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Asymptotics on the general fractionally integrated processes with applications to unit root testing

Qiying Wang
University of Wollongong

ASYMPTOTICS ON THE GENERAL FRACTIONALLY INTEGRATED PROCESSES WITH APPLICATIONS TO UNIT ROOT TESTING

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Qiying WANG

(BSc, Anhui University, MSc, University of Science and Technology of China)

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Abstract

As a basic tool, asymptotic theory (in particular, functional limit theorem) plays a key role in characterizing the limit distribution of various statistics arising from statistical inferences in economic time series, such as testing for unit roots, testing for stationarity, and time series regression. Asymptotics on the fractional processes and the summable linear processes have been studied by many people. However, the results in the literature are quite restrictive on both the processes themselves and the conditions used in deriving the results. For example, a functional limit theorem is only available for a general fractional process with innovations being iid $N(0, \sigma^2)$ or a simple fractional process under at least fourth moment conditions.

The aim of this work is to investigate systematically asymptotics of the general fractionally integrated processes and the summable linear processes with dependent or independent innovations under quite general conditions. We derive the functional limit theorem on the general fractionally integrated processes with and without "pre-historical influence". We give sufficient conditions so that the partial sum process of a summable linear process converges to a standard Brownian motion, and discuss asymptotics of sample autocovariances and autocorrelations based on nonstationary fractionally integrated processes. In particular, the result for the functional limit theorem on the general fractionally integrated processes provides a unified treatment for

previous studies on the functional limit theorem for fractional processes and nonstationary fractionally integrated processes. Additionally, only finite second moments are required for most of the results established in this dissertation. Such a condition is the best possible moment condition in the literature and it is interesting from the theoretical point of view.

Also, we discuss applications of the results established in this dissertation to testing for unit roots, testing for stationarity, and time series regression. The limit distributions of Dickey and Fuller test statistics and KPSS (Kwiatkowski, Phillips, Schmidt and Shin) test statistics are derived for more general models under very weak conditions.

List of Symbols

The following notation will be used in this thesis without further explanations.

AR	autoregressive
ARMA	autoregressive moving average
ARIMA	autoregressive integrated moving average
ARFIMA	autoregressive fractionally integrated moving average
B	backshift operator
iid	independent and identically distributed
iid (μ, σ^2)	iid random variables with mean μ and variance σ^2
iid $N(\mu, \sigma^2)$	independent and normally distributed random variables with mean μ and variance σ^2
$D[0,1]$	the metric space with the Skorohod topology of all real-valued right continuous functions having finite left-hand limits on the interval $[0,1]$
$W(t)$	the standard Brownian motion

$\to_d$	convergence in distribution
$\to_P$	convergence in probability
$\to_{a.s.}$	convergence almost surely
$\Rightarrow$	weak convergence of probability measures
[a]	integer part of the real number a
$a_n \sim b_n$	$\lim_{n\to\infty} a_n/b_n = 1$
$a_n = o(b_n)$	$\lim_{n\to\infty} a_n/b_n = 0$
$a_n = O(b_n)$	$\limsup_{n\to\infty} a_n/b_n < \infty$
$X_n = o_P(b_n)$	$X_n/b_n \to_P 0$, as $n \to \infty$
$X_n = O_P(b_n)$	$X_n/b_n < \infty$, in probability
$X_n = o(b_n)$, $a.s.$	$X_n/b_n \to_{a.s.} 0$, as $n \to \infty$
$X_n = O(b_n)$, $a.s.$	$X_n/b_n < \infty$, almost surely

Chapter 1

Introduction and summary

Many time series data, especially those found in business and economics, exhibit nonstationary behaviour. Some time series like gross domestic product and industrial product grow in a secular way over long periods of time. Other time series, such as interest rates, exchange rates, stock returns as well as asset prices, seem to display random wandering behaviour. Because of the nonstationarity, one cannot be expected to apply the usual regression methods (see Davidson and Mackinnon, 1993, Chapter 20) in modeling these time series. Instead, we have to trend or difference them prior to use. Explicitly, trending a time series y_t will be appropriate if the data generating process (DGP) for y_t can be written as

$$y_t = \gamma_0 + \gamma_1 t + u_t, \tag{1.0.1}$$

where u_t is a stationary process. On the other hand, differencing is appropriate if the DGP for y_t can be written as

$$y_t = \gamma_1 + y_{t-1} + u_t, \tag{1.0.2}$$

where u_t again is a stationary process. A model of the form (1.0.1) is known as a trend-stationary time series. $\gamma_1 t$ is a deterministic trend which may be more complex

than a simple polynomial. The output of (1.0.2) can be written as a accumulated process $y_t = \gamma_1 t + \sum_{j=1}^{t} u_j + y_0$, and hence it has a stochastic trend by virtue of the fact that the stochastic element $\sum_{j=1}^{t} u_j$ is of random order $O_P(t^{1/2})$. We also call the process y_t in (1.0.2) as difference stationary with drift γ_1 because $\Delta y_t = \gamma_1 + u_t$ where u_t is a stationary process and $\Delta = 1 - B$ is a differencing operator.

As is well known, characteristics of time series with stochastic trends attributable to the form (1.0.2) are quite different from those of time series with deterministic trends such as in the form (1.0.1). Therefore, the importance of determining the type of trends in time series has long been recognized, especially since the work of Nelson and Plosser (1982), which includes detailed discussions about the difference between the types of trends.

There is more than one way of determining the type of trends. The obvious way to choose between (1.0.1) and (1.0.2) is to nest them both within a more general model. As an example, we consider the following DGP advocated by Bhargava (1986)

$$y_t = \gamma_0 + \gamma_1 t + v_t, \qquad v_t = \alpha v_{t-1} + u_t, \tag{1.0.3}$$

where u_t is a stationary process. After a simple calculation, we can rewrite (1.0.3) as

$$y_t = [\gamma_0(1-\alpha) + \gamma_1\alpha] + \gamma_1(1-\alpha)t + \alpha y_{t-1} + u_t. \tag{1.0.4}$$

It is clear that, when $|\alpha| < 1$, (1.0.3) is equivalent to the trend-stationary model (1.0.1); when $\alpha = 1$, it reduces to (1.0.2). These facts, together with (1.0.4), imply that determining for the presence of a stochastic trend in a model like (1.0.3) is equivalent to testing the null hypothesis that the autoregressive parameter $\alpha = 1$ in the following autoregressive model:

$$y_t = \beta_0 + \beta_1 t + \alpha y_{t-1} + u_t, \tag{1.0.5}$$

where u_t is a stationary process. Such a test and those related are commonly called unit root tests in the literature. If the null hypothesis that $\alpha = 1$ holds, we also say that the model (1.0.5) has a unit root. Since the alternative hypothesis that $|\alpha| < 1$ corresponds to trend stationarity, the unit root test can be represented as a test of difference stationarity versus trend-stationarity in a time series. This property makes unit root tests play a key role in economic theory and practice.

In the development of all unit root tests, asymptotic theory is necessary. Because of the nonstationarity under the null hypothesis, limit distributions of test statistics in these unit root tests (also called unit root distributions) generally involve functionals of a Brownian motion, some of which are stochastic integrals, and hence are very different from those in the standard statistical test problems in which the limit distribution usually is a standard normal distribution. On the other hand, limit theory forms the basis of unit root tests, and in particular functional limit theorems (weak convergence of probability measures on $D[0,1]$, also called invariance principle) are basic tools in characterizing unit root distribution.

To illustrate this, let us consider the simplest and the widest used Dickey-Fuller (DF) test (see Dickey-Fuller, 1979, 1981). The simplest version of this test is based on model (1.0.5) with $\beta_0 = \beta_1 = 0$. We rewrite it as follows:

$$y_t = \alpha y_{t-1} + u_t, \quad y_0 = 0, \tag{1.0.6}$$

where u_t are iid $(0, \sigma_u^2)$. Denote the ordinary least square (OLS) estimator of α by $\widehat{\alpha}_n = \sum_{t=1}^n y_t y_{t-1} / \sum_{t=1}^n y_{t-1}^2$. To test the null hypothesis that $\alpha = 1$ (i.e., model (1.0.6) has a unit root), the DF test statistic is

$$n(\hat{\alpha}_n - 1) = \left\{ n^{-1} \sum_{t=1}^n y_{t-1}(y_t - y_{t-1}) \right\} \Big/ \left\{ n^{-2} \sum_{t=1}^n y_{t-1}^2 \right\}. \tag{1.0.7}$$

By noting that under the null hypothesis $\alpha = 1$, $y_t = \sum_{j=1}^{t} u_j$, where u_j are iid $(0, \sigma_u^2)$, it follows from Donsker's Theorem (see Billingsley, 1968, p137) and the law of large numbers that

$$\frac{1}{\sqrt{n}\sigma_u} y_{[nt]} = \frac{1}{\sqrt{n}\sigma_u} \sum_{j=1}^{[nt]} u_j \Rightarrow W(t), \qquad 0 \le t \le 1, \tag{1.0.8}$$

$$\text{and} \quad \frac{1}{n} \sum_{t=1}^{n} u_t^2 \rightarrow_P \sigma_u^2, \quad \text{as } n \rightarrow \infty. \tag{1.0.9}$$

These estimates, together with the continuous mapping theorem (Theorem A.2, see Appendix A), imply that

$$\frac{1}{n^2} \sum_{t=1}^{n} y_{t-1}^2 = \sum_{j=1}^{n-1} \int_{(j-1)/n}^{j/n} \left(\frac{1}{\sqrt{n}} y_{[ns]} \right)^2 ds \Rightarrow \sigma_u^2 \int_0^1 W^2(s) ds,$$

$$\begin{aligned} \frac{2}{n} \sum_{t=1}^{n} y_{t-1} (y_t - y_{t-1}) &= \frac{1}{n} \sum_{t=1}^{n} \left(y_t^2 - y_{t-1}^2 - u_t^2 \right) \\ &= \left(\frac{1}{\sqrt{n}} y_n \right)^2 - \frac{1}{n} \sum_{t=1}^{n} u_t^2 \Rightarrow \sigma_u^2 \left(W^2(1) - 1 \right). \end{aligned}$$

Hence, under the null hypothesis that $\alpha = 1$ (i.e., model (1.0.6) has a unit root),

$$n(\hat{\alpha}_n - 1) \Rightarrow \frac{1}{2} \left(W^2(1) - 1 \right) \Big/ \int_0^1 W^2(s) ds \, . \tag{1.0.10}$$

This simple example shows explicitly the importance of a functional limit theorem (here Donsker's Theorem, see (1.0.8)) in deriving the asymptotic distribution of the DF test statistic $n(\hat{\alpha}_n - 1)$. For more general models, such as model (1.0.5) where the error u_t is an ARMA process or a strong mixing sequence, similar unit root distributions can also be obtained by using the corresponding functional limit theorem. For some references on unit root tests, see Said and Dickey (1984), Phillips (1987), Hall (1989), Sowell (1990), Chan and Wei (1988), Chan and Terrin (1995) as well as

Tanaka (1999). Further references can be found in Phillips and Xiao (1998), where authors present a survey of unit root theory with an emphasis on testing principles and recent development.

In addition to applying to unit root tests, as a basic tool, functional limit theorems are also quite successful in characterizing the limit distribution of various statistics arising from other inferences in economic time series, such as spurious regression and testing for stationarity and cointegration. Indeed, these applications have been developed by many statisticians and economists such as Kwiatkowski, et al (1992), Phillips (1991), Lee and Schmidt (1996), Cheung and Lai (1993), Cappuccio and Lubian (1997) as well as Jeganathan (1999).

The research on functional limit theorems has a long history. Celebrated results have been obtained in many interesting fields, such as martingale differences, strong mixing sequences (more generally, mixingale sequences), linear processes and long memory processes. The literature is immense. Here we only cite a basic textbook by Billingsley (1968) for a fundamental contribution in weak convergence of probability measures on $D[0,1]$; and a review paper for mixing sequences by Peligrad (1986). For linear processes and long memory processes that are close to the topic of this dissertation, we refer to Davydov (1970), Gorodetskii(1977), Hannan (1979), Avram and Taqqu (1987), Phillips and Solo (1992) as well as Mielniczuk (1997).

Motivated in establishing basic tools that apply to statistical inferences related economic time series, this dissertation will discuss systematically functional limit theorems and some other asymptotic properties for general fractionally integrated processes and linear processes with dependent or independent innovations. A general

fractionally integrated process X_t is defined by

$$(1-B)^{d_0+m}X_t = u_t, \qquad u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad t = 1, 2, ..., \tag{1.0.11}$$

where $m \geq 0$ is an integer and $d_0 \in (-1/2, 1/2]$; ϵ_t are iid $(0, E\epsilon_0^2)$ and $\psi_j, j \geq 0$, are a sequence of real numbers satisfying

$$\sum_{j=0}^{\infty} |\psi_j| < \infty \quad \text{and} \quad b_\psi \equiv \sum_{j=0}^{\infty} \psi_j \neq 0;$$

and the fractional difference operator $(1-B)^\gamma$ is defined by its Maclaurin series (by its binomial expansion, if γ is an integer):

$$(1-B)^\gamma = \sum_{j=0}^{\infty} \frac{\Gamma(-\gamma+j)}{\Gamma(-\gamma)\Gamma(j+1)} B^j \quad \text{where} \quad \Gamma(z) = \begin{cases} \int_0^\infty s^{z-1} e^{-z} ds & \text{if } z > 0 \\ \infty & \text{if } z = 0. \end{cases}$$

If $z < 0$, $\Gamma(z)$ is defined by the recursion formula $z\Gamma(z) = \Gamma(z+1)$.

It is well-known that if u_t is an ARMA(p,q) process (i.e., there exist polynomials $\phi(B)$ and $\theta(B)$ with order p and q respectively such that $\phi(B)u_t = \theta(B)\epsilon_t$, where both $\phi(B)$ and $\theta(B)$ have only roots outside the unit circle), then Brockwell and Davis (Theorem 3.1.1, 1987, p 85) showed that u_t can be expressed as $u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ with $\sum_{k=0}^{\infty} \psi_k = \theta(1)/\phi(1)$. Therefore, the process X_t defined by (1.0.11) covers a number of important economic time series as special examples, such as the summable linear process ($d_0 = m = 0$), the ARIMA process ($d_0 = 0$, $m \geq 1$ is a integer and u_t is an ARMA(p,q) process), and the ARFIMA process ($d_0 \in (-1/2, 1/2]$, $m \geq 1$ is a integer and u_t is an ARMA(p,q) process). These facts make general fractionally integrated processes play a very important role in economic theory. The research of their properties is therefore quite interesting from the point of theory and practice.

The content of this dissertation is divided into six chapters. This chapter is the introduction. In Chapters 2-5, we shall investigate asymptotics, mainly for general

fractionally integrated processes and linear processes with dependent or independent innovations. The main results in Chapters 2-5 will be used in Chapter 6, where we discuss unit root tests, stationarity test and time series regression.

Below, we briefly introduce the contents of Chapters 2-6 and give a review on the literature.

Chapter 2 mainly contributes to establish the functional limit theorem for the partial sum process of the X_t defined by (1.0.11). In earlier research, Sowell (1990) first derived this kind of result for a simple stationary fractional process (i.e., the process defined by (1.0.11) with $m = 0$, $\psi_0 = 1$ and $\psi_j = 0$ for $j \geq 1$) under $E|\epsilon_0|^r < \infty$, where $r \geq \max\{4, -8d_0/(1+2d_0)\}$. The result given by Sowell (1990) was later extended to $m \geq 1$ by Liu (1998) without improvement of the moment conditions. With innovations being a class of stationary Gaussian processes, Chan and Terrin (1995) also discussed a functional limit theorem for general nonstationary fractionally integrated processes. Chan and Terrin's results extend those given by Chan and Wei (1988), Parks and Phillips (1988, 1989) and Sims, et al. (1990) from the domain of integer m's (i.e., $d_0 = 0$) to fractional $d_0 + m$'s. We will extend the results cited, mainly given by Sowell (1990) and Liu (1998), to the more general process X_t defined by (1.0.11) and establish the results only under $E|\epsilon_0|^r < \infty$, where $r = \max\{2, 2/(1+2d_0)\}$. This provides a unified treatment for previous studies on the functional limit theorem for summable limit processes, fractional processes and nonstationary fractionally integrated processes. On the other hand, the moment conditions used in Chapter 2 are also quite weak, and in particular we give the best possible moment condition $E\epsilon_0^2 < \infty$ when $d_0 \geq 0$. The functional limit theorem for summable linear processes can be found in Hannan (1979), Phillips and Solo (1992)

as well as Chan and Tasy (1996).

Functional limit theorems for linear process with square summable weights have been proven by Davydov (1970), Gorodetsskii (1977), Taqqu (1975), Avram and Taqqu (1987) as well as Mielniczuk (1997). We note that a simple stationary fractional process can be explicitly denoted as a linear process with square summable weights. Indeed, Sowell (1990) used this fact in establishing his result. However, the results cited cannot be applied directly to the general fractionally integrated process X_t defined by (1.0.11) even with $m = 0$ because of its complexity. Here, we give a new proof for our results in Chapter 2.

By using established functional limit theorems, Chapter 2 also discusses asymptotics of sample autocovariance and autocorrelation based on the nonstationary fractionally integrated processes. Related results can be found in Hosking (1996), Hasza (1980) and Hassler (1997).

In Chapter 3, we discuss the asymptotics of the following process:

$$Z_t = \sum_{k=0}^{t-1} \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} u_{t-k}, \qquad t = 1, 2, ..., \tag{1.0.12}$$

where $d > 1/2$, ϵ_j are iid random variables and

$$u_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}, \quad \sum_{k=0}^{\infty} |\psi_k| < \infty, \quad b_\psi \equiv \sum_{k=0}^{\infty} \psi_k \neq 0.$$

This process is closely related to X_t defined by (1.0.11). According to the definition of the fractional difference operator $(1-B)^\gamma$, the process X_t defined by (1.0.11) satisfies (let $d = d_0 + m$)

$$\sum_{j=0}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)} X_{t-j} = u_t, \qquad t = 1, 2, ... \tag{1.0.13}$$

From (1.0.13), it is clear that the process X_t depends on a term that is usually called prehistorical influence:

$$\sum_{j=t}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)} X_{t-j}.$$

In practice, if we only consider the process X_t defined by (1.0.11) starting at a given initial date, such as $t = 1$, we may assume that $X_j = 0$ for $j \leq 0$. In this case, after some algebra (for details, see Chapter 3), it can be shown the process X_t defined by (1.0.11) is a special case of the process defined by (1.0.12).

The asymptotic behaviour of the process Z_t was first investigated by Aknom and Gourieroux (1987) with $u_t = \epsilon_t$ (i.e., $\psi_0 = 1$ and $\psi_k = 0$ for $k \geq 1$ in (1.0.12)) under $E|\epsilon_0|^r < \infty$, where $r > \max\{2, 2/(2d-1)\}$. The results of Aknom and Gourieroux (1987) later were extended to the multivariate case by Marinucci and Robission (2000) without any improvements on the moment condition. More recently, Tanaka (1999) discussed a functional limit theorem for a more general process $Z_{[nt]}$ where the ψ_k satisfy $\sum_{k=0}^{\infty} k|\psi_k| < \infty$. The proof of the result given by Tanaka depends on the functional limit theorem for martingale differences. Unfortunately, the process Z_t itself is not a martingale. Therefore, the proof in Tanaka (1999) fails in this case. In Chapter 3, we give a different proof for the case. For more general models, we establish a similar result only under the moment condition $E|\epsilon_0|^{\max\{2,2/(2d-1)\}} < \infty$. It should be pointed out that the limit process of $Z_{[nt]}/Var^{1/2}(Z_n)$ is different from those established in Chapter 2 because of the "prehistorical influence". By using established results, in the same chapter, we also consider the asymptotic behaviour of sample autocovariances and autocorrelations based on the process Z_t. These results do not appear in the previous literature.

As mentioned before, several authors, such as Davydov (1970), Gorodetsskii (1977),

Taqqu (1975), Avram and Taqqu (1987) as well as Mielniczuk (1997), have discussed functional limit theorems for linear processes with square summable weights. However, all of the research cited has been confined to the cases where the innovations are independent with common variance. Frequently, it is assumed that innovations are iid $(0, \sigma^2)$ or, further, that they are iid $N(0, \sigma^2)$. Few results show what would happen if innovations are dependent random variables.

In Chapter 4, functional limit theorems for linear processes with dependent innovations will be investigated. For the long memory linear process, the innovations are assumed to be a sequence of stationary ergodic martingale differences. As a corollary, we derive a functional limit theorem for the process X_t defined by (1.0.11) with the innovations u_t being a sequence of stationary ergodic martingale differences. We also give quite general sufficient conditions so that the partial sum process of a summable linear process converges to a standard Brownian motion. These conditions are quite different from those given by Stadtmuler and Trauter (1985). By using these general results, the functional limit theorem is derived for summable linear processes with innovations being martingale differences and mixing sequences.

In Chapter 5, we continue to discuss sufficient conditions so that the partial sum process of a linear process converges to a standard Brownian motion. We establish two basic results. The first result is under the condition that innovations are iid random variables, but does not require that the weight of a linear process be summable. We note that, to make the partial sum process of a linear process converge to standard Brownian motion, the condition that the weight is summable is commonly used in the previous research. The second result is for the situation where the innovations form a martingale difference. For this result, the commonly used assumption of equal

variance of innovations is weakened. References can be found in Hannan (1979), Phillips and Solo (1992), Stadtmuler and Trauter (1985) and Yokoyama (1995).

Finally, in Chapter 6, we apply the results established in Chapters 2-5 to several examples in economic time series, namely, testing for unit roots, testing for stationarity and time series regression. Such applications have been studied before by many authors in recent years. Here we cite Said and Dickey (1984), Phillips (1987), Hall (1989), Sowell (1990), Chan and Wei (1988), Kwiatkowski, et al. (1992) Chan and Terrin (1995), Dehlhaus (1995), Deo (1997), Phillips and Xiao (1998) as well as Tanaka (1999). More references can be found in Chapter 6.

As shown in Chapter 6, applications of the results established in the previous chapters to the related statistics can lead to better results under weak conditions. Explicitly, by applying Theorems 2.2.1, 2.2.2 and 2.3.1, Chapter 6 derives the limit distribution of the Dickey-Fuller test statistic when the error process is a general fractionally integrated process. Theorem 6.1.4, under quite weak moment conditions, provides a unified treatment of the previous cited results on the summable linear processes and fractional processes. By applying Theorem 5.2.1, we give the limit distributions of the Dickey-Fuller test statistic and the KPSS test statistic when the error process is a linear process that does not necessarily have absolutely summable coefficients.

In Chapter 6, we also derive that "long-run variance", σ^2, can be consistently estimated by a nonparametric method with a lag-truncation parameter l_n of $o(n)$. In the previous research, it was usually assumed to be of $o(n^{1/2})$. This result provides more choice for the estimate of σ^2 and is theoretically interesting.

Chapter 2

Asymptotics for general fractionally integrated processes

In this chapter, functional limit theorems for general fractional processes and nonstationary fractionally integrated processes, under quite weak conditions, are derived. Asymptotic distributions of sample autocovariances and autocorrelations based on nonstationary fractionally integrated processes are also discussed.

2.1 Introduction

Consider a ARFIMA process $\{X_t\}$ defined by

$$(1-B)^{d_0+m}X_t = u_t, \qquad \phi(B)u_t = \theta(B)\epsilon_t, \tag{2.1.1}$$

where $m \geq 0$ is an integer and $d_0 \in (-1/2, 1/2)$; ϵ_t are iid random variables with zero mean and finite variance; $\phi(B)$ and $\theta(B)$ are polynomial functions of B with order p and q respectively and both of them only have roots outside the unit circle, i.e., the ARMA(p,q) process u_t is stationary and invertible. The fractional difference operator $(1-B)^\gamma$ is defined by its Maclaurin series (by its binomial expansion, if γ

is an integer):

$$(1-B)^{\gamma}=\sum_{j=0}^{\infty}\frac{\Gamma(-\gamma+j)}{\Gamma(-\gamma)\Gamma(j+1)}B^{j}, \tag{2.1.2}$$

where $\Gamma(z) = \begin{cases} \int_0^{\infty} s^{z-1}e^{-z}ds & \text{if } z>0 \\ \infty & \text{if } z=0 \end{cases}$. If $z<0$, $\Gamma(z)$ is defined by the recursion formula $z\Gamma(z)=\Gamma(z+1)$.

Since model (2.1.1) was introduced by Granger and Joyeux (1980) and Hosking (1981), it has become very popular in applications. It nests the usual Box-Jenkins ARIMA model and has an ability to capture both short term dynamics and a wide variety of low-frequency behaviour at the same time. Also, there is considerable evidence on the success of applying ARFIMA model to describe financial data such as forward premiums, interest rate differentials, and inflation rates. Illustrations can be found in the survey and review papers of Robinson (1994) and Baillie (1996).

Because of their applications in economics and finance, ARFIMA processes have been studied quite extensively in recent years. In model (2.1.1), it is well-known that the process is stationary and invertible when $m=0$ (Hosking, 1981 and Odaki, 1993); when $m \geq 1$, the process is nonstationary; in particular, when $d_0=0$ and m is an integer, the process becomes a usual unit root process. For estimates of the parameter d_0+m and other related statistical inference, because the literature is rather extensive, we here only refer to Hosking (1984), Li and Mcleod (1986), Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990), Beran (1995) and Beran et al (1998). For more results, see the references cited in these papers and a review book of Beran (1994).

As to the asymptotics of the ARFIMA processes, Sowell (1990) first derived a result that the partial sum process of a fractional process (i.e., $m=0$ in model (2.1.1))

converges weakly to a "type I" fractional Brownian motion[1] on $D[0,1]$ instead of the standard Brownian motion. As a basic tool, combined with the continuous mapping theorem, Sowell's result is quite useful in characterizing the limit distributions of various statistics arising from statistical inference in economic time series related with fractional processes, such as spurious regression and testing for unit roots, stationary and cointegration. Indeed, these applications have been developed by many statisticians and economists such as Sowell (1990), Cheung and Lai (1993), Lee and Schmidt (1996) as well as Cappuccio and Lubian (1997). The extensions of Sowell's result to nonstationary fractionally integrated processes (i.e., $m \geq 1$) can be found in Chan and Terrin (1995) and Liu (1998), where these authors also apply their results to nonstationary fractional unit root tests. Further details are given in Chapter 6.

Despite the well-known works which have be done in connection with Sowell's original results, Sowell (1990) only provided a weak convergence result on simple fractional processes (i.e., $m = 0$, $\phi(B) \equiv \theta(B) \equiv 1$ in model (2.1.1)). This shortcoming limits the applicability of Sowell's result to statistical inference in economic time series. For example, by Sowell's result, it is impossible to consider a unit root test for a model with error being a fractional ARMA process (i.e., $m = 0$, but $\phi(B), \theta(B) \neq 1$ in model (2.1.1)). As is well-known, this problem is important from a practical point of view.

Motivated by characterizing the unit root distribution in a more general model, this chapter extends the weak convergence result given by Sowell (1990) to the general fractional processes and the general nonstationary fractionally integrated processes. Instead of assuming the innovations u_t in model (2.1.1) being an ARMA(p, q) process,

[1] Definition can be found later in this section. For a correction of Sowell's Theorem 2, see Theorem 1.1 given by Liu (1998).

we allow it to be a more general linear process. Therefore, this chapter provides a unified treatment for previous studies on weak convergence for summable linear processes, fractional processes and nonstationary fractionally integrated processes. The weak convergence results for summable linear processes can be found in Hannan (1979), Phillips and Solo (1992) as well as Chan and Tsay (1996).

In Chapter 6, the results given in this chapter will be used to derive the limit distribution of the least square estimate (LSE) of the coefficient for a AR(1) model when true coefficient is 1 (i.e., the true model has a unit root) and the error process is a general fractional process or a general nonstationary fractionally integrated process. The result also provides a unified treatment of the unit root test with the errors being a summable linear process, a fractional process and a nonstationary fractionally integrated process.

The main results of this chapter hold under quite weak moment conditions for the innovations ϵ_t. For example, the weak convergence for nonstationary fractionally integrated processes is derived whenever the innovations ϵ_t have finite second moment. Such a condition is the best possible moment condition in the literature and it is interesting from a theoretical point of view.

This chapter is organized as follows. In the next section, we first give our main results without proof and compare them to the previous related results in the literature. In Section 2.3, applications of these results to nonstationary fractionally integrated processes will be presented. In Section 2.4, we extend the main results to $d_0 = 1/2$ and $m \geq 0$. As shown in Liu (1998), the behavior of $\{X_t\}$ in (2.1.1) is different for $d_0 = 1/2$ and $d_0 \in (-1/2, 1/2)$. In Section 2.5, we discuss the asymptotics of sample autocovariances and autocorrelations based on nonstationary fractionally integrated

processes. Finally in Section 2.6, we give the proofs of the main results.

We end this section with some further notation. We denote positive constants by $C, C_1, \ldots$, which may take on different values in different places. The "type I" fractional Brownian motion with $-1/2 < d_0 < 1/2$ on $D[0,1]$ is defined as follows:

$$W_{d_0}(t) = \frac{1}{A(d_0)} \int_{-\infty}^{0} \left[(t-s)^{d_0} - (-s)^{d_0}\right] dW(s) + \int_0^t (t-s)^{d_0} dW(s),$$

where $W(s)$ is a standard Brownian motion and

$$A(d_0) = \left(\frac{1}{2d_0+1} + \int_0^{\infty} \left[(1+s)^{d_0} - s^{d_0}\right]^2 ds \right)^{1/2}.$$

Clearly, $W_{d_0}(t)$ is a self-similar Gaussian process with covariance

$$EW_{d_0}(s)W_{d_0}(t) = \frac{1}{2}\left\{s^{1+2d_0} + t^{1+2d_0} - |s-t|^{1+2d_0}\right\}, \qquad \text{for } 0 \leq s,t \leq 1.$$

A more general definition of fractional Brownian motion can be found in Mandelbrot and Van Ness (1968), Samorodnitsky and Taqqu (1994) and Marinucci and Robinson (1999).

2.2 General stationary fractional processes

From here on, we discuss the following general fractionally integrated process X_t defined by

$$(1-B)^{d_0+m} X_t = u_t, \qquad u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad t = 1, 2, \ldots, \tag{2.2.1}$$

where $m \geq 0$ is an integer and $d_0 \in (-1/2, 1/2)$; $(1-B)^{d_0+m}$ is defined by (2.1.2); $\epsilon_j, j = 0, \pm 1, \ldots$ are iid random variables with $E\epsilon_0 = 0$ and $\psi_j, j \geq 0$, is a sequence of real numbers to be specified later.

The two theorems in this section derive results on the weak convergence of general stationary fractional processes (i.e., the process X_t defined by (2.2.1) with $m = 0$). They provide a unified treatment for the cases of fractional processes and summable linear processes, and basic tools for later discussion.

Theorem 2.2.1. *Let X_j satisfy (2.2.1) with $m = 0$, and $\psi_j, j \geq 0$, satisfy*

$$\sum_{j=0}^{\infty} |\psi_j| < \infty \quad \text{and} \quad b_\psi \equiv \sum_{j=0}^{\infty} \psi_j \neq 0. \tag{2.2.2}$$

Assume that $E\epsilon_0^2 < \infty$. Then, for $0 \leq d_0 < 1/2$,

$$\frac{1}{\kappa(d_0) n^{1/2+d_0}} \sum_{j=1}^{[nt]} X_j \Rightarrow W_{d_0}(t), \quad 0 \leq t \leq 1, \tag{2.2.3}$$

where $\kappa^2(d_0) = \frac{b_\psi^2 \, E\epsilon_0^2 \, \Gamma(1-2d_0)}{(1+2d_0)\Gamma(1+d_0)\Gamma(1-d_0)}$ and $W_{d_0}(t)$ is a "type I" fractional Brownian motion on $D[0,1]$.

If in addition $E|\epsilon_0|^{(2+\delta)/(1+2d_0)} < \infty$, where $\delta > 0$, then (2.2.3) still holds for $-1/2 < d_0 < 0$.

For $0 \leq d_0 < 1/2$, Theorem 2.2.1 gives the result under the best possible moment condition $E\epsilon_0^2 < \infty$. If there is a slightly stronger restriction for ψ_k, for $-1/2 < d_0 < 0$, the moment condition in Theorem 2.2.1 can be weakened to $E|\epsilon_0|^{2/(1+2d_0)} < \infty$. Explicitly, we have the following Theorem 2.2.2.

Theorem 2.2.2. *Let X_j satisfy (2.2.1) with $m = 0$, and $\psi_j, j \geq 0$, satisfy*

$$\sum_{j=0}^{\infty} j^{1/2-d_0} |\psi_j| < \infty \quad \text{and} \quad b_\psi \equiv \sum_{j=0}^{\infty} \psi_j \neq 0. \tag{2.2.4}$$

Assume that $E|\epsilon_0|^{\max\{2, 2/(1+2d_0)\}} < \infty$. Then (2.2.3) holds for $d_0 \in (-1/2, 1/2)$.

The proofs of Theorems 2.2.1-2.2.2 are postponed to Section 2.6.

If u_t is a process satisfying $\phi(B)u_t = \theta(B)\epsilon_t$, where polynomials $\phi(B)$ and $\theta(B)$ with order p and q respectively, have only roots outside the unit circle, Theorem 3.1.1 of Brockwell and Davis (1987, p 85) implies that $u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ with $|\psi_k| \le Ca^{-k}, k \ge 0$, where $a > 1$, and $\sum_{k=0}^{\infty} \psi_k = \theta(1)/\phi(1)$. Therefore, the following Corollary 2.2.3 is a direct consequence of Theorem 2.2.2.

Corollary 2.2.3. *Let X_j satisfy (2.1.1) with $m = 0$. If $E|\epsilon_0|^{\max\{2,2/(1+2d_0)\}} < \infty$, then (2.2.3) follows with $b_\psi = \theta(1)/\phi(1)$ for $d_0 \in (-1/2, 1/2)$.*

Remark 2.2.1. If $\phi(B) \equiv \theta(B) \equiv 1$ in model (2.1.1), Corollary 2.2.3 reduces to Theorem 2 in Sowell (1990), where the author derived (2.2.3) provided $E|\epsilon_t|^r < \infty$ for $r \ge \max\{4, -8d_0/(1+2d_0)\}$. If $d_0 = 0$, then $X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ with $\sum_{j=0}^{\infty} |\psi_j| < \infty$. In this case $\kappa^2(0) = b_\psi^2 E\epsilon_0^2$ and $W_0(t)$ is a standard Brownian motion on $D[0,1]$. Thus, Hannan's (1979) result becomes a special case of Theorem 2.2.1. Theorem 3.4 of Phillips and Solo (1992) and Theorem 2.5 of Chan and Tsay (1996) also gave similar results but imposing more restrictions on ψ_j.

Remark 2.2.2. Davydov (1970) and later Gorodetsskii (1977), Taqqu (1975), Avram and Taqqu (1987) as well as Mielniczuk (1997) gave results on the functional limit theorem for linear process with square summable weights. Theorems 2.2.1-2.2.2 are not a direct consequence of the papers cited above in terms of complexity of the processes satisfying (2.2.1). In fact, we give a totally new proof for our results.

2.3 General nonstationary fractionally integrated processes

As mentioned before, Theorems 2.2.1-2.2.2 are quite useful in characterizing limit distribution of various statistics arising from statistical inference related with general fractional processes. In this section, Theorems 2.2.1-2.2.2 are used to derive functional limit theorem on the general nonstationary fractionally integrated processes (i.e., the process X_t defined by (2.2.1) with $m \geq 1$). It is interesting to note that the results (Theorem 2.3.1 below) are established only under $E|\epsilon_0|^{\max\{2,2/(1+2d_0)\}} < \infty$. Specially, we obtain the best possible moment condition when $d_0 \geq 0$.

In previous research, the functional limit theorem on the general nonstationary fractionally integrated processes has been discussed in a very general framework by Chan and Terrin (1995) under the assumption that u_t defined in (2.2.1) is a class of stationary Gaussian processes. The result of Chan and Terrin (1995) extends the results of Chan and Wei (1988), Parks and Phillips (1988, 1989), and Sims et al.(1990) from the domain of integer m's (i.e., $d_0 = 0$) to fractional $d_0 + m$'s. More recently, Liu (1998) derived a functional limit theorem on the simple nonstationary fractionally integrated processes (i.e., the process X_t defined by (2.2.1) with $m \geq 1$ and $u_t = \epsilon_t$) provided $E|\epsilon_t|^r < \infty$ for $r \geq \max\{4, -8d_0/(1+2d_0)\}$. Theorem 2.3.1 below gives an essential improvement of the results cited before.

