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An elementary approach to C^* -algebras associated to topological graphs

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An elementary approach to C^* -algebras associated to topological graphs

Abstract

We develop notions of a representation of a topological graph E and of a covariant representation of a topological graph E which do not require the machinery of C^* -correspondences and Cuntz-Pimsner algebras. We show that the C^* -algebra generated by a universal representation of E is isomorphic to the Toeplitz algebra of Katsura's topological-graph bimodule, and that the C^* -algebra generated by a universal covariant representation of E is isomorphic to Katsura's topological graph C^* -algebra. We exhibit our results by constructing the isomorphism between the C^* -algebra of the row-finite directed graph E with no sources and the C^* -algebra of the topological graph arising from the shift map acting on the infinite-path space E (figure 8).

Keywords

associated, approach, topological, graphs, C^* , algebras, elementary

Disciplines

Engineering | Science and Technology Studies

Publication Details

Li, H., Pask, D. & Sims, A. (2014). An elementary approach to C^* -algebras associated to topological graphs. *The New York Journal of Mathematics*, 20 447-469.

An elementary approach to C^* -algebras associated to topological graphs

Hui Li, David Pask and Aidan Sims

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1. Introduction

Let E be a countable directed graph with vertex set E^0 , edge set E^1 and range and source maps $r, s : E^1 \rightarrow E^0$. The Toeplitz–Cuntz–Krieger algebra $\mathcal{TC}^*(E)$ is the universal C^* -algebra generated by a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a family of partial isometries $\{s_e : e \in E^1\}$ such that:

- (1) $s_e^* s_e = p_{s(e)}$ for every $e \in E^1$.

Received May 15, 2012; revised May 5, 2014.

2010 *Mathematics Subject Classification.* 46L05.

Key words and phrases. C^* -algebra; graph algebra; C^* -correspondence; topological graph; Cuntz–Pimsner algebra.

This research was supported by the Australian Research Council.

$$(2) \quad p_v \geq \sum_{e \in F} s_e s_e^* \text{ for every } v \in E^0 \text{ and finite } F \subset r^{-1}(v).$$

The graph algebra $C^*(E)$ is universal for families as above satisfying the additional relation that $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever $r^{-1}(v)$ is nonempty and finite.

These C^* -algebras have been studied very extensively over the last fifteen years, part of their appeal being precisely that they can be defined, as above, in the space of a paragraph. Combined with key structure theorems like the gauge-invariant uniqueness theorem [1] and the Cuntz–Krieger uniqueness theorem [11], or Fowler and Raeburn’s analogue of Coburn’s Theorem for the Toeplitz algebra of a directed graph [4], the above elementary presentation often makes it easy to verify that a given C^* -algebra is isomorphic to $C^*(E)$ or $\mathcal{TC}^*(E)$ as appropriate.

Some years after the introduction of graph algebras, Katsura introduced topological graphs and their C^* -algebras [5]. His construction is based on that of [6], which in turn was a modification of Pimsner’s construction of C^* -algebras associated to C^* -correspondences in [9]. Roughly speaking, a topological graph is like a directed graph except that E^0 and E^1 are locally compact Hausdorff spaces rather than countable discrete sets, and r and s are required to be topologically well-behaved. Building on Fowler and Raeburn’s construction of a C^* -correspondence from a directed graph given in [4], Katsura associated to each topological graph E a C^* -correspondence $X(E)$, and defined the C^* -algebra of the topological graph E to be the C^* -algebra $\mathcal{O}_{X(E)}$ he had associated to the module $X(E)$ in [6].

The drawback of this approach is that the relations defining the C^* -algebra are complicated and cannot be stated without first introducing at least the rudiments of C^* -correspondences, which is quite a bit of overhead — see Section 3. Clearly it would be handy to be able to circumvent all this technical overhead when defining and dealing with the C^* -algebras of topological graphs. This paper takes the first step in this direction. Our main theorems show that if E is a topological graph, then the Toeplitz algebra $\mathcal{T}(E)$ and Katsura’s topological graph algebra $\mathcal{O}(E)$ of [5] can be described as C^* -algebras universal for relatively elementary relations involving elements of $C_0(E^0)$ and $C_c(E^1)$. In particular, our presentations can be stated without any reference to C^* -correspondences. We state our results so that they apply to arbitrary topological graphs, but we also show how our crucial covariance condition simplifies when the range map $r : E^1 \rightarrow E^0$ is a local homeomorphism. Even this special situation is interesting: it includes, for example, Cantor minimal systems, and all crossed products of abelian C^* -algebras by \mathbb{Z} .

To emphasise how little background is needed to present our definition of a representation of a topological graph and our covariance condition, we begin by stating all our key definitions and main theorems in Section 2. In Section 3 we recall Katsura’s construction of a C^* -correspondence from a topological graph and his definitions of the associated C^* -algebras. In

Section 4, we prove our main results. We finish in Section 5 by considering the topological graph \widehat{E} obtained from a row-finite directed graph E with no sources by taking $\widehat{E}^0 = \widehat{E}^1 = E^\infty$ and defining the range map to be the identity map, and the source map to be the left-shift map σ . We apply our results to provide a relatively elementary proof that $\mathcal{O}(\widehat{E})$ is canonically isomorphic to $C^*(E)$. This result could be recovered from [2, 3], but working out the details provides a good example of the efficacy of our results.

2. Main results

Definition 2.1 ([5, Definition 2.1]). A quadruple $E = (E^0, E^1, r, s)$ is called a *topological graph* if E^0, E^1 are locally compact Hausdorff spaces, $r : E^1 \rightarrow E^0$ is a continuous map, and $s : E^1 \rightarrow E^0$ is a local homeomorphism.

We think of E^0 as a space of vertices, and we think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. If E^0, E^1 are both countable and discrete, then E is a directed graph in the sense of [8, 10].

Given $x \in C_c(E^1)$ and $f \in C_0(E^0)$, we define $x \cdot f$ and $f \cdot x$ in $C_c(E^1)$ by

$$(2.1) \quad (x \cdot f)(e) = x(e)f(s(e)) \quad \text{and} \quad (f \cdot x)(e) = f(r(e))x(e).$$

To define our notion of a representation of a topological graph, we need a couple of preliminary ideas. An *s-section* in a topological graph E is a subset $U \subset E^1$ such that $s|_U$ is a homeomorphism. An *r-section* is defined similarly, and a *bisection* is a set which is both an *s-section* and an *r-section*.

If $x \in C_c(E^1)$ then $\text{osupp}(x)$ denotes the precompact open set

$$\{e \in E^1 : x(e) \neq 0\},$$

and $\text{supp}(x)$ is the closure of $\text{osupp}(x)$. If $x \in C_c(E^1)$ and $\text{osupp}(x)$ is an *s-section*, we define $\widehat{x} : E^0 \rightarrow \mathbb{C}$ by

$$(2.2) \quad \widehat{x}(v) := \begin{cases} |x(e)|^2 & \text{if } v = s(e) \text{ for } e \in \text{osupp}(x), \\ 0 & \text{for } v \notin s(\text{osupp}(x)). \end{cases}$$

Definition 2.2. Let E be a topological graph. A *Toeplitz representation* of E in a C^* -algebra B is a pair (ψ, π) where $\psi : C_c(E^1) \rightarrow B$ is a linear map, $\pi : C_0(E^0) \rightarrow B$ is a homomorphism, and:

- (1) $\psi(f \cdot x) = \pi(f)\psi(x)$, for all $x \in C_c(E^1)$, $f \in C_0(E^0)$;
- (2) for $x \in C_c(E^1)$ such that $\text{supp}(x)$ is contained in an open *s-section*, $\pi(\widehat{x}) = \psi(x)^*\psi(x)$;
- (3) for $x, y \in C_c(E^1)$ such that $\text{supp}(x)$ and $\text{supp}(y)$ are contained in disjoint open *s-sections*, $\psi(x)^*\psi(y) = 0$.

We say that a Toeplitz representation (ψ, π) of E in B is *universal* if for any Toeplitz representation (ψ', π') of E in C , there exists a homomorphism $h : B \rightarrow C$, such that $h \circ \psi = \psi'$ and $h \circ \pi = \pi'$.

Remark 2.3. Suppose that $x \in C_c(E^1)$ and $\text{supp}(x)$ is contained in an open s -section U . Since E^1 is locally compact, we may cover $\text{supp}(x)$ with precompact open sets $\{V_i : i \in I\}$, and since $\text{supp}(x)$ is compact, we may pass to a finite subcover $\{V_i : i \in F\}$. Then each $V_i \cap U$ is precompact and open, so $\bigcup_{i \in F} (V_i \cap U)$ is a precompact open s -section containing $\text{supp}(x)$. Thus we may assume in Conditions (2) and (3) that the open s -sections containing $\text{supp}(x)$ and $\text{supp}(y)$ are precompact.

To state our first main theorem, recall that if E is a topological graph, then $\mathcal{T}(E)$ denotes the Toeplitz algebra of Katsura's topological-graph bi-module (see Notation 3.6).

Theorem 2.4. *Let E be a topological graph. Then there is a universal Toeplitz representation (i_1, i_0) of E which generates $\mathcal{T}(E)$. Moreover the C^* -algebra generated by the image of any universal Toeplitz representation of E is isomorphic to $\mathcal{T}(E)$.*

Remark 2.5. Since the map ψ occurring in a Toeplitz representation of E is not a homomorphism, it will not usually be norm-decreasing with respect to the supremum norm. So it is not clear that one can just extend by continuity a linear map ψ_0 defined on a dense subspace of $C_c(E^1)$. We show in Proposition 4.12 how to get around this difficulty: the map ψ is norm-decreasing with respect to the supremum norm when applied to functions supported on s -sections.

