \in \mathcal{Z}(\mu)$, and that $i \le k$ satisfies $d(\pi(\lambda))_i > d(\pi(\mu))_i$. Then $\pi(\lambda) \in \mathcal{Z}(\pi(\mu))$, and $d(\mu)_i = d(\pi(\mu))_i$.*

Proof. Suppose that $\lambda = [x;(m, m+d(\lambda))]$. Then $\mu = [x;(m, m+d(\mu))]$, so

$$\begin{aligned} &\pi(\lambda) = [x; (m \wedge d(x), (m+d(\lambda)) \wedge d(x))], \text{ and} \\ &\pi(\mu) = [x; (m \wedge d(x), (m+d(\mu)) \wedge d(x))]. \end{aligned}$$

Since $d(\pi(\lambda))_i > d(\pi(\mu))_i$, we have

$$\min\{m_i + d(\lambda)_i, d(x)_i\} > \min\{m_i + d(\mu)_i, d(x)_i\}.$$

It then follows that $m_i \leq d(x)_i$. If $m_i + d(\lambda)_i \leq d(x)_i$, then $d(\mu) \leq d(\lambda)$ implies that $m_i + d(\mu)_i \leq d(x)_i$. If $m_i + d(\lambda) > d(x_i)$, then $d(x)_i > \min\{m_i + d(\mu)_i, d(x)_i\}$. So $d(x)_i > m_i + d(\mu)_i$, and

$$\begin{aligned} d(\pi(\mu))_i &= \min\{m_i + d(\mu)_i, d(x)_i\} - \min\{m_i, d(x)_i\} \\ &= m_i + d(\mu)_i - m_i \\ &= d(\mu)_i, \end{aligned}$$

as required. □

Lemma 3.2.19. *Let Λ be a k-graph. Let $\mu, \nu \in \widetilde{\Lambda}$. Then*

$$\pi(\mathrm{MCE}(\mu, \nu)) \subset \mathrm{MCE}(\pi(\mu), \pi(\nu)).$$

Proof. Suppose that $\lambda \in \mathrm{MCE}(\mu, \nu)$. Then $\lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$ and $d(\lambda) = d(\mu) \vee d(\nu)$. By Lemma 3.2.18 we have $\pi(\lambda) \in \mathcal{Z}(\pi(\mu)) \cap \mathcal{Z}(\pi(\nu))$, hence $d(\pi(\lambda)) \geq d(\pi(\mu)) \vee d(\pi(\nu))$.

It remains to prove that $d(\pi(\lambda)) = d(\pi(\mu)) \vee d(\pi(\nu))$. Suppose, for a contradiction, that there is some $i \leq k$ such that $d(\pi(\lambda))_i > \max\{d(\pi(\mu))_i, d(\pi(\nu))_i\}$. Then $d(\pi(\lambda))_i > d(\pi(\mu))_i$ and $d(\pi(\lambda))_i > d(\pi(\nu))_i$. Then by Lemma 3.2.18, we have $d(\pi(\mu))_i = d(\mu)_i$ and $d(\pi(\nu))_i = d(\nu)_i$. It then follows that

$$d(\lambda)_i \geq d(\pi(\lambda))_i > \max\{d(\mu)_i, d(\nu)_i\},$$

contradicting that $d(\lambda) = d(\mu) \vee d(\nu)$. □

Lemma 3.2.20. *Let Λ be a k-graph, and let $\mu, \lambda \in \iota(\Lambda^0)\widetilde{\Lambda}$ be such that $d(\lambda) = d(\mu)$ and $\pi(\lambda) = \pi(\mu)$. Then $\lambda = \mu$.*

Proof. Since $\mu, \lambda \in \iota(\Lambda^0)\widetilde{\Lambda}$ and $d(\lambda) = d(\mu)$, we can write $\lambda = [x; (0, n)]$ and $\mu = [y; (0, n)]$ for some $x, y \in \partial\Lambda$ and $n \in \mathbb{N}^k$. We will show that $(x; (0, n)) \sim (y; (0, n))$. Conditions (P2) and (P3) is trivially satisfied. Then

$$\begin{aligned} [x; (0, n \wedge d(x))] &= [x; (0 \wedge d(x), n \wedge d(x))] \\ &= \pi(\lambda) \\ &= \pi(\mu) \\ &= [y; (0 \wedge d(y), n \wedge d(y))] \\ &= [y; (0, n \wedge d(y))]. \end{aligned}$$

So $(x;(0,n\wedge d(x))) \sim (y;(0,n\wedge d(y)))$. Hence $x(0,n\wedge d(x)) = y(0,n\wedge d(y))$, verifying (P1). $\square$

REMARK 3.2.21. Let Λ be a k-graph. Suppose $[x;m] \in \widetilde{\Lambda}^0$, then $[x;(0,m)] \in \iota(\Lambda)^0\widetilde{\Lambda}[x;m]$. In particular, $\iota(\Lambda)^0\widetilde{\Lambda}v \neq \emptyset$ for all $v \in \widetilde{\Lambda}^0$.

THEOREM 3.2.22. *Let Λ be a finitely aligned k-graph. Then the extension $\widetilde{\Lambda}$ is finitely aligned. Furthermore, if Λ is row-finite, then so is $\widetilde{\Lambda}$.*

PROOF. Fix $\mu,\nu \in \widetilde{\Lambda}$, and let $\alpha \in \iota(\Lambda)^0\widetilde{\Lambda}r(\mu)$. It's easy to see that $\lambda \in \mathrm{MCE}(\mu,\nu)$ if and only if $\alpha\lambda \in \mathrm{MCE}(\alpha\mu,\alpha\nu)$. So $|\mathrm{MCE}(\mu,\nu)| = |\mathrm{MCE}(\alpha\mu,\alpha\nu)|$. Since Λ is finitely aligned, $|\mathrm{MCE}(\pi(\alpha\mu),\pi(\alpha\nu))|$ is finite, so it suffices to show that

$$|\mathrm{MCE}(\alpha\mu,\alpha\nu)| = |\mathrm{MCE}(\pi(\alpha\mu),\pi(\alpha\nu))|.$$

It follows from Lemma 3.2.19 that

$$|\mathrm{MCE}(\alpha\mu,\alpha\nu)| \geq |\mathrm{MCE}(\pi(\alpha\mu),\pi(\alpha\nu))|.$$

Suppose λ,β are distinct elements of $\mathrm{MCE}(\alpha\mu,\alpha\nu)$. Then $d(\lambda) = d(\beta)$. Since $r(\alpha\mu), r(\alpha\nu) \in \iota(\Lambda)^0$, Lemma 3.2.20 implies that $\pi(\lambda) \neq \pi(\beta)$. So $|\mathrm{MCE}(\alpha\mu,\alpha\nu)| = |\mathrm{MCE}(\pi(\alpha\mu),\pi(\alpha\nu))|$.

To show that if Λ is row-finite, then $\widetilde{\Lambda}$ is row-finite, we prove the contrapositive. Suppose $\widetilde{\Lambda}$ is not row-finite. Let $[x;m]\widetilde{\Lambda}^0$ and $i \leq k$ be such that $|[x;m]\widetilde{\Lambda}^{e_i}| = \infty$. Then for each $[y;(n,n+e_i)] \in [x;m]\widetilde{\Lambda}^{e_i}$ we have $[y;n] = [x;m]$, so $[x;(m,m+e_i)] \neq [y;(n,n+e_i)]$ only if (P1) fails. That is,

$$x(m\wedge d(x),(m+e_i)\wedge d(x)) \neq y(n\wedge d(y),(n+e_i)\wedge d(y)). \tag{3.2.13}$$

Since $|[x;m]\widetilde{\Lambda}^{e_i}| = \infty$, there are infinitely many $[y;(n,n+e_i)] \in [x;m]\widetilde{\Lambda}^{e_i}$ satisfying (3.2.13). Hence $|x(m\wedge d(x))\Lambda^{e_i}| = \infty$. $\square$

REMARK 3.2.23. Suppose that Λ is a finitely-aligned k-graph, that $x \in \partial\Lambda$ and that $E \subset x(0)\Lambda$. Since $\iota : \Lambda \to \iota(\Lambda)$ is a bijective k-graph morphism, we have $E \in x(0)\mathcal{FE}(\Lambda)$ if and only if $\iota(E) \in [x;0]\mathcal{FE}(\iota(\Lambda))$.

The following results show how sets of minimal common extensions and finite exhaustive sets in a k-graph Λ relate to those in $\widetilde{\Lambda}$.

PROPOSITION 3.2.24. *Suppose that Λ is a finitely-aligned k-graph, and that $v \in \iota(\Lambda)^0$. Then $E \in v\mathcal{FE}(\iota(\Lambda))$ implies that $E \in v\mathcal{FE}(\widetilde{\Lambda})$.*

PROOF. Since $v \in \iota(\Lambda)^0$, we have $v = [x;0]$ for some $x \in \partial\Lambda$. Remark 3.2.23 implies that $\iota^{-1}(E) \in x(0)\mathcal{FE}(\Lambda)$. Then $\lambda = [y;(0,p)]$ for some $y \in \partial\Lambda$ and $p \in \mathbb{N}^k$, and there exists $t \leq d(y)$ such that $y(0,t) \in \iota^{-1}(E)$. Hence $\iota(y(0,t)) = [y;(0,t)] \in E$. Then trivially $[y;0(t\vee p)] \in \mathrm{MCE}([y;(0,p)],[y;(0,t)])$. So $E \in v\mathcal{FE}(\widetilde{\Lambda})$. $\square$

LEMMA 3.2.25. *Let Λ be a finitely-aligned k-graph and $\mu, \nu \in \iota(\Lambda)$. Then* $\mathrm{MCE}_{\iota(\Lambda)}(\mu, \nu) = \mathrm{MCE}_{\widetilde{\Lambda}}(\mu, \nu)$.

PROOF. Since $\iota(\Lambda) \subset \widetilde{\Lambda}$, we have $\mathrm{MCE}_{\iota(\Lambda)}(\mu, \nu) \subset \mathrm{MCE}_{\widetilde{\Lambda}}(\mu, \nu)$. Suppose that $\lambda \in \mathrm{MCE}_{\widetilde{\Lambda}}(\mu, \nu)$. It suffices to show that $\lambda \in \iota(\Lambda)$. Write $\mu = [x; (0, n)], \nu = [y; (0, q)]$ and $\lambda = [z; (0, n \vee q)]$. We must show that $d(z) \geq n \vee q$. Since $\lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$, we have $[z; (0, n)] = [x; (0, n)]$ and $[z; (0, q)] = [y; (0, q)]$. This implies that $d(z) \geq n$ and $d(z) \geq q$. So $d(z) \geq n \vee q$, so $\lambda = [z; (0, n \vee q)] \in \iota(\Lambda)$. $\square$

REMARK 3.2.26. Since there is a bijection between $\Lambda^{\min}(\mu, \nu)$ and $\mathrm{MCE}(\mu, \nu)$ which maps (α, β) to $\mu\alpha = \nu\beta$, and since ι is a k-graph morphism, Lemma 3.2.25 implies that $\widetilde{\Lambda}^{\min}(\mu, \nu) = \iota(\Lambda)^{\min}(\mu, \nu)$ for all $\mu, \nu \in \iota(\Lambda)$.

3.3. Topology of Path Spaces under Desourcification

Recall from Theorem 3.1.1 that for a finitely-aligned k-graph Λ, a basis for the topology on W_Λ is given by the sets

$$\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in g} \mathcal{Z}(\mu\nu),$$

where $\mu \in \Lambda$ and $G \subset \bigcup_{i=1}^{k} \Lambda^{e_i}$.

We aim to extend the projection π defined in (3.2.12) on page 67 to the set of infinite paths in $\widetilde{\Lambda}$, and prove that its restriction to $\iota(\Lambda)^0\widetilde{\Lambda}^\infty$ is a homeomorphism onto $\iota(\partial\Lambda)$. For $x \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$, let

$$p_x = \bigvee \{p \in \mathbb{N}^k : x(0, p) \in \iota(\Lambda)\},$$

and define $\pi(x)$ to be the composition of x with the inclusion of Ω_{k,p_x} in $\Omega_{k,d(x)}$. Then $\pi(x)$ is a k-graph morphism. Our goal for this section is the following theorem.

THEOREM 3.3.1. *Let Λ be a row-finite k-graph. Then $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is a homeomorphism.*

We first show that π has range in $\iota(\Lambda)$.