More applications of Theorems 2.2.1-2.2.2 can be found in Section 2.6 and Chapter 6. In there, we discuss asymptotics of sample autocovariances and autocorrelations, and testing for unit root respectively.

We now turn to our main result. For convenience of reading, we introduce the

following conditions.

Condition A: $\psi_j, j \geq 0$, satisfy (2.2.2) and $E|\epsilon_0|^p < \infty$, where $p = 2$, for $0 \leq d_0 < 1/2$; $p = (2+\delta)/(1+2d_0) < \infty, \delta > 0$, for $-1/2 < d_0 < 0$.

Condition B: $\psi_j, j \geq 0$, satisfy (2.2.4) and $E|\epsilon_0|^{\max\{2,2/(1+2d_0)\}} < \infty$.

Theorem 2.3.1. *Let X_j satisfy (2.2.1) with $m \geq 1$. Let Condition A or Condition B hold. Then, for $d_0 \in (-1/2, 1/2)$,*

$$\frac{1}{\kappa(d_0)n^{-1/2+d_0+m}} X_{[nt]} \Rightarrow W_{d_0,m}(t), \qquad 0 \leq t \leq 1, \tag{2.3.1}$$

$$\frac{1}{\kappa(d_0)n^{1/2+d_0+m}} \sum_{j=1}^{[nt]} X_j \Rightarrow W_{d_0,m+1}(t), \qquad 0 \leq t \leq 1. \tag{2.3.2}$$

Furthermore, we have that, for any fixed integer $k \geq 0$,

$$\frac{1}{\kappa^2(d_0)n^{2(d_0+m)}} \sum_{j=1}^{n-k} X_j^2 \Rightarrow \int_0^1 \left(W_{d_0,m}(s)\right)^2 ds, \tag{2.3.3}$$

$$\frac{1}{\kappa(d_0)n^{-1/2+d_0+m}} \sum_{j=1}^{n-k} (X_{j+k} - X_j) \Rightarrow k\ W_{d_0,m}(1), \tag{2.3.4}$$

$$\frac{1}{n} \sum_{j=1}^{n-k} (X_{j+k} - X_j)^2 \rightarrow_{a.s.} E\left(X_{k+1} - X_1\right)^2, \quad if \quad m = 1, \tag{2.3.5}$$

$$\frac{1}{n^{-1+2(d_0+m)}} \sum_{j=1}^{n-k} (X_{j+k} - X_j)^2 = o_P(1), \quad if \quad m \geq 2. \tag{2.3.6}$$

where $\kappa(d_0)$, $W_{d_0}(t)$ are defined as in Theorem 2.2.1 and

$$W_{d_0,m}(t) = \begin{cases} W_{d_0}(t), & if \quad m = 1, \\ \int_0^t \int_0^{t_{m-1}} \ldots \int_0^{t_2} W_{d_0}(t_1)dt_1 dt_2 \ldots dt_{m-1}, & if \quad m \geq 2. \end{cases}$$

Proof. Put $Y_j = X_j - X_{j-1}, j = 2, 3, \ldots$. Then,

$$X_j \;=\; X_1 + \sum_{i=2}^{j} Y_i, \qquad j = 2, 3, \ldots, \tag{2.3.7}$$

and by the definition of the lag operator,

$$(1-B)^{d_0+m-1} Y_t \;=\; (1-B)^{d_0+m} X_t = u_t, \quad t = 2, 3, \ldots \tag{2.3.8}$$

We first prove (2.3.1). Since X_1 is a random variable, clearly we have that, for $m \geq 1$,

$$|X_1| / \left(\kappa(d_0) n^{-1/2+d_0+m}\right) \to_P 0, \quad \text{as } n \to \infty. \tag{2.3.9}$$

This, together with Theorems 2.2.1-2.2.2, implies that if $m = 1$, then for $0 \leq t \leq 1$,

$$\begin{aligned} \frac{1}{\kappa(d_0) n^{1/2+d_0}} X_{[nt]} \;&=\; \frac{1}{\kappa(d_0) n^{1/2+d_0}} X_1 + \frac{1}{\kappa(d_0) n^{1/2+d_0}} \sum_{i=2}^{[nt]} Y_i \\ &\Rightarrow\; W_{d_0}(t) = W_{d_0,1}(t), \end{aligned}$$

i.e., (2.3.1) holds for $m = 1$.

Let us assume (2.3.1) hold for $m = k$. By induction, it suffices to show (2.3.1) also holds for $m = k + 1$. In terms of (2.3.8) and the assumption that (2.3.1) holds for $m = k$, it is clear that, for $m = k + 1$,

$$\frac{1}{\kappa(d_0) n^{-1/2+d_0+k}} Y_{[nt]} \;\Rightarrow\; W_{d,k}(t), \qquad 0 \leq t \leq 1, \tag{2.3.10}$$

From (2.3.10) and the continuous mapping theorem (Theorem A.2, see Appendix A), we obtain that, for $m = k + 1$,

$$\begin{aligned} \frac{1}{\kappa(d_0) n^{1/2+d_0+k}} \sum_{j=2}^{[nt]} Y_j \;&=\; \int_{2/n}^{t} \left(\frac{1}{\kappa(d_0) n^{-1/2+d_0+k}} Y_{[ns]} \right) ds \\ &\Rightarrow\; \int_0^t W_{d_0,k}(t_k) dt_k = W_{d_0,k+1}(t). \end{aligned} \tag{2.3.11}$$

Now it follows from (2.3.7), (2.3.9) and (2.3.11) that, for $m = k+1$,

$$\frac{1}{\kappa(d_0)n^{1/2+d_0+k}} X_{[nt]} = \frac{1}{\kappa(d_0)n^{1/2+d_0+k}} \left(X_1 + \sum_{i=2}^{[nt]} Y_i \right) \Rightarrow W_{d_0,k+1}(t).$$

This gives (2.3.1) for $m = k+1$ and hence the proof of (2.3.1) is complete.

By using (2.3.1), the proofs of (2.3.2) and (2.3.3) are similar to (2.3.11) and hence details are omitted. The proof of (2.3.4) follows easily from (2.3.1), (2.3.9) and for each fixed $k \geq 0$,

$$\sum_{j=1}^{n-k} (X_{j+k} - X_k) = \sum_{j=n-k+1}^{n} X_j - \sum_{j=1}^{k} X_j.$$

To prove (2.3.5) and (2.3.6), we note that, for each fixed $k \geq 0$,

$$X_{j+k} - X_j = \sum_{i=j+1}^{j+k} Y_i, \quad j = 1, 2, \ldots. \tag{2.3.12}$$

If $m = 1$, it follows from (2.3.8) and Lemma 2.6.3 (see Section 2.6) that $Y_i, i \geq 1$ is a stationary ergodic random sequence. Therefore, for each fixed $k \geq 0$, $X_{j+k} - X_j, j \geq 1$ still is a stationary ergodic random sequence (Theorem B.1, see Theorem B). Now the stationary ergodic theorem (Theorem B.2, see Appendix B) implies that

$$\frac{1}{n} \sum_{j=1}^{n-k} (X_{j+k} - X_j)^2 \to_{a.s.} E \left(\sum_{i=2}^{k+1} Y_i \right)^2 = E\left(X_{k+1} - X_1\right)^2.$$

This gives (2.3.5).

Recalling (2.3.12) and Hölder's inequality, we have that

$$\sum_{j=1}^{n-k} (X_{j+k} - X_j)^2 \leq k \sum_{j=1}^{n-k} \sum_{i=j+1}^{j+k} Y_i^2 = k \sum_{i=1}^{k} \sum_{j=1}^{n-k} Y_{i+j}^2.$$

If $m \geq 2$, it follows from (2.3.8) and (2.3.3) that, for each fixed $1 \leq i \leq k$,

$$\frac{1}{\kappa^2(d_0)n^{2(d_0+m-1)}} \sum_{j=1}^{n-k} Y_{i+j}^2 \Rightarrow \int_0^1 (W_{d,m-1}(s))^2 \, ds.$$

This implies (2.3.6) clearly. This completes the proof of Theorem 2.3.1. $\square$

Remark 2.3.1. If $m = 0$, the process X_t defined in model (2.2.1) has a moving average expression that depends on the innovations u_k (see part (a) in Lemma 2.6.3):

$$X_t = \sum_{k=-\infty}^{\infty} c_{t-k} u_k, \quad t = 1, 2, ...,$$

where $c_k = \frac{\Gamma(d_0+k)}{\Gamma(d_0)\Gamma(k+1)}$ for $k \geq 0$ and $c_k = 0$ for $k < 0$. For general $m \geq 1$, if the initial values $X_1, X_2, ..., X_m$ of the process X_t are known, we can obtain a moving average expression of the process X_t by using (2.3.7) and (2.3.8) repeatedly. For example, for $m = 1$, it follows from (2.3.7) that $X_t = X_1 + \sum_{i=2}^{t} Y_i$, where Y_t satisfies (2.3.8). Therefore,

$$X_t = X_1 + \sum_{j=2}^{t} \sum_{k=-\infty}^{\infty} c_{j-k} u_k = X_1 + \sum_{k=-\infty}^{\infty} \left(\sum_{j=2}^{t} c_{j-k} \right) u_k, \quad t \geq 2.$$

Similarly, for general $m \geq 2$, we have

$$\begin{aligned} X_t &= f(t, X_1, X_2, ..., X_m) \\ &\quad + \sum_{k=-\infty}^{\infty} \left(\sum_{j_m=m}^{t} \sum_{j_{m-1}=m}^{j_m} \cdots \sum_{j_1=m}^{j_2} c_{j_1-k} \right) u_k, \quad t \geq m+1, \end{aligned} \tag{2.3.13}$$

where $f(.)$ is a linear function of its variate and by induction it can be easily proved that for $d_0 \in (-1/2, 1/2)$,

$$\frac{1}{j^{m+1/2+d_0}} \left| f(j, X_1, X_2, ..., X_m) \right| \to_P 0, \quad j \to \infty. \tag{2.3.14}$$

As implied in Theorem 2.3.1, the relation (2.3.14) also shows that the finite initial value of the process X_t does not affect its asymptotics.

We notice that, for $m \geq 1$, the moving average expression of the process X_t cannot be obtained by using

$$X_t = (1 - B)^{-d_0 - m} u_t, \tag{2.3.15}$$

where $(1-B)^{-d_0-m} = \sum_{k=0}^{\infty} \frac{\Gamma(d_0+m+k)}{\Gamma(d_0+m)\Gamma(k+1)} B^k$. The main reason for this is that

$$\frac{\Gamma(d_0+m+k)}{\Gamma(d_0+m)\Gamma(k+1)} \sim \frac{1}{\Gamma(d_0+m)} k^{d_0+m-1},$$

and hence the infinite sum $\sum_{k=0}^{\infty} \frac{\Gamma(d_0+m+k)}{\Gamma(d_0+m)\Gamma(k+1)} u_{t-k}$ does not exist.

However, if in (2.3.15) we use u_t^* instead of u_t, where $u_t^* = u_t$ for $t \geq 1$ and $u_t^* = 0$ for $t \leq 0$, we will obtain the following process:

$$\begin{aligned} Y_t &= (1-B)^{-d_0-m} u_t^* \\ &= \sum_{k=0}^{t-1} \frac{\Gamma(d_0+m+k)}{\Gamma(d_0+m)\Gamma(k+1)} u_{t-k} = \sum_{k=1}^{t} c_{t-k}^* u_k, \quad t \geq 1, \end{aligned} \tag{2.3.16}$$

where $c_k^* = \frac{\Gamma(d_0+m+k)}{\Gamma(d_0+m)\Gamma(k+1)}$. Weak convergence of such processes will be discussed in the next Chapter.

2.4 Extensions to $d_0 = 1/2$ and $m \geq 0$

In this section, we discuss weak convergence of the process X_t defined by

$$(1-B)^{1/2+m} X_t = u_t, \qquad u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad t = 1, 2, ..., \tag{2.4.1}$$

where $m \geq 0$ is an integer, $\epsilon_j, j = 0, \pm 1, ...$ are iid random variables with $E\epsilon_0 = 0$ and $\psi_j, j \geq 0$ are a sequence of real numbers specified later. This process is nonstationary and an important complement of the process X_t defined by (2.2.1). For a simple case of the process X_t defined by (2.4.1) (i.e., $\psi_0 = 1$ and $\psi_k = 0$ for $k \geq 0$), Liu (1998) discussed weak convergence for its partial sums process. Liu's results will be extended to general cases in this section. It is interesting to note that behaviour of the process defined by (2.4.1) is quite different from that of the process defined by (2.2.1).

Theorem 2.4.1. *Let X_j satisfy (2.4.1) and $\psi_j, j \geq 0$ satisfy*

$$\sum_{j=0}^{\infty}\left(\sum_{k=j+1}^{\infty}\psi_k\right)^2 < \infty \quad and \quad b_\psi \equiv \sum_{j=0}^{\infty}\psi_j \neq 0. \tag{2.4.2}$$

Assume that $E\epsilon_0^2 < \infty$. Then, for all $m \geq 0$,

$$\frac{(1+m)!}{Kn^{1+m}\log^{1/2}n}\sum_{j=1}^{[nt]}X_j \Rightarrow t^{1+m}W(1), \quad 0 \leq t \leq 1, \tag{2.4.3}$$

where $K^2 = \frac{1}{\pi}b_\psi^2 E\epsilon_0^2$.

Proof. We first prove (2.4.3) for $m = 0$. Let $\Psi(B) = \sum_{j=0}^{\infty}\psi_j B^j$. It follows from Lemma 2.1 given in Phillips and Solo (1992) that

$$\Psi(B) = b_\psi - (1-B)\widetilde{\Psi}(B),$$

where $\widetilde{\Psi}(B) = \sum_{j=0}^{\infty}\widetilde{\psi_j}B^j$ and $\widetilde{\psi_j} = \sum_{k=j+1}^{\infty}\psi_k$. Since ψ_k satisfies (2.4.2), for all t,

$$E\left|\widetilde{\Psi}(B)\epsilon_t\right|^2 = \sum_{j=0}^{\infty}\widetilde{\psi_j}^2 E\epsilon_{t-j}^2 = E\epsilon_0^2\sum_{j=0}^{\infty}\widetilde{\psi_j}^2 < \infty. \tag{2.4.4}$$

This implies that $\widetilde{\epsilon_t} \equiv \widetilde{\Psi}(B)\epsilon_t = \sum_{j=0}^{\infty}\widetilde{\psi_j}\epsilon_{t-j}$ is well-defined. Hence,

$$u_t = \Psi(B)\epsilon_t = b_\psi\epsilon_t - (1-B)\widetilde{\epsilon_t}.$$

Define a process $Y_t = X_t + (1-B)^{1/2}\widetilde{\epsilon_t}$. Then we have that

$$(1-B)^{1/2}Y_t = (1-B)^{1/2}X_t + (1-B)\widetilde{\epsilon_t} = b_\psi\epsilon_t, \qquad t = 1, 2, \ldots.$$

It follows from Theorem 2.2 given by Liu (1998) that

$$\frac{1}{Kn\log^{1/2}n}\sum_{j=1}^{[nt]}Y_j \Rightarrow t\,W(1), \quad \text{where } K^2 = \frac{1}{\pi}b_\psi^2 E\epsilon_0^2.$$

Therefore, by Theorem A.4 (see Appendix A), we only need to show that

$$\sup_{0\le t\le 1}\left|\sum_{j=1}^{[nt]} X_j - \sum_{j=1}^{[nt]} Y_j\right| = \max_{1\le m\le n}\left|\sum_{j=1}^{m}(1-B)^{1/2}\widetilde{\epsilon}_j\right| = o_P\left(n\log^{1/2} n\right). \tag{2.4.5}$$

Recalling (Hosking, 1981)

$$(1-B)^{1/2}\widetilde{\epsilon}_j = \sum_{k=0}^{\infty} c_k \widetilde{\epsilon}_{j-k},$$

where $c_0 = 1$ and $|c_k| \le Ck^{-3/2}$ for $k \ge 1$, it follows that

$$\begin{aligned} E\max_{1\le m\le n}\left|\sum_{j=1}^{m}(1-B)^{1/2}\widetilde{\epsilon}_j\right| &\le C\sum_{j=1}^{n}\left(E|\widetilde{\epsilon}_j| + \sum_{k=1}^{\infty} k^{-3/2}\, E|\widetilde{\epsilon}_{j-k}|\right) \\ &\le C_1 n = o\left(n\log^{1/2} n\right), \end{aligned}$$

where we use (2.4.4) and the following estimate: for all $j, k \ge 0$,

$$E|\widetilde{\epsilon}_{j-k}| \le \left(E|\widetilde{\epsilon}_0|^2\right)^{1/2} < \infty.$$

Now (2.4.5) follows from Markov's inequality. This proves (2.4.3) for $m = 0$. For general $m \ge 1$, the proof of (2.4.3) is similar to that of (2.3.1) by induction and details are omitted. □

2.5 Sample autocovariance and autocorrelation of nonstationary fractionally integrated processes

Let $X_t, t = 1, 2, \ldots$, be a process with mean $EX_t = u$ and lag-k autocovariance $\gamma_k = E(X_t - u)(X_{t+k} - u)$. It is well-known that lag autocovariance γ_k is an important factor in describing the properties of the process X_t; for example we can say that X_t has short memory (long memory) according to $\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty$ $(= \infty)$.

The standard sample autocovariances and related autocorrelations based on a process X_t can be defined as, for $k = 0, 1, 2, ..., n-1$,

$$\begin{aligned}
\widehat{r}_k &= \frac{1}{n}\sum_{t=1}^{n-k}(X_t - \overline{X})(X_{t+k} - \overline{X}), \quad \overline{X} = \frac{1}{n}\sum_{t=1}^{n} X_t \\
\widehat{\rho}_k &= \widehat{r}_k / \widehat{r}_0, \quad \text{or} \quad \widehat{\rho}^*_k = \widehat{r}_k \Big/ \frac{1}{n}\sum_{j=1}^{n-k}(X_j - \overline{X})^2.
\end{aligned}$$

It is well-known that the sample autocorrelation $\widehat{\rho}^*_k$ is the OLS estimator of the coefficient β in the auxiliary regression $X_t - \overline{X} = \beta(X_{t-k} - \overline{X}) + v_t$. The advantage of $\widehat{\rho}_k$ over $\widehat{\rho}^*_k$ is that $\widehat{\rho}_k$ is bounded between -1 and 1.

Asymptotic distributions of sample autocovariances and autocorrelations have been studied extensively in recent years under different sets of assumptions. Here we focus on the situation in which the process X_t is a general fractionally integrated process defined by (2.2.1). In this case, Hannan (1976) and Hosking (1996) discussed asymptotics of $\widehat{r}_k$, $\widehat{\rho}_k$ and $\widehat{\rho}^*_k$ for linear processes and stationary $ARIMA(p, d_0, q)$ processes (i.e., the process X_t defined by (2.1.1) with $m = 0$) respectively. For the non-stationary fractionally integrated process, the first result can be found in Hastz (1980), where the author considered the simplest nonstationary process X_t defined by (2.2.1) with $d_0 = 0$, $m = 1$ and $u_t = \epsilon_t, t \geq 1$. The result given by Hartz (1980) was later extended by Bierens (1993) to the case that u_t is a mixing sequence. More recently, Hassler (1994, 1997) derived asymptotic distribution of $\widehat{r}_k$, $\widehat{\rho}_k$ and $\widehat{\rho}^*_k$ when the process X_t satisfies (2.2.1) with $d_0 \in (-1/2, 1/2)$, $m = 1$ and $u_t = \epsilon_t, t \geq 1$.

By using Theorems 2.2.1-2.3.1, in this section, the results given by Hassler (1994 1997) will be extended to more general processes under quite general conditions.

We continue to use the notation $W_{d_0,m}(t)$ and Conditions A and B defined in

Section 2.3. For convenience of reading, we rewrite them as follows:

$$W_{d_0,m}(t) = \begin{cases} W_{d_0}(t), & \text{if } m = 1, \\ \int_0^t \int_0^{t_{m-1}} \dots \int_0^{t_2} W_{d_0}(t_1) dt_1 dt_2 \dots dt_{m-1}, & \text{if } m \geq 2, \end{cases}$$

where $W_{d_0}(t)$ is a "type I" fractional Brownian motion on $D[0,1]$.

Condition A: $\psi_j, j \geq 0$, satisfy (2.2.2) and $E|\epsilon_0|^p < \infty$, where $p = 2$, for $0 \leq d_0 < 1/2$; $p = (2+\delta)/(1+2d_0) < \infty, \delta > 0$, for $-1/2 < d_0 < 0$.

Condition B: $\psi_j, j \geq 0$, satisfy (2.2.4) and $E|\epsilon_0|^{\max\{2,2/(1+2d_0)\}} < \infty$.

Theorem 2.5.1. *Let X_j satisfy (2.2.1) with $m \geq 1$. Assume that Condition A or Condition B holds, $d_0 \in (-1/2, 1/2)$ and $k \geq 0$ is a integer. Then,*

$$\kappa^{-2}(d_0) n^{1-2(d_0+m)} \, \widehat{r}_k \;\Rightarrow\; \widetilde{W}_{d_0,m}(1), \tag{2.5.1}$$

where $\widetilde{W}_{d_0,m}(1) = \int_0^1 W_{d_0,m}^2(s)ds - \left(\int_0^1 W_{d_0,m}(s)ds\right)^2$.

Furthermore, if $m \geq 2$ or $m = 1$ and $0 < d_0 < 1/2$, then

$$n(\widehat{\rho}_k^* - 1) \;\Rightarrow\; \frac{k}{2} \, \frac{W_{d_0,m}^2(1) - 2W_{d_0,m}(1)\, W_{d_0,m+1}(1)}{\widetilde{W}_{d_0,m}(1)}, \tag{2.5.2}$$

$$n(\widehat{\rho}_k - 1) \;\Rightarrow\; -\frac{k}{2} \, \frac{W_{d_0,m+1}^2(1) + \left(W_{d_0,m}(1) - W_{d_0,m+1}(1)\right)^2}{\widetilde{W}_{d_0,m}(1)}; \tag{2.5.3}$$

if $m = 1$ and $d_0 = 0$, then

$$n(\widehat{\rho}_k^* - 1) \;\Rightarrow\; -\frac{E(X_{k+1} - X_1)^2}{2\kappa^2(0)\, \widetilde{W}_{0,1}(1)} + \frac{k}{2} \, \frac{W_{0,1}^2(1) - 2W_{0,1}(1)\, W_{0,2}(1)}{\widetilde{W}_{0,1}(1)}, \tag{2.5.4}$$

$$n(\widehat{\rho}_k - 1) \;\Rightarrow\; -\frac{E(X_{k+1} - X_1)^2}{2\kappa^2(0)\, \widetilde{W}_{0,1}(1)} - \frac{k}{2} \, \frac{W_{0,2}^2(1) + \left(W_{0,1}(1) - W_{0,2}(1)\right)^2}{\widetilde{W}_{0,1}(1)}; \tag{2.5.5}$$

if $m = 1$ *and* $-1/2 < d_0 < 0$, *then*

$$n^{1+2d_0}(\widehat{\rho}_k^* - 1) \;\Rightarrow\; -\frac{E(X_{k+1} - X_1)^2}{2\kappa^2(d_0)\,\widetilde{W}_{d_0,1}(1)}, \tag{2.5.6}$$

$$n^{1+2d_0}(\widehat{\rho}_k - 1) \;\Rightarrow\; -\frac{E(X_{k+1} - X_1)^2}{2\kappa^2(d_0)\,\widetilde{W}_{d_0,1}(1)}. \tag{2.5.7}$$

Proof. We can write

$$\begin{aligned}
&\sum_{t=1}^{n-k}(X_t - \overline{X})(X_{t+k} - \overline{X}) \\
&= \sum_{t=1}^{n-k} X_t(X_{t+k} - X_t) - \overline{X}\sum_{t=1}^{n-k}(X_{t+k} - X_t) + \sum_{t=1}^{n-k}(X_t - \overline{X})^2.
\end{aligned} \tag{2.5.8}$$

Under the conditions of Theorem 2.5.1, by using Theorem 2.3.1 and the continuous mapping theorem, it can be easily shown that (recalling (2.3.2)-(2.3.4) and k is fixed),

$$\begin{aligned}
&\frac{1}{\kappa^2(d_0)n^{-1+2(d_0+m)}}\;\overline{X}\sum_{t=1}^{n-k}(X_{t+k} - X_t) \\
&\qquad \Rightarrow\; k\; W_{d_0,m}(1)W_{d_0,m+1}(1),
\end{aligned} \tag{2.5.9}$$

and

$$\begin{aligned}
&\frac{1}{\kappa^2(d_0)n^{2(d_0+m)}}\sum_{t=1}^{n-k}(X_t - \overline{X})^2 \\
&= \frac{1}{\kappa^2(d_0)n^{2(d_0+m)}}\left(\sum_{t=1}^{n-k}X_t^2 - 2\overline{X}\sum_{t=1}^{n-k}X_t + (n-k)\overline{X}^2\right) \\
&\Rightarrow \int_0^1 W_{d_0,m}^2(s)ds - \left(\int_0^1 W_{d_0,m}(s)ds\right)^2 \\
&= \widetilde{W}_{d_0,m}(1).
\end{aligned} \tag{2.5.10}$$

On the other hand, by noting

$$\begin{aligned}
\sum_{t=1}^{n-k} X_t(X_{t+k} - X_t) &= \frac{1}{2}\sum_{t=1}^{n-k}\left(X_{t+k}^2 - X_t^2 - (X_{t+k} - X_t)^2\right) \\
&= \frac{1}{2}\sum_{t=n-k+1}^{n} X_t^2 - \frac{1}{2}\sum_{t=1}^{k}X_t^2 - \frac{1}{2}\sum_{t=1}^{n-k}(X_{t+k} - X_t)^2,
\end{aligned}$$

we obtain that if $m \geq 2$ or $m = 1$ and $d_0 > 0$ (recalling (2.3.1) and (2.3.6)), then

$$\frac{1}{\kappa^2(d_0)n^{-1+2(d_0+m)}} \sum_{t=1}^{n-k} X_t(X_{t+k} - X_t) \Rightarrow \frac{k}{2}\, W^2_{d_0,m}(1); \tag{2.5.11}$$

if $m = 1$ and $-1/2 < d_0 < 0$ (recalling (2.3.1) and (2.3.5)), then

$$\frac{1}{n}\sum_{t=1}^{n-k} X_t(X_{t+k} - X_t) \quad \to_{a.s.} \quad -\frac{1}{2}\, E(X_{k+1} - X_1)^2; \tag{2.5.12}$$

if $m = 1$ and $d_0 = 0$ (recalling (2.3.1) and (2.3.5)), then

$$\frac{1}{\kappa^2(0)n} \sum_{t=1}^{n-k} X_t(X_{t+k} - X_t) \Rightarrow \frac{k}{2}\, W^2_{d_0,m}(1) - \frac{1}{2\kappa^2(0)}\, E(X_{k+1} - X_1)^2. \tag{2.5.13}$$

In terms of (2.5.8)-(2.5.13) and

$$\widehat{\rho}^*_k - 1 = \left(\sum_{t=1}^{n-k} X_t(X_{t+k} - X_t) - \overline{X} \sum_{t=1}^{n-k}(X_{t+k} - X_t)\right) \Big/ \sum_{t=1}^{n-k}(X_t - \overline{X})^2,$$

(2.5.1), (2.5.2), (2.5.4) and (2.5.6) follow easily from the continuous mapping theorem.

To prove (2.5.3), (2.5.5) and (2.5.7), write

$$d_k = \sum_{t=1}^{n-k}(X_t - \overline{X})^2 \Big/ \sum_{t=1}^{n}(X_t - \overline{X})^2.$$

Similar methods to those used in (2.5.9) and (2.5.10) show that

$$\begin{aligned} n(d_k - 1) &= -n \sum_{t=n-k+1}^{n} (X_t - \overline{X})^2 \Big/ \sum_{t=1}^{n}(X_t - \overline{X})^2 \\ &= \frac{n}{\sum_{t=1}^{n}(X_t - \overline{X})^2}\left(2\overline{X} \sum_{t=n-k+1}^{n} X_t - \sum_{t=n-k+1}^{n} X_t^2 - k\overline{X}^2\right) \\ &\Rightarrow \frac{k}{\widetilde{W}_{d_0,m}(1)}\left(2W_{d_0,m}(1)W_{d_0,m+1}(1) - W^2_{d_0,m}(1) - W^2_{d_0,m+1}(1)\right) \\ &= -\frac{k}{\widetilde{W}_{d_0,m}}\, (W_{d_0,m}(1) - W_{d_0,m+1}(1))^2. \end{aligned} \tag{2.5.14}$$

By noting that (2.5.14) implies $d_k \to_P 1$, as $n \to \infty$, (2.5.3), (2.5.5) and (2.5.7) follow easily from the continuous mapping theorem, relation

$$\widehat{\rho}_k - 1 = d_k(\widehat{\rho}_k^* - 1) + d_k - 1,$$

and (2.5.2), (2.5.4) and (2.5.6) respectively. □

2.6 Proofs of the main results

In this section, the proofs of Theorems 2.2.1 and 2.2.2 are given. To do this, we need some preliminary results.

2.6.1 Preliminary lemmas

In this section, we derive several preliminary lemmas which will be used in the proofs of the main results. These lemmas are also interesting in their own right. Let $\{v_j, j = 0, \pm 1, ...\}$ be a sequence of iid random variables with $Ev_0 = 0$ and $Ev_0^2 = \sigma^2$. $\{a_{n,k}, k = 0, \pm 1, \pm 2, ...\}$ is a triangular array of constants. For reading convenience, we give the following basic assumptions for $a_{n,k}$.

Assumption 1. $0 < A_n^2 \equiv \sum_{k=-\infty}^{\infty} a_{n,k}^2 < \infty$, for every fixed $n \geq 1$.

Assumption 2. $A_n \to \infty$ and $\max_k |a_{n,k}|/A_n \to 0$ as $n \to \infty$.

Assumption 3. There exists a positive constant C such that

$$\sup_{n\geq 1} \frac{1}{A_n} \sum_{j=-\infty}^{\infty} |a_{n,j} - a_{n,j-1}| \leq C. \tag{2.6.1}$$

Lemma 2.6.1. *Let Assumptions 1-3 hold, $b_0 \equiv \sigma \sum_{j=0}^{\infty} \psi_j \neq 0$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$.*

Then, as $n \to \infty$,

$$\frac{1}{\sigma_n}\sum_{j=0}^{\infty}\psi_j Y_{nj} \to_d N(0,1) \quad \textit{and} \quad \sigma_n^2 \equiv Var\left(\sum_{j=0}^{\infty}\psi_j Y_{nj}\right) \sim A_n^2\, b_0^2, \tag{2.6.2}$$

where $Y_{nj} = \sum_{k=-\infty}^{\infty} a_{nk} v_{k-j}$.

Proof. Since $\sum_{i=0}^{\infty}|\psi_i| < \infty$, there exists a sequence of positive increasing constants λ_n such that

$$\sum_{j=\lambda_n}^{\infty}|\psi_j| \le A_n^{-3/2}, \quad n = 1, 2, \ldots. \tag{2.6.3}$$

For this λ_n, we rewrite

$$\sum_{j=0}^{\infty}\psi_j Y_{nj} = \sum_{j=0}^{\lambda_n}\psi_j Y_{nj} + \sum_{j=\lambda_n+1}^{\infty}\psi_j Y_{nj}.$$

By a simple calculation, to prove (2.6.2), it suffices to show that

$$E\left(\sum_{j=\lambda_n+1}^{\infty}|\psi_j Y_{nj}|\right)^2 = o(1); \quad \text{and} \tag{2.6.4}$$

$$\frac{1}{\sigma_n^*}\sum_{j=0}^{\lambda_n}\psi_j Y_{nj} \to_d N(0,1), \quad \text{where } \sigma_n^{*2} = Var\left(\sum_{j=0}^{\lambda_n}\psi_j Y_{nj}\right) \sim A_n^2\, b_0^2, \tag{2.6.5}$$

Note that relation (2.6.4) also implies that $\sum_{j=0}^{\infty}\psi_j Y_{nj}$ is well defined almost surely.

In the following, we will contribute to the proofs of (2.6.4) and (2.6.5). From Assumption 1, it is clear that

$$EY_{nj}^2 = \sigma^2\sum_{k=-\infty}^{\infty}a_{nk}^2 = \sigma^2\, A_n^2, \quad \text{for } j = 0, 1, 2, \ldots. \tag{2.6.6}$$

By Hölder's inequality, it follows from (2.6.3) and (2.6.6) that

$$\begin{aligned} E\left(\sum_{j=\lambda_n+1}^{\infty}|\psi_j Y_{nj}|\right)^2 &\le \sum_{j=\lambda_n+1}^{\infty}|\psi_j|\sum_{j=\lambda_n+1}^{\infty}|\psi_j| EY_{nj}^2 \\ &\le \sigma^2 A_n^2\left(\sum_{j=\lambda_n+1}^{\infty}|\psi_j|\right)^2 \le \sigma^2 A_n^{-1}. \end{aligned}$$

It implies (2.6.4) because $A_n \to \infty$ from Assumption 2.