We now describe the covariance condition for a Toeplitz representation of a topological graph. The condition is somewhat technical, but we will indicate how it simplifies under additional hypotheses in Corollary 2.15.

We first need a little notation from [5].

Definition 2.6 ([5, Definition 2.6]). Let E be a topological graph. We define:

- (1) $E_{\text{sce}}^0 = E^0 \setminus \overline{r(E^1)}$.
- (2) $E_{\text{fin}}^0 = \{v \in E^0 : \text{there exists a neighbourhood } N \text{ of } v \text{ such that } r^{-1}(N) \text{ is compact}\}$.
- (3) $E_{\text{rg}}^0 = E_{\text{fin}}^0 \setminus \overline{E_{\text{sce}}^0}$.

Remark 2.7. The set E_{fin}^0 is open in E^0 . To see this, fix $v \in E_{\text{fin}}^0$. Then there exists a neighbourhood U of v such that $r^{-1}(U)$ is compact and there exists an open neighbourhood V of v contained in U . So $V \subset E_{\text{fin}}^0$, whence E_{fin}^0 is open. It follows that $E_{\text{sce}}^0, E_{\text{fin}}^0, E_{\text{rg}}^0$ are all open in E^0 . Moreover E_{rg}^0 is the intersection of E_{fin}^0 with the interior of $\overline{r(E^1)}$. Finally, as proved by Katsura (see [5, Lemma 1.23]), for any compact subset $K \subset E_{\text{fin}}^0$, the set $r^{-1}(K)$ is compact in E^1 .

Notation 2.8. Let X be a locally compact Hausdorff space and U be an open subset of X . Then the standard embedding ι_U of $C_0(U)$ as an ideal of

$C_0(X)$ is given by

$$\iota_U(f)(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

We will usually suppress the map ι_U and just identify each of $C_0(U)$ and $C_c(U)$ with their images in $C_0(X)$ under ι_U . That is, we think of $C_0(U)$ as an ideal of $C_0(X)$, and we regard $C_c(U)$ as an (algebraic) ideal of $C_0(X)$.

Remark 2.9. Let X be a Hausdorff space, and let $K \subset X$ be compact. Fix a finite cover \mathcal{N} of K by open subsets of X . By [7, Chapter 5.W], there exists a partition of unity $\{h_N : N \in \mathcal{N}\}$ on K subordinate to $\{N \cap K : N \in \mathcal{N}\}$, where $h_N(x) \in [0, 1]$, for all $N \in \mathcal{N}$, $x \in K$. Since X may not be normal, we cannot necessarily extend this to a partition of unity on X . Nevertheless, if $f \in C_c(X)$ with $\text{supp}(f) \subset K$, then the functions $f_N : X \rightarrow \mathbb{C}$ given by

$$(2.3) \quad f_N(x) := \begin{cases} f(x)h_N(x) & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

belong to $C_c(X)$. Hence each $\text{osupp}(f_N) \subset N \cap K$, and $\sum_{N \in \mathcal{N}} f_N = f$.

Definition 2.10. Let E be a topological graph, and let (ψ, π) be a Toeplitz representation of E in a C^* -algebra B . We call (ψ, π) *covariant* if there exists a collection $\mathcal{G} \subset C_c(E_{\text{rg}}^0)$ of nonnegative functions which generates $C_0(E_{\text{rg}}^0)$ as a C^* -algebra, and for each $f \in \mathcal{G}$ there exist a finite cover \mathcal{N}_f of $r^{-1}(\text{supp}(f))$ by open s -sections, and a collection of nonnegative functions $\{f_N : N \in \mathcal{N}_f\} \in C_c(E^1)$ such that:

- (1) $\text{osupp}(f_N) \subset N \cap r^{-1}(\text{supp}(f))$ for all $N \in \mathcal{N}_f$.
- (2) $\sum_{N \in \mathcal{N}_f} f_N = f \circ r$.
- (3) $\pi(f) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$.

We say that a covariant Toeplitz representation (ψ, π) of E in B is *universal* if for any covariant Toeplitz representation (ψ', π') of E in C , there exists a homomorphism $h : B \rightarrow C$, such that $h \circ \psi = \psi'$ and $h \circ \pi = \pi'$.

Remark 2.11. Definition 2.10 is formulated so as to make it easy to check that a given pair (ψ, π) is covariant. However, when *using* covariance of a pair (ψ, π) it is helpful to know that (3) of Definition 2.10 holds for every $f \in C_c(E_{\text{rg}}^0)$, every \mathcal{N}_f , and every $\{f_N : N \in \mathcal{N}_f\}$ satisfying (1) and (2) of Definition 2.10. We prove this in Proposition 4.8.

Remark 2.12. In Definition 2.10, we made implicit use of Remark 2.9 because the preimage $r^{-1}(\text{supp}(f))$ is compact by Remark 2.7. Observe that the covariance condition for a Toeplitz representation of E only involves functions in $C_0(E_{\text{rg}}^0) \subset C_0(E^0)$.

Recall that if E is a topological graph, then $\mathcal{O}(E)$ denotes Katsura's topological graph C^* -algebra (see Notation 3.6).

Theorem 2.13. *Let E be a topological graph. Then there is a universal covariant Toeplitz representation (j_1, j_0) of E which generates $\mathcal{O}(E)$. Moreover the C^* -algebra generated by the image of any universal covariant Toeplitz representation of E is isomorphic to $\mathcal{O}(E)$.*

Although Definition 2.10 looks complicated, the hypotheses are relatively easy to check in specific instances. To give some intuition, we indicate how the definition simplifies if the range map $r : E^1 \rightarrow E^0$ is a local homeomorphism. This situation still includes many interesting examples.

Definition 2.14. Let E be a topological graph. A pair (\mathcal{U}, V) is called a *local r -fibration* if V is a subset of E^0 , and \mathcal{U} is a finite collection of mutually disjoint bisections such that $r^{-1}(V) = \bigcup_{U \in \mathcal{U}} U$, and $r(U) = V$ for each $U \in \mathcal{U}$. A local r -fibration is *precompact* if each $U \in \mathcal{U}$ is precompact and V is precompact. A local r -fibration is *open* if each $U \in \mathcal{U}$ is open and V is open.

Suppose that (\mathcal{U}, V) is an open local r -fibration. Suppose that $U \in \mathcal{U}$ and that $f \in C_c(V)$. We write $r_U^* f \in C_c(U)$ for the function

$$r_U^* f : e \mapsto \begin{cases} f(r(e)) & \text{if } e \in U \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.15. *Let E be a topological graph. Suppose that r is a local homeomorphism. Let (ψ, π) be a Toeplitz representation of E . Then (ψ, π) is covariant if and only if there exists a collection $\mathcal{G} \subset C_c(E_{rg}^0)$ of nonnegative functions which generates $C_0(E_{rg}^0)$ as a C^* -algebra, and for each $f \in \mathcal{G}$ there exists an open local r -fibration (\mathcal{U}, V) such that $\text{supp}(f) \subset V$, and*

$$(2.4) \quad \pi(f) = \sum_{U \in \mathcal{U}} \psi(\sqrt{r_U^* f}) \psi(\sqrt{r_U^* f})^*.$$

If (ψ, π) is covariant, then Equation (2.4) holds for every $f \in C_c(E_{rg}^0)$ and open local r -fibration (\mathcal{U}, V) with $\text{supp}(f) \subset V$.

Remark 2.16. Let E be a topological graph. Fix $f \in C_c(E_{rg}^0)$ and suppose that (\mathcal{U}, V) is an open local r -fibration such that $\text{supp}(f) \subset V$. By Remark 2.7, $r^{-1}(\text{supp}(f))$ is compact. Since \mathcal{U} is an open cover of $r^{-1}(\text{supp}(f))$ by disjoint open bisections, Remark 2.9 gives functions $\{f_U : U \in \mathcal{U}\}$ such that $\text{osupp}(f_U) \subset U \cap r^{-1}(\text{supp}(f))$, and $\sum_{U \in \mathcal{U}} f_U = f \circ r$, for all $U \in \mathcal{U}$. We then have $r_U^* f = f_U$, for all $U \in \mathcal{U}$.

3. C^* -correspondences, C^* -algebras and Katsura's construction

We recall some background on Hilbert C^* -modules. For more detail see [12].

Definition 3.1. Let A be a C^* -algebra. A *right Hilbert A -module* is a right A -module X equipped with a map $\langle \cdot, \cdot \rangle_A : X \times X \rightarrow A$ such that for $x, y \in X, a \in A$:

- (1) $\langle x, x \rangle_A \geq 0$ with equality only if $x = 0$.
- (2) $\langle x, y + z \rangle_A = \langle x, y \rangle_A + \langle x, z \rangle_A$.
- (3) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$.
- (4) $\langle x, y \rangle_A = \langle y, x \rangle_A^*$.
- (5) X is complete in the norm¹ defined by $\|x\|_A^2 = \|\langle x, x \rangle_A\|$.

Recall from, for example, [10, Page 72], that a right Hilbert A -module X is a C^* -correspondence over A if there is a left action of A on X such that

$$(3.1) \quad \langle a \cdot y, x \rangle_A = \langle y, a^* \cdot x \rangle_A \quad \text{for all } a \in A, x, y \in X.$$

An operator $T : X \rightarrow X$ is *adjointable* if there exists $T^* : X \rightarrow X$ such that $\langle T \cdot x, y \rangle_A = \langle y, T^* x \rangle_A$, for all $x, y \in X$. The adjoint T^* is unique, and T is automatically bounded and linear. The set $\mathcal{L}(X)$ of adjointable operators on X is a C^* -algebra. Equation (3.1) implies that there is a homomorphism $\phi : A \rightarrow \mathcal{L}(X)$ such that $\phi(a)x = a \cdot x$ for all $a \in A$ and $x \in X$.

