

1-1-2009

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Recommended Citation

Hill, James M.; Spencer, Anthony J.; and McCue, Scott W.: Application of a continuum theory to vertical vibrations of a layer of granular material 2009, 1216-1231.
<https://ro.uow.edu.au/infopapers/1460>

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Keywords

Application, continuum, theory, vertical, vibrations, layer, granular, material

Disciplines

Physical Sciences and Mathematics

Publication Details

Hill, J. M., Spencer, A. J.. & McCue, S. W. (2009). Application of a continuum theory to vertical vibrations of a layer of granular material. *International Journal of Engineering Science*, 47 (11-12), 1216-1231.

Application of a continuum theory to vertical vibrations of a layer of granular material

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Abstract

Many interesting phenomena have been observed in layers of granular materials subjected to vertical oscillations; these include the formation of a variety of standing wave patterns, and the occurrence of isolated features called oscillons, which alternately form conical heaps and craters oscillating at one-half of the forcing frequency. No continuum-based explanation of these phenomena has previously been proposed. We apply a continuum theory, termed the double-shearing theory, which has had success in analyzing various problems in the flow of granular materials, to the problem of a layer of granular material on a vertically vibrating rigid base undergoing vertical oscillations in plane strain. There exists a trivial solution in which the layer moves as a rigid body. By investigating linear perturbations of this solution, we find that at certain amplitudes and frequencies this trivial solution can bifurcate. The time dependence of the perturbed solution is governed by Mathieu's equation, which allows stable, unstable and periodic solutions, and the observed period-doubling behaviour. Several solutions for the spatial velocity distribution are obtained; these include one in which the surface undergoes vertical velocities that have sinusoidal dependence on the horizontal space dimension, which corresponds to the formation of striped standing waves, and is one of the observed patterns. An alternative continuum theory of granular material mechanics, in which the principal axes of stress and rate-of-deformation are coincident, is shown to be incapable of giving rise to similar instabilities.

Key words: Granular material, vertical vibrations, double-shearing theory, pattern formation, oscillon.

1. Introduction

There is currently an extensive interest in patterns formed in thin layers of granular materials, such as sand, subjected to vertical oscillations. Interest in the topic has its origin in a paper by Faraday [9], although it has been recognized that Faraday's observations reflect properties of the vibrating base rather than those

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of the granular material. As a result of a recent revival of interest in the dynamical behaviour of granular materials, there have been several recent general descriptions of the phenomena observed, including [5,40,41]. Detailed accounts of many aspects of the behaviour of vertically vibrated granular layers are contained in, for example, [7,8,10,20,23,27,42,43]; these papers also include references to other work.

The main features of the phenomena are as follows. A thin layer of granular material lies on a rigid horizontal base that undergoes vertical oscillations so that its vertical displacement has the form $a(t) = a_0 \cos \omega t$. For sufficiently small values of the displacement amplitude a_0 and acceleration amplitude $a_0 \omega^2$ the surface of the layer remains plane, and the layer moves essentially as a rigid body. However at critical values of a_0 and $a_0 \omega^2$ a pattern of standing waves forms on the surface of the layer. Initially these waves form a square pattern. At higher acceleration this becomes a striped pattern. At yet higher acceleration the pattern becomes hexagonal, and waves whose period is double that of the period of the base vibrations are seen. For still higher amplitude and acceleration, so-called “oscillons” are observed; these are isolated features which are essentially fixed in position, have axial symmetry, and oscillate vertically so that they alternately form conical heaps and craters. Other complex patterns, including spirals, “labyrinths” and “worms” have also been observed under various conditions. Furthermore, a range of pattern formations can co-exist under the same vibration.

The most important non-dimensional parameters governing the phenomena are the relative acceleration amplitude $a_0 \omega^2 / g$, where g is the gravitational acceleration, and the relative displacement amplitude a_0 / h , where h is the thickness of the undisturbed granular layer. Parameters which have a lesser, but not negligible, effect are d/h , where d is the particle diameter, and the shape of the container.

Attempts to explain these phenomena have been made at various levels. The related but simpler problem of a single particle bouncing on a vertically oscillating plate has received considerable attention. For example, [15] gives extensive calculations and references to other work on this problem. In studies of aggregates of granular materials, such as [2,24,30], large scale numerical simulations are carried out to describe the motion of layers composed of discrete particles. These simulations reproduce some features of the observed behaviour of vibrated granular layers, but do not fully explain them.

A natural approach is to adopt a continuum model for the granular material, and to analyze the response of such a model to vertical oscillations. The difficulty here is that there is no generally accepted continuum model for the mechanics of granular materials; many such models have been proposed, but none has found general acceptance. Analyses based on the principal symmetries of the problem and key experimental observations of granular materials have been carried out by [3,4,39,41]. Tsimring & Aronson [39] constructed a phenomenological theory that reproduces some of the experimental effects but lacks a continuum basis in granular material mechanics. The corresponding problem for a perfect fluid has been exhaustively studied; an extensive list of references is given in [26].

In the present study we apply a continuum model, termed the *double-shearing* theory, which has been quite successful in describing a variety of problems of flow of granular materials, in both quasi-static and dynamic situations. These include flow in hoppers ([35]-[37]), shear band formation ([33]), and dynamic shear flows ([16]). The double-shearing theory differs from most other model for mechanics of granular materials in that the constitutive equation involves the rate of stress as well as the stress; this feature seems to be sufficient (and possibly even crucial) to the description of the instabilities observed in vibrations of granular layers. Note that other more recent (but related) models, such as those present in [14,13,45,46], also share this property, as do the class of hypoplastic models described and reviewed in [21,22,44] and the references therein. Indeed, as argued in [25,34], the double-shearing theory may be regarded as a hypoplastic constitutive theory. Further,

comparisons between hypoplastic models and generalised double-shearing models are given in [47]. For a more general discussion on theories that describe the flow of granular materials, see the review paper [18], for example.

The relevant aspects of the double-shearing theory for plane strain are outlined in Section 2. In Section 3 we state the trivial solution in which a layer of granular material undergoes vertical vibrations as a rigid body, and then proceed to formulate perturbations (in plane strain) of this solution. The time dependence of the perturbation is considered in Section 4, where it is shown that linear perturbations are essentially governed by Mathieu's equation, whose properties are consistent with many of the observed effects. Spatial behaviour is investigated in Sections 5 to 7, where it is shown that the boundary conditions at the top and bottom of the layer give rise to an eigenvalue problem that determines the parameters in the Mathieu equation, and so the values of the acceleration and vibration amplitude at which bifurcation from the trivial solution is possible. In the plane strain theory only solutions with variation in one horizontal space direction can be studied, but several solutions of this type are obtained. These include one in which the surface undergoes vertical velocities that have sinusoidal dependence on the horizontal space dimension, which corresponds to the formation of striped standing waves, and is one of the patterns that has been observed experimentally. In Section 8 it is shown that an alternative theory of the mechanics of granular media, in which the principal axes of stress and rate-of-strain are assumed to be coincident, cannot give rise to instabilities of the type predicted by the double-shearing theory. Finally, in Section 9 we close with a discussion.

2. Plane strain theory

Initially all quantities are referred to a fixed system of rectangular Cartesian coordinates $Oxyz$ in which the z -axis points vertically upwards. The components of the velocity vector are denoted by (u, v, w) . For the present study we consider plane strain in the (x, z) planes, so that $v = 0$ and the relevant stress components are σ_{xx} , σ_{xz} , and σ_{zz} , all of which depend only on x , z and t . We write

$$p = -\frac{1}{2}(\sigma_{xx} + \sigma_{zz}), \quad q = \left\{ \frac{1}{4}(\sigma_{xx} - \sigma_{zz})^2 + \sigma_{xz}^2 \right\}^{\frac{1}{2}} \geq 0, \quad (1)$$

so that p and q are stress invariants that represent the mean in-plane hydrostatic pressure and the maximum shear stress, respectively. The stress angle ψ is defined by

$$\tan 2\psi = \frac{2\sigma_{xz}}{\sigma_{xx} - \sigma_{zz}}; \quad (2)$$

physically it is the angle that the principal stress axis associated with the algebraically greater principal stress makes with the x -axis (tensile stress is taken to be positive). Then the relevant stress components can be expressed as

$$\sigma_{xx} = -p + q \cos 2\psi, \quad \sigma_{zz} = -p - q \cos 2\psi, \quad \sigma_{xz} = q \sin 2\psi. \quad (3)$$

In soil mechanics terminology, the case $\cos 2\psi > 0$ corresponds to passive lateral pressure while $\cos 2\psi < 0$ corresponds to active lateral pressure.