To prove (2.6.5), put

$$B_n^2 = \sum_{k=-\infty}^{\infty} b_{nk}^2, \quad \text{where } b_{nk} = \sum_{j=0}^{\lambda_n} \psi_j a_{n,k+j}.$$

It is easy to show that for each fixed $n \geq 1$ (recalling Y_{nj} are well defined),

$$\begin{aligned}\sum_{j=0}^{\lambda_n} \psi_j Y_{nj} &= \sum_{j=0}^{\lambda_n} \psi_j \sum_{k=-\infty}^{\infty} a_{nk} v_{k-j} \\ &= \sum_{k=-\infty}^{\infty} v_k \sum_{j=0}^{\lambda_n} \psi_j a_{n,k+j} = \sum_{k=-\infty}^{\infty} v_k b_{nk}, \quad (2.6.7) \\ \max_k |b_{nk}|/B_n &\leq \sum_{j=0}^{\infty} |\psi_j| \max_k |a_{n,k}|/B_n, \quad (2.6.8)\end{aligned}$$

and similar to (2.6.6)

$$\sigma_n^{*2} = Var\left(\sum_{j=0}^{\lambda_n} \psi_j Y_{nj}\right) = \sigma^2 B_n^2. \quad (2.6.9)$$

Because of (2.6.7)-(2.6.9) and Assumptions 1-2, tracing the proof of Lemma 2.6.1 given in Robinson (1997), (2.6.5) holds if we can prove, as $n \to \infty$,

$$\frac{\sigma^2 B_n^2 - b_0^2 A_n^2}{A_n^2} \to 0, \quad i.e., \quad \sigma^2 B_n^2 \sim A_n^2\, b_0^2. \quad (2.6.10)$$

Since Hölder's inequality implies that for each $n \geq 1, i, j \geq 0$,

$$\sum_{k=-\infty}^{\infty} |a_{n,k+i} a_{n,k+j}| \leq \sum_{k=-\infty}^{\infty} a_{n,k}^2 < \infty,$$

elementary calculation shows that

$$\begin{aligned}B_n^2 &= \sum_{k=-\infty}^{\infty} \sum_{i,j=0}^{\lambda_n} \psi_i \psi_j a_{n,k+i} a_{n,k+j} \\ &= \sum_{i,j=0}^{\lambda_n} \psi_i \psi_j \sum_{k=-\infty}^{\infty} a_{n,k+i} a_{n,k+j} = \sum_{i,j=0}^{\lambda_n} \psi_i \psi_j \sum_{k=-\infty}^{\infty} a_{n,k} a_{n,k+j-i}.\end{aligned}$$

Writing

$$b_0^* = \sum_{i=0}^{\lambda_n} \psi_i \quad \text{and} \quad \eta_n^2 = A_n / \max_k |a_{n,k}|,$$

it follows that

$$\begin{aligned}
B_n^2 - b_0^{*2} A_n^2 &= \sum_{i,j=0}^{\lambda_n} \psi_i \psi_j \sum_{k=-\infty}^{\infty} a_{n,k} \left(a_{n,k+j-i} - a_{n,k}\right) \\
&= \left(\sum_{|j-i|>\eta_n} + \sum_{|j-i|\le\eta_n} \right) \psi_i \psi_j \sum_{k=-\infty}^{\infty} a_{n,k} \left(a_{n,k+j-i} - a_{n,k}\right) \\
&= \Delta_{n1} + \Delta_{n2}, \quad \textit{say}. \qquad (2.6.11)
\end{aligned}$$

In terms of Assumptions 2-3, we have $\eta_n \to \infty$ and

$$\begin{aligned}
|\Delta_{n1}| &\le \sum_{|j-i|>\eta_n} |\psi_i \psi_j| \left(\sum_{k=-\infty}^{\infty} a_{n,k}^2 \right)^{1/2} \left(\sum_{k=-\infty}^{\infty} |a_{n,k+j-i} - a_{n,k}|^2 \right)^{1/2} \\
&\le 4 \sum_{j=\eta_n}^{\infty} |\psi_j| \sum_{i=0}^{\infty} |\psi_i| \sum_{k=-\infty}^{\infty} a_{n,k}^2 = o\left(A_n^2\right). \qquad (2.6.12)
\end{aligned}$$

In terms of the following inequality

$$\max_{|j-i|\le\eta_n} |a_{n,k+j-i} - a_{n,k}| \le \sum_{t=-\eta_n}^{\eta_n} |a_{n,k+t} - a_{n,k+t-1}|,$$

$\max_k |a_{n,k}| = A_n/\eta_n^2$, and Assumption 3, we have

$$\begin{aligned}
|\Delta_{n2}| &\le \sum_{|j-i|\le\eta_n} |\psi_i \psi_j| \sum_{k=-\infty}^{\infty} |a_{n,k}| \sum_{t=-\eta_n}^{\eta_n} |a_{n,k+t} - a_{n,k+t-1}| \\
&\le \left(\sum_{i=0}^{\infty} |\psi_i| \right)^2 \sum_{t=-\eta_n}^{\eta_n} \max_k |a_{n,k}| \sum_{k=-\infty}^{\infty} |a_{n,k+t} - a_{n,k+t-1}| \\
&\le 2\eta_n^{-1} \left(\sum_{i=0}^{\infty} |\psi_i| \right)^2 A_n \sum_{k=-\infty}^{\infty} |a_{n,k} - a_{n,k-1}| = o\left(A_n^2\right). \qquad (2.6.13)
\end{aligned}$$

Therefore, from (2.6.11)- (2.6.13), we obtain that

$$\frac{B_n^2 - b_0^{*2} A_n^2}{A_n^2} \to 0, \quad \textit{i.e.}, \quad \sigma^2 B_n^2 \sim \sigma^2\, b_0^{*2}\, A_n^2. \qquad (2.6.14)$$

Now, relation (2.6.10) follows immediately from (2.6.14) and

$$\left|b_0^2 - \sigma^2 b_0^{*2}\right| \le 2\sigma^2 \sum_{j=\lambda_n}^{\infty} |\psi_j| \sum_{j=0}^{\infty} |\psi_j| = o(1).$$

The proof of Lemma 2.6.1 is complete. □

Lemma 2.6.2. *Let* $c_k = \frac{\Gamma(d_0+k)}{\Gamma(d_0)\Gamma(k+1)}$ *for* $k \ge 0$ *and* $c_k = 0$ *for* $k < 0$, *where* $-1/2 < d_0 < 1/2$. *Then,*

$$c_0 = 1, \qquad |c_k| \le C k^{d_0 - 1}; \tag{2.6.15}$$

$$\max_k \sum_{i=1}^{n} |c_{i+k}| \le C_1 \max\left\{1, n^{d_0}\right\}, \quad \textit{for } d_0 \ne 0; \tag{2.6.16}$$

$$|c_{n+k} - c_k| \le C_2 n k^{d_0 - 2}, \quad \textit{for all } 1 \le n \le k; \tag{2.6.17}$$

$$\sum_{k=-\infty}^{\infty} \left(\sum_{j=[ns]+1}^{[nt]} c_{j-k} \right)^2 \sim \frac{n^{1+2d_0}\Gamma(1-2d_0)(t-s)^{1+2d_0}}{(1+2d_0)\Gamma(1+d_0)\Gamma(1-d_0)}, \tag{2.6.18}$$

for $0 \le s < t \le 1$.

Proof. For the proof of (2.6.15), see Theorem 1 in Hosking (1981). The proof of (2.6.16) follows easily from (2.6.15). By noting $\Gamma(z+1) = z\Gamma(z)$ for all z, we have that, for $1 \le n \le k$ and $d_0 \in (-1/2, 1/2)$,

$$\begin{aligned}
|c_{n+k} - c_k| &= |c_k| \left(1 - \frac{(k+n+d_0-1)...(k+d_0)}{(k+n)...(k+1)}\right) \\
&\le |c_k| \left(1 - \frac{(k+d_0)(k+d_0+1)}{(k+n)(k+n-1)}\right) \\
&\le \frac{|c_k|}{k^2} \left\{(k+n)^2 - (k+d_0)^2\right\} \le C_2 n k^{d_0-2},
\end{aligned}$$

which implies (2.6.17).

In order to prove (2.6.18), let $\zeta_k, k = 0, \pm 1, \pm 2, ...$, be iid $N(0,1)$ random variables and $Y_j = \sum_{k=0}^{\infty} c_k \zeta_{j-k}$. Since $c_k = 0$ for $k < 0$ and hence

$$\sum_{j=[ns]+1}^{[nt]} Y_j = \sum_{k=-\infty}^{\infty} \zeta_k \sum_{j=[ns]+1}^{[nt]} c_{j-k},$$

clearly, we have that for $0 \le s < t \le 1$

$$\sum_{k=-\infty}^{\infty}\left(\sum_{j=[ns]+1}^{[nt]} c_{j-k}\right)^2 = E\left(\sum_{j=[ns]+1}^{[nt]} Y_j\right)^2.$$

By noting that $Y_j, j \ge 1$, are stationary random variables, on the other hand, it follows from Theorem 1 given in Sowell (1990) that

$$E\left(\sum_{j=[ns]+1}^{[nt]} Y_j\right)^2 = E\left(\sum_{j=1}^{[nt]-[ns]} Y_j\right)^2 \sim \frac{n^{1+2d_0}\Gamma(1-2d_0)(t-s)^{1+2d_0}}{(1+2d_0)\Gamma(1+d_0)\Gamma(1-d_0)},$$

where we use the estimate: $\frac{[nt]-[ns]}{n} \sim t-s$. Therefore, (2.6.18) follows. The proof of Lemma 2.6.2 is complete. □

Lemma 2.6.3. *Let X_t satisfy (2.2.1) with $m = 0$. Assume that $E\epsilon_0^2 < \infty$. Then,*

$$(a)\quad X_t = \sum_{k=-\infty}^{\infty} c_{t-k}u_k = \sum_{j=0}^{\infty}\psi_j Z_{t-j}, \tag{2.6.19}$$

where c_k is as in Lemma 2.6.2 and $Z_t = \sum_{k=0}^{\infty} c_k\epsilon_{t-k}$.

(b) $\{X_t, t \ge 1\}$ is a stationary ergodic random sequence with zero mean and

$$EX_1^2 = \frac{E\epsilon_0^2}{2\pi}\int_{-\pi}^{\pi}\left|1-e^{i\lambda}\right|^{-2d_0}\left|\psi(e^{-i\lambda})\right|^2\, d\lambda < \infty, \tag{2.6.20}$$

where $\psi(e^{-i\lambda}) = \sum_{k=0}^{\infty}\psi_k e^{-ik\lambda}$; in particular, $EX_1^2 = E\epsilon_0^2\sum_{k=0}^{\infty}\psi_k^2 < \infty$ if $d_0 = 0$.

Proof. Writing $X_t = \Psi(B)\epsilon_t$, we have $\Psi(z) = (1-z)^{-d_0}\psi(z)$, where $\psi(z) = \sum_{j=0}^{\infty}\psi_j z^j$. Since $\sum_{j=0}^{\infty}|\psi_j| < \infty$, similar to the proof of Theorem 2 part (a) given by Hosking (1981), the power series expansion of $\Psi(z)$ converges for all $|z| \le 1$ when $d_0 < 1/2$. Thus, if $-1/2 < d_0 < 1/2$, we have

$$X_t = (1-B)^{-d_0}\psi(B)\epsilon_t = \psi(B)(1-B)^{-d_0}\epsilon_t.$$

Now (2.6.19) follows from the Binomial expansion of $(1-z)^{-d_0}$ (see Hosking, 1981).

As well-known (see, for example, Hosking (1981)), $\{Z_t, t \geq 1\}$ is a stationary ergodic random sequence with mean zero. It follows from Theorem B.1 (see Appendix B) that $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ has the same properties as Z_t. To prove (2.6.20), let $f_Z(.)$ be the spectral density of $\{Z_t\}$. According to Theorem 12.4.1 in Brockwell and Davis (1987, p466),

$$f_Z(\lambda) = \frac{E\epsilon_0^2}{2\pi} \left|1 - e^{i\lambda}\right|^{-2d_0}, \qquad \text{for } -\pi \leq \lambda \leq \pi.$$

This, together with the second equality of (2.6.19) and (4.4.3) in Brockwell and Davis (1987, p121), shows that $\{X_t\}$ has a spectral density $f_X(.)$ and $f_X(\lambda) = \left|\psi(e^{-i\lambda})\right|^2 f_Z(\lambda)$, where $\psi(e^{-i\lambda}) = \sum_{k=0}^{\infty} \psi_k e^{-ik\lambda}$. In terms of $\sum_{j=0}^{\infty} |\psi_j| < \infty$, we obtain

$$\begin{aligned} EX_1^2 &= \frac{E\epsilon_0^2}{2\pi} \int_{-\pi}^{\pi} \left|1 - e^{i\lambda}\right|^{-2d_0} \left|\psi(e^{-i\lambda})\right|^2 d\lambda \\ &\leq \frac{E\epsilon_0^2}{2\pi} \left(\sum_{j=0}^{\infty} |\psi_j|\right)^2 \int_{-\pi}^{\pi} \left|1 - e^{i\lambda}\right|^{-2d_0} d\lambda < \infty. \end{aligned}$$

If $d_0 = 0$, the result is obvious since $X_1 = \sum_{k=0}^{\infty} \psi_k \epsilon_{1-k}$. The proof of Lemma 2.6.3 is complete. □

Lemma 2.6.4. *Let $Y_j, j \geq 1$ be a sequence of stationary random variables with zero mean and finite variance. Let*

$$Z_n(t) = \frac{S_{[nt]}}{n^H L^{1/2}(n)}, \quad S_n = \sum_{j=1}^{n} Y_j, \quad 0 < H < 1 \tag{2.6.21}$$

where $L(n)$ is a function varying slowly at infinity. Assume that

(i) $ES_n^2 = O\left(n^{2H} L(n)\right)$, *as $n \to \infty$;*

(ii) $E|S_n|^{2\alpha} = O\left((ES_n^2)^{\alpha}\right)$, *for some $\alpha > 1/(2H)$;*

(iii) for each fixed $l \geq 1$ and real constants $0 \leq t_1 \neq t_2 \neq ... \neq t_l \leq 1$,

$$\tau_1 Z_n(t_1) + ... + \tau_l Z_n(t_l) \to_d N(0, \sigma^2) \tag{2.6.22}$$

where $\tau_1, \tau_2, ..., \tau_l$ are any real constants, $\sigma^2 = \sum_{i,j=1}^{l} \tau_i \tau_j B_H(t_i, t_j)$ and

$$B_H(s,t) = \frac{1}{2}\left\{s^{2H} + t^{2H} - |s-t|^{2H}\right\}.$$

Then, for $0 \leq t \leq 1$,

$$Z_n(t) \Rightarrow W_{H-1/2}(t), \tag{2.6.23}$$

where $W_{d_0}(t)$ is a "type I" fractional Brownian motion.

Proof. Under the conditions of Lemma 2.6.4, Theorem 2.1 in Taqqu(1975)[3] implies that $Z_n(t)$ converges weakly to a stationary increments, self-similar and a.s. continuous Gaussian process $Z(t)$ with the covariance

$$EZ(s)Z(t) = \frac{1}{2}\left\{s^{2H} + t^{2H} - |s-t|^{2H}\right\}, \quad for\ 0 \leq s, t \leq 1.$$

Now Lemma 2.6.4 follows directly from Proposition 3.8 given in Mandelbrot and Van Ness (1968). □

Using these results, we can give the proofs of the main results as follows.

2.6.2 Proof of Theorem 2.2.1

If $d_0 = 0$, the model (2.2.1) reduces to summable linear processes. In this case, (2.2.3) follows from Hannan (1979). So, we assume that $d_0 \neq 0$ in the sequel. Put

$$V_n(t) = \frac{1}{n^{1/2+d_0}} \sum_{j=1}^{[nt]} X_j, \qquad B_{d_0}(s,t) = \frac{1}{2}\left\{s^{1+2d_0} + t^{1+2d_0} - |s-t|^{1+2d_0}\right\}.$$

[3]In Taqqu(1975), $Z_n(t)$ was written as $S_{[nt]}/n^{2H}L(n)$ by error.

In terms of part (b) of Lemma 2.6.3 and Lemma 2.6.4 with $H = d_0 - 1/2$, it suffices to show:

(i) for each fixed $l \geq 1$ and real constants $0 \leq t_1 \neq t_2 \neq ... \neq t_l \leq 1$,

$$\tau_1 V_n(t_1) + ... + \tau_l V_n(t_l) \to_d N(0, \sigma_1^2), \tag{2.6.24}$$

where $\tau_1, \tau_2, ..., \tau_l$ are any real constants and $\sigma_1^2 = \kappa^2(d_0) \sum_{i,j=1}^{l} \tau_i \tau_j B_{d_0}(t_i, t_j)$;

(ii) for some $a > 1/(1 + 2d_0)$,

$$ES_n^2 = O\left(n^{1+2d_0}\right) \quad \text{and} \quad E|S_n|^{2a} = O\left((ES_n^2)^a\right), \tag{2.6.25}$$

where $S_n = \sum_{j=1}^{n} X_j$.

We first prove part (i). From part (a) in Lemma 2.6.3, we have that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad \text{where} \quad Z_t = \sum_{k=0}^{\infty} c_k \epsilon_{t-k} \tag{2.6.26}$$

and c_k are defined as in Lemma 2.6.2. Let $m_i = [nt_i], i = 1, ..., l$. It follows from (2.6.26) that

$$\tau_1 V_n(t_1) + ... + \tau_l V_n(t_l) = \frac{1}{n^{1/2+d_0}} \sum_{i=1}^{l} \tau_i \sum_{t=1}^{m_i} X_t = \frac{1}{n^{1/2+d_0}} \sum_{j=0}^{\infty} \psi_j Y_{nj}, \tag{2.6.27}$$

where, by a elementary calculation (recalling $c_k = 0$ if $k < 0$),

$$\begin{aligned} Y_{nj} &= \sum_{i=1}^{l} \tau_i \sum_{t=1}^{m_i} Z_{t-j} \\ &= \sum_{i=1}^{l} \tau_i \sum_{t=1}^{m_i} \sum_{k=0}^{\infty} c_k \epsilon_{t-k-j} = \sum_{i=1}^{l} \tau_i \sum_{t=1}^{m_i} \sum_{k=-\infty}^{\infty} c_{t-k} \epsilon_{k-j} \\ &= \sum_{k=-\infty}^{\infty} b_{n,k} \epsilon_{k-j} \end{aligned} \tag{2.6.28}$$

with $b_{n,k} = \sum_{i=1}^{l} \tau_i \sum_{t=1}^{m_i} c_{t-k}$.

In order to apply Lemma 2.6.1 to (2.6.27), we first show that

$$B_n^2 \equiv \sum_{k=-\infty}^{\infty} b_{n,k}^2 \sim \frac{n^{1+2d_0}\Gamma(1-2d_0)}{(1+2d_0)\Gamma(1+d_0)\Gamma(1-d_0)} \sum_{i,j=1}^{l} \tau_i\tau_j B_{d_0}(t_i,t_j). \qquad (2.6.29)$$

In fact, (2.6.29) follows immediately from Lemma 2.6.2 (see (2.6.18)) and

$$\begin{aligned} b_{n,k}^2 &= \sum_{i,j=1}^{l} \tau_i\tau_j \left(\sum_{s=1}^{m_i} c_{s-k}\right)\left(\sum_{t=1}^{m_j} c_{t-k}\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{l} \tau_i\tau_j \left\{\left(\sum_{s=1}^{m_i} c_{s-k}\right)^2 + \left(\sum_{t=1}^{m_j} c_{t-k}\right)^2 - \left(\sum_{s=m_i}^{m_j} c_{s-k}\right)^2\right\}. \end{aligned}$$

From Lemma 2.6.2 (i.e., (2.6.15)-(2.6.17)), on the other hand, we have that for $d_0 \neq 0$,

$$\max_k |b_{n,k}| \leq \sum_{i=1}^{l} |\tau_i| \max_k \sum_{t=0}^{n} |c_{t-k}| \leq C\max\{1, n^{d_0}\}; \qquad (2.6.30)$$

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |b_{n,k} - b_{n,k-1}| &= \sum_{k=-\infty}^{\infty} \left|\sum_{i=1}^{l} \tau_i \left(\sum_{t=1}^{m_i} c_{t-k} - \sum_{t=1}^{m_i} c_{t-k+1}\right)\right| \\ &\leq \sum_{i=1}^{l} |\tau_i| \sum_{k=-\infty}^{\infty} |c_{1-k} - c_{m_i-k+1}| \\ &\leq \sum_{i=1}^{l} |\tau_i| \left\{\sum_{|k|\leq n} (|c_{1-k}| + |c_{m_i+1-k}|) + \sum_{k=n+1}^{\infty} |c_{k+1} - c_{m_i+k+1}|\right\} \\ &\leq \sum_{i=1}^{l} |\tau_i| \left\{C\max\{1, n^{d_0}\} + C_1\, n \sum_{k=n+1}^{\infty} k^{d_0-2}\right\} \\ &\leq C_2 \max\{1, n^{d_0}\}. \end{aligned} \qquad (2.6.31)$$

In terms of (2.6.29)-(2.6.31), the conditions of Lemma 2.6.1 hold for $b_{n,k}$ defined in

(2.6.27) and (2.6.28). By applying (2.6.27), (2.6.29) and Lemma 2.6.1, we obtain that

$$\frac{1}{B_n^2 b_0^2} \sum_{j=0}^{\infty} \psi_j Y_{nj} \to_d N(0,1), \tag{2.6.32}$$

$$\begin{aligned} E\left(\sum_{i=1}^{l} \tau_i \sum_{j=1}^{m_i} X_j\right)^2 &\sim B_n^2 \, b_0^2 \\ &\sim n^{1+2d_0} \kappa^2(d_0) \sum_{i,j=1}^{l} \tau_i \tau_j B_{d_0}(t_i, t_j). \end{aligned} \tag{2.6.33}$$

The relation (2.6.32), together with (2.6.27) and (2.6.33), implies (2.6.24). This completes the proof of part (i).

Secondly, we prove part (ii). By applying (2.6.27) and (2.6.28) with $l = \tau_1 = t_1 = 1$, we obtain that

$$S_n = \sum_{j=1}^{n} X_j = \sum_{j=0}^{\infty} \psi_j Y_{nj}, \qquad \text{where} \quad Y_{nj} = \sum_{k=-\infty}^{\infty} b_{n,k} \epsilon_{k-j}$$

with $b_{n,k} = \sum_{t=1}^{n} c_{t-k}$. Furthermore, it follows from (2.6.33) that

$$ES_n^2 \sim b_0^2 \sum_{k=-\infty}^{\infty} b_{n,k}^2 \sim n^{1+2d_0} \kappa^2(d_0) \tag{2.6.34}$$

Therefore, the first relation of (2.6.25) holds.

If $0 < d_0 < 1/2$, the second relation of (2.6.25) is obvious by letting $a = 1$. To establish the second relation of (2.6.25) for $-1/2 < d_0 < 0$, we let $2a = (2 + \delta)/(1 + 2d_0)$. By noting $\delta > 0$, obviously, we have that $a > 1/(1 + 2d_0) > 1$ and $E|\epsilon_0|^{2a} < \infty$ when $d_0 < 0$. By Burkholder's inequality (see Hall and Heyde, 1980, p23) and Hölder's inequality, there exists a constants C_a depending only on a such

that for all integers j and $s \leq h$,

$$
\begin{aligned}
E\left|\sum_{k=s}^{h} \epsilon_k b_{n,k+j}\right|^{2a} &\leq C_a E\left(\sum_{k=s}^{h} \epsilon_k^2 b_{n,k+j}^2\right)^a \\
&= C_a E\left(\sum_{k=s}^{h} \epsilon_k^2 |b_{n,k+j}|^{2/a} |b_{n,k+j}|^{(2a-2)/a}\right)^a \\
&\leq C_a E\left\{\sum_{k=s}^{h} |\epsilon_k|^{2a} b_{n,k+j}^2 \left(\sum_{k=s}^{h} b_{n,k+j}^2\right)^{a-1}\right\} \\
&\leq C_a \left(\sum_{k=-\infty}^{\infty} b_{n,k}^2\right)^a E|\epsilon_0|^{2a} \qquad (2.6.35)
\end{aligned}
$$

Because of (2.6.34) and (2.6.35), it follows from Fatou's Lemma that for all j,

$$
\begin{aligned}
E|Y_{nj}|^{2a} &= E\left|\sum_{k=-\infty}^{\infty} \epsilon_k b_{n,k+j}\right|^{2a} \\
&\leq E\left|\lim_{h\to\infty} \sum_{k=1}^{h} \epsilon_k b_{n,k+j}\right|^{2a} + E\left|\lim_{s\to\infty} \sum_{k=-s}^{0} \epsilon_k b_{n,k+j}\right|^{2a} \\
&\leq \limsup_{h\to\infty} E\left|\sum_{k=1}^{h} \epsilon_k b_{n,k+j}\right|^{2a} + \limsup_{s\to\infty} E\left|\sum_{k=-s}^{0} \epsilon_k b_{n,k+j}\right|^{2a} \\
&\leq C\left(\sum_{k=-\infty}^{\infty} b_{nk}^2\right)^a = O\left((ES_n^2)^a\right).
\end{aligned}
$$

Hence, by Hölder's inequality again, we obtain that

$$
\begin{aligned}
E|S_n|^{2a} &\leq E\left(\sum_{j=0}^{\infty} |\psi_j|^{(2a-1)/(2a)} |\psi_j|^{1/(2a)} |Y_{nj}|\right)^{2a} \\
&\leq \left(\sum_{j=0}^{\infty} |\psi_j|\right)^{2a-1} \sum_{j=0}^{\infty} |\psi_j| E|Y_{nj}|^{2a} = O\left((ES_n^2)^a\right),
\end{aligned}
$$

which implies the desired result. The proof of Theorem 2.2.1 is complete.

2.6.3 Proof of Theorem 2.2.2

We only need to consider the case that $-1/2 < d_0 < 0$. In this case, since ϵ_k are iid random variables with $E\epsilon_0 = 0$ and $E|\epsilon_0|^{2/(1+2d_0)} < \infty$, by applying Theorem C.1 (see Appendix C), on a suitable probability space, we can construct $\eta_k, k = 0, \pm 1, \pm 2, \cdots$, which are iid $N(0, E\epsilon_0^2)$ random variables such that, as $n \to \infty$,

$$\max_{1\le m\le n}\left|\sum_{j=1}^{m}\epsilon_j - \sum_{j=1}^{m}\eta_j\right| = o\left(n^{1/2+d_0}\right), \quad a.s., \tag{2.6.36}$$

$$\max_{1\le m\le n}\left|\sum_{j=0}^{m}\epsilon_{-j} - \sum_{j=0}^{m}\eta_{-j}\right| = o\left(n^{1/2+d_0}\right), \quad a.s.. \tag{2.6.37}$$

Let Y_t satisfy

$$(1-B)^{d_0}\, Y_t = u_t, \qquad u_t = \sum_{j=0}^{\infty}\psi_j\,\eta_{t-j}, \quad t = 1, 2, ...,$$

where $-1/2 < d_0 < 0$ and ψ_k satisfies (2.2.4). Because η_k are iid $N(0, E\epsilon_0^2)$, by applying Theorem 2.2.1, we have that, for $0 \le t \le 1$,

$$\frac{1}{\kappa(d_0)n^{1/2+d_0}}\sum_{j=1}^{[nt]} Y_j \Rightarrow W_{d_0}(t). \tag{2.6.38}$$

In terms of (2.6.38) and Theorem A.1 (see Appendix A), to prove Theorem 2.2.2, it suffices to show that

$$\sup_{0\le t\le 1}\left|\sum_{j=1}^{[nt]} X_j - \sum_{j=1}^{[nt]} Y_j\right| = o_P\left(n^{1/2+d_0}\right). \tag{2.6.39}$$

By applying part (a) of Lemma 2.6.3, for any $m \ge 1$, we can write

$$\begin{aligned}
\sum_{j=1}^{m}(X_j - Y_j) &= \sum_{j=1}^{m}\sum_{s=0}^{\infty}\psi_s\sum_{k=0}^{\infty}c_k(\epsilon_{j-k-s} - \eta_{j-k-s}) \\
&= \sum_{j=1}^{m}\sum_{s=0}^{\infty}\psi_s\sum_{k=s}^{\infty}c_{k-s}(\epsilon_{j-k} - \eta_{j-k}) \\
&= A_{1n}^m + A_{2n}^m + A_{3n}^m + A_{4n}^m,
\end{aligned} \tag{2.6.40}$$

where $c_k = \frac{\Gamma(d_0+k)}{\Gamma(d_0)\Gamma(k+1)}$ for $k \geq 0$ and $c_k = 0$ for $k < 0$;

$$
\begin{aligned}
A_{1n}^m &= \sum_{j=1}^{m} \sum_{s=n+1}^{\infty} \psi_s \sum_{k=s}^{\infty} c_{k-s}(\epsilon_{j-k} - \eta_{j-k}), \\
A_{2n}^m &= \sum_{j=1}^{m} \sum_{s=0}^{n} \psi_s \sum_{k=s}^{2n} c_{k-s}(\epsilon_{j-k} - \eta_{j-k}), \\
A_{3n}^m &= \sum_{j=1}^{m} \sum_{s=0}^{n} \psi_s \sum_{k=2n+1}^{n^2} c_{k-s}(\epsilon_{j-k} - \eta_{j-k}), \\
A_{4n}^m &= \sum_{j=1}^{m} \sum_{s=0}^{n} \psi_s \sum_{k=n^2+1}^{\infty} c_{k-s}(\epsilon_{j-k} - \eta_{j-k}).
\end{aligned}
$$

Clearly, (2.6.39) follows if

$$
\max_{1\leq m\leq n} \left|A_{jn}^m\right| = o_P\left(n^{1/2+d_0}\right), \qquad \text{for } j = 1, 2, 3, 4. \tag{2.6.41}
$$

We next prove (2.6.41). Write

$$
V_{l,j,s} = \sum_{k=l}^{\infty} c_{k-s}(\epsilon_{j-k} - \eta_{j-k}).
$$

By noting $c_0 = 0$ and $|c_k| \leq Ck^{d_0-1}$ for $k \geq 1$ (see Lemma 2.6.2), it can be easily shown that for all $l \geq s \geq 0$ and $j \geq 1$,

$$
E|V_{l,j,s}| \leq \left(EV_{l,j,s}^2\right)^{1/2} \leq C\left(\sum_{k=l}^{\infty} c_{k-s}^2\right)^{1/2} \leq C_1 \min\{1, l^{d_0-1/2}\}.
$$

Therefore, it follows that

$$
E \max_{1\leq m\leq n} |A_{1n}^m| \leq \sum_{j=1}^{n} \sum_{s=n+1}^{\infty} |\psi_s|\, E|V_{s,j,s}| \leq Cn^{1/2+d_0} \sum_{s=n+1}^{\infty} s^{1/2-d_0}|\psi_s|. \tag{2.6.42}
$$

Similarly, we obtain that

$$
\begin{aligned}
E \max_{1\leq m\leq n} |A_{4n}^m| &\leq \sum_{j=1}^{n} \sum_{s=0}^{n} |\psi_s|\, E|V_{n^2+1,j,s}| \\
&\leq Cn\left(n^2\right)^{d_0-1/2} \sum_{s=0}^{\infty} |\psi_s| \leq C_1 n^{2d_0}.
\end{aligned} \tag{2.6.43}
$$

In terms of (2.2.4), (2.6.42)-(2.6.43) and $d_0 < 0$, Markov's inequality implies that (2.6.41) holds for $j = 1$ and 4.

On the other hand, it follows from (2.6.36) and (2.6.37) that

$$\begin{aligned}\max_{1\le m\le n} |A_{2n}^m| &\le \sum_{s=0}^{n} |\psi_s| \sum_{k=s}^{2n} |c_{k-s}| \max_{\substack{1\le m\le n\\ 0\le k\le 2n}} \left|\sum_{j=1}^{m} (\epsilon_{j-k} - \eta_{j-k})\right| \\ &\le C\left(\max_{1\le m\le n}\left|\sum_{j=1}^{m}(\epsilon_j - \eta_j)\right| + \max_{0\le k\le n}\left|\sum_{j=0}^{2k}(\epsilon_{-j} - \eta_{-j})\right|\right) \\ &= o\left(n^{1/2+d_0}\right), \qquad a.s..\end{aligned}$$

This implies that (2.6.41) holds for $j = 2$.

We now prove (2.6.41) for $j = 3$. For convenience, write $S_k = \sum_{i=0}^{k}(\epsilon_{-i} - \eta_{-i})$. We have that

$$\begin{aligned}A_{3n}^m &= \sum_{s=0}^{n} \psi_s \sum_{k=2n+1}^{n^2} c_{k-s} \sum_{j=1}^{m} (S_{k-j} - S_{k-j-1}) \\ &= \sum_{s=0}^{n} \psi_s \sum_{k=2n+1}^{n^2} c_{k-s} \left(S_{k-1} - S_{k-m-1}\right). \end{aligned} \tag{2.6.44}$$

Clearly, it follows that

$$\begin{aligned}&\sum_{k=2n+1}^{n^2} c_{k-s}\left(S_{k-1} - S_{k-m-1}\right) \\ &= \sum_{k=2n}^{n^2-1} c_{k+1-s} S_k - \sum_{k=2n-m}^{n^2-m-1} c_{k+1+m-s} S_k \\ &= \sum_{k=n^2-m}^{n^2-1} c_{k+1-s} S_k - \sum_{k=2n-m}^{2n-1} c_{k+1+m-s} S_k + \sum_{k=2n}^{n^2-m-1} \left(c_{k+1-s} - c_{k+1+m-s}\right) S_k \\ &= I_{1ns}^m + I_{2ns}^m + I_{3ns}^m, \qquad \text{say}. \end{aligned} \tag{2.6.45}$$

Recalling (2.6.37), $|c_k| \le Ck^{d_0-1}$ and $d_0 < 0$, we get that

$$\begin{aligned}\max_{\substack{1\le m\le n\\0\le s\le n}} |I^m_{1ns}| &\le C \sum_{k=n^2-n}^{n^2-1} (k-n)^{d_0-1} \max_{0\le k\le n^2} |S_k| \\ &\le Cn\left(n^2\right)^{d_0-1}\left(n^2\right)^{1/2+d_0} = o\left(n^{1/2+d_0}\right), \quad a.s.. \end{aligned} \tag{2.6.46}$$

Similarly, we have

$$\max_{\substack{1\le m\le n\\0\le s\le n}} |I^m_{2ns}| = o\left(n^{1/2+d_0}\right), \quad a.s.. \tag{2.6.47}$$

On the other hand, by applying Lemma 2.6.2, we know that (noting $m \le k-s$)

$$|c_{k+1-s} - c_{k+1+m-s}| \le Cm(k-s)^{d_0-2}.$$

Therefore, it follows from (2.6.37) that (recalling $d_0 < 0$)

$$\max_{\substack{1\le m\le n\\0\le s\le n}} |I^m_{3ns}| \le Cn\sum_{k=2n}^{n^2}(k-n)^{d_0-2}k^{1/2+d_0} = o\left(n^{1/2+d_0}\right), \quad a.s.. \tag{2.6.48}$$

In term of (2.6.44)-(2.6.48), we have that

$$\max_{1\le m\le n} |A^m_{3n}| \le \max_{\substack{1\le m\le n\\0\le s\le n}} \left(|I^m_{1ns}| + |I^m_{2ns}| + |I^m_{3ns}|\right)\sum_{s=0}^{n}|\psi_s| = o\left(n^{1/2+d_0}\right), \quad a.s..$$

This implies (2.6.41) for $j=3$. We finish the proof of Theorem 2.2.2.

Chapter 3

Asymptotics for nonstationary fractionally integrated processes without prehistorical influence

In this chapter, we discuss a functional limit theorem for the nonstationary fractionally integrated processes having no influence from prehistory. Asymptotic distributions of sample autocovariances and autocorrelations based on these processes are also investigated. The problem arises naturally in discussing fractionally integrated processes when the processes start at a given initial date.

3.1 Introduction

In the last chapter, we discussed systematically asymptotics of the general fractionally integrated processes X_t defined by

$$(1-B)^d X_t = u_t, \qquad u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad t = 1, 2, \cdots, \tag{3.1.1}$$

where $d = d_0 + m > -1/2$ and $\epsilon_k, k = 0, \pm 1, \pm 2, \cdots$, are iid random variables. By using the definition of the fractional difference operator $(1-B)^{\gamma}$ (see (2.1.2)), the

process X_t satisfies, if $d \neq 0, 1, 2, \cdots$,

$$\sum_{j=0}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)} X_{t-j} = u_t, \qquad t = 1, 2, \cdots ; \tag{3.1.2}$$

and if $d = 0, 1, 2, \cdots$,

$$\sum_{j=0}^{d} (-1)^j \frac{d!}{j!(d-j)!} X_{t-j} = u_t, \qquad t = 1, 2, \cdots . \tag{3.1.3}$$

From (3.1.2) and (3.1.3), it is clear that the process X_t defined by (3.1.1) depends on a term that is usually called "prehistorical influence":

$$\sum_{j=t}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)} X_{t-j} \quad \text{or} \quad \sum_{j=t}^{d} (-1)^j \frac{d!}{j!(d-j)!} X_{t-j}.$$

In practice, if we only consider the process X_t defined by (3.1.1) starting at a given initial time, such as $t = 1$, we may assume that $X_j = 0$ for $j \leq 0$. In this case, after some algebra (see Section 3.5), it can be checked that the process X_t defined by (3.1.1) is a special case of the process Z_t defined by

$$Z_t = \sum_{k=0}^{t-1} c_k^{(d)} \, u_{t-k}, \qquad u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \qquad t = 1, 2, \cdots , \tag{3.1.4}$$

where $\epsilon_k, k = 0, \pm 1, \pm 2, \cdots$, are iid random variables, $d > -1/2$ and

$$\begin{aligned} c_0^{(0)} &= 1, \quad c_k^{(0)} = 0, \quad k \geq 1; \\ c_k^{(\alpha)} &= \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)}, \quad k \geq 0 \text{ and } \alpha \neq 0, -1, \cdots . \end{aligned}$$

Asymptotics of the process Z_t were first investigated in Aknom and Gourieroux (1987) with $d > 1/2$ and $u_t = \epsilon_t$ (i.e., $\psi_0 = 1$ and $\psi_k = 0$ for $k \geq 1$ in (3.1.4)) under $E|\epsilon_0|^r < \infty$, where $r > \max\{2, 2/(2d-1)\}$. The results of Aknom and Gourieroux (1987) were extended to the multivariate case by Marinucci and Robisson (1998)

without any improvements on the moment conditions. More recently, Tanaka (1999) discussed the functional limit theorem for the more general process $Z_{[ns]}$ where the ψ_k satisfy $\sum_{k=0}^{\infty} k|\psi_k| < \infty$. However, the proof of Tanaka depends on the functional limit theorem for martingale differences. Unfortunately, the process Z_t itself (even as $u_t = \epsilon_t, t = 1, 2, \cdots$) is not a martingale. Therefore, the proof of Tanaka is not applicable in this case. This chapter gives a different proof of the functional limit theorem for the process $Z_{[nt]}$. It shows that the main results given by Tanaka (1999) still hold. In fact, we establish a similar result for the more general model only under the moment condition $E|\epsilon_0|^{\max\{2,2/(2d-1)\}} < \infty$. It should be pointed out that the limit process of $Z_{[nt]}/Var^{1/2}(Z_n)$ is different from those established in Chapter 2 because of the "prehistorical influence".