Definition 3.2. Let A be a C^* -algebra and let X be a right Hilbert A -module. Fix $x, y \in X$, we define $\Theta_{x,y} : X \rightarrow X$ by $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_A$, for all $z \in X$. We define $\mathcal{K}(X) := \overline{\text{span}}\{\Theta_{x,y} : x, y \in X\}$.

A calculation shows that $T\Theta_{x,y} = \Theta_{Tx,y}$, and $\Theta_{x,y}^* = \Theta_{y,x}$, for all $x, y \in X, T \in \mathcal{L}(X)$. Hence $\mathcal{K}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Toeplitz representations and Toeplitz algebras of C^* -correspondences were introduced and studied in [9]. Cuntz–Pimsner algebras were also introduced in [9], but the definition was later modified by Katsura in [6, Definition 3.5] so as to include graph algebras as a special case [10, Example 8.13]. We use Katsura’s definition in this paper.

Definition 3.3 ([6, Definition 2.1]). Let A be a C^* -algebra and let X be a C^* -correspondence over A . A *Toeplitz representation* of X in a C^* -algebra B is a pair (ψ, π) , where $\psi : X \rightarrow B$ is a linear map, $\pi : A \rightarrow B$ is a homomorphism, and for any $x, y \in X$ and $a \in A$:

- (1) $\psi(a \cdot x) = \pi(a)\psi(x)$.
- (2) $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A)$.

We say that a Toeplitz representation (ψ, π) of X in B is *universal* if for any Toeplitz representation (ψ', π') of X in C , there exists a homomorphism $h : B \rightarrow C$, such that $h \circ \psi = \psi'$, and $h \circ \pi = \pi'$.

Remark 3.4 ([6, Page 370]). Let X be a C^* -correspondence over a C^* -algebra A and let (ψ, π) be a Toeplitz representation of X . For $x \in X, a \in A$, a calculation using Definition 3.3(2) and that π is a homomorphism

¹It is not supposed to be immediately obvious that this defines a norm, but it is true (see [12]).

shows that $(\psi(xa) - \psi(x)\pi(a))^*(\psi(xa) - \psi(x)\pi(a)) = 0$, and hence that $\psi(x \cdot a) = \psi(x)\pi(a)$. Definition 3.3(2) show that ψ is bounded with $\|\psi\| \leq 1$.

Proposition 1.3 of [4] implies that there exists a C^* -algebra \mathcal{T}_X generated by the image of a universal Toeplitz representation (i_X, i_A) of X . This C^* -algebra is unique up to canonical isomorphism and we call it the *Toeplitz algebra* of X . Given another Toeplitz representation (ψ, π) of X in a C^* -algebra B , we write $h_{\psi, \pi}$ for the induced homomorphism from \mathcal{T}_X to B .

Recall from [9, Page 202] (see also [4, Proposition 1.6]) that if (ψ, π) is a Toeplitz representation of a C^* -correspondence X in a C^* -algebra B , then there is a unique homomorphism $\psi^{(1)} : \mathcal{K}(X) \rightarrow B$ such that $\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$ for all $x, y \in X$. Recall also that given a C^* -algebra A and a closed two-sided ideal J of A , we define $J^\perp := \{a \in A : ab = 0 \text{ for all } b \in J\}$. Then J^\perp is also a closed two-sided ideal of A .

Definition 3.5 ([6, Definitions 3.4 and 3.5]). Let A be a C^* -algebra, let X be a C^* -correspondence over A , and write $\phi : A \rightarrow \mathcal{L}(X)$ for the homomorphism implementing the left action. A Toeplitz representation (ψ, π) of X is *covariant* if $\psi^{(1)}(\phi(a)) = \pi(a)$ for all $a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^\perp$.

A covariant Toeplitz representation (ψ, π) of X in B is *universal* if for any covariant Toeplitz representation (ψ', π') of X in C , there exists a homomorphism h from B into C , such that $h \circ \psi = \psi'$, and $h \circ \pi = \pi'$.

Recall from [10, Page 75] that there exists a C^* -algebra \mathcal{O}_X generated by the image of a universal covariant Toeplitz representation (j_X, j_A) of X . This C^* -algebra is unique up to canonical isomorphism. Given another covariant Toeplitz representation (ψ, π) of X in a C^* -algebra B , we write $\psi \times \pi$ for the induced homomorphism from \mathcal{O}_X to B .

Let E be a topological graph. As in [5], define actions of $C_0(E^0)$ on $C_c(E^1)$ by Equation (2.1). For $x_1, x_2 \in C_c(E^1)$, define $\langle x_1, x_2 \rangle_{C_0(E^0)} : E^0 \rightarrow \mathbb{C}$ by

$$\langle x_1, x_2 \rangle_{C_0(E^0)}(v) = \sum_{s(e)=v} \overline{x_1(e)} x_2(e).$$

(By convention an empty sum is 0.) If $\text{osupp}(x)$ is an s -section, then the function \hat{x} of Equation (2.2) is equal to $\langle x, x \rangle_{C_0(E^0)}$. As in [5] (see also [10, Page 79]), $\langle \cdot, \cdot \rangle_{C_0(E^0)}$ defines a $C_0(E^0)$ -valued inner product on $C_c(E^1)$, and the completion $X(E)$ of $C_c(E^1)$ in the norm $\|x\|_{C_0(E^0)}^2 = \|\langle x, x \rangle_{C_0(E^0)}\|$ is a C^* -correspondence over $C_0(E^0)$. We call $X(E)$ the *graph correspondence* of E .

Notation 3.6. We denote by $\mathcal{T}(E)$ [5, Definition 2.2] the Toeplitz algebra $\mathcal{T}_{X(E)}$, and we denote by $\mathcal{O}(E)$ [5, Definition 2.10] the C^* -algebra $\mathcal{O}_{X(E)}$.

4. Proofs of the main results

To prove Theorem 2.4, we must show that $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_{C_0(E^0)})$ for all $x, y \in C_c(E^1)$. We establish this formula for x and y supported on

a common s -section in Lemma 4.1, and then extend it to arbitrary $x, y \in C_c(E^1)$ in Proposition 4.3.

Let U, V be complex vector spaces. Then any sesquilinear form $\varphi : V \times V \rightarrow U$ which is conjugate linear in the first variable satisfies the polarisation identity

$$\varphi(v_1, v_2) = \frac{1}{4} \sum_{n=0}^3 (-i)^n \varphi(v_1 + i^n v_2, v_1 + i^n v_2).$$

(To prove this, just expand the sum.)

Lemma 4.1. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E . Fix $x_1, x_2 \in C_c(E^1)$. Suppose that $\text{supp}(x_1) \cup \text{supp}(x_2)$ is contained in an open s -section. Then $\pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) = \psi(x_1)^* \psi(x_2)$.*

Proof. The polarisation identity for $\langle \cdot, \cdot \rangle_{C_0(E^0)}$ and Definition 2.2(2) give

$$\begin{aligned} \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) &= \frac{1}{4} \sum_{n=0}^3 (-i)^n \pi(\langle x_1 + i^n x_2, x_1 + i^n x_2 \rangle_{C_0(E^0)}) \\ &= \frac{1}{4} \sum_{n=0}^3 (-i)^n \psi(x_1 + i^n x_2)^* \psi(x_1 + i^n x_2) \\ &= \psi(x_1)^* \psi(x_2). \end{aligned} \quad \square$$

In the following and throughout the rest of the paper we write U^c for the complement $X \setminus U$ of a subset U of a set X .

Remark 4.2. Let X be locally compact Hausdorff space. Fix a compact subset $K \subset X$ and an open neighbourhood U of K . Since U^c is closed and disjoint from K , there is a function $f \in C(X, [0, 1])$ which is identically 1 on K and identically 0 on U^c (see for example [14, Theorem 37.A]). So $V := f^{-1}((1/2, 3/2))$ is open and satisfies $K \subset V \subset \overline{V} \subset U$.

Proposition 4.3. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E . Fix open s -sections $U_1, U_2 \subset E^1$, and $x_1, x_2 \in C_c(E^1)$ with $\text{supp}(x_i) \subset U_i$, for $i = 1, 2$. Then $\pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) = \psi(x_1)^* \psi(x_2)$.*

Proof. Let $K_i = \text{supp}(x_i)$, for $i = 1, 2$, and $K = K_1 \cup K_2$. Since E^1 is locally compact and Hausdorff, Remark 4.2 implies that there exist open sets $U'_i \subset E^1$ such that $K_i \subset U'_i \subset \overline{U'_i} \subset U_i$, for $i = 1, 2$.

For each $e \in K$ and $W \in \{U_i, U'_i, \overline{U'_i}^c : i = 1, 2\}$ such that $e \in W$, fix an open neighbourhood $V(e, W)$ of e such that $\overline{V(e, W)} \subset W$. Define

$$N_e = \left(\bigcap_{e \in U_i} V(e, U_i) \right) \cap \left(\bigcap_{e \in K_i} V(e, U'_i) \right) \cap \left(\bigcap_{e \in U_i^c} V(e, \overline{U'_i}^c) \right).$$

Then for $i = 1, 2$:

- (1) If $e \in U_i$, then $\overline{N_e} \subset U_i$.
- (2) If $e \in K_i$, then $\overline{N_e} \subset U'_i$.

(3) If $e \notin U_i$, then $\overline{N_e} \cap \overline{U'_i} = \emptyset$.

