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N. A. Balonin

Saint-Petersburg State University Of Aerospace Instrumentation, korbendfs@mail.ru

Jennifer Seberry

University of Wollongong, jennie@uow.edu.au

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Keywords

matrices, conference, symmetric, review

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A Review and New Symmetric Conference Matrices

N. A. Balonin ^{*}and Jennifer Seberry [†]

Email: [korbendfs@mail.ru] and [jennifer_seberry]@uow.edu.au

Abstract

We consider symmetric conference matrices which were first highlighted by Vitold Belevitch, who showed that such matrices mapped to lossless telephone connections. We give the known properties of symmetric conference matrices, known orders and illustrations for some elementary and some interesting cases. We restrict our attention in this note to *symmetric conference* matrices. Web addresses are given for other illustrations.

Key Words and Phrases: Conference matrices, Hadamard matrices, symmetric balanced incomplete block designs (SBIBD), circulant difference sets, symmetric difference sets, constructions, telephony.

AMS Subject Classification: 05B20; 20B20.

1 Introduction

Symmetric Conference matrices are a particularly important class of $\{0, \pm 1\}$ matrices. Usually written as, C , they are $n \times n$ matrices with elements 0, +1 or -1 which satisfy

$$CC^T = C^T C = nI_n,$$

where “ T ” denotes the matrix transpose and I_n is the identity matrix of order n . We say that a conference matrix is an *orthogonal matrix*.

In this paper we use $-$ for -1 which corresponds to the usual Hadamard or weighing matrix notation.

A *circulant matrix* $C = (c_{ij})$ of order n satisfies $c_{ij} = c_{1, j-i+1(\text{mod } n)}$.

2 Properties of Symmetric Conference Matrices.

We note the following properties of a conference matrix

- the order of a conference matrix must be $\equiv 2(\text{mod } 4)$;
- $n - 1$, where n is the order of a conference matrix, must be the sum of two squares;
- if there is a conference matrix of order n then there is a symmetric conference matrix of order n with zero diagonal. The two forms are equivalent as one can be transformed into the other by (i) interchanging rows (columns) or (ii) multiplying rows (columns) by -1 ;
- a conference matrix is said to be normalized if it has first row and column all plus ones.
- $C^{-1} = nC^T$.

^{*}Saint Petersburg State University of Aerospace Instrumentation, 67, B. Morskaya St., 190000, St. Petersburg, Russian Federation

[†]Centre for Computer Security Research, EIS, University of Wollongong, NSW, 2522, Australia

3 Known Conference Matrix Orders

Conference matrices are known for the following orders

Key	Method	Explanation	References
c1	$p^r + 1$	$p^r \equiv 1 \pmod{4}$ is a prime power	[11, 6]
c2	$q^2(q + 2) + 1$	$q \equiv 3 \pmod{4}$ is a prime power $q + 2$ is a prime power	[10]
c3	46		[10]
c4	$5 \times 9^{2t+1} + 1$	$t \geq 0$ an integer	[14]
c5	$(n - 1)^s + 1$	$s \geq 2$ is an integer n the order of a conference matrix	[16, 13]
c6	$(h - 1)^{2s} + 1$	$s \geq 1$ is an integer, h the order of a skew-Hadamard matrix	[16, 13]

We now describe the examples of the $C(46)$ which differ from that of Mathon. We will observe and use three types of cells.

1. **type 0:** 0-circulant (zero shift, every row is equal to each-other);
2. **type 1:** circulant (circulant shift every new row right);
3. **type 2:** back-circulant (circulant shift every new row left).

We will say that a matrix has a *rich structure*, if it consists several different types of cells. Such notation allows us to describe special matrix structures for different $C(46)$.

3.1 Rich Structures and Families of Mathon Structure

The Mathon $C(46)$ [10] has as its core the usual symmetry block-circulant matrices, where every block has 9 little 3x3-cells. We write

$$W = \text{circ}(A, B, C, C^\top, B^\top)$$

where all cells of type 0 are situated inside of C .