PROPOSITION 3.3.2. *Let Λ be a finitely-aligned k-graph. Let $x \in \widetilde{\Lambda}^\infty$, and let p_x and $\pi(x)$ be as above. Suppose that $\{y_n : n \in \mathbb{N}^k\} \subset \partial\Lambda$ satisfy $[y_n; (0, n)] = x(0, n)$. Then*

(i) $\lim_{n \in \mathbb{N}^k} \iota(y_n) = \pi(x)$ *in $W_{\widetilde{\Lambda}}$; and*
(ii) *There exists $y \in \partial\Lambda$ such that $\pi(x) = \iota(y)$, and for $m, n \in \mathbb{N}^k$ with $m \leq n \leq p_x$ we have $\pi(x)(m, n) = \iota(y(m, n))$.*

PROOF. For part (i), fix a basic open set $\mathcal{Z}(\mu \setminus G) \subset W_{\widetilde{\Lambda}}$ containing $\pi(x)$. Let $N = \bigvee_{\nu \in G} d(\mu\nu)$. We will show that $n \geq N$ implies that $\iota(y_n) \in \mathcal{Z}(\mu \setminus G)$. Fix $n \geq N$. We first show that $\iota(y_n) \in \mathcal{Z}(\mu)$. Since $\pi(x) \in \mathcal{Z}(\mu)$, we have $d(\mu) \leq d(\pi(x)) = p_x$, and hence

$$\mu = \pi(x)(0, d(\mu)) = x(0, d(\mu)) \in \iota(\Lambda).$$

Since $n \geq d(\mu)$, we have

$$\begin{aligned}[y_n; (0, d(\mu))] &= [y_n; (0, n)](0, d(\mu)) \\ &= (x(0, n))(0, d(\mu)) \\ &= x(0, d(\mu)) \\ &= \mu.\end{aligned}$$

Let $\alpha = \iota^{-1}(\mu)$. Then $\mu = [\alpha z; (0, d(\mu))]$ for any $z \in s(\alpha)\Lambda^{\leq\infty}$. This implies that $[y_n; (0, d(\mu))] = [\alpha z; (0, d(\mu))]$. Then (P1) gives

$$\iota(y_n(0, d(\mu) \wedge d(y_n))) = \iota((\alpha z)(0, d(\mu))) = \iota(\alpha) = \mu.$$

We now show that $\iota(y_n) \notin \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu)$. Fix $\nu \in G$. If $d(y_n) \not\geq d(\mu\nu)$, then trivially we have $\iota(y_n) \notin \mathcal{Z}(\mu\nu)$. Suppose that $d(y_n) \geq d(\mu\nu)$. Since $n \geq N \geq d(\mu\nu)$ by definition,

$$\begin{aligned}x(0, d(\mu\nu)) &= (x(0, n))(0, d(\mu\nu)) \\ &= [y_n; (0, n)](0, d(\mu\nu)) \\ &= \iota(y_n(0, d(\mu\nu))) \\ &= \iota(y_n)(0, d(\mu\nu)) \in \iota(\Lambda).\end{aligned}$$

So $\iota(y_n)(0, d(\mu\nu)) = x(0, d(\mu\nu)) = \pi(x)(0, d(\mu\nu)) \neq \mu\nu$, as required.

For part (ii), we construct $y \in W_\Lambda$ such that $\pi(x) = \iota(y)$ and prove that $y \in \partial\Lambda$. For $m, n \in \mathbb{N}^k$ with $m \leq n \leq p_x$, we have $\pi(x)(m, n) \in \iota(\Lambda)$, and thus the range of $\pi(x)$ is a subset of $\iota(\Lambda)$. Since ι is injective, we can define $y : \Omega_{k,p_x} \to \Lambda$ by $\iota(y(m, n)) = \pi(x)(m, n)$. So $\iota(y) = \pi(x)$.

We now show that $y \in \partial\Lambda$. To do this, fix $m \in \mathbb{N}^k$ such that $m \leq d(y)$ and fix $E \in y(m)\mathcal{FE}(\Lambda)$. We seek $t \in \mathbb{N}^k$ such that $y(m, m+t) \in E$. Let $p := m + \bigvee_{\mu \in E} d(\mu)$.

Then

$$\begin{aligned}
[y_p;(0,m)] &= [y_p;(0,p)](0,m) && \text{since } m \le p \\
&= x(0,p)(0,m) \\
&= x(0,m) \\
&= \pi(x)(0,m) && \text{since } m \le p_x \\
&= \iota(y(0,m)) && \text{by definition of } y \\
&= [y(0,m)y';(0,m)] && \text{for some } y' \in y(m)\partial\Lambda.
\end{aligned}$$

Then (P1) implies that

$$y_p(0, m \wedge d(y_p)) = (y(0,m)y')(0, m \wedge d(y(0,m)y')) = y(0,m).$$

In particular, this implies that $y_p(m) = y(m)$. Since $y_p \in \partial\Lambda$, there exists $t \in \mathbb{N}^k$ such that $y_p(m, m+t) \in E$. So $m + t \le p$, and we have

$$\iota(y_p(m, m+t)) = [y_p;(0,p)](m, m+t) = x(0,p)(m, m+t) = x(m, m+t).$$

So $x(m, m+t) \in \iota(\Lambda)$, giving

$$\iota(y_p(m, m+t)) = x(m, m+t) = \pi(x)(m, m+t) = \iota(y(m, m+t)).$$

Finally, injectivity of ι gives

$$y(m, m+t) = y_p(m, m+t) \in E,$$

as required. □

We must check that our definition of π on $\widetilde{\Lambda}^\infty$ is compatible with (3.2.12) when we regard finite paths as k-graph morphisms. To do so, we use a few lemmas. The following lemma is also crucial in showing that π is injective on $\iota(\Lambda^0)\widetilde{\Lambda}^\infty$.

Lemma 3.3.3. *Let Λ be a finitely-aligned k-graph. Let $x \in \widetilde{\Lambda}^\infty$ and $r(x) \in \iota(\Lambda^0)$. Suppose that $w \in \partial\Lambda$ satisfies $\pi(x) = \iota(w)$. Then $x(0,n) = [w;(0,n)]$ for all $n \in \mathbb{N}^k$.*

Proof. Fix $n \in \mathbb{N}^k$. Let $z \in \partial\Lambda$ be such that $x(0,n) = [z;(0,n)]$. We aim to show that $(z;(0,n)) \sim (w;(0,n))$. That (P2) and (P3) hold follows immediately from their definitions. It remains to verify condition (P1):

$$z(0, n \wedge d(z)) = w(0, n \wedge d(\omega)). \tag{3.3.1}$$

Since $\pi(x) = \iota(w)$ we have $d(w) = p_x$. Thus

$$\begin{aligned}[][w;(0, n \wedge p_x)] &= \iota(w(0, n \wedge p_x)) \\ &= \pi(x)(0, n \wedge p_x) \\ &= x(0, n \wedge p_x) \\ &= [z;(0, n \wedge p_x)].\end{aligned}$$

So $(w;(0, n \wedge p_x)) \sim (z;(0, n \wedge p_x))$. Then (P1) implies that

$$w(0,(n \wedge p_x) \wedge d(w)) = z(0,(n \wedge p_x) \wedge d(z)).$$

Since $d(w) = p_x$, we have

$$w(0, n \wedge p_x) = z(0, n \wedge p_x). \tag{3.3.2}$$

So $d(z) \geq n \wedge p_x$, hence $n \wedge d(z) \geq n \wedge p_x$. Furthermore,

$$x(0, n \wedge d(z)) = [z;(0, n \wedge d(z))] = \iota(z(0, n \wedge d(z)) \in \iota(\Lambda).$$

This implies that $p_x \geq n \wedge d(z)$, hence $n \wedge p_x \geq n \wedge d(z)$. So $n \wedge d(z) = n \wedge p_x$. Since $d(w) = p_x$, equation (3.3.2) becomes (3.3.1) and we are done. □

Lemma 3.3.4. *Let Λ be a finitely-aligned k-graph. Suppose that $y \in \partial\Lambda$ and $m, n \in \mathbb{N}^k$ satisfy $m \leq n \leq d(y)$. Then*

$$[y;(m, n)] = \iota(y(m, n)).$$

Proof. We first show that $(y;(m, n)) \sim (\sigma^m(y);(0, n - m))$. That conditions (P1)–(P3) hold follows easily: since $m, n \leq d(y)$, we have

(P1) $m \leq n \leq d(y)$ implies $n - m \leq d(y) - m = d(\sigma^m(y))$, so we have

$$\begin{aligned}y(m \wedge d(y), n \wedge d(y)) &= y(m, n) \\ &= \sigma^m(y)(0, n - m) \\ &= \sigma^m(y)(0 \wedge d(\sigma^m(y)),(n - m) \wedge d(\sigma^m(y)));\end{aligned}$$

(P2) $m - m \wedge d(y) = m - m = 0 = 0 - 0 \wedge d(\sigma^m(y))$;

(P3) $n - m = (n - m) - 0$.

Then

$$[y;(m, n)] = [\sigma^m(y);(0, n - m)] = \iota(\sigma^m(y)(0, n - m)) = \iota(y(m, n)).$$ □

We can now show that our definitions of π for finite and infinite paths are compatible:

Proposition 3.3.5. *Let Λ be a finitely-aligned k-graph. Suppose that $x \in \widetilde{\Lambda}^\infty$, and $m \leq n \in \mathbb{N}^k$. Then $\pi(x(m, n)) = \pi(x)(m \wedge p_x, n \wedge p_x)$.*

PROOF. Fix $y \in \partial\Lambda$ such that $\pi(x) = \iota(y)$. Then

$$\begin{aligned}
\pi(x(m,n)) &= \pi([y;(m,n)] && \text{by Lemma 3.3.3}\\
&= [y;(m \wedge p_x, n \wedge p_x)] && \text{since } d(y) = p_x\\
&= \iota(y(m \wedge p_x, n \wedge p_x)) && \text{by Lemma 3.3.4}\\
&= \pi(x)(m \wedge p_x, n \wedge p_x) && \text{by Proposition 3.3.2(ii).}
\end{aligned}$$

as required. $\square$

We can now show that π restricts to a homeomorphism of $\iota(\Lambda)^0\widetilde{\Lambda}^\infty$ onto $\iota(\partial\Lambda)$. We first show that it is a bijection, then show it is continuous. Openness is the trickiest part, and the proof of it completes this section.

PROPOSITION 3.3.6. *Let Λ be a finitely-aligned k-graph. Then the map $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is a bijection.*

PROOF. We first show that π is injective. Fix $x, y \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$ such that $\pi(x) = \pi(y)$, and $w \in \partial\Lambda$ such that $\pi(x) = \pi(y) = \iota(w)$. Then Lemma 3.3.3 implies that

$$x(0,n) = [w;(0,n)] = y(0,n)$$

for every $n \in \mathbb{N}^k$. This implies that $x = y$, and so π is injective.

To see that π is onto $\iota(\partial\Lambda)$, let $w \in \partial\Lambda$ and define $x : \Omega_k \to \widetilde{\Lambda}$ by $x(p,q) = [w;(p,q)]$. Then $p_x = d(w)$, and $r(x) = x(0,0) = [w;(0,0)] = \iota(w(0,0)) \in \iota(\Lambda)$. To see that $\pi(x) = \iota(w)$, fix $m, n \in \mathbb{N}^k$ with $m \le n \le p_x = d(w)$. Then

$$\begin{aligned}
\pi(x)(m,n) &= x(m,n) && \text{by Proposition 3.3.5}\\
&= [w;(m,n)] && \text{by Lemma 3.3.3}\\
&= \iota(w(m,n)) && \text{by Lemma 3.3.4}\\
&= \iota(w)(m,n) && \text{by Proposition 3.2.14.}
\end{aligned}$$

Thus $\pi(x) = \iota(w)$, and π is onto. $\square$

PROPOSITION 3.3.7. *Let Λ be a finitely-aligned k-graph. Then $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is continuous.*

PROOF. Fix a basic open set $\mathcal{Z}(\mu \setminus G) \subset W_{\widetilde{\Lambda}}$. If $\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda) = \emptyset$, then $\pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)) = \emptyset$ is open. Suppose that $\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda) \neq \emptyset$, and fix $y \in \mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)$. Let $F = G \cap \iota(\Lambda)$. We will show that

$$\pi^{-1}(y) \in \mathcal{Z}(\mu \setminus F) \cap \big(\widetilde{\Lambda}^\infty \cap r^{-1}(\iota(\Lambda))\big) \subset \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)). \tag{3.3.3}$$

Let $x = \pi^{-1}(y)$. We first show that $x \in \mathcal{Z}(\mu)$. Since $\pi(x) = y \in \mathcal{Z}(\mu) \cap \iota(\partial\Lambda)$, we have $d(\mu) \le p_x = \pi(x)$. This implies that

$$x(0, d(\mu)) = \pi(x)(0, d(\mu)) = y(0, d(\mu)) = \mu.$$

Now we will show that $x \notin \bigcup_{\beta \in F} \mathcal{Z}(\mu\beta)$. Fix $\beta \in F$. Suppose that $d(\mu\beta) \not\le d(y) = p_x$. Then by definition of p_x, we have $x(0, d(\mu\beta)) \notin \iota(\Lambda)$. Since $\mu\beta \in \iota(\Lambda)$, we must have $x(0, d(\mu\beta)) \neq \mu\beta$. Now suppose that $d(\mu\beta) \le d(y) = p_x$, then

$$x(0, d(\mu\beta)) = \pi(x)(0, d(\mu\beta)) = y(0, d(\mu\beta)) \neq \mu\beta$$

as required. So $\pi^{-1}(y) = x \in \mathcal{Z}(\mu \setminus F)$.

We now show that

$$\mathcal{Z}(\mu \setminus F) \cap \iota(\Lambda)^0\widetilde{\Lambda}^\infty \subset \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)).$$

Let $z \in \mathcal{Z}(\mu \setminus F) \cap \iota(\Lambda)^0\widetilde{\Lambda}^\infty$. Since $z \in \mathrm{dom}(\pi)$, we just have to show that $\pi(z) \in \mathcal{Z}(\mu \setminus G)$. Recall that $y \in \mathcal{Z}(\mu) \cap \iota(\partial\Lambda)$. This implies that $\mu \in \iota(\Lambda)$, so $z(0, d(\mu)) = \mu \in \iota(\Lambda)$, and thus $\pi(z)(0, d(\mu)) = z(0, d(\mu)) = \mu$. So $\pi(z) \in \mathcal{Z}(\mu)$.

Now we show that $\pi(z) \notin \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu)$. Fix $\nu \in G$. If $d(\mu\nu) \not\le d(\pi(z))$, then trivially $\pi(z) \notin \mathcal{Z}(\mu\nu)$. So suppose that $d(\mu\nu) \le d(\pi(z)) = p_z$. If $\nu \notin \iota(\Lambda)$, then since $\mathrm{range}(\pi(z)) \subset \iota(\Lambda)$, we have $\pi(z)(0, d(\mu\nu)) \neq \mu\nu$. So suppose that $\nu \in \iota(\Lambda)$. Then $\nu \in F$ and since $z \in \mathcal{Z}(\mu \setminus F)$, we have

$$\pi(z)(0, d(\mu\nu)) = z(0, d(\mu\nu)) \neq \mu\nu.$$

So $\pi(z) \in \mathcal{Z}(\mu \setminus G)$, and thus $z \in \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda))$. So (3.3.3) holds, and hence π is continuous. $\square$

To show that π is open, and hence a homeomorphism, we use the following results.

