2005

A time-varying generalised minimum variance controller

Z. Li

Christian Schmid

Ruhr-Universität Bochum
Lehrstuhl für Automatisierungstechnik
und Prozessinformatik

Forschungsbericht

Z. Li, Chr. Schmid

A Time-Varying Generalised Minimum Variance Controller

Juni 2005

Bericht Nr. 2005.16

A Time-Varying Generalised Minimum Variance Controller

ZHENG LI
School of Electrical, Computer and Telecom. Eng.
University of Wollongong
Wollongong NSW 2522
AUSTRALIA
zheng@elec.uow.edu.au

CHRISTIAN SCHMID
Institute of Automation and Computer Control
Ruhr-Universität Bochum
D-44801 Bochum
GERMANY
cs@atp.rub.de

Abstract: - A generalised minimum variance controller is developed for linear time-varying systems. The plants to be controlled are described using a controlled autoregressive moving average model and are exponentially stable. Both, the plant parameters and the noise variance are time-varying. The generalised minimum variance cost functional is the sum of a tracking error variance between a filtered plant output and a filtered reference plus a penalty term of filtered plant input. Time-varying filters are introduced for weighting to make the controller more flexible.

Key-Words: - Adaptive control; minimum variance control; predictive control; time-varying systems.

1 Introduction

The generalised minimum variance controller (GMVC) was developed by Clarke and Gawthrop [1], [2] for single-input and single-output (SISO) linear time-invariant (LTI) systems described using a controlled autoregressive moving average (CARMA) model. The basic feature of the cost functional of a GMVC is a tracking error variance plus a penalty term of the control signal. Because large control action will be penalised the LTI GMVC extends the LTI minimum variance controller of Aström [3] by removing the minimum phase condition for the CARMA model. It is a very useful controller in stochastic adaptive control and has seen many applications.

Because one of the prime motivations of adaptive control is for achieving optimal performance or maintaining stability under unforeseeable time variations in plant dynamics and environment, a GMVC for linear time-varying (LTV) plants is needed in many applications of stochastic adaptive control. The basic difficulty in extending the LTI GMVC for LTV plants is the noncommutativity of transfer operators with respect to multiplication.

Noncommutativity is well known for LTI multi-input and multi-output (MIMO) transfer functions. The first GMVC for MIMO LTI plants was developed by Koivo [4] using a pseudocommutation technique for MIMO LTI transfer functions [5].

A pseudocommutation technique called left pseudocommutation was developed in order to overcome noncommutativity for SISO LTV transfer operators [6]. This technique was later used in the development of a GMVC for SISO LTV plants [7]. The technique requires that determinants of some left Sylvester matrices of the plants to be controlled are uniformly bounded away from zero. A right pseudocommutation technique was also developed for the introduction of time-varying filters for the plant input and output [7].

The left pseudocommutability was later removed in the case of a simple LTV GMVC, where the LTV CARMA model is exponentially stable and the cost functional is the sum of the output tracking error variance plus a squared current control variable weighted using a time-varying function [8].

In this paper this LTV GMVC will be extended to the general case where the cost functional is the sum of a tracking error variance between a filtered plant output and a filtered reference plus a penalty term of filtered plant input. The time-varying filters are introduced in order to make the new GMVC more flexible for applications and widen its applicability for more LTV systems. The right pseudocommutation technique will be used for the introduction of the LTV filters.

Simulation studies will also be presented, which demonstrates the design steps using a simple example. Properties of the new LTV GMVC will also be studied based on simulation results. It will be shown that the control signal varies not only in response to feedback but also in accordance with variations in plant parameters in order to maintain closed-loop optimality.

The structure of the remainder of this paper is the following. Section 2 introduces the LTV CARMA model, the generalised minimum variance control cost functional and basics of LTV operators in LTV transfer operator framework. Section 3 develops the LTV GMVC and studies its closed-loop behaviour. A simulation example will be presented and studied in Section 4. Section 5 concludes this paper.

2 LTV plants and operators

The plants to be controlled are described using the standard SISO LTV CARMA model

$$A(k,q^{-1})y(k+d) \\ = B(k,q^{-1})u(k) + C(k,q^{-1})w(k+d)\,, \tag{1}$$

where $u(k)$ and $y(k)$ are the plant input and output and $d>0$ is the integral time delay between them, $w(k)$ is an independent Gaussian noise of zero mean and possibly time-varying variance. The variance is assumed to be uniformly bounded away from infinite. However, it is not necessary to know the function or the value of the variance. The LTV moving average operators (MAOs) in the CARMA model have the forms

$$\begin{aligned} A(k,q^{-1}) &= 1 + a_1(k)q^{-1} + \ldots + a_{n_a}(k)q^{-n_a} \\ B(k,q^{-1}) &= b_0(k) + b_1(k)q^{-1} + \ldots + b_{n_b}(k)q^{-nb} \\ C(k,q^{-1}) &= 1 + c_1(k)q^{-1} + \ldots + c_{n_c}(k)q^{-n_c}\,, \end{aligned} \tag{2}$$

where $a_{i_a}(k), i_a = 1,2,\ldots,n_a$, $b_{i_b}(k), i_b = 0,1,\ldots,n_b$ and $c_{i_c}(k), i_c = 1,2,\ldots,n_c$ are time-varying parameters and q is the one-step-advance operator.