Since K is compact, there is a finite $F \subset K$ such that $\{N_e : e \in F\}$ covers K .

Fix $e, f \in F$. Suppose that $\overline{N_e} \cap \overline{N_f} \neq \emptyset$. We claim that

$$(4.1) \quad \text{either } \overline{N_e} \cup \overline{N_f} \subset U_1, \text{ or } \overline{N_e} \cup \overline{N_f} \subset U_2.$$

When $e = f$, this is trivial. Suppose $e \neq f$. Assume without loss of generality that $e \in K_1$. Then (2) forces $\overline{N_e} \subset U'_1$, so $\overline{N_e} \cap \overline{N_f} \neq \emptyset$ forces $\overline{N_f} \cap \overline{U'_1} \neq \emptyset$. Condition (3) then forces $f \in U_1$, so (1) forces $\overline{N_f} \subset U_1$ and hence $\overline{N_e} \cup \overline{N_f} \subset U_1$ as required.

Since $\{N_e : e \in F\}$ is a finite open cover of K , Remark 2.9 implies that there are finite collections of functions

$$\{x_{1,e} : e \in F\}, \{x_{2,e} : e \in F\} \subset C_c(E^1)$$

such that $\text{osupp}(x_{i,e}) \subset N_e \cap K$ for all $e \in F$, $i = 1, 2$, and $\sum_{e \in F} x_{1,e} = x_1$, $\sum_{e \in F} x_{2,e} = x_2$. Linearity of ψ gives $\psi(x_1)^* \psi(x_2) = \sum_{e,f \in F} \psi(x_{1,e})^* \psi(x_{2,f})$. Fix $e, f \in F$ such that $\overline{N_e} \cap \overline{N_f} = \emptyset$. By Remark 4.2 there are disjoint open s -sections $O_e, O_f \subset E^1$, such that $\overline{N_e} \subset O_e$, and $\overline{N_f} \subset O_f$. Thus condition (3) of Definition 2.2 gives $\psi(x_{1,e})^* \psi(x_{2,f}) = 0$ since $\text{supp}(x_{i,e}) \subset \overline{N_e}$ for all $e \in F$, $i = 1, 2$. It follows that $\psi(x_1)^* \psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \psi(x_{1,e})^* \psi(x_{2,f})$. Whenever $\overline{N_e} \cap \overline{N_f} \neq \emptyset$, Equation (4.1) implies that $\overline{N_e} \cup \overline{N_f}$ is contained in an open s -section, so Lemma 4.1 gives

$$\psi(x_1)^* \psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \pi(\langle x_{1,e}, x_{2,f} \rangle_{C_0(E^0)}) = \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}). \quad \square$$

Proposition 4.4. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E in B . Then*

$$\psi : (C_c(E^1), \|\cdot\|_{C_0(E^0)}) \rightarrow B$$

is a bounded linear map. Let $\tilde{\psi}$ be the unique extension of ψ to $X(E)$. Then the pair $(\tilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$.

Proof. Fix $x_1, x_2 \in C_c(E^1)$. Let $K = \text{supp}(x_1) \cup \text{supp}(x_2)$. For each $e \in K$, there exists an open s -section N_e containing e . Remark 4.2 yields an open neighbourhood N'_e of e such that $\overline{N'_e} \subset N_e$. Since K is compact, there is a finite subset $F \subset K$, such that $\{N'_e : e \in F\}$ covers K . By Remark 2.9, there exist $\{x_{1,e} : e \in F\}, \{x_{2,e} : e \in F\} \subset C_c(E^1)$ such that $\text{osupp}(x_{i,e}) \subset N'_e \cap K$ for all $e \in F$ and $i = 1, 2$, and such that $\sum_{e \in F} x_{1,e} = x_1$ and $\sum_{e \in F} x_{2,e} = x_2$. Proposition 4.3 implies that

$$\begin{aligned} \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) &= \sum_{e,f \in F} \pi(\langle x_{1,e}, x_{2,f} \rangle_{C_0(E^0)}) \\ &= \sum_{e,f \in F} \psi(x_{1,e})^* \psi(x_{2,f}) = \psi(x_1)^* \psi(x_2). \end{aligned}$$

Then

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x \rangle_{C_0(E^0)})\| \leq \|\langle x, x \rangle_{C_0(E^0)}\| = \|x\|_{C_0(E^0)}^2$$

for all $x \in C_c(E^1)$. So ψ is bounded, and hence extends uniquely to a bounded linear map $\tilde{\psi}$ on $X(E)$. By continuity, $(\tilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$. \square

Remark 4.5. Let (ψ, π) be a Toeplitz representation of the graph correspondence $X(E)$. Then the pair $(\psi|_{C_c(E^1)}, \pi)$ is a Toeplitz representation of E . So Proposition 4.4 implies that $(\psi, \pi) \mapsto (\psi|_{C_c(E^1)}, \pi)$ is a bijection between Toeplitz representations of $X(E)$ and Toeplitz representations of E , with inverse described by Proposition 4.4.

Proof of Theorem 2.4. Let (i_X, i_A) be the universal Toeplitz representation of $X(E)$ in $\mathcal{T}(E)$. Then $(i_1, i_0) := (i_X|_{C_c(E^1)}, i_A)$ is a Toeplitz representation of E . Fix another Toeplitz representation (ψ, π) of E in a C^* -algebra B . By Proposition 4.4, ψ extends to $\tilde{\psi} : X(E) \rightarrow B$ such that $(\tilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$. By the universal property of (i_X, i_A) , there exists a homomorphism $h_{\tilde{\psi}, \pi} : \mathcal{T}(E) \rightarrow B$, such that $h_{\tilde{\psi}, \pi} \circ i_X = \tilde{\psi}$, and $h_{\tilde{\psi}, \pi} \circ i_A = \pi$. In particular $h_{\tilde{\psi}, \pi} \circ i_1 = \psi$. Hence (i_1, i_0) is a universal Toeplitz representation of E which generates $\mathcal{T}(E)$. The second statement follows easily. \square

Our next task is to prove Theorem 2.13. We first need some background results.

Remark 4.6. Let E be a topological graph. Fix $v \in E_{\text{fin}}^0$. There exists a neighbourhood U of v such that $r^{-1}(U)$ is compact, and there exists an open neighbourhood V of v such that $V \subset U$. By Remark 4.2, there exists an open neighbourhood W of v such that $\overline{W} \subset V$. Since $r^{-1}(\overline{W})$ is closed and is contained in $r^{-1}(U)$, it is compact. Hence $v \in E_{\text{fin}}^0$ if and only if there exists an open neighbourhood N of v such that $r^{-1}(\overline{N})$ is compact.

Let E be a topological graph. Recall from Definition 3.5 that Katsura's covariance condition for a Toeplitz representation of the correspondence $X(E)$ involves the ideal $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker \phi)^\perp$. Katsura computed this ideal in [5]. We quote his result and give a simple proof.

Observe that $\ker \phi = \{f \in C_0(E^0) : f(\overline{r(E^1)}) \equiv 0\}$. Hence

$$(\ker \phi)^\perp = \{f \in C_0(E^0) : f(\overline{E_{\text{sce}}^0}) \equiv 0\}.$$

Proposition 4.7 ([5, Proposition 1.24]). *Let E be a topological graph. Then*

$$\phi^{-1}(\mathcal{K}(X(E))) = C_0(E_{\text{fin}}^0).$$

Moreover $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker \phi)^\perp = C_0(E_{\text{rg}}^0)$.

Proof. The final statement follows from the previous one and definition of E_{rg}^0 . So we just need to show that $\phi^{-1}(\mathcal{K}(X(E))) = C_0(E_{\text{fin}}^0)$.

Fix $f \in C_0(E^0) \setminus C_0(E_{\text{fin}}^0)$. We must show that $\phi(f) \notin \mathcal{K}(X(E))$. Fix $v_0 \in (E_{\text{fin}}^0)^c$, such that $f(v_0) \neq 0$. Let $m = |f(v_0)|$ and let

$$N_0 = \{v \in E^0 : |f(v)| > m/2\},$$

so N_0 is an open neighbourhood of v_0 . By Remark 4.6, $r^{-1}(\overline{N_0})$ is not compact. Fix x_1, \dots, x_n and y_1, \dots, y_n in $C_c(E^1)$. Let

$$K = \bigcup_{i=1}^n \text{supp}(x_i) \cup \text{supp}(y_i).$$

Then K is compact, so $r^{-1}(\overline{N_0})$ is not contained in K . So there exists $e_0 \in r^{-1}(\overline{N_0}) \setminus K$. By Remark 4.2 there exists $x_0 \in C_c(E^1)$ such that $x_0(e_0) = 1$. Hence

$$\begin{aligned} & \left\| \phi(f) - \sum_{i=1}^n \Theta_{x_i, y_i} \right\| \\ & \geq \left\| \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i, y_i}(x_0) \right\|_{C_0(E^0)} \\ & \geq \left| \left\langle \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i, y_i}(x_0), \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i, y_i}(x_0) \right\rangle_{C_0(E^0)} (s(e_0)) \right|^{1/2} \\ & \geq m/2. \end{aligned}$$

Thus $\|\phi(f) - a\| \geq m/2$ for all $a \in \text{span}\{\Theta_{x,y} : x, y \in C_c(E^1)\}$. Since $\overline{\text{span}\{\Theta_{x,y} : x, y \in C_c(E^1)\}} = \mathcal{K}(X(E))$, it follows that $\phi(f) \notin \mathcal{K}(X(E))$.