Lemma 3.3.8. *Let Λ be a finitely-aligned k-graph. Let $\mu \in \iota(\Lambda^0)\widetilde{\Lambda}$ and let G be a finite subset of $s(\mu)\widetilde{\Lambda}$. Then*

$$\pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty) \subset \mathcal{Z}(\pi(\mu) \setminus \pi(G)) \cap \iota(\partial\Lambda).$$

Proof. Suppose that $\pi(y) \in \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty)$. Trivially $\pi(y) \in \iota(\partial\Lambda)$. We will show that $\pi(y) \in \mathcal{Z}(\pi(\mu) \setminus \pi(G))$. First we show that $\pi(y) \in \mathcal{Z}(\pi(\mu))$. Since $y(0, d(\mu)) = \mu$, we have

$$\pi(\mu) = \pi(y(0, d(\mu))) = \pi(y)(0, d(\mu) \wedge p_y).$$

So $\pi(y) \in \mathcal{Z}(\pi(\mu))$. Furthermore, $d(\pi(\mu)) = d(\mu) \wedge p_y$.

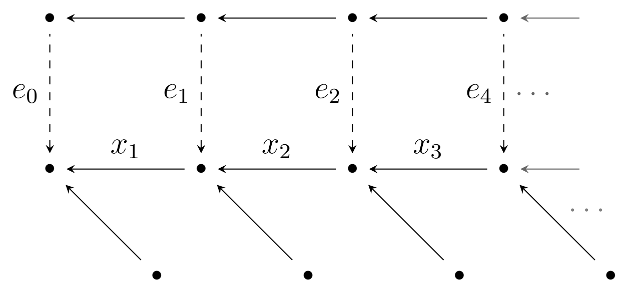
Fix $\nu \in G$. We will show that $\pi(y) \notin \mathcal{Z}(\pi(\mu\nu))$. Since $y \in \mathcal{Z}(\mu \setminus G$, we have $y(0, d(\mu\nu)) \neq \mu\nu$. Since $d(y(0, d(\mu\nu))) = d(\mu\nu)$ and $r(y) = r(\mu\nu) \in \iota(\Lambda^0)$, Lemma 3.2.20 implies that

$$\pi(\mu\nu) \neq \pi(y(0, d(\mu\nu))) = \pi(y)(0, d(\mu\nu) \wedge p_y). \qquad \square$$

The following discussion and example arose in preliminary work on a proof that π is open when Λ is row-finite and locally convex. Though we have now found a proof that works for row-finite k-graphs in general, we have retained this example since it helps illustrate some of the issues surrounding the map π.

Recall that, for a finitely aligned k-graph, the sets $\mathcal{Z}(\mu \setminus G)$ ranging over $\mu \in \Lambda$ and finite $G \subset \bigcup_{i=1}^{k} \Lambda^{e_i}$ form a basis for a locally compact Hausdorff topology on W_Λ, hereafter referred to as τ_1. The collection $\{\mathcal{Z}(\mu) : \mu \in \Lambda\}$ of cylinder sets also form a basis for a topology: they cover W_Λ, and if $x \in \mathcal{Z}(\lambda) \cap \mathcal{Z}(\nu)$, then $x \in \mathcal{Z}(x(0, d(\lambda) \vee d(\nu))) \subset \mathcal{Z}(\lambda) \cap \mathcal{Z}(\nu)$. This topology, denoted τ_2, is not necessarily Hausdorff: we cannot separate any edge from its range: if $r(f) \in \mathcal{Z}(\mu)$ then $\mu = r(f)$, and thus $f \in \mathcal{Z}(\mu)$.

It may seem reasonable to expect that $\{\mathcal{Z}(\mu) \cap \partial\Lambda : \mu \in \Lambda\}$ is a basis for the restriction of τ_1 to $\partial\Lambda$. However, this is not so. To see why, consider the following 2-graph



Let $x = x_1x_2x_3\ldots$ and let y be the boundary path beginning with e_0. So $x, y \in \partial\Lambda$. Let μ be such that $x \in \mathcal{Z}(\mu)$. Then $\mu = x_1 \ldots x_n$ for some $n \in \mathbb{N}$, so $y \in \mathcal{Z}(\mu)$ also. So the topology τ_1 is not Hausdorff even when restricted to $\partial\Lambda$. Endowed with τ_2, it is easy to see how to separate these two points: $y \in \mathcal{Z}(e_0) \cap \partial\Lambda$ and $x \in \mathcal{Z}(r(x) \setminus \{e_0\}) \cap \partial\Lambda$, and these two sets are disjoint.

If we restrict ourselves to locally convex k-graphs, τ_1 and τ_2 do restrict to the same topology on $\partial\Lambda$: certainly, for each $\mu \in \Lambda$, we can realise a cylinder set $\mathcal{Z}(\mu)$ as a set of the form $\mathcal{Z}(\mu \setminus G)$ by taking $G = \emptyset$. Now suppose that $x \in \mathcal{Z}(\mu \setminus G) \cap \partial\Lambda$. We claim that

$$\nu_x := x(0, \big(\bigvee_{\alpha \in G} d(\mu\alpha)\big) \wedge d(x))$$

satisfies

$$x \in \mathcal{Z}(\nu_x) \cap \partial\Lambda \subset \mathcal{Z}(\mu \setminus G) \cap \partial\Lambda.$$

By definition of ν_x, we have $x \in \mathcal{Z}(\nu_x) \cap \partial\Lambda$. The containment requires a little more work. Let $y \in \mathcal{Z}(\nu_x) \cap \partial\Lambda$. Then since $x \in \mathcal{Z}(\mu)$ and $d(\mu) \le d(\nu_x)$, we have

$$y(0, d(\mu)) = \nu_x(0, d(\mu))) = x(0, d(\mu)) = \mu.$$

So $y \in \mathcal{Z}(\mu)$. Fix $\alpha \in G$. We will show that $y \notin \mathcal{Z}(\mu\alpha)$. If $d(y) \not\ge d(\mu\alpha)$, then trivially $y \notin \mathcal{Z}(\mu\alpha)$. Suppose that $d(y) \ge d(\mu\alpha)$. We claim that $d(x) \ge d(\mu\alpha)$ also: suppose, for a contradiction, that $d(x) \not\ge d(\mu\alpha)$. Then there exists $i \le k$ such that $d(x)_i < d(\mu\alpha)_i$. Then $d(x)_i = d(\nu_x)_i$. Since $x \in \partial\Lambda$, we must have $x(d(\nu_x))\Lambda^{e_i} \notin x(d(\nu_x))\mathcal{FE}(\Lambda)$. Since Λ is locally convex, Lemma 3.0.19 implies that

$$y(d(\nu_x))\Lambda^{e_i} = x(d(\nu_x))\Lambda^{e_i} = \emptyset.$$

So $d(y)_i = d(\nu_x)_i = d(x)_i < d(\mu\alpha)_i$, a contradiction. Hence $d(x) \ge d(\mu\alpha)$. This implies that $d(\nu_x) \ge d(\mu\alpha)$. So $y(0, d(\mu\alpha)) = v_x(0, d(\mu\alpha)) = x(0, d(\mu\alpha)) \ne \mu\alpha$.

Although $\pi|_{\iota(\Lambda^0)\widetilde{\Lambda}^\infty}$ is in fact open for all row-finite k-graphs (Proposition 3.3.10), it behaves particularly well with respect to cylinder sets for locally convex k-graphs:

PROPOSITION 3.3.9. *Suppose that Λ is a row-finite, locally convex k-graph, and let $\mu \in \iota(\Lambda^0)\widetilde{\Lambda}$. Then*

$$\pi(\mathcal{Z}(\mu) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty) = \mathcal{Z}(\pi(\mu)) \cap \iota(\partial\Lambda).$$

In particular, π is open.

PROOF. Lemma 3.3.8 implies that $\pi(\mathcal{Z}(\mu) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty) \subset \mathcal{Z}(\pi(\mu)) \cap \iota(\partial\Lambda)$, so we need only show the opposite containment. Suppose that $x \in \mathcal{Z}(\pi(\mu)) \cap \iota(\partial\Lambda)$. By Proposition 3.3.6, there exists a unique element $y \in \iota(\Lambda)^0$ such that $\pi(y) = x$. Then $y \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$. We claim that $y \in \mathcal{Z}(\mu)$. Write $\mu = [z; (0, d(\mu))]$ and $\pi(y) = \iota(w)$ for some $z, w \in \partial\Lambda$. Then $\pi(\mu) = [z; (0, d(\mu) \wedge d(z))]$ and $y(0, d(\mu)) = [w; (0, d(\mu))]$. We claim that $(z; (0, d(\mu))) \sim (w; (0, d(\mu)))$. We must verify conditions (P1)–(P3). That (P2) and (P3) hold follows immediately from their definition. To show that (P1) is satisfied, we must show that $z(0, d(\mu) \wedge d(z)) = w(0, d(\mu) \wedge d(w))$. Since $\pi(y) = x \in \mathcal{Z}(\mu)$, we have $y \in \mathcal{Z}(\pi(\mu))$. Then

$$[w; (0, d(\pi(\mu)))] = y(0, d(\pi(\mu))) = \pi(\mu) = [z; (0, d(\mu) \wedge d(z))].$$

Since $d(w) = d(x) \ge d(\pi(\mu))$, equation (P1) from the equivalence $(w; (0, d(\pi(\mu)))) \sim (z; (0, d(\mu) \wedge d(z)))$ implies that

$$w(0, d(\pi(\mu))) = w(0, d(\pi(\mu)) \wedge d(w)) = z(0, d(\mu) \wedge d(z)).$$

Furthermore, this yields $d(\pi(\mu)) = d(\mu) \wedge d(z)$. We will show $d(\mu) \wedge d(w) = d(\pi(\mu))$. Fix $i \le k$. We argue the following cases separately:

(1) $d(\pi(\mu))_i < d(\mu)_i$, and

(2) $d(\pi(\mu))_i = d(\mu)_i$.

Case (1): we have $\min\{d(\mu)_i, d(z)_i\} = d(\pi(\mu))_i < d(\mu)_i$. So $d(z)_i < d(\mu)_i$, and thus $d(\pi(\mu))_i = d(z)_i$. Since $z \in \partial\Lambda$, this implies that $z(d(\pi(\mu)))\Lambda^{e_i} \notin z(d(\pi(\mu)))\mathcal{FE}(\Lambda)$. Then by Lemma 3.0.19, we have $z(d(\pi(\mu)))\Lambda^{e_i} = \emptyset$. Furthermore, $z(0, d(\pi(\mu))) = w(0, d(\pi(\mu)))$ implies that $w(d(\pi(\mu)))\Lambda^{e_i} = \emptyset$. So $d(w)_i = d(\pi(\mu))_i$. Then $d(\mu)_i > d(w)_i$, hence $(d(\mu) \wedge d(w))_i = d(w)_i = d(\pi(\mu))_i$.

Case (2): we have $d(w) = d(x) \geq d(\pi(\mu))$, so $d(\pi(\mu))_i = d(\mu)_i$ implies that $d(w)_i \geq d(\mu)_i$. Hence $(d(\mu) \wedge d(w))_i = d(\mu)_i = d(\pi(\mu))_i$. So $d(\mu) \wedge d(w) = d(\pi(\mu))$.

Now we have

$$w(0, d(\mu) \wedge d(w)) = w(0, d(\pi(\mu))) = z(0, d(\mu) \wedge d(z)),$$

verifying equation (P1). $\square$

We now prove that $\pi|_{\iota(\Lambda^0)\widetilde{\Lambda}^\infty}$ is open for all row-finite k-graphs. Note, however, that in this generality it does not carry basic open sets to basic open sets as it does under the additional hypothesis of local convexity.

Proposition 3.3.10. *Let Λ be a row-finite k-graph. Then $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is open.*

Proof. Fix $\pi(y) \in \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty)$, and let $\omega \in \partial\Lambda$ be such that $\pi(y) = \iota(\omega)$. Let $\lambda = y(0, \bigvee_{\nu \in G} d(\mu\nu))$, and define

$$F := \bigcup\{s(\pi(\lambda))\iota(\Lambda^{e_i}) : d(\lambda)_i > d(\pi(y))_i\}.$$

We claim that

$$\pi(y) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda) \subset \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty)).$$

First we will show that $\pi(y) \in \mathcal{Z}(\pi(\lambda))$. By Lemma 3.3.3, we can write $\lambda = [\omega; (0, d(\lambda))]$, then $\pi(\lambda) = [\omega; (0, d(\lambda) \wedge d(\omega))]$. Since $d(\omega) = d(\pi(y)) = p_y$, we have

$$\begin{aligned}\pi(y)(0, d(\pi(\lambda))) &= \pi(y)(0, d(\lambda) \wedge d(\omega))\\ &= \pi(y)(0, d(\lambda) \wedge p_y)\\ &= \pi(y(0, d(\lambda))) \qquad \text{by Proposition 3.3.5}\\ &= \pi(\lambda).\end{aligned}$$

Now we show that $\pi(y) \notin \bigcup_{f \in F} \mathcal{Z}(\pi(\lambda)f)$. Fix $f \in F$; say $d(f) = e_i$. Then $d(\pi(y))_i < d(\lambda)_i$ by definition of F. Since $d(\pi(y)) = d(\omega)$ we have $d(\omega)_i < d(\lambda)_i$, and thus

$$d(\pi(\lambda))_i = \min\{d(\lambda)_i, d(\omega)_i\} = d(\omega)_i = d(\pi(y))_i.$$

So $d(\pi(y)) \not\geq d(\pi(\lambda)f)$, and hence $\pi(y) \notin \mathcal{Z}(\pi(\lambda)f)$. So $\pi(y) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda)$ as required.

