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## A note on Devaney's definition of chaos

### Abstract

Devaney defines a function to be chaotic if it satisfies three conditions: transitivity, having dense set of periodic points and sensitive dependence on initial conditions. Banks et al prove that if the function is continuous then the third condition is implied from the first two and therefore is redundant. However, if the function is not assumed to be continuous, then it is not known if the third condition is redundant or not.

In this note, without assuming the function is continuous, we prove that the third condition is redundant if the underlying topological space is not precompact.

### Keywords

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# A Note on Devaney's Definition of Chaos

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## Abstract

Devaney [5] defines a function to be chaotic if it satisfies three conditions: transitivity, having dense set of periodic points and sensitive dependence on initial conditions. Banks et al [2] prove that if the function is continuous then the third condition is implied from the first two and therefore it is redundant. However, if the function is not assumed to be continuous, then it is not known if the third condition is redundant or not.

In this note, without assuming the function is continuous, we prove that the third condition is redundant if the underlying topological space is not precompact.

## 1 Introduction

**Devaney's Definition of Chaos** [5]. Let  $(\mathcal{X}, d)$  be a metric space. A function  $f : \mathcal{X} \rightarrow \mathcal{X}$  is called *chaotic* if and only if it satisfies the following three conditions:

- (D1)  $f$  is *topological transitive*, that is, for any two open sets  $U$  and  $V$ , there exists  $k$  such that  $f^{[k]}(U) \cap V \neq \emptyset$
- (D2) *The set of periodic points of  $f$  is dense*. A point  $x$  is called periodic if  $f^{[k]}(x) = x$  for some  $k \geq 1$ .
- (D3)  $f$  has *sensitive dependence on initial conditions*, that is,  $\exists \delta > 0$  such that for any open set  $U$  and for any point  $x \in U$ , there exists a point  $y \in U$  such that  $d(f^{[k]}(x), f^{[k]}(y)) > \delta$  for some  $k$ . The positive number  $\delta$  is called a *sensitivity constant*, it only depends on the space  $\mathcal{X}$  and the function  $f$ .

Devaney's definition of chaos is the most widely known and accepted. It defines a function  $f : \mathcal{X} \rightarrow \mathcal{X}$  on a metric space  $\mathcal{X}$  to be *chaotic* if it satisfies three conditions: *topological transitive*, having dense set of periodic points, and *sensitive dependence on initial conditions*. Essentially, transitivity says that the function  $f$  draws open sets close together. Sensitivity, on the other hand, says  $f$  draws things apart. The condition that  $f$  has a dense set of periodic

points is referred to as an “element of regularity” ([5, p. 50]). Among these three conditions, the third condition, sensitive dependence on initial conditions, is widely understood as the central idea in chaos.

However, Banks et al [2] prove that if  $f$  is continuous then the third condition (D3) is redundant, that is the first two conditions (D1) and (D2) imply the third. Assaf et al [1] show that this is the only redundancy. That is, (D1) and (D3) do not imply (D2), and (D2) and (D3) do not imply (D1). Transitivity condition (D1) was studied in [7, 4]. Specially in the real line, Vellekoop and Berglund [7] prove that if a continuous function  $f$  on an interval is transitive then it has a dense set of periodic points and also sensitive dependence on initial conditions. See [3] for further details on one dimensional dynamic systems. Touhey [6] also proposed another definition of chaos concerning about orbits of periodic points, which proved to be equivalent to Devaney’s definition.

In this note, we consider the redundancy of (D3) without assuming the continuity of the function. It is not known if the third condition (D3) remains redundant if we do not assume the function is continuous. An example of non-continuous chaotic function is the *baker function*  $B(x)$  [5, p. 52] defined on the interval  $[0, 1]$  by  $B(x) = 2x$  if  $0 \leq x < 1/2$  and  $B(x) = 2x - 1$  if  $1/2 \leq x \leq 1$ . The main theorem in this note says that if the space  $\mathcal{X}$  is not *precompact* then the condition (D3) is implied from the first two conditions (D1) and (D2) regardless whether the function is continuous or not. It follows that in an unbounded metric space (D3) is redundant.