Now fix a nonnegative function $f \in C_c(E_{\text{fin}}^0)$. We must show that $\phi(f) \in \mathcal{K}(X(E))$. Let $K' = \text{supp}(f)$. For any $e \in r^{-1}(K')$, there exists an open s -section N_e containing e . Remark 2.7 shows that $r^{-1}(K')$ is compact. Hence there exists a finite subset $F \subset r^{-1}(K')$ such that $\{N_e\}_{e \in F}$ covers $r^{-1}(K')$. Since $\text{supp}(f \circ r) \subset r^{-1}(K')$, Remark 2.9 yields a finite collection of functions $\{f_e : e \in F\} \subset C_c(E^1)$ such that each $\text{osupp}(f_e) \subset N_e \cap r^{-1}(K')$ and $\sum_{e \in F} f_e = f \circ r$. Since the f_e are supported on the s -sections N_e , we have $\theta_{\sqrt{f_e}, \sqrt{f_e}}(x)(e') = f_e(e')x(e')$ for all $e' \in E^1$. Hence

$$(4.2) \quad \phi(f) = \sum_{e \in F} \Theta_{\sqrt{f_e}, \sqrt{f_e}} \in \mathcal{K}(X(E)). \quad \square$$

Proposition 4.8. *Let E be a topological graph and let (ψ, π) be a covariant Toeplitz representation of E . Then the Toeplitz representation $(\tilde{\psi}, \pi)$ of $X(E)$ from Proposition 4.4 is also covariant. For any nonnegative function $f \in C_c(E_{\text{rg}}^0)$, any finite cover \mathcal{N} of $r^{-1}(\text{supp}(f))$ by open s -sections and*

any collection of nonnegative functions $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$, such that $\text{osupp}(f_N) \subset N \cap r^{-1}(\text{supp}(f))$, and $\sum_{N \in \mathcal{N}} f_N = f \circ r$, we have

$$\pi(f) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*.$$

Proof. Choose collections \mathcal{G} , $\{\mathcal{N}_f : f \in \mathcal{G}\}$, and $\{f_N : N \in \mathcal{N}_f\}$ as in Definition 2.10. By Corollary 4.7, to prove that $(\tilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$, must show that $\tilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$, for all $f \in \mathcal{G}$. Fix $f \in \mathcal{G}$. Since $\phi(f) = \sum_{N \in \mathcal{N}_f} \Theta_{\sqrt{f_N}, \sqrt{f_N}}$, we have $\tilde{\psi}^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$. Hence $\tilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$ by Definition 2.10. Therefore $(\tilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$.

For the second statement observe that since $(\tilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$,

$$\pi(f) = \tilde{\psi}^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}_f} \tilde{\psi}^{(1)}(\Theta_{\sqrt{f_N}, \sqrt{f_N}}) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*. \quad \square$$

Proposition 4.9. *Let E be a topological graph and let (ψ, π) be a covariant Toeplitz representation of $X(E)$. Then $(\psi|_{C_c(E^1)}, \pi)$ is a covariant Toeplitz representation of E .*

Proof. Remark 4.5 implies that $(\psi|_{C_c(E^1)}, \pi)$ is a Toeplitz representation of E . Let \mathcal{G} be the set of all nonnegative functions in $C_c(E_{\text{rg}}^0)$. Fix $f \in \mathcal{G}$. By Equation (4.2), there exists a finite cover \mathcal{N} of $r^{-1}(\text{supp}(f))$ by open s -sections and a finite collection of functions $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$, such that $\text{osupp}(f_N) \subset N \cap r^{-1}(\text{supp}(f))$, $\sum_{N \in \mathcal{N}} f_N = f \circ r$, and $\phi(f) = \sum_{N \in \mathcal{N}} \Theta_{\sqrt{f_N}, \sqrt{f_N}}$. Since (ψ, π) is a covariant Toeplitz representation of $X(E)$, we have

$$\begin{aligned} \pi(f) &= \psi^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^* \\ &= \sum_{N \in \mathcal{N}} \psi|_{C_c(E^1)}(\sqrt{f_N})\psi|_{C_c(E^1)}(\sqrt{f_N})^*. \end{aligned}$$

Hence $(\psi|_{C_c(E^1)}, \pi)$ is covariant. \square

Proof of Theorem 2.13. Propositions 4.8 and 4.9 provide a bijective map from covariant Toeplitz representations of E to covariant Toeplitz representations of $X(E)$. The result now follows from the same argument as Theorem 2.4. \square

We still need to prove Corollary 2.15. We must first show that under the hypotheses of the corollary, there are plenty of local r -fibrations (see Definition 2.14).

Lemma 4.10. *Let E be a topological graph. Suppose that r is a local homeomorphism. Then for any $v \in E_{\text{rg}}^0$, there exists a precompact open local r -fibration (\mathcal{U}, V) , such that $v \in V \subset \overline{V} \subset E_{\text{rg}}^0$.*

Proof. Fix $v \in E_{\text{rg}}^0$. Since $E_{\text{rg}}^0 = E_{\text{fin}}^0 \setminus \overline{E_{\text{sce}}^0} \subset E_{\text{fin}}^0 \setminus E_{\text{sce}}^0$, Lemma 1.22 of [5] shows that $r^{-1}(v)$ is nonempty. Lemma 1.23 of [5] shows that it is also compact. Since r is a local homeomorphism, $r^{-1}(v)$ is finite. Since E^1 is locally compact Hausdorff, we can separate points in $r^{-1}(v)$ by mutually disjoint precompact open bisections $\{U_e : e \in r^{-1}(v)\}$. We can assume, by shrinking if necessary, that the U_e have common range N with $\overline{N} \subset E_{\text{rg}}^0$. Now $|r^{-1}(w)| \geq |r^{-1}(v)|$ for all $w \in N$.

We claim there exists an open neighbourhood V of v such that $V \subset N$, and $|r^{-1}(w)| = |r^{-1}(v)|$ for all $w \in V$. Suppose for a contradiction there exists a convergent net $(v_\alpha)_{\alpha \in \Lambda} \subset N$ with limit v satisfying $|r^{-1}(v_\alpha)| > |r^{-1}(v)|$ for all $\alpha \in \Lambda$. Then for any $\alpha \in \Lambda$, there exists $e_\alpha \notin \bigcup_{e \in r^{-1}(v)} U_e$ such that $r(e_\alpha) = v_\alpha$. Since $r^{-1}(\overline{N})$ is compact, [13, Theorem IV.3] implies that there exists a convergent subnet $(e_{h(a)})_{a \in A}$ of $(e_\alpha)_{\alpha \in \Lambda}$ with limit $e \notin \bigcup_{f \in r^{-1}(v)} U_f$. By continuity of r , we have $r(e) = v$, which is a contradiction.

Hence there exists an open neighbourhood V of v satisfying $V \subset N$, such that $|r^{-1}(w)| = |r^{-1}(v)|$ for all $w \in V$. So with

$$\mathcal{U} = \{U_e \cap r^{-1}(V) : e \in r^{-1}(v)\},$$

the pair (\mathcal{U}, V) is a precompact open local r -fibration. \square

Lemma 4.11. *Let E be a topological graph. Suppose that r is a local homeomorphism. Let \mathcal{G} be the set of all nonnegative functions f in $C_c(E_{\text{rg}}^0)$ such that $\text{supp}(f) \subset V$ for some open local r -fibration (\mathcal{U}, V) . Then \mathcal{G} generates $C_0(E_{\text{rg}}^0)$.*

Proof. By Lemma 4.10, for all $v \neq w \in E_{\text{rg}}^0$, there is a local r -fibration (\mathcal{U}, V) such that $v \in V$ and $w \notin V$. So there exists $f \in \mathcal{G}$ such that $f(v) = 1$ and $f(w) = 0$. Now the result follows from the Extended Stone–Weierstrass Theorem. \square

Proof of Corollary 2.15. Let (ψ, π) be a covariant Toeplitz representation of E . Let \mathcal{G} be the set of all nonnegative functions f in $C_c(E_{\text{rg}}^0)$ such that $\text{supp}(f) \subset V$ for some open local r -fibration (\mathcal{U}, V) . Lemma 4.11 implies that \mathcal{G} generates $C_0(E_{\text{rg}}^0)$. Fix $f \in \mathcal{G}$ and an open local r -fibration (\mathcal{U}, V) with $\text{supp}(f) \subset V$. Since \mathcal{U} is a finite cover of $r^{-1}(\text{supp}(f))$ by open s -sections, each $\text{osupp}(r_U^* f) \subset U \cap r^{-1}(\text{supp}(f))$, and $\sum_{U \in \mathcal{U}} r_U^* f = f \circ r$. By Proposition 4.8, we have $\pi(f) = \sum_{U \in \mathcal{U}} \psi(\sqrt{r_U^* f}) \psi(\sqrt{r_U^* f})^*$. The converse of the first statement follows from Definition 2.10 and Remark 2.16. The second statement follows easily from the construction of \mathcal{G} . \square

4.1. The C^* -algebra generated by a Toeplitz representation. In this subsection we provide some technical results which may prove useful in using our descriptions of the C^* -algebras associated to topological graphs. Proposition 4.12 aids in constructing representations; and Proposition 4.16 provides a well-behaved collection of spanning elements for the image of any Toeplitz representation of E , and also a formula for computing products of these spanning elements.

To construct Toeplitz representations of a topological graph, one needs to build linear maps $\psi : C_c(E^1) \rightarrow B$ that are $\|\cdot\|_{C_0(E^0)}$ -bounded. The following technical result reduces the task to defining ψ on functions that are dense in supremum norm on $C_0(U)$ for a suitable family of open s -sections U .