We consider a horizontal layer of ideal granular material of uniform thickness h and density ρ , and suppose that at rest in equilibrium the surfaces of the layer lie in the planes $z = 0$ and $z = h$. The material is assumed to be cohesionless and to conform to the Coulomb-Mohr yield condition

$$q \leq p \sin \phi, \quad (4)$$

where ϕ is the angle of internal friction, assumed to be constant, and (4) holds as an equality whenever the material is undergoing deformation. In physical terms, (4) states that flow can only take place when the maximum shear stress q reaches the critical value $p \sin \phi$. It follows from (1)-(3) that the critical shear stress acts on the surfaces defined by

$$\frac{dz}{dx} = \tan(\psi \pm \frac{1}{4}\pi \pm \frac{1}{2}\phi). \quad (5)$$

The equations of motion are, in plane strain,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right), \quad (6)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left(g + \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right), \quad (7)$$

where g is the gravitational acceleration and t denotes time. To complete the material description it is necessary to specify a ‘flow rule’ that relates the stress to the deformation. Whereas (4) is generally accepted as a reasonable constitutive assumption for the description of stress in dry granular materials, the formulation of an appropriate flow rule is still controversial. Many proposals have been made, some of which are compared in, for example, [6,11,32]. In the present paper we adopt the *non-dilatant double-shearing* model [31,32] which is based on the physical assumption that flow occurs by simultaneous shearing on the two families of surfaces on which the critical shear stress is mobilized (given by (5)), and is by shear in the direction parallel to these surfaces. This theory has a number of attractive features. The equations of the plane strain double-shearing theory are formulated in [31,32] and are

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (8)$$

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \cos 2\psi - \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \sin 2\psi + \sin \phi \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} + 2\Omega \right) = 0, \quad (9)$$

where

$$\Omega = \dot{\psi} = \frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} + w \frac{\partial \psi}{\partial z}, \quad (10)$$

with the notation $\dot{\psi}$ here used to denote the material derivative of ψ . Equation (8) is the condition that the flow is isochoric, and (9) expresses the condition that the flow consists of simultaneous shears on the critical surfaces (5). Equation (10) defines Ω as the spin of the principal axes of the stress tensor through a generic material particle. Because Ω appears in the equations, the formulation involves the stress-rate as well as the stress and the velocity gradients. Note that while Ω itself is not a frame-indifferent quantity, the combination

$$\frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \Omega = \omega_{12} + \dot{\psi}$$

is (here ω_{12} denotes the material spin $(\partial u/\partial z - \partial w/\partial x)/2$), so the principle of material frame indifference ([38]) is not violated. With appropriate stress and velocity boundary conditions the above equations represent a complete description for plane granular flow of a cohesionless material.

We emphasise that the double-shearing requires only that deformation takes place on surfaces on which the critical stress (according to the yield condition (4)) is mobilised (these surfaces are described by (5)). It happens that these surfaces coincide with the stress characteristics in the quasi-static case (for which

the inertial terms in (6)-(7) are neglected), but this consequence is not part of the physical argument in deriving the theory. Indeed the double shearing mechanism is a constitutive assumption that is not related to the ideas of equilibrium or momentum. It is, however, worth mentioning that the system (4), (6)-(9) is elliptic, while the quasi-static equations are hyperbolic. This property of the double-shearing equations can have implications for initial-value problems, although in the present context it is not relevant. We also note that the system (4), (6)-(9) is linearly ill-posed with regard to initial conditions ([12,28]), a fact which has consequences for the question of stability. This point is discussed further in Section 9.

3. Plane strain - vertical vibrations

It is now supposed that the layer of granular material rests on a rigid base that undergoes periodic vertical vibrations so that at time t its vertical displacement is $a(t)$, and correspondingly its velocity and acceleration are in the vertical direction and have magnitudes da/dt and d^2a/dt^2 respectively. It is assumed that $a(t)$ is a periodic function with period $2\pi/\omega$ and maximum amplitude a_0 . For harmonic vibrations we have $a(t) = a_0 \cos \omega t$, but the form of $a(t)$ is left general at this stage. The motion is referred to a fixed reference frame, so that the base of the layer lies in the surface $z = a(t)$, and u, w are velocity components relative to the fixed frame. (6)-(10) then admit the trivial solution in which the granular mass moves as a rigid body

$$u = 0, \quad w = \frac{da}{dt}, \quad \sigma_{xz} = 0, \quad \sigma_{zz} = -\rho \left(g + \frac{d^2a}{dt^2} \right) (h - z + a), \quad (11)$$

where a constant of integration has been chosen so that the upper surface $z = h + a(t)$ is traction-free. Correspondingly, if the confining lateral stress σ_{xx} is such that $q = p \sin \phi$, then

$$\psi = 0, \quad q = q_0 = \frac{\rho \sin \phi}{1 + \sin \phi} \left(g + \frac{d^2a}{dt^2} \right) (h - z + a). \quad (12)$$

We now investigate small perturbations of this motion, and seek solutions of the form

$$u = \varepsilon u_1(x, z, t), \quad w = \frac{da}{dt} + \varepsilon w_1(x, z, t), \quad \psi = \varepsilon \psi_1(x, z, t), \quad q = q_0 + \varepsilon q_1(x, z, t), \quad (13)$$

where ε is a small parameter. It follows from (3) and (13) that, to terms of order ε

$$\begin{aligned} \sigma_{xx} &= -(q_0 + \varepsilon q_1) \frac{(1 - \sin \phi)}{\sin \phi}, \quad \sigma_{zz} = -(q_0 + \varepsilon q_1) \frac{(1 + \sin \phi)}{\sin \phi}, \quad \sigma_{xz} = 2\varepsilon q_0 \psi_1, \\ \Omega &= \varepsilon \left(\frac{\partial \psi_1}{\partial t} + \frac{da}{dt} \frac{\partial \psi_1}{\partial z} \right). \end{aligned} \quad (14)$$

By inserting these into (6), (8)-(9) we obtain, again to order ε ,

$$-\frac{(1 - \sin \phi)}{\sin \phi} \frac{\partial q_1}{\partial x} + 2 \frac{\partial(q_0 \psi_1)}{\partial z} = \rho \left(\frac{\partial u_1}{\partial t} + \frac{da}{dt} \frac{\partial u_1}{\partial z} \right), \quad (15)$$

$$-\frac{(1 + \sin \phi)}{\sin \phi} \frac{\partial q_1}{\partial z} + 2 \frac{\partial(q_0 \psi_1)}{\partial x} = \rho \left(\frac{\partial w_1}{\partial t} + \frac{da}{dt} \frac{\partial w_1}{\partial z} \right), \quad (16)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \quad (17)$$

$$\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} + \left\{ \frac{\partial u_1}{\partial z} - \frac{\partial w_1}{\partial x} + 2 \frac{\partial \psi_1}{\partial t} + 2 \frac{da}{dt} \frac{\partial \psi_1}{\partial z} \right\} \sin \phi = 0. \quad (18)$$

We also make the substitution $z = \zeta + a(t)$, so that ζ denotes vertical distance above the vibrating base, and $\zeta = 0$ and $\zeta = h$ represent the undeformed lower and upper surfaces of the granular layer at any time. In terms of the independent variables x, ζ and t , (15)-(16) simplify to

$$-\frac{(1 - \sin \phi)}{\sin \phi} \frac{\partial q_1}{\partial x} + 2 \frac{\partial(q_0 \psi_1)}{\partial \zeta} = \rho \frac{\partial u_1}{\partial t}, \quad -\frac{(1 + \sin \phi)}{\sin \phi} \frac{\partial q_1}{\partial \zeta} + 2 \frac{\partial(q_0 \psi_1)}{\partial x} = \rho \frac{\partial w_1}{\partial t}, \quad (19)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial \zeta} = 0, \quad \frac{\partial u_1}{\partial \zeta} + \frac{\partial w_1}{\partial x} + \left\{ \frac{\partial u_1}{\partial \zeta} - \frac{\partial w_1}{\partial x} + 2 \frac{\partial \psi_1}{\partial t} \right\} \sin \phi = 0. \quad (20)$$

It is also convenient to introduce a stream function $\chi(x, \zeta, t)$ such that

$$u_1 = \partial \chi / \partial \zeta, \quad w_1 = -\partial \chi / \partial x, \quad (21)$$

and to denote

$$k^2 = \frac{1 - \sin \phi}{1 + \sin \phi}. \quad (22)$$

Then (20)₁ is satisfied automatically and (20)₂ becomes

$$\frac{\partial^2 \chi}{\partial \zeta^2} - k^2 \frac{\partial^2 \chi}{\partial x^2} + (1 - k^2) \frac{\partial \psi_1}{\partial t} = 0. \quad (23)$$

Also, by eliminating q_1 from (19), there follows

$$\frac{\partial^2(q_0 \psi_1)}{\partial \zeta^2} - k^2 \frac{\partial^2(q_0 \psi_1)}{\partial x^2} = \frac{\rho}{2} \left(\frac{\partial^2 u_1}{\partial t \partial \zeta} - k^2 \frac{\partial^2 w_1}{\partial t \partial x} \right), \quad (24)$$

which, by introducing (21), gives

$$\frac{\partial^2(q_0 \psi_1)}{\partial \zeta^2} - k^2 \frac{\partial^2(q_0 \psi_1)}{\partial x^2} = \frac{\rho}{2} \left(\frac{\partial^3 \chi}{\partial t \partial \zeta^2} + k^2 \frac{\partial^3 \chi}{\partial t \partial x^2} \right). \quad (25)$$