For the inverse of an MAO the autoregressive process

$$A(k,q^{-1})z(k) = v(k) \tag{3}$$

is considered. The solution of the autoregression can be the written as

$$z(k) = A^{-1}(k,q^{-1})v(k) + z_0(k)\ , \tag{4}$$

where $z_0(k)$ is the zero input solution,

$$z_v(k) = A^{-1}(k,q^{-1})v(k) \tag{5}$$

is the zero initial condition solution and $A^{-1}(k,q^{-1})$ is defined as the inverse of $A(k,q^{-1})$. $A^{-1}(k,q^{-1})$ is called an LTV autoregressive operator (ARO). From the definition it can be verified that

$$A(k,q^{-1})A^{-1}(k,q^{-1}) = 1\,. \tag{6}$$

However

$$A^{-1}(k,q^{-1})A(k,q^{-1})z(k) = z(k) - z_0(k) \tag{7}$$

with

$$A(k,q^{-1})z_0(k) = 0\ . \tag{8}$$

The stability of the LTV ARO is defined using the state transition matrix in the n_a-step-reachable form of the LTV state space equation of (3) which has the form

$$\Phi(k+1,k) = \begin{bmatrix} -a_1(k) & -a_2(k) & \cdots & -a_{n-1}(k) & -a_{n_a}(k) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{9}$$

The LTV ARO is exponentially stable if and only if there are constants $C>0$ and $c>0$ such that

$$\left\|\Phi(k,s)\right\| \le C\exp\left[-c(k-s)\right] \tag{10}$$

for all $k \ge s \ge 0$. The maximum of all possible c is called the rate of exponential stability for the LTV ARO. For simplicity of presentation when an LTV ARO is exponentially stable, its zero input solution will be ignored in the rest of this paper, because $z_0(k)$ will decay exponentially to zero. In particular, when $A^{-1}(k,q^{-1})$ is exponentially stable it can be cancelled using

$$A^{-1}(k,q^{-1})A(k,q^{-1}) = 1 \tag{11}$$

regardless of its initial condition and modelling inaccuracy.

The LTV MAOs and AROs are noncommutative with respect to multiplication. For example, when $n_a=n_b=1$, one gets

$$\begin{aligned} A(k,q^{-1})B(k,q^{-1}) &= [1+a_1(k)q^{-1}][b_0(k)+b_1(k)q^{-1}] \\ &= b_0(k) + b_1(k)q^{-1} + a_1(k)q^{-1}b_0(k) \\ &\quad + a_1(k)q^{-1}b_1(k)q^{-1} \\ &= b_0(k) + [b_1(k) + a_1(k)b_0(k-1)]q^{-1} \\ &\quad + a_1(k)b_1(k-1)q^{-2} \end{aligned} \tag{12}$$

and similarly

$$\begin{aligned} &B(k,q^{-1})A(k,q^{-1}) \\ &\quad = [b_0(k)+b_1(k)q^{-1}][1+a_1(k)q^{-1}] \\ &\quad = b_0(k) + [b_1(k) + b_0(k)a_1(k)]q^{-1} \\ &\qquad + b_1(k)a_1(k-1)q^{-2}\quad . \end{aligned} \tag{13}$$

It follows that

$$\begin{aligned} &A(k,q^{-1})B(k,q^{-1}) - B(k,q^{-1})A(k,q^{-1}) \\ &\quad = [a_1(k)b_0(k-1) - b_0(k)a_1(k)]q^{-1} \\ &\qquad + [a_1(k)b_1(k-1) - b_1(k)a_1(k-1)]q^{-2}\quad . \end{aligned} \tag{14}$$

Apparently, the difference will be zero if the MAOs are LTI. However, for LTV MAOs it is not always zero. In order to indicate the difference, the product $A(k,q^{-1})B(k,q^{-1})$ is referred to as $B(k,q^{-1})$ left multiplied by $A(k,q^{-1})$, or $A(k,q^{-1})$ left multiplies $B(k,q^{-1})$. It can also be referred to as $A(k,q^{-1})$ right multiplied by $B(k,q^{-1})$, or $B(k,q^{-1})$ right multiplies $A(k,q^{-1})$.

For the CARMA model to be controlled it is assumed that

a) all the coefficients of $A(k,q^{-1})$, $B(k,q^{-1})$ and $C(k,q^{-1})$ are uniformly bounded away from infinite and $b_0(k)$ is also uniformly bounded away from zero, and
b) the LTV AROs $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable.

Both assumptions are the same as those for the previous LTV GMVC [8] and are natural extensions of the corresponding assumptions of LTI systems for LTV plants.