In next section, we establish the main results and give some corollaries. In particular, we derive the functional limit theorem for the partial sum process of the process X_t defined by (3.1.1) with $d > -1/2$ and $X_t = 0, t \leq 0$. The proof of the basic result (Theorem 3.2.1) will be postponed to Section 3.3. In Section 3.4, by using established results, we study the asymptotics of sample autocovariances and sample autocorrelations based on the process Z_t in (3.1.4). These results do not appear in the existing literature. Finally, in Section 3.5, we prove that the process X_t defined by (3.1.1) with $X_t = 0$ for $t \leq 0$ is a special case of the process Z_t defined by (3.1.4).

Throughout this chapter, we denote positive constants by C with or without subscript, which might have different values in different places. A "type II" fractional Brownian motion $B_d(t), d > 1/2$, is defined as

$$B_d(0) = 0, \quad B_d(t) = \int_0^t (t-s)^{d-1} dW(s), \quad 0 \leq t \leq 1,$$

where $W(t)$ is a standard Brownian motion. Comparison between "type I" (see

Chapter 2) and "type II" fractional Brownian motions can be found in Marinucci and Robinson (1999).

3.2 Main results

We first give the following basic result. Its proof will be postponed to the next section.

Theorem 3.2.1. *Let Z_t satisfy (3.1.4) with $\epsilon_k, k = 0, \pm 1, \cdots$, iid $(0, \sigma^2)$,*

$$\sum_{j=0}^{\infty} |\psi_j| < \infty \quad \text{and} \quad b_\psi \equiv \sum_{j=0}^{\infty} \psi_j \neq 0. \tag{3.2.1}$$

Then, for $d \geq 1$,

$$\frac{1}{n^{d-1/2}} Z_{[nt]} \Rightarrow \kappa_1(d)\ B_d(t), \quad 0 \leq t \leq 1, \tag{3.2.2}$$

where $\kappa_1^2(d) = b_\psi^2 \sigma^2 / \Gamma^2(d)$ and $B_d(t)$ is a "type II" fractional Brownian motion.

If, in addition, $E|\epsilon_0|^{2/(2d-1)} < \infty$ and $\sum_{k=0}^{\infty} k|\psi_k| < \infty$, then (3.2.2) still holds for $1/2 < d < 1$.

As a direct consequence of Theorems 3.2.1 and the continuous mapping theorem (Theorem B.2, see Appendix B), the following corollary gives the asymptotic distribution of the partial sum process of the process Z_t.

Corollary 3.2.2. *Let Z_t satisfy (3.1.4) with $\epsilon_k, k = 0, \pm 1, \cdots$, iid $(0, \sigma^2)$, and $\psi_j, j \geq 0$, satisfy (3.2.1). Then, for $d > 1/2$,*

$$\frac{1}{n^{d+1/2}} \sum_{j=1}^{[nt]} Z_j \Rightarrow \kappa_1(d+1) B_{d+1}(t), \qquad 0 \leq t \leq 1; \tag{3.2.3}$$

for $d \geq 1$,

$$\frac{1}{n^{2d}} \sum_{j=1}^{[nt]} Z_j^2 \;\Rightarrow\; \kappa_1^2(d) \int_0^t B_d^2(s)ds, \qquad 0 \leq t \leq 1, \tag{3.2.4}$$

$$\frac{1}{n^{d-1/2}} \sum_{j=1}^{n-k} (Z_{j+k} - Z_j) \;\Rightarrow\; k\; \kappa_1(d)\; B_d(1), \qquad 0 \leq t \leq 1, \tag{3.2.5}$$

where $\kappa_1(d)$ *is defined as in Theorem 3.2.1 and* k *is a fixed integer.*

If, in addition, $E|\epsilon_0|^{2/(2d-1)} < \infty$ *and* $\sum_{k=0}^{\infty} k|\psi_k| < \infty$, *then (3.2.4) and (3.2.5) still hold for* $1/2 < d < 1$.

Proof. Recalling $c_j^{(\alpha)} = \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}$ and $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, we have that, for any integer $m \geq 1$ and $\alpha \neq 0, -1, -2, \cdots$,

$$\sum_{j=0}^{m} c_j^{(\alpha)} = 1 + \frac{1}{\alpha\Gamma(\alpha)} \left[\frac{\Gamma(1+m+\alpha)}{\Gamma(1+m)} - \frac{\Gamma(1+\alpha)}{\Gamma(1)} \right] = c_m^{(1+\alpha)} \tag{3.2.6}$$

(see Lemma given in Sowell (1990, p502) with $a = \alpha$ and $b = 1$). This equality implies that, for $d > 1/2$,

$$\begin{aligned} \sum_{j=1}^{[nt]} Z_j &= \sum_{j=1}^{[nt]} \sum_{k=1}^{j} c_{j-k}^{(d)}\, u_k = \sum_{k=1}^{[nt]} u_k \sum_{j=k}^{[nt]} c_{j-k}^{(d)} \\ &= \sum_{k=1}^{[nt]} u_k \sum_{j=0}^{[nt]-k} c_j^{(d)} = \sum_{k=1}^{[nt]} u_k\, c_{[nt]-k}^{(1+d)} = \sum_{k=0}^{[nt]-1} c_k^{(1+d)}\, u_{[nt]-k}. \end{aligned} \tag{3.2.7}$$

By using Theorem 3.2.1 with $1+d$, we obtain the desired (3.2.3).

To prove (3.2.4), we note that (let $\sum_{i=1}^{[ns]} Z_i = 0$ if $s < 1/n$)

$$\frac{1}{n^{2d}} \sum_{j=1}^{[nt]} Z_j^2 = \int_0^t \left(\frac{1}{n^{d-1/2}} Z_{[ns]} \right)^2 ds,$$

and then use the continuous mapping theorem.

Using Theorem 3.2.1, the continuous mapping theorem and

$$\sum_{j=1}^{n-k} (Z_{j+k} - Z_j) = \sum_{j=n-k+1}^{n} Z_j - \sum_{j=1}^{k} Z_j,$$

(3.2.5) follows easily. The proof of Corollary 3.2.2 is complete. □

In the next corollary, we consider asymptotics for stationary and nonstationary fractionally integrated processes without prehistorical influence.

Corollary 3.2.3. *Let* $\alpha > -1/2$,

$$\begin{aligned} (1-B)^\alpha X_t &= u_t, \quad u_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}, \quad t = 1, 2, \cdots, \\ X_t &= 0, \quad t \le 0, \end{aligned} \tag{3.2.8}$$

where $\sum_{k=0}^{\infty} |\psi_k| < \infty$ *and* $b_\psi = \sum_{k=0}^{\infty} \psi_k \neq 0$.

(a) If $\epsilon_k, k = 0, \pm 1, \cdots$, *are iid* $(0, \sigma^2)$, *then, for* $\alpha \ge 0$,

$$\frac{1}{n^{1/2+\alpha}} \sum_{j=1}^{[nt]} X_j \Rightarrow \kappa_2(\alpha) \int_0^t (t-s)^\alpha dW(s), \qquad 0 \le t \le 1, \tag{3.2.9}$$

where $\kappa_2^2(\alpha) = b_\psi^2 \sigma^2 / \Gamma^2(1+\alpha)$.

(b) If, in addition, $E|\epsilon_0|^{2/(2\alpha+1)} < \infty$ *and* $\sum_{k=0}^{\infty} k|\psi_k| < \infty$, *then (3.2.9) still holds for* $-1/2 < \alpha < 0$.

Proof. In Section 3.5, it will be shown that the process X_t defined by (3.2.8) can be rewritten as

$$X_t = \sum_{k=0}^{t-1} c_k^{(\alpha)} u_{t-k}, \qquad t = 1, 2, \cdots,$$

where $c_k^{(\alpha)} = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)}$ is defined as in (3.1.4). If $\alpha = 0$, then $X_t = u_t$ and the result is obvious by using Theorem 3.2.1 with $d = 1$. If $\alpha \neq 0$ and $\alpha > -1/2$, similar to (3.2.7), we obtain that

$$\sum_{j=1}^{[nt]} X_j = \sum_{k=0}^{[nt]-1} c_k^{(1+\alpha)} u_{[nt]-k}.$$

Since $1 + \alpha > 1/2$ when $\alpha > -1/2$, the results follow from Theorem 3.2.1 with $d = 1 + \alpha$. The proof of Corollary 3.2.3 is complete. □

3.3 Proofs of the main results

In Section 3.3.1, some preliminary lemmas required in the proof of Theorem 3.2.1 are given. In Section 3.3.2, using these lemmas, the proof of Theorem 3.2.1 is given.

3.3.1 Preliminary lemmas

To prove Theorem 3.2.1, we start with the following lemmas. For convenience, we always assume $c_j^{(d)} = 0$, if $j < 0$. Otherwise, we recall that $c_j^{(d)} = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}$, for $j \geq 0$, and $\epsilon_j, j = 0, \pm 1, .., $ are iid $(0, \sigma^2)$.

Lemma 3.3.1. *Let $E|\epsilon_0|^\alpha < \infty$, where $\alpha \geq 2$. Then,* $\max_{-n \leq j \leq n} |\epsilon_j| = o_P(n^{1/\alpha})$.

Lemma 3.3.1 is straightforward by using the following equality: for any $\delta > 0$,

$$P\left(\max_{-n \leq j \leq n} |\epsilon_j| \geq \delta\right) = P\left(\sum_{j=-n}^{n} |\epsilon_j|^\alpha I_{(|\epsilon_j| \geq \delta)} \geq \delta^\alpha\right).$$

Lemma 3.3.2. *If $d > 1/2$, then,*

$$\left|c_j^{(d)} - \frac{1}{\Gamma(d)} j^{d-1}\right| \leq C j^{d-2}, \qquad j = 1, 2, \cdots,$$

and

$$\sum_{j=0}^{n} |c_j^{(d)} - c_{j-1}^{(d)}| \leq C n^{\max\{d-1, 0\}}, \qquad n = 1, 2, \cdots.$$

Proof. The first inequality follows from Abramowitz and Stegun, 1970, formula 6.1.47 (also see Akonom and Gourieroux, 1987, Lemma 3). The second one is obvious by using the first inequality (if $d = 1$, the equality comes directly from $c_j^{(1)} = 1, j \geq 0$) and details are omitted. □

Lemma 3.3.3. *Assume that $\eta_j, j \geq 1$, are iid $N(0,1)$ random variables. Then, for $d > 1/2$,*

$$\frac{1}{n^{d-1/2}} \sum_{j=1}^{[nt]} c_{[nt]-j}^{(d)} \eta_j \Rightarrow \frac{1}{\Gamma(d)} B_d(t), \quad 0 \leq t \leq 1. \tag{3.3.1}$$

Proof. It follows from Proposition 4 given by Akonom and Gourieroux (1987) that, for $d > 1/2$,

$$\frac{1}{n^{1/2}} \sum_{j=1}^{[nt]} \left(t - \frac{j}{n}\right)^{d-1} \eta_j \Rightarrow B_d(t), \quad 0 \le t \le 1.$$

This, together with Theorem A.1 (see Appendix A), implies that Lemma 3.3.3 follows easily by noting that

$$\begin{aligned} &\sup_{0\le t\le 1} \left| \sum_{j=1}^{[nt]} c_{[nt]-j}^{(d)} \eta_j - \frac{1}{\Gamma(d)} \sum_{j=1}^{[nt]} (nt-j)^{d-1} \eta_j \right| \\ \le\ & \max_{1\le m\le n} |\eta_m| \sup_{0\le t\le 1} \sum_{j=1}^{[nt]} \left| c_j^{(d)} - \frac{1}{\Gamma(d)} (j + nt - [nt])^{d-1} \right| \\ =\ & o_P\left(n^{\min\{\frac{1}{2}, d-\frac{1}{2}\}+\max\{d-1,0\}}\right) = o_P\left(n^{d-1/2}\right), \end{aligned}$$

where we use Lemma 3.3.2, Lemma 3.3.1 with $\alpha = \max\{2, 2/(2d-1)\}$ and the following well-known bounds: $0 \le nt - [nt] < 1$ and for all $0 \le \theta \le 1$,

$$\left|(j+\theta)^{d-1} - j^{d-1}\right| \le C j^{d-2}, \qquad j = 1, 2, \cdots.$$

The proof of Lemma 3.3.3 is complete. □

Lemma 3.3.4. *Assume that* $E|\epsilon_0|^p < \infty$, *where* $p = \max\{2, 2/(2d-1)\}$. *Then, for* $d > 1/2$,

$$\frac{1}{n^{d-1/2}} \sum_{j=1}^{[nt]} c_{[nt]-j}^{(d)}\, \epsilon_j \Rightarrow \frac{\sigma}{\Gamma(d)} B_d(t), \quad 0 \le t \le 1. \tag{3.3.2}$$

Proof. By using Appendix C.1 (see Appendix C), on a rich enough probability space, there exists a sequence of random variables $\eta_j, j \ge 1$, which are iid $N(0,1)$ such that, for $d > 1/2$,

$$\max_{1\le m\le n} \left| \sum_{j=1}^{m} \epsilon_j - \sigma \sum_{j=1}^{m} \eta_j \right| = o_P\left(n^{\min\{(2d-1)/2, 1/2\}}\right). \tag{3.3.3}$$

It follows from Lemma 3.3.3 that, for $d > 1/2$,

$$\frac{\sigma}{n^{d-1/2}} \sum_{j=1}^{[nt]} c_{[nt]-j}^{(d)} \eta_j \Rightarrow \frac{\sigma}{\Gamma(d)} B_d(t), \quad 0 \le t \le 1.$$

Therefore, by using Theorem A.1 (see Appendix A), Lemma 3.3.4 follows from

$$\begin{aligned}
&\sup_{0\le t\le 1} \left| \sum_{j=1}^{[nt]} c_{[nt]-j}^{(d)} \epsilon_j - \sigma \sum_{j=1}^{[nt]} c_{[nt]-j}^{(d)} \eta_j \right| \\
&= \max_{1\le m\le n} \left| \sum_{j=1}^{m} (c_{m-j}^{(d)} - c_{m-j-1}^{(d)}) \sum_{k=1}^{j} (\epsilon_k - \sigma\eta_k) \right| \\
&\le \sum_{j=0}^{n-1} |c_j^{(d)} - c_{j-1}^{(d)}| \max_{1\le m\le n} \left| \sum_{k=1}^{m} (\epsilon_k - \sigma\eta_k) \right| \\
&= o_P\left(n^{\max\{d-1,0\}+\min\{(2d-1)/2,1/2\}}\right) = o_P\left(n^{d-1/2}\right),
\end{aligned}$$

where we use Lemma 3.3.2 and the relation (3.3.3). The proof of Lemma 3.3.4 is complete. □

3.3.2 Proof of Theorem 3.2.1

Using the lemmas derived in the previous section, the proof of Theorem 3.2.1 is given in this section.

First it is shown that (3.2.2) holds for $d \ge 1$. Let

$$C(B,l) = \sum_{k=0}^{l} \psi_k B^k, \quad C^*(B,l) = \sum_{i=0}^{l-1} \left(\sum_{k=i+1}^{l} \psi_k \right) B^i,$$

where B is a backshift operator. From Lemma 2.1 in Phillips and Solo (1992), we have that

$$C(B,l) = C(1,l) - C^*(B,l)(1-B). \tag{3.3.4}$$

It follows from (3.3.4) that, for all $m \geq 1$ and $l \geq 1$,

$$\begin{aligned}\sum_{j=1}^{m} c_{m-j}^{(d)} \sum_{k=0}^{l} \psi_k \epsilon_{j-k} &= \sum_{j=1}^{m} c_{m-j}^{(d)}\, C(B,l)\epsilon_j \\ &= C(1,l) \sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_j - C^*(B,l) \sum_{j=1}^{m} c_{m-j}^{(d)} (\epsilon_j - \epsilon_{j-1}). \qquad (3.3.5)\end{aligned}$$

Therefore, we can write that, for all $m \geq 1$,

$$\begin{aligned}\sum_{j=1}^{m} c_{m-j}^{(d)} Z_j &= \sum_{j=1}^{m} c_{m-j}^{(d)} \left(\sum_{k=0}^{l} \psi_k \epsilon_{j-k} + \sum_{k=l+1}^{\infty} \psi_k \epsilon_{j-k} \right) \\ &= C(1,l) \sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_j - C^*(B,l) \sum_{j=1}^{m} c_{m-j}^{(d)} (\epsilon_j - \epsilon_{j-1}) \\ &\qquad + \sum_{k=l+1}^{\infty} \psi_k \sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_{j-k} \\ &= C(1,l) \sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_j + R_1(m,l) + R_2(m,l), \quad \text{say.} \qquad (3.3.6)\end{aligned}$$

Because $C(1,l) \to b_\psi$, as $l \to \infty$, by using Theorem A.1 (see Appendix A) and Lemma 3.3.4, it suffices to show, for $d \geq 1$,

$$\lim_{l \to \infty} \lim_{n \to \infty} P\left(\sup_{0 \leq t \leq 1} |R_j([nt], l)| \geq n^{d-1/2} \right) = 0, \qquad j = 1, 2. \qquad (3.3.7)$$

It can be easily shown that, for all $m \geq 1$,

$$\begin{aligned}R_1(m,l) &= \sum_{i=0}^{l-1} \left(\sum_{k=i+1}^{l} \psi_k \right) Y_{m,i}, \qquad \text{where} \\ Y_{m,i} &\equiv \sum_{j=1}^{m} c_{m-j}^{(d)} (\epsilon_{j-i} - \epsilon_{j-i-1}) \\ &= c_0^{(d)} \epsilon_{m-i} - c_{m-1}^{(d)} \epsilon_{-i} + \sum_{j=1}^{m-1} (c_{m-j}^{(d)} - c_{m-j-1}^{(d)}) \epsilon_{j-i}.\end{aligned}$$

Since

$$\max_{1 \leq m \leq n} |Y_{m,i}| \leq \left(2 + \sum_{j=1}^{n-1} |c_j^{(d)} - c_{j-l}^{(d)}| \right) \max_{1 \leq j \leq n} |\epsilon_{j-i}|,$$

it follows from Lemma 3.3.2 and Lemma 3.3.1 with $\alpha = 2$ that, as $n \to \infty$,

$$\begin{aligned}
\sup_{0\le t\le 1} |R_1([nt], l)| &\le \sum_{i=0}^{l-1}\sum_{k=i+1}^{l} |\psi_k| \sup_{0\le t\le 1} \left|Y_{[nt],i}\right| \\
&\le Cn^{d-1} \max_{-l\le j\le n} |\epsilon_j| \sum_{i=0}^{l-1}\sum_{k=i+1}^{l} |\psi_k| = o_P\left(n^{d-1/2}\right). \qquad (3.3.8)
\end{aligned}$$

This proves (3.3.7) for $j = 1$.

To prove (3.3.7) for $j = 2$, we note that, for all $k \ge 1$,

$$\begin{aligned}
E \max_{1\le m\le n} \left|\sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_{j-k}\right| &= E \max_{1\le m\le n} \left|\sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_j\right| \\
&= E \max_{1\le m\le n} \left|\sum_{j=1}^{m} \left(c_{m-j}^{(d)} - c_{m-j-1}^{(d)}\right) \sum_{i=1}^{j} \epsilon_i\right| \\
&\le \sum_{j=1}^{n} \left|c_j^{(d)} - c_{j-l}^{(d)}\right| E \max_{1\le j\le n} \left|\sum_{i=1}^{j} \epsilon_i\right| \\
&\le Cn^{d-1/2},
\end{aligned}$$

where we use the following well-known result:

$$E \max_{1\le j\le n} \left|\sum_{i=1}^{j} \epsilon_i\right| \le \left(E \max_{1\le j\le n} \left|\sum_{i=1}^{j} \epsilon_i\right|^2\right)^{1/2} \le Cn^{1/2}.$$

Therefore, Markov's inequality implies that

$$\begin{aligned}
P\left(\sup_{0\le t\le 1} |R_2([nt], l)| \ge n^{d-1/2}\right) &\le \frac{1}{n^{d-1/2}} \sum_{k=l+1}^{\infty} |\psi_k| \, E \max_{1\le m\le n} \left|\sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_{j-k}\right| \\
&\le C \sum_{k=l+1}^{\infty} |\psi_k|. \qquad (3.3.9)
\end{aligned}$$

Let $n \to \infty$ and then $l \to \infty$, we get (3.3.7) for $j = 2$. This proves (3.2.2) when $d \ge 1$.

Next, the proof of (3.2.2), for $1/2 < d < 1$, is given. We still use (3.3.6), but here we choose $l = n$. Recalling $1/2 < d < 1$ and $E|\epsilon_0|^{2/(2d-1)} < \infty$, it follows from Lemma 3.3.2 and Lemma 3.3.1 with $\alpha = 2/(2d-1)$ that

$$\sum_{j=1}^{n} |c_j^{(d)} - c_{j-l}^{(d)}| \leq C, \quad \max_{-n \leq j \leq n} |\epsilon_j| = o_P(n^{d-1/2}).$$

By noting $\sum_{k=0} k|\psi_k| < \infty$, similar to (3.3.8), we have that

$$\sup_{0 \leq t \leq 1} |R_1([nt], n)| \leq C \max_{-n \leq j \leq n} |\epsilon_j| \sum_{i=0}^{n-1} \sum_{k=i+1}^{n} |\psi_k| = o_P\left(n^{d-1/2}\right). \tag{3.3.10}$$

On the other hand, similar to (3.3.9), we get that

$$\begin{aligned} & P\left(\sup_{0 \leq t \leq 1} |R_2([nt], n)| \geq n^{d-1/2}\right) \\ \leq\ & \frac{1}{n^{d-1/2}} \sum_{k=n+1}^{\infty} |\psi_k|\, E \max_{1 \leq m \leq n} \left|\sum_{j=1}^{m} c_{m-j}^{(d)} \epsilon_{j-k}\right| \\ \leq\ & C n^{1-d} \sum_{k=n+1}^{\infty} |\psi_k| \leq C \sum_{k=n+1}^{\infty} k|\psi_k| = o(1). \end{aligned} \tag{3.3.11}$$

Using (3.3.10) - (3.3.11), Lemma 3.3.4 and Theorem A.1 (see Appendix A), (3.2.2) follows for $1/2 < d < 1$. □

3.4 Sample autocovariances and autocorrelations

The standard sample autocovariances and related autocorrelations based on the process Z_t can be defined as, for $k = 0, 1, 2, \cdots, n-1$,

$$\begin{aligned} \widehat{r}_k &= \frac{1}{n} \sum_{t=1}^{n-k} (Z_t - \overline{Z})(Z_{t+k} - \overline{Z}), \quad \overline{Z} = \frac{1}{n} \sum_{t=1}^{n} Z_t \\ \widehat{\rho}_k &= \widehat{r}_k / \widehat{r}_0, \quad \text{or} \quad \widehat{\rho}_k^* = \widehat{r}_k \Big/ \frac{1}{n} \sum_{j=1}^{n-k} (Z_j - \overline{Z})^2. \end{aligned}$$

In this section, we discuss the asymptotics of $\widehat{r}_k$ and $\widehat{\rho}_k$, which do not appear in previous research. We get the following results.

Theorem 3.4.1. *Let Z_t satisfy (3.1.4) with $d \geq 1$, $\epsilon_k, k = 0, \pm 1, \cdots$, iid $(0, \sigma^2)$, and $\psi_j, j \geq 0$, satisfy (3.2.1). Then, for any fixed integer number k, we have that*

$$n^{1-2d}\ \widehat{r}_k \ \Rightarrow\ \kappa_1^2(d)\ \widetilde{B_d}(1); \tag{3.4.1}$$

$$\frac{n}{k}\ (\widehat{\rho}_k^* - 1) \ \Rightarrow\ \frac{1}{2\widetilde{B_d}(1)}\left(B_d^2(1) - \frac{2}{d}\ B_d(1)B_{d+1}(1)\right), \quad if\ d > 1, \tag{3.4.2}$$

$$\Rightarrow\ \frac{1}{2\widetilde{B_d}(1)}\left(B_d^2(1) - 2B_d(1)B_{d+1}(1) - \frac{1}{kb_\psi^2\sigma^2}E\left(\sum_{j=1}^{k} u_j\right)^2\right), \quad if\ d = 1; \tag{3.4.3}$$

$$\frac{n}{k}(\widehat{\rho}_k - 1) \ \Rightarrow\ -\frac{1}{2\widetilde{B_d}(1)}\left(\frac{B_{d+1}^2(1)}{d^2} + \left(B_d(1) - \frac{1}{d}B_{d+1}(1)\right)^2\right), \quad if\ d > 1, \tag{3.4.4}$$

$$\Rightarrow\ -\frac{1}{2\widetilde{B_d}(1)}\left(B_{d+1}^2(1) + (B_d(1) - B_{d+1}(1))^2 - \frac{1}{kb_\psi^2\sigma^2}E\left(\sum_{j=1}^{k} u_j\right)^2\right), \quad if\ d = 1, \tag{3.4.5}$$

where $\kappa_1(d)$ is defined as in Theorem 3.2.1 and

$$\widetilde{B_d}(1) = \int_0^1 B_d^2(t)dt - \frac{1}{d^2}\ B_{d+1}^2(1).$$

Theorem 3.4.2. *Let Z_t satisfy (3.1.4) with $\epsilon_k, k = 0, \pm 1, \cdots$, iid $(0, \sigma^2)$. Assume that $E|\epsilon_0|^{2/(2d-1)} < \infty$, $\sum_{k=0}^{\infty} k|\psi_k| < \infty$ and $b_\psi \equiv \sum_{k=0}^{\infty} \psi_k < \infty$. Then, for any fixed integer number k and $1/2 < d < 1$, (3.4.1) still holds and*

$$n^{-1+2d}\ (\widehat{\rho}_k^* - 1) \ \Rightarrow\ -\frac{I_k}{2\kappa_1^2(d)\widetilde{B_d}(1)}, \tag{3.4.6}$$

$$n^{-1+2d}(\widehat{\rho}_k - 1) \ \Rightarrow\ -\frac{I_k}{2\kappa_1^2(d)\widetilde{B_d}(1)}, \tag{3.4.7}$$

where $\kappa_1(d)$, $\widetilde{B_d}(1)$ *are defined as in Theorem 3.4.1 and (let* $c_j^{(d)} = 0$ *if* $j < 0$*)*

$$I_k = E\left(\sum_{l=0}^{\infty}(c_l^{(d)} - c_{l-1}^{(d)})\sum_{i=1}^{k} u_{i-l}\right)^2, \quad c_l^{(d)} = \frac{\Gamma(l+d)}{\Gamma(d)\Gamma(l+1)}, \quad l \geq 0.$$

To prove Theorems 3.4.1-3.4.2, we need the following Proposition 3.4.3. The proof of this proposition is quite tedious and will be postponed to later in this section.

Proposition 3.4.3. *Let* Z_t *satisfy (3.1.4) with* $d > 1/2$, $\epsilon_k, k = 0, \pm 1, \cdots$, *be iid* $(0, \sigma^2)$ *and* $\psi_j, j \geq 0$, *satisfy (3.2.1). Then, for any fixed integer number k, we have that*

$$\frac{1}{n}\sum_{j=1}^{n-k}(Z_{j+k} - Z_j)^2 - I_k = o_P(1), \quad for\ 1/2 < d < 1, \tag{3.4.8}$$

where I_k *is defined as in Theorem 3.4.2;*

$$\frac{1}{n}\sum_{j=1}^{n-k}(Z_{j+k} - Z_j)^2 \rightarrow_{a.s.} E\left(\sum_{j=1}^{k} u_j\right)^2, \quad for\ d = 1; \tag{3.4.9}$$

$$\frac{1}{n^{-1+2d}}\sum_{j=1}^{n-k}(Z_{j+k} - Z_j)^2 = o_P(1), \quad for\ d > 1. \tag{3.4.10}$$

In the following, we give the proofs of Theorems 3.4.1-3.4.2.

Proof of Theorem 3.4.1. The idea of the proof Theorem 3.4.1 is similar to that of the proof of Theorem 2.4.1 by using Corollary 3.2.2 and Proposition 3.4.3.

We can write

$$\begin{aligned}&\sum_{t=1}^{n-k}(Z_t - \overline{Z})(Z_{t+k} - \overline{Z}) \\ &= \sum_{t=1}^{n-k} Z_t(Z_{t+k} - Z_t) - \overline{Z}\sum_{t=1}^{n-k}(Z_{t+k} - Z_t) + \sum_{t=1}^{n-k}(Z_t - \overline{Z})^2.\end{aligned} \tag{3.4.11}$$

By using Corollary 3.2.2, Proposition 3.4.3 and the continuous mapping theorem, it can be easily shown that (recalling k is fixed and $\kappa_1(d+1) = \frac{1}{d}\kappa_1(d)$)

$$\frac{1}{\kappa_1^2(d)n^{-1+2d}} \overline{Z} \sum_{t=1}^{n-k}(Z_{t+k} - Z_t) \Rightarrow \frac{k}{d} B_d(1)\, B_{d+1}(1), \tag{3.4.12}$$

$$\begin{aligned}
\frac{1}{\kappa_1^2(d)n^{2d}} \sum_{t=1}^{n-k}(Z_t - \overline{Z})^2 &= \frac{1}{\kappa_1^2(d)n^{2d}} \left(\sum_{t=1}^{n-k} Z_t^2 - 2\overline{Z} \sum_{t=1}^{n-k} Z_t + (n-k)\overline{Z}^2 \right) \\
&\Rightarrow \int_0^1 B_d^2(s)ds - \frac{1}{d^2} B_{d+1}^2(1) \\
&= \widetilde{B_d}(1).
\end{aligned} \tag{3.4.13}$$

On the other hand, by noting

$$\begin{aligned}
\sum_{t=1}^{n-k} Z_t(Z_{t+k} - Z_t) &= \frac{1}{2} \sum_{t=1}^{n-k} \left(Z_{t+k}^2 - Z_t^2 - (Z_{t+k} - Z_t)^2\right) \\
&= \frac{1}{2} \sum_{t=n-k+1}^{n} Z_t^2 - \frac{1}{2} \sum_{t=1}^{k} Z_t^2 - \frac{1}{2} \sum_{t=1}^{n-k} (Z_{t+k} - Z_t)^2,
\end{aligned}$$

we obtain (using Theorem 3.2.1 and Proposition 3.4.3) that, if $d > 1$, then

$$\frac{1}{\kappa_1^2(d)n^{-1+2d}} \sum_{t=1}^{n-k} Z_t(Z_{t+k} - Z_t) \Rightarrow \frac{k}{2} B_d^2(1); \tag{3.4.14}$$

if $d = 1$, then

$$\frac{1}{n\kappa_1^2(1)} \sum_{t=1}^{n-k} Z_t(Z_{t+k} - Z_t) \Rightarrow \frac{k}{2} B_d^2(1) - \frac{1}{2\kappa_1^2(1)} E\left(\sum_{i=1}^{k} u_i\right)^2. \tag{3.4.15}$$

In terms of (3.4.11)-(3.4.15) and

$$\widehat{\rho_k^*} - 1 = \left(\sum_{t=1}^{n-k} Z_t(Z_{t+k} - Z_t) - \overline{Z} \sum_{t=1}^{n-k} (Z_{t+k} - Z_t) \right) \Big/ \sum_{t=1}^{n-k} (Z_t - \overline{Z})^2, \tag{3.4.16}$$

(3.4.1)-(3.4.3) follow easily from the continuous mapping theorem.

To prove (3.4.4) and (3.4.5), write

$$d_k = \sum_{t=1}^{n-k}(Z_t - \overline{Z})^2 \Big/ \sum_{t=1}^{n}(Z_t - \overline{Z})^2 .$$

By using methods similar to those methods used before, we have that

$$\begin{aligned}
n(d_k - 1) &= -n \sum_{t=n-k+1}^{n} (Z_t - \overline{Z})^2 \Big/ \sum_{t=1}^{n}(Z_t - \overline{Z})^2 \\
&= \frac{n}{\sum_{t=1}^{n}(Z_t - \overline{Z})^2}\left(2\overline{Z} \sum_{t=n-k+1}^{n} Z_t - \sum_{t=n-k+1}^{n} Z_t^2 - k\overline{Z}^2\right) \\
&\Rightarrow \frac{k}{\widetilde{B_d}(1)}\left(\frac{2}{d}B_d(1)B_{d+1}(1) - B_d^2(1) - \frac{1}{d^2}B_{d+1}^2(1)\right) \\
&= -\frac{k}{\widetilde{B_d}(1)}\left(B_d(1) - \frac{1}{d}B_{d+1}(1)\right)^2 . \qquad (3.4.17)
\end{aligned}$$

Since (3.4.17) implies that $d_n \to_P 1$, (3.4.4) and (3.4.5) follow easily from (3.4.2), (3.4.3), (3.4.17), the relation

$$\widehat{\rho}_k - 1 = d_k(\widehat{\rho}_k^* - 1) + d_k - 1, \qquad (3.4.18)$$

and the continuous mapping theorem. □

Proof of Theorem 3.4.2. Under the conditions of Theorem 3.4.2, we note that (3.4.12), (3.4.13) and (3.4.17) still hold for $1/2 < d < 1$ by using Theorem 3.2.1. Therefore, for $1/2 < d < 1$, we have that

$$\begin{aligned}
\frac{1}{n}\overline{Z}\sum_{t=1}^{n-k}(Z_{t+k} - Z_t) &= o_P(1), \qquad (3.4.19) \\
n^{-1+2d}(d_k - 1) &= o_P(1). \qquad (3.4.20)
\end{aligned}$$

On the other hand, similar to (3.4.15), it follows from (3.4.8) that, for $1/2 < d < 1$,

$$\frac{1}{n}\sum_{t=1}^{n-k} Z_t(Z_{t+k} - Z_t) - \frac{1}{2}\, I_k = o_P(1). \qquad (3.4.21)$$

In terms of (3.4.13), (3.4.16), (3.4.19) and (3.4.21), we obtain that

$$n^{-1+2d}\left(\widehat{\rho}_k^* - 1\right) \;\Rightarrow\; -\frac{I_k}{2\kappa_1^2(d)\widetilde{B_d}(1)}. \tag{3.4.22}$$

This implies (3.4.6). Finally, (3.4.7) follows from (3.4.22), (3.4.18) and (3.4.20). □

At the end of this section, we give the proof of Proposition 3.4.3.

Proof. We write

$$\begin{aligned} Z_{j+k} - Z_j &= \sum_{l=j+1}^{j+k} c_{j+k-l}^{(d)} u_l + \sum_{l=1}^{j} (c_{j+k-l}^{(d)} - c_{j-l}^{(d)}) u_l \\ &= I_{1j}^{(k)} + I_{2j}^{(k)}, \quad say. \end{aligned} \tag{3.4.23}$$

Clearly, we have that

$$I_{1j}^{(k)} = \sum_{i=0}^{\infty} \psi_i \sum_{l=j+1}^{j+k} c_{j+k-l}^{(d)}\, \epsilon_{l-i} = \sum_{i=0}^{\infty} \psi_i \sum_{l=1}^{k} c_{k-l}^{(d)}\, \epsilon_{l+j-i}.$$

This, together with Hölder's inequality, implies that, for $d > 1/2$,

$$\begin{aligned} E\left(I_{1j}^{(k)}\right)^2 &\le \sum_{i=0}^{\infty} |\psi_i| \sum_{i=0}^{\infty} |\psi_i|\, E\left(\sum_{l=1}^{k} c_{k-l}^{(d)} \epsilon_{l+j-i}\right)^2 \\ &\le \left(\sum_{i=0}^{\infty} |\psi_i|\right)^2 \sum_{l=0}^{k-1} \left(c_l^{(d)}\right)^2 E\epsilon_0^2 \le C k^{\max\{2d-1,0\}}. \end{aligned} \tag{3.4.24}$$

Similarly, we have that, for $d > 1/2$,

$$\begin{aligned} E\left(I_{2j}^{(k)}\right)^2 &\le \sum_{i=0}^{\infty} |\psi_i| \sum_{i=0}^{\infty} |\psi_i|\, E\left(\sum_{l=1}^{j} (c_{j+k-l}^{(d)} - c_{j-l}^{(d)})\, \epsilon_{l-i}\right)^2 \\ &\le \left(\sum_{i=0}^{\infty} |\psi_i|\right)^2 \sum_{l=0}^{j-1} \left(c_{k+l}^{(d)} - c_l^{(d)}\right)^2 E\epsilon_0^2 \\ &\le C j^{\max\{2d-3,0\}}, \end{aligned} \tag{3.4.25}$$

where we use the inequality that (recalling Lemma 3.3.2), for any fixed k,

$$|c_{k+l}^{(d)} - c_l^{(d)}| \leq Ck\ l^{d-2}, \quad l = 1, 2, \cdots. \tag{3.4.26}$$

In terms of (3.4.23)-(3.4.25), it follows that, for $d > 1/2$,

$$\begin{aligned}\sum_{j=1}^{n-k} E\left(Z_{j+k} - Z_j\right)^2 &\leq 2\sum_{j=1}^{n-k}\left(E\left(I_{1j}^{(k)}\right)^2 + E\left(I_{2j}^{(k)}\right)^2\right) \\ &\leq C\ n^{\max\{1,2d-2\}}. \end{aligned}\tag{3.4.27}$$

By noting $2d - 2 > 1$, if $d > 1$, (3.4.10) follows immediately from (3.4.27) and Markov's inequality.