Thus the problem reduces to solving (23) and (25) for the variables χ and ψ_1 . From (12) and (22), q_0 can be expressed as

$$q_0 = \frac{\rho(1 - k^2)}{2} \left(g + \frac{d^2 a}{dt^2} \right) (h - \zeta) \quad (26)$$

and (25) becomes

$$(1 - k^2) \left(g + \frac{d^2 a}{dt^2} \right) \left\{ (h - \zeta) \left(\frac{\partial^2 \psi_1}{\partial \zeta^2} - k^2 \frac{\partial^2 \psi_1}{\partial x^2} \right) - 2 \frac{\partial \psi_1}{\partial \zeta} \right\} - \frac{\partial^3 \chi}{\partial t \partial \zeta^2} - k^2 \frac{\partial^3 \chi}{\partial t \partial x^2} = 0. \quad (27)$$

We note also that (19) may be written as

$$\frac{\partial q_1}{\partial x} = \frac{\rho(1 - k^2)}{2k^2} \left\{ (1 - k^2) \left(g + \frac{d^2 a}{dt^2} \right) \frac{\partial}{\partial \zeta} \{ (h - \zeta) \psi_1 \} - \frac{\partial^2 \chi}{\partial t \partial \zeta} \right\}, \quad (28)$$

$$\frac{\partial q_1}{\partial \zeta} = \frac{\rho(1 - k^2)}{2} \left\{ (1 - k^2) \left(g + \frac{d^2 a}{dt^2} \right) \frac{\partial}{\partial x} \{ (h - \zeta) \psi_1 \} + \frac{\partial^2 \chi}{\partial t \partial x} \right\}. \quad (29)$$

We seek separated solutions of (23) and (27) of the form

$$\chi = F(x, \zeta)G(t), \quad \psi_1 = H(x, \zeta)K(t). \quad (30)$$

By introducing (30) into (23) it follows that

$$\frac{1}{H} \left\{ \frac{\partial^2 F}{\partial \zeta^2} - k^2 \frac{\partial^2 F}{\partial x^2} \right\} = -\frac{1}{G} (1 - k^2) \frac{dK}{dt} = c_1, \quad (31)$$

where c_1 is a separation constant. Similarly introducing (30) into (27) gives

$$\left[\frac{\partial^2 F}{\partial \zeta^2} + k^2 \frac{\partial^2 F}{\partial x^2} \right] \left[(h - \zeta) \left(\frac{d^2 H}{d\zeta^2} - k^2 \frac{d^2 H}{dx^2} \right) - 2 \frac{\partial H}{\partial \zeta} \right]^{-1} = (1 - k^2) \left\{ \left(g + \frac{d^2 a}{dt^2} \right) K \right\} \left[\frac{dG}{dt} \right]^{-1} = c_2 \quad (32)$$

where c_2 is also a separation constant. Hence in this solution the time dependence is determined by the equations

$$(1 - k^2) \frac{dK}{dt} + c_1 G = 0, \quad (1 - k^2) \left(g + \frac{d^2 a}{dt^2} \right) K - c_2 \frac{dG}{dt} = 0. \quad (33)$$

and the spatial dependence by

$$\left\{ \frac{\partial^2 F}{\partial \zeta^2} - k^2 \frac{\partial^2 F}{\partial x^2} \right\} - c_1 H = 0, \quad (34a)$$

$$\left\{ \frac{\partial^2 F}{\partial \zeta^2} + k^2 \frac{\partial^2 F}{\partial x^2} \right\} - c_2 \left\{ (h_0 - \zeta) \left(\frac{\partial^2 H}{\partial \zeta^2} - k^2 \frac{\partial^2 H}{\partial x^2} \right) - 2 \frac{\partial H}{\partial \zeta} \right\} = 0. \quad (34b)$$

Now, using (30) and (33)₂, equations (28) and (29) can be written as

$$\frac{\partial q_1}{\partial x} = \frac{\rho(1 - k^2)}{2k^2} \frac{\partial}{\partial \zeta} \{ c_2 (h - \zeta) H - F \} \frac{dG}{dt}, \quad \frac{\partial q_1}{\partial \zeta} = \frac{\rho(1 - k^2)}{2} \frac{\partial}{\partial x} \{ c_2 (h - \zeta) H + F \} \frac{dG}{dt}.$$

Hence q_1 has the form

$$q_1 = Q(x, \zeta) \frac{dG}{dt}, \quad (35)$$

where

$$\frac{\partial Q}{\partial x} = \frac{\rho(1 - k^2)}{2k^2} \left\{ c_2 \left\{ (h - \zeta) \frac{\partial H}{\partial \zeta} - H \right\} - \frac{\partial F}{\partial \zeta} \right\}, \quad \frac{\partial Q}{\partial \zeta} = \frac{\rho(1 - k^2)}{2} \left\{ c_2 (h - \zeta) \frac{\partial H}{\partial x} + \frac{\partial F}{\partial x} \right\}. \quad (36)$$

We emphasise that in this study we are not dealing with a traditional initial value problem. Instead, we consider an initial period solution and seek periodic perturbations of this solution. In this case the temporal behaviour is effectively uncoupled from the spatial behaviour, and this is made explicit by applying separation of variables. The system (4), (6)-(9) is elliptic, however the equations (34) that govern the spatial perturbation have four families of real characteristics, two of which are coincident.

4. Time dependence of perturbed solution

By eliminating $G(t)$ from (33)₂ it follows immediately that

$$\frac{d^2 K}{dt^2} + \frac{c_1}{c_2} \left(g + \frac{d^2 a}{dt^2} \right) K = 0. \quad (37)$$

Since $a(t)$ is a periodic function, (37) is a case of Hill's equation, which has a canonical form ([19])

$$\frac{d^2 f}{dt^2} + p(t)f = 0,$$

where $p(t)$ is an even integrable periodic function of t . Many properties of this equation are well known; in particular, for appropriate values of the Fourier coefficients of $p(t)$ it has solutions of the form $f = e^{\mu t} q(t)$

where $q(t)$ is periodic. The parameter μ may be positive or negative depending on the Fourier coefficients of $p(t)$. Thus f may decay to zero, undergo unbounded oscillations, or, exceptionally, be periodic. These properties of solutions of (37) are entirely consistent with the observed behaviour of vibrated layers of granular material.

For the remainder of this paper we consider the special case in which the base of the granular layer undergoes harmonic vibrations of amplitude a_0 and angular frequency ω , so that $a(t) = a_0 \cos \omega t$. Then (37) becomes

$$\frac{d^2 K}{dt^2} + \frac{c_1}{c_2} (g - \omega^2 a_0 \cos \omega t) K = 0. \quad (38)$$

and

$$G = -\frac{1 - k^2}{c_1} \frac{dK}{dt}. \quad (39)$$

Equation (38) is Mathieu's equation, which is obviously a special case of Hill's equation and has been extensively studied and tabulated (see [1], for example). A standard form is

$$\frac{d^2 f}{d\tau^2} + (\alpha - 2\beta \cos 2\tau) f = 0, \quad (40)$$

where the parameters α and β are constants. To express (38) in this standard form we substitute $f(\tau) = K(t)$, $t = 2\tau/\omega$ so that (38) becomes

$$\frac{d^2 f}{d\tau^2} + \frac{4c_1}{c_2} \left(\frac{g}{\omega^2} - a_0 \cos 2\tau \right) f = 0$$

which is Mathieu's equation with the parameters α and β as

$$\alpha = \frac{4c_1}{c_2} \frac{g}{\omega^2}, \quad \beta = \frac{2c_1}{c_2} a_0. \quad (41)$$

Solutions of Mathieu's equation (40) have a variety of behaviours, depending on the values of the parameters α and β . They may represent stable or unstable motion and, for particular values of the parameters, they have periodic solutions (Mathieu functions) of periods π and 2π . Qualitatively, this is consistent with observed behaviour of vertically vibrating granular layers, which also show both stable and unstable behaviours and, for special values of $a_0 \omega^2 / g$ (that is, β/α) and the geometrical parameters, exhibit a variety of periodic behaviours. In particular, it has been observed ([7,10,40]) that granular layers often vibrate at one-half of the exciting frequency of the base vibrations, which is a property of certain solutions of Mathieu's equation. We concentrate on the periodic solutions, which arise when α is one of a certain family of functions of β . The Mathieu functions are denoted $\text{ce}_m(\tau, \beta)$, which are even periodic functions of period π if m is even and of period 2π if m is odd, and $\text{se}_m(\tau, \beta)$, which are odd periodic functions of period π if m is even and of period 2π if m is odd. For definiteness we adopt the solution

$$K(t) = \text{ce}_m(\omega t/2, \beta), \quad (42)$$

where β is given by (41), and any multiplicative arbitrary constant is absorbed into $F(x, z)$. It then follows from (33)₁ that

$$G(t) = -\frac{1 - k^2}{c_1} \frac{d \text{ce}_m(\omega t/2, \beta)}{dt}. \quad (43)$$

It is of interest to note ([26]) that systems of Mathieu's equations also arise in the corresponding problem of irrotational motion of an ideal fluid.