3.1.1 The Basic Mathon $C(46)$ has cells only of types 0 and 1

Cells have type 1 inside of $A = \text{circ}(a, b, b^\top)$, cells have type 1 inside of $B = \text{backcirc}(c, d, c^\top)$, cells have type 0 inside of $C = \text{crosscirc}(e)$.

The $C = \text{crosscirc}(e)$ consists of $m=3$ columns (m – size of e), every column has $m=3$ rows – circulant shifted cell of type 0. We will call it as cross-shifted matrix (or cross-matrix, in short).

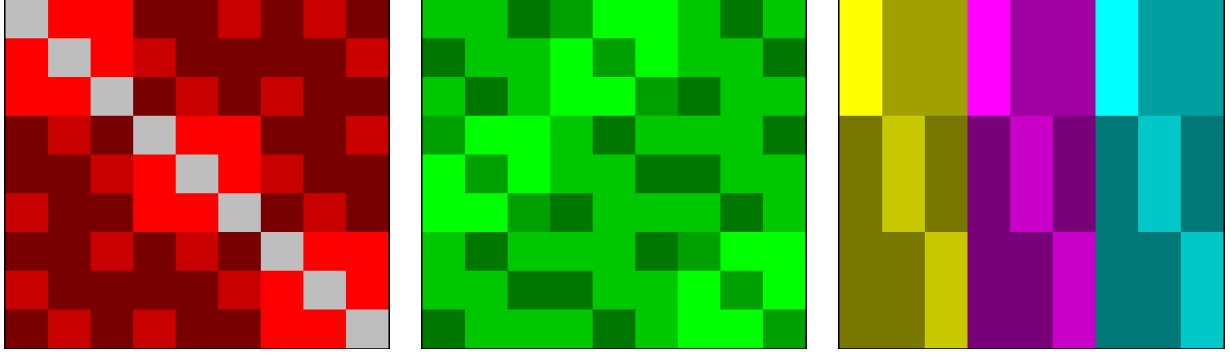


Figure 1: Matrices A , B , C of old cell-structure

3.1.2 The new Balonin-Seberry $C(46)$ is based on cells of all types 0, 1 and 2 (that is there are richer cells)

The different structures that appear have cells with

1. type 1: $A = \text{circ}(a, b, b^\top)$,
2. type 2: $B = \text{circ}(c, d, d^*)$,
3. type 0: $C = \text{crosscirc}(e)$.

Now let $d = [d_1 d_2 d_3]$ then cell $d^* = [d_3 d_1 d_2]$ is used instead of d^\top with back-circulant cells.

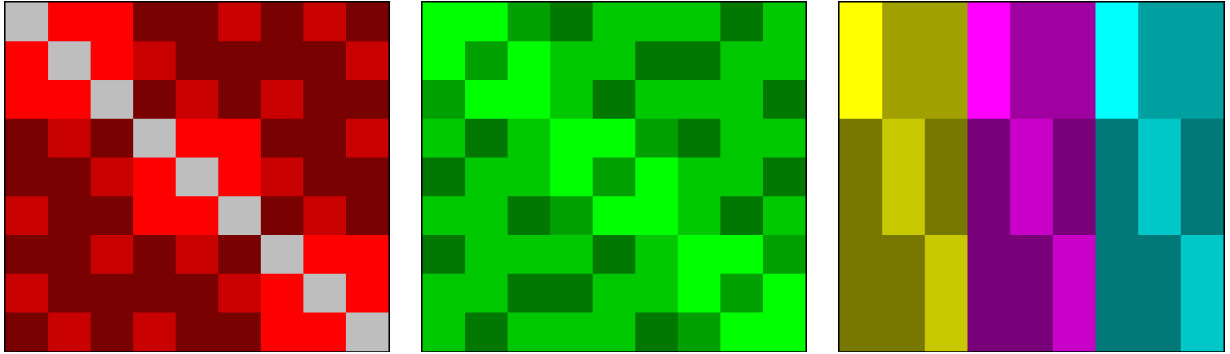


Figure 2: Matrices A , B , C of new cell-structure

This the most compact description of Mathon's matrix based on the term: "rich structure".