If $d = 1$, then $Z_{j+k} - Z_j = \sum_{l=1}^{k} u_{l+j}, j \geq 1$, is a stationary linear process (see Theorem B.1, Appendix B). (3.4.9) follows from the stationary ergodic theorem (Theorem B.2, see Appendix B).

The proof of (3.4.8) is more laborious . Write $c_l^* = c_l^{(d)} - c_{l-1}^{(d)}, l \geq 0$ for convenience. By noting $c_0^* = 1$, $|c_l^*| \leq Cl^{d-2}, l \geq 1$ (see (3.4.26)) and $1/2 < d < 1$, it follows that, for every $t \geq 1$,

$$\sum_{l=0}^{\infty} |c_l^*|\ E|u_{t-l}| \leq \left(1 + C\sum_{l=1}^{\infty} l^{d-2}\right)\sum_{k=0}^{\infty} |\psi_k| E|\epsilon_0| < \infty.$$

This implies that, for every $t \geq 1$, $\sum_{l=0}^{\infty} c_l^*\ u_{t-l}$ is well defined. Therefore, we can write (let $c_j^{(d)} = 0$ if $j < 0$)

$$\begin{aligned}Z_{j+k} - Z_j &= \sum_{i=1}^{k}\left(Z_{j+i} - Z_{j+i-1}\right) = \sum_{i=1}^{k}\sum_{l=1}^{j+i}\left(c_{j+i-l}^{(d)} - c_{j+i-l-1}^{(d)}\right)\ u_l \\ &= \sum_{i=1}^{k}\sum_{l=0}^{j+i-1}\left(c_l^{(d)} - c_{l-1}^{(d)}\right)\ u_{j+i-l} \\ &= \sum_{i=1}^{k}\sum_{l=0}^{\infty} c_l^*\ u_{j+i-l} - \sum_{i=1}^{k}\sum_{l=j+i}^{\infty} c_l^*\ u_{j+i-l} \\ &= I_{3j}^{(k)} + I_{4j}^{(k)}, \qquad \textit{say}. \end{aligned}\tag{3.4.28}$$

By Hölder's inequality, it can be easily shown that

$$\begin{aligned}
\sum_{j=1}^{n} E\left(I_{4j}^{(k)}\right)^2 &\leq k \sum_{j=1}^{n}\sum_{i=1}^{k} E\left(\sum_{l=0}^{\infty} c_{l+j+i}^{*}\, u_{-l}\right)^2 \\
&\leq k^2 \sum_{j=1}^{n}\sum_{l=0}^{\infty} |c_{l+j}^{*}| \sum_{l=0}^{\infty} |c_{l+j}^{*}|\, Eu_{-l}^2 \\
&\leq C \sum_{j=1}^{n}\left(\sum_{l=j}^{\infty} l^{d-2}\right)^2 = O\left(n^{2d-1}\right), \qquad (3.4.29)
\end{aligned}$$

where we use the bound: $Eu_t^2 = \sum_{k=0}^{\infty}\psi_k^2 E\epsilon_0^2 < \infty$, for any $t = 0, \pm 1, \pm 2, \cdots$. Similarly, we get that

$$\sum_{j=1}^{n} E\left(I_{3j}^{(k)}\right)^2 \leq k^2 \sum_{j=1}^{n}\left(\sum_{l=0}^{\infty}|c_l^{*}|\right)^2 Eu_0^2 = O\left(n\right). \qquad (3.4.30)$$

In terms of (3.4.29), (3.4.30) and

$$(Z_{j+k}-Z_j)^2 - \left(I_{3j}^{(k)}\right)^2 = I_{4j}^{(k)}\left(Z_{j+k}-Z_j+I_{3j}^{(k)}\right) = I_{4j}^{(k)}\left(2I_{3j}^{(k)}+I_{4j}^{(k)}\right),$$

it follows that

$$\begin{aligned}
&\sum_{j=1}^{n} E\left|(Z_{j+k}-Z_j)^2 - \left(I_{3j}^{(k)}\right)^2\right| \\
\leq\ & \sum_{j=1}^{n}\left[E\left(I_{4j}^{(k)}\right)^2\right]^{1/2}\left[E\left(2I_{3j}^{(k)}+I_{4j}^{(k)}\right)^2\right]^{1/2} \\
\leq\ & \left(\sum_{j=1}^{n} E\left(I_{4j}^{(k)}\right)^2\right)^{1/2}\left(\sum_{j=1}^{n} E\left(2I_{3j}^{(k)}+I_{4j}^{(k)}\right)^2\right)^{1/2} \\
=\ & O(n^d). \qquad (3.4.31)
\end{aligned}$$

Recalling $1/2 < d < 1$, (3.4.31) implies that

$$\frac{1}{n}\sum_{j=1}^{n-k}\left[(Z_{j+k}-Z_j)^2 - \left(I_{3j}^{(k)}\right)^2\right] = o_P(1).$$

Now (3.4.8) follows if

$$\frac{1}{n}\sum_{j=1}^{n-k}\left(I_{3j}^{(k)}\right)^2 - E\left(\sum_{i=1}^{k}\sum_{l=0}^{\infty} c_l^* \, u_{i-l}\right)^2 = o_P(1). \qquad (3.4.32)$$

Recalling that $u_t, t = 0, \pm 1, \cdots$, is a stationary ergodic linear process, by using Theorem B.1 (see Appendix B), $I_{3j}^{(k)}, j \geq 1$, still has same properties as those of the process u_t. Therefore, (3.4.32) follows from the stationary ergodic theorem (Theorem B.2, see Appendix B). We finish the proof of Proposition 3.4.3. □

3.5 A complementary proposition

In this section, we give a complementary proposition. The proposition is used to show that the process X_t, which is defined by (3.1.1) and has no prehistorical influence, is a special case of the process Z_t defined by (3.1.4).

Proposition 3.5.1. *Let $d > -1/2$,*

$$\begin{aligned} (1-B)^d X_t &= v_t, \qquad t = 1, 2, \cdots, \\ X_t &= 0, \qquad t \leq 0, \end{aligned} \qquad (3.5.1)$$

where $v_t, t \geq 1$ is an arbitrary well-defined process, B is a backshift operator and the fractional difference operator $(1-B)^\gamma$ is defined as in (2.1.2).

Then, we have that $X_t = 0, t \leq 0$, and

$$X_t = \sum_{k=0}^{t-1} c_k^{(d)} v_{t-k}, \qquad t = 1, 2, \cdots, \qquad (3.5.2)$$

where $c_0^{(0)} = 1, c_k^{(0)} = 0, k \geq 1$, and $c_k^{(\alpha)} = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)}, k \geq 0$, for $\alpha \neq 0, -1, -2, \cdots$.

Proof. At first, we assume that $d > -1/2$ and $d \neq 0, 1, 2, \cdots$. Under this assumption, we first show that

$$c_t^{(d)} = -\sum_{k=1}^{t} c_{t-k}^{(d)}\, c_k^{(-d)}, \qquad t = 1, 2, \cdots . \tag{3.5.3}$$

Recalling $\Gamma(z+1) = z\Gamma(z)$ (for all $z \neq 0, -1, \cdots$, by definition), it is obvious that

$$c_1^{(d)} = \frac{\Gamma(1+d)}{\Gamma(d)\Gamma(2)} = d, \qquad c_1^{(-d)} = \frac{\Gamma(1-d)}{\Gamma(-d)\Gamma(2)} = -d.$$

Hence, (3.5.3) holds for $t = 1$. We next assume that (3.5.3) holds for $t = n$, i.e.,

$$c_t^{(d)} = -\sum_{k=1}^{t} c_{t-k}^{(d)}\, c_k^{(-d)}, \qquad t = 1, 2, \cdots, n. \tag{3.5.4}$$

By induction, it suffices to show that

$$c_{n+1}^{(d)} = -\sum_{k=1}^{n+1} c_{n+1-k}^{(d)}\, c_k^{(-d)} = -\sum_{k=0}^{n} c_k^{(d)}\, c_{n+1-k}^{(-d)}. \tag{3.5.5}$$

To prove (3.5.5), by summing each term of (3.5.4), we obtain that

$$\begin{aligned} \sum_{t=1}^{n} c_t^{(d)} &= -\sum_{t=1}^{n}\sum_{k=1}^{t} c_{t-k}^{(d)}\, c_k^{(-d)} \\ &= -\sum_{k=1}^{n} c_k^{(-d)} \sum_{t=k}^{n} c_{t-k}^{(d)} = -\sum_{k=1}^{n} c_k^{(-d)} \sum_{t=0}^{n-k} c_t^{(d)} \\ &= -\sum_{k=0}^{n-1} c_{n-k}^{(-d)} \sum_{t=0}^{k} c_t^{(d)}. \end{aligned} \tag{3.5.6}$$

By using (3.2.6) and the definition of $c_k^{(\alpha)}$, it can be easily shown that, for all $k \geq 1$ and $\alpha \neq 0, -1, -2, \cdots$,

$$\sum_{t=0}^{k} c_t^{(\alpha)} = c_k^{(1+\alpha)} = \frac{k+1}{\alpha} c_{k+1}^{(\alpha)}. \tag{3.5.7}$$

In terms of (3.5.6), $c_0^{(d)} = 1$ and (3.5.7) with $\alpha = d$, it follows that, for $d > -1/2$ and $d \neq 0, 1, 2, \cdots$,

$$\frac{n+1}{d} c_{n+1}^{(d)} - 1 = -\frac{1}{d} \sum_{k=0}^{n-1} (k+1) c_{n-k}^{(-d)} \, c_{k+1}^{(d)} = -\frac{1}{d} \sum_{k=1}^{n} k \; c_k^{(d)} \; c_{n+1-k}^{(-d)}. \tag{3.5.8}$$

On the other hand, (3.5.4) also implies that (recalling $c_0^{(d)} = c_0^{(-d)} = 1$)

$$-c_t^{(-d)} = \sum_{k=0}^{t-1} c_{t-k}^{(d)} \, c_k^{(-d)} = \sum_{k=1}^{t} c_k^{(d)} \, c_{t-k}^{(-d)}, \qquad t = 1, 2, \cdots, n. \tag{3.5.9}$$

By summing each term of (3.5.9), it follows that

$$-\sum_{t=1}^{n} c_t^{(-d)} = \sum_{k=1}^{n} c_k^{(d)} \sum_{t=0}^{n-k} c_t^{(-d)}. \tag{3.5.10}$$

In terms of (3.5.10), $c_0^{(-d)} = 1$ and (3.5.7) with $\alpha = -d$, we obtain that, for $d > -1/2$ and $d \neq 0, 1, 2, \cdots$,

$$\frac{n+1}{d} c_{n+1}^{(-d)} + 1 = \frac{1}{d} \sum_{k=1}^{n} (k - n - 1) \; c_k^{(d)} \; c_{n+1-k}^{(-d)}. \tag{3.5.11}$$

Now (3.5.5) follows immediately from summing the two sides of (3.5.8) and (3.5.11). This gives (3.5.3) by induction.

Because of (3.5.3), we can give the proof of Proposition 3.5.1 for $d > -1/2$ and $d \neq 0, 1, 2, \cdots$. Clearly, we only need to consider the case of $t \geq 1$. Recalling the definition of the fractional difference operator $(1-B)^\gamma$ (see (3.1.2)) and $X_t = 0, t \leq 0$, we rewrite (3.5.1) as

$$\sum_{j=0}^{t-1} c_j^{(-d)} \; X_{t-j} = v_t, \qquad t = 1, 2, \cdots. \tag{3.5.12}$$

It follows from (3.5.12) that, if $t = 1$, then $X_1 = v_1$, i.e., (3.5.2) holds for $t = 1$. Next we assume that (3.5.2) holds for $t = 2, \cdots, n$, i.e.,

$$X_j = \sum_{k=0}^{j-1} c_k^{(d)} v_{j-k}, \qquad j = 1, 2, \cdots, n. \tag{3.5.13}$$

By induction, it suffices to show that (3.5.13) also holds for $j = n+1$. To do this, we use (3.5.12) with $t = n+1$. In this case, recalling (3.5.13), (3.5.3) and $c_0^{(-d)} = 1$, we have that

$$\begin{aligned}
X_{n+1} &= v_{n+1} - \sum_{j=1}^{n} c_j^{(-d)} X_{n+1-j} \\
&= v_{n+1} - \sum_{j=1}^{n} c_j^{(-d)} \sum_{k=0}^{n-j} c_k^{(d)} v_{n+1-j-k} \\
&= v_{n+1} - \sum_{j=1}^{n} c_j^{(-d)} \sum_{k=j}^{n} c_{k-j}^{(d)} v_{n+1-k} \\
&= v_{n+1} - \sum_{k=1}^{n} v_{n+1-k} \sum_{j=1}^{k} c_{k-j}^{(d)} c_j^{(-d)} \\
&= v_{n+1} - \sum_{k=1}^{n} c_k^{(d)} v_{n+1-k} \\
&= -\sum_{k=0}^{n} c_k^{(d)} v_{n+1-k}. \qquad (3.5.14)
\end{aligned}$$

This implies that (3.5.13) holds for $j = n+1$ and hence the proof of Proposition 3.5.1 is complete for $d > -1/2$ and $d \neq 0, 1, 2, \cdots$.

Next we show that Proposition 3.5.1 holds for $d = 0, 1, 2, \cdots$. Recalling the definition of $c_k^{(0)}, k \geq 0$, Proposition 3.5.1 is obvious for $d = 0$. In the following, we assume that Proposition 3.5.1 holds for $d = m$. We will prove that (3.5.1) for $d = m+1$, i.e.,

$$\begin{aligned}
(1-B)^{m+1} X_t &= v_t, \qquad t = 1, 2, \cdots, \\
X_t &= 0, \qquad t \leq 0, \qquad (3.5.15)
\end{aligned}$$

implies that

$$X_t = \sum_{k=0}^{t-1} c_k^{(m+1)} v_{t-k}, \qquad t = 1, 2, \cdots, \qquad (3.5.16)$$

and then Proposition 3.5.1 follows by induction. To do this, let $Y_t = 0, t \le 0$, and

$$Y_t = (1-B)X_t, \qquad t = 1, 2, \cdots .$$

It can be easily checked that $X_t = \sum_{j=1}^{t} Y_j$ and, by using (3.5.15),

$$(1-B)^m Y_t = (1-B)^{m+1} X_t = v_t, \quad t = 1, 2, \cdots . \tag{3.5.17}$$

(3.5.17) implies that Y_t satisfy (3.5.1) for $d = m$. By using the assumption that Proposition 3.5.1 holds for $d = m$, we obtain

$$Y_t = \sum_{k=0}^{t-1} c_k^{(m)} v_{t-k}, \qquad t = 1, 2, \cdots ,$$

and therefore,

$$\begin{aligned} X_t &= \sum_{j=1}^{t} Y_j \\ &= \sum_{j=1}^{t} \sum_{k=0}^{j-1} c_k^{(m)} v_{j-k} = \sum_{j=1}^{t} \sum_{k=1}^{j} c_{j-k}^{(m)} v_k \\ &= \sum_{k=1}^{t} v_k \sum_{j=k}^{t} c_{j-k}^{(m)} = \sum_{k=0}^{t-1} v_{t-k} \sum_{j=0}^{k} c_j^{(m)} \\ &= \sum_{k=0}^{t-1} v_{t-k} c_k^{(m+1)}, \qquad t = 1, 2, \cdots , \end{aligned}$$

where we use the well-known equality[1]: for all $m \ge 1$,

$$\sum_{j=0}^{k} c_j^{(m)} = c_k^{(m+1)}, \qquad k = 0, 1, 2, \cdots .$$

The proof of Proposition 3.5.1, for $d = 0, 1, 2, \cdots$, is complete. $\square$

[1]This equality can be easily proved by induction. In fact, equality is obvious for $k = 0$. If $\sum_{j=0}^{k} c_j^{(m)} = c_k^{(m+1)}$, then

$$\sum_{j=0}^{k+1} c_j^{(m)} = c_{k+1}^{(m)} + \sum_{j=0}^{k} c_j^{(m)} = c_{k+1}^{(m)} + c_k^{(m+1)} = c_{k+1}^{(m+1)}.$$

By induction, equality follows immediately.

Chapter 4

Asymptotics for linear processes with dependent innovations

Let X_t be a linear process defined by $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$, $t = 1, 2, \cdots$, where $\{\epsilon_k\}$ is a sequence of random variables with mean zero and $\{\psi_k\}$ is a sequence of real numbers. Under conditions on $\{\psi_k\}$ which entail that $\{X_t\}$ is either a long memory process or a linear process with summable weights, we study asymptotics of the partial sum process $\sum_{t=1}^{[ns]} X_t$ in this chapter. For long memory processes with the innovations being stationary ergodic martingale differences, the functional limit theorem of $\sum_{t=1}^{[ns]} X_t$ is derived. For linear processes with summable weights, we give rather general sufficient conditions for $\sum_{t=1}^{[ns]} X_t$ (properly normalized) converging weakly to a standard Brownian motion. The applications to fractionally integrated processes and other processes with innovations satisfying certain mixing conditions are also discussed in this chapter.

4.1 Introduction

Consider a linear process X_t defined by

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}, \qquad t = 1, 2, \cdots, \tag{4.1.1}$$

where $\{\epsilon_k\}$ is a sequence of random variables with mean zero and $\{\psi_k\}$ is a sequence of real numbers. In time series analysis, this process has great importance. Many important time series models, such as the causal ARMA process (Brockwell and Davis, p89), have the type (4.1.1) with $\sum_{k=0}^{\infty} |\psi_k| < \infty$. Let Z_t, $t = 1, 2, \cdots$ denote a covariance-stationary, purely non-deterministic time series with mean zero and autocovariance function $\gamma_Z(k) = cov(Z_t, Z_{t-k})$. More generally, as shown by Mcleod (1998) (also see Beran, 1994, Lemma 5.1), the process (4.1.1) with $\psi_k \sim c_\psi k^{-\alpha}$ covers long memory processes, such as the fractional Gaussian noise process (Mandelbrot, 1983) and the simple fractional process (Granger and Joyeux 1980; Hosking, 1981, also see Chapter 2), which are characterized by the property that $\gamma_Z(k) \sim c_\gamma k^{-\alpha}$ for some $\alpha \in (1/2, 1)$ and $c_\gamma > 0$.

On the asymptotics of the process X_t, assuming the innovations ϵ_k are iid $(0, 1)$, Avram and Taqqu (1987) proved that, if $\psi_k \sim k^{-\alpha} l(k)$, where $1/2 < \alpha < 1$ and $l(k)$ is a slowly varying function at infinity, then $\sum_{t=1}^{[ns]} X_t$, properly normalized, converges weakly to a fractional Brownian motion. Theorem 2 of Avram and Taqqu (1987) improved the previous results given by Davydov (1970) and it was extended later by Mielniczuk (1997). However, few results show what would happen for $\sum_{t=1}^{[ns]} X_t$ when innovations ϵ_k are dependent random variables.

In this chapter, the previous cited results are extended to dependent innovations. Explicitly, we derive the functional limit theorem of the partial sum process of the

X_t when the innovations ϵ_k form a martingale difference sequence. These results will be stated in next section. We also discuss the limiting behaviour of $\sum_{t=1}^{[ns]} X_t$ when X_t satisfy model (4.1.1) with $\sum_{k=0}^{\infty} |\psi_k| < \infty$, and give rather general sufficient conditions so that $\sum_{t=1}^{[ns]} X_t$ (properly normalized) converges weakly to a standard Brownian motion. The proofs of these results will be given in Section 4.3.

For reading convenience, we end this section by introducing some notation. Throughout the chapter, $C, C_1, C_2 \cdots$ are positive constants, which might take on different values at different places. $W(t)$ denotes a standard Brownian motion and $W_d(t)$, $d \in (-1/2, 1/2)$ is a "type I" fractional Brownian motion on $D[0,1]$ as defined in Chapter 2, i.e.,

$$W_d(t) = \frac{1}{A(d)} \int_{-\infty}^{0} \left[(t-s)^d - (-s)^d\right] dW(s) + \int_0^t (t-s)^d dW(s),$$

where

$$A(d) = \left(\frac{1}{2d+1} + \int_0^{\infty} \left[(1+s)^d - s^d\right]^2 ds\right)^{1/2}.$$

As shown in Chapter 2, $W_d(t)$ is a self-similar Gaussian process with covariance

$$E\left(W_d(s) W_d(t)\right) = \frac{1}{2}\left\{s^{1+2d} + t^{1+2d} - |s-t|^{1+2d}\right\}, \qquad \text{for } 0 \le s, t \le 1.$$

4.2 Main Results

We now present our main results. Let X_t satisfy model (4.1.1). Put

$$\psi_k = \begin{cases} k^{-\alpha} l(k), & \text{if } k \ge 0, \\ 0, & \text{if } k < 0, \end{cases} \qquad \text{where } 1/2 < \alpha < 1, \tag{4.2.1}$$

and $l(x)$ ($l(0)/0 \equiv 1$) is a positive function slowly varying at infinity. In this section, we call X_t a linear process with summable weights or a long memory process[1] if

[1] For a more general definition of long memory process, see Beran (1994), page 42.

$\sum_{k=0}^{\infty} |\psi_k| < \infty$ or ψ_k satisfies (4.2.1). Since the results for a linear process with summable weights differ from those in the case of a long memory process, for more clarity, we shall deal with the two cases separately.

4.2.1 Long memory process

In this section, we derive a functional limit theorem for the partial sum process of the X_t, where $X_t, t \geq 1$, is a long memory process with innovations ϵ_k forming a martingale difference sequence. This extends the previous results given by Davydov (1970), Avram and Taqqu (1987) and Mielniczuk (1997), who discussed a similar question with the innovations ϵ_k being iid random variables. We start with the following definition.

Definition 4.2.1. Sequence $\{\epsilon_k\}$ is said to be a martingale difference sequence if

$$E\left(\epsilon_k | \mathcal{F}_{k-1}\right) = 0, \quad a.s. \;\; k = 0, \pm 1, \pm 2, \cdots,$$

where $\mathcal{F}_k = \sigma\{\epsilon_i, i \leq k\}$, i.e., $\mathcal{F}_k$ is the σ-field generated by $\{\epsilon_i, i \leq k\}$.

The following theorem gives sufficient conditions for the derivation of the functional limit theorem. The proof will be given in Section 4.3.2.

Theorem 4.2.1. *Let $X_t, t \geq 1$, satisfy model (4.1.1). Assume that*

(a) $\{\epsilon_k\}$ is a stationary ergodic martingale difference sequence with $0 < E\epsilon_0^2 < \infty$;

(b) $\{\psi_k\}$ satisfies (4.2.1) and there exist positive constants A_1 and A_2 such that

$$\left|\frac{l(m+n)}{l(n)} - 1\right| \leq C\,\frac{m}{n}, \quad \text{if } A_1 \leq m \leq A_2 n. \tag{4.2.2}$$

Then, for $1/2 < \alpha < 1$,

$$\frac{1}{n^H l(n)} \sum_{t=1}^{[ns]} X_t \Rightarrow \Delta_\alpha \, W_{1-\alpha}(s), \qquad 0 \le s \le 1, \tag{4.2.3}$$

where

$$H = \frac{1}{2}(3-2\alpha), \quad \Delta_\alpha^2 = \frac{\chi_\alpha E\epsilon_0^2}{(1-\alpha)(3-2\alpha)} \quad \textit{with} \quad \chi_\alpha = \int_0^\infty x^{-\alpha}(x+1)^{-\alpha} dx.$$

Remark 4.2.1. The relation (4.2.2) is a weak assumption, which is satisfied by a large class of slowly varying functions such as $\log^\beta x$, $(\log\log x)^\beta$ and $e^{\log^\gamma x}$, where β is real and $0 < \gamma < 1$. For example, to show that $\log^\beta x$ satisfies (4.2.2), by noting

$$\log(1+x) \le x \quad \text{and} \quad (1+x)^\beta \le 1 + \max\{2^\beta, 1\}x,$$

for $0 \le x \le 1$, we obtain that, if $\beta \ge 0$, then

$$\begin{aligned} \left|\frac{\log^\beta(m+n)}{\log^\beta n} - 1\right| &= \left|\left(1 + \frac{\log(1+m/n)}{\log n}\right)^\beta - 1\right| \\ &\le \max\{2^\beta, 1\}\frac{\log(1+m/n)}{\log n} \le \max\{2^\beta, 1\} m/n, \end{aligned}$$

for all $n \ge m \ge 3$.

Similarly, if $\beta < 0$, then

$$\begin{aligned} \left|\frac{\log^\beta(m+n)}{\log^\beta n} - 1\right| &= \left|\frac{\log^\beta(m+n)}{\log^\beta n}\left(\frac{\log^{-\beta}(m+n)}{\log^{-\beta} n} - 1\right)\right| \\ &\le \max\{2^{-\beta}, 1\} m/n, \end{aligned}$$

for all $n \ge m \ge 3$. Therefore, $\log^\beta x$, where β is real, satisfies (4.2.2).

Define a fractionally integrated process by

$$(1-B)^d Y_t = \epsilon_t, \quad t = 1, 2, \cdots, \tag{4.2.4}$$

where the fractional difference operator $(1-B)^d$ is defined as in (2.1.2). As mentioned in Section 4.1, model (4.1.1) covers the process defined by (4.2.4). As a direct consequence of Theorem 4.2.1, we have the following Corollary 4.2.2, which extends Theorem 2 given by Sowell (1990) to martingale difference sequence.

Corollary 4.2.2. *Let Y_t satisfy model (4.2.4) with $0 < d < 1/2$, where $\{\epsilon_k\}$ is a stationary ergodic martingale difference sequence with $0 < E\epsilon_0^2 < \infty$. Then,*

$$\frac{1}{n^{1/2+d}} \sum_{j=1}^{[nt]} Y_j \Rightarrow \kappa_3(d)\; W_d(t), \quad 0 \le t \le 1, \tag{4.2.5}$$

where $\kappa_3^2(d) = \frac{E\epsilon_0^2 \Gamma(1-2d)}{(1+2d)\Gamma(1+d)\Gamma(1-d)}$.

Proof. Similar to Hosking (1981) (also see Lemma 2.6.3), we have that

$$Y_t = \sum_{k=0}^{\infty} c_k \epsilon_{t-k}, \quad \text{where } c_k = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}.$$

Write

$$\psi_0 = 1, \quad \psi_k = k^{d-1} l(k),\ k \ge 1, \quad \text{where} \quad l(k) = k^{1-d}\, c_k.$$

It follows from Lemma 4.3.6 that $l(k)$ is a slowly varying function at infinity satisfying (4.2.2) and $l(k) \sim 1$. Hence, Theorem 4.2.1 with $\alpha = 1-d$ implies that

$$\frac{1}{n^{1/2+d}} \sum_{j=1}^{[nt]} Y_j \Rightarrow \frac{\chi_{1-d} E\epsilon_0^2}{d(1+2d)}\, W_d(t), \quad 0 \le t \le 1.$$

Therefore, (4.2.5) follows immediately from $\chi_{1-d} = \Gamma(1-2d)/\left(\Gamma(d)\Gamma(1-d)\right)$ (see Mielniczuk, 1997) and $\Gamma(1+d) = d\,\Gamma(d)$. □

Remark 4.2.2. By a method similar to that used in Theorem 2.3.1, Corollary 4.2.2 can be extended to $d = d_0 + m$, where $0 < d_0 < 1/2$ and $m \ge 1$ is a integer. The details are omitted.

4.2.2 Linear process with summable weights

In this section, we derive the functional limit theorem for the partial sum process of the X_t, where X_t is a linear process with summable weights. The following Theorem 4.2.3 gives rather general sufficient conditions for $\sum_{t=1}^{[ns]} X_t$ (properly normalized) to converge weakly to a standard Brownian motion. These conditions differ from Theorem 6 given by Stadtmüller and Trautner (1985), who established a similar result by assuming

$$E\epsilon_i = 0 \quad E\epsilon_i\epsilon_j = \delta_{ij} \quad \text{for } i, j = 0, \pm 1, \pm 2, \cdots$$

$$E\epsilon_i^4 \leq C \quad \text{for all } i = 0, \pm 1, \pm 2, \cdots$$

$$E(\epsilon_i\epsilon_j\epsilon_k\epsilon_l) = E(\epsilon_i^2\epsilon_k\epsilon_l) = 0 \quad \text{if } i, j, k, l \text{ are different indices,}$$

where $\delta_{ij} = 0$, if $i \neq j$; 1 if $i = j$.

Theorem 4.2.3. *Let X_t satisfy model (4.1.1) and $d(n)$ be a positive sequence of real numbers satisfying $d(n) \to \infty$, as $n \to \infty$. Assume that $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and $\{\epsilon_k\}$ is a sequence of arbitrary random variables satisfying*

$$\sup_j E \max_{1\leq m\leq n} \left(\sum_{k=1}^{m} \epsilon_{k+j} \right)^2 \leq C\, d^2(n), \tag{4.2.6}$$

for every $n \geq 1$, and as $n \to \infty$,

$$\frac{1}{d(n)} \max_{-n\leq k\leq n} |\epsilon_k| \to_P 0. \tag{4.2.7}$$

Then, for $0 \leq s \leq 1$,

$$\frac{1}{d(n)} \sum_{k=1}^{k_n(s)} \epsilon_k \Rightarrow W(s) \quad \textit{implies} \quad \frac{1}{d(n)} \sum_{t=1}^{k_n(s)} X_t \Rightarrow b_\psi\, W(s) \tag{4.2.8}$$

where $k_n(s) = \sup\{m:\ d^2(m) \leq s d^2(n)\}$ and $b_\psi = \sum_{k=0}^{\infty} \psi_k$.

The proof of this theorem will be given in Section 4.3.2.

Theorem 4.2.3 can be applied to many important cases, such as where innovations ϵ_k form a martingale difference or a mixing sequence. In the following, we will derive several corollaries of Theorem 4.2.3. We note that Corollary 4.2.4 below improves the previous results given by Hannan (1979), Phillips and Solo (1992) and Yokoyama (1995); Corollary 4.2.5 is a new result.

Corollary 4.2.4. *Let X_t satisfy model (4.1.1). Assume that $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and $\{\epsilon_k\}$ is a martingale difference sequence satisfying one of the following two conditions:*

(i) $\{\epsilon_k^2\}$ is uniformly integrable and $E\left(\epsilon_k^2 | \mathcal{F}_{k-1}\right) = \sigma^2 > 0$ for all $k \geq 1$;

(ii) $\{\epsilon_k^2\}$ is strongly uniformly integrable (s.u.i.) (i.e., there exists a random variable η such that $P(|\epsilon_k| \geq x) \leq A\ P(|\eta| \geq x)$, for each $x \geq 0$ and $k \geq 1$) and

$$\frac{1}{n} \sum_{k=1}^{n} E\left(\epsilon_k^2 | \mathcal{F}_{k-1}\right) \to_P \sigma^2 > 0.$$

Then, $\frac{1}{\sqrt{n}\ \sigma} \sum_{t=1}^{[ns]} X_t \Rightarrow b_\psi\ W(s),\ 0 \leq s \leq 1$, where $b_\psi = \sum_{k=0}^{\infty} \psi_k$.

Proof. Since $\{\epsilon_k^2\}$ is uniformly integrable and $\{\epsilon_{k+j}\}$ still is a martingale difference for each fixed j, by Lemma 4.3.2 and part (i) of Lemma 4.3.4, we obtain that

$$\sup_j E \max_{1 \leq m \leq n} \left(\sum_{k=1}^{n} \epsilon_{k+j} \right)^2 \leq C \sup_j \sum_{k=1}^{n} E\epsilon_{k+j}^2 \leq C_1\ n.$$

On the other hand, it follows from part (iii) of Lemma 4.3.4 that

$$\frac{1}{\sqrt{n}} \max_{-n \leq m \leq n} |\epsilon_k| \to_P 0.$$

Therefore, using Theorem 4.2.3 with $d(n) = \sqrt{n}\sigma$, it suffices to show that, under one of the conditions (i) and (ii),

$$\frac{1}{\sqrt{n}\sigma} \sum_{k=1}^{[ns]} \epsilon_k \Rightarrow W(s), \quad 0 \leq s \leq 1. \tag{4.2.9}$$

If the condition (ii) holds, (4.2.9) follows directly from the classical results for martingale difference, for example, Corollary 3.3.3 given in Tanaka (1996, pages 82).

If the condition (i) holds, then $E\epsilon_k^2 = E\left(\epsilon_k^2|\mathcal{F}_{k-1}\right) = \sigma^2$. This, together with part (iv) of Lemma 4.3.4, implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\epsilon_k^2 - E\epsilon_k^2\right) \to_P 0. \tag{4.2.10}$$

In terms of (4.2.10) and part (ii) of Lemma 4.3.4, (4.2.9) follows by using Theorem 3.3.8 given in Tanaka (1996, pages 80). $\square$

To state Corollary 4.2.5, we first introduce the definitions of ϕ-mixing and ρ-mixing.

Definition 4.2.2. A sequence $\{\epsilon_k\}$ is said to be a ϕ-mixing or ρ-mixing sequence if as $n \to \infty$,

$$\begin{aligned}
\phi(n) &= \sup_k \sup_{A\in\mathcal{F}_k,\ B\in\mathcal{F}_{k+n}^{\infty}} |P(B|A) - P(B)| \to 0; \quad \textit{or} \\
\rho(n) &= \sup_k \sup_{X\in L_2(\mathcal{F}_k),\ Y\in L_2\left(\mathcal{F}_{k+n}^{\infty}\right)} \frac{cov(X,Y)}{\|X\|_2\ \|Y\|_2} \to 0
\end{aligned}$$

where $\mathcal{F}_k = \sigma\{\epsilon_i, i \le k\}$, $\mathcal{F}_{k+n}^{\infty} = \sigma\{\epsilon_i, i \ge k+n\}$ and $\|X\|_2 = (EX^2)^{1/2}$.

Corollary 4.2.5. *Let X_t satisfy model (4.1.1). Assume that $\sum_{k=0}^{\infty}|\psi_k| < \infty$ and $\{\epsilon_k\}$ is a ϕ-mixing or ρ-mixing sequence such that*

(a) $\{\epsilon_k^2\}$ is uniformly integrable;

(b) $E\epsilon_k = 0$ for all k and $\lim_{n\to\infty}\frac{1}{n}E\left(\sum_{k=1}^{n}\epsilon_k\right)^2 = \sigma^2 > 0$;

(c) $\sum_{i=1}^{\infty}\phi^{1/2}(2^i) < \infty$; or

(c') $\sum_{i=1}^{\infty} \rho^{1/2}(2^i) < \infty$.

Then, $\frac{1}{\sqrt{n}\,\sigma} \sum_{t=1}^{[ns]} X_t \Rightarrow b_\psi\, W(s),\ 0 \le s \le 1,$ *where* $b_\psi = \sum_{k=0}^{\infty} \psi_k$.

Proof. If the conditions (a), (b) and (c) hold, it follows from Corollary 2.4 given by Peligrad (1982) that

$$\frac{1}{\sqrt{n}\sigma} \sum_{k=1}^{[ns]} \epsilon_k \Rightarrow W(s), \quad 0 \le s \le 1.$$

On the other hand, by using Lemma 4.3.4 below and Theorem 1 in Shao (1988), we have that

$$\frac{1}{\sqrt{n}} \max_{-n \le m \le n} |\epsilon_k| \to_P 0, \quad \text{and for every } n \ge 1$$

$$\sup_j E \max_{1 \le m \le n} \left(\sum_{k=1}^{m} \epsilon_{k+j} \right)^2 \le C\ n\ \sup_j \sup_k E\epsilon_{k+j}^2 \le C_1\ n.$$

Therefore, Theorem 4.2.3 with $d(n) = \sqrt{n}\ \sigma$ implies Corollary 4.2.5.

