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Bounds for the Best Constant in an Improved Hardy-Sobolev Inequality

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Abstract. *In this note we show that the best constant C in the improved Hardy-Sobolev inequality of Adimurthi, Chaudhuri and Ramaswamy* [1] *for* $2 \leq p < n$ *is bounded by* $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \leq C \leq \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}$.

Keywords: *Hardy-Sobolev inequality, best constant in inequality*

AMS subject classification: 35P15, 35J20

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with $0 \in \Omega$. Adimurthi, Chaudhuri and Ramaswamy in [1] have obtained the following improved Hardy-Sobolev inequality. Let $1 < p < n$ and let $R \geq e^{\frac{2}{p}} \sup_\Omega |x|$. Then there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}\, dx + C \int_\Omega \frac{|u|^p}{|x|^p}\left(\log \frac{R}{|x|}\right)^{-2} dx \qquad (1.1)$$

holds for all $u \in W_0^{1,p}(\Omega)$. In his book on *Sobolev Spaces* [14: Section 2.1.6] Maz'ja discovered that the classical multi-dimensional Hardy-type inequalities with sharp constant can be improved by adding different additional positive integrals. However, the above inequality have applications in proving existence, non-existence and regularity of solutions for differential equations involving the potential $\frac{1}{|x|^p}$ (see [1, 3, 10 - 12, 15]). Adimurthi and Esteban [2] extended the above inequality for $W^{1,p}$ functions and found interesting applications to the Schrödinger operator. However, finding the *best constant* in inequality (1.1) remains open. In this article we find interesting bounds for the best

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constant $C(n,p,R,\Omega)$, defined in (1.4) below. In [1: Theorem 1.2] it has been shown that for $0 < \mu < \left(\frac{n-p}{p}\right)^p$ the eigenvalue problem

$$\left.\begin{aligned} -\Big(\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \frac{\mu}{|x|^p}|u|^{p-2}u\Big) &= \lambda \frac{|u|^{p-2}}{|x|^p\big(\log\frac{R}{|x|}\big)^2}u \quad \text{in } \Omega \\ u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{aligned}\right\} \tag{1.2}$$

admits a positive weak solution $u \in W_0^{1,p}(\Omega)$ corresponding to the eigenvalue $\lambda = \lambda_\mu^1 > 0$. Moreover, $\lambda_\mu^1 \to C(n,p,R,\Omega)$ as $\mu \to \left(\frac{n-p}{p}\right)^p$. Thus the bounds on the best constant in inequality (1.1) gives bounds on the limiting behaviour of the first eigenvalue for the eigenvalue problem (1.2).

In [1], the following n-dimensional version of the Hardy-Sobolev inequality also has been established. For any bounded domain $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ with $0 \in \Omega$,

$$\int_\Omega |\nabla u|^n dx \geq \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n}\left(\log\frac{R}{|x|}\right)^{-n} dx \tag{1.3}$$

holds for every $u \in W_0^{1,n}(\Omega)$. Adimurthi and Sandeep [3] proved that the best constant herein is indeed $\left(\frac{n-1}{n}\right)^n$. For some interesting improvements of the classical Hardy-Sobolev inequality and their applications see [5 - 9, 13].

Before stating our theorem we define the *best constant* $C(n,p,R,\Omega)$ in inequality (1.1) by

$$C(n,p,R,\Omega) = \inf_{0\neq u \in W_0^{1,p}(\Omega)} Q_{\Omega,R}(u) \tag{1.4}$$

where

$$Q_{\Omega,R}(u) = \frac{\int_\Omega |\nabla u|^p dx - \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}dx}{\int_\Omega \frac{|u(x)|^p}{|x|^p}\left(\log\frac{R}{|x|}\right)^{-2}dx}.$$

It is also known (see [1]) that the best constant in $C(n,p,R,\Omega)$ is not achieved.

2. Result

In this article we prove the following

Theorem. *The constant $C(n,p,R,\Omega)$ defined by (1.4) is independent of the domain Ω and the choice of the constant $R \geq e^{\frac{2}{p}}\sup_\Omega |x|$. For $2 \leq p < n$,*

$$\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \leq C(n,p) \leq \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}.$$

It appears to the author that, for the case $2 \leq p < n$, the constant $C(n,p)$ herein is indeed $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$.