Proposition 4.12. *Let E be a topological graph, let \mathcal{B} be an open base for the topology on E^1 consisting of s -sections, and let $\mathcal{F} \subset C_c(E^1)$ be a collection of nonnegative functions such that $\text{osupp}(x)$ is an s -section for all $x \in \mathcal{F}$. Suppose that for each $U \in \mathcal{B}$,*

$$(4.3) \quad \text{span}\{x \in \mathcal{F} : \text{osupp}(x) \subset U\} \text{ is } \|\cdot\|_\infty\text{-dense in } C_0(U).$$

Then $X_0 := \text{span } \mathcal{F}$ is dense in $X(E)$. Let B be a C^ -algebra. Suppose that $\psi_0 : X_0 \rightarrow B$ is a linear map, that $\pi : C_0(E^0) \rightarrow B$ is a homomorphism, and that the pointwise product xy satisfies*

$$(4.4) \quad \pi(\widehat{\sqrt{xy}}) = \psi_0(x)^* \psi_0(y) \quad \text{for all } x, y \in \mathcal{F}.$$

Then ψ_0 extends uniquely to a bounded linear map ψ on $C_c(E^1)$. Suppose that $\mathcal{G} \subset C_0(E^0)$ generates $C_0(E^0)$. If the extension ψ satisfies

$$(4.5) \quad \psi(f \cdot x) = \pi(f) \psi_0(x) \quad \text{for all } f \in \mathcal{G} \text{ and } x \in \mathcal{F},$$

then (ψ, π) is a Toeplitz representation of E .

Proof. Fix $x \in C_c(E^1)$. Let $K = \text{supp}(x)$. For each $e \in K$, there exists an open s -section N_e containing e , such that $N_e \in \mathcal{B}$. Since K is compact, there is a finite subset $F \subset K$, such that $\{N_e : e \in F\}$ covers K . By Remark 2.9, there exists a finite collection of functions $\{x_e : e \in F\} \subset C_c(E^1)$, such that $\text{osupp}(x_e) \subset N_e \cap K$ for all $e \in F$, and $\sum_{e \in F} x_e = x$. Fix $e \in F$. Since $\text{osupp}(x_e) \subset N_e$, there exists a sequence $(x_{e,n}) \subset X_0 \cap C_0(N_e)$ converging to x_e in supremum norm. That $(x_{e,n})$ and x_e vanish off the s -section N_e imply that $\|x_{e,n} - x_e\|_{C_0(E^0)} = \sup_{e \in E^1} |x_{e,n} - x_e|$. Hence $\sum_{e \in F} x_{e,n} \rightarrow x$ in $\|\cdot\|_{C_0(E^0)}$ norm. Therefore X_0 is dense in $X(E)$.

Fix $x, y \in \mathcal{F}$. Since x, y are nonnegative, $\widehat{\sqrt{xy}} = \langle x, y \rangle_{C_0(E^0)}$. Hence (4.4) implies that $\pi(\langle x, y \rangle_{C_0(E^0)}) = \psi_0(x)^* \psi_0(y)$. Linearity of ψ_0 and π gives $\pi(\langle x, y \rangle_{C_0(E^0)}) = \psi_0(x)^* \psi_0(y)$, for all $x, y \in X_0$. By Remark 3.4, ψ_0 is bounded, so extends uniquely to $\tilde{\psi} : X(E) \rightarrow B$. Then $\psi := \tilde{\psi}|_{C_c(E^1)}$ is the required map. Continuity and (4.5) show that (ψ, π) is a Toeplitz representation of E . \square

Remark 4.13. To prove Proposition 4.12 we showed that Equation (4.3) implies that X_0 is dense in $X(E)$ under the $\|\cdot\|_{C_0(E^0)}$ norm, and then deduced that (ψ, π) extends to a Toeplitz representation of E . So replacing Equation (4.3) with the hypothesis that X_0 is dense in $X(E)$ would yield a formally stronger result. However, Equation (4.3) is in many instances easier to check.

Our next proposition provides a description of the C^* -algebra generated by a Toeplitz representation of E in terms of a spanning family which captures many of the key properties of the usual spanning family in the Toeplitz algebra of a directed graph. We first need some notation and two technical lemmas.

Recall that E^n denotes the space

$$\{\mu = \mu_1 \dots \mu_n : \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$$

of paths of length n in E . We define $r, s : E^n \rightarrow E^0$ by $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$, and we give E^n the relative topology inherited from the product space $\prod_{i=1}^n E^1$. For $x \in C_c(E^n)$ and $f \in C_0(E^0)$ we define $f \cdot x, x \cdot f \in C_c(E^n)$ by $(f \cdot x)(\mu) = f(r(\mu))x(\mu)$ and $(x \cdot f)(\mu) = x(\mu)f(s(\mu))$.

For $x_1, \dots, x_n \in C_c(E^1)$, we define $x_1 \diamond \dots \diamond x_n \in C_c(E^n)$ by

$$(x_1 \diamond \dots \diamond x_n)(\mu) = \prod_{i=1}^n x_i(\mu_i) \quad \text{for } \mu = \mu_1 \dots \mu_n \in E^n.$$

We use the symbol \diamond for this operation to distinguish it from the pointwise product of elements of $C_c(E^1)$ appearing in, for example, Equation (4.4).

The second assertion of the following technical result follows from the discussion preceding [9, Proposition 3.3] together with [5, Proposition 1.27] (see also [10, Proposition 9.7]). We include the result and a simple proof here for completeness.

Suppose that $x, y \in C_c(E^n)$ are supported on s -sections. Then there is a unique $H(x, y) \in C_c(E^0)$ that vanishes on $E^0 \setminus \{s(\mu) : x(\mu)y(\mu) \neq 0\}$ and satisfies

$$H(x, y)(s(\mu)) = \overline{x(\mu)}y(\mu) \text{ whenever } x(\mu)y(\mu) \neq 0.$$

Lemma 4.14. *Let E be a topological graph. Suppose that x_1, \dots, x_n and y_1, \dots, y_n are supported on s -sections. Let (ψ, π) be a Toeplitz representation of E . Let $x = x_1 \diamond \dots \diamond x_n$ and $y = y_1 \diamond \dots \diamond y_n$. Then*

$$\pi(H(x, y)) = \psi(x_n)^* \dots \psi(x_1)^* \psi(y_1) \dots \psi(y_n).$$

If $x = y$ then $\prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(y_i)$.

Proof. By Proposition 4.4, (ψ, π) extends to a Toeplitz representation of $X(E)$. We have $H(x, y) = \langle x_n, H(x_1 \diamond \dots \diamond x_{n-1}, y_1 \diamond \dots \diamond y_{n-1}) \cdot y_n \rangle$, and hence

$$\pi(H(x, y)) = \psi(x_n)^* \pi(H(x_1 \diamond \dots \diamond x_{n-1}, y_1 \diamond \dots \diamond y_{n-1}) \psi(y_n).$$

The first assertion now follows by induction. For the second assertion, we use the first to see that

$$\begin{aligned} \left(\prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i) \right)^* \left(\prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i) \right) \\ = \pi(H(x, x) - H(x, y) - H(y, x) + H(y, y)), \end{aligned}$$

which is equal to zero since $x = y$. \square

Lemma 4.15. *Let E be a topological graph. Suppose that $x_1, \dots, x_n \in C_c(E^1)$ are supported on s -sections, and fix $f \in C_0(E^0)$. Then there exists $\tilde{f} \in C_c(E^0)$ such that*

$$(4.6) \quad \tilde{f}(s(\mu)) = f(r(\mu)) \quad \text{whenever } (x_1 \diamond \dots \diamond x_n)(\mu) \neq 0.$$

For any such \tilde{f} , we have $f \cdot (x_1 \diamond \dots \diamond x_n) = (x_1 \diamond \dots \diamond x_n) \cdot \tilde{f}$, and $\pi(f) \prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(x_i) \pi(\tilde{f})$ for any Toeplitz representation (ψ, π) of E .

Proof. The second assertion will follow from the first by definition of $f \cdot (x_1 \diamond \dots \diamond x_n)$ and $(x_1 \diamond \dots \diamond x_n) \cdot \tilde{f}$. The final assertion will then follow from Lemma 4.14. So we just need to prove the first assertion. Let $x := x_1 \diamond \dots \diamond x_n$. Fix $f \in C_0(E^0)$. Since $K := \text{supp}(x) \subseteq E^n$ is an s -section, there is a well-defined continuous function from $s(K)$ to $r(K)$ given by $s(\mu) \mapsto r(\mu)$ for $\mu \in K$. So there is a continuous function $\tilde{f}_0 \in C(s(K))$ given by $\tilde{f}_0(s(\mu)) = f(r(\mu))$ for all $\mu \in K$. Since $s(K)$ is compact, an application of the Tietze extension theorem shows that \tilde{f} has an extension $\tilde{f} \in C_0(E^0)$, which satisfies (4.6) by definition. \square

Proposition 4.16. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E . Let \mathcal{B} be a base for the topology on E^1 consisting of open s -sections, and let $\mathcal{F} \subseteq C_c(E^1)$ be a collection of nonnegative functions such that $\text{supp}(f)$ is an s -section for each $f \in \mathcal{F}$. Then:*

- (1) $C^*(\psi, \pi)$ is densely spanned by elements of the form

$$\psi(x_1) \dots \psi(x_n) \pi(f) \psi(y_m)^* \dots \psi(y_1)^*$$

where $m, n \geq 0$, $f \in C_c(E^0)$, the x_i, y_j all belong to \mathcal{F} , each of the sets $s(\text{osupp}(x_i)) \cap r(\text{osupp}(x_{i+1}))$ and $s(\text{osupp}(y_i)) \cap r(\text{osupp}(y_{i+1}))$ is nonempty, and $s(\text{osupp}(x_n)) \cap s(\text{osupp}(y_m)) \cap \text{osupp}(f) \neq \emptyset$.