Now we will show that $\mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda) \subset \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty))$. Let $\pi(\beta) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda)$. We aim to show that $\beta \in \mathcal{Z}(\mu \setminus G)$. Since $\mathcal{Z}(\lambda) \subset \mathcal{Z}(\mu \setminus G)$, it suffices to show that $\beta \in \mathcal{Z}(\lambda)$.

We first show that $\beta \in \mathcal{Z}(\pi(\lambda) \setminus F)$. We have $\beta \in \mathcal{Z}(\pi(\beta)) \subset \mathcal{Z}(\pi(\lambda))$. We need to show that $\beta \notin \bigcup_{f \in F} \mathcal{Z}(\pi(\lambda)f)$. Fix $f \in F$. Since $\pi(\lambda)f \in \iota(\Lambda)$, we have

$$\alpha \in \mathcal{Z}(\pi(\lambda)f) \implies \pi(\alpha) \in \mathcal{Z}(\pi(\lambda)f).$$

Since $\pi(\beta) \notin \mathcal{Z}(\pi(\lambda)f)$, we have $\beta \notin \mathcal{Z}(\pi(\lambda)f)$. This is true for all $f \in F$, and thus $\beta \in \mathcal{Z}(\pi(\lambda) \setminus F)$.

If $d(\lambda) = d(\pi(\lambda))$ then $\pi(\lambda) = \lambda$ and the preceding paragraph implies that $\beta \in \mathcal{Z}(\lambda)$. So suppose that $d(\lambda) > d(\pi(\lambda))$, and let $\tau = \beta(d(\pi(\lambda)), d(\lambda))$. We know that $\beta(0, d(\pi(\lambda))) = \lambda(0, d(\pi(\lambda))) = \pi(\lambda)$, so we aim to use Lemma 3.2.20 to show that $\tau = \lambda(d(\pi(\lambda)), d(\lambda))$. Fix $i \leq k$ such that $d(\lambda)_i > d(\pi(\lambda))_i$, or equivalently that $d(\tau)_i > 0$. Then $d(\pi(\lambda)) = d(\lambda) \wedge d(\omega)$ implies that $d(\lambda)_i > d(\omega)_i = d(\pi(y))_i$. Furthermore, $\beta \in \mathcal{Z}(\pi(\lambda) \setminus F)$ implies that $\tau(0, e_i) \notin F$. In particular, $\tau(0, e_i) \notin \iota(\Lambda)$. We claim that $d(\pi(\tau)) = 0$. Suppose, for a contradiction, that $d(\pi(\tau))_j > 0$ for some $j \leq k$. Then

$$\pi(\tau)(0, e_j) = \tau(0, e_j) \notin \iota(\Lambda).$$

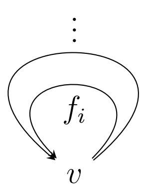
But $\pi(\tau) \in \iota(\Lambda)$ by definition of π. So we must have $d(\pi(\tau)) = 0$. This implies that

$$\pi(\tau) = r(\tau) = s(\pi(\lambda)) = r(\lambda(d(\pi(\lambda)), d(\lambda))) = \pi(\lambda(d(\pi(\lambda)), d(\lambda))).$$

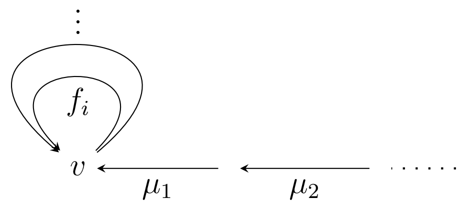
Now Lemma 3.2.20 implies that $\tau = \lambda(d(\pi(\lambda)), d(\lambda))$. Then

$$\beta(0, \lambda) = \beta(0, d(\pi(\lambda)))\tau = \pi(\lambda)\lambda(d(\pi(\lambda)), d(\lambda)) = \lambda. \qquad \square$$

EXAMPLE 3.3.11. We can see that π is not open for non-row-finite graphs by considering an earlier 1-graph example from subsection 3.2.1. Consider the graph E with $E^0 = \{v\}$ and $E^1 = \{f_i : i \in \mathbb{N}\}$ where $s(f_i) = r(f_i) = v$ for all $i \in \mathbb{N}$. That is, an infinite number of loops on a single vertex v:



Setting $\mathrm{Mor}(\Lambda) = E^*$, $\mathrm{Obj}(\Lambda) = E^0$ and $d(\lambda) = |\lambda|$ yields a 1-graph Λ. We can apply the construction §3.2 to Λ to yield the 1-graph $\widetilde{\Lambda}$ with the following skeleton.



Then $\mathcal{Z}(\mu_1) \cap \iota(\Lambda)^0\widetilde{\Lambda}^\infty = \{\mu_1\mu_2\cdots\}$ is open in $\widetilde{\Lambda}$, and $\pi(\mathcal{Z}(\mu_1) \cap \iota(\Lambda)^0\widetilde{\Lambda}^\infty) = \{v\}$. Since $\partial\Lambda = \Lambda$, any basic open set in $\partial\Lambda$ containing v is of the form $\mathcal{Z}(v\setminus G)$ for some finite $G \subset \Lambda^1$. Since Λ^1 is infinite, there is no finite $G \subset \Lambda^1$ such that $\mathcal{Z}(v\setminus G) \subset \{v\}$. Hence $\{v\}$ is not open in Λ, and π is not an open map.

Proof of Theorem 3.3.1. Propositions 3.3.6, 3.3.7 and 3.3.10 say precisely that π is a bijection, is continuous, and is open. $\square$

3.4. High-Rank Graph C^*-algebras

Definition 3.4.1. Let Λ be a finitely aligned k-graph. As is standard in the literature (for example [**8, 22**]), a *Cuntz-Krieger Λ-family* in a C^*-algebra B is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

(CK1) $\{s_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $s_\mu^* s_\nu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} s_\alpha s_\beta^*$ for all $\mu, \nu \in \Lambda$; and
(CK4) $\prod_{\mu\in E}(s_v - s_\mu s_\mu^*) = 0$ for every $v \in \Lambda^0$ and $E \in v\mathcal{FE}(\Lambda)$.

The C^*-algebra $C^*(\Lambda)$ of a k-graph Λ is the C^*-algebra generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$, which is universal in the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ family in a C^*-algebra B, then there exists a C^*-homomorphism $\pi : C^*(\Lambda) \to B$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

Remark 3.4.2. The following Theorem is stated as [**7**, Theorem 2.28]. Farthing alerted us to an issue in the proof of the theorem. It contains a claim which is proved in cases, and in the proof of Case 1 of the claim (on page 189), there is an error when i_0 is such that $m_{i_0} = d(x)_{i_0} + 1$. Then $a_{i_0} = d(x)_{i_0}$, and [**7**, Equation (2.13)] gives $t_{i_0} \leq d(z)_{i_0}$; not $t_{i_0} \geq d(z)_{i_0}$ as stated.

Theorem 3.4.3. *Let Λ be a row-finite k-graph. Let $C^*(\Lambda)$ and $C^*(\widetilde{\Lambda})$ be generated by the Cuntz-Krieger families $\{s_\lambda : \lambda \in \Lambda\}$ and $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$. Then the sum $\sum_{v\in\iota(\Lambda)^0} t_v$ converges strictly to a full projection $p \in M(C^*(\widetilde{\Lambda}))$, $pC^*(\widetilde{\Lambda})p = C^*(\{t_{\iota(\lambda)} : \lambda \in \Lambda\})$, and $s_\lambda \mapsto t_{\iota(\lambda)}$ determines an isomorphism $\varsigma : C^*(\Lambda) \cong pC^*(\widetilde{\Lambda})p$.*

Before proving Theorem 3.4.3, we need the following results.

PROPOSITION 3.4.4. *Let Λ be a finitely aligned k-graph. If $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ is a Cuntz-Krieger $\widetilde{\Lambda}$-family, then $\{t_\lambda : \lambda \in \iota(\Lambda)\}$ is a Cuntz-Krieger $\iota(\Lambda)$-family.*

PROOF. Conditions (CK1) and (CK2) hold because $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ is a Cuntz-Krieger $\widetilde{\Lambda}$-family. Conditions (CK3) and (CK4) follow from Remark 3.2.26 and Proposition 3.2.24 and the corresponding relations in $C^*(\widetilde{\Lambda})$. $\square$

REMARK 3.4.5. Let $C^*(\Lambda)$ and $C^*(\iota(\Lambda))$ be generated by the Cuntz-Krieger families $\{s_\lambda : \lambda \in \Lambda\}$ and $\{t_\lambda : \lambda \in \iota(\Lambda)\}$ respectively. Since ι is a k-graph isomorphism from Λ onto $\iota(\Lambda)$, it follows that $\{t_{\iota(\lambda)} : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in $C^*(\iota(\Lambda))$. Hence the universal property of $C^*(\Lambda)$ gives a C^*-homomorphism $C^*(\Lambda) \to C^*(\iota(\Lambda))$ such that $s_{\iota(\lambda)} \mapsto t_\lambda$ for each $\lambda \in \Lambda$. Similarly, there exists a C^*-homomorphism $C^*(\iota(\Lambda)) \to C^*(\Lambda)$ such that $t_\lambda \mapsto s_{\iota^{-1}(\lambda)}$ for each $\lambda \in \iota(\Lambda)$. Hence $C^*(\Lambda) \cong C^*(\iota(\Lambda))$.

Let Λ be a finitely aligned k-graph. Following [**22**, §3], we denote by γ the *gauge action* $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$, which is a strongly continuous action determined by $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ where $z^m = z_1^{m_1} \dots z_k^{m_k} \in \mathbb{T}$.

PROPOSITION 3.4.6. *Let Λ be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ be the universal Cuntz-Krieger $\widetilde{\Lambda}$-family which generates $C^*(\widetilde{\Lambda})$. Then $C^*(\Lambda)$ is isomorphic to the subalgebra of $C^*(\widetilde{\Lambda})$ generated by $\{t_\lambda : \lambda \in \iota(\Lambda)\}$.*

PROOF. Remark 3.4.5 tells us that $C^*(\Lambda) \cong C^*(\iota(\Lambda))$, so it suffices to prove that $C^*(\iota(\Lambda))$ is isomorphic the subalgebra of $C^*(\widetilde{\Lambda})$ generated by $\{t_\lambda : \lambda \in \iota(\Lambda)\}$.

Let $\{s_\lambda : \lambda \in \iota(\Lambda)\}$ be the universal Cuntz-Krieger $\iota(\Lambda)$-family which generates $C^*(\iota(\Lambda))$. Let A be the subalgebra of $C^*(\widetilde{\Lambda})$ generated by $\{t_\lambda : \lambda \in \iota(\Lambda)\}$. By Proposition 3.4.4, $\{t_\lambda : \lambda \in \iota(\Lambda)\}$ is a Cuntz-Krieger $\iota(\Lambda)$-family and the universal property of $C^*(\iota(\Lambda))$ gives a $*$-homomorphism $\pi : C^*(\iota(\Lambda)) \to C^*(\widetilde{\Lambda})$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \iota(\Lambda)$. Since π maps the generators of $C^*(\iota(\Lambda))$ onto the generators of A, we have $\pi(C^*(\iota(\Lambda))) = A$. It follows from [**22**, Proposition 2.12] that $t_v \neq 0$ for all $v \in \widetilde{\Lambda}^0$. It then follows that $\pi(s_v) = t_v \neq 0$ for all $v \in \iota(\Lambda)^0$.

We now show that π is injective. Let $\theta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\widetilde{\Lambda}))$ and $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\iota(\Lambda)))$ denote the gauge actions on $C^*(\widetilde{\Lambda})$ and $C^*(\iota(\Lambda))$ respectively. For all $z \in \mathbb{T}^k$ and $\lambda \in \iota(\Lambda)$ we have

$$\begin{aligned}(\theta_z \circ \pi)(s_\lambda) &= \theta_z(t_\lambda)\\ &= z^{d(\lambda)} t_\lambda\\ &= \pi(z^{d(\lambda)} s_\lambda)\\ &= (\pi \circ \gamma_z)(s_\lambda).\end{aligned}$$

So $\theta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}^k$. Hence, by [**22**, Theorem 4.2], π is injective, and thus we have $\pi(C^*(\iota(\Lambda))) \cong A$, as required. $\square$

LEMMA 3.4.7. *Let Λ be a finitely-aligned k-graph. Let $\lambda \in \widetilde{\Lambda}$, and let $\lambda' = \lambda(d(\pi(\lambda)), d(\lambda))$ so that $\lambda = \pi(\lambda)\lambda'$. Suppose that $x \in \partial\Lambda$ satisfies $\iota(r(x)) = r(\lambda')$ and $d(x) \wedge d(\lambda') = 0$. Then $\lambda' = [x; (0, d(\lambda'))]$.*

PROOF. Write $\lambda = [y; (0, d(\lambda))]$, then $\lambda' = [y; (d(\lambda) \wedge d(y), d(\lambda))]$. We must show that $(y; (d(\lambda) \wedge d(y), d(\lambda)) \sim (x; (0, d(\lambda'))$. That conditions (P2) and (P3) hold follows immediately from their definitions. It remains to show that (P1) is satisfied. That is, that $y(d(\lambda) \wedge d(y), d(\lambda) \wedge d(y)) = x(0, d(\lambda') \wedge d(x))$. Since $d(x) \wedge d(\lambda') = 0$, it suffices to show that $y(d(\lambda) \wedge d(y)) = x(0)$. We have

$$\iota(x(0)) = \iota(r(x)) = r(\lambda') = [y; d(\lambda) \wedge d(y)] = \iota(y(d(\lambda) \wedge d(y))).$$

Injectivity of ι then gives $y(d(\lambda) \wedge d(y)) = x(0)$. $\square$

LEMMA 3.4.8. *Let $\lambda \in \widetilde{\Lambda}$. Let $\lambda' = \lambda(d(\pi(\lambda)), d(\lambda))$ so that $\lambda = \pi(\lambda)\lambda'$, and define*

$$G_\lambda := \bigcup_{i=1}^{k} \{\alpha \in s(\pi(\lambda))\iota(\Lambda)^{e_i} : \mathrm{MCE}(\alpha, \lambda') = \emptyset\}.$$

Then $G_\lambda \cup \{\lambda'\} \in s(\pi(\lambda))\mathcal{FE}(\widetilde{\Lambda})$.