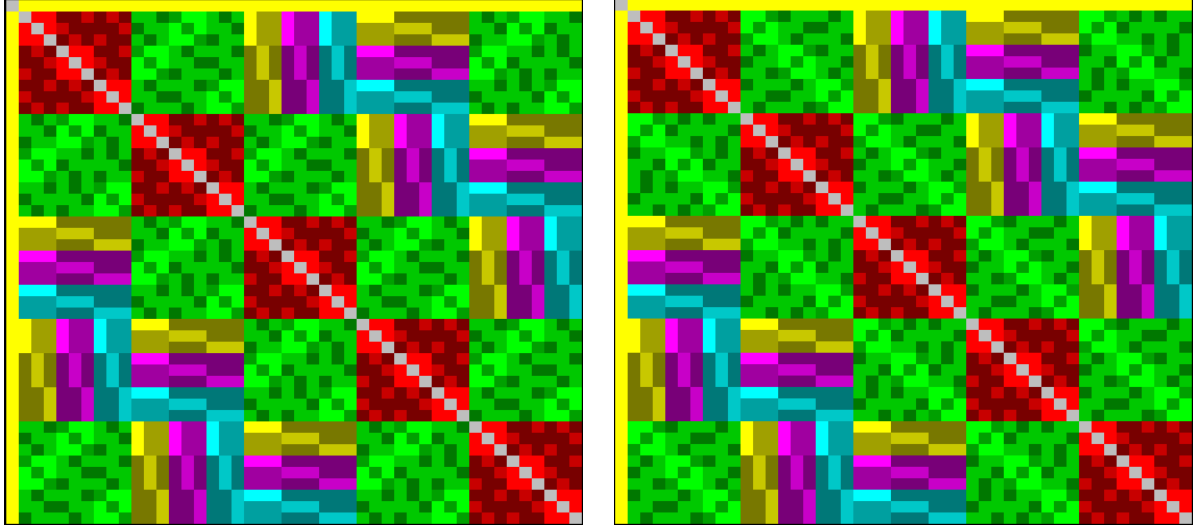


Figure 3: Matrice C46 of old (poor) and new (rich) cell-structures

The old structure has 2 types of cells and 3 types of matrices A , B , C . The new structure has 3 types of cells and 2 types of matrices. There is an important structural invariant: the common quantity of types (cells and matrices) is equal to 5.

3.2 Easy to use Conference Matrix Forms

When used in real world mechanical applications it may be useful to have them in one of a few main forms: a *conference matrix with circulant core* or a *conference matrix constructed from two circulant matrices* the latter matrices will not be normalized.

Key	Method	Explanation	References
c1a	$p + 1$	$p \equiv 1(\text{mod } 4)$ is a prime	[11, 6]
c1b	$p + 1$	$p \equiv 1(\text{mod } 4)$ is a prime (power?)	[]
c7	4 circulant matrices with two borders.		

The conference matrix (actually an $OD(13;4,9)$) found by D. Gregory of Queens University, Kingston, Canada [7] given here is of the type c7.

$$\left[\begin{array}{cc|cccccccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - \\ \hline 1 & 1 & 0 & - & - & 1 & - & - & 1 & 1 & - & 1 \\ 1 & 1 & - & 0 & - & - & 1 & - & - & 1 & 1 & - \\ 1 & 1 & - & - & 0 & - & - & 1 & 1 & - & 1 & - \\ \hline 1 & - & 1 & - & - & 0 & 1 & 1 & - & 1 & 1 & - \\ 1 & - & - & 1 & - & 1 & 0 & 1 & 1 & - & 1 & - \\ 1 & - & - & - & 1 & 1 & 1 & 0 & 1 & 1 & - & 1 \\ \hline 1 & 1 & 1 & - & 1 & - & 1 & 1 & 0 & - & - & - \\ 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 0 & - & 1 \\ 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 0 & - \\ \hline 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & 0 & 1 \\ 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 0 \\ 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & 0 \end{array} \right]$$

4 Conference matrices with cores and from two block matrices.

We particularly identify conferences matrices, of order n , which are normalized and can be written in one of the two forms: conference matrices with core or conference matrices made from two blocks.