5. Spatial dependence of perturbed solution. Boundary conditions

In this and the following two sections we consider solutions of (34) and (36), which determine the spatial variation of χ , ψ_1 and q_1 . Various boundary conditions on these variables are possible, but we shall make the following assumptions which seem applicable to the physical problem of vertical vibrations of a granular layer.

At the upper surface $\zeta = h$ we assume that the surface is free from traction, so that $\sigma_{zz} = 0$ and $\sigma_{xz} = 0$ at $\zeta = h$. It follows from (14) that

$$q_0 = 0, \quad q_1 = 0, \quad q_0\psi_1 = 0 \quad \text{at} \quad \zeta = h. \quad (44)$$

The first of these is satisfied by (12). The other two of (44) can be expressed, from (12), (30) and (35), as

$$Q(x, h) = 0, \quad \text{and} \quad (\zeta - h)H(x, \zeta) \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow h. \quad (45)$$

At the lower surface $\zeta = 0$, it is first assumed that contact between the vibrating base and the granular layer is maintained throughout the motion. This may not always obtain in practice (see the discussion in Section 9), but we consider the case in which it does. Hence we impose the condition $w_1 = 0$ at $\zeta = 0$, which gives, from (21) and (30)

$$\frac{\partial F(x, 0)}{\partial x} = 0. \quad (46)$$

The assumption that the particles remain in contact with the plate is equivalent to assuming that the relative acceleration amplitude $\Gamma = a_0\omega^2/g$ remains less than unity. It is well known that many of the interesting patterns and isolated features such as oscillons only occur when the layer loses contact with the plate, which happens for values of the relative acceleration amplitude Γ exceeding unity. Accordingly, the analysis presented here cannot account for those patterns and isolated features occurring for $\Gamma > 1$; nevertheless, interesting pattern formation arises for $\Gamma < 1$ and this phenomenon is encompassed by the subsequent analysis.

It is possible to apply a further condition at $\zeta = 0$. We consider two alternative cases:

(i) *Perfectly rough base.* In this case it is assumed that there is no slip between the base and the granular layer, and hence that $u_1 = 0$ at $\zeta = 0$. From (21) and (30) it follows that

$$\frac{\partial F(x, 0)}{\partial \zeta} = 0. \quad (47)$$

(ii) *Perfectly smooth base.* Alternatively it may be assumed that the base is perfectly smooth, and therefore that $\sigma_{xz} = 0$ at $\zeta = 0$. Then, from (14), $\psi_1 = 0$ and it follows from (30) that in this case

$$H(x, 0) = 0. \quad (48)$$

Deriving a general solution of (34) and (36) subject to these conditions is clearly difficult, but it is possible to identify a few special solutions of interest.

6. Solutions polynomial in x

6.1. Symmetric velocity fields

It can be seen by inspection that equations (34) have solutions of the form

$$F(x, \zeta) = x^{2n+1}F_1(\zeta) + x^{2n-1}F_3(\zeta) + x^{2n-3}F_5(\zeta) + \dots + xF_{2n+1}(\zeta), \quad (49)$$

$$H(x, \zeta) = x^{2n+1}H_1(\zeta) + x^{2n-1}H_3(\zeta) + x^{2n-3}H_5(\zeta) + \dots + xH_{2n+1}(\zeta), \quad (50)$$

for any integer n ; these represent velocity fields that are symmetrical about $x = 0$. The corresponding solution for $Q(x, \zeta)$ has the form

$$Q(x, \zeta) = x^{2n+2}Q_1(\zeta) + x^{2n}Q_3(\zeta) + x^{2n-2}Q_5(\zeta) + \dots + x^2Q_{2n+1}(\zeta). \quad (51)$$

By substituting (49)-(50) into (34) and equating coefficients of powers of x , there follows

$$\frac{d^2 F_1}{d\zeta^2} - c_1 H_1 = 0, \quad (52)$$

$$\frac{d^2 F_{2m+1}}{d\zeta^2} - (2n - 2m + 3)(2n - 2m + 2)k^2 F_{2m-1} - c_1 H_{2m+1} = 0, \quad (53)$$

$$\frac{d^2 F_1}{d\zeta^2} - c_2 \frac{d^2}{d\zeta^2} \{(h - \zeta)H_1\} = 0, \quad (54)$$

$$\frac{d^2 F_{2m+1}}{d\zeta^2} - c_2 \frac{d^2}{d\zeta^2} \{(h - \zeta)H_{2m+1}\} + (2n - 2m + 3)(2n - 2m + 2)k^2 \{F_{2m-1} + c_2(h - \zeta)H_{2m-1}\} = 0, \quad (55)$$

for $m = 1, 2, \dots, n$. Similarly, substituting (49)-(50) and (51) in (36)₁ and equating powers gives

$$Q_{2m+1} = \frac{\rho(1 - k^2)}{4k^2(n - m + 1)} \frac{d}{d\zeta} \{c_2(h - \zeta)H_{2m+1} - F_{2m+1}\}, \quad (56)$$

for $m = 0, 1, 2, \dots, n$. In particular, in the case $m = 0$ (56) gives

$$Q_1 = \frac{\rho(1 - k^2)}{4k^2(n + 1)} \frac{d}{d\zeta} \{c_2(h - \zeta)H_1 - F_1\}, \quad (57)$$

and it follows from (54) that Q_1 is constant. No new information is obtained by substituting (49)-(50) and (51) in (36)₂ and equating powers. We note that F_1 and H_1 do not depend on n , whereas F_{2m+1} and H_{2m+1} depend on n for $m \geq 1$.

(a) *Solution of form (49)-(50) and (51) with $n = 0$.*

This is the simplest case, whereby

$$F(x, \zeta) = xF_1(\zeta), \quad H(x, \zeta) = xH_1(\zeta), \quad Q(x, \zeta) = x^2Q_1(\zeta), \quad (58)$$

where Q_1 is constant. By substituting these in (34) and (36), or from (52), (54) and (57), we obtain

$$\frac{d^2 F_1}{d\zeta^2} - c_1 H_1 = 0, \quad \frac{d^2 F_1}{d\zeta^2} - c_2 \left((h - \zeta) \frac{d^2 H_1}{d\zeta^2} - 2 \frac{dH_1}{d\zeta} \right) = 0, \quad (59)$$

$$Q_1 = \frac{\rho(1 - k^2)}{4k^2} \frac{d}{d\zeta} \{c_2(h - \zeta)H_1 - F_1\}. \quad (60)$$

We proceed by eliminating F_1 from (59), which gives

$$(h - \zeta) \frac{d^2 H_1}{d\zeta^2} - 2 \frac{dH_1}{d\zeta} + \lambda^2 H_1 = 0, \quad (61)$$

where $\lambda^2 = -c_1/c_2$. The solution of (61) is

$$H_1(\zeta) = \frac{A_1}{\sqrt{h-\zeta}} J_1 \left(2\lambda\sqrt{h-\zeta} \right) + \frac{A_2}{\sqrt{h-\zeta}} Y_1 \left(2\lambda\sqrt{h-\zeta} \right), \quad (62)$$

where $J_1(\xi)$ and $Y_1(\xi)$ denote Bessel functions of order one of the first and second kinds, respectively, and A_1 and A_2 are arbitrary constants. If c_1/c_2 is positive, then λ is pure imaginary and (62) is replaced by corresponding solutions in terms of the modified Bessel functions $I_1(\xi)$ and $K_1(\xi)$. In the case when λ is real it follows from (59) that

$$F_1(\zeta) = c_2 A_1 \sqrt{h-\zeta} J_1 \left(2\lambda\sqrt{h-\zeta} \right) + c_2 A_2 \sqrt{h-\zeta} Y_1 \left(2\lambda\sqrt{h-\zeta} \right) - A_3 \zeta - A_4, \quad (63)$$

where A_3 and A_4 are also arbitrary constants. Thus, for solutions which are bounded as $\zeta \rightarrow h$, it is necessary that $A_2 = 0$, meaning

$$H_1(\zeta) = \frac{A_1}{\sqrt{h-\zeta}} J_1 \left(2\lambda\sqrt{h-\zeta} \right), \quad F_1(\zeta) = c_2 A_1 \sqrt{h-\zeta} J_1 \left(2\lambda\sqrt{h-\zeta} \right) - A_3 \zeta - A_4, \quad (64)$$

and

$$Q_1 = \frac{\rho(1-k^2)}{4k^2} A_3. \quad (65)$$

Now consider the boundary conditions. The first of (45) requires $Q_1 = 0$ and therefore $A_3 = 0$. Further, from (46) we have that

$$A_4 = c_2 A_1 \sqrt{h} J_1 \left(2\lambda\sqrt{h} \right).$$

If the no-slip base condition (47) is adopted it follows that

$$J_0 \left(2\lambda\sqrt{h} \right) = 0, \quad (66)$$

and for the perfectly smooth condition (48) that

$$J_1(2\lambda\sqrt{h}) = 0. \quad (67)$$

Thus χ and ψ_1 take the forms

$$\begin{aligned} \chi &= \frac{(1-k^2)}{\lambda^2} A_1 x \left\{ \sqrt{h-\zeta} J_1 \left(2\lambda\sqrt{h-\zeta} \right) - \sqrt{h} J_1 \left(2\lambda\sqrt{h} \right) \right\} \frac{dce_m(\omega t/2, \beta)}{dt}, \\ \psi_1 &= \frac{A_1 x}{\sqrt{h-\zeta}} J_1 \left(2\lambda\sqrt{h-\zeta} \right) ce_m(\omega t/2, \beta). \end{aligned} \quad (68)$$

The corresponding velocity components are

$$u_1 = -\frac{(1-k^2)}{\lambda} A_1 x J_0 \left(2\lambda\sqrt{h-\zeta} \right) \frac{dce_m(\omega t/2, \beta)}{dt}, \quad (69)$$

$$w_1 = -\frac{(1-k^2)}{\lambda^2} A_1 \left\{ \sqrt{h-\zeta} J_1 \left(2\lambda\sqrt{h-\zeta} \right) - \sqrt{h} J_1 \left(2\lambda\sqrt{h} \right) \right\} \frac{dce_m(\omega t/2, \beta)}{dt}. \quad (70)$$

Equations (69) and (70) describe a motion in which the surface $\zeta = h$ undergoes vertical oscillations superposed on the oscillations of the layer as a whole, but remains plane.