Proof of the theorem. We prove the independence and the bounds for the best constant through the following steps.

Step 1. We first prove that if B_i $(i = 1, 2)$ are concentric balls centered at origin of radii T_i, then $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$, where $R_i = \alpha T_i$ with $\alpha \geq e^{\frac{2}{p}}$. For this take $u \in W_0^{1,p}(B_2)$ and define $v(x) = u(\frac{T_2}{T_1}x)$ for $|x| < T_1$. Then

$$\begin{aligned} Q_{B_1,R_1}(v) &= \frac{\int_{B_1} |\nabla v|^p dx - (\frac{n-p}{p})^p \int_{B_1} \frac{|v|^p}{|x|^p} dx}{\int_{B_1} \frac{|v|^p}{|x|^p} \big(\log \frac{\alpha T_1}{|x|}\big)^{-2} dx} \\ &= \frac{\int_{B_2} |\nabla u|^p dx - (\frac{n-p}{p})^p \int_{B_2} \frac{|u|^p}{|x|^p} dx}{\int_{B_2} \frac{|u|^p}{|x|^p} \big(\log \frac{\alpha T_2}{|x|}\big)^{-2} dx} \\ &= Q_{B_2,R_2}(u) \end{aligned}$$

and hence $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$.

Step 2. Now we prove that $C(n,p,R,\Omega) = C(n,p,R,\Omega^*)$, where $\Omega^* = B(0,T)$ is the ball of radius $T = \big(\frac{|\Omega|}{|B(0,1)|}\big)^{1/n}$, $|\cdot|_n$ denoting the n-dimensional Lebesgue measure. Indeed, for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(\Omega^*)$, where $|u|^*$ is the symmetric decreasing rearrangement of $|u|$. By standard symmetrization arguments (see [4]) we conclude that, for any $u \in W_0^{1,p}(\Omega)$, $Q_{\Omega,R}(u) \geq Q_{\Omega^*,R}(u^*)$ and hence

$$C(n,p,R,\Omega) \geq C(n,p,R,\Omega^*).$$

To prove the other inequality, take $s > 0$ such that $B_s = B(0,s) \subseteq \Omega$. Then, clearly, $C(n,p,R,\Omega) \leq C(n,p,R,B_s)$ and hence, by Step 1, $C(n,p,R,\Omega) = C(n,p,R,\Omega^*)$.

Now if Ω_i $(i = 1, 2)$ are two bounded domains with $R_i \geq e^{\frac{2}{p}} \sup_{\Omega_i} |x|$, by Steps 1 and 2, $C(n,p,R_1,\Omega_1) = C(n,p,R_2,\Omega_2)$ and hence the constant is independent of the domain and the choice of R. We shall denote this constant simply by $C(n,p)$.

Step 3. Lower Bound: The lower bound for the best constant $C(n,p)$ essentially follows from [1: Proof of Theorem 1.1], but for sake of completeness we include a proof. Since $C(n,p)$ is independent of the domain, without loss of generality we assume Ω to be the unit ball $B = B(0,1)$. Let $R \geq e^{\frac{2}{p}}$. For $0 < u \in C_0^2(B)$ radially non-increasing we define

$$v(r) = u(r)\, r^{\frac{n-p}{p}} \qquad (r = |x|). \tag{2.1}$$

Here without loss of generality we as well assume $u'(r) < 0$ (replacing u by $u + \varepsilon(1-r)$ for $\varepsilon > 0$ sufficiently small). Now we observe that

$$\begin{aligned}
&\int_B |\nabla u|^p dx - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u(x)|^p}{|x|^p} dx \\
&\quad = \omega_n \int_0^1 \left|\frac{n-p}{p} r^{-\frac{n}{p}} v(r) - r^{1-\frac{n}{p}} v'(r)\right|^p r^{n-1} dr \\
&\qquad - \left(\frac{n-p}{p}\right)^p \omega_n \int_0^1 \frac{v^p(r)}{r}\, dr \\
&\quad = \omega_n \left(\frac{n-p}{p}\right)^p \int_0^1 v^p(r) \left\{\left|1 - \frac{pv'(r)r}{(n-p)v(r)}\right|^p - 1\right\} \frac{dr}{r}
\end{aligned}$$