These two forms look like

$$\frac{1}{e^\top} \left| \begin{array}{c} e \\ B \end{array} \right. \quad \text{and} \quad \left[\begin{array}{cc} A & B \\ B^\top & -A^\top \end{array} \right]$$

It is not necessary for A or B in either case to be circulant. However, in the form written they must commute. A variation of the second matrix can be used if A and B are amicable.

Then we say we have a *conference matrix with circulant core* or a *conference matrix constructed from two circulant matrices* the latter matrices will not be normalized.

Example 1

$$\left[\begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{array} \right] \quad \text{and} \quad \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & - & 1 & 1 \\ 1 & 0 & 1 & 1 & - & 1 \\ 1 & 1 & 0 & 1 & 1 & - \\ - & 1 & 1 & 0 & - & - \\ 1 & - & 1 & - & 0 & - \\ 1 & 1 & - & - & - & 0 \end{array} \right]$$

In this example the two matrices are in fact equivalent.

5 A Classification to Differentiate between Symmetric Conference Matrices

We classify these by whether they

1. have a circulant core;

2. are constructed from two circulant blocks;
3. have a core but it is not circulant;
4. are constructed from two blocks but they are not circulant;
5. Mathon's type;
6. from skew Hadamard matrices;
7. are constructed from four blocks with two borders;
8. any other pattern we see;
9. ad hoc.

6 Useful URLs and Webpages for this Study

Some useful url's include:

1. <http://mathscinet.ru/catalogue/0D/>
2. <http://mathscinet.ru/catalogue/artifact22/>
3. <http://mathscinet.ru/catalogue/conference/blocks/>

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8 Conclusion and Future Work

Comment In order to consider other matrices with these kinds of cells we consider the condition $n = p^2(q + 2) + 1$ as this allows many more little cells.

Version $n = 9 \times 9 + 1$ is very well known and class c1 [11]; Versions $n = 5 \times 9 \times 9 + 1$ and in general $n = 5 \times 9^{2t+1}$ is class c4 [14];

Version $n = 9 \times 9 \times 9 + 1$ is very well known and class c1 [11]; To continue to look at the versions $mp^r + 1$ we would next have to consider version $n = 13 \times 9 \times 9 + 1$ and so on.

Henceforth we consider the Mathon matrix as **oscillations** motivated by the Fourier basis.

Then the new Balonin-Seberry C(46) reflects phases “shift right” - “0-shift” - “shift-left” - “shift-left” - “0-shift”.