If the conditions (a), (b) and (c') hold, Corollary 4.2.5 can be proved similarly by using Theorem 2.5 given by Peligrad (1982), Lemma 4.3.4 and Theorem 1.1 in Shao (1995, a). □

4.3 Proofs of the main results

This section contributes to the proofs of the main results given in Section 4.2. We start with some preliminary lemmas.

4.3.1 Preliminary lemmas

In this section, we give several lemmas which are used in the proofs of the main results. Some of these lemmas also are of interest in themselves. Notation used in Section 4.2 is still valid in this section.

Lemma 4.3.1. *Let* $\{\epsilon_k\}$ *be a stationary ergodic martingale difference sequence with* $0 < E\epsilon_0^2 < \infty$. *Let* $\{a_{nk}, k = 0, \pm 1, \pm 2, \cdots,\}$ *be a triangular array of constants and* $A_n^2 = \sum_{k=-\infty}^{\infty} a_{nk}^2$. *Assume that*

(i) for every fixed $n \geq 1$, $0 < A_n < \infty$, $\sup_k |k| a_{nk}^2 < \infty$;

(ii) $\sup_k |a_{nk}|/A_n \to 0$, *as* $n \to \infty$;

(iii) there exists a positive constant M *such that*

$$\sup_{n\geq 1} \frac{1}{A_n^2} \sum_{k=-\infty}^{\infty} |k| \left| a_{nk}^2 - a_{n(k+1)}^2 \right| \leq M. \tag{4.3.1}$$

Then,

$$\frac{1}{A_n\ \sigma} \sum_{k=-\infty}^{\infty} a_{nk}\epsilon_k \to_d N(0,1), \qquad \textit{where } \sigma^2 = E\epsilon_0^2. \tag{4.3.2}$$

Remark 4.3.1. Peligrad and Utev (1997) established similar results for pairwise mixing martingale differences. We note that Lemma 4.3.1 will fail if a_{nk} do not satisfy (4.3.1). For a counterexample, see Peligrad and Utev (1997, page 444).

Proof. Let

$$S_{ni} = \frac{1}{A_n\ \sigma} \sum_{k=-\infty}^{i} a_{nk}\epsilon_k, \quad \mathcal{F}_{ni} = \mathcal{F}_i, \quad -\infty < i \leq n.$$

Under the conditions of Lemma 4.3.1, as is well-known, $\{S_{ni}, \mathcal{F}_{ni},\ -\infty < i \leq n\}$ is a zero-mean, square-integrable martingale with difference $\frac{1}{A_n\sigma} a_{ni}\epsilon_i$ for each $n \geq 1$, and we have that $S_{n,-\infty} = 0$, *a.s.* and $\sup_{n,i} ES_{n,i}^2 \leq 1$. To prove (4.3.2), by applying Theorem 3.6 given in Hall and Heyde (1980, page 77), it suffices to show that

(a) $\sup_k |a_{nk}\epsilon_k|/(A_n\sigma) \to_P 0$ and $E \sup_k a_{nk}^2\epsilon_k^2/(A_n^2\sigma^2)$ is bounded in n.

(b) $$\frac{1}{A_n^2} \sum_{k=-\infty}^{\infty} a_{nk}^2\epsilon_k^2 \to_P \sigma^2, \quad \text{as } n \to \infty. \tag{4.3.3}$$

Under the condition (ii), it is clear that for every $\delta > 0$, as $n \to \infty$,

$$\frac{1}{A_n^2} \sum_{k=-\infty}^{\infty} a_{nk}^2 E\epsilon_k^2 I_{(|a_{nk}\epsilon_k| \geq \delta A_n)} \leq E\epsilon_0^2 I_{(|\epsilon_0| \geq \delta A_n / \max_k |a_{nk}|)} \to 0, \tag{4.3.4}$$

This, together with Lemma 4.3.3, implies part (a).

Next let us prove part (b). To prove part (b), i.e., (4.3.3), put

$$Z_j = \sum_{k=0}^{j} (\epsilon_k^2 - E\epsilon_k^2), \qquad j = 0, 1, 2, \cdots.$$

It is easy to show that for every $N \geq 1$ (let $Z_k \equiv 0, if\ k < 0$),

$$\begin{aligned} &\sum_{k=0}^{\infty} a_{nk}^2(\epsilon_k^2 - E\epsilon_k^2) \\ &\quad = \sum_{k=0}^{N} a_{nk}^2 (Z_k - Z_{k-1}) + \sum_{k=N+1}^{\infty} a_{nk}^2(\epsilon_k^2 - E\epsilon_k^2) \\ &\quad = \sum_{k=0}^{N-1} (a_{nk}^2 - a_{n(k+1)}^2) Z_k + a_{nN}^2 Z_N + \sum_{k=N+1}^{\infty} a_{nk}^2(\epsilon_k^2 - E\epsilon_k^2). \end{aligned} \tag{4.3.5}$$

By applying the stationary ergodic theorem (Theorem B.2, see Appendix B), we have that $Z_k/k \to 0,\ a.s.$ as $k \to \infty$. This, together with the Toeplitz lemma (Stout, 1974, page 120) and (4.3.1), implies that, as $n \to \infty$,

$$\begin{aligned} &\frac{1}{A_n^2} \left| \sum_{k=0}^{N-1} (a_{nk}^2 - a_{n(k+1)}^2) Z_k \right| \\ &\qquad \leq \frac{1}{A_n^2} \sum_{k=0}^{\infty} k \left| a_{nk}^2 - a_{n(k+1)}^2 \right| (|Z_k|/k) \to_{a.s.} 0. \end{aligned} \tag{4.3.6}$$

On the other hand, it follows from the condition (i) that

$$|a_{nN}^2 Z_N| \leq \left(\sup_k |k| a_{nk}^2 \right) |Z_N|/N \to 0, \quad a.s. \quad \text{as } N \to \infty. \tag{4.3.7}$$

By noting $\sum_{k=0}^{\infty} a_{nk}^2 E\left|\epsilon_k^2 - E\epsilon_k^2\right| \leq 2E\epsilon_0^2 A_n^2 < \infty$, we have

$$\sum_{k=N}^{\infty} a_{nk}^2 \left(\epsilon_k^2 - E\epsilon_k^2 \right) \to_P 0, \quad \text{as} \quad N \to \infty. \tag{4.3.8}$$

In terms of (4.3.6)-(4.3.8), let $N \to \infty$ first and then $n \to \infty$ in (4.3.5), we obtain that

$$\frac{1}{A_n^2} \sum_{k=0}^{\infty} a_{nk}^2 (\epsilon_k^2 - E\epsilon_k^2) \to_P 0. \tag{4.3.9}$$

Similarly, we have that

$$\frac{1}{A_n^2} \left| \sum_{k=0}^{\infty} a_{n(-k)}^2 (\epsilon_{-k}^2 - E\epsilon_{-k}^2) \right| \to_P 0. \tag{4.3.10}$$

(4.3.3) follows immediately from (4.3.9) and (4.3.10). The proof of Lemma 4.3.1 is complete. □

Lemma 4.3.2. *Let $\{\eta_n, \mathcal{F}_n, s \le n \le t\}$ be a martingale difference sequence, then there exists a constant K such that, for any constant sequence α_k,*

$$E \max_{s \le n \le t} \left(\sum_{k=s}^{n} \alpha_k \eta_k \right)^2 \le K \sum_{k=s}^{t} \alpha_k^2 E\eta_k^2. \tag{4.3.11}$$

It is straightforward by applying Doob and Burkholder's inequality (see, for example, Hall and Heyde, 1980, pages 15 and 23 respectively).

Lemma 4.3.3. *Let $\{\eta_{nk}, k = 0, \pm 1, \pm 2, ..., n \ge 1\}$ be any triangular random variable sequence and d_n be a positive real numbers series. If, as $n \to \infty$, $d_n \to \infty$ and*

$$\frac{1}{d_n^2} \sum_{k=-\infty}^{\infty} E\eta_{nk}^2 I_{(|\eta_{nk}| \ge \delta d_n)} \to 0, \quad \text{for any } \delta > 0,$$

then $\frac{1}{d_n} \sup_k |\eta_{nk}| \to_P 0$.

Proof. It is well-known that

$$P\left(\frac{1}{d_n} \sup_k |\eta_{nk}| \ge \delta \right) = P\left(\frac{1}{d_n^2} \sum_{k=-\infty}^{\infty} \eta_{nk}^2 I_{(|\eta_{nk}| \ge \delta d_n)} \ge \delta^2 \right). \tag{4.3.12}$$

By Markov's inequality, the result follows immediately. □

Lemma 4.3.4. *Let $\{\eta_k, k = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of arbitrary random variables. If $\{\eta_k^2\}$ is uniformly integrable, then*

(i) $\sup_k E\eta_k^2 < \infty$.

(ii) $\frac{1}{n}\sum_{k=-n}^{n} E\eta_k^2 I_{(|\eta_k| \geq \delta\sqrt{n})} \to 0$, *for any* $\delta > 0$;

(iii) $\frac{1}{\sqrt{n}} \max_{-n \leq k \leq n} |\eta_k| \to_P 0$.

(iv) $\frac{1}{n}\sum_{k=1}^{n} \left(\eta_k^2 - E\left(\eta_k^2 | \mathcal{F}^*{}_{k-1}\right)\right) \to_P 0$, *where* $\mathcal{F}^*{}_k = \sigma\{\eta_j, j \leq k\}$.

Proof. By the definition of uniform integrability, we have that $\sup_k E\eta_k^2 I_{(|\eta_k| \geq \delta_n)} \to 0$, for any $\delta_n \to \infty$. Hence, $\sup_k E\eta_k^2 < \infty$, i.e, part (i) holds. Furthermore, we have

$$\frac{1}{n}\sum_{k=-n}^{n} E\eta_k^2 I_{(|\eta_k| \geq \delta\sqrt{n})} \leq 2 \sup_k E\eta_k^2 I_{(|\eta_k| \geq \delta\sqrt{n})} \to 0,$$

which implies part (ii). Part (iii) follows from Lemma 4.3.3 with $\eta_{nk} = \eta_k$, $k = -n, \cdots, n$, otherwise $\eta_{nk} = 0$. To prove part (iv), let

$$\begin{aligned}
\xi_{1k} &= \eta_k^2 I_{(|\eta_k| \leq n^{1/4})} - E\left(\eta_k^2 I_{(|\eta_k| \leq n^{1/4})} | \mathcal{F}^*{}_{k-1}\right), \\
\xi_{2k} &= \eta_k^2 I_{(|\eta_k| > n^{1/4})} - E\left(\eta_k^2 I_{(|\eta_k| > n^{1/4})} | \mathcal{F}^*{}_{k-1}\right).
\end{aligned}$$

Since $\xi_{1k} + \xi_{2k} = \eta_k^2 - E\left(\eta_k^2 | \mathcal{F}^*{}_{k-1}\right)$, it suffices to show that

$$\frac{1}{n}\sum_{k=1}^{n} \xi_{jk} \to_P 0, \qquad j = 1, 2. \tag{4.3.13}$$

For $j = 1$, by noting that $\{\xi_{1k}\}$ is a martingale difference sequence, it follows from Markov's inequality and Lemma 4.3.2 that, for any $\delta > 0$,

$$\begin{aligned}
P\left(\frac{1}{n}\left|\sum_{k=1}^{n} \xi_{1k}\right| \geq \delta\right) &\leq \frac{C}{\delta^2 n^2}\sum_{k=1}^{n} E\xi_{1k}^2 \leq \frac{C}{n} \sup_k E\eta_k^4 I_{(|\eta_k| \leq n^{1/4})} \\
&\leq C\, n^{-1/2} \sup_k E\eta_k^2 \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

This implies (4.3.13) for $j = 1$. For $j = 2$, (4.3.13) follows from Markov's inequality and

$$\frac{1}{n}\sum_{k=1}^{n} E|\xi_{2k}| \le \frac{2}{n}\sum_{k=1}^{n} E\eta_k^2 I_{(|\eta_k|>n^{1/4})} \to 0.$$

The proof of Lemma 4.3.4 is complete. □

Lemma 4.3.5. *Let ψ_k be defined as in (4.2.1). Then, for $1/2 < \alpha < 1$ and $0 \le s < t \le 1$,*

$$\sum_{k=-\infty}^{\infty}\left(\sum_{j=[ns]+1}^{[nt]} \psi_{j-k}\right)^2 \sim \frac{\chi_\alpha n^{3-2\alpha}(t-s)^{3-2\alpha} l^2(n)}{(1-\alpha)(3-2\alpha)}, \tag{4.3.14}$$

where χ_α is defined as in Theorem 4.2.1.

Proof. Let $\{\eta_k, k = 0, \pm 1, \pm 2, \cdots\}$ be iid $N(0,1)$ and $Y_j = \sum_{k=0}^{\infty}\psi_k\eta_{j-k}$. Recalling $\psi_k = 0, k < 0$, we have that

$$\sum_{j=[ns]+1}^{[nt]} Y_j = \sum_{k=-\infty}^{\infty}\eta_k \sum_{j=[ns]+1}^{[nt]}\psi_{j-k},$$

and clearly

$$E\left(\sum_{j=[ns]+1}^{[nt]} Y_j\right)^2 = \sum_{k=-\infty}^{\infty}\left(\sum_{j=[ns]+1}^{[nt]}\psi_{j-k}\right)^2. \tag{4.3.15}$$

Write $r(j) = EY_1Y_{1+j}$. Since ψ_k satisfies (4.2.1), it follows from Hall (1992, page 118) that

$$r(j) \sim \chi_\alpha\, j^{1-2\alpha} l^2(j), \qquad 1/2 < \alpha < 1. \tag{4.3.16}$$

In terms of (4.3.16) and stationarity of the Y_j, it follows from Horvath and Shao (1996, page 134-135) that

$$\begin{aligned} E\left(\sum_{t=1}^{m} Y_t\right)^2 &= mr(0) + 2\sum_{j=1}^{m-1}(m-j)r(j) \\ &\sim \frac{\chi_\alpha m^{3-2\alpha} l^2(m)}{(1-\alpha)(3-2\alpha)}. \end{aligned} \tag{4.3.17}$$

Using (4.3.17) and stationarity of the Y_j again, we obtain that, for $1/2 < \alpha < 1$ and $0 \le s < t \le 1$,

$$E\left(\sum_{j=[ns]+1}^{[nt]} Y_j\right)^2 = E\left(\sum_{j=1}^{[nt]-[ns]} Y_j\right)^2 \sim \frac{\chi_\alpha n^{3-2\alpha} l^2(n)(t-s)^{3-2\alpha}}{(1-\alpha)(3-2\alpha)}, \qquad (4.3.18)$$

where we use the estimate: $\frac{[nt]-[ns]}{n} \sim t-s$ and the property of the function $l(x)$ slowly varying at infinity: $l(\beta n) \sim l(n)$ for any $\beta > 0$ (see Feller, 1971, page 275). Hence (4.3.14) follows from (4.3.15) and (4.3.18). □

Lemma 4.3.6. *Let $c_k = \frac{k^{1-d}\Gamma(k+d)}{\Gamma(k+1)}, k \ge 0$, and $0 < d < 1/2$. Then $c_k \sim 1$, i.e. c_k is a slowly varying function at infinity, and we have that for all $1 \le m \le n$,*

$$|c_{m+n}/c_n - 1| \le C\ m/n.$$

Proof. It follows from (2.48) in Beran (1994, page 65) (also see Hosking, 1981) that $c_k \sim 1$. By applying Corollary given in Feller (1971, p282), c_k is a slowly varying function at infinity. Noting $\Gamma(z+1) = z\Gamma(z)$, we have that for all $0 < d < 1/2$ and $1 \le m \le n$,

$$\begin{aligned}\left|\frac{c_{m+n}}{c_n} - 1\right| &= \left|\left(\frac{m+n}{n}\right)^{1-d} \frac{\Gamma(n+m+d)\Gamma(n+1)}{\Gamma(n+d)\Gamma(n+m+1)} - 1\right| \\ &\le \left|\left(\frac{m+n}{n}\right)^{1-d} - 1\right| + 2\left(1 - \frac{(n+m+d-1)\cdots(n+d)}{(n+m)\cdots(n+1)}\right) \\ &\le \frac{m}{n} + 2\left(1 - \frac{n+d}{n+m}\right) \le C\,\frac{m}{n},\end{aligned}$$

Hence, the result follows. □

Lemma 4.3.7. *Let $l(x)$ be a positive function slowly varying at infinity. Then, for any $0 < \beta < 1$,*

$$\sum_{k=1}^{n} k^{-\beta} l(k) \sim \frac{1}{1-\beta}\, n^{1-\beta} l(n) \quad \textit{and} \quad \sum_{k=n}^{\infty} k^{-1-\beta} l(k) \sim \frac{1}{\beta}\, n^{-\beta} l(n).$$

Proof. See Bingham et al. (1987, page 26). □

Using these lemmas, we can give the proof of Theorems 4.2.1 and 4.2.3 as follows.

4.3.2 Proofs of the main theorems

Proof of Theorem 4.2.1. For $H = \frac{1}{2}(3 - 2\alpha)$, put

$$Z_n(s) = \frac{1}{n^H l(n)} \sum_{t=1}^{[ns]} X_t, \qquad B_H(s,t) = \frac{1}{2}\left\{s^{2H} + t^{2H} - |s-t|^{2H}\right\}.$$

Using the usual approach in deriving weak convergence in $D[0,1]$ (see Theorem A.3, Appendix A), it suffices to show:

(1) for each fixed $l \geq 1$ and real constants $0 < t_1 \neq t_2 \neq \cdots \neq t_l \leq 1$,

$$\tau_1 Z_n(t_1) + \cdots + \tau_l Z_n(t_l) \to_d N(0, \sigma_1^2) \tag{4.3.19}$$

where $\tau_1, \tau_2, \cdots, \tau_l$ are any real constants and $\sigma_1^2 = \Delta_\alpha^2 \sum_{i,j=1}^{l} \tau_i \tau_j B_H(t_i, t_j)$, where Δ_α^2 is defined as in (4.2.3);

(2) the tightness of $Z_n(s)$.

We first prove part (1).

Let $m_j = [nt_j], j = 1, \cdots, l$. By noting $X_i = \sum_{k=-\infty}^{i} \psi_{i-k}\epsilon_k = \sum_{k=-\infty}^{\infty} \psi_{i-k}\epsilon_k$ (recall $\psi_k = 0$, if $k < 0$), we obtain that

$$\tau_1 Z_n(t_1) + \cdots + \tau_l Z_n(t_l) = \frac{1}{n^H l(n)} \sum_{j=1}^{l} \tau_j \sum_{t=1}^{m_j} X_t = \frac{1}{n^H l(n)} \sum_{k=-\infty}^{\infty} d_{k,n}\epsilon_k, \tag{4.3.20}$$

where $d_{k,n} = \sum_{j=1}^{l} \tau_j \sum_{i=1}^{m_j} \psi_{i-k}$. By the definition (4.2.1) of the ψ_k and Lemma 4.3.7, it is clear that

$$\sum_{i=0}^{n} \psi_{i-k} = 0, \quad \text{if } k \geq n+1 \tag{4.3.21}$$

and

$$\sum_{i=0}^{n} \psi_{i-k} \leq C \begin{cases} n^{1-\alpha} l(n), & \text{if } |k| \leq n, \\ n|k|^{-\alpha} l(|k|), & \text{if } k \leq -n-1. \end{cases} \tag{4.3.22}$$

It follows from (4.3.21) and (4.3.22) that $d_{k,n} = 0$, if $k \geq n+1$ and

$$\begin{aligned} |d_{k,n}| + |d_{k+1,n}| &\leq \sum_{j=1}^{l} |\tau_j| \sum_{i=0}^{n} (\psi_{i-k} + \psi_{i-k-1}) \\ &\leq C_1 \begin{cases} n^{1-\alpha} l(n), & \text{if } |k| \leq n, \\ n|k|^{-\alpha} l(|k|), & \text{if } k \leq -n-1. \end{cases} \end{aligned} \tag{4.3.23}$$

On the other hand, by condition (4.2.2), we have that, if $k \leq -n-1$,

$$\begin{aligned} |d_{k,n} - d_{k+1,n}| &= \left| \sum_{j=1}^{l} \tau_j \left(\sum_{i=1}^{m_j} \psi_{i-k} - \sum_{i=1}^{m_j} \psi_{i-k-1} \right) \right| \\ &\leq \sum_{j=1}^{l} |\tau_j| \left| \psi_{m_j-k} - \psi_{-k} \right| \\ &\leq \sum_{j=1}^{l} |\tau_j| \left| \frac{l(m_j-k)}{(m_j-k)^\alpha} - \frac{l(-k)}{(-k)^\alpha} \right| \\ &\leq l(|k|) \sum_{j=1}^{l} |\tau_j| \left\{ \left| \frac{1}{(m_j+|k|)^\alpha} - \frac{1}{|k|^\alpha} \right| \right. \\ &\qquad \left. + \frac{1}{(m_j+|k|)^\alpha} \left| \frac{l(m_j+|k|)}{l(|k|)} - 1 \right| \right\} \\ &\leq C_2\, n|k|^{-1-\alpha} l(|k|), \end{aligned} \tag{4.3.24}$$

where we use the following estimate: for $j = 1, \cdots, l$,

$$m_j \leq n, \quad \text{and} \quad \left| \frac{1}{(m_j+|k|)^\alpha} - \frac{1}{|k|^\alpha} \right| \leq n|k|^{-1-\alpha},$$

for any $|k| \geq n$. From (4.3.23), (4.3.24) and $1/2 < \alpha < 1$, it can be easily shown that

for every fixed $n \geq 1$, $\sup_k |k| d_{k,n}^2 < \infty$, $\sum_{k=0}^{\infty} d_{k,n}^2 < \infty$ and

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} |k| \left| d_{k,n}^2 - d_{k+1,n}^2 \right| \\
\leq \; & \sum_{k=-\infty}^{\infty} |k| \left| d_{k,n} - d_{k+1,n} \right| \left(|d_{k,n}| + |d_{k+1,n}| \right) \\
\leq \; & C_1 n^{2-\alpha} l(n) \sum_{|k| \leq n} \sum_{j=1}^{l} |\tau_j| \left(\psi_{m_j - k} + \psi_{-k} \right) + C_1 C_2 n^2 \sum_{k=-\infty}^{-n-1} |k|^{-2\alpha} l^2(|k|) \\
\leq \; & C_3 n^{2-\alpha} \sum_{k=0}^{2n} \psi_k + A_3 n^{3-2\alpha} l^2(n) \leq C_4 n^{3-2\alpha} l^2(n).
\end{aligned}$$

Therefore, conditions of Lemma 4.3.1 hold for $d_{k,n}$ defined in (4.3.20). From Lemma 4.3.1 and (4.3.20), part (i) follows if

$$\frac{1}{n^{2H} l^2(n)} \sum_{k=-\infty}^{\infty} d_{k,n}^2 \to \frac{\Delta_\alpha^2}{\sigma^2} \sum_{i,j=1}^{l} \tau_i \tau_j B_H(t_i, t_j), \quad as \quad n \to \infty. \tag{4.3.25}$$

Since

$$d_{k,n}^2 = \sum_{i,j=1}^{l} \tau_i \tau_j \left(\sum_{s=1}^{m_i} \psi_{s-k} \right) \left(\sum_{t=1}^{m_j} \psi_{t-k} \right)$$

and using $xy = \frac{1}{2}\left[x^2 + y^2 - (x-y)^2\right]$, relation (4.3.25) holds if for $0 \leq s < t \leq 1$,

$$\frac{1}{n^{2H} l^2(n)} \sum_{k=-\infty}^{\infty} \left(\sum_{j=[ns]+1}^{[nt]} \psi_{j-k} \right)^2 \to \frac{\Delta_\alpha^2}{\sigma^2} (t-s)^{3-2\alpha}, \quad as \quad n \to \infty. \tag{4.3.26}$$

This follows immediately from Lemma 4.3.5. We finish the proof of part (1).

Secondly, we prove part (2).

Similar to (4.3.20), we have

$$\sum_{t=1}^{n} X_t = \sum_{k=-\infty}^{\infty} \left(\sum_{t=1}^{n} \psi_{t-k} \right) \epsilon_k.$$

Hence, by Lemma 4.3.5 and noting that $\{\epsilon_k\}$ is a stationary martingale difference sequence, it can be easily shown that

$$E\left(\sum_{t=1}^{n} X_t \right)^2 = \sum_{k=-\infty}^{\infty} \left(\sum_{t=1}^{n} \psi_{t-k} \right)^2 E\epsilon_0^2 = O\left(n^{(3-2\alpha)/2} l^2(n) \right).$$

On the other hand, we have that $EX_t = 0$ for all $t \geq 1$ and $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$ still is a stationary random variable sequence (Theorem B.1, see Appendix B). By Lemma 2.1 given by Taqqu (1975), part (2) follows immediately. The proof of Theorem 4.2.1 is complete. □

Proof of Theorem 4.2.3. For every fixed $l \geq 1$, put

$$Z_{1j}^{(l)} = \sum_{k=0}^{l} \psi_k \epsilon_{j-k} \quad \textit{and} \quad Z_{2j}^{(l)} = \sum_{k=l+1}^{\infty} \psi_k \epsilon_{j-k}.$$

From Fuller (1996, page 320), we obtain that for any $m \geq 1$,

$$\begin{aligned}
\sum_{j=1}^{m} Z_{1j}^{(l)} &= \sum_{j=1}^{m} \sum_{k=0}^{l} \psi_k \epsilon_{j-k} \\
&= \sum_{k=0}^{l} \psi_k \sum_{j=1}^{m} \epsilon_j + \sum_{s=1}^{l} \epsilon_{1-s} \sum_{j=s}^{l} \psi_j + \sum_{s=0}^{l-1} \epsilon_{m-s} \sum_{j=s+1}^{l} \psi_j \\
&= \sum_{k=0}^{l} \psi_k \sum_{j=1}^{m} \epsilon_j + R(m, l), \quad \textit{say}.
\end{aligned}$$

Therefore, it follows that for every fixed $l \geq 1$,

$$\frac{1}{d(n)} \sum_{t=1}^{k_n(s)} X_t = \left(\sum_{k=0}^{l} \psi_k \right) \frac{1}{d(n)} \sum_{j=1}^{k_n(s)} \epsilon_j + \frac{1}{d(n)} R(k_n(s), l) + \frac{1}{d(n)} \sum_{j=1}^{k_n(s)} Z_{2j}^{(l)}. \quad (4.3.27)$$

To prove (4.2.8), by (4.3.27), Theorem A.1 (see Appendix A) and noting $\sum_{k=0}^{l} \psi_k \to b_\psi$ as $l \to \infty$, it suffices to show that for any $\delta > 0$,

$$\limsup_{n \to \infty} P \left\{ \sup_{0 \leq t \leq 1} |R(k_n(t), l)| \geq \delta \, d(n) \right\} = 0, \quad (4.3.28)$$

for every fixed $l \geq 1$ and

$$\lim_{l \to \infty} \limsup_{n \to \infty} P \left\{ \sup_{0 \leq t \leq 1} \left| \sum_{j=1}^{k_n(t)} Z_{2j}^{(l)} \right| \geq \delta \, d(n) \right\} = 0. \quad (4.3.29)$$

By condition (4.2.7), (4.3.28) holds since $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and hence

$$\begin{aligned}\frac{1}{d(n)} \sup_{0\le t\le 1} |R(k_n(t), l)| &\le \frac{1}{d(n)} \max_{-l\le j\le n} |\epsilon_j| \sum_{s=0}^{l} \left(\sum_{j=s}^{l} |\psi_j| + \sum_{j=s+1}^{l} |\psi_j| \right) \\ &\to_P 0.\end{aligned}$$

We next prove (4.3.29). Noting

$$\sum_{j=1}^{m} Z_{2j}^{(l)} = \sum_{k=l+1}^{\infty} \psi_k \sum_{j=1}^{m} \epsilon_{j-k}, \quad \text{for any } m \ge 1,$$

by applying Hölder's inequality and (4.2.6), we have that

$$\begin{aligned} E \sup_{0\le t\le 1} \left(\sum_{j=1}^{k_n(t)} Z_{2j}^{(l)} \right)^2 &\le \sum_{k=l+1}^{\infty} |\psi_k| \sum_{k=l+1}^{\infty} |\psi_k| E \max_{1\le m\le n} \left(\sum_{j=1}^{m} \epsilon_{j-k} \right)^2 \\ &\le C\, d^2(n) \left(\sum_{k=l+1}^{\infty} |\psi_k| \right)^2 . \end{aligned}$$

Now, (4.3.29) follows immediately from Markov's inequality and $\sum_{k=l+1}^{\infty} |\psi_k| \to 0$ as $l \to \infty$. The proof of Theorem 4.2.3 is complete. □

Chapter 5

Further results for linear processes

Let X_t be a linear process defined by $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$, where $\{\epsilon_k\}$ is a sequence of random variables with mean zero and $\{\psi_k\}$ is a sequence of real numbers. In this chapter, we derive two basic results on the functional limit theorem for the partial sum process $\sum_{t=1}^{[ns]} X_t$. In the first result, we assume that innovations ϵ_k are iid random variables, but the conditions requiring absolute summability of coefficients for the linear process (i.e., $\sum_{k=0}^{\infty} |\psi_k| < \infty$) is weakened. We note that, for the partial sum process of the X_t converging to a standard Brownian motion, the condition $\sum_{k=0}^{\infty} |\psi_k| < \infty$ or a stronger condition is commonly used in previous research. The second result is for the situation where the innovations ϵ_k form a martingale difference sequence, where the commonly used assumption of equal variance is relaxed.

5.1 Introduction

Consider a linear process X_t defined by

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}, \qquad t = 1, 2, \cdots, \tag{5.1.1}$$

where $\{\epsilon_k\}$ is a sequence of random variables with mean zero and $\{\psi_k\}$ is a sequence of real numbers. In the last chapter, we have pointed out the importance of such processes in time series analysis, and discussed the functional limit theorem for the partial sum process $\sum_{t=1}^{[ns]} X_t$ when the process X_t has dependent innovations. In particular, Theorem 4.2.3 gave quite general sufficient conditions so that the asymptotic distribution of the partial sum process $\sum_{t=1}^{[ns]} X_t$ (properly normalized) is a standard Brownian motion.

This chapter continues to discuss such sufficient conditions so that $\sum_{t=1}^{[ns]} X_t$ (properly normalized) converges weakly to a standard Brownian motion. We present two basic results in this chapter. In the first result, we assume that the innovations ϵ_k are iid random variables, but the condition $\sum_{k=0}^{\infty} |\psi_k| < \infty$, used in Theorem 4.2.3 and in similar research (cf. Hannan (1979), Phillips and Solo (1992) (also see Tanaka, 1996) as well as Yokoyama (1995)), is weakened. This relaxation of this condition is interesting since some linear processes do not have absolutely summable coefficients. In particular, a linear process with $\psi_k = k^{-1}l(k)$ where $\sum_{k=0}^{\infty} k^{-1}l(k) = \infty$ is important since it is expressed by neither a finite order autoregressive moving average process nor a fractional process. Only finite second moments for the ϵ_k's are required in the present chapter, which also gives a essential improvement of similar results given by Davydov (1970).

Our second result is for the situation in which the innovations ϵ_k form a martingale difference sequence. For this result, the commonly used assumption of equal variance of the innovations ϵ_k is weakened, which will be of practical interest to researchers.

We give the statements of the main theorems and detailed remarks on the preceding results in the next section. In Section 5.3, we give the proofs of the main

theorems.

As used before, we denote $C, C_1, C_2, \ldots$, for positive constants.

5.2 Main results and remarks

We now present our main results. In Theorem 5.2.1 below, we assume that the innovations ϵ_k are iid random variables, but, to cover some interesting cases, the ψ_k's are rather general. Write for $j = 1, 2, 3, \ldots$,

$$v_j = \sum_{k=0}^{j-1} \psi_k \quad \text{and} \quad s_n^2 = \sum_{j=1}^{n} v_j^2.$$

Theorem 5.2.1. *Let $\epsilon_k, k = 0, \pm 1, \pm 2, \ldots$ be i.i.d. random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = 1$. Assume that $\psi_0 \neq 0$,*

$$\frac{1}{s_n} \max_{1 \le j \le n} |v_j| \to 0 \quad \text{and} \quad \sum_{j=0}^{n} \left(\sum_{k=j}^{\infty} \psi_k^2 \right)^{1/2} = o(s_n). \tag{5.2.1}$$

Then, we have that

$$\frac{1}{s_n} \sum_{j=1}^{k_n(t)} X_j \Rightarrow W(t), \quad 0 \le t \le 1, \tag{5.2.2}$$

where $k_n(t) = \sup\{m : s_m^2 \le t s_n^2\}$.

In particular, if $\psi_k = k^{-1} l(k)$, where the positive function $l(k)$ ($l(0)/0 \equiv 1$) is slowly varying at infinity, satisfying $\sum_{k=1}^{\infty} k^{-1} l(k) = \infty$, then

$$\frac{1}{\sqrt{n} \sum_{k=1}^{n} k^{-1} l(k)} \sum_{j=1}^{k_n(t)} X_j \Rightarrow W(t), \quad 0 \le t \le 1. \tag{5.2.3}$$

If $0 < |\sum_{k=0}^{\infty} \psi_k| < \infty$ and $\sum_{k=1}^{\infty} k\psi_k^2 < \infty$ or $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and $\sum_{k=0}^{\infty} \psi_k \neq 0$, then

$$\frac{1}{\sigma_n} \sum_{j=1}^{[nt]} X_j \Rightarrow W(t), \quad 0 \le t \le 1, \tag{5.2.4}$$

where $\sigma_n^2 = n\left(\sum_{k=0}^{\infty}\psi_k\right)^2$.

Remark 5.2.1. It follows from Hall (1992, page 118) that

$$\sigma_n^2 = Var\left(\sum_{j=1}^{n} X_j\right) \sim n\left(\sum_{k=1}^{n} k^{-1}l(k)\right)^2,$$

provided $\psi_k = k^{-1}l(k)$ and $\sum_{k=1}^{\infty} k^{-1}l(k) = \infty$. Hence, we can replace $\sqrt{n}\,\sum_{k=1}^{n} k^{-1}l(k)$ by σ_n in (5.2.3). It is unclear whether or not s_n in (5.2.2) can be replaced by σ_n.

Remark 5.2.2. Let $\psi_k = k^{-\alpha}$, where $1/2 < \alpha < 1$. It is easy to show that the second condition of (5.2.1) fails to hold. In this case, we also know that $\frac{1}{\sigma_n}\sum_{j=1}^{k_n(t)} X_j$ fails to converge to $W(t)$. In fact, by applying Liu (1998) (also see Marinucci and Robinson, 1998), $\frac{1}{\sigma_n}\sum_{j=1}^{[nt]} X_j$ converges to a fractional Brownian motion with $d = 1-\alpha$. Therefore, to make the partial sum process of the X_j converge to a standard Brownian motion, the condition (5.2.1) is close to the necessary condition.

Remark 5.2.3. The conditions given in this theorem are different from those given by Davydov (1970). Specifically, Theorem 5.2.1 does nor require the condition $E\epsilon_0^4 < \infty$. It is an essential improvement of Davydov's result for the moment condition.

In the next theorem, the iid assumption for innovations ϵ_k is weakened and is replaced by a martingale difference sequence. In this case, an excellent result is given by Hannan (1979), where it is required that $E\epsilon_k^2 = \sigma^2$ for all k and $\lim_{n\to\infty} E(\epsilon_k^2|\mathcal{F}_{k-n}) = \sigma^2, a.s.$ ($\mathcal{F}_k$ is defined as in Theorem 5.2.2 below). In Theorem 5.2.2, these conditions are modified. Our Corollary 5.2.3 also improves Theorem 3.15 in Phillip and Solo (1992), where the authors assumed $\sum_{k=0}^{\infty} k|\psi_k| < \infty$ and the innovations ϵ_k are a s.u.i. (strongly uniformly integrable, see Corollary 4.2.4 for a definition) martingale difference sequence.