Given a uniformly bounded reference $s(k)$, the objective of the GMVC is to minimise the cost functional

$$J(k+d) = E\left\{\left|P(k,q^{-1})y(k+d) - Q(k,q^{-1})s(k)\right|^2 + \left|R(k,q^{-1})u(k)\right|^2 \middle| D(k)\right\}, \tag{15}$$

where E is the operator for mathematical expectation conditioned on $D(k)=\{y(k), y(k\text{-}1), ..., u(k), u(k\text{-}1),...\}$, which is the set of input and output data up to and including time k. In the above cost functional MAOs $P(k,q^{-1})$, $Q(k,q^{-1})$ and $R(k,q^{-1})$ are LTV filters for the output, reference and input. They are introduced as weightings and are chosen to have the forms

$$\begin{aligned} P(k,q^{-1}) &= 1 + p_1(k)q^{-1} + ... + p_{n_p}(k)q^{-n_p} \\ Q(k,q^{-1}) &= q_0(k) + q_1(k)q^{-1} + ... + q_{n_q}(k)q^{-n_q} \\ R(k,q^{-1}) &= r_0(k) + r_1(k)q^{-1} + ... + r_{n_r}(k)q^{-n_r} \;, \end{aligned} \tag{16}$$

where $p_{i_p}(k), i_p = 1,2,\ldots,n_p$, $q_{i_q}(k), i_q = 0,1,\ldots,n_q$ and $r_{i_r}(k), i_r = 0,1,\ldots,n_r$ are time-varying parameters of the filters.

It is assumed that these LTV MAOs are chosen such that all their coefficients are uniformly bounded away from infinite and $r_0(k)$ is also uniformly bounded away from zero. In addition, $P^{-1}(k,q^{-1})$ is chosen to be exponentially stable.

For the introduction of the LTV filters as weightings in the LTV GMVC cost functional the right time-varying Sylvester matrix

$$\tilde{S}_{AP}(k) = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_1(k-1) & 1 & \cdots & \vdots & p_1(k-1) & 1 & \cdots & \vdots \\ a_2(k-1) & a_1(k-2) & \ddots & 0 & p_2(k-1) & p_1(k-2) & \ddots & 0 \\ \vdots & \vdots & \cdots & 1 & \vdots & \vdots & \cdots & 1 \\ a_{n_a}(k-1) & a_{n_a-1}(k-2) & \cdots & a_1(k-n_p) & p_{n_p}(k-1) & p_{n_p-1}(k-2) & \cdots & p_1(k-n_a) \\ 0 & a_{n_a}(k-2) & \cdots & \vdots & 0 & p_{n_p}(k-2) & \cdots & \vdots \\ 0 & \vdots & \cdots & a_{n_a-1}(k-n_p) & \vdots & \vdots & \cdots & p_{n_p-1}(k-n_a) \\ 0 & 0 & \cdots & a_{n_a}(k-n_p) & 0 & 0 & \cdots & p_{n_p}(k-n_a) \end{array}\right] \tag{17}$$

is used for the LTV MAO pair $[A(k,q^{-1}), P(k,q^{-1})]$. It can be verified that when the MAO pair is LTI the above right Sylvester matrix reduces to the Sylvester matrix for LTI systems.

It is further assumed that the determinant of this right time-varying Sylvester matrix is uniformly bounded away from zero. All the above assumptions related to the LTV filters are not restrictive because the choice of the weighting MAOs is in the hands of the designer. It will be shown that the introduction of the weighting LTV MAOs in the cost functional will make the LTV GMVC not only more flexible for more complicated control tasks but also capable of stabilising more LTV systems.

3 Generalised Minimum Variance Control

Without losing generality it is assumed that all initial conditions are independent of the process disturbance $\{w(k)\}$. Applying the right pseudocommutation the LTV MAOs $\tilde{A}(k,q^{-1})$ and $\tilde{P}(k,q^{-1})$ can be determined from [7]

$$\tilde{A}(k,q^{-1})P(k,q^{-1}) = \tilde{P}(k,q^{-1})A(k,q^{-1}) \;. \tag{18}$$

Both, $\tilde{A}^{-1}(k,q^{-1})$ and $\tilde{P}^{-1}(k,q^{-1})$ are exponentially stable and the LTV MAOs have the forms

$$\begin{aligned} \tilde{A}(k,q^{-1}) &= 1 + \tilde{a}_1(k)q^{-1} + \tilde{a}_2(k)q^{-2} + ... + \tilde{a}_{n_a}(k)q^{-n_a} \\ \tilde{P}(k,q^{-1}) &= 1 + \tilde{p}_1(k)q^{-1} + \tilde{p}_2(k)q^{-2} + ... + \tilde{p}_{n_p}(k)q^{-n_p} \end{aligned} \tag{19}$$

Left multiplying by $\tilde{P}(k,q^{-1})$ on both sides of the CARMA model it becomes

$$\begin{aligned} \tilde{P}(k,q^{-1})A(k,q^{-1})y(k+d) &= \tilde{P}(k,q^{-1})B(k,q^{-1})u(k) \\ &+ \tilde{P}(k,q^{-1})C(k,q^{-1})w(k+d) \;. \end{aligned} \tag{20}$$