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Appendix

A Known Conference Matrix Orders Less than 1000

Order	Exist?	Type	Order	Exist?	Type	Order	Exist?	Type	Order	Exist?	Type
6	✓	c1, c1a	254	NE		506	?		758	✓	c1, c1a
10	✓	c1a, c6	258	✓	c1, c1a	510	✓	c1, c1a	762	✓	c1, c1a
14	✓	c1, c1a	262	?		514	NE		766	?	
18	✓	c1, c1a	266	?		518	NE		770	✓	c1, c1a
22	NE		270	✓	c1, c1a	522	✓	c1, c1a	774	✓	c1, c1a
26	✓	c1	274	NE		526	NE		778	NE	
30	✓	c1, c1a	278	✓	c1, c1a	530	✓	c1, c6	782	NE	
34	NE		282	✓	c1, c1a	534	?		786	?	
38	✓	c1, c1a	286	NE		538	NE		790	NE	
42	✓	c1, c1a	290	✓	c1	542	✓	c1, c1a	794	?	
46	✓	c2, c3, c3	294	✓	c1, c1a	546	?		798	✓	c1, c1a
50	✓	c1, c6	298	NE		550	?		802	?	
54	✓	c1, c1a	302	NE		554	NE		806	NE	
58	NE		306	?		558	✓	c1, c1a	810	✓	c1, c1a
62	✓	c1, c1a	310	NE		562	NE		814	NE	
66	?		314	✓	c1, c1a	566	?		818	NE	
70	NE		318	✓	c1, c1a	570	✓	c1, c1a	822	✓	c1, c1a
74	✓	c1, c1a	322	NE		574	NE		826	NE	
78	NE		326	?		578	✓	c1, c1a	830	✓	c1, c1a
82	✓	c1, c6	330	NE		582	NE		834	?	
86	?		334	?		586	?		838	NE	
90	✓	c1, c1a	338	✓	c1, c1a	590	NE		842	✓	c1
94	NE		346	NE		594	✓	c1, c1a	846	?	
98	✓	c1, c1a	350	✓	c1, c1a	598	NE		850	NE	
102	✓	c1, c1a	354	✓	c1, c1a	602	✓	c1, c1a	854	✓	c1, c1a
106	NE		358	NE		606	?		858	✓	c1, c1a
110	✓	c1, c1a	362	✓	c1, c6	610	NE		862	NE	
114	✓	c1, c1a	366	?		614	✓	c1, c1a	866	?	
118	?		370	?		618	✓	c1, c1a	870	NE	
122	✓	c1, c6	374	✓	c1, c1a	622	NE		874	?	
126	✓	c1	378	?		626	✓	c1	878	✓	c1, c1a
130	NE		382	NE		630	?		882	✓	c1, c1a
134	NE		386	NE		634	NE		886	NE	
138	✓	c1, c1a	390	✓	c1, c1a	638	?		890	NE	
142	NE		394	NE		642	✓	c1, c1a	894	NE	
146	?		398	✓	c1, c1a	646	NE		898	NE	
150	✓	c1, c1a	402	✓	c1, c1a	650	NE		902	?	
154	?		406	?		654	✓	c1, c1a	906	?	
158	✓	c1, c1a	410	✓	c1, c1a	658	?		910	?	
162	NE		414	NE		662	✓	c1, c1a	914	NE	
166	NE		418	NE		666	NE		918	NE	
170	✓	c1	422	✓	c1, c1a	670	NE		922	NE	
174	✓	c1, c1a	426	?		674	✓	c1, c1a	926	?	
178	NE		430	NE		682	NE		930	✓	c1, c1a
182	✓	c1, c1a	434	✓	c1, c1a	686	?		934	NE	
186	?		438	NE		690	?		938	✓	c1, c1a
190	NE		442	✓	c2	694	NE		942	✓	c1, c1a
194	✓	c1, c1a	446	?		698	?		946	NE	
198	✓	c1, c1a	450	✓	c1, c1a	702	✓	c1, c1a	950	?	
202	NE		454	NE		706	NE		954	✓	c1, c1a
206	?		458	✓	c1, c1a	710	✓	c1, c1a	958	NE	
210	NE		462	✓	c1, c1a	714	NE		962	✓	c1, c6
214	NE		466	NE		718	NE		966	?	
218	NE		470	NE		722	NE		970	NE	
222	?		474	NE		726	?		974	NE	
226	?		478	?		730	✓	c1, c6	978	✓	c1, c1a
230	✓	c1, c1a	482	?		734	✓	c1, c1a	982	?	
234	✓	c1, c1a	486	?		738	NE		986	?	
238	NE		490	NE		742	NE		990	NE	
242	✓	c1, c1a	494	?		746	?		994	NE	
246	?		498	NE		750	NE		998	✓	c1, c1a
250	NE		502	NE		754	NE		1002	?	