Theorem 5.2.2. *Let ψ_k satisfy*

$$b_\psi \equiv \sum_{k=0}^{\infty} \psi_k \neq 0, \quad \sum_{k=0}^{\infty} |\psi_k| < \infty \quad and \quad \sum_{k=1}^{\infty} k\psi_k^2 < \infty.$$

Let ϵ_k be random variables such that

$$E\left(\epsilon_k | \mathcal{F}_{k-1}\right) = 0, \quad a.s. \quad k = 0, \pm 1, \pm 2, ...,$$

where $\mathcal{F}_k$ is the σ-field generated by $\{\epsilon_j, j \leq k\}$. If

$$\sup_{n\geq 1} \frac{1}{n} \sum_{k=-n}^{n} E\epsilon_k^2 < \infty, \qquad \inf_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} E\epsilon_k^2 > 0; \tag{5.2.5}$$

and, as $n \to \infty$,

$$\frac{1}{n} \sum_{k=1}^{n} \left(\epsilon_k^2 - E\epsilon_k^2\right) \to_P 0; \tag{5.2.6}$$

$$\frac{1}{n} \sum_{k=-n}^{n} E\epsilon_k^2 I_{(|\epsilon_k| \geq \delta\sqrt{n})} \to 0, \quad for\ any\ \delta > 0, \tag{5.2.7}$$

then

$$\frac{1}{\sigma_n^*} \sum_{j=1}^{k_n^*(t)} X_j \Rightarrow W(t), \quad 0 \leq t \leq 1, \tag{5.2.8}$$

where $\sigma_n^{*2} = b_\psi^2 \sum_{k=1}^n E\epsilon_k^2$, and $k_n^*(t) = \sup\left\{j : \sum_{k=1}^j E\epsilon_k^2 \leq t \sum_{k=1}^n E\epsilon_k^2\right\}$.

Remark 5.2.4. Theorem 5.2.2 is not a consequence of Theorem 4.2.3. In fact, we can not derive the condition (4.2.6) needed in Theorem 4.2.3 by using (5.2.5)-(5.2.7).

From Theorem 5.2.2, we may obtain the following Corollary 5.2.3, which is similar to Corollary 4.2.4.

Corollary 5.2.3. *If the conditions (5.2.5), (5.2.6) and (5.2.7) in Theorem 5.2.2 are replaced by one of the following conditions (a)-(c), then (5.2.8) still holds.*

(a) $\{\epsilon_k^2\}$ *is uniformly integrable and* $E(\epsilon_k^2|\mathcal{F}_{k-1}) = \sigma^2 > 0$ *for all* $k \geq 1$*;*

(b) $\{\epsilon_k^2\}$ *is s.u.i. and* $\frac{1}{n}\sum_{k=1}^n E(\epsilon_k^2|\mathcal{F}_{k-1}) \to_P \sigma^2 > 0$*;*

(c) $E(\sup_k \epsilon_k^2) < \infty$ *and* $\frac{1}{n}\sum_{k=1}^n E(\epsilon_k^2|\mathcal{F}_{k-1}) \to_P \sigma^2 > 0$.

Proof. If the condition (a) holds, then $E\epsilon_k^2 = E(\epsilon_k^2|\mathcal{F}_{k-1}) = \sigma^2$. (5.2.8) follows immediately from Theorem 5.2.2 by using Lemma 4.3.4.

If the condition (b) holds, it follows from Lemma 4.3.4 that,

$$\frac{1}{n}\sum_{k=-n}^{n} E\epsilon_k^2 I_{(|\epsilon_k|\geq\delta\sqrt{n})} \to 0 \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} \epsilon_k^2 \to_P \sigma^2. \tag{5.2.9}$$

On the other hand, it is known that $\{\frac{1}{n}\sum_{k=1}^n \epsilon_k^2\}$ is s.u.i. if $\{\epsilon_k^2\}$ is s.u.i. (Chow and Teicher, 1988, page 102). This fact, together with the second relation of (5.2.9), implies that

$$\frac{1}{n}\sum_{k=1}^{n} E\epsilon_k^2 \to \sigma^2 \tag{5.2.10}$$

(Chow and Teicher, 1988, page 100). In terms of (5.2.9) and (5.2.10), it is easy to check that all conditions of Theorem 5.2.2 are satisfied and hence (5.2.8) holds.

Finally, if the condition (c) holds, (5.2.8) follows obviously because $E(\sup_k \epsilon_k^2) < \infty$ implies that $\{\epsilon_k^2\}$ is s.u.i.. □

5.3 Proofs of the main results

Proofs of Theorems 5.2.1 and 5.2.2 along with some preliminary lemmas are given in this section.

5.3.1 Preliminary lemmas

In this section, we provide some lemmas which will be needed in the proofs of the main results. Some lemmas are also of interest in their own right.

Lemma 5.3.1. *Let $\{\eta_k, k \geq 0\}$ be a sequence of arbitrary random variables and $\{b_i, i \geq 0\}$ is a sequence of real numbers. Assume that*

$$\psi_0^2 + \sum_{k=1}^{\infty} k\psi_k^2 < \infty, \qquad \sup_{n\geq 1} \frac{1}{n} \sum_{i=0}^{n} b_i^2 < \infty$$

and there exists a positive constant C such that

$$E\left(\sum_{k=0}^{\infty} \psi_{j+k}\eta_k\right)^2 \leq C \sum_{k=0}^{\infty} \psi_{j+k}^2 b_k^2, \quad \text{for } j \geq 0. \tag{5.3.1}$$

Then, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \max_{0\leq m\leq n} \left|\sum_{j=0}^{m}\sum_{k=0}^{\infty} \psi_{j+k}\eta_k\right| \to_P 0. \tag{5.3.2}$$

Proof. By using $E|Y| \leq (EY^2)^{1/2}$ for any random variable Y, it follows from (5.3.1) that

$$\begin{aligned} E\max_{0\leq m\leq n} \left|\sum_{j=0}^{m}\sum_{k=0}^{\infty} \psi_{j+k}\eta_k\right| &\leq \sum_{j=0}^{n} E\left|\sum_{k=0}^{\infty} \psi_{j+k}\eta_k\right| \\ &\leq C\sum_{j=0}^{n}\left(\sum_{k=0}^{\infty}\psi_{j+k}^2 b_k^2\right)^{1/2}. \end{aligned} \tag{5.3.3}$$

Put $\alpha_j = \sum_{k=0}^{\infty} \psi_{j+k}^2 b_k^2$. For any $0 \leq l \leq m$, we have that

$$\begin{aligned} \sum_{j=l}^{m} \alpha_j &= \sum_{j=l}^{m}\sum_{k=0}^{\infty} \psi_{j+k}^2 b_k^2 \\ &= \sum_{k=l}^{\infty} \psi_k^2 \sum_{j=l}^{\min\{k,m\}} b_{k-j}^2 \leq \left(\sup_{k\geq 1}\frac{1}{k}\sum_{i=0}^{k} b_i^2\right) \sum_{k=l}^{\infty} (k+1)\psi_k^2. \end{aligned}$$

This inequality, together with Hölder's inequality, implies that

$$
\begin{aligned}
\left(\sum_{j=0}^{n} \alpha_j^{1/2}\right)^2 &= \left\{\left(\sum_{j=0}^{[\sqrt{n}]} + \sum_{j=[\sqrt{n}]+1}^{n}\right) \alpha_j^{1/2}\right\}^2 \\
&\le 2\left\{\left(\sum_{j=0}^{[\sqrt{n}]} \alpha_j^{1/2}\right)^2 + \left(\sum_{j=[\sqrt{n}]}^{n} \alpha_j^{1/2}\right)^2\right\} \\
&\le 2\left([\sqrt{n}] \sum_{j=0}^{[\sqrt{n}]} \alpha_j + n \sum_{j=[\sqrt{n}]}^{n} \alpha_j\right) \\
&\le 2\left(\sup_{k\ge 1} \frac{1}{k}\sum_{i=0}^{k} b_i^2\right)\left([\sqrt{n}] \sum_{k=0}^{\infty}(k+1)\psi_k^2 + n \sum_{k=[\sqrt{n}]}^{\infty} (k+1)\psi_k^2\right) \\
&= o(n).
\end{aligned}
$$

Now (5.3.2) follows from Markov's inequality, (5.3.3) and the bound established above. The proof of Lemma 5.3.1 is complete. □

Lemma 5.3.2. *Let $\{\eta_n, \mathcal{F}_n, -\infty < n < \infty\}$ be a martingale difference sequence satisfying $\sup_{k\ge 1} \frac{1}{k}\sum_{i=-k}^{k} E\eta_i^2 < \infty$. Assume that*

$$\sum_{k=0}^{\infty} |\psi_k| < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} k\psi_k^2 < \infty.$$

Then, for any $\delta > 0$,

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\left\{\max_{1\le m\le n} \left|\sum_{j=1}^{m} u_j^{(l)}\right| \ge \delta\sqrt{n}\right\} = 0, \tag{5.3.4}$$

where $u_j^{(l)} = \sum_{k=l+1}^{\infty} \psi_k \eta_{j-k}$ and $l \ge -1$.

Proof. We first note that $\sum_{k=0}^{\infty} \psi_k \eta_{j-k} < \infty, a.s.$, for every fixed $j \ge 1$, i.e., $u_j^{(l)}$ is well defined. In fact, by applying Lemma 4.3.2, there exists a constant K such that

for any $j \le m \le m'$,

$$E \max_{m\le n\le m'} \left(\sum_{k=m}^{n} \psi_k \eta_{j-k}\right)^2 \le K \sum_{k=m}^{m'} \psi_k^2 E\eta_{j-k}^2 \le K \left(\sup_{k\ge 1} \frac{1}{k} \sum_{i=-k}^{k} E\eta_i^2\right) \sum_{k=m}^{\infty} (k+1)\psi_k^2. \quad (5.3.5)$$

From (5.3.5) and Markov's inequality, it follows that for any $\delta > 0$, as $n \to \infty$,

$$P\left(\sup_{i\ge 1}\left|\sum_{k=n}^{n+i} \psi_k \eta_{j-k}\right| \ge \delta\right) \le 2K\delta^{-2}\left(\sup_{k\ge 1} \frac{1}{k} \sum_{i=-k}^{k} E\eta_i^2\right) \sum_{k=n}^{\infty} k\psi_k^2 \to 0.$$

So we conclude by the Cauchy criterion that $\sum_{k=0}^{n} \psi_k \eta_{j-k}$ converges almost surely, i.e., for every fixed $j \ge 1$, $\sum_{k=0}^{\infty} \psi_k \eta_{j-k} < \infty, a.s.$.

In terms of $\sum_{k=0}^{\infty} \psi_k \eta_{j-k} < \infty, a.s.$, it is easy to show (let $\sum_{j=1}^{l} \cdot = 0$ for $l \le 0$) that for $m \ge l+1$,

$$\begin{aligned}
\sum_{j=1}^{m} u_j^{(l)} &= \sum_{j=1}^{m} \sum_{k=-\infty}^{j-(l+1)} \psi_{j-k}\eta_k \\
&= \left(\sum_{j=l+2}^{m} \sum_{k=1}^{j-(l+1)} + \sum_{j=1}^{m} \sum_{k=-\infty}^{0} - \sum_{j=1}^{l} \sum_{k=j-l}^{0}\right) \psi_{j-k}\eta_k \\
&= \sum_{k=1}^{m-(l+1)} \eta_k \sum_{j=l+1}^{m-k} \psi_j + \sum_{j=1}^{m}\sum_{k=0}^{\infty} \psi_{j+k}\eta_{-k} - \sum_{j=1}^{l}\sum_{k=0}^{l-j} \psi_{j+k}\eta_{-k} \\
&= \Delta_{m1}^{(l)} + \Delta_{m2}^{(l)} + \Delta_{m3}^{(l)}, \quad \text{say,}
\end{aligned} \quad (5.3.6)$$

where $\Delta_{m3}^{(l)} \equiv 0$ for $l = 0$ and -1. Now (5.3.4) follows if for any $\delta > 0$,

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\left\{\max_{1\le m\le n} \left|\Delta_{mt}^{(l)}\right| \ge \delta\sqrt{n}\right\} = 0, \quad t = 1, 2, 3. \quad (5.3.7)$$

To prove (5.3.7) for $t = 1$, put $S_k = \sum_{i=1}^{k} \eta_i$ and $S_0 = 0$. We obtain that

$$\Delta_{m1}^{(l)} = \sum_{k=1}^{m-(l+1)} (S_k - S_{k-1}) \sum_{j=l+1}^{m-k} \psi_j = \sum_{k=1}^{m-(l+1)} \psi_{m-k} S_k$$

and hence,

$$\max_{1\le m\le n}|\Delta_{m1}^{(l)}|\le\sum_{k=l+1}^{n}|\psi_k|\max_{1\le m\le n}|S_m|.$$

Again, it follows from Markov's inequality and Lemma 4.3.2 that

$$\begin{aligned}P\left\{\max_{1\le m\le n}\left|\Delta_{m1}^{(l)}\right|\ge\delta\sqrt{n}\right\} &\le \delta^{-2}n^{-1}\left(\sum_{k=l+1}^{n}|\psi_k|\right)^2 E\max_{1\le m\le n}S_m^2\\ &\le K\delta^{-2}\left(\sum_{k=l+1}^{\infty}|\psi_k|\right)^2\sup_{k\ge1}\frac{1}{k}\sum_{i=0}^{k}E\eta_i^2.\end{aligned}$$

Since $\sum_{k=0}^{\infty}|\psi_k|<\infty$, we conclude (5.3.7) holds for $t=1$.

For every fixed $j\ge0$, it follows from Fatou's Lemma and Lemma 4.3.2 that

$$\begin{aligned}E\left(\sum_{k=0}^{\infty}\psi_{j+k}\eta_{-k}\right)^2 &= E\lim_{n\to\infty}\left(\sum_{k=j}^{n}\psi_k\eta_{j-k}\right)^2\\ &\le \lim_{n\to\infty}E\left(\sum_{k=j}^{n}\psi_k\eta_{j-k}\right)^2\le K\sum_{k=j}^{\infty}\psi_k^2E\eta_{j-k}^2\\ &= K\sum_{k=0}^{\infty}\psi_{j+k}^2E\eta_{-k}^2.\end{aligned}$$

By applying Lemma 5.3.1 (choosing $b_k^2=E\eta_{-k}^2$), (5.3.7) holds for $t=2$.

That (5.3.7) holds for $t=3$ is obvious because the term $\Delta_{m3}^{(l)}$ does not include n. The proof of Lemma 5.3.2 is complete. □

5.3.2 Proofs of the main theorems

In this section, we provide the proofs of main results.

Proof of Theorem 5.2.1. According to (5.3.6) (for $l=-1$), for any $0\le t\le1$,

$$\sum_{j=1}^{k_n(t)}X_j=\sum_{k=1}^{k_n(t)}\epsilon_k\sum_{j=0}^{k_n(t)-k}\psi_j+\sum_{j=1}^{k_n(t)}\sum_{k=0}^{\infty}\psi_{j+k}\epsilon_{-k}.$$

Similar to (5.3.3), we have that

$$E \sup_{0\le t\le 1} \left| \sum_{j=1}^{k_n(t)} \sum_{k=0}^{\infty} \psi_{j+k}\epsilon_{-k} \right| \le C \sum_{j=1}^{n} \left(\sum_{k=j}^{\infty} \psi_k^2 \right)^{1/2} = o(s_n).$$

This, together with Markov's inequality, implies that

$$\frac{1}{s_n} \sup_{0\le t\le 1} \left| \sum_{j=1}^{k_n(t)} \sum_{k=0}^{\infty} \psi_{j+k}\epsilon_{-k} \right| \to_P 0.$$

On the other hand, it is well-known (noting the ϵ_k are iid random variables) that for any $0 \le t \le 1$,

$$\sum_{k=1}^{k_n(t)} \epsilon_k \sum_{j=0}^{k_n(t)-k} \psi_j \stackrel{d}{=} \sum_{k=1}^{k_n(t)} \epsilon_k \sum_{j=0}^{k-1} \psi_j \tag{5.3.8}$$

where $\stackrel{d}{=}$ denotes the same in distribution. Therefore, by applying Theorem 1.4.1 given in Billingsley (1968, page 25), (5.2.2) follows if

$$\frac{1}{s_n} \sum_{k=1}^{k_n(t)} \epsilon_k \sum_{j=0}^{k-1} \psi_j \Rightarrow W(t), \quad 0 \le t \le 1. \tag{5.3.9}$$

Recall $v_k = \sum_{j=0}^{k-1} \psi_j$. Since $\max_{1\le k\le n} |v_k|/s_n \to 0$, we see that for any $\delta > 0$,

$$\frac{1}{s_n^2} \sum_{k=1}^{n} v_k^2 E\epsilon_k^2 I_{(|v_k\epsilon_k| \ge \delta s_n)} \le E\epsilon_1^2 I_{(|\epsilon_1| \ge \delta s_n / \max_{1\le k\le n} |v_k|)} \to 0. \tag{5.3.10}$$

It follows from Lemma 4.3.3 that

$$\frac{1}{s_n} \max_{1\le k\le n} |v_k\epsilon_k| \to_P 0. \tag{5.3.11}$$

In terms of (5.3.10) and (5.3.11), (5.3.9) follows from Prokhorov's Theorem (Rao, 1984, page 343). This completes the proof of (5.2.2).

If $\psi_k = k^{-1}l(k)$ where the positive function $l(k)$ is slowly varying at infinity, it is easy to check that $\sum_{k=0}^{n} k^{-1}l(k)$ still is slowly varying at infinity. When $\sum_{k=0}^{\infty} k^{-1}l(k) =$

∞, we obtain (see Bingham et al., 1987, p26) that

$$s_n^2 = \sum_{j=1}^{n}\left(\sum_{k=0}^{j-1} k^{-1}l(k)\right)^2 \sim n\left(\sum_{k=1}^{n} k^{-1}l(k)\right)^2,$$

$$\sum_{j=0}^{n}\left(\sum_{k=j}^{\infty} k^{-2}l^2(k)\right)^{1/2} \sim \sum_{j=1}^{n} j^{-1/2}l(j) \sim 2n^{1/2}l(n) = o(s_n).$$

Hence (5.2.3) follows from (5.2.2).

If $0 < |\sum_{k=0}^{\infty}\psi_k| < \infty$ and $\sum_{k=1}^{\infty} k\psi_k^2 < \infty$, by applying Lemma 5.3.1 and methods similar to the proof used in (5.2.2), it suffices to show that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\epsilon_k\sum_{j=0}^{k-1}\psi_j \Rightarrow \left(\sum_{j=0}^{\infty}\psi_j\right)W(t), \quad 0 \le t \le 1. \tag{5.3.12}$$

By noting

$$\sum_{k=1}^{[nt]}\epsilon_k\sum_{j=0}^{k-1}\psi_j = \left(\sum_{j=0}^{\infty}\psi_j\right)\sum_{k=1}^{[nt]}\epsilon_k - \sum_{k=1}^{[nt]}\epsilon_k\sum_{j=k}^{\infty}\psi_j,$$

(5.3.12) follows from Donsker's Theorem (Billingsley, 1968, page 137), Theorem A.1 (see Appendix A) and as $n \to \infty$,

$$\begin{aligned} P\left(\sup_{0\le t\le 1}\left|\sum_{k=1}^{[nt]}\epsilon_k\sum_{j=k}^{\infty}\psi_j\right| \ge \delta\sqrt{n}\right) &\le C(\delta^2 n)^{-1}E\left(\sum_{k=1}^{n}\epsilon_k\sum_{j=k}^{\infty}\psi_j\right)^2 \\ &\le \frac{C_1}{n}\sum_{k=1}^{n}\left(\sum_{j=k}^{\infty}\psi_j\right)^2 \to 0, \end{aligned}$$

where we use the estimate: $\sum_{j=k}^{\infty}\psi_j \to 0$ as $k \to \infty$.

If $\sum_{k=0}^{\infty}|\psi_k| < \infty$ and $\sum_{k=1}^{\infty}\psi_k \ne 0$, the result follows from Hannan (1979). The proof of Theorem 5.2.1 is complete. □

Proof of Theorem 5.2.2. Generally speaking, (5.3.8) fails to hold for martingale differences. To prove Theorem 5.2.2, we need a new method.

For every fixed $l \geq 1$, put

$$Z_{1j}^{(l)} = \sum_{k=0}^{l} \psi_k \epsilon_{j-k} \quad \text{and} \quad Z_{2j}^{(l)} = \sum_{k=l+1}^{\infty} \psi_k \epsilon_{j-k}.$$

From Fuller (1996, page 320), we obtain that for any $m \geq 1$,

$$\begin{aligned}
\sum_{j=1}^{m} Z_{1j}^{(l)} &= \sum_{j=1}^{m}\sum_{k=0}^{l} \psi_k \epsilon_{j-k} \\
&= \sum_{k=0}^{l} \psi_k \sum_{j=1}^{m} \epsilon_j + \sum_{s=1}^{l} \epsilon_{1-s} \sum_{j=s}^{l} \psi_j - \sum_{s=0}^{l-1} \epsilon_{m-s} \sum_{j=s+1}^{l} \psi_j \\
&= \sum_{k=0}^{l} \psi_k \sum_{j=1}^{m} \epsilon_j + R(m,l), \quad \textit{say.}
\end{aligned}$$

Therefore, it follows that for every fixed $l \geq 1$,

$$\frac{1}{\sigma_n^*} \sum_{j=1}^{k_n^*(t)} X_j = \left(\frac{1}{\sigma_n^*} \sum_{k=0}^{l} \psi_k \right) \sum_{j=1}^{k_n^*(t)} \epsilon_j + \frac{1}{\sigma_n^*} R(k_n^*(t), l) + \frac{1}{\sigma_n^*} \sum_{j=1}^{k_n^*(t)} Z_{2j}^{(l)}. \tag{5.3.13}$$

Noting $\sum_{k=0}^{l} \psi_k \to b_\psi$, as $l \to \infty$, and existing positive constants A_1 and A_2 such that $A_1 n \leq \sigma_n^{*2} \leq A_2 n$, by applying Theorem A.1 (see Appendix A), it only need to show for any $\delta > 0$,

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\left\{ \sup_{0\leq t\leq 1} \left| \sum_{j=1}^{k_n^*(t)} Z_{2j}^{(l)} \right| \geq \delta\sqrt{n} \right\} = 0; \tag{5.3.14}$$

$$\limsup_{n\to\infty} P\left\{ \sup_{0\leq t\leq 1} |R(k_n^*(t), l)| \geq \delta\sqrt{n} \right\} = 0, \tag{5.3.15}$$

for every fixed $l \geq 1$; and

$$\frac{1}{s_n^*} \sum_{j=1}^{k_n^*(t)} \epsilon_j \Rightarrow W(t), \quad 0 \leq t \leq 1, \tag{5.3.16}$$

where $s_n^{*2} = \sum_{j=1}^{n} E\epsilon_j^2$.

In fact, (5.3.14) follows directly from Lemma 5.3.2 since $\{\epsilon_k, \mathcal{F}_n, -\infty < n < \infty\}$ is a martingale difference sequence.

In terms of (5.2.7) and Lemma 4.3.3 with $\eta_{nk} = \epsilon_k, k = -n, \cdots, n$, otherwise $\eta_{nk} = 0$, we have that

$$\frac{1}{\sqrt{n}} \max_{-n \leq j \leq n} |\epsilon_j| \rightarrow_P 0. \tag{5.3.17}$$

By (5.3.17), (5.3.15) holds because $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and hence

$$\begin{aligned} \frac{1}{\sqrt{n}} \sup_{0 \leq t \leq 1} |R(k_n^*(t), l)| &\leq \frac{1}{\sqrt{n}} \max_{-l \leq j \leq n} |\epsilon_j| \sum_{s=0}^{l} \left(\sum_{j=s}^{l} |\psi_j| + \sum_{j=s+1}^{l} |\psi_j| \right) \\ &\rightarrow_P 0. \end{aligned}$$

Finally, (5.3.16) follows Brown (1971) (see also Tanaka, 1996, page 80, Theorem 3.3.8) by using (5.2.6) and (5.2.7). The proof of Theorem 5.2.2 is complete. □

Chapter 6

Applications to unit root testing and time series regression

In this chapter, we apply the main results presented in the previous chapters to several well-known examples, namely, testing for unit roots, testing for stationarity, and time series regression. The limit distribution of the test statistics arising from statistical inference in economic time series related with fractional processes and linear processes have been studied by many statisticians and economists under various conditions in recent years. By using the results given in this thesis, we can see that the conditions under which the test statistics have desirable distributions can be essentially weaker than those previously discussed in the literature.

6.1 Testing for unit roots

Let $\{y_t\}$ be a stochastic process generated according to:

$$y_t = \alpha y_{t-1} + X_t, \quad t = 1, 2, \cdots, \tag{6.1.1}$$

where y_0 is a constant with probability one or has a certain specified distribution, and $\{X_t\}$ is a sequence of errors. Denote the ordinary least squares (OLS) estimator

of α by $\hat{\alpha}_n = \sum_{t=1}^{n} y_t y_{t-1} / \sum_{t=1}^{n} y_{t-1}^2$. We have that

$$n(\hat{\alpha}_n - 1) = \left\{ \sum_{t=1}^{n} y_{t-1}(y_t - y_{t-1}) \right\} \Big/ \left\{ n^{-1} \sum_{t=1}^{n} y_{t-1}^2 \right\}. \tag{6.1.2}$$

The statistic $n(\hat{\alpha}_n - 1)$ is commonly called the DF (Dickey and Fuller) test statistic. When model (6.1.1) has a unit root (i.e., the null hypothesis $\alpha = 1$ holds), the limit distribution of the DF test statistic was first considered by Dickey and Fuller (1979) under the assumption that the error processes X_t are iid random variables. Since then, considerable attention has been focused on how to weaken or avoid the iid assumption. For these discussions, see Said and Dickey (1984), Phillips (1987), Hall (1989) and Chan and Tasy (1996). In these articles, the unit root distribution is obtained but only for the situation that the error process is a short memory process, such as strong mixing sequence, ARMA processes or linear processes. For similar results, more references can be found in Phillips and Xiao (1998), where the authors presented a survey of unit root theory with an emphasis on testing principles and recent developments.

On weakening the iid assumption, another important contribution is made by Sowell (1990). By assuming that the error process is a simple fractional process Sowell (1990) established a well-known fractional unit root distribution[1] and pointed out that asymptotics in this case significantly differ from that when error processes are short memory processes. The results of Sowell (1990) were later extended to nonstationary fractionally integrated processes by Chan and Terrin (1995). With a Gaussian innovation, Chan and Terrin (1995) studied the general unstable AR unit root test, which extended results given by Chan and Wei (1988), Parks and Phillips

[1] The fractional Brownian motion used as limiting process is insufficiently defined in Sowell (1990). For a correction of Sowell's Theorem 3, see Marinucci and Robinson(1999). Also, it can be found in following Theorem 3.

(1988, 1989), and Sims et al. (1990) to fractional cases. Recently, Tanaka (1999) considered another situation where the error process satisfies (3.1.4).

By applying the main results in the previous chapters, in this section, we shall derive the limit distribution of the DF test statistic $n(\hat{\alpha}_n - 1)$ while the error process X_t is a linear process or a general fractionally integrated process. It will be shown that applications of the theorems established in previous chapters to the related statistics can lead to the results similar to those cited in previous papers under quite weak conditions. Since the properties of linear processes and general fractionally integrated processes are quite different, the limit distribution of $n(\hat{\alpha}_n - 1)$ will be studied separately when the error process X_t is a linear process and when it is a general fractionally integrated process.

6.1.1 Linear processes

In this section, the main results in Chapter 5 are applied to unit root testing.

At first, we assume that $\{y_t\}$ is a process generated by (6.1.1) with $\alpha = 1$.

Phillips (1987) investigated the limit behavior of the DF test statistic $n(\hat{\alpha}_n - 1)$ defined by (6.1.2) provided the error process $\{X_t\}$ is a strong mixing sequence with appropriate mixing conditions. Here, we assume that the error process $\{X_t\}$ is a linear process, i.e.,

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}, \qquad t = 1, 2, \cdots, \tag{6.1.3}$$

where $\{\psi_k\}$ is a sequence of real numbers and $\{\epsilon_k\}$ is a sequence of random variables specified later. Under quite general conditions for both the ψ_k and the innovations ϵ_k, this section shows that the DF test statistic $n(\hat{\alpha}_n - 1)$ has a distribution similar

to that in Phillips and Xiao (1998), where the authors obtained the limit distribution of the $n(\hat{\alpha}_n - 1)$ provided $\sum_{k=0}^{\infty} k^{1/2}|\psi_k| < \infty$.

Theorem 6.1.1. *Let $\epsilon_k, k = 0, \pm 1, \pm 2, \cdots$ be iid random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = \sigma^2$. If $0 < |\sum_{k=0}^{\infty} \psi_k| < \infty$ and $\sum_{k=1}^{\infty} k\psi_k^2 < \infty$ or $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and $\sum_{k=0}^{\infty} \psi_k \neq 0$, then as $n \to \infty$,*

(a) $\frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2 \Rightarrow \sigma^2 b_\psi^2 \int_0^1 W(r)^2 dr$;

(b) $\frac{1}{n}\sum_{t=1}^{n} y_{t-1}(y_t - y_{t-1}) \Rightarrow (\sigma^2 b_\psi^2/2)\,(W(1)^2 - \gamma)$;

(c) $n(\hat{\alpha}_n - 1) \Rightarrow (1/2)\,(W(1)^2 - \gamma) \Big/ \int_0^1 W(r)^2 dr$;

(d) $\hat{\alpha}_n \to_P 1$;

(e) $t_\alpha \Rightarrow (\frac{1}{2}\gamma^{-1/2})\,(W(1)^2 - \gamma) \Big/ \left\{\int_0^1 W(r)^2 dr\right\}^{1/2}$,

where

$$\begin{aligned} b_\psi &= \sum_{k=0}^{\infty} \psi_k, \quad \gamma = \sum_{k=0}^{\infty} \psi_k^2 \Big/ b_\psi^2, \\ \hat{\alpha}_n &= \sum_{t=1}^{n} y_t y_{t-1} \Big/ \sum_{t=1}^{n} y_{t-1}^2, \quad \textit{and} \\ t_\alpha &= \left(\sum_{t=1}^{n} y_{t-1}^2\right)^{1/2} (\hat{\alpha}_n - 1)/\delta_n \quad \textit{with} \quad \delta_n^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{\alpha}_n y_{t-1})^2. \end{aligned}$$

As in Phillips (1987), the proof of Theorem 6.1.1 can be obtained by applying (5.2.4) appeared in Theorem 5.2.1. The details are omitted.

If $\sum_{k=0}^{\infty} \psi_k = \infty$, the results differ from those in Theorem 6.1.1. The limit distribution of the DF test statistic $n(\hat{\alpha}_n - 1)$ is free from the unknown parameters b_ψ and γ, but t_α diverges to ∞ in probability. This result is given by Theorem 6.1.2 below.

Theorem 6.1.2. *Let $\epsilon_k, k = 0, \pm 1, \pm 2, \cdots$, be iid random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = \sigma^2$. If $\psi_k = k^{-1}l(k)$, where $l(0)/0 \equiv 1$ and positive function $l(k)$ is slowly varying at infinity satisfying $\sum_{k=1}^{\infty} k^{-1}l(k) = \infty$, then*

(a) $(nv_n)^{-2}\sum_{t=1}^{n} y_{t-1}^2 \Rightarrow \sigma^2 \int_0^1 W(r)^2 dr$;

(b) $(\sqrt{n}v_n)^{-2}\sum_{t=1}^{n} y_{t-1}(y_t - y_{t-1}) \Rightarrow (\sigma^2/2)\, W(1)^2$;

(c) $n(\hat{\alpha}_n - 1) \Rightarrow (1/2)W(1)^2 \Big/ \int_0^1 W(r)^2 dr$;

(d) $t_\alpha \to \infty$, *in probability,*

where $v_n = \sum_{k=1}^{n} k^{-1}l(k)$, $\hat{\alpha}_n$ and t_α are defined as in Theorem 6.1.1.

Proof. Recall

$$s_n^2 = \sum_{j=1}^{n} v_j^2, \qquad k_n(t) = \sup\{m : \; s_m^2 \le ts_n^2\}$$

and the process $\{y_t\}$ is defined by (6.1.1). If $\alpha = 1$, then $y_t = \sum_{j=1}^{t} X_j$ (without loss of generality, here and below, we assume $y_0 = 0$), and hence

$$y_{t-1}^2 = \frac{s_n^2}{v_t^2}\int_{s_{t-1}^2/s_n^2}^{s_t^2/s_n^2} \left(\sum_{j=1}^{k_n(r)} X_j\right)^2 dr.$$

Therefore, we obtain that

$$\begin{aligned}
\sum_{t=1}^{n} y_{t-1}^2 &= \sum_{t=1}^{n} \frac{v_t^2}{v_n^2} y_{t-1}^2 + \sum_{t=1}^{n}\left(\frac{1}{v_t^2} - \frac{1}{v_n^2}\right) v_t^2 y_{t-1}^2 \\
&= \frac{s_n^2}{v_n^2}\sum_{t=1}^{n}\int_{s_{t-1}^2/s_n^2}^{s_t^2/s_n^2}\left(\sum_{j=1}^{k_n(r)} X_j\right)^2 dr + R_n, \quad say, \\
&= \frac{s_n^4}{v_n^2}\int_0^1\left(\frac{1}{s_n}\sum_{j=1}^{k_n(r)} X_j\right)^2 dr + R_n.
\end{aligned}$$

It follows from (5.2.3) in Theorem 5.2.1 and the continuous mapping theorem (Theorem A.2, see Appendix A) that

$$\int_0^1 \left(\frac{1}{s_n} \sum_{j=1}^{k_n(r)} X_j \right)^2 dr \Rightarrow \sigma^2 \int_0^1 W(r)^2 dr.$$

This fact, together with Theorem A.1 (see Appendix A), implies that (a) follows if

$$s_n^2 = \sum_{j=1}^{n} v_j^2 \sim n v_n^2 \quad \text{and} \quad \frac{1}{n^2 v_n^2} R_n \to_P 0. \tag{6.1.4}$$

Since $v_n = \sum_{k=1}^{n} k^{-1} l(k)$ is still a slowly varying function, the first relation of (6.1.4) follows from Bingham et al. (1987, p26).

Since $Ey_t^2 \sim t v_t^2$ (recalling Remark 5.2.1), we have that

$$\frac{1}{n^2 v_n^2} \sum_{t=1}^{n} E y_{t-1}^2 \sim \frac{1}{2} \quad \text{and} \quad \frac{1}{n^2 v_n^4} \sum_{t=1}^{n} v_t^2 E y_{t-1}^2 \sim \frac{1}{2}$$

by using the slowly varying properties of v_t. Hence, by noting $v_t \uparrow$, as $t \uparrow \infty$, it follows that

$$\frac{1}{n^2 v_n^2} E|R_n| = \frac{1}{n^2 v_n^2} \sum_{t=1}^{n} E y_{t-1}^2 - \frac{1}{n^2 v_n^4} \sum_{t=1}^{n} v_t^2 E y_{t-1}^2 \to 0.$$

The second relation in (6.1.4) follows from using Markov's inequality. The proof of (a) is complete.

The proof of (b) follows directly from (5.2.3) in Theorem 5.2.1 and (a) in Theorem 6.1.3 by noting that $\sum_{t=1}^{n} y_{t-1}(y_t - y_{t-1}) = \frac{1}{2} y_n^2 - \frac{1}{2} \sum_{t=1}^{n} X_t^2$.

The proof of (c) is straightforward by applying (a), (b) and the continuous mapping theorem.