PROOF. Fix $\mu \in s(\pi(\lambda))\widetilde{\Lambda}$, and suppose that $\mathrm{MCE}(\mu, \alpha) = \emptyset$ for all $\alpha \in G_\lambda$. We will show that $\mathrm{MCE}(\mu, \lambda') \neq \emptyset$. Fix $\nu \in s(\mu)\widetilde{\Lambda}^{d(\mu)\vee d(\lambda')-d(\mu)}$. Then $d(\mu\nu) \geq d(\lambda')$. It suffices to show that $\mathrm{MCE}(\mu\nu, \lambda') \neq \emptyset$. Write $\mu\nu = [z; (0, d(\mu\nu))]$.

We first show that $d(\lambda') \wedge d(\pi(\mu\nu)) = 0$. We suppose that $d(\lambda') \wedge d(\pi(\mu\nu)) > 0$ and seek a contradiction. Since $d(\pi(\mu\nu)) = d(\mu\nu) \wedge d(z)$, we have $d(\lambda') \wedge d(\mu\nu) \wedge d(z) > 0$. So there exists $i \leq k$ such that $d(\lambda')_i, d(\mu\nu)_i$, and $d(z)_i$ are all greater than zero. Then

$$(\mu\nu)(0, e_i) = [z; (0, e_i)] = \iota(z)(0, e_i) = \iota(z(0, e_i)) \in \iota(\Lambda).$$

Since $\pi|_{\iota(\Lambda)} = \mathrm{id}_{\iota(\Lambda)}$ and $\pi(\lambda') = s(\pi(\lambda)) \neq \lambda'$, we have $\lambda' \notin \iota(\Lambda)$. This implies $(\mu\nu)(0, e_i) \neq \lambda'(0, e_i)$. So $\mathrm{MCE}((\mu\nu)(0, e_i), \lambda') = \emptyset$, and thus $(\mu\nu)(0, e_i) \in G_\lambda$. But $\mathrm{MCE}(\mu\nu(0, e_i), \mu\nu) \neq \emptyset$, which implies that $\mathrm{MCE}(\mu, \mu\nu(0, e_i)) \neq \emptyset$. This contradicts our supposition that $\mathrm{MCE}(\mu, \alpha) = \emptyset$ for all $\alpha \in G_\lambda$. So $d(\lambda') \wedge d(\pi(\mu\nu)) = 0$.

Since we chose ν such that $d(\mu\nu) \geq d(\lambda')$, we have

$$d(z) \wedge d(\lambda') = d(z) \wedge d(\mu\nu) \wedge d(\lambda') = d(\pi(\mu\nu)) \wedge d(\lambda') = 0$$

Since $r(\lambda') = r(\mu\nu) = \iota(r(z))$, Lemma 3.4.7 implies that $\lambda' = [z; (0, \lambda')]$. Thus $\mu\nu = [z; (0, \mu\nu)] \in \mathrm{MCE}(\mu\nu, \lambda')$. $\square$

Proof of Theorem 3.4.3. Let $A := C^*(\{t_\lambda : \lambda \in \iota(\Lambda)\})$. Then $A \cong C^*(\Lambda)$ by Proposition 3.4.6. We will show that A is a full corner of $C^*(\widetilde{\Lambda})$.

Following the argument for [**19**, Lemma 2.10], the sum $\sum_{v \in \iota(\Lambda)^0} t_v$ converges strictly in $M(C^*(\widetilde{\Lambda}))$ to a projection p satisfying

$$pt_\lambda t_\mu^* p = \begin{cases} t_\lambda t_\mu^* & \text{if } \widetilde{r}(\lambda), \widetilde{r}(\mu) \in \iota(\Lambda)^0; \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1}$$

We first show that p is a full projection in $M(C^*(\widetilde{\Lambda}))$, suppose that J is an ideal in $C^*(\widetilde{\Lambda})$ such that $pC^*(\widetilde{\Lambda})p \subset J$. Let $v \in \widetilde{\Lambda}^0$, and $\alpha \in \pi(v)\widetilde{\Lambda}v$. Then $t_\alpha = t_{r(\alpha)}t_\alpha$. Since $r(\alpha) = \pi(v) \in \iota(\Lambda)$, we have $t_{r(\alpha)} \in pC^*(\widetilde{\Lambda})p \subset J$. So $t_\alpha \in J$, and hence $t_v = t_\alpha^* t_\alpha \in J$. So for any $\lambda \in \widetilde{\Lambda}$, we have $t_\lambda = t_{r(\lambda)}t_\lambda \in J$. So $\{t_\lambda : \lambda \in \widetilde{\Lambda}\} \subset J$, and thus $J = C^*(\widetilde{\Lambda})$.

We now show that $A = pC^*(\widetilde{\Lambda})p$. It follows from (3.4.1) that $A \subset pC^*(\widetilde{\Lambda})p$. Now suppose that $\lambda, \mu \in \iota(\Lambda)^0\widetilde{\Lambda}$. We will show that $pt_\lambda t_\mu^* p \in A$. In order to show this, we first show that

$$\lambda(d(\pi(\lambda)), d(\lambda)) = \mu(d(\pi(\mu)), d(\mu)). \tag{3.4.2}$$

Let $x, y \in \partial\Lambda$ such that $\lambda = [x; (0, d(\lambda))]$ and $\mu = [y; (0, d(\mu))]$. Let

$$\lambda' = \lambda(d(\pi(\lambda)), d(\lambda)) \text{ and } \mu' = \mu(d(\pi(\mu)), d(\mu)),$$

so $\lambda = \pi(\lambda)\lambda'$ and $\mu = \pi(\mu)\mu'$. We have

$$\begin{aligned} \lambda' &= [x; (d(\lambda) \wedge d(x), d(\lambda)] \qquad \text{and} \\ \mu' &= [y; (d(\mu) \wedge d(y), d(\mu))]. \end{aligned}$$

We claim $\lambda' = \mu'$. Condition (P2) is trivially satisfied. Since $[x; d(\lambda)] = \widetilde{s}(\lambda) = \widetilde{s}(\mu) = [y; d(\mu)]$, (V1) and (V2) imply that

$$\begin{aligned} x(d(\lambda) \wedge d(x)) &= y(d(\mu) \wedge d(y)) \text{ and} \\ d(\lambda) - d(\lambda) \wedge d(x) &= d(\mu) - d(\mu) \wedge d(x), \end{aligned}$$

which are precisely equations (P1) and (P3). Hence $\lambda' = \mu'$.

Claim 3.4.8.1. *Let* $G_\lambda := \bigcup_{i=1}^k \{\alpha \in s(\pi(\lambda))\iota(\Lambda)^{e_i} : \mathrm{MCE}(\alpha, \lambda') = \emptyset\}$. *Then*

$$t_{\lambda'} t_{\lambda'}^* = \prod_{\alpha \in G_\lambda} \left(t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*\right)$$

Proof. Lemma 3.4.8 implies that $G_\lambda \cup \{\lambda'\}$ is finite exhaustive, so (CK4) implies that

$$\prod_{\beta \in G_\lambda \cup \{\lambda'\}} \left(t_{s(\pi(\lambda))} - t_\beta t_\beta^*\right) = 0.$$

Furthermore,

$$\prod_{\beta\in G_\lambda\cup\{\lambda'\}}\big(t_{s(\pi(\lambda))}-t_\beta t_\beta^*\big)=\Big(\prod_{\alpha\in G_\lambda}\big(t_{s(\pi(\lambda))}-t_\alpha t_\alpha^*\big)\Big)(t_{s(\pi(\lambda))}-t_{\lambda'}t_{\lambda'}^*)$$
$$=\Big(\prod_{\alpha\in G_\lambda}(t_{s(\pi(\lambda))}-t_\alpha t_\alpha^*)\Big)-\Big(t_{\lambda'}t_{\lambda'}^*\prod_{\alpha\in G_\lambda}(t_{s(\pi(\lambda))}-t_\alpha t_\alpha^*)\Big).$$

Fix $\alpha\in G_\lambda$. By [**22**, Lemma 2.7(i)],

$$\begin{aligned}t_{\lambda'}t_{\lambda'}^*(t_{s(\pi(\lambda))}-t_\alpha t_\alpha^*)&=t_{\lambda'}t_{\lambda'}^*-\sum_{\gamma\in\mathrm{MCE}(\lambda',\alpha)}t_\gamma t_\gamma^*\\&=t_{\lambda'}t_{\lambda'}^*.\end{aligned}$$

So

$$0=\prod_{\beta\in G_\lambda\cup\{\lambda'\}}\big(t_{s(\pi(\lambda))}-t_\beta t_\beta^*\big)=\prod_{\alpha\in G_\lambda}\big(t_{s(\pi(\lambda))}-t_\alpha t_\alpha^*\big)-t_{\lambda'}t_{\lambda'}^*,$$

as required. $\square_{\text{Claim}}$

Now we put the pieces together:

$$\begin{aligned}pt_\lambda t_\mu^*p&=t_\lambda t_\mu^*\\&=t_{\pi(\lambda)}t_{\lambda'}t_{\lambda'}^*t_{\pi(\mu)}^*\qquad\text{by (3.4.2)}\\&=t_{\pi(\lambda)}\prod_{\alpha\in G_\lambda}\big(t_{s(\pi(\lambda))}-t_\alpha t_\alpha^*\big)t_{\pi(\mu)}^*\qquad\text{by Claim 3.4.8.1.}\end{aligned}$$

which belongs to A since $\pi(\lambda),\pi(\mu),\alpha\in\iota(\Lambda)$ for all $\alpha\in G_\lambda$. So $A=pC^*(\widetilde{\Lambda})p$. $\square$

3.5. The Diagonal and the Spectrum

For k-graph Λ, we call $C^*\{s_\mu s_\mu^*:\mu\in\Lambda\}\subset C^*(\Lambda)$ the *diagonal* C^*-algebra of Λ and denote it D_Λ, dropping the subscript when confusion is unlikely.

THEOREM 3.5.1. *Let Λ be a row-finite higher-rank graph. Let $p\in M(C^*(\widetilde{\Lambda}))$ and $\varsigma:C^*(\Lambda)\to pC^*(\widetilde{\Lambda})p$ be from Theorem 3.4.3. Then the restriction $\varsigma|_{D_\Lambda}=:\rho$ is an isomorphism of D_Λ onto $pD_{\widetilde{\Lambda}}p$. Let $\pi:\iota(\Lambda)^0\widetilde{\Lambda}^\infty\to\iota(\partial\Lambda)$ be the homeomorphism from Theorem 3.3.1, then there exist homeomorphisms $h_\Lambda:\partial\Lambda\to\Delta(D_\Lambda)$ and $\eta:\iota(\Lambda)^0\widetilde{\Lambda}^\infty\to\Delta(pD_{\widetilde{\Lambda}}p)$ such that the following diagram commutes.*

$$\begin{tikzcd}\iota(\Lambda)^0\widetilde{\Lambda}^\infty \arrow[r,"\pi"] \arrow[d,"\eta"'] & \iota(\partial\Lambda) \arrow[d,"h_\Lambda\circ\iota^{-1}"]\\ \Delta(pD_{\widetilde{\Lambda}}p) \arrow[r,"\rho^*"'] & \Delta(D_\Lambda)\end{tikzcd}$$

To prove this, we use several technical results. As in [**20**], for a finite subset $F \subset \Lambda$, define $\vee F := \bigcup_{G \subset F} \mathrm{MCE}(G) = \bigcup_{G \subset F} \big\{\lambda \in \bigcap_{\mu \in G} \mu\Lambda : d(\lambda) = \bigvee_{\mu \in G} d(\mu)\big\}$.

Lemma 3.5.2. *Let Λ be a finitely aligned k-graph and let F be a finite subset of Λ. Suppose that $r(\lambda) \in F$ for each $\lambda \in F$. For $\mu \in F$, define*

$$q_\mu^{\vee F} := s_\mu s_\mu^* \prod_{\mu\mu' \in \vee F \setminus \{\mu\}} (s_\mu s_\mu^* - s_{\mu\mu'} s_{\mu\mu'}^*).$$

Then the $q_\mu^{\vee F}$ are mutually orthogonal projections in $\mathrm{span}\{s_\mu s_\mu^* : \mu \in \vee F\}$*, and for each $\nu \in \vee F$*

$$s_\nu s_\nu^* = \sum_{\nu\nu' \in \vee F} q_{\nu\nu'}^{\vee F} \tag{3.5.1}$$

Proof. Since

$$s_\mu s_\mu^* \prod_{\mu\mu' \in \vee F \setminus \{\mu\}} (s_\mu s_\mu^* - s_{\mu\mu'} s_{\mu\mu'}^*) = s_\mu s_\mu^* \prod_{\mu\mu' \in \vee F, d(\mu') \neq 0} (s_{r(\mu)} - s_{\mu\mu'} s_{\mu\mu'}^*),$$

[**20**, Proposition 8.6] says precisely that the $q_\mu^{\vee F}$ are mutually orthogonal projections. That

$$s_\nu s_\nu^* = \sum_{\nu\nu' \in \vee F} q_{\nu\nu'}^{\vee F}$$

is established in the proof of [**20**, Proposition 8.6] on page 421. □

Remark 3.5.3. Replacing F with $\vee F$ in Remark 2.4.3, the same argument gives

$$q_\mu^{\vee F} = s_\mu \Big(\prod_{\substack{\mu' \in s(\mu)\Lambda \setminus \{s(\mu)\} \\ \mu\mu' \in \vee F}} (s_{s(\mu)} - s_{\mu'} s_{\mu'}^*) \Big) s_\mu^*.$$

Proposition 3.5.4. *Let Λ be a finitely aligned k-graph. Then $D = \overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in \Lambda\}$, and for each $x \in \partial\Lambda$ there exists a unique $h(x) \in \Delta(D)$ such that*

$$h(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x = \mu\mu' \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $x \mapsto h(x)$ is a homeomorphism $h : \partial\Lambda \to \Delta(D)$.