- (2) Consider a pair of spanning elements

$$\begin{aligned} \psi(w_1) \dots \psi(w_m) \pi(f) \psi(x_n)^* \dots \psi(x_1)^*, \\ \psi(y_1) \dots \psi(y_p) \pi(g) \psi(z_q)^* \dots \psi(z_1)^* \end{aligned}$$

as in (1) with $p \geq n$. Let $x = x_1 \diamond \dots \diamond x_n$ and $y = y_1 \diamond \dots \diamond y_n$. Fix $k \in C_0(E^0)$ such that $(fH(x, y)) \cdot (y_{n+1} \diamond \dots \diamond y_p) = (y_{n+1} \diamond \dots \diamond y_p) \cdot k$

as in Lemma 4.15. Then

$$\begin{aligned} & (\psi(w_1) \dots \psi(w_m) \pi(f) \psi(x_n)^* \dots \psi(x_1)^*) \\ & \quad \cdot (\psi(y_1) \dots \psi(y_p) \pi(g) \psi(z_q)^* \dots \psi(z_1)^*) \\ & = \psi(w_1) \dots \psi(w_m) \psi(y_{n+1}) \dots \psi(y_p) \pi(kg) \psi(z_q)^* \dots \psi(z_1)^*. \end{aligned}$$

Remark 4.17. Consider the situation of Proposition 4.16(2) but with $p < n$. Let $x = x_1 \diamond \dots \diamond x_p$ and $y = y_1 \diamond \dots \diamond y_p$, and fix $k' \in C_0(E^0)$ such that $(H(x, y)g) \cdot (x_{p+1} \diamond \dots \diamond x_n) = (x_{p+1} \diamond \dots \diamond x_n) \cdot k'$. Taking adjoints in Proposition 4.16(2) gives

$$\begin{aligned} & (\psi(w_1) \dots \psi(w_m) \pi(f) \psi(x_n)^* \dots \psi(x_1)^*) \\ & \quad \cdot (\psi(y_1) \dots \psi(y_p) \pi(g) \psi(z_q)^* \dots \psi(z_1)^*) \\ & = \psi(w_1) \dots \psi(w_m) \pi(fk') \psi(x_n)^* \dots \psi(x_{p+1}) \psi(z_q)^* \dots \psi(z_1)^*. \end{aligned}$$

Proof of Proposition 4.16. (1) By Proposition 4.4, $(\tilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$, where $\tilde{\psi}$ is the unique extension of ψ to $X(E)$. The argument of [6, Proposition 2.7] shows that $C^*(\psi, \pi)$ is densely spanned by elements of the form $\psi(x_1) \dots \psi(x_n) \pi(f) \psi(y_m)^* \dots \psi(y_1)^*$ where each $x_i, y_i \in C_c(E_1)$ and $f \in C_c(E^0)$. Fix $x_1, x_2 \in C_c(E^1)$ with

$$s(\text{osupp}(x_1)) \cap r(\text{osupp}(x_2)) = \emptyset.$$

Then

$$\begin{aligned} \|\psi(x_1)\psi(x_2)\|^2 &= \|\psi(x_2)^*\psi(x_1)^*\psi(x_1)\psi(x_2)\| \\ &= \|\psi(x_2)^*\pi(\langle x_1, x_1 \rangle_{C_0(E^0)})\psi(x_2)\| \\ &= \|\psi(x_2)^*\psi(\langle x_1, x_1 \rangle_{C_0(E^0)} \cdot x_2)\| = 0. \end{aligned}$$

Similarly, $\psi(x_1)\psi(x_2)^* = 0$ whenever $s(\text{osupp}(x_1)) \cap s(\text{osupp}(x_2)) = \emptyset$. So elements of the form

$$\psi(x_1) \dots \psi(x_m) \pi(f) \psi(y_m)^* \dots \psi(y_1)^*$$

where

- $f \in C_c(E^0)$,
- each $s(\text{osupp}(x_i)) \cap r(\text{osupp}(x_{i+1})) \neq \emptyset$,
- each $s(\text{osupp}(y_i)) \cap r(\text{osupp}(y_{i+1})) \neq \emptyset$, and
- $s(\text{osupp}(x_n)) \cap s(\text{osupp}(y_m)) \cap \text{osupp}(f) \neq \emptyset$

span a dense subspace of $C^*(\psi, \pi)$. Since Proposition 4.12 implies that $X_0 = \text{span } \mathcal{F}$ is dense in $X(E)$ and hence in $C_c(E^1)$, the first assertion follows.

(2) Lemma 4.14 implies that

$$\pi(f) \psi(x_n)^* \dots \psi(x_1)^* \psi(y_1) \dots \psi(y_p) = \pi(fH(x, y)) \psi(y_{n+1}) \dots \psi(y_p).$$

The result now follows from Lemma 4.15. \square

Example 4.18. Let E be a directed graph regarded as a topological graph under the discrete topology. Let $\mathcal{G} = \{\delta_v : v \in E^0\}$ and $\mathcal{F} = \{\delta_e : e \in E^1\}$. Then \mathcal{G} and \mathcal{F} satisfy the hypotheses of Proposition 4.12, so we recover as an immediate consequence the isomorphism of the Toeplitz algebra $\mathcal{T}(E)$ of the graph bimodule (see [4] and [5]) with the Toeplitz algebra $\mathcal{TC}^*(E)$ of the graph E . The usual spanning family and multiplication rule for $\mathcal{TC}^*(E)$ follows from Proposition 4.16 applied to the same \mathcal{F} .

Remark 4.19. The multiplication formula of Proposition 4.16(2) has the drawback that the element k has no explicit formula in terms of the x_i the y_i and the function f ; it is obtained by an application of the Tietze extension theorem (see Lemma 4.15). However, in practise there will frequently be a natural choice for k . Suppose, for example, that E^1 is totally disconnected. Then \mathcal{F} can be taken to consist of characteristic functions of compact open s -sections. We can then take k to be the function that is identically zero off $s(\text{supp}(y_{n+1} \diamond \cdots \diamond y_p))$ and satisfies $k(s(\mu)) = f(r(\mu))H(x, y)(r(\mu))$ whenever $\mu \in \text{supp}(y_{n+1} \diamond \cdots \diamond y_p)$; this k is continuous because $s(\text{supp}(y_{n+1} \diamond \cdots \diamond y_p))$ is clopen.

5. The topological graph arising from the shift map on the infinite-path space

In this section we discuss how our results apply to the topological graph \widehat{E} arising from the shift map on the infinite-path space E^∞ of a row-finite directed graph E with no sources. It is known that $\mathcal{O}_{X(\widehat{E})}$ is isomorphic to $C^*(E)$ (it could be recovered from [2, 3]) but existing proofs use the universal property of $C^*(E)$ to induce a homomorphism from $C^*(E)$ to $\mathcal{O}_{X(\widehat{E})}$, invoke the gauge-invariant uniqueness theorem for $C^*(E)$ to establish injectivity, and then argue surjectivity by hand. It takes some work to show using the universal property of $\mathcal{O}_{X(\widehat{E})}$ that there is a homomorphism going in the other way.

Let E be a row-finite directed graph with no sources. Let $E^* = \bigcup_{n \geq 0} E^n$ and let $E^\infty = \{z \in \prod_{i=1}^\infty E^1 : s(z_i) = r(z_{i+1}), \text{ for all } i = 1, 2, \dots\}$. We view E^∞ as a topological space under the subspace topology coming from the ambient space $\prod_{i=1}^\infty E^1$. For any $\mu \in E^* \setminus E^0$, we define the cylinder set $Z(\mu) = \{z \in E^\infty : z_1 = \mu_1, \dots, z_{|\mu|} = \mu_{|\mu|}\}$. For $v \in E^0$ we define $Z(v) = \{z \in E^\infty : r(z_1) = v\}$. Since E has no sources, each $Z(\mu)$ is nonempty. The space E^∞ is a locally compact Hausdorff space with a base of compact open sets $\{Z(\mu) : \mu \in E^*\}$ ([8, Corollary 2.2]).

We construct a topological graph $\widehat{E} = (\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s})$. Let $\widehat{E}^0 = \widehat{E}^1 = E^\infty$. Define \widehat{r} to be the identity map, and define $\widehat{s}(z) = (z_2, z_3, \dots)$ for all $z \in \widehat{E}^1$. Since $Z(\mu)$ is a compact open \widehat{s} -section whenever $\mu \notin E^0$, the map \widehat{s} is a local homeomorphism and hence \widehat{E} is a topological graph.

For the following result recall that a Cuntz–Krieger E -family is a family $\{p_v, s_e\}$ satisfying the relations described in the first paragraph of the introduction: that is, the numbered relations (1) and (2) and the additional relation $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever $0 < |r^{-1}(v)| < \infty$.