where ω_n is the volume of the $(n-1)$-dimensional sphere. Since u is a decreasing function, from (2.1) we have $v'(r) - \frac{(n-p)v(r)}{pr} < 0$. We set $x(r) = -\frac{pv'(r)r}{(n-p)v(r)}$ so that $x(r) > -1$. By using the inequality $(1+x)^p \geq 1 + px + (p-1)x^2$ for all $x \geq -1$ and for all $p \geq 2$ we obtain

$$\begin{aligned}
&\int_B |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u(x)|^p}{|x|^p} \\
&\quad \geq \omega_n (p-1) \left(\frac{n-p}{p}\right)^{p-2} \int_0^1 v^{p-2}(r) |v'(r)|^2 r dr \\
&\qquad - \omega_n p \left(\frac{n-p}{p}\right)^{p-1} \int_0^1 v^{p-1}(r) v'(r) dr \\
&\quad = \frac{4\omega_n(p-1)}{p^2} \left(\frac{n-p}{p}\right)^{p-2} \int_0^1 \left|(v^{p/2}(r))'\right|^2 r\, dr
\end{aligned}$$

since $v \in C_0^1(0,T)$. By applying the n-dimensional Hardy inequality (1.3) with $n = 2$ for the function $v^{\frac{p}{2}}$, we obtain

$$\begin{aligned}
\int_0^1 \left|(v^{p/2}(r))'\right|^2 r\, dr &\geq \frac{1}{4} \int_0^1 \left(\frac{v^{p/2}(r)}{r \log \frac{R}{r}}\right)^2 r\, dr \\
&= \frac{1}{4} \int_0^1 \frac{u^p(r)}{r^p} \left(\log \frac{R}{r}\right)^{-2} r^{n-1} dr \\
&= \frac{1}{4\omega_n} \int_B \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx.
\end{aligned}$$

Hence for all radially non-increasing functions $0 < u \in C_0^2(B)$ we have

$$\begin{aligned}
&\int_B |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u(x)|^p}{|x|^p} \\
&\quad \geq \frac{p-1}{p^2} \left(\frac{n-p}{p}\right)^{p-2} \int_B \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx.
\end{aligned}$$

Now by standard approximation and symmetrization this inequality holds for all $u \in W_0^{1,p}(B)$ and hence $C(n,p) \geq \frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$.

Step 3. Upper Bound: Here our idea is to construct a family of functions $\{u_{\varepsilon,k}\}_{0<\varepsilon<1}$ in $W_0^{1,p}(B)$, where $B = B(0,1)$ is the unit ball, and then to estimate $Q_{B,R}$ for this family. Similar to the family found in [1], for any $0<\varepsilon<1$ and for $2 \leq k \in \mathbb{N}$ we define

$$u_{\varepsilon,k}(r) = \begin{cases} 0 & \text{for } r \leq \varepsilon^k \\ \dfrac{\log \frac{r}{\varepsilon^k}}{(k-1)r^{\frac{n-p}{p}} \log \frac{1}{\varepsilon}} & \text{for } \varepsilon^k \leq r \leq \varepsilon \\ \dfrac{\log \frac{1}{r}}{r^{\frac{n-p}{p}} \log \frac{1}{\varepsilon}} & \text{for } \varepsilon \leq r \leq 1. \end{cases}$$

Clearly, $u_{\varepsilon,k} \in W_0^{1,p}(B)$ is continuous and differentiable a.e., and its derivative is given by

$$u'_{\varepsilon,k}(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq \varepsilon^k \\ \dfrac{1}{(k-1)r^{\frac{n}{p}} \log \frac{1}{\varepsilon}}\left[1 - \dfrac{n-p}{p}\log \dfrac{r}{\varepsilon^k}\right] & \text{for } \varepsilon^k \leq r \leq \varepsilon \\ -\dfrac{1}{r^{\frac{n}{p}} \log \frac{1}{\varepsilon}}\left[1 + \dfrac{n-p}{p}\log \dfrac{1}{r}\right] & \text{for } \varepsilon \leq r \leq 1. \end{cases}$$