Proof. We will first show that $D = \overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in \Lambda\}$. Let $\mu, \nu \in \Lambda$. Then by multiplying $s_\mu^* s_\nu$ on the left by s_μ and on the right by s_ν^*, it follows from (CK3) that

$$(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} s_\lambda s_\lambda^*.$$

So $\mathrm{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$ is closed under multiplication and is thus is a $*$-subalgebra of $C^*(\Lambda)$. Hence the closed span is a C^*-algebra. Since D is the smallest C^*-subalgebra of $C^*(\Lambda)$ containing the generators $\{s_\mu s_\mu^*\}$, we have $D = \overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in \Lambda\}$.

Fix $x \in \partial\Lambda$ and $\sum_{\mu \in F} b_\mu s_\mu s_\mu^* \in \operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$. By setting extra coefficients to zero we can assume that each path in F has its range in F, and write

$$\sum_{\mu \in F} b_\mu s_\mu s_\mu^* = \sum_{\mu \in \vee F} b_\mu s_\mu s_\mu^*.$$

Let $n = \bigvee\{p \in \mathbb{N}^k : x(0,p) \in \vee F\}$. Since $\vee F$ is a finite set of finite paths, n is finite. Since $\vee F$ is closed under minimal common extensions, $x(0,n) \in \vee F$. Furthermore, since $x \in \partial\Lambda$, we have

$$F_x := \{\mu' \in x(n)\Lambda \setminus \{x(n)\} : x(0,n)\mu' \in \vee F\} \notin x(n)\mathcal{FE}(\Lambda).$$

So there exists $\nu \in x(n)\Lambda$ such that for each $\mu' \in F_x$, $\operatorname{MCE}(\nu,\mu') = \emptyset$. Then $s_\nu s_\nu^* s_{\mu'} s_{\mu'}^* = 0$ for all $\mu' \in F_x$. Applying Lemma A.0.7 with $p = s_{x(n)}$, $q_0 = s_\nu s_\nu^*$ and $Q = \{s_{\mu'} s_{\mu'}^* : \mu' \in F_x\}$, we have $\prod_{\mu' \in F_x}(s_{x(n)} - s_{\mu'} s_{\mu'}^*) \neq 0$. So

$$q^F_{x(0,n)} = s_{x(0,n)} \prod_{\mu' \in F_x} (s_{x(n)} - s_{\mu'} s_{\mu'}^*) s_{x(0,n)}^* \neq 0.$$

Then by Lemma A.0.6 we have

$$\begin{aligned}
\Big\| \sum_{\nu \in \vee F} b_\mu s_\mu s_\mu^* \Big\| &= \Big\| \sum_{\nu \in \vee F} \Big(\sum_{\substack{\mu \in \vee F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big) q_\nu^{\vee F} \Big\| && \text{by Lemma 3.5.1} \\
&= \max_{\{\nu \in \vee F : q_\nu^{\vee F} \neq 0\}} \Big| \sum_{\substack{\mu \in \vee F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big| && \text{by Lemma A.0.6} \\
&\geq \Big| \sum_{\substack{\mu \in \vee F \\ x(0,n) \in \mathcal{Z}(\mu)}} b_\mu \Big| && \text{since } q^{\vee F}_{x(0,n)} \neq 0 \\
&= \Big| \sum_{\substack{\mu \in F \\ x(0,n) \in \mathcal{Z}(\mu)}} b_\mu \Big| && \text{since } b_\mu = 0 \text{ for } \mu \in \vee F \setminus F.
\end{aligned}$$

Hence the formula

$$h(x)\Big(\sum_{\mu \in F} b_\mu s_\mu s_\mu^* \Big) = \sum_{\substack{\mu \in F \\ x \in \mathcal{Z}(\mu)}} b_\mu, \tag{3.5.2}$$

determines a norm-decreasing linear map on $\operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$.

We now show $h(x)$ is a homomorphism. Since $h(x)$ is continuous and linear, it suffices to show that

$$h(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) = h(x)(s_\mu s_\mu^*)\ h(x)(s_\alpha s_\alpha^*). \tag{3.5.3}$$

Calculating the right hand side of (3.5.3) yields

$$h(x)(s_\mu s_\mu^*)\ h(x)(s_\alpha s_\alpha^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the left hand side of (3.5.3) gives

$$h(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) = h(x)\Big(\sum_{\lambda\in\mathrm{MCE}(\mu,\alpha)} s_\lambda s_\lambda^*\Big).$$

Since $d(\lambda) = d(\mu) \vee d(\alpha)$ for all $\lambda \in \mathrm{MCE}(\mu,\alpha)$, there exists at most one $\lambda \in \mathrm{MCE}(\mu,\alpha)$ such that $x \in \mathcal{Z}(\lambda)$. Such a λ exists if and only if $x \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\alpha)$, so we have

$$h(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\alpha) \cap \mathcal{Z}(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have established (3.5.3), and hence $h(x)$ is a homomorphism. Since $h(x)$ is a nonzero bounded linear map on a dense subspace of D, and since multiplication is continuous, $h(x)$ extends uniquely to a nonzero homomorphism $h(x) : D \to \mathbb{C}$.

We claim the map $h : \partial\Lambda \to \Delta(D)$ is a homeomorphism. The trickiest part is to show h is onto:

Claim 3.5.4.1. *The map h is surjective.*

Proof. Fix $\phi \in \Delta(D)$. We seek $x \in \partial\Lambda$ such that $h(x) = \phi$.

We know $\phi(p) \in \{0,1\}$ for any projection $p \in D$, and that for each $n \in \mathbb{N}^k$, $\{s_\mu s_\mu^* : d(\mu) = n\}$ are mutually orthogonal projections. It follows that for each $n \in \mathbb{N}^k$ there exists at most one $\nu^n \in \Lambda^n$ such that $\phi(s_{\nu^n} s_{\nu^n}^*) = 1$.

Let S denote the set of n for which such ν^n exist. If $\nu = \mu\nu'$ and $\phi(s_\nu s_\nu^*) = 1$, then

$$1 = \phi(s_\nu s_\nu^*) = \phi(s_\nu s_\nu^* s_\mu s_\mu^*) = \phi(s_\nu s_\nu^*)\phi(s_\mu s_\mu^*) = \phi(s_\mu s_\mu^*).$$

This implies that if $n \in S$ and $m \le n$, then $m \in S$ and $\nu^m = \nu^n(0,m)$. Set $N := \vee S$, and define $x : \Omega_{k,N} \to \Lambda$ by $x(p,q) = \nu^q(p,q)$. Then x is a k-graph morphism because each ν^q is.

We now show $x \in \partial\Lambda$. Fix $n \in \mathbb{N}^k$ such that $n \le d(x)$, and $E \in x(n)\mathcal{FE}(\Lambda)$. We seek $m \in \mathbb{N}^k$ such that $x(n, n+m) \in E$. Since E is finite exhaustive, (CK4) says that

$$\prod_{\lambda\in E} (s_{x(n)} - s_\lambda s_\lambda^*) = 0.$$

Multiplying on the left by $s_{x(0,n)}$ and on the right by $s_{x(0,n)}^*$ yields

$$\prod_{\lambda\in E} (s_{x(0,n)} s_{x(0,n)}^* - s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*) = 0.$$

Since ϕ is a homomorphism, this implies that

$$\prod_{\lambda\in E} (\phi(s_{x(0,n)} s_{x(0,n)}^*) - \phi(s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*)) = 0.$$

So there exists $\lambda \in E$ such that

$$\begin{aligned} 0 &= \phi(s_{x(0,n)}s^*_{x(0,n)}) - \phi(s_{x(0,n)\lambda}s^*_{x(0,n)\lambda}) \\ &= \phi(s_{\nu^n}s^*_{\nu^n}) - \phi(s_{x(0,n)\lambda}s^*_{x(0,n)\lambda}) \\ &= 1 - \phi(s_{x(0,n)\lambda}s^*_{x(0,n)\lambda}) \end{aligned}$$

Hence $\phi(s_{x(0,n)\lambda}s^*_{x(0,n)\lambda}) = 1$, giving $x(0,n)\lambda = \nu^{n+d(\lambda)} = x(0, n + d(\lambda))$. Hence $x \in \partial\Lambda$.

Now we must show that $h(x) = \phi$. For each $\mu \in \Lambda$ we have

$$\begin{aligned} \phi(s_\mu s^*_\mu) = 1 &\iff d(\mu) \in S \text{ and } \nu^{d(\mu)} = \mu \\ &\iff x(0, d(\mu)) = \mu \\ &\iff h(x)(s_\mu s^*_\mu) = 1. \end{aligned}$$

Since both $\phi(s_\mu s^*_\mu)$ and $h(x)(s_\mu s^*_\mu)$ only take values in $\{0, 1\}$, it follows that $h(x) = \phi$. $\square_{\text{Claim}}$

To see that h is injective, suppose that $h(x) = h(y)$. Then for each $n \in \mathbb{N}^k$, we have

$$h(y)(s_{x(0,n\wedge d(x))}s^*_{x(0,n\wedge d(x))}) = h(x)(s_{x(0,n\wedge d(x))}s^*_{x(0,n\wedge d(x))}) = 1.$$

Hence $y(0, n \wedge d(x)) = x(0, n \wedge d(x))$. By symmetry, we also have $y(0, n \wedge d(y)) = x(0, n \wedge d(y))$ for all n. In particular, $d(x) = d(y)$ and $y(0, n) = x(0, n)$ for all $n \leq d(x)$. Thus $x = y$.

We now show that h is continuous. Suppose $x^n \to x$. We must show that $h(x^n)(a) \to h(x)(a)$ for each $a \in D$. We will first show that for each $\mu \in \Lambda$, there exists N such that $n \geq N$ implies that $h(x^n)(s_\mu s^*_\mu) = h(x)(s_\mu s^*_\mu)$. Since $x^n \to x$, there exists N_0 such that $n \geq N_0$ implies that $x^n(0, d(\mu) \wedge d(x)) = x(0, d(\mu) \wedge d(x))$. Fix $n \geq N_0$. Suppose that $h(x)(s_\mu s^*_\mu) = 1$. Then $x(0, d(\mu)) = \mu$. In particular, $x^n(0, d(\mu)) = x(0, d(\mu)) = \mu$, so $h(x^n)(s_\mu s^*_\mu) = 1$. Now suppose that $h(x)(s_\mu s^*_\mu) = 0$. Then $x(0, d(\mu) \wedge d(x)) \neq \mu$, so $x^n(0, d(\mu) \wedge d(x)) \neq \mu$, and thus $h(x^n)(s_\mu s^*_\mu) = 0$. Since $h(x)$ and the $h(x^n)$ are linear, $h(x^n)$ converges to $h(x)$ in $\operatorname{span}\{s_\mu s^*_\mu : \mu \in E\}$. An $\varepsilon/3$ argument similar to that on page 33 shows that h is continuous on D.

Finally, we show that h is open. Since h is a bijection, it suffices to show that h^{-1} is continuous. Suppose that $h(x^n) \to h(x)$. We will show $x^n \to x$. Fix a basic open set $\mathcal{Z}(\mu)$ containing x, so $h(x)(s_\mu s^*_\mu) = 1$. We seek $N \in \mathbb{N}^k$ such that $n \geq N$ implies that $x^n \in \mathcal{Z}(\mu)$. Since $h(x^n) \to h(x)$ in $\Delta(D)$ and $h(x^n)(s_\mu s^*_\mu) \in \{0, 1\}$ for all n, there exists $N \in \mathbb{N}^k$ such that $n \geq N$ implies $h(x^n)(s_\mu s^*_\mu) = 1$. So $x^n \in \mathcal{Z}(\mu)$ as required. $\square$

We can now prove our main result.

Proof of Theorem 3.5.1. Let Λ be a row-finite k-graph, and $\widetilde{\Lambda}$ be the desourcification described in Proposition 3.2.10. Let $\{s_\lambda : \lambda \in \Lambda\}$ and $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ be universal Cuntz-Krieger families in $C^*(\Lambda)$ and $C^*(\widetilde{\Lambda})$. Let A be the C^*-subalgebra of $C^*(\widetilde{\Lambda})$ generated by $\{t_\lambda : \lambda \in \iota(\Lambda)\}$, and define the diagonal subalgebra of A by $D_A := \overline{\operatorname{span}}\{t_\lambda t_\lambda^* : \lambda \in \iota(\Lambda)\}$. Replacing $t_\lambda t_\mu^*$ with $t_\lambda t_\lambda^*$ in the proof Theorem 3.4.3 yields $D_A \cong pD_{\widetilde{\Lambda}}p$. Since $A \cong C^*(\Lambda)$, it follows that $D_A \cong D_\Lambda$. Thus $D_\Lambda \cong pD_{\widetilde{\Lambda}}p$ as required.