Substituting (18) it follows that

$$\begin{aligned} \tilde{A}(k,q^{-1})P(k,q^{-1})y(k+d) &= \tilde{P}(k,q^{-1})B(k,q^{-1})u(k) \\ &+ \tilde{P}(k,q^{-1})C(k,q^{-1})w(k+d) \;. \end{aligned} \tag{21}$$

Left dividing $\tilde{P}(k,q^{-1})C(k,q^{-1})$ using $\tilde{A}(k,q^{-1})$, the noise term in equation (21) can be separated using

$$\tilde{A}^{-1}(k,q^{-1})\tilde{P}(k,q^{-1})C(k,q^{-1}) = F(k,q^{-1}) + \tilde{A}^{-1}(k,q^{-1})G(k,q^{-1})q^{-d}\,, \quad (22)$$

where

$$F(k,q^{-1}) = 1 + f_1(k)q^{-1} + f_2(k)q^{-2} + \ldots + f_{d-1}(k)q^{-d+1}\,. \quad (23)$$

Substituting (22) into (21) one gets

$$\tilde{A}(k,q^{-1})P(k,q^{-1})y(k+d) = \tilde{P}(k,q^{-1})B(k,q^{-1})u(k) + \tilde{A}(k,q^{-1})F(k,q^{-1})w(k+d) + G(k,q^{-1})w(k)\,. \quad (24)$$

It follows that

$$\tilde{A}(k,q^{-1})P(k,q^{-1})y(k+d) - \tilde{A}(k,q^{-1})F(k,q^{-1})w(k+d) = \tilde{P}(k,q^{-1})B(k,q^{-1})u(k) + G(k,q^{-1})w(k)\,. \quad (25)$$

In the above equation all the input and noise up to and including current time k are on the right hand side of the equation. All the plant output and future noise are on the left side of the equation. Defining the filtered plant output as

$$\psi(k+d) = P(k,q^{-1})y(k+d)\,. \quad (26)$$

If the current and past noise can be estimated the filtered output can be predicted using

$$\tilde{A}(k,q^{-1})\hat{\psi}(k+d) = \tilde{P}(k,q^{-1})B(k,q^{-1})u(k) + G(k,q^{-1})w(k)\,, \quad (27)$$

where

$$\hat{\psi}(k+d) = \psi(k+d) - F(k,q^{-1})w(k+d) \quad (28)$$

is the d-step-ahead prediction of the filtered plant output.

GMVC Theorem

If the plant to be controlled is described by the CARMA model (1), where the LTV AROs $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable, the LTV GMVC is given by

$$\hat{w}(k) = C^{-1}(k-d,q^{-1})\left[A(k-d,q^{-1})y(k) - B(k-d,q^{-1})u(k-d)\right] \quad (29a)$$

$$u(k) = T^{-1}(k,q^{-1})[\tilde{A}(k,q^{-1})Q(k,q^{-1})s(k) - G(k,q^{-1})\hat{w}(k)]\,, \quad (29b)$$

where $\hat{w}(k)$ is an estimate of $w(k)$ and

$$T(k,q^{-1}) = \tilde{A}(k,q^{-1})\frac{r_0(k)}{b_0(k)}R(k,q^{-1}) + \tilde{P}(k,q^{-1})B(k,q^{-1})\,. \quad (30)$$

Proof

Substituting (28) into the cost functional (15) it follows that

$$J(k+d) = \left|\hat{\psi}(k+d) - Q(k,q^{-1})s(k)\right|^2 + \left|R(k,q^{-1})u(k)\right|^2 + E\left\{\left|F(k,q^{-1})w(k+d)\right|^2 \middle| D(k)\right\}. \quad (31)$$

From (1), (2), (16), (26) and (28) it is known that

$$\frac{\partial\hat{\psi}(k+d)}{\partial u(k)} = \frac{\partial y(k+d)}{\partial u(k)} = b_0(k)\,. \quad (32)$$

Thus

$$\frac{\partial J(k+d)}{\partial u(k)} = 2b_0(k)\hat{\psi}(k+d) - 2b_0(k)Q(k,q^{-1})s(k) + 2r_0(k)R(k,q^{-1})u(k) \quad (33)$$

and

$$\frac{\partial^2 J(k+d)}{\partial u^2(k)} = 2b_0^2(k) + 2r_0^2(k)\,. \quad (34)$$

It follows that the control signal $u(k)$, which minimises the cost functional (15), does exist and satisfy the following equation:

$$\frac{r_0(k)}{b_0(k)}R(k,q^{-1})u(k) = Q(k,q^{-1})s(k) - \hat{\psi}(k+d)\,. \quad (35)$$