(b) *Solution of form (49)-(50) and (51) with $n = 1$.*

In this case

$$F(x, \zeta) = x^3 F_1(\zeta) + x F_3(\zeta), \quad H(x, \zeta) = x^3 H_1(\zeta) + x H_3(\zeta), \quad Q(x, \zeta) = x^4 Q_1(\zeta) + x^2 Q_3(\zeta).$$

By substituting these in (34) and (36) and equating powers of x , or from (52)-(55) with $n = 1$, it follows that

$$\frac{d^2 F_1}{d\zeta^2} - c_1 H_1 = 0, \quad \frac{d^2 F_1}{d\zeta^2} - c_2 \left((h - \zeta) \frac{d^2 H_1}{d\zeta^2} - 2 \frac{dH_1}{d\zeta} \right) = 0, \quad (71)$$

$$Q_1 = \frac{\rho(1 - k^2)}{8k^2} \frac{d}{d\zeta} \{c_2(h - \zeta)H_1 - F_1\}, \quad (72)$$

where Q_1 is constant, and

$$\frac{d^2 F_3}{d\zeta^2} - c_1 H_3 - 6k^2 F_1 = 0, \quad (73)$$

$$\frac{d^2 F_3}{d\zeta^2} - c_2 \left((h - \zeta) \frac{d^2 H_3}{d\zeta^2} - 2 \frac{dH_3}{d\zeta} \right) + 6k^2 \{c_2(h - \zeta)H_1 + F_1\} = 0, \quad (74)$$

$$Q_3 = \frac{\rho(1 - k^2)}{4k^2} \frac{d}{d\zeta} \{c_2(h - \zeta)H_3 - F_3\}. \quad (75)$$

Except for the coefficient $1/8$ in (72), equations (71)-(72) are the same as (59)-(60) and thus the solution that is finite as $\zeta \rightarrow h$ is

$$H_1(\zeta) = \frac{A_1}{\sqrt{h - \zeta}} J_1 \left(2\lambda \sqrt{h - \zeta} \right), \quad (76)$$

$$F_1(\zeta) = c_2 A_1 \sqrt{h - \zeta} J_1 \left(2\lambda \sqrt{h - \zeta} \right) - A_3 \zeta - A_4, \quad (77)$$

$$Q_1 = \frac{\rho(1 - k^2)}{8k^2} A_3. \quad (78)$$

By inserting (76)-(78) in (73)-(74) there follows

$$\begin{aligned} \frac{d^2 F_3}{d\zeta^2} - c_1 H_3 &= 6k^2 \left\{ c_2 A_1 \sqrt{h - \zeta} J_1 \left(2\lambda \sqrt{h - \zeta} \right) - A_3 \zeta - A_4 \right\}, \\ \frac{d^2}{d\zeta^2} \{F_3 - c_2(h - \zeta)H_3\} &= -6k^2 \left\{ 2c_2 A_1 \sqrt{h - \zeta} J_1 \left(2\lambda \sqrt{h - \zeta} \right) - A_3 \zeta - A_4 \right\}. \end{aligned} \quad (79)$$

Hence, by eliminating F_3 we arrive at

$$(h - \zeta) \frac{d^2 H_3}{d\zeta^2} - 2 \frac{dH_3}{d\zeta} + \lambda^2 H_3 = \frac{6k^2}{c_2} \left\{ 3c_2 A_1 \sqrt{h - \zeta} J_1 \left(2\lambda \sqrt{h - \zeta} \right) - 2A_3 \zeta - 2A_4 \right\}, \quad (80)$$

which has the solution

$$H_3(\zeta) = \frac{A_5}{\sqrt{h - \zeta}} J_1 \left(2\lambda \sqrt{h - \zeta} \right) + \frac{6k^2}{c_2} \left\{ \frac{A_1 c_2}{\lambda} (h - \zeta) J_2 \left(2\lambda \sqrt{h - \zeta} \right) - 2A_3 \left(\frac{\zeta}{\lambda^2} + \frac{2}{\lambda^4} \right) - \frac{2A_4}{\lambda^2} \right\}. \quad (81)$$

In (81) a term involving $Y_1(2\lambda\sqrt{h - \zeta})$ has been set equal to zero to ensure that $H_3(\zeta)$ remains finite as $\zeta \rightarrow h$. Further, integrating (79) once gives

$$\frac{d}{d\zeta} \{F_3 - c_2(h - \zeta)H_3\} = 6k^2 \left\{ \frac{2c_2 A_1}{\lambda} (h - \zeta) J_2 \left(2\lambda \sqrt{h - \zeta} \right) + \frac{A_3 \zeta^2}{2} + A_4 \zeta + A_7 \right\}, \quad (82)$$

and therefore, from (75),

$$Q_3 = \frac{3}{2} \rho(1 - k^2) \left\{ \frac{2c_2 A_1}{\lambda} (h - \zeta) J_2 \left(2\lambda \sqrt{h - \zeta} \right) + \frac{A_3 \zeta^2}{2} + A_4 \zeta + A_7 \right\}. \quad (83)$$

A further integration of (82) then gives

$$F_3 - c_2(h - \zeta)H_3 = -6k^2 \left\{ \frac{2c_2 A_1}{\lambda^2} (h - \zeta)^{3/2} J_3 \left(2\lambda \sqrt{h - \zeta} \right) - \frac{A_3 \zeta^3}{6} - \frac{A_4 \zeta^2}{2} - A_7 \zeta - A_8 \right\}, \quad (84)$$

and hence

$$F_3 = c_2 A_5 \sqrt{h - \zeta} J_1 \left(2\lambda \sqrt{h - \zeta} \right) - 6k^2 \left\{ \frac{2c_2 A_1}{\lambda^2} (h - \zeta)^{3/2} J_3 \left(2\lambda \sqrt{h - \zeta} \right) - \frac{A_3 \zeta^3}{6} - \frac{A_4 \zeta^2}{2} - A_7 \zeta - A_8 \right\} \\ + 6k^2 (h - \zeta) \left\{ \frac{c_2 A_1}{\lambda} (h - \zeta) J_2 \left(2\lambda \sqrt{h - \zeta} \right) - 2A_3 \left(\frac{\zeta}{\lambda^2} + \frac{2}{\lambda^4} \right) - \frac{2A_4}{\lambda^2} \right\}, \quad (85)$$

where A_5 , A_6 and A_7 are further arbitrary constants.

The first boundary condition (45) now requires that $Q_1 = 0$ and $Q_3 = 0$ at $\zeta = h$, and hence from (72) and (83)

$$A_3 = 0, \quad A_7 = -A_4 h. \quad (86)$$

The second of (45) is satisfied by (75) and (81). The condition (46) requires only that $F_1 = 0$ at $\zeta = 0$, which gives, from (77) and (86),

$$A_4 = c_2 A_1 \sqrt{h} J_1 \left(2\lambda \sqrt{h} \right). \quad (87)$$

Hence, collecting these results, the solution of this form is

$$H(x, \zeta) = \frac{(A_1 x^3 + A_5 x)}{\sqrt{h_0 - \zeta}} J_1 \left(2\lambda \sqrt{h - \zeta} \right) \\ + \frac{6k^2 A_1 x}{\lambda^2} \left\{ \lambda (h - \zeta) J_2 \left(2\lambda \sqrt{h - \zeta} \right) - 2\sqrt{h} J_1 \left(2\lambda \sqrt{h} \right) \right\}, \quad (88)$$

$$F(x, \zeta) = c_2 (A_1 x^3 + A_5 x) \sqrt{h - \zeta} J_1 \left(2\lambda \sqrt{h - \zeta} \right) \\ - \frac{12k^2 c_2 A_1 x}{\lambda^2} (h - \zeta)^{3/2} J_3 \left(2\lambda \sqrt{h - \zeta} \right) \\ + \frac{6k^2 c_2 A_1 x}{\lambda} (h - \zeta)^2 J_2 \left(2\lambda \sqrt{h - \zeta} \right) \\ - c_2 A_1 \left\{ x^3 + 3k^2 x (-\zeta^2 + 2\zeta h + 4\lambda^{-2} (h - \zeta)) \right\} \sqrt{h} J_1 \left(2\lambda \sqrt{h} \right) + 12k^2 A_8 x, \quad (89)$$

$$Q(x, \zeta) = \frac{\rho(1 - k^2) c_2 A_1 x^2 (h - \zeta)}{\lambda} \left\{ 2J_2 \left(2\lambda \sqrt{h - \zeta} \right) - \lambda \sqrt{h} J_1 \left(2\lambda \sqrt{h} \right) \right\}, \quad (90)$$