We now construct η and show that it is a homeomorphism. That p commutes with $D_{\widetilde{\Lambda}}$ implies that $pD_{\widetilde{\Lambda}}p$ is an ideal in $D_{\widetilde{\Lambda}}$. Then [**23**, Propositions A26(a) and A27(b)] imply that map $k : \phi \mapsto \phi|_{pD_{\widetilde{\Lambda}}p}$ is a homeomorphism of $\{\phi \in \Delta(D_{\widetilde{\Lambda}}) : \phi|_{pD_{\widetilde{\Lambda}}p} \neq 0\}$ onto $\Delta(pD_{\widetilde{\Lambda}}p)$. Since $\widetilde{\Lambda}$ is row finite with no sources, $\partial\widetilde{\Lambda} = \widetilde{\Lambda}^\infty$. Let $h_{\widetilde{\Lambda}} : \widetilde{\Lambda}^\infty \to \Delta(D_{\widetilde{\Lambda}})$ be the homeomorphism obtained from Proposition 3.5.4. Fix $x \in \iota(\Lambda)^0\widetilde{\Lambda}^\infty$. Then there exists $\lambda \in \widetilde{\Lambda}$ such that $h_{\widetilde{\Lambda}}(x)|_{pD_{\widetilde{\Lambda}}p}(t_\lambda t_\lambda^*) \neq 0$, so $h_{\widetilde{\Lambda}}(x) \in \operatorname{dom}(k)$ for all $x \in \iota(\Lambda)^0\widetilde{\Lambda}^\infty$. We define $\eta := k \circ h_{\widetilde{\Lambda}}|_{\iota(\Lambda)^0\widetilde{\Lambda}^\infty} : \iota(\Lambda)^0\widetilde{\Lambda}^\infty \to \Delta(pD_{\widetilde{\Lambda}}p)$.

Let $\pi : \iota(\Lambda)^0\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ be the homeomorphism from Theorem 3.3.1, and let ρ be the isomorphism which maps D_Λ onto $pD_{\widetilde{\Lambda}}p$. Let $\rho^* : \Delta(pD_{\widetilde{\Lambda}}p) \to \Delta(D_\Lambda)$ be the $*$-homomorphism given by $\rho^*(\phi) = \phi \circ \rho$.

We now show that diagram on page 85 commutes. Since $(h_\Lambda \circ \iota^{-1} \circ \pi)(x)$ and $\eta(x)$ are homomorphisms, and since ρ is an isomorphism, it suffices to fix $x \in \iota(\Lambda)^0\widetilde{\Lambda}^\infty$ and $\mu \in \Lambda$ and show that

$$(h_\Lambda \circ \iota^{-1} \circ \pi)(x)(s_\mu s_\mu^*) = (\rho^* \circ \eta)(x)(s_\mu s_\mu^*). \tag{3.5.4}$$

Let $\omega \in \partial\Lambda$ be the element such that $\pi(x) = \iota(\omega)$. Then the left-hand side of (3.5.4) becomes

$$(h_\Lambda \circ \iota^{-1} \circ \pi)(x)(s_\mu s_\mu^*) = h_\Lambda(w)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \omega \in \mathcal{Z}(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

The right-hand side of (3.5.4) simplifies to

$$\begin{aligned}(\rho^* \circ \eta)(x)(s_\mu s_\mu^*) &= \eta(x)(\rho(s_\mu s_\mu^*)) \\ &= h_{\widetilde{\Lambda}}(x)(t_{\iota(\mu)} t_{\iota(\mu)}^*) \qquad\qquad \text{since } r(x) \in \iota(\Lambda)^0 \\ &= \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\iota(\mu)) \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

We claim that $x \in \mathcal{Z}(\iota(\mu))$ if and only if $\omega \in \mathcal{Z}(\mu)$. Suppose that $x \in \mathcal{Z}(\iota(\mu))$. Since $\mu \in \Lambda$ and $\pi(x) = \iota(\omega)$, we have $\pi(x(0, d(\mu))) = \pi(\iota(\mu)) = \iota(\mu)$. So

$d(\pi(x(0, d(\mu)))) = d(\mu)$, and thus $d(x) \wedge d(w) \geq d(\mu)$. So $d(\omega) \geq d(\mu)$. Then

$$\begin{aligned} x \in \mathcal{Z}(\iota(\mu)) &\iff x(0, d(\mu)) = \iota(\mu) && \text{since } \iota \text{ preserves degree} \\ &\iff [\omega; (0, d(\mu))] = \iota(\mu) && \text{by Lemma 3.3.3} \\ &\iff \iota(\omega(0, d(\mu))) = \iota(\mu) && \text{by Lemma 3.3.4} \\ &\iff \omega(0, d(\mu)) = \mu && \text{since } \iota \text{ is a injective} \\ &\iff \omega \in \mathcal{Z}(\mu). \end{aligned}$$

So equation (3.5.4) holds, and we are done. □

3.6. From k-Coloured to Rank-k

In [**11**] Hazlewood introduced the notion of a k-coloured graph as a formal realisation of the 1-skeleton of a k-graph (see Remark 3.0.8 and Theorem 3.6.1). He showed how to build a k-graph from a k-coloured graph together with a set of factorisation rules for bi-coloured paths, providing a very elegant and concrete proof of a folklore result first asserted in [**10**]. In this section we show how to think of the path-space of the k-graph as a quotient of the path-space of the k-coloured graph, and investigate how the topologies on the two are related. For row-finite k-graphs, it turns out that the topology on the k-graph is precisely the quotient topology inherited from its k-coloured skeleton. However, we show in Example 3.6.5 that for non-row-finite k-graphs, the quotient map from the path space of the k-coloured graph to the path space of the k-graph need not even be continuous.

Let $\mathbb{F}_k^+$ denote the free semigroup with k generators $\{c_1, \dots, c_k\}$. A *k-coloured graph* is a directed graph E together with a map $c : E^1 \to \{c_1, \dots, c_k\}$. The map c extends to a functor $c : E^* \to \mathbb{F}_k^+$. Write π_k for the quotient map $\pi_k : \mathbb{F}_k^+ \to \mathbb{N}^k$ determined by $\pi_k(c_i) = e_i$. Then the *degree* of a path $x \in E^*$ is $d(x) := \pi_k(c(x))$. So for example a $c_1c_2c_2$-coloured path in a 3-coloured graph has degree $(1, 0, 0) + (0, 1, 0) + (0, 1, 0) = (1, 2, 0)$; and a $c_2c_2c_1$-coloured path has a different colouring but the same degree.

A *coloured-graph morphism* is a graph morphism ψ between k-coloured graphs which preserves colour. That is, $c(\psi(x)) = c(x)$ for every $x \in E^*$.

For $m \in (\mathbb{N} \cup \{\infty\})^k$, we define a coloured graph $E_{k,m}$ by

$$\begin{aligned} E^0_{k,m} &= \{n \in \mathbb{N}^k : 0 \le n \le m\}, \\ E^1_{k,m} &= \{n + v_i : n \in E^0_{k,m}, i \in \{1, \dots, k\}, n + e_i \in E^0_{k,m}\}, \\ r(n + v_i) &= n \\ s(n + v_i) &= n + e_i \\ c(n + v_i) &= c_i. \end{aligned}$$

For $e \in E^1$ with $c(e) = c_j$, it is unambiguous and often useful to write $v_{c(e)} := v_j$. For a coloured graph morphism $\lambda : E_{k,m} \to E$ we say λ has degree m, and define $r(\lambda) := \lambda(0)$ and $s(\lambda) := \lambda(m)$.

The following definitions, up to the statement of Theorem 3.6.1, are as given by Hazlewood in [**11**]. Given a k-coloured graph E and distinct $i, j \in \{1, \dots, k\}$, a $\{i, j\}$-*square* in E is a coloured-graph morphism $\lambda : E_{k,e_i+e_j} \to E$. When i, j are not important we simply call λ a *square* in E. If $\lambda : E_{k,m} \to E$ is a coloured graph morphism and ψ is a square in E, then ψ *occurs in* λ if there exists $n \in \mathbb{N}^k$ such that $\psi(x) = \lambda(x + n)$ for all $x \in E_{k,e_i+e_j}$.

Let E be a k-coloured graph. A *complete collection of squares* is a collection $\mathcal{C}$ of squares in E such that for each $x \in E^*$ with $c(x) = c_i c_j$, there exists a unique $\mu \in \mathcal{C}$ such that $x = \mu(v_i)\mu(e_i + v_j)$. We write $\mu(v_i)\mu(e_i + v_j) \sim_{\mathcal{C}} \mu(v_j)\mu(e_j + v_i)$, so for each $c_i c_j$-coloured path $x \in E^*$, there is a unique $c_j c_i$-coloured path y such that $x \sim_{\mathcal{C}} y$. If there is only one complete collection of squares around we will simply write $x \sim y$. A coloured-graph morphism λ is $\mathcal{C}$-*compatible* if every square occurring in λ belongs to $\mathcal{C}$.

For $p, q, m \in \mathbb{N}^k$ with $p \le q \le m$, define $E_{k,[p,q]}$ to be the subgraph of $E_{k,m}$ such that

$$\begin{aligned} E^0_{k,[p,q]} &= \{n \in \mathbb{N}^k : p \le n \le q\}, \\ E^1_{k,[p,q]} &= \{x \in E^1_{k,m} : s(x), r(x) \in E^0_{k,[p,q]}\}. \end{aligned}$$

Given a coloured-graph morphism $\lambda : E_{k,m} \to E$ and $p, q \in \mathbb{N}^k$ such that $p \le q \le m$, define $\left(\lambda|^*_{E_{k,[p,q]}}\right) : E_{k,q-p} \to E$ to be the coloured-graph morphism such that

$$\lambda|^*_{E_{k,[p,q]}}(a) = \lambda(p + a) \tag{3.6.1}$$

for every $a \in E_{k,q-p}$. We put the * there to indicate that this is not normal restriction, and includes a translation. This notation is useful for factorising graph morphisms. In particular it is convenient for picking out squares embedded in λ.

We say a complete collection of squares $\mathcal{C}$ in a k-coloured graph E is *associative* if for every path fgh in E such that f, g, h are edges of distinct colour, the edges

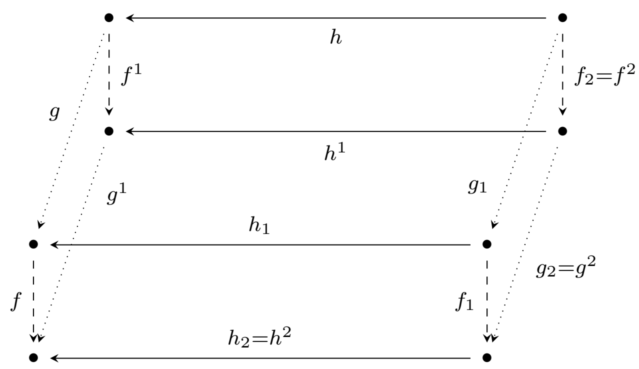


FIGURE 2. Associativity in a 3-graph.

$f_1, f_2, g_1, g_2, h_1, h_2$ determined by

$$gh \sim h_1 g_1, \qquad f h_1 \sim h_2 f_1, \quad \text{and} \quad f_1 g_1 \sim g_2 f_2$$

and the edges $f^1, f^2, g^1, g^2, h^1, h^2$ determined by

$$fg \sim g^1 f^1, \qquad f^1 h \sim h^1 f^2, \quad \text{and} \quad g^1 h^1 \sim h^2 g^2$$

satisfy $f^2 = f_2, g^2 = g_2$ and $h^2 = h_2$. This is pictured in Figure 2.

Let E be a k-coloured graph, and $m \in \mathbb{N}^k \setminus \{0\}$. For a path $x \in E^{|m|}$, and a coloured-graph morphism λ with domain $E_{k,m}$, we say that x *traverses* λ if $d(x) = d(\lambda)$ and $\lambda(d(x_1 \dots x_{l-1}) + v_{c(x_l)}) = x_l$ for all $0 < l \leq |m|$. If $m = 0$ and $x \in E^{|m|} = E^0$, and if λ is a colored-graph morphism with domain $E_{k,0} = \{0\}$, then we say that x traverses λ if $x = \lambda(0)$.

THEOREM 3.6.1 (From [**11**, Theorem 4.11]). *Suppose that E is a k-coloured graph, and that $\mathcal{C}$ is a complete collection of squares in E satisfying the associativity condition. Define $\Lambda^0 = E^0$ and define Λ^* to be the set of all $\mathcal{C}$-compatible coloured-graph morphisms $\lambda : E_{k,m} \to E$. Then $\Lambda = (\Lambda^0, \Lambda^*)$ is the unique (up to isomorphism) k-graph with 1-skeleton E.*

We extend the notion of traversing a coloured-graph morphism to infinite paths: let $x \in E^\infty$ and $\lambda : E_{k,p} \to E$ be a coloured-graph morphism of non-finite degree (so $p \in (\mathbb{N} \cup \{\infty\})^k \setminus \mathbb{N}^k$). Then we say that x *traverses* λ if $x_1 \dots x_n$ traverses $\lambda|_{E_{k,d(x_1 \dots x_n)}}$ for every $n \in \mathbb{N}$.

PROPOSITION 3.6.2. *Suppose that E is a k-colored graph, and that $\mathcal{C}$ is a complete collection of squares in E satisfying the associativity condition. Then for every path $x \in E^* \cup E^\infty$ there exists a unique $\mathcal{C}$-compatible coloured graph morphism*

$$\lambda_x : E_{k,d(x)} \to E$$

that is traversed by x.

Proof. If $x \in E^*$ then [**11**, Proposition 4.7] gives the result. Fix $x \in E^\infty$. By [**11**, Proposition 4.7], for each n there exists a unique $\mathcal{C}$-compatible colored-graph morphism $\lambda_x^{(n)}$ traversed by x^n. Regarding the $\lambda_x^{(n)}$ as paths in the k-graph obtained from Theorem 3.6.1, we apply Lemma 3.1.4 to the sequence $\{\lambda_x^{(n)}\}$ to obtain a unique $\lambda_x \in W$ such that $d(\lambda_x) = d(x)$ and $\lambda_x(0, d(x^n)) = \lambda_x^{(n)}$.