## 2 Main Result

**Definition 1** A metric space  $(\mathcal{X}, d)$  is called *precompact* (or *totally bounded*) if  $\forall \epsilon > 0$  there exists a finite set  $S$  such that  $\mathcal{X} = \cup_{x \in S} \mathcal{B}(x, \epsilon)$  where  $\mathcal{B}(x, \epsilon)$  denotes the open ball with center  $x$  and radius  $\epsilon$ .

Note that if a metric space is precompact then it is bounded. Especially, a subspace of the Euclidean space  $\mathbf{R}^n$  is precompact if and only if it is bounded.

**Theorem 1 (Main Theorem)** Let  $(\mathcal{X}, d)$  be a non-precompact metric space and  $f : \mathcal{X} \rightarrow \mathcal{X}$ . If  $f$  is transitive and has a dense set of periodic points then  $f$  has sensitive dependence on initial conditions.

The above theorem is derived from the following Lemma

**Lemma 1** Let  $(\mathcal{X}, d)$  be a metric space and  $f : \mathcal{X} \rightarrow \mathcal{X}$ . If  $f$  satisfies the two conditions (D1) and (D2) but does not satisfy (D3) then  $\forall \delta > 0$ , there exists a periodic point  $p$  such that

$$\bigcup_{i=0}^{n-1} \mathcal{B}(f^{[i]}(p), \delta) = \mathcal{X}$$

where  $n$  denotes the period of  $p$ . Consequently,  $\mathcal{X}$  must be a precompact space.

*Proof.* Assume  $f$  satisfies (D1) and (D2), but does not satisfy (D3). Take  $\delta > 0$ , and let  $\delta' = \delta/4$ .

First of all,  $f$  does not satisfy (D3) implies that there exists an open set  $U$  and a point  $x \in U$  such that

$$\forall y \in U, \forall k, \quad d(f^{[k]}(x), f^{[k]}(y)) \leq \delta'. \quad (1)$$

From (D2), the set of periodic points of  $f$  is dense, there exists a periodic point  $p \in U$ . By (1), we have

$$d(f^{[k]}(x), f^{[k]}(p)) \leq \delta'$$

and hence,

$$\forall y \in U, \forall k, \quad d(f^{[k]}(y), f^{[k]}(p)) \leq 2\delta' < 3\delta'.$$

Thus,  $\forall k, f^{[k]}(U) \subset \mathcal{B}(f^{[k]}(p), 3\delta')$ . Let  $n$  be the period of  $p$  then

$$\bigcup_{k=0}^{\infty} f^{[k]}(U) \subset \bigcup_{i=0}^{n-1} \mathcal{B}(f^{[i]}(p), 3\delta') \quad (2)$$

Now since  $f$  satisfies (D1),  $\forall z \in \mathcal{X}$ , apply the transitivity condition of  $f$  on the open sets  $U$  and  $V = \mathcal{B}(z, \delta')$ , we have

$$\left( \bigcup_{k=0}^{\infty} f^{[k]}(U) \right) \cap \mathcal{B}(z, \delta') \neq \emptyset.$$

It follows from (2) that

$$\left( \bigcup_{i=0}^{n-1} \mathcal{B}(f^{[i]}(p), 3\delta') \right) \cap \mathcal{B}(z, \delta') \neq \emptyset.$$

Therefore,  $\exists z' \in \mathcal{B}(z, \delta')$  such that  $z' \in \mathcal{B}(f^{[i]}(p), 3\delta')$  for some  $i$ ,  $0 \leq i \leq n-1$ . We have,

$$d(z', z) < \delta'$$

and

$$d(z', f^{[i]}(p)) < 3\delta',$$

so

$$d(z, f^{[i]}(p)) < 4\delta' = \delta.$$

Hence,

$$z \in \bigcup_{i=0}^{n-1} \mathcal{B}(f^{[i]}(p), \delta).$$

Since, this is true for all  $z \in \mathcal{X}$ , we have

$$\bigcup_{i=0}^{n-1} \mathcal{B}(f^{[i]}(p), \delta) = \mathcal{X}.$$

**Corollary 1** *In an unbounded metric space, the sensitivity condition is redundant regardless the function is continuous or not.*

It is interesting to see an example of a non-continuous function in a precompact space that is transitive and has a dense set of periodic points but does not have sensitive dependence on initial conditions.

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