Since $\varepsilon > 0$ is sufficiently small, after a change of variables and the use of Neumann series we have the estimates

$$\begin{aligned} \int_B |\nabla u_{\varepsilon,k}|^p dx &= \frac{\omega_n}{\left(\log \frac{1}{\varepsilon}\right)^p}\left[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\left|\frac{n-p}{p}\log\frac{r}{\varepsilon^k} - 1\right|^p \frac{dr}{r}\right. \\ &\quad \left. + \int_{\varepsilon}^{1}\left|1 + \frac{n-p}{p}\log\frac{1}{r}\right|^p \frac{dr}{r}\right] \\ &= \frac{\lambda_{n,p}\omega_n}{p+1}\log\frac{1}{\varepsilon}\left[(k-1)\left(1 - \frac{p}{(k-1)(n-p)\log\frac{1}{\varepsilon}}\right)^{p+1}\right. \\ &\quad \left. + \left(1 + \frac{p}{(n-p)\log\frac{1}{\varepsilon}}\right)^{p+1}\right] \\ &= \frac{\lambda_{n,p}\omega_n}{p+1}\log\frac{1}{\varepsilon}\left[(k-1) - \frac{p(p+1)}{(n-p)\log\frac{1}{\varepsilon}}\right. \\ &\quad + \frac{p(p+1)}{2(k-1)}\left(\frac{p}{(n-p)\log\frac{1}{\varepsilon}}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&+O\Big(\frac{1}{(k-1)^2\big(\log\frac{1}{\varepsilon}\big)^3}\Big)+1+\frac{p(p+1)}{(n-p)\log\frac{1}{\varepsilon}}\\
&+\frac{p(p+1)}{2}\Big(\frac{p}{(n-p)\log\frac{1}{\varepsilon}}\Big)^2+O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^3\Big]\\
&=\frac{k\lambda_{n,p}\omega_n}{p+1}\log\frac{1}{\varepsilon}+\frac{kp\omega_n}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}\Big(\log\frac{1}{\varepsilon}\Big)^{-1}\\
&+O\Big(\frac{1}{(k-1)\log\frac{1}{\varepsilon}}\Big)^2+O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2.
\end{aligned}
\tag{2.2}
$$

Then we have

$$
\begin{aligned}
\int_B\frac{|u_{\varepsilon,k}|^p}{|x|^p}dx&=\frac{\omega_n}{\big(\log\frac{1}{\varepsilon}\big)^p}\Big[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\Big(\log\frac{r}{\varepsilon^k}\Big)^p\frac{dr}{r}+\int_{\varepsilon}^{1}\Big(\log\frac{1}{r}\Big)^p\frac{dr}{r}\Big]\\
&=\frac{\omega_n}{(p+1)\big(\log\frac{1}{\varepsilon}\big)^p}\Big[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\frac{d}{dr}\Big(\log\frac{r}{\varepsilon^k}\Big)^{p+1}dr\\
&\quad-\int_{\varepsilon}^{1}\frac{d}{dr}\Big(\log\frac{1}{r}\Big)^{p+1}dr\Big]\\
&=\frac{k\omega_n}{(p+1)}\log\frac{1}{\varepsilon}.
\end{aligned}
\tag{2.3}
$$

Thus (2.2) - (2.3) yield

$$
\begin{aligned}
&\int_B|\nabla u_{\varepsilon,k}|^p-\Big(\frac{n-p}{p}\Big)^p\int_B\frac{|u_{\varepsilon,k}|^p}{|x|^p}\\
&\quad=\frac{kp\omega_n}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}\Big(\log\frac{1}{\varepsilon}\Big)^{-1}+O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2.
\end{aligned}
\tag{2.4}
$$