The constant A_8 makes no contribution to the displacement and may be omitted. In this solution it is not possible to choose λ so as to satisfy either of the conditions (47) or (48) at $\zeta = 0$ for all values of x . However the choice $J_0 \left(2\lambda \sqrt{h} \right) = 0$ ensures that u_1 is independent of x at $\zeta = 0$, and the choice $J_1 \left(2\lambda \sqrt{h} \right) = 0$ makes ψ_1 independent of x at $\zeta = 0$. In this solution the vertical component of the velocity perturbation is quadratic in x , and so the free surface assumes an oscillating parabolic shape.

It is clearly possible to proceed to develop solutions of the forms (49)-(50) and (51) for values of n greater than one, but it is equally clear that the algebraic complexity of these solutions will increase rapidly as n increases.

6.2. Antisymmetric velocity fields

In the same way as done in the previous subsection, we could note that equations (34) have solutions of the form

$$F(x, \zeta) = x^{2n} F_0(\zeta) + x^{2n-2} F_2(\zeta) + x^{2n-4} F_4(\zeta) + \dots + F_{2n}(\zeta), \quad (91)$$

$$H(x, \zeta) = x^{2n} H_0(\zeta) + x^{2n-2} H_2(\zeta) + x^{2n-4} H_4(\zeta) + \dots + H_{2n}(\zeta), \quad (92)$$

$$Q(x, \zeta) = x^{2n+1} Q_0(\zeta) + x^{2n-1} Q_2(\zeta) + x^{2n-3} Q_4(\zeta) + \dots + x Q_{2n}(\zeta), \quad (93)$$

for any integer n , and proceed by fixing n and solving for F_m , H_m and Q_m . The resulting solutions represent velocity fields that are anti-symmetrical about $x = 0$.

It turns out that the mathematical details are very similar, but the interpretations different. For $n = 0$ the solutions of the form (91)-(93) describe a time-dependent horizontal shearing motion; this is of limited interest from the point of view of observed vibrations of granular layers, since it involves no vertical deformation of the surface and in practice may be inhibited by lateral constraints on the layer. However, it is instructive as a comparatively simple example which exhibits several of the features observed in the vibration of granular layers. For $n = 1$ the ansatz (91)-(93) leads to solutions for which the vertical component of velocity at the surface $\zeta = h$ is linear in x , while the horizontal component is quadratic in x . Thus in this case the material appears to undergo a ‘sloshing’ type of periodic motion. Due to space restrictions we are unable to include further details here.

7. Solutions harmonic in x

Also by inspection, it is seen that (34) and (36) have solutions of the form

$$F(x, \zeta) = \hat{F}(\zeta) \cos px, \quad H(x, \zeta) = \hat{H}(\zeta) \cos px, \quad Q(x, \zeta) = \hat{Q}(\zeta) \sin px. \quad (94)$$

By substituting these in (34) and (36) there follows

$$\frac{d^2 \hat{F}}{d\zeta^2} - c_1 \hat{H} + p^2 k^2 \hat{F} = 0, \quad (95)$$

$$\frac{d^2 \hat{F}}{d\zeta^2} - c_2 \frac{d^2}{d\zeta^2} \left\{ (h - \zeta) \hat{H} \right\} - p^2 k^2 \left\{ \hat{F} + c_2 (h - \zeta) \hat{H} \right\} = 0, \quad (96)$$

$$\frac{\rho(1 - k^2)}{2k^2} \left[c_2 \left\{ (h - \zeta) \frac{d\hat{H}}{d\zeta} - \hat{H} \right\} - \frac{d\hat{F}}{d\zeta} \right] = p \hat{Q}. \quad (97)$$

It is significant that (95) and (96) reduce to (52) and (54) (whose solution is given above) in the limit $p \rightarrow 0$; that is when the wavelength of the perturbation is large compared to the layer thickness. From the solution (64)₂ this suggests introducing the substitutions

$$c_1 = -\lambda^2 c_2, \quad \hat{L} = c_2 (h - \zeta) \hat{H}, \quad \xi = \sqrt{1 - \frac{\zeta}{h}}, \quad \hat{\lambda}^2 = 4h\lambda^2, \quad \epsilon = 2hpk. \quad (98)$$

In terms of these variables (95)-(97) become

$$\frac{d^2 \hat{F}}{d\xi^2} - \frac{1}{\xi} \frac{d\hat{F}}{d\xi} + \hat{\lambda}^2 \hat{L} + \epsilon^2 \xi^2 \hat{F} = 0, \quad (99)$$

$$\frac{d^2 \hat{F}}{d\xi^2} - \frac{1}{\xi} \frac{d\hat{F}}{d\xi} - \frac{d^2 \hat{L}}{d\xi^2} + \frac{1}{\xi} \frac{d\hat{L}}{d\xi} - \epsilon^2 \xi^2 (\hat{F} + \hat{L}) = 0, \quad (100)$$

$$\frac{\rho(1 - k^2)}{4k^2 \xi} \frac{d}{d\xi} (\hat{F} - \hat{L}) = hp \hat{Q}, \quad (101)$$

and the boundary conditions (45) and (46) reduce to

$$\frac{1}{\xi} \frac{d}{d\xi} (\hat{L} - \hat{F}) = 0, \quad \hat{L} = 0, \quad \text{at } \xi = 0; \quad \hat{F} = 0 \quad \text{at } \xi = 1. \quad (102)$$

The condition (47) for no slip at the base becomes

$$\frac{d\hat{F}}{d\xi} = 0 \quad \text{at} \quad \xi = 1, \quad (103)$$

and the condition (48) for a smooth base is

$$\hat{L} = 0 \quad \text{at} \quad \xi = 1. \quad (104)$$

The main objective is to determine the eigenvalue $\hat{\lambda} = 2\lambda\sqrt{h} = 2\sqrt{-hc_1/c_2}$ so as to obtain non-trivial solutions that satisfy the homogeneous boundary conditions (102) and (103) or (104). This then determines the parameters α and β defined in (41) which define the time dependence of the perturbed solution, and in particular discriminate between stable, unstable and periodic solutions of the perturbed equations.

When $\epsilon = 0$, the solution of (99)-(100), subject to (102) is

$$\hat{L} = \hat{L}_0 = A\xi J_1(\hat{\lambda}\xi), \quad \hat{F} = \hat{F}_0 = A \left\{ \xi J_1(\hat{\lambda}\xi) - J_1(\hat{\lambda}) \right\}, \quad (105)$$

which gives $d\hat{F}_0/d\xi = A\lambda\xi J_0(\hat{\lambda}\xi)$.

For $\epsilon \neq 0$, it does not seem to be possible to solve (99)-(101) in closed form, and recourse must be made to numerical or approximate methods. We first describe an approximate solution. We observe that k^2 lies between 0 and 1, and for typical granular materials has a value of about 1/3, and that $hp/2\pi$ is the ratio of the thickness of the granular layer to the wavelength of the velocity perturbations. Hence in practice ϵ is often a small number, and it is reasonable to seek solutions of the form

$$\hat{L} = \hat{L}_0 + \epsilon^2 \hat{L}_1 + O(\epsilon^4), \quad \hat{F} = \hat{F}_0 + \epsilon^2 \hat{F}_1 + O(\epsilon^4). \quad (106)$$

By substituting (106) in (99)-(101), there follows

$$\frac{d^2 \hat{F}_1}{d\xi^2} - \frac{1}{\xi} \frac{d\hat{F}_1}{d\xi} + \hat{\lambda}^2 \hat{L}_1 + \xi^2 \hat{F}_0 = 0, \quad (107)$$

$$\frac{d^2 \hat{F}_1}{d\xi^2} - \frac{1}{\xi} \frac{d\hat{F}_1}{d\xi} - \frac{d^2 \hat{L}_1}{d\xi^2} + \frac{1}{\xi} \frac{d\hat{L}_1}{d\xi} - \xi^2 (\hat{F}_0 + \hat{L}_0) = 0. \quad (108)$$