Noting the exponential stability of $\tilde{A}^{-1}(k,q^{-1})$ equation (27) is left multiplied using $\tilde{A}^{-1}(k,q^{-1})$ and substituted into (35). It follows that

$$\frac{r_0(k)}{b_0(k)}R(k,q^{-1})u(k) = Q(k,q^{-1})s(k) - \tilde{A}^{-1}(k,q^{-1})[\tilde{P}(k,q^{-1})B(k,q^{-1})u(k) + G(k,q^{-1})w(k)]\,. \quad (36)$$

Left multiplying by $\tilde{A}(k,q^{-1})$ on both sides of (36) one receives

$$\left[\tilde{A}(k,q^{-1})\frac{r_0(k)}{b_0(k)}R(k,q^{-1}) + \tilde{P}(k,q^{-1})B(k,q^{-1})\right]u(k) = \tilde{A}(k,q^{-1})Q(k,q^{-1})s(k) - G(k,q^{-1})w(k)\,. \quad (37)$$

Thus, if the past and current noise can be estimated the control action can be computed using the above equation.

In the LTV GMVC the noise is estimated using (29a). Subtracting (1) and (29a) one gets

$$C(k-d,q^{-1})\tilde{w}(k)=0\ , \tag{38}$$

where

$$\tilde{w}(k)=w(k)-\hat{w}(k) \tag{39}$$

is the error between the noise and its estimate. Because of exponential stability of $C^{-1}(k,q^{-1})$ this estimation error will decay to zero exponentially as shown by autoregressive equation (38). Replacing $w(k)$ in (37) using its estimate $\hat{w}(k)$ one receives (29b). Noting (39) it can be further rewritten as

$$\begin{aligned}T(k,q^{-1})u(k)&=\tilde{A}(k,q^{-1})Q(k,q^{-1})s(k)\\&\quad+G(k,q^{-1})\tilde{w}(k)-G(k,q^{-1})w(k)\,.\end{aligned} \tag{40}$$

Equations (1), (38) and (40) together describe the closed loop behaviour for the LTV GMVC control system. They can be rewritten in the compact form,

$$\begin{bmatrix}C(k-d,q^{-1}) & 0 & 0\\ -G(k,q^{-1}) & T(k,q^{-1}) & 0\\ 0 & -B(k-d,q^{-1})q^{-d} & A(k-d,q^{-1})\end{bmatrix}\begin{bmatrix}\tilde{w}(k)\\ u(k)\\ y(k)\end{bmatrix}=\begin{bmatrix}0 & 0\\ -G(k,q^{-1}) & \tilde{A}(k,q^{-1})Q(k,q^{-1})\\ C(k-d,q^{-1}) & 0\end{bmatrix}\begin{bmatrix}w(k)\\ s(k)\end{bmatrix}. \tag{41}$$

The left most matrix in the above closed loop equation is a lower triangular matrix where the inverses of both the first and the last diagonal LTV MAOs are exponentially stable. The closed-loop stability is, therefore, determined by the second diagonal LTV MAO, $T(k,q^{-1})$. If the ARO of this MAO is exponentially stable then the closed-loop system will be exponentially stable.

Remarks

1. The new LTV GMVC given by (29) includes the LTV GMVC developed in [8] as a special case, where the LTV weightings are

$$\begin{aligned}&P(k,q^{-1})=Q(k,q^{-1})=1\\&R(k,q^{-1})=r_0(k)\quad .\end{aligned} \tag{42}$$

2. Noting (30) and (42) one knows that the introduction of the weighting LTV MAOs to replace the single time-varying weighting function in [8] makes it much more likely for $T^{-1}(k,q^{-1})$ to be stabilised because in the simple weighting case there is only one parameter $r_0(k)$ to choose in order to make the closed loop stable. While for the new GMVC one has both $P(k,q^{-1})$ and $Q(k,q^{-1})$ in the hands to choose in order to make the closed loop stable.

3. The LTV GMVC has two components. (29a) estimates the noise using exponential stability of $C^{-1}(k,q^{-1})$. (29b) computes the control action based on the estimate. When the initial conditions are considered, the noise estimate given by (29a) will converge exponentially to its true value as is indicated by equation (38) and by the exponential stability of $C^{-1}(k,q^{-1})$. Consequently, when $T^{-1}(k,q^{-1})$ is exponentially stable the control action determined by equation (29b) will also converge exponentially to the optimal control.

4. Noting equation (28) one knows that $\hat{\psi}(k+d)$ is the minimum variance d-step-ahead prediction of the filtered output. Using the estimated noise given by (29a) in (27) one gets

$$\begin{aligned}\hat{\psi}(k+d)&=\tilde{A}^{-1}(k,q^{-1})[\tilde{P}(k,q^{-1})B(k,q^{-1})u(k)\\&\quad+G(k,q^{-1})\hat{w}(k)]\,,\end{aligned} \tag{43}$$

which will converge exponentially to the minimum variance prediction because of the exponential stability of $\tilde{A}^{-1}(k,q^{-1})$ and of the exponential decay of the estimation error for $\hat{w}(k)$.

5. Remarks 3 and 4 show that the actual cost functional achieved using the LTV GMVC will exponentially converge to the optimal performance because both $\hat{\psi}(k+d)$ and $u(k)$ will converge exponentially to the optimal value.