To see that λ_x is $\mathcal{C}$-compatible, suppose that ψ is a square embedded in λ_x. Then exists $N \in \mathbb{N}^k$ such that $\psi(y) = \lambda_x(y+N)$ for all $y \in \operatorname{dom}(\psi)$. Applying Claim 3.1.4.1 gives $M = N_{N+d(\psi)}$ such that $d(\lambda_x^{(M)}) \geq N + d(\psi)$. This implies that

$$\lambda_x^{(M)}(y+N) = \lambda_x(y+N) = \psi(y)$$

for all $y \in \operatorname{dom}(\psi)$. So ψ occurs in $\lambda_x^{(M)}$. Since $\lambda_x^{(M)}$ is $\mathcal{C}$-compatible, ψ is a square in $\mathcal{C}$, so λ_x is $\mathcal{C}$-compatible. $\square$

Remark 3.6.3. Suppose that E is a k-coloured directed graph, that $\mathcal{C}$ is a complete associative collection of squares in E, and that Λ is the k-graph with skeleton E and squares $\mathcal{C}$ obtained from Theorem 3.6.1. Let $q : E^* \cup E^\infty \to W$ be the map which takes x to the unique $\mathcal{C}$-compatible coloured-graph morphism λ_x traversed by x.

If Λ is a row-finite, we claim that the collection $\{\mathcal{Z}(\mu) : \mu \in \Lambda\} \cup \{\{\mu\} : \mu \in \Lambda\}$ is a basis for the topology on Λ. Denote by τ' the topology generated by $\{\mathcal{Z}(\mu) : \mu \in \Lambda\} \cup \{\{\mu\} : \mu \in \Lambda\}$, and our usual topology by τ. To see $\tau = \tau'$ we show their open sets coincide. Let $\mathcal{Z}(\mu \setminus G)$ be a basic open set in τ, and $\nu \in \mathcal{Z}(\mu \setminus G)$. Then $\{\nu\} \subset \mathcal{Z}(\mu \setminus G)$, so $\mathcal{Z}(\mu \setminus G)$ is open in τ', and $\tau \subset \tau'$. On the other hand, let B be a basic open set of τ', and $\nu \in B$. Since Λ is row-finite, $G := \bigcup_{i=1}^k s(\nu)\Lambda^{e_i}$ is finite. Then $\mathcal{Z}(\nu \setminus G) = \{\nu\} \subset B$, so B is open in τ, hence $\tau' \subset \tau$.

Proposition 3.6.4. *Let E be a row-finite, k-coloured graph, and let $\mathcal{C}$ be a complete associative collection of squares in E, let Λ be the k-graph with skeleton E and squares $\mathcal{C}$ obtained from Theorem 3.6.1. Then U is open in W if and only if $q^{-1}(U)$ is open in $E^* \cup E^\infty$.*

Proof. First suppose that U is open in W, and fix $x \in q^{-1}(U)$. We seek a basic open set B_x in $E^* \cup E^\infty$ such that $x \in B_x \subset q^{-1}(U)$. If $x \in E^*$, then $B_x = \{x\}$ does the trick. Now suppose that $x \in E^\infty$. Since $x \in E^\infty \cap q^{-1}(U)$, we have $q(x) \in U$. Since U is open, there exists $\mu \in \Lambda$ such that $q(x) \in \mathcal{Z}(\mu) \subset U$. In particular, $d(x) > d(\mu) \in \mathbb{N}^k$, and hence $d(x_1 \dots x_n) > d(\mu)$ for some $n \in \mathbb{N}$. We then have $q(x_1 \dots x_n) \in \mathcal{Z}(\mu)$. Let $y_x = x_1 \dots x_n$. Then $x \in \mathcal{Z}(y_x)$. To see $\mathcal{Z}(y_x) \subset q^{-1}(U)$, fix $y \in \mathcal{Z}(y_x)$; say $y = y_x y'$. Then $q(y) = q(x_1 \dots x_n y') \in \mathcal{Z}(\mu) \subset U$, so $y \in q^{-1}(U)$ as required.

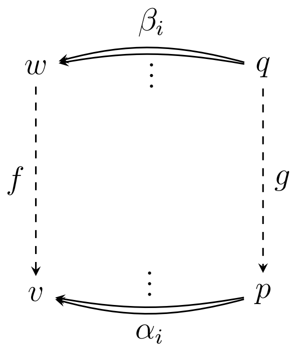


FIGURE 3. A picture of E

For the reverse implication, suppose that $q^{-1}(U)$ is open in $E^* \cup E^\infty$, and fix $\lambda \in U$. We seek a basic open set B_λ such that $\lambda \in B_\lambda \subset U$. If $\lambda \in \Lambda$, then $B_\lambda = \{\lambda\}$ suffices. Suppose that $\lambda \in W \setminus \Lambda$, so $|d(\lambda)| = \infty$. Fix $x \in E^\infty$ which traverses λ. Then $x \in q^{-1}(U)$, which is open, so there exists a basic open set $B_x \in E^* \cup E^\infty$ such that $x \in B_x \subset q^{-1}(U)$. Since $|d(\lambda)| = |x| = \infty$, B_x is not equal to $\{\lambda\}$ for any $\lambda \in E^*$, and hence $B_x = \mathcal{Z}(y_x)$ for some $y_x \in E^*$. Then

$$\lambda = q(x) = q(y_x x') = q(y_x)q(x') \in \mathcal{Z}(q(y_x)).$$

To see $\mathcal{Z}(q(y_x)) \subset U$, let $\mu \in \mathcal{Z}(q(y_x))$. Write $\mu = q(y_x)\mu'$, and let $x_{\mu'}$ be a path in $E^\star$ which traverses μ'. Then $y_x x_{\mu'} \in \mathcal{Z}(y_x) \subset q^{-1}(U)$, which implies $\mu = q(y_x x_{\mu'}) \in U$, as required. $\square$

Proposition 3.6.4 says that when E is row-finite, the topology on W is precisely the quotient topology inherited from $E^* \cup E^\infty$ under q. In particular, q is continuous. This is not true in general.

EXAMPLE 3.6.5. Let E be the 2-colored graph of Figure 3; so

$$\begin{aligned} E^0 &= \{v, w, p, q\}, & E^1 &= \{f, g\} \cup \Big(\bigcup_{i \in \mathbb{N}} \{\alpha_i, \beta_i\}\Big) \\ r(f) = r(\alpha_i) &= v, & s(f) &= r(\beta_i) = w \\ r(g) = s(\alpha_i) &= p, & s(g) &= s(\beta_i) = q \\ c(\alpha_i) = c(\beta_i) &= c_1, & c(f) &= c(g) = c_2. \end{aligned}$$

We call c_1 blue and c_2 red. Since not everyone can easily print in colour, we draw them as solid and dashed lines respectively.

Let $\mathcal{C}$ be the collection of graph morphisms $\lambda_i : E_{2,(1,1)} \to E$ such that

$$\begin{aligned} \lambda_i((0,0)) &= v, & \lambda_i((0,1)) &= w \\ \lambda_i((1,0)) &= p, & \lambda_i((1,1)) &= q \\ \lambda_i((0,0)+v_1) &= \alpha_i, & \lambda_i((0,0)+v_2) &= f \\ \lambda_i((1,0)+v_2) &= g, & \lambda_i((0,1)+v_1) &= \beta_i. \end{aligned}$$

So each λ_i is a square, and $\mathcal{C}$ is a complete collection of squares, and $\alpha_i g \sim_{\mathcal{C}} f\beta_i$ for all i. Since E has only 2 colours, $\mathcal{C}$ is automatically associative.

The 2-graph Λ determined by $(E, \mathcal{C})$ has, by definition, skeleton E with factorisations $\alpha_i g = f\beta_i$ for each $i \in \mathbb{N}$. Moreover, by definition, $\Lambda^{(1,1)} = \{\lambda_i : i \in \mathbb{N}\}$ and $q(\alpha_i g) = \lambda_i = q(f\beta_i)$.

To see that q is not continuous, we show that $\{\alpha_i g\}_{i\in\mathbb{N}}$ converges to v in E, but $\{\lambda_i\}$ converges to $f \neq q(v)$ in Λ.

To see that $\alpha_i g \to v$ in E, fix a basic open subset $\mathcal{Z}(y \setminus F) \subset E$ containing v. Then $y = v$, and since F is finite, only finitely many of the α_i may be in F. So there exists $N_0 \in \mathbb{N}$ such that $n \geq N_0 \implies \alpha_n \notin F$. Then

$$n \geq N_0 \implies \alpha_n g \in \mathcal{Z}(v \setminus F).$$

So $\alpha_i g \to v$ as $i \to \infty$.

To see that $\lambda_i \to f$ in Λ, fix a basic open subset $\mathcal{Z}(\mu \setminus G) \subset \Lambda$ containing f. We consider 2 cases:

(i) $\mu = f$, or
(ii) $\mu = v$.

For case (i), suppose that $\mu = f$. Then G is a finite collection of $q(\beta_i)$. Let $N_1 = \max\{i : q(\beta_i) \in G\}$. Then

$$n \geq N_1 \implies \lambda_n = q(f\beta_n) \in \mathcal{Z}(f \setminus G).$$

For case (ii), suppose that $y = v$. Then since $f \notin G$, G is a finite collection of $q(\alpha_i)$. Let $N_2 = \max\{i : q(\alpha_i) \in G\}$. Then

$$n \geq N_2 \implies \lambda_n = q(\alpha_n g) \in \mathcal{Z}(v \setminus G).$$

So for any neighbourhood U of f in Λ, the λ_i eventually belong to U. Hence $q(\alpha_i g) \to f \neq v$ as $i \to \infty$. Hence $q(\lim \alpha_i g) = q(v) \neq q(f) = \lim \lambda_i = \lim q(\alpha_i g)$, so q is not continuous.

APPENDIX A

C^*-algebras

Here we provide a brief recap of some general C^*-algebraic results we apply in both the directed and k-graph settings.

Given a C^*-algebra A, the *multiplier algebra* $M(A)$ consists of pairs (L,R) of maps from A to itself such that $aL(b) = R(a)b$. This is a C^*-algebra with $\|(L,R)\| = \|L\| = \|R\|, (L_1,R_1)(L_2,R_2) = (L_1 \circ L_2, R_2 \circ R_1)$ and $(L,R)^* = (R^{\#}, L^{\#})$, where $R^{\#}(a) = R(a^*)^*$. The multiplier algebra $M(A)$ is the largest unital C^*-algebra which contains A as an essential ideal, and is unique up to isomorphism. If p is a projection in $M(A)$, then the C^*-subalgebra $B = pAp$ of A is called a *corner* of A. If B is not contained in any proper two-sided ideal in A, we call B a *full corner*.

Lemma A.0.6. *If P and Q are non-zero mutually orthogonal projections on a Hilbert space H and $\lambda, \mu \in \mathbb{C}$, then $\|\lambda P + \mu Q\| = \max\{|\lambda|, |\mu|\}$.*

Proof. Recall that for any orthogonal projection S on H we have $\|P\| = \|P^2\| = \|P^*P\| = \|P\|^2$, and thus $\|P\| \in \{0,1\}$. Furthermore, that P and Q are mutually orthogonal implies that $P+Q$ is a projection. Hence

$$\begin{aligned}
\|(\lambda P + \mu Q)h\|^2 &= \|\lambda Ph + \mu Qh\|^2 \\
&= |\lambda|^2\|Ph\|^2 + |\mu|^2\|Qh\|^2 \quad \text{by Pythagoras' Theorem} \\
&\le \max\{|\lambda|^2, |\mu|^2\}(\|Ph\|^2 + \|Qh\|^2) \\
&= \max\{|\lambda|^2, |\mu|^2\}(\|Ph + Qh\|^2) \\
&= \max\{|\lambda|^2, |\mu|^2\}(\|(P+Q)h\|^2) \\
&\le \max\{|\lambda|^2, |\mu|^2\}(\|P+Q\|^2\|h\|^2) \\
&\le \max\{|\lambda|^2, |\mu|^2\}\|h\|^2
\end{aligned}$$

So $\|\lambda P + \mu Q\| \le \max\{|\lambda|, |\mu|\}$. For $h \in PH$, we have $\|(\lambda P + \mu Q)h\| = \|\lambda Ph\| = |\lambda|\|h\|$. Similarly, $h \in QH$ gives $\|(\lambda P + \mu Q)h\| = \|\mu Qh\| = |\mu|\|h\|$. So $\|\lambda P + \mu Q\| \ge |\lambda|$ and $\|\lambda P + \mu Q\| \ge |\mu|$. Thus $\|\lambda P + \mu Q\| = \max\{|\lambda|, |\mu|\}$. □

Lemma A.0.7. *Let A be a C^*-algebra, let p be a projection in A, let Q be a finite set of commuting subprojections of p and let q_0 be a nonzero subprojection of p. Then $\prod_{q \in Q}(p-q)$ is a projection. If q_0 is orthogonal to each $q \in Q$, then $q_0 \prod_{q\in Q}(p-q) = q_0$, so in particular, $\prod_{q\in Q}(p-q) \ne 0$.*

PROOF. Since $(p-q)^* = p^* - q^* = p-q$ for each $q \in Q$, and since the projections in Q commute, the $(p-q)$ commute, so

$$\Big(\prod_{q\in Q}(p-q)\Big)^* = \prod_{q\in Q}(p-q)^* = \prod_{q\in Q}(p-q).$$

Since $(p-q)^2 = p^2 - 2pq + q^2 = p - 2q + q = p - q$, we have

$$\Big(\prod_{q\in Q}(p-q)\Big)^2 = \prod_{q\in Q}(p-q)^2 = \prod_{q\in Q}(p-q).$$

So $\prod_{q\in Q}(p-q)$ is a projection.

Suppose q_0 is orthogonal to each $q \in Q$. Then for each $q \in Q$, we have

$$q_0(p-q) = q_0p - q_0q = q_0 - 0 = q_0,$$

so

$$q_0\Big(\prod_{q\in Q}(p-q)\Big) = q_0(p-q')\Big(\prod_{q\in Q\setminus\{q'\}}(p-q)\Big) = q_0\Big(\prod_{q\in Q\setminus\{q'\}}(p-q)\Big).$$

Now an induction on $|Q|$ shows that $q_0\prod_{q\in Q}(p-q) = q_0$. Since $q_0 \neq 0$, then $q_0\prod_{q\in Q}(p-q) \neq 0$, so $\prod_{q\in Q}(p-q) \neq 0$. $\square$

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