Hence, by eliminating \hat{F}_1 and inserting (105),

$$\frac{d^2 \hat{L}_1}{d\xi^2} - \frac{1}{\xi} \frac{d\hat{L}_1}{d\xi} + \hat{\lambda}^2 \hat{L}_1 = -A \left\{ 3\xi^3 J_1(\hat{\lambda}\xi) - 2\xi^2 J_1(\hat{\lambda}) \right\}, \quad (109)$$

which has the solution

$$\hat{L}_1 = B\xi J_1(\hat{\lambda}\xi) - A \left\{ \frac{1}{2\hat{\lambda}} \xi^4 J_2(\hat{\lambda}\xi) - \frac{2}{\hat{\lambda}^2} \xi^2 J_1(\hat{\lambda}) \right\}. \quad (110)$$

The term $B\xi J_1(\hat{\lambda}\xi)$ can be absorbed into \hat{L}_0 and so without loss of generality we set $B = 0$, and we note that condition (102)₂ is satisfied. Then (105) and (108) give

$$\frac{d}{d\xi} \left(\frac{1}{\xi} \left\{ \frac{d\hat{F}_1}{d\xi} - \frac{d\hat{L}_1}{d\xi} \right\} \right) = A \left\{ 2\xi^2 J_1(\hat{\lambda}\xi) - \xi J_1(\hat{\lambda}) \right\}. \quad (111)$$

Hence, by integrating and applying the boundary conditions (102)₁

$$\frac{d\hat{F}_1}{d\xi} - \frac{d\hat{L}_1}{d\xi} = A \left\{ \frac{2}{\hat{\lambda}} \xi^3 J_2(\hat{\lambda}\xi) - \frac{1}{2} \xi^3 J_1(\hat{\lambda}) \right\}. \quad (112)$$

Hence, from (110) and (112)

$$\frac{d\hat{F}_1}{d\xi} = A \left\{ \frac{1}{\hat{\lambda}} \xi^3 J_2(\hat{\lambda}\xi) - \frac{1}{2} \xi^4 J_1(\hat{\lambda}\xi) - \left(\frac{1}{2} \xi^3 - \frac{4}{\hat{\lambda}^2} \xi \right) J_1(\hat{\lambda}) \right\}, \quad (113)$$

and therefore, integrating and applying the condition (102)₃

$$\hat{F}_1 = A \left\{ \frac{2 \left(\xi^3 J_3(\hat{\lambda}\xi) - J_3(\hat{\lambda}) \right)}{\hat{\lambda}^2} - \frac{\xi^4 J_2(\hat{\lambda}\xi) - J_2(\hat{\lambda})}{2\hat{\lambda}} + \frac{(\xi^2 - 1) \{16 - \hat{\lambda}^2(\xi^2 + 1)\} J_1(\hat{\lambda})}{8\hat{\lambda}^2} \right\}. \quad (114)$$

Hence, to this approximation, the no-slip base boundary condition (103) becomes

$$\frac{d\hat{F}_0}{d\xi} + \epsilon^2 \frac{d\hat{F}_1}{d\xi} = 0, \quad \xi = 1,$$

which, from (105) and (113) determines $\hat{\lambda}$ to be a root of

$$\hat{\lambda} J_0(\hat{\lambda}) + \epsilon^2 \left\{ \frac{1}{\hat{\lambda}} J_2(\hat{\lambda}) + \left(\frac{4}{\hat{\lambda}^2} - 1 \right) J_1(\hat{\lambda}) \right\} = 0. \quad (115)$$

Similarly, for the smooth base boundary condition (104),

$$\hat{L}_0 + \epsilon^2 \hat{L}_1 = 0, \quad \xi = 1$$

and in this case, from (105) and (110), $\hat{\lambda}$ is a root of

$$J_1(\hat{\lambda}) - \epsilon^2 \left\{ \frac{1}{2\hat{\lambda}} J_2(\hat{\lambda}) - \frac{2}{\hat{\lambda}^2} J_1(\hat{\lambda}) \right\} = 0. \quad (116)$$

In general, each of (115) and (116) has an infinite number of real roots for a given value of hpk . The variation of the smallest root $\hat{\lambda}$ with hpk is shown in Figure 1(a) for the rough base boundary condition (115), and in Figure 1(b) for the smooth base boundary condition (116); these results are applicable for small values of the thickness to wavelength ratio. Clearly, in the ranges illustrated, the values of $\hat{\lambda}$ that satisfy (115) and (116) are not sensitive to the values of hpk . The lowest zeros of $J_0(\hat{\lambda})$ and $J_1(\hat{\lambda})$, which correspond to the limits $hpk \rightarrow 0$, are $\hat{\lambda} = 2.4048$ and $\hat{\lambda} = 3.8137$ respectively.

An alternative procedure is to solve (99)-(100) numerically to determine the eigenvalue $\hat{\lambda}$ for given values of the parameter hpk , subject to the homogeneous boundary conditions (102) and (103) in the case of no slip at the base, or (102) and (104) in the case of a perfectly smooth base. In the case of a smooth base, the numerical method employed was to replace (99)-(100) by the equivalent system of first-order equations

$$\frac{d\hat{F}}{d\xi} = r, \quad (117)$$

$$\frac{d\hat{L}}{d\xi} = r + s, \quad (118)$$

$$\frac{dr}{d\xi} = -\epsilon^2 \xi^2 \hat{F} - \hat{L} \hat{\lambda}^2 + \frac{r}{\xi}, \quad (119)$$

$$\frac{ds}{d\xi} = -\epsilon^2 \xi^2 (\hat{F} + \hat{L}) + \frac{s}{\xi}. \quad (120)$$

This system was solved numerically by a MAPLE routine with initial conditions

$$\hat{F}(0) = \hat{\mu}, \quad \hat{L}(0) = 0, \quad r(0) = 1, \quad s(0) = 0, \quad (121)$$

initially with estimated values of $\hat{\mu}$ and $\hat{\lambda}$ based on the approximation (116). These values were then iteratively adjusted by a simple interpolation procedure until the boundary conditions $\hat{F}(1) = 0$ and $\hat{L}(1) = 0$

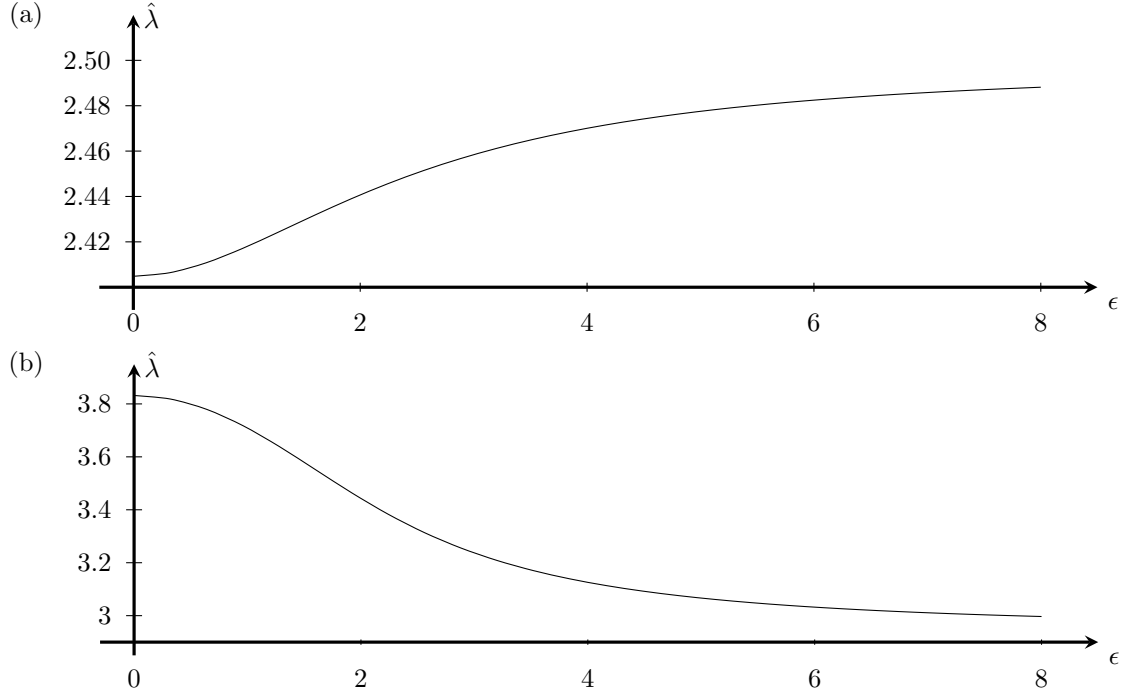


Fig. 1. Variation of the eigenvalue $\hat{\lambda}$ with ϵ for sinusoidal dependence of velocity on x . Part (a) corresponds to the no-slip base condition, according to approximate equation (115), while part (b) is for the smooth base condition, according to (116).

were satisfied to a specified accuracy. In practice the iteration converged rapidly. A similar procedure was used for the no-slip boundary condition (103). Some numerical results are shown in Tables 1 and 2. We observe that the approximate formulae (115) and (116) determine $\hat{\lambda}$ to within an error of 1% for $hpk < 1$ (or $\epsilon < 2$).