4 Simulation

The plant to be controlled has the form

$$\begin{aligned}y(k+2)+a(k)y(k+1)&=u(k)+b(k)u(k-1)\\&\quad+w(k+2)+c(k)w(k+1),\end{aligned} \tag{44}$$

where the plant has a delay of 2 sampling periods and $w(k)$ is a white Gaussian noise that has zero mean and unit variance. The variance is assumed unknown for controller design. As there is only one time-varying parameter for each of the three LTV MAOs of the above CARMA model the subscripts for these parameters are suppressed for simplicity. The subscripts of the plant and controller parameters will be suppressed in this section whenever it can be applied for simplicity of presentation. The time-varying plant parameters are

$$a(k)=\begin{cases}0.3(1-0.9e^{-k-2}) & 20i-2<k\le 20i+8\\ 0.3(0.9e^{-k-2}-1) & 20i-12<k\le 20i-2\end{cases} \tag{45a}$$

$$b(k)=2.3\{2+\sin[0.2\pi(k+2)]\} \tag{45b}$$

$$c(k)=\begin{cases}0.9\dfrac{k+2}{k+3} & 40i-17<k\le 40i+3\\ -0.9\dfrac{k+2}{k+3} & 40i+3<k\le 40i+23\end{cases} \qquad (45c)$$

for i=0, 1, 2,… . In the CARMA model $B^{-1}(k,q^{-1})$ is exponentially unstable because the absolute value of $b(k)$ is uniformly greater than unit. However, $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable because both $|b(k)|$ and $|c(k)|$ are uniformly less than unit. The weighting MAOs are chosen as

$$\begin{aligned}P(k,q^{-1})&=1+p(k)q^{-1}=1+0.04q^{-1}\\ Q(k,q^{-1})&=q_0(k)=5\\ R(k,q^{-1})&=r_0(k)=\sqrt{30}\quad .\end{aligned} \qquad (46)$$

For the right pseudocommutation one has

$$\begin{aligned}\tilde{A}(k,q^{-1})P(k,q^{-1})&=[1+\tilde{a}(k)q^{-1}][1+p(k)q^{-1}]\\ &=1+[\tilde{a}(k)+p(k)]q^{-1}+\tilde{a}(k)p(k-1)q^{-2}\\ \tilde{P}(k,q^{-1})A(k,q^{-1})&=[1+\tilde{p}(k)q^{-1}][1+a(k)q^{-1}]\\ &=1+[\tilde{p}(k)+a(k)]q^{-1}+\tilde{p}(k)a(k-1)q^{-2}\,.\end{aligned} \qquad (47)$$

Noting (18) and comparing the coefficients in (47) one gets

$$\begin{aligned}\tilde{a}(k)+p(k)&=\tilde{p}(k)+a(k)\\ \tilde{a}(k)p(k-1)&=\tilde{p}(k)a(k-1)\,.\end{aligned} \qquad (48)$$

Solving for the right pseudocommutation it follows that

$$\begin{aligned}\tilde{a}(k)&=\frac{a(k)-p(k)}{a(k-1)-p(k-1)}a(k-1)\\ &=\frac{a(k)-0.04}{a(k-1)-0.04}a(k-1)\\ \tilde{p}(k)&=\frac{a(k)-p(k)}{a(k-1)-p(k-1)}p(k-1)\\ &=\frac{a(k)-0.04}{a(k-1)-0.04}0.04\quad .\end{aligned} \qquad (49)$$

Substituting (49) into (30) one receives

$$\begin{aligned}T(k,q^{-1})&=[1+\tilde{a}(k)q^{-1}]r_0^2(k)/b_0(k)\\ &\quad+[1+\tilde{p}(k)q^{-1}][1+b(k)q^{-1}]\\ &=1+r_0^2(k)/b_0(k)\\ &\quad+[r_0^2(k-1)/b_0(k-1)\tilde{a}(k)+\tilde{p}(k)+b(k)]q^{-1}\\ &\quad+\tilde{p}(k)b(k-1)q^{-2}\\ &=31+[30\tilde{a}(k)+\tilde{p}(k)+b(k)]q^{-1}\\ &\quad+\tilde{p}(k)b(k-1)q^{-2}\quad .\end{aligned} \qquad (50)$$

Substituting (49) into (22) in order to solve the equation for the controller parameters one has

$$\begin{aligned}&[1+\tilde{p}(k)q^{-1}][1+c(k)q^{-1}]\\ &\quad=[1+\tilde{a}(k)q^{-1}][1+f(k)q^{-1}]+g(k)q^{-2}\end{aligned} \qquad (51)$$

and it follows that

$$\begin{aligned}&1+(\tilde{p}(k)+c(k))q^{-1}+\tilde{p}(k)c(k-1)q^{-2}\\ &\quad=(1+[\tilde{a}(k)+f(k)]q^{-1}\\ &\qquad+[\tilde{a}(k)f(k-1)+g(k)]q^{-2}\quad .\end{aligned} \qquad (52)$$