Finally, let us try to find a "good" estimate of the integral

$$
\begin{aligned}
I_p&=\int_B\frac{|u_{\varepsilon,k}|^p}{|x|^p}\Big(\log\frac{R}{|x|}\Big)^{-2}dx\\
&=\frac{\omega_n}{\big(\log\frac{1}{\varepsilon}\big)^p}\Big[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\frac{\big(\log\frac{r}{\varepsilon^k}\big)^p}{r\big(\log\frac{R}{r}\big)^2}dr+\int_{\varepsilon}^{1}\frac{\big(\log\frac{1}{r}\big)^p}{r\big(\log\frac{R}{r}\big)^2}dr\Big].
\end{aligned}
$$

By change of variable $r\mapsto\log\frac{R}{r}$ and denoting $a_\varepsilon=\log\frac{R}{\varepsilon}$, $b_\varepsilon=\log\frac{R}{\varepsilon^k}$ and $c=\log R$ we get

$$
\begin{aligned}
I_p&=\frac{\omega_n}{\big((k-1)\log\frac{1}{\varepsilon}\big)^p}\int_{a_\varepsilon}^{b_\varepsilon}\frac{\big(\log\frac{Re^{-r}}{\varepsilon^k}\big)^p}{r^2}dr\\
&\quad+\frac{\omega_n}{\big(\log\frac{1}{\varepsilon}\big)^p}\int_{c}^{a_\varepsilon}\frac{\big(\log\frac{e^r}{R}\big)^p}{r^2}dr\\
&=:I_p^1+I_p^2.
\end{aligned}
$$

For the integrals I_p^1 and I_p^2 we get the estimations

$$\begin{aligned}
I_p^1 &= \int_{a_\varepsilon}^{b_\varepsilon} \left(\log \frac{R}{\varepsilon^k} - r\right)^p \frac{dr}{r^2} \\
&= b_\varepsilon^p \int_{a_\varepsilon}^{b_\varepsilon} \left(1 - \frac{r}{b_\varepsilon}\right)^p \frac{dr}{r^2} \\
&\geq b_\varepsilon^p \int_{a_\varepsilon}^{b_\varepsilon} \left(1 - \frac{pr}{b_\varepsilon} + \frac{(p-1)r^2}{b_\varepsilon^2}\right) \frac{dr}{r^2} \\
&= \frac{b_\varepsilon^p}{a_\varepsilon}\left[\left(1 - \frac{a_\varepsilon}{b_\varepsilon}\right)\left(1 + (p-1)\frac{a_\varepsilon}{b_\varepsilon}\right) - \frac{pa_\varepsilon}{b_\varepsilon}\log\frac{b_\varepsilon}{a_\varepsilon}\right]
\end{aligned}$$

and

$$\begin{aligned}
I_p^2 &= \int_c^{a_\varepsilon} (r - \log R)^p \frac{dr}{r^2} \\
&= \int_c^{a_\varepsilon} r^{p-2}\left(1 - \frac{c}{r}\right)^p dr \\
&\geq \int_c^{a_\varepsilon} r^{p-2}\left(1 - \frac{pc}{r} + \frac{(p-1)c^2}{r^2}\right) dr \\
&= \begin{cases}
a_\varepsilon\left[\left(1 - \dfrac{c}{a_\varepsilon}\right) - 2\dfrac{c}{a_\varepsilon}\log\dfrac{a_\varepsilon}{c} + o(1)\right] & \text{for } p = 2 \\
a_\varepsilon^2\left[\dfrac{1}{2}\left(1 - \left(\dfrac{c}{a_\varepsilon}\right)^2\right) + 2\left(\dfrac{c}{a_\varepsilon}\right)^2\log\dfrac{a_\varepsilon}{c} + o(1)\right] & \text{for } p = 3 \\
a_\varepsilon^{p-1}\left[\dfrac{1}{p-1}\left(1 - \left(\dfrac{c}{a_\varepsilon}\right)^{p-1}\right) + o(1)\right] & \text{for } p \neq 2, p \neq 3
\end{cases} \\
&= a_\varepsilon^{p-1}\left[\frac{1}{p-1} + o(1)\right]
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. From these estimations for I_p^1 and I_p^2 we obtain