ϵ	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0
$\hat{\lambda}$	2.41	2.41	2.42	2.44	2.47	2.50	2.55	2.60	2.66	2.80	3.55	3.37	3.21	3.16	3.16	3.15	3.20	3.26	3.29	3.28	3.26
$\hat{\lambda}$	3.83	3.80	3.71	3.59	3.45	3.31	3.20	3.11	3.04	2.99	3.04	3.17	3.33	3.41	3.37	3.31	3.29	3.31	3.36	3.40	3.40

Table 1

Values of $\hat{\lambda}$ for various values of ϵ with sinusoidal dependence of velocity on x by numerical solution of (99)-(100) with boundary conditions (102) and either (103) for the no-slip base condition (the second row of data) or (104) for the smooth base condition (the third row of data).

8. Coaxial theory

The double-shearing theory employed above is one of many theories that have been proposed to describe the mechanics of granular materials. Another class of theories adopts as a constitutive assumption the postulate that the principal axes of stress and rate-of-strain are coincident; this was proposed by Hill [17] and is also an ingredient of ‘critical state’ theories which are described in, for example, Schofield & Wroth [29]. This coaxiality condition is often stated to be a condition for isotropy, but this is not the case if, as in the double-shearing theory, the deformation depends on stress-rate as well as stress. In the double shearing theory, it follows from (9) that the principal axes of stress and rate-of-strain are coincident only if

$$\sin \phi = 0, \quad \text{or} \quad \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} + 2\Omega \right) = 0,$$

neither of which holds in general.

We consider the coaxial theory formulated by Hill [17], which is the simplest theory based on coaxiality. In this theory (4), (7) and (8) still apply, but (9) is replaced by the coaxiality condition

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \cos 2\psi - \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \sin 2\psi = 0.$$

The trivial solution expressed by (11)-(12) is also valid in this theory. As in Section 3, we seek perturbations of the form (13), which for the coaxial theory leads to

$$\begin{aligned} -\frac{(1 - \sin \phi)}{\sin \phi} \frac{\partial q_1}{\partial x} + 2 \frac{\partial(q_0 \psi_1)}{\partial \zeta} &= \rho \frac{\partial u_1}{\partial t}, & -\frac{(1 + \sin \phi)}{\sin \phi} \frac{\partial q_1}{\partial \zeta} + 2 \frac{\partial(q_0 \psi_1)}{\partial x} &= \rho \frac{\partial w_1}{\partial t}, \\ \frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial \zeta} &= 0, & \frac{\partial u_1}{\partial \zeta} + \frac{\partial w_1}{\partial x} &= 0. \end{aligned} \quad (122)$$

The first three of these are the same as (19)-(20)₁, but (20)₂ in the double-shearing theory is replaced by (122)₂ in the coaxial theory.

We introduce the stream function χ as in (21) and, following the same procedure as in Section 3, obtain (25). However, in the coaxial theory, (23) is replaced by

$$\frac{\partial^2 \chi}{\partial \zeta^2} - \frac{\partial^2 \chi}{\partial x^2} = 0.$$

Hence, seeking separated solutions of the form (30) it follows that either

$$G(t) = 0 \quad \text{or} \quad \frac{\partial^2 F}{\partial \zeta^2} - \frac{\partial^2 F}{\partial x^2} = 0.$$

If the latter is true, only trivial deformations are admitted. If $G(t) = 0$ it follows from (33)₂ that $K(t) = 0$. Therefore the coaxial theory does not give rise to non-trivial solutions of the form (13), and so does not provide a basis for explaining the observed behaviour of vibrating granular layers. Consequently, inclusion of dependence of deformation on the stress-rate as well as on the stress, as in the double-shearing theory, seems to be an essential ingredient of any continuum theory capable of describing the observed phenomena.

9. Discussion

It has been shown that the double-shearing theory of the mechanics of granular materials is capable of explaining some of the surprising features that are observed in experiments on vertical vibrations of layers of granular materials. In particular, the time dependence of the deformation has been shown to be governed by Mathieu's equation, which has many properties that feature in the experimental observation, such as regions in the parameter space of stability and instability, periodic solutions for certain values of the parameters, and 'period-doubling' solutions. Thus as far as the time-dependence is concerned, the theory seems to be a promising tool for the description of vibrations of granular layers. It is emphasized that the double-shearing theory was originally formulated as a model to describe quasi-static flows of granular material (for example in silos and hoppers) which it does quite successfully ([35]-[37]) and that it also qualitatively explains features of dynamic shear flow of granular materials ([16]) and shear band formation ([33]) in granular materials. It is therefore quite a versatile theory, and is by no means restricted to the problems discussed in this paper.

In the present study we have considered only plane strain problems, and so the theory as presented cannot explain the diverse and complicated two-dimensional surface patterns that have been observed experimentally. Essentially it is limited to describing the formation of ‘sloshing’ motions, time-periodic ridges and valleys, and parallel striped patterns. However it will be shown in future publications that the analysis can be extended to axially symmetric deformations and possibly to general deformations in two horizontal space dimensions.

The analysis clearly has limitations and does not give a full description of the observed phenomena. Firstly, as it is based on a continuum theory, it takes no account of the particulate nature of the material; in practice it is often seen that individual particles of the granular material are in free flight and the theory does not allow for this. However provided that the layer thickness is large compared to the particle dimensions this may not be significant. More importantly, the analysis presented is essentially a linear stability analysis, and does not consider the post-bifurcation behaviour. It is likely that a non-linear analysis of the equations formulated in Section 2 would be needed to provide a full description of many features (one of which is probably the oscillon) of the behaviour of oscillating granular layers; such an analysis, although desirable, would be difficult to undertake. Photographs of standing waves in the layers show that the surface slope is often quite steep, and so probably outside the limits of the linearized analysis.

It has been assumed in this paper that the granular layer remains in contact with the rigid base during the deformation. This is not the case in many of the experimental studies, which show that the layer is often in free flight during part of the cycle. Clearly the present analysis applies only during that portion of the cycle in which contact is maintained. It is possible to construct periodic solutions in which contact is lost in part of the cycle, but the boundary conditions at the base of the layer become more complicated in this case. If the experiment is conducted in a vacuum (as, for example, in [42,43]) then contact with the base is lost when σ_{zz} becomes zero at $\zeta = 0$, and subsequently the layer is stress-free in free flight under gravity until contact is resumed. For experiments carried out at atmospheric pressure, the pressure on the lower surface will depend on the speed at which the pressure on the upper and lower surfaces is equalized after contact is lost; this equalization may occur either by an influx of air from the sides of the container or by diffusion of air through the layer. The particle size of the granules has a strong effect on the diffusivity of the layer.

The main purpose of this paper has been to demonstrate that the continuum double-shearing theory of mechanics of granular material is capable of a qualitative description of many aspects of the behaviour of vibrating granular layers. Because of the limitations noted above, we consider it premature to attempt seriously a quantitative comparison with experiment. However, as a very crude estimate, if we take the long wavelength limit $\hat{\lambda} = 3.8137$ for a layer with a smooth base, and an amplitude ratio $a_0/h = 0.2$, then $\beta = -\hat{\lambda}^2 a_0/2h \approx -1.46$. For the lowest mode Mathieu function the corresponding characteristic value of α is (from [1]), $\alpha \approx -0.82$. From (41), the ratio of the acceleration amplitude to g is

$$\Gamma = \frac{a_0 \omega^2}{g} = \frac{2\beta}{\alpha} \approx 3.56.$$

Umbanhowar & Swinney [43] report values of $\Gamma = 2.2$ for the flat layer to wave transition, and Umbanhowar [40] quotes as typical values $\Gamma = 3.0$ for the transition to a square pattern and $\Gamma = 3.2$ for the transition to a striped pattern. Fauve et al. [10] describe the appearance of parallel surface standing waves at $\Gamma = 3.5\Gamma_c$, where for large enough grains Γ_c is slightly greater than one. This reasonable agreement is no doubt fortuitous (in particular the predicted value of Γ is sensitive to the choice of a_0/h) but it is encouraging in that it shows that at least the theory is capable of predicting results of the correct order of magnitude, and suggests that further study may be fruitful.

We close with some brief comments regarding the ill-posedness of (4), (6)-(9). We have not dealt with an initial-value problem, but instead with perturbations from a periodic solution that is certainly stable for sufficiently small amplitude and frequency. We have shown, quite explicitly, that linear perturbations from this solution can be stable, unstable, or periodic, depending on the parameters defined in (41). For these reasons the ill-posedness of (4), (6)-(9) is not relevant here. Nevertheless, it would be interesting to explore whether or not similar behaviours are predicted by recently developed well-posed models such as those presented in [14,13].

Additional note: The first draft of this paper was prepared by Tony Spencer during the first six months of 2001. Although Tony provided the formal analysis, he insisted that our names appear alphabetically. In fact two papers were planned: the present one on plane strain; and another on the axially symmetric analysis of the oscillon, which I drafted and on which Tony was to appear as the lead author. Only the first paper was submitted for publication. The reviews contained some negative comments, and while Tony expressed numerous times his intention to revise the manuscript, this never happened. The present manuscript is a fully revised version of Tony's original draft by SWM, incorporating the commentary of the original referees and some of Tony's own notes. The second part will be submitted for publication in the near future. JMH.

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