Solving for the controller parameters the results are

$$\begin{aligned}F(k,q^{-1})&=1+f(k)q^{-1}\\ &=1+[\tilde{p}(k)+c(k)-\tilde{a}(k)]q^{-1}\end{aligned} \qquad (53)$$

and

$$\begin{aligned}G(k,q^{-1})&=g(k)\\ &=\tilde{p}(k)c(k-1)-\tilde{a}(k)f(k-1)\\ &=\tilde{p}(k)c(k-1)-\tilde{a}(k)[\tilde{p}(k-1)\\ &\qquad+c(k-1)-\tilde{a}(k-1)]\quad .\end{aligned} \qquad (54)$$

Noting (29) the LTV GMVC for plant (44) is

$$\begin{aligned}\hat{w}(k)&=y(k)+a(k-2)y(k-1)-u(k-2)\\ &\quad-b(k-2)u(k-3)-c(k-2)\hat{w}(k-1)\end{aligned} \qquad (55a)$$

$$\begin{aligned}u(k)&=\tfrac{1}{31}\{q_0(k)s(k)+q_0(k-1)\tilde{a}(k)s(k-1)\\ &\quad-g(k)\hat{w}(k)\\ &\quad-[30\tilde{a}(k)+\tilde{p}(k)+b(k)]u(k-1)\\ &\quad-\tilde{p}(k)b(k-1)u(k-2)\}\\ &=\tfrac{1}{31}\{5s(k)+5\tilde{a}(k)s(k-1)-g(k)\hat{w}(k)\\ &\quad-[30\tilde{a}(k)+\tilde{p}(k)+b(k)]u(k-1)\\ &\quad-\tilde{p}(k)b(k-1)u(k-2)\}\quad .\end{aligned} \qquad (55b)$$

The three time-varying parameters in CARMA model (44) are shown in Fig. 1, where $a(k)$ is the solid line, $b(k)$ is the dotted line and $c(k)$ is the dashed line. It can be clearly seen that the time variation in each parameter is quite rapid. While $b(k)$ is continuous the other two time-varying parameters jump frequently for every 10 or 20 sampling periods. $A^{-1}(k,q^{-1})$ has a high exponential stability rate because the absolute value of $a(k)$ is much less than unit as shown in Fig. 1. $B^{-1}(k,q^{-1})$ is highly exponentially unstable because $b(k)$ is always much higher than unit. The exponential stability rate for $C^{-1}(k,q^{-1})$ is low because $c(k)$ converges slowly to a square wave that has an amplitude of 0.9.

The effects of the low exponential stability rate are illustrated in Fig. 2, where the noise is represented by the dots, and its estimate is the solid line. Fig. 2 shows that the estimate converges to its true value quite rapidly in this case even when the exponential stability rate is rather low.

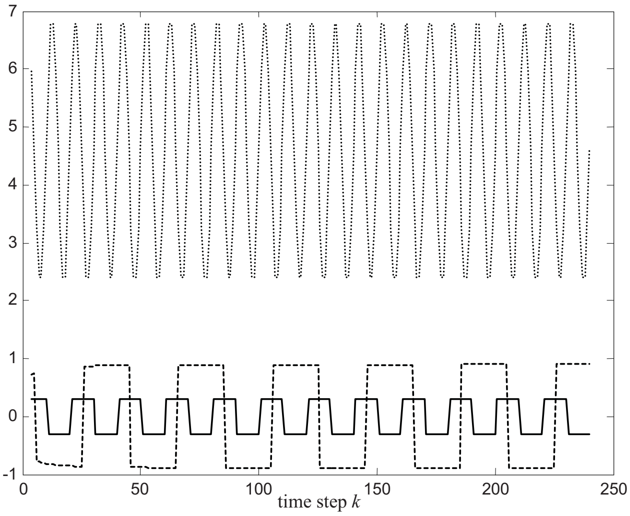

Fig. 1 Behaviour of the time-varying parameters, $a(k)$ is the solid line, $b(k)$ is the dotted line and $c(k)$ is the dashed line.

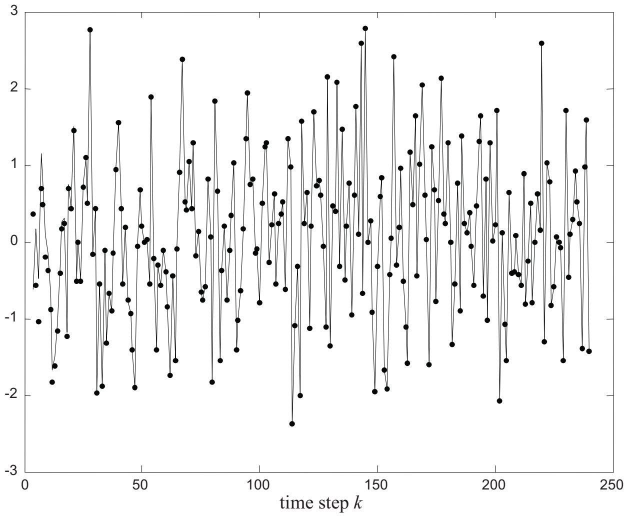

Fig. 2 Noise and its estimate, the noise is represented by the dots and its estimate is the solid line.