$$\begin{aligned}
I_p &\geq J_{k,\varepsilon} \\
&:= \frac{\omega_n}{\left((k-1)\log\frac{1}{\varepsilon}\right)^p}\frac{b_\varepsilon^p}{a_\varepsilon}\left[\left(1 - \frac{a_\varepsilon}{b_\varepsilon}\right)\left(1 + (p-1)\frac{a_\varepsilon}{b_\varepsilon}\right) - \frac{pa_\varepsilon}{b_\varepsilon}\log\frac{b_\varepsilon}{a_\varepsilon}\right] \\
&\quad + \frac{\omega_n}{\left(\log\frac{1}{\varepsilon}\right)^p} a_\varepsilon^{p-1}\left[\frac{1}{p-1} + o(1)\right].
\end{aligned}$$

Hence from (2.4) we obtain

$$\begin{aligned}
Q_{B,R}(u_{\varepsilon,k}) &\leq \frac{pk}{2(k-1)}\left(\frac{n-p}{p}\right)^{p-2}\left(\log\frac{1}{\varepsilon}\right)^{p-1} \\
&\quad \times \left[\frac{b_\varepsilon^p}{(k-1)^p a_\varepsilon}\left\{\left(1 - \frac{a_\varepsilon}{b_\varepsilon}\right)\left(1 + (p-1)\frac{a_\varepsilon}{b_\varepsilon}\right)\right\}\right.
\end{aligned}$$

$$+ a_\varepsilon^{p-1}\Big\{\frac{1}{p-1}+o(1)\Big\}\Big]^{-1} + J_{k,\varepsilon}^{-1}\Big[O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2\Big]$$

$$= \frac{pk}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}$$

$$\times\Bigg[\frac{(k-1)^{-p}b_\varepsilon^p}{a_\varepsilon\big(\log\frac{1}{\varepsilon}\big)^{p-1}}\Big\{\Big(1-\frac{a_\varepsilon}{b_\varepsilon}\Big)\Big(1+(1-p)\frac{a_\varepsilon}{b_\varepsilon}\Big)\Big\}$$

$$+\Big(\frac{a_\varepsilon}{\log\frac{1}{\varepsilon}}\Big)^{p-1}\Big\{\frac{1}{p-1}+o(1)\Big\}\Bigg]^{-1} + J_{k,\varepsilon}^{-1}\Big[O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2\Big].$$

Here we note that $\frac{b_\varepsilon^p}{a_\varepsilon}\big(\log\frac{1}{\varepsilon}\big)^{p-1} \to k^p$ as $\varepsilon \to 0$ and hence $J_{k,\varepsilon}^{-1}\big[O\big(\frac{1}{\log\frac{1}{\varepsilon}}\big)^2\big] \to 0$ as either $\varepsilon \to 0$ or $k \to \infty$. Thus

$$\begin{aligned} Q_{B,R}(u_{\varepsilon,k}) &\to \frac{pk}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2} \\ &\quad\times\Bigg[\Big(\frac{k}{k-1}\Big)^p\Big\{\Big(1-\frac{1}{k}\Big)\Big(1+\frac{p-1}{k}\Big) \\ &\quad+\frac{p}{k}\log\frac{1}{k}\Big\}+\frac{1}{p-1}\Bigg]^{-1} \quad (\varepsilon\to 0) \\ &\to \frac{p}{2}\Big(\frac{n-p}{p}\Big)^{p-2}\Big[1+\frac{1}{p-1}\Big]^{-1} \quad (k\to\infty) \\ &= \frac{p-1}{2}\Big(\frac{n-p}{p}\Big)^{p-2}. \end{aligned}$$

Since $C(n,p) \le Q_{B,R}(u_{\varepsilon,k})$ for all $k \ge 2$ and for any sufficiently small $\varepsilon > 0$, by passing through the limits as $\varepsilon \to 0$ and $k \to \infty$ we get $C(n,p) \le \frac{p-1}{2}\big(\frac{n-p}{p}\big)^{p-2}$ and hence the theorem is proved ∎

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