Proposition 5.1. *Let E be a row-finite directed graph with no sources, and let \widehat{E} be the topological graph described above.*

- (1) *Let (ψ, π) be a covariant Toeplitz representation of \widehat{E} . For $v \in E^0$ define $q_v := \pi(\chi_{Z(v)})$ and for $e \in E^1$ define $t_e := \psi(\chi_{Z(e)})$. Then the q_v and the t_e form a Cuntz–Krieger E -family.*
- (2) *Let $\{q_v : v \in E^0\}$, $\{t_e : e \in E^1\}$ be a Cuntz–Krieger E -family in a C^* -algebra B . Then there is a unique covariant Toeplitz representation (ψ, π) of \widehat{E} such that $\psi(\chi_{Z(e\mu)}) = t_{e\mu} t_\mu^*$, and $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ for all $e \in E^1$, $\mu \in E^*$.*
- (3) *Let (j_1, j_0) be the universal covariant Toeplitz representation of \widehat{E} in $\mathcal{O}(\widehat{E})$, and let $\{p_v, s_e : v \in E^0, e \in E^1\}$ be the Cuntz–Krieger E -family generating $C^*(E)$. There is an isomorphism $\mathcal{O}(\widehat{E}) \cong C^*(E)$ which carries each $j_1(\chi_{Z(e\mu)})$ to $s_{e\mu} s_\mu^*$, and each $j_0(\chi_{Z(\mu)})$ to $s_\mu s_\mu^*$.*

Proof of Proposition 5.1(1). The q_v are mutually orthogonal projections because the $\chi_{Z(v)}$ are. For $\mu \in E^* \setminus E^0$, the set $Z(\mu)$ is a compact open s -section. Thus for $e \in E^1$, relation (2) of Definition 2.2 implies that $t_e^* t_e = \pi(\widehat{\chi_{Z(e)}}) = \pi(\chi_{Z(s(e))}) = q_{s(e)}$. For $\mu \in E^* \setminus E^0$, $(\{Z(\mu)\}, Z(\mu))$ is an open local \widehat{r} -fibration. Since $\text{supp}(\chi_{Z(\mu)}) = Z(\mu)$ and (ψ, π) is covariant, Corollary 2.15 implies that

$$(5.1) \quad \pi(\chi_{Z(\mu)}) = \psi\left(\sqrt{r_{Z(\mu)}^* \chi_{Z(\mu)}}\right) \psi\left(\sqrt{r_{Z(\mu)}^* \chi_{Z(\mu)}}\right)^* = \psi(\chi_{Z(\mu)}) \psi(\chi_{Z(\mu)})^*.$$

So for $v \in E^0$, we have

$$q_v = \pi(\chi_{Z(v)}) = \sum_{r(e)=v} \pi(\chi_{Z(e)}) = \sum_{r(e)=v} \psi(\chi_{Z(e)}) \psi(\chi_{Z(e)})^* = \sum_{r(e)=v} t_e t_e^*. \quad \square$$

Proof of Proposition 5.1(2). Let $\mathcal{G} := \{\chi_{Z(\mu)} : \mu \in E^* \setminus E^0\} \subseteq C_0(\widehat{E}^0)$. Then $\text{span } \mathcal{G}$ is a dense $*$ -subalgebra of $C_0(\widehat{E}^0)$. We aim to define a map $\pi_0 : \text{span } \mathcal{G} \rightarrow B$ by $\pi_0(\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)}) = \sum_{i=1}^n \alpha_i t_{\mu_i} t_{\mu_i}^*$. We check that π_0 is well-defined. It suffices to prove that $\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} = 0$ implies $\sum_{i=1}^n \alpha_i t_{\mu_i} t_{\mu_i}^* = 0$, where the μ_i are distinct. Since E is row-finite and has no sources,

$$\begin{aligned} \pi_0\left(\sum_{e \in r^{-1}(s(\mu))} \chi_{Z(\mu e)}\right) &= \sum_{e \in r^{-1}(s(\mu))} t_{\mu e} t_{\mu e}^* \\ &= t_\mu \left(\sum_{e \in r^{-1}(s(\mu))} t_e t_e^*\right) t_\mu^* = t_\mu t_\mu^* = \pi_0(\chi_{Z(\mu)}), \end{aligned}$$

so we can assume that the μ_i have the same length. It follows that the $\chi_{Z(\mu_i)}$ are mutually orthogonal nonzero projections and hence each $\alpha_i = 0$. So π_0 is well-defined. It is obvious that π_0 is a linear map preserving the involution. [10, Corollary 1.15] implies that π_0 is a homomorphism. Now we show that π_0 is norm decreasing. Fix μ_1, \dots, μ_n . We can assume that the μ_i are distinct and have the same length. Then

$$\begin{aligned} \left\| \pi_0 \left(\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} \right) \right\|^2 &= \left\| \sum_{i=1}^n |\alpha_i|^2 t_{\mu_i} t_{\mu_i}^* \right\|^2 \\ &\leq (\max_i |\alpha_i|^2) \left\| \sum_{i=1}^n t_{\mu_i} t_{\mu_i}^* \right\|^2 \\ &\leq \max_i |\alpha_i|^2 \left\| \sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} \right\|^2, \end{aligned}$$

so π_0 is norm decreasing. Thus we obtain a unique homomorphism

$$\pi : C_0(\widehat{E}^0) \rightarrow C^*(E)$$

by $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ for all $\mu \in E^*$.

We next aim to define a linear map $\psi : C_c(\widehat{E}^1) \rightarrow B$ by extension of the formula $\psi(\chi_{Z(e\mu)}) = t_{e\mu} t_\mu^*$, and to show that the pair (ψ, π) is a Toeplitz representation of \widehat{E} . To do so, we will apply Proposition 4.12, so we need to set up the rest of the elements of the statement. Let

$$\mathcal{B} := \{Z(\mu) : \mu \in E^* \setminus E^0\}$$

and let $\mathcal{F} := \{\chi_{Z(e\mu)} : e \in E^1, \mu \in E^*\} \subseteq C_c(E^1)$. Certainly \mathcal{F} and \mathcal{B} satisfy Equation (4.3). Similarly to the construction of π_0 , there is a well-defined linear map $\psi_0 : \text{span } \mathcal{F} \rightarrow B$ satisfying

$$\psi_0 \left(\sum_{i=1}^n \alpha_i \chi_{Z(e_i \mu_i)} \right) = \sum_{i=1}^n \alpha_i t_{e_i \mu_i} t_{\mu_i}^*.$$

Fix $x = \chi_{Z(e\mu)}$ and $y = \chi_{Z(f\nu)}$ in \mathcal{F} . We verify Equation (4.4). For this, observe that

$$\widehat{\sqrt{xy}} = \begin{cases} \chi_{Z(\mu)} & \text{if } e\mu = f\nu\mu' \\ \chi_{Z(\nu)} & \text{if } f\nu = e\mu\nu' \\ 0 & \text{otherwise.} \end{cases}$$

Then calculate:

$$(t_{e\mu} t_\mu^*)^* t_{f\nu} t_\nu^* = t_\mu t_{e\mu}^* t_{f\nu} t_\nu^* = \begin{cases} t_\nu t_\nu^* & \text{if } f\nu = e\mu\nu' \\ t_\mu t_\mu^* & \text{if } e\mu = f\nu\mu' \\ 0 & \text{otherwise.} \end{cases}$$

This establishes Equation (4.4). So Proposition 4.12 shows that ψ_0 extends uniquely to a linear map $\psi : C_c(\widehat{E}^1) \rightarrow B$. A similar calculation establishes

Equation (4.5). Proposition 4.12 implies that (ψ, π) is a Toeplitz representation of E .

It remains to check covariance. Since the range map is a homeomorphism onto \widehat{E}^0 we can apply Corollary 2.15 with \mathcal{G} as in the proof of part (2) and the local \widehat{r} -fibrations $\{(\{Z(\mu)\}, Z(\mu)) : \mu \in E^* \setminus E^0\}$ to see that (ψ, π) is covariant. \square

Proof of Proposition 5.1(3). We show $C^*(E)$ has the universal property of $\mathcal{O}(\widehat{E})$ and then invoke Theorem 2.13. Proposition 5.1(2) yields a covariant Toeplitz representation (θ_1, θ_0) of \widehat{E} in $C^*(E)$ such that

$$\begin{aligned}\theta_1(\chi_{Z(e\mu)}) &= s_{e\mu}s_\mu^* && \text{for all } e \in E^1, \mu \in E^*, \\ \theta_0(\chi_{Z(\mu)}) &= s_\mu s_\mu^* && \text{for all } \mu \in E^*.\end{aligned}$$

Fix a covariant Toeplitz representation (ψ, π) of \widehat{E} in a C^* -algebra B . Then Proposition 5.1(1) gives a Cuntz–Krieger E -family

$$\{q_v := \pi(\chi_{Z(v)}), t_e := \psi(\chi_{Z(e)}) : v \in E^0, e \in E^1\}$$

in B . So [10, Proposition 1.21] gives a homomorphism $\rho : C^*(E) \rightarrow B$ such that $\rho(p_v) = q_v$, and $\rho(s_e) = t_e$. An induction on the length of μ using Equation (5.1) shows that $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$. For $e \in E^1$, and $\mu \in E^*$, we have $\rho \circ \theta_1(\chi_{Z(e\mu)}) = t_e t_\mu t_\mu^* = \psi(\chi_{Z(e)}) \pi(\chi_{Z(\mu)}) = \psi(\chi_{Z(e\mu)})$. For $\mu \in E^* \setminus E^0$, we have $\rho \circ \theta_0(\chi_{Z(\mu)}) = t_\mu t_\mu^* = \pi(\chi_{Z(\mu)})$. Hence $\rho \circ \theta_1 = \psi$, and $\rho \circ \theta_0 = \pi$. Since the image of (θ_1, θ_0) generates $C^*(E)$, Theorem 2.13 implies that there is an isomorphism $\mathcal{O}(E) \cong C^*(E)$ which carries each $j_1(\chi_{Z(e\mu)})$ to $s_{e\mu}s_\mu^*$, and each $j_0(\chi_{Z(\mu)})$ to $s_\mu s_\mu^*$. \square

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This paper is available via <http://nyjm.albany.edu/j/2014/20-26.html>.