The reference used in this simulation is a periodic square wave with zero mean and a peak to peak value of 10. The reference is presented using dots and the plant output is the solid line. The simulation result in Fig. 3 shows that the LTV GMVC is able to drive the output of the LTV CARMA model to follow the reference.

The control action is shown in Fig. 4. It can be clearly seen that the control action varies not only in response to the reference but also in accordance with the changes in plant parameters. However, the former is due to feedback control and the latter is due to the variation of controller parameters. The control signal jumps when a plant parameter jumps in this simulation.

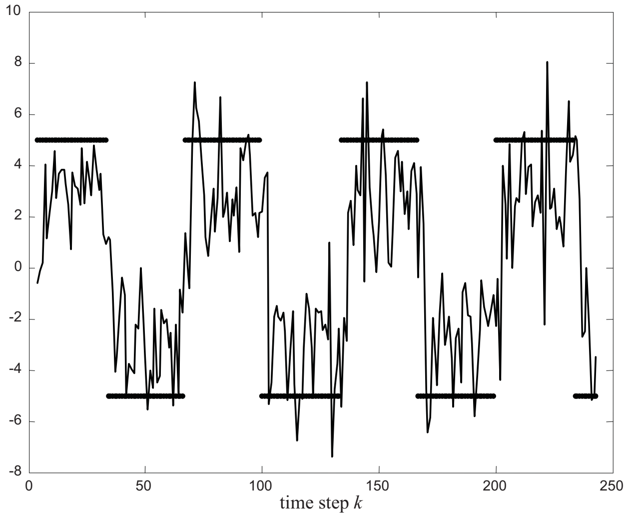

Fig. 3 The square wave reference and the plant output. The reference is represented by the thick lines and the plant output is the thin line.

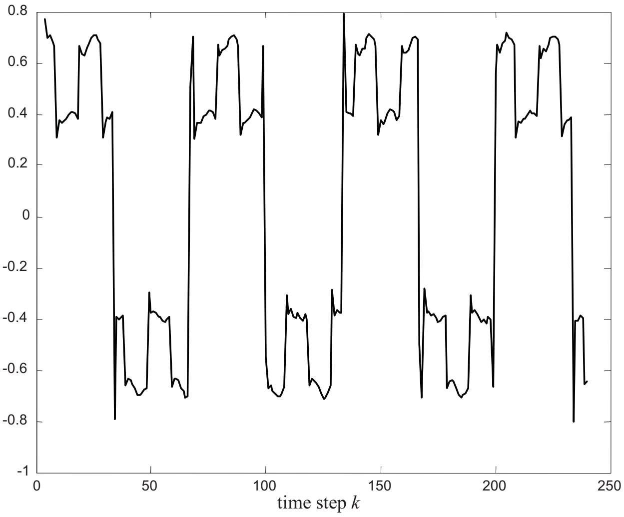

Fig. 4 Control signal in response to not only the reference but also the variation in plant parameters.

5 Conclusion

An LTV GMVC has been developed based on a generalised minimum variance cost functional. The cost functional employs time-varying filters for weighting. In this new LTV GMVC the left pseudocommutation for the plants itself is removed in order to avoid restriction of plants to be controlled. However, the right pseudocommutation is used for the introduction of the weighting LTV MAOs. This right pseudocommutation is not restrictive because the choice of the weighting MAOs is in the hands of the designer. The introduction of the three weighting MAOs makes the new GMVC more flexible in such a way that it is not only capable to control more LTV plants but also flexible for more complicated control tasks.

References:

[1] Clarke, D. W. and P. J. Gawthrop, Self-tuning controller, *Proc. IEE Part. D,* Vol.122, 1975, pp.929-934

[2] Clarke, D. W. and P. J. Gawthrop, Self-tuning control, *Proc. IEE Part. D,* Vol.126, 1979, pp.633-640

[3] Aström, K. J., *Introduction to Stochastic Control Theory*, New York: Academic Press, 1970.

[4] Koivo, H. N., A multivariable self-tuning controller, *Automatica,* Vol.16, 1980, pp.351-366

[5] Borison, U., Self-tuning regulators for a class of multivariable systems, *Automatica*, Vol.15, 1979, pp.209-215

[6] Li, Z., Evans, R. J. and Wittenmark, B., Minimum variance prediction for linear time-varying systems, *Automatica,* Vol.33, 1997, pp.607-618

[7] Li, Z. and R. J. Evans, Generalised minimum variance control of linear time-varying systems, *IEE Proc.-Control Theory Appl.,* Vol. 149, 2002, pp.111-116.

[8] Z. Li and C. Schmid, A generalised minimum variance controller for time-varying systems, To appear in *Proc. of the 16th IFAC World Congress*, Prague, Czech, July 2005.