To prove (d), we rewrite

$$\begin{aligned}\delta_n^2 &= \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{\alpha}_n y_{t-1})^2 \\ &= \frac{1}{n}\sum_{t=1}^{n} X_t^2 - \frac{2(\hat{\alpha}_n - 1)}{n}\sum_{t=1}^{n} y_{t-1}X_t + \frac{(\hat{\alpha}_n - 1)^2}{n}\sum_{t=1}^{n} y_{t-1}^2.\end{aligned}$$

Since $n^{-\delta}v_n \to 0$, for any $\delta > 0$ (see Feller, 1971, page 277), and $v_n \to \infty$, it follows from (a)-(c) that for $\forall\, \epsilon > 0$, as $n \to \infty$,

$$\begin{aligned} P\left(\left|\frac{\hat{\alpha}_n - 1}{n}\sum_{t=1}^{n} y_{t-1}X_t\right| \geq \epsilon\right) &\leq P\left(n|\hat{\alpha}_n - 1| \geq \epsilon v_n\right) \\ &\quad + P\left(\frac{1}{n v_n^2}\left|\sum_{t=1}^{n} y_{t-1}X_t\right| \geq n v_n^{-3}\right) \to 0, \qquad (6.1.5) \\ P\left(\frac{(\hat{\alpha}_n - 1)^2}{n}\sum_{t=1}^{n} y_{t-1}^2 \geq \epsilon\right) &\leq P\left(n|\hat{\alpha}_n - 1| \geq \epsilon v_n\right) \\ &\quad + P\left(\frac{1}{n^2 v_n^2}\sum_{t=1}^{n} y_{t-1}^2 \geq n v_n^{-4}\right) \to 0. \qquad (6.1.6) \end{aligned}$$

In terms of (6.1.5), (6.1.6) and the part (a) in Theorem 6.1.3 below, we have that

$$\delta_n^2 \to_P \sigma^2 \sum_{k=0}^{\infty} \psi_k^2 < \infty.$$

Therefore, (d) follows easily by applying (a) and (c). The proof of Theorem 6.1.2 is complete. □

Remark 6.1.1. When the error process X_t satisfies (6.1.3), where the innovations ϵ_k form a martingale difference or a mixing sequence, results similar to Theorems 6.1.1 and 6.1.2 can also be obtained by using Theorem 5.2.2, Corollaries 4.2.4 and 4.2.5. The details will not be discussed here.

It is well-known that the limit distribution given in Theorem 6.1.1 depends on the unknown parameter

$$\gamma = \sum_{k=0}^{\infty} \psi_k^2 \Big/ \left(\sum_{k=0}^{\infty} \psi_k \right)^2 .$$

As in Phillips (1987, page 285), we can construct an estimate of γ as follows:

$$\hat{\gamma} = \hat{\sigma}_n^{*2} / \hat{\sigma}_n^2 , \quad \text{where } \hat{\sigma}_n^{*2} = \frac{1}{n} \sum_{t=1}^{n} X_t^2$$

and $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{t=1}^n X_t^2 + \frac{2}{n} \sum_{r=1}^{l_n} \sum_{t=r+1}^n X_t X_{t-r}$. Here and below, $\{l_n, n \geq 1\}$ denotes a sequence of positive real numbers. The following Theorem 6.1.3 shows that $\hat{\gamma}$ is a consistent estimate of γ for any l_n satisfying $l_n = o(n)$ and $l_n \to \infty$. This result also is useful for the testing of stationarity discussed in Section 6.2.

Theorem 6.1.3. *Let $\epsilon_k, k = 0, \pm 1, \pm 2, \cdots$, be iid random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = \sigma^2$.*

(a) If $\sum_{k=0}^{\infty} \psi_k^2 < \infty$, then $\hat{\sigma}_n^{*2} / \sigma^2 \to \sum_{k=0}^{\infty} \psi_k^2$, a.s..

(b) If $\sum_{k=0}^{\infty} |\psi_k| < \infty$, then for any l_n satisfying $l_n = o(n)$ and $l_n \to \infty$, $\hat{\sigma}_n^2 / \sigma^2 \to_P \left(\sum_{k=0}^{\infty} \psi_k\right)^2$.

Proof. Noting that $\{X_t, t \geq 1\}$ is a stationary ergodic sequence (Theorem B.3, see Appendix B) and $EX_1^2 = \sigma^2 \sum_{k=0}^{\infty} \psi_k^2 < \infty$, it follows from the stationary ergodic theorem (Theorem B.2, see Appendix B) that $\hat{\sigma}_n^{*2} \to \sigma^2 \sum_{k=0}^{\infty} \psi_k^2$, *a.s.*. This proves (a).

It is well-known that (Brockwell and Davis, 1987, page 212)

$$\frac{1}{n} E \left(\sum_{t=1}^{n} X_t \right)^2 = \frac{1}{n} \sum_{t=1}^{n} EX_t^2 + \frac{2}{n} \sum_{r=1}^{n-1} \sum_{t=r+1}^{n} EX_t X_{t-r} \to \sigma^2 \left(\sum_{k=0}^{\infty} \psi_k \right)^2 . \tag{6.1.7}$$

This fact, together with (a), implies that, to prove

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n} X_t^2 + \frac{2}{n}\sum_{r=1}^{l_n}\sum_{t=r+1}^{n} X_t X_{t-r} \to_P \sigma^2\left(\sum_{k=0}^{\infty}\psi_k\right)^2,$$

it suffices to show

$$\frac{1}{n}\sum_{r=1}^{l_n}\sum_{t=r+1}^{n}(X_tX_{t-r} - EX_tX_{t-r}) \to_P 0 \quad \text{and} \tag{6.1.8}$$

$$\frac{1}{n}\sum_{r=l_n+1}^{n-1}\sum_{t=r+1}^{n} EX_tX_{t-r} \to 0. \tag{6.1.9}$$

Next we give the proofs of (6.1.8) and (6.1.9).

Proof of (6.1.8). Recalling (6.1.3), we have (noting $E\epsilon_j = 0$ for all j)

$$\begin{aligned} A_n &\equiv \frac{1}{n}E\left|\sum_{r=1}^{l_n}\sum_{t=r+1}^{n}(X_tX_{t-r} - EX_tX_{t-r})\right| \\ &= \frac{1}{n}E\left|\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\psi_k\psi_s\sum_{r=1}^{l_n}\sum_{t=r+1}^{n}(\epsilon_{t-k}\epsilon_{t-r-s} - E\epsilon_{t-k}\epsilon_{t-r-s})\right| \\ &\leq \frac{1}{n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}|\psi_k\psi_s|E\left|\sum_{t=\lambda}^{n}(\epsilon_{t-k}^2 - E\epsilon_{t-k}^2)\right| \\ &\quad + \frac{1}{n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}|\psi_k\psi_s|E\left|\sum_{\substack{r\neq k-s \\ r=1}}^{l_n}\sum_{t=r+1}^{n}\epsilon_{t-k}\epsilon_{t-r-s}\right|, \end{aligned}$$

where $\lambda = \max\{2, k-s+1\}$. By Markov's inequality, to prove (6.1.8), it suffices to show that $A_n \to 0$, as $n \to \infty$. This follows from

$$B_n \equiv \sup_{k,\ s\geq 0}\frac{1}{n}E\left|\sum_{\substack{r\neq k-s \\ r=1}}^{l_n}\sum_{t=r+1}^{n}\epsilon_{t-k}\epsilon_{t-r-s}\right| \to 0, \tag{6.1.10}$$

where $l_n = o(n)$; and

$$C_n \equiv \sup_{k,\ s\geq 0}\frac{1}{n}E\left|\sum_{t=\lambda}^{n}(\epsilon_{t-k}^2 - E\epsilon_{t-k}^2)\right| \to 0, \tag{6.1.11}$$

where $\lambda = \max\{2, k - s + 1\}$.

In the following, we give the proofs of (6.1.10) and (6.1.11).

Since ϵ_k are iid random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 < \infty$, it follows that

$$E\left(\sum_{\substack{r \neq k-s \\ r=1}}^{l_n} \sum_{t=r+1}^{n} \epsilon_{t-k}\epsilon_{t-r-s}\right)^2 = \sum_{\substack{r \neq k-s \\ r=1}}^{l_n} \sum_{t=r+1}^{n} E\epsilon_{t-k}^2\epsilon_{t-r-s}^2 \le nl_n \left(E\epsilon_0^2\right)^2.$$

Hence, (6.1.10) follows from, as $n \to \infty$,

$$\begin{aligned} B_n &\le \frac{1}{n} \sup_{k,\ s \ge 0} \left(E\Big(\sum_{\substack{r \neq k-s \\ r=1}}^{l_n} \sum_{t=r+1}^{n} \epsilon_{t-k}\epsilon_{t-r-s}\Big)^2\right)^{1/2} \\ &\le (l_n/n)^{1/2}\left(E\epsilon_0^2\right)^2 \to 0. \end{aligned}$$

To prove (6.1.11), for every j, let

$$\epsilon_{1,j}^* = \epsilon_j^2 I_{(|\epsilon_j| \le n^{1/4})} - E\epsilon_j^2 I_{(|\epsilon_j| \le n^{1/4})} \quad \text{and} \quad \epsilon_{2,j}^* = \epsilon_j^2 I_{(|\epsilon_j| > n^{1/4})} - E\epsilon_j^2 I_{(|\epsilon_j| > n^{1/4})}.$$

After some algebra, we obtain

$$I_{n,k,s} \equiv E\left(\sum_{t=\lambda}^{n} \epsilon_{1,t-k}^*\right)^4 \le A\left\{n^2\left(E\epsilon_0^4 I_{(|\epsilon_0| \le n^{1/4})}\right)^2 + nE\epsilon_0^8 I_{(|\epsilon_0| \le n^{1/4})}\right\}. \quad (6.1.12)$$

The relation (6.1.12) implies that, as $n \to \infty$,

$$\begin{aligned} H_{n1} &\equiv \sup_{k,\ s \ge 0} \frac{1}{n} E\left|\sum_{t=\lambda}^{n} \epsilon_{1,t-k}^*\right| \\ &\le \frac{1}{n} \sup_{k,s \ge 0} (I_{n,k,s})^{1/4} \le A\left\{n^{-1/4}(E\epsilon_0^2)^2 + n^{-3/8} E\epsilon_0^2\right\} \to 0, \end{aligned}$$

where the inequality $E|X| \le (EX^4)^{1/4}$ for any X is employed. Therefore, it follows that, as $n \to \infty$,

$$C_n = \sup_{k,\ s \ge 0} \frac{1}{n} E\left|\sum_{t=\lambda}^{n} \left(\epsilon_{1,t-k}^* + \epsilon_{2,t-k}^*\right)\right| \le H_{n1} + 2E\epsilon_0^2 I_{(|\epsilon_0| > n^{1/4})} \to 0.$$

The proof of (6.1.8) is complete.

Proof of (6.1.9). Since $E\epsilon_j\epsilon_k = 0$ for $j \neq k$, we have that

$$\begin{aligned}\sum_{r=l_n+1}^{n-1}\sum_{t=r+1}^{n} EX_tX_{t-r} &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\psi_k\psi_s \sum_{r=l_n+1}^{n}\sum_{t=r+1}^{n} E\epsilon_{t-k}\epsilon_{t-r-s} \\ &= \sum_{s=0}^{\infty}\sum_{k=s+l_n}^{\infty}\psi_k\psi_s \sum_{t=k-s+1}^{n} E\epsilon_{t-k}^2.\end{aligned}$$

Therefore, as $n \to \infty$,

$$\frac{1}{n}\left|\sum_{r=l_n+1}^{n-1}\sum_{t=r+1}^{n} EX_tX_{t-r}\right| \leq E\epsilon_1^2\left(\sum_{k=l_n}^{\infty}|\psi_k|\right)\left(\sum_{s=0}^{\infty}|\psi_s|\right) \to 0.$$

This gives (6.1.9).

The proof of Theorem 6.1.3 is complete. □

6.1.2 General fractionally integrated processes

In this section, we discuss the applications of main results in Chapter 2 to unit root testing. We assume that the process $\{y_t\}$ is generated by (6.1.1) with $\alpha = 1$. In previous research, Sowell (1990) derived the limit distribution of the DF test statistic $n(\hat{\alpha}_n - 1)$ defined by (6.1.2) provided the error process $\{X_t\}$ is a simple fractional process (i.e., $m = 0$ and $u_t = \epsilon_t$ in (6.1.13)). With a Gaussian innovation, Chan and Terrin (1995) extended Sowell's result to the general unstable AR processes. Here, we consider a further general situation by assuming that the error process $\{X_t\}$ is a general fractionally integrated process, i.e.,

$$(1-B)^{d_0+m}X_t = u_t, \qquad u_t = \sum_{j=0}^{\infty}\psi_j\epsilon_{t-j}, \quad t = 1, 2, ..., \tag{6.1.13}$$

where $m \geq 0$ is an integer and $d_0 \in (-1/2, 1/2)$; $(1-B)^{d_0+m}$ is defined by (2.1.2); $\epsilon_j, j = 0, \pm 1, \ldots$ are iid random variables with $E\epsilon_0 = 0$ and $\psi_j, j \geq 0$, is a sequence of real numbers to be specified later. By using Theorems 2.2.1, 2.2.2 and 2.3.1, under quite general moment conditions, Theorem 6.1.4 below derives the limit distribution of the DF test statistic $n(\hat{\alpha}_n - 1)$ defined by (6.1.2), which provides a unified treatment of the previous cited results. In particular, we point out that the limit distribution of $n(\hat{\alpha_n} - 1)$ is free of the choice of the weights ψ_k of the u_t in model (6.1.13) if $d_0 + m > 0$.

We continue to use the notation $W_{d_0,m}(t)$ and Conditions A and B defined in Chapter 2. For convenience of reading, we rewrite them as follows:

$$W_{d_0,m}(s) = \begin{cases} W_{d_0}(t), & \text{if } m = 1, \\ \int_0^t \int_0^{t_{m-1}} \ldots \int_0^{t_2} W_{d_0}(t_1) dt_1 dt_2 \ldots dt_{m-1}, & \text{if } m \geq 2, \end{cases}$$

where $W_{d_0}(t), d_0 \in (-1/2, 1/2)$, is a "type I" fractional Brownian motion on $D[0, 1]$.

Condition A: $E|\epsilon_0|^p < \infty$, where $p = 2$, for $0 \leq d_0 < 1/2$; $p = (2+\delta)/(1+2d_0) < \infty, \delta > 0$, for $-1/2 < d_0 < 0$, and $\psi_j, j \geq 0$, satisfy

$$\sum_{j=0}^{\infty} |\psi_j| < \infty \quad \text{and} \quad b_\psi \equiv \sum_{j=0}^{\infty} \psi_j \neq 0.$$

Condition B: $E|\epsilon_0|^{\max\{2, 2/(1+2d_0)\}} < \infty, d_0 \in (-1/2, 1/2)$, and $\psi_j, j \geq 0$, satisfy

$$\sum_{j=0}^{\infty} j^{1/2-d_0} |\psi_j| < \infty \quad \text{and} \quad b_\psi \equiv \sum_{j=0}^{\infty} \psi_j \neq 0.$$

Theorem 6.1.4. *Let the process y_t be generated by (6.1.1) with $\alpha = 1$ and the error process X_t satisfies (6.1.13). Assume that Condition A or Condition B holds.*

(a) If $m \geq 1$ or $m = 0$ and $0 < d_0 < 1/2$, then

$$n(\hat{\alpha_n} - 1) \;\Rightarrow\; \frac{1}{2} [W_{d_0,m+1}(1)]^2 \Big/ \int_0^1 [W_{d_0,m+1}(s)]^2 \, ds\,. \tag{6.1.14}$$

(b) If $m = 0$ and $d_0 = 0$, then

$$n(\hat{\alpha_n} - 1) \Rightarrow \frac{1}{2}\left[W(1)^2 - \gamma\right] \Big/ \int_0^1 [W(s)]^2\, ds\,, \tag{6.1.15}$$

where $\gamma = \sum_{k=0}^{\infty} \psi_k^2 / b_\psi^2$.

(c) If $m = 0$ and $d_0 \in (-1/2, 0)$, then

$$n^{1+2d_0}(\hat{\alpha_n} - 1) \Rightarrow -\frac{M(d_0, \psi)}{2\kappa^2(d_0) \int_0^1 [W_{d_0}(s)]^2\, ds}, \tag{6.1.16}$$

where $\kappa^2(d_0)$ is defined as in Theorem 2.2.1 and

$$M(d_0, \psi) = \frac{E\epsilon_0^2}{2\pi} \int_{-\pi}^{\pi} \left|1 - e^{i\lambda}\right|^{-2d_0} \left|\psi(e^{-i\lambda})\right|^2\, d\lambda$$

with $\psi(e^{-i\lambda}) = \sum_{k=0}^{\infty} \psi_k e^{-ik\lambda}$.

Proof. By noting

$$\begin{aligned} \sum_{t=1}^{n} y_{t-1}(y_t - y_{t-1}) &= \frac{1}{2}\sum_{t=1}^{n} \{y_t^2 - y_{t-1}^2 - (y_t - y_{t-1})^2\} \\ &= \frac{1}{2}\, y_n^2 - \frac{1}{2}\sum_{j=1}^{n} X_j^2, \end{aligned}$$

we can rewrite $n(\hat{\alpha_n} - 1)$ as

$$\begin{aligned} & n(\hat{\alpha_n} - 1) \\ = & \frac{1}{2\kappa^2(d_0) n^{1+2(d_0+m)}} \left\{ y_n^2 - \sum_{j=1}^{n} X_j^2 \right\} \Big/ \left\{ \frac{1}{n} \sum_{t=1}^{n-1} \left(\frac{y_t}{\kappa(d_0) n^{1/2+d_0+m}} \right)^2 \right\}. \end{aligned} \tag{6.1.17}$$

Since $y_t = y_0 + \sum_{j=1}^{t} X_j$ (We assume $y_0 = 0$ below, clearly it does not affect the proofs of the main results), where X_j satisfy (6.1.13), it follows from Theorem 2.3.1

that

$$\left(\frac{1}{\kappa(d_0)n^{1/2+d_0+m}}\, y_{[nt]}\right)^2 \Rightarrow [W_{d_0,m+1}(t)]^2, \qquad \text{for } m \geq 0; \quad (6.1.18)$$

$$\frac{1}{n}\sum_{t=1}^{n-1}\left(\frac{y_t}{\kappa(d_0)n^{1/2+d_0+m}}\right)^2 \Rightarrow \int_0^1 [W_{d_0,m+1}(s)]^2 ds, \quad \text{for } m \geq 0; \quad (6.1.19)$$

$$\frac{1}{\kappa^2(d_0)n^{2(d_0+m)}}\sum_{j=1}^{[nt]} X_j^2 \Rightarrow \int_0^t [W_{d_0,m}(s)]^2 ds, \qquad \text{for } m \geq 1. \quad (6.1.20)$$

Because of (6.1.20), it is clear that

$$\frac{1}{n^{1+2(d_0+m)}}\sum_{j=1}^{n} X_j^2 \to_P 0, \quad \text{if } m \geq 1. \qquad (6.1.21)$$

Therefore, (6.1.14) holds for $m \geq 1$ by using (6.1.17)-(6.1.19), (6.1.21) and the continuous mapping theorem.

On the other hand, if $m = 0$ in (6.1.13), it follows from Lemma 2.6.3 that $\{X_t, t \geq 1\}$ is a stationary ergodic linear process with

$$EX_1^2 = \frac{E\epsilon_0^2}{2\pi}\int_{-\pi}^{\pi} \left|1 - e^{i\lambda}\right|^{-2d_0} \left|\psi(e^{-i\lambda})\right|^2 \, d\lambda < \infty,$$

where $\psi(e^{-i\lambda}) = \sum_{k=0}^{\infty} \psi_k e^{-ik\lambda}$. By using the stationary ergodic theorem (Theorem B.2, see Appendix B), we obtain that

$$\frac{1}{n}\sum_{j=1}^{n} X_j^2 \to EX_1^2 = M(d_0, \psi) < \infty, \quad a.s.. \qquad (6.1.22)$$

In particular, we point out that $EX_1^2 = E\epsilon_0^2 \sum_{k=0}^{\infty} \psi_k^2 < \infty$ if $d_0 = 0$.

Since (6.1.22) implies (6.1.21) when $m = 0$ and $0 < d_0 < 1/2$, (6.1.14) still holds for $m = 0$ and $0 < d_0 < 1/2$ by using (6.1.17)-(6.1.19), (6.1.22) and the continuous mapping theorem.

Similarly, (6.1.15) and (6.1.16) follows easily from (6.1.17)-(6.1.19), (6.1.22) and the fact that $W_{d_0,1}(s) = W_{d_0}(s)$ and $W_0(s) = W(s)$. The proof of Theorem 6.1.4 is complete. □

6.2 Testing for stationarity

In this section, we discuss another application of the main results in chapter 5. Let us consider the model:

$$y_t = \psi + r_t + z_t, \quad t = 1, 2, \cdots . \tag{6.2.1}$$

Here ψ is a constant, z_t is a stationary error and r_t is a random walk:

$$r_t = r_{t-1} + u_t \quad \text{with} \quad r_0 = 0, \tag{6.2.2}$$

where u_t are iid random variables with $Eu_t = 0$ and $Eu_t^2 = \sigma_u^2$. To test $\sigma_u^2 = 0$, i.e., to test whether the data generating process is stationary, the commonly used test statistic (known as KPSS test statistic) is

$$\hat{\eta}_u = n^{-2} \sum_{t=1}^{n} S_t^2 / s^2(l_n), \quad \text{where } S_t = \sum_{j=1}^{t} e_j, \tag{6.2.3}$$

$$s^2(l_n) = \frac{1}{n} \sum_{t=1}^{n} e_t^2 + \frac{2}{n} \sum_{s=1}^{l_n} \sum_{t=s+1}^{n} e_t e_{t-s}$$

and $e_t = y_t - \frac{1}{n} \sum_{t=1}^{n} y_t$ is the residual from the regression of y on the intercept ψ. Kwiatkowski et al. (1992) discussed the asymptotic distribution of the $\hat{\eta}_u$ provided that $l_n = 0\left(n^{1/2}\right)$ and z_t satisfies the (strong mixing) regularity conditions given by Phillips and Perron (1988, page 336) or the linear process conditions given by Phillips and Solo (1992, Theorems 3.4 and 3.15). Among Phillips and Solo's conditions, one is $\sum_{k=0}^{\infty} k^{1/2} |\psi_k| < \infty$. This condition is weakened and replaced by $\sum_{k=0}^{\infty} |\psi_k| < \infty$ in this section. In particular, we only need l_n satisfying $l_n = o(n)$ and $l_n \to \infty$, which, in practice, provides more choice for $s^2(l_n)$. Therefore, our results are an extension of theirs.

Theorem 6.2.1. *Let $\epsilon_k, k = 0, \pm 1, \pm 2, \cdots$ be iid random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = \sigma^2$. Assume that the data generating process is given by (6.2.1) with*

$$z_t = X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}.$$

If $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and $\sum_{k=0}^{\infty} \psi_k \neq 0$, then for any l_n satisfying $l_n = o(n)$ and $l_n \to \infty$,

$$\hat{\eta}_u \Rightarrow \int_0^1 V(r)^2 dr, \quad \textit{where } V(r) = W(r) - rW(1). \tag{6.2.4}$$

Proof. Under the hypothesis $\sigma_u^2 = 0$, it is well known that $e_t = X_t - \frac{1}{n}\sum_{t=1}^{n} X_t$. By applying Theorem 5.2.1, we have that, for any $0 \le r \le 1$,

$$\frac{1}{\sigma_n} S_{[nr]} = \frac{1}{\sigma_n} \sum_{t=1}^{[nr]} X_t - \frac{[nr]}{n\sigma_n} \sum_{t=1}^{n} X_t \Rightarrow W(r) - rW(1) = V(r),$$

where $\sigma_n^2 = n\sigma^2 \left(\sum_{k=0}^{\infty} \psi_k\right)^2$. Hence, it follows from the continuous mapping theorem that

$$n^{-2} \sum_{t=1}^{n} S_t^2 = \frac{1}{n} \int_0^1 S_{[nr]}^2 dr \Rightarrow \sigma^2 \left(\sum_{k=0}^{\infty} \psi_k\right)^2 \int_0^1 V(r)^2 dr. \tag{6.2.5}$$

On the other hand, we have that

$$\begin{aligned} s^2(l_n) &= \frac{1}{n}\sum_{t=1}^{n} e_t^2 + \frac{2}{n}\sum_{s=1}^{l_n}\sum_{t=s+1}^{n} e_t e_{t-s} \\ &= \frac{1}{n}\sum_{t=1}^{n} X_t^2 + \frac{2}{n}\sum_{s=1}^{l_n}\sum_{t=s+1}^{n} X_t X_{t-s} + R_{1n}, \end{aligned} \tag{6.2.6}$$

where, after a simple calculation,

$$\begin{aligned} |R_{1n}| &\le \frac{4l_n}{n^2}\left(\sum_{j=1}^{n} X_j\right)^2 + \frac{2}{n^2}\left|\sum_{s=1}^{l_n}\sum_{t=s+1}^{n} (X_t + X_{t-s})\right| \left|\sum_{j=1}^{n} X_j\right| \\ &\le \frac{Cl_n}{n^2}\left(\sum_{j=1}^{n} X_j\right)^2. \end{aligned}$$

Recalling (6.1.7), Markov's inequality implies that for any $l_n = o(n)$, $|R_{1n}| \to_P 0$. Therefore, by using (b) in Theorem 6.1.3, we obtain that for any l_n satisfying $l_n = o(n)$ and $l_n \to \infty$,

$$s^2(l_n) \to_P \sigma^2 \left(\sum_{k=0}^{\infty} \psi_k \right)^2 . \tag{6.2.7}$$

(6.2.4) follows immediately from (6.2.5) and (6.2.7). The proof of Theorem 6.2.1 is complete. □

6.3 Time series regression

In this section, we consider another application of Theorems 2.2.1 and 2.2.2. Let the observed process z_t follow the following regression model

$$z_t = x_t'\beta + y_t, \quad t = 1, 2, \cdots, \tag{6.3.1}$$

where $x_t = (x_{t1}, x_{t2}, \cdots, x_{tp})$ is a $1 \times p$ vector of nonstochastic regressors, $\beta = (\beta_1, \beta_2, \cdots, \beta_p)$ is a vector of unknown regression parameters and the sequence of errors $\{y_t\}$ is a long memory process. For the model (6.3.1), Yajima (1988, 1991) derived necessary and sufficient conditions for the least squares estimate (LSE) to be asymtotically effective relative to the best linear unbiased estimator (BLUE). This result extended the work of Grenander (1954) in the short memory case. Yajima's results were extended later by Robision and Hidalgo (1997) to the presence of long-range dependence in both errors and stochastic regressors. Robision and Hidalgo (1997) established a central limit theorem for time series regression estimates which include generalized least squares.

However, as shown in Yajima (1988), the LSE is no longer asymtotically efficient in the case of polynomial regression. For a Gaussian long-range dependent polynomial

regression model, Dahlhaus (1995) constructed a weighted LSE and proved asymptotic normality and efficiency of the weighted LSE. By applying Theorem 2.2.1, we give the following Theorem 6.3.1 and derive the asymptotic distribution of Dahlhaus's weighted LSE when the error satisfies (6.1.13) with $m = 0$ and $0 < d_0 < 1/2$ under the best possible second moment conditions.

We first introduce the following notation:

$\omega_{k,d_0}(x) = x^k\{x(1-x)\}^{-d_0}, k = 0, 1, \cdots;$

W is an $n \times n$ diagonal matrix with ith element $\omega_{0,d_0}(i/(n+1))$;

$Y = (y_1, \cdots, y_n)'$, $P_n = \text{diag}(1, n^{-1}, \cdots, n^{-(p-1)})$ and $X = ((x_{ij}))_{n\times p}$, where $x_{ij} = i^{j-1}$. Define

$$\hat{\beta}_n - \beta = (X'WX)^{-1} X'WY. \tag{6.3.2}$$

We obtain the following theorem.

Theorem 6.3.1. *Let (6.3.1) hold with $x_t' = (1, t, \cdots, t^{p-1})$ and the errors $y_t \equiv X_t$, where X_t satisfies (6.1.13) with $m = 0$ and $0 < d_0 < 1/2$. Assume that $E\epsilon_0^2 < \infty$. Then,*

$$n^{1/2-d_0} P_n^{-1}\left(\hat{\beta}_n - \beta\right) \Rightarrow A^{-1}E \tag{6.3.3}$$

where

$$E = \kappa(d_0)\left(\int_0^1 \omega_{0,d_0}(s)dW_{d_0}(s), \cdots, \int_0^1 \omega_{p-1,d_0}(s)dW_{d_0}(s)\right)',$$

$$A = (a_{ij})_{p\times p} \quad \text{with} \quad a_{ij} = \int_0^1 \omega_{i+j-2,d_0}(x)dx,$$

$\kappa(d_0)$ and $W_{d_0}(s)$ are defined as in Theorem 2.2.1 and $\int_0^1 \omega_{r,d}dW_d(s)$, $0 \le r \le p-1$, are defined as the limit distribution of $\int_0^1 g_m(r,s)dW_d(s)$, in mean square, where

$g_m(r,s)$ is a sequence of non-decreasing left continuous step functions that converges to $\omega_{r,d}(s)$ everywhere (noting that $g_m(r,s)$ exists according to Theorem 1.17 of Rudin (1986)).

Following the proofs of Theorem 1 and Corollary 1 in Deo (1997), the proof of Theorem 6.3.1 is straightforward by (2.2.3) (see Theorem 2.2.1) instead of (4) given in Deo (1997). The details are omitted.

Remark 6.3.1. Deo (1997) assumed the error is a linear process: $y_t = \sum_{i=0}^{\infty} \alpha_i e_{t-i}$ satisfying $\sum_{i=0}^{\infty} \alpha_i^2 < \infty$ and $Var\left(\sum_{t=1}^{n} y_t\right) \sim n^{2H} L(n)$ for some $1/2 < H < 1$, where $L(x)$ is a slowly varying function at ∞, but he imposes a strict restriction on the moment conditions for innovations ϵ_i.

Appendix A: Weak convergence of probability measure

This appendix gives some basic results on the weak convergence of probability measure. The context is mainly from Billingsley (1968).

Let S be a metric space and $\mathcal{B}$ be the σ-field generated by the open sets in S. A probability measure P on $\mathcal{B}$ is a nonnegative, countably additive set function with $P(S) = 1$. Let P_n and P be probability measures on $\mathcal{B}$. We say that P_n converges weakly to P and write $P_n \Rightarrow P$ if $\int_S f dP_n \to \int_S f dP$ for every bounded, continuous real function f on S.

Let X be a mapping from a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ into a metric space S. If X is measurable (i.e., $X^{-1}A \subset \mathcal{F}$, for each A in $\mathcal{B}$), we call it a random element. In particular, if $S = D[0,1]$, we also call X a random function.

The distribution of X is the probability measure $P = \mathrm{P}X^{-1}$ on $(S, \mathcal{B})$:

$$P(A) = \mathrm{P}(X^{-1}A) = P(X \in A), \qquad A \in \mathcal{B}.$$

We say a sequence X_n of random elements converges weakly (converges in distribution called in Billingsley, 1968) to the random element X, and write $X_n \Rightarrow X$, if the distributions P_n of the X_n converges weakly to the distribution P of X: $P_n \Rightarrow P$.

Denote the metric on S by $\rho(x, y)$ and let S be separable. The following Theorems A.1 and Theorem A.2 come from Theorem 4.2 and Corollary 5.1 given in Billingsley (1968) respectively.

Theorem A.1. *Suppose that, for each* u, $X_{un} \Rightarrow X_u$ *as* $n \to \infty$ *and that* $X_u \Rightarrow X$ *as* $u \to \infty$. *Suppose further that*

$$\lim_{u\to\infty} \limsup_{n\to\infty} P\{\rho(X_{un}, Y_n) \geq \epsilon\} = 0$$

for each positive ϵ. *Then* $Y_n \Rightarrow X$.

Theorem A.2. *Assume that* h *is a measurable mapping of* S *into another metric space* S'. *If* $X_n \Rightarrow X$ *and* $P\{X \in D_h\} = 0$, *where* D_h *is the set of discontinuities of* h, *then* $h(X_n) \Rightarrow h(X)$.

In the literature, Theorem A.2 is commonly called the continuous mapping theorem.

In the following, we consider weak convergence of probability measures (also named as the functional limit theorem or invariance principle) in $D[0,1]$. $D[0,1]$ denotes a metric space of all real-valued right continuous functions having finite left-hand limits on the $[0,1]$ with the Skorohod topology, i.e., the metric on $D[0,1]$ is defined to be the infimum of those positive ϵ for which there exists in Λ a $\lambda(t)$ such that

$$\sup_t |\lambda(t) - t| \leq \epsilon \quad \text{and} \quad \sup_t |x(t) - y(\lambda(t))| \leq \epsilon,$$

where Λ denotes the class of strictly increasing, continuous mapping of $[0,1]$ to itself and $x(t), y(t) \in D[0,1]$.

Let $X_n(t)$ and $X(t)$ be mappings from a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ into $D[0,1]$, i.e., $X_n(t)$ and $X(t)$ are random functions of $D[0,1]$. Let $\{X_{n'}(t)\} \subset \{X_n(t)\}$. If

each $\{X_{n'}(t)\}$ contains a further subsequence $\{X_{n''}(t)\}$ such that, for some $X(t)$, $X_{n''}(t) \Rightarrow X(t), 0 \leq t \leq 1$, then $\{X_n(t)\}$ is called tight. The following result comes from Theorem 15.1 given in Billingsley (1968).

Theorem A.3. *Assume that* $\{X_n(t)\}$ *is tight and*

$$(X_n(t_1), \cdots, X_n(t_k)) \rightarrow_d (X(t_1), \cdots, X(t_k))$$

holds for any $k \geq 1$ *and* $0 \leq t_1 \neq t_2 \neq \cdots \neq t_k \leq 1$. *Then* $X_n(t) \Rightarrow X(t)$, $0 \leq t \leq 1$.

In many places of this dissertation, we use the following Theorem A.4, which is a direct consequence of Theorem A.1.

Theorem A.4. *Let* $X_{un}(t)$, $X_u(t)$, $Y_n(t)$ *as well as* $X(t)$ *be random functions on* $D[0,1]$. *Suppose that, as* $n \to \infty$,

$$X_{un}(t) \Rightarrow X_u(t), \quad 0 \leq t \leq 1, \quad \textit{for each} \quad u,$$

and that, as $u \to \infty$,

$$X_u(t) \Rightarrow X(t), \quad 0 \leq t \leq 1.$$

Suppose further that

$$\lim_{u \to \infty} \limsup_{n \to \infty} P\left\{ \sup_{0 \leq t \leq 1} |X_{un}(t) - Y_n(t)| \geq \epsilon \right\} = 0$$

for each positive ϵ. *Then* $Y_n(t) \Rightarrow X(t)$, $0 \leq t \leq 1$.

Appendix B: Stationary ergodic theorems

This appendix gives the properties of a stationary ergodic sequence. The context is mainly from Chapter 3 of Stout (1974).

Let R_∞ be the infinite-dimensional Euclidean space and $\mathbb{R}_\infty$ be the smallest σ-field of set of R_∞ which contains all the measurable finite dimensional product cylinders $\prod_{i=1}^{\infty} C_i$. A stochastic sequence $\{X_i, i \geq 1\}$ is said to be stationary if $X_1, X_2, \cdots$ has the same distribution as $X_k, X_{k+1}, \cdots$ for each $k \geq 1$; i.e., if for each $k \geq 1$,

$$P\left[(X_1, X_2, \cdots) \in C\right] = P\left[(X_{k+1}, X_{k+2}, \cdots) \in C\right]$$

for every $C \in \mathbb{R}_\infty$. Let $\{X_i, i \geq 1\}$ be stationary and

$$A = \left[(X_1, X_2, \cdots) \in C\right] \qquad \text{for some } C \in \mathbb{R}_\infty.$$

Then A is said to be invariant if

$$A = \left[(X_k, X_{k+1}, \cdots) \in C\right] \qquad \text{for all } k \geq 1.$$

We call a stationary sequence $\{X_i, i \geq 1\}$ to be ergodic if every invariant event has probability zero or one. The following theorems come from Theorems 3.5.3, 3.5.7 and 3.5.8 given in Stout (1974) respectively.

Theorem B.1. *Let $\{X_i, i \geq 1\}$ be stationary or stationary ergodic and ϕ be a measurable function ϕ: $R_\infty \to R_1$. Let $Y_i = \phi(X_i, X_{i+1}, \cdots), i \geq 1$. Then $\{Y_i, i \geq 1\}$ is stationary or stationary ergodic.*

Theorem B.2. *Let $\{X_i, i \geq 1\}$ be stationary ergodic with $E|X_1| < \infty$. Then $\frac{1}{n}\sum_{j=1}^n X_j \to_{a.s.} EX_1$.*

Theorem B.2. is commonly called the stationary ergodic theorem. Since an iid sequence $\{\epsilon_i\}$ is stationary ergodic, by Theorems B.1 and B.2, we have that

Theorem B.3. *Let $\{\epsilon_i, i = 0, \pm 1, \pm 2, \cdots\}$ be iid $(0, \sigma^2)$ and $\{\psi_k, k \geq 0\}$ be a sequence of real numbers satisfying $\sum_{k=0}^\infty \psi_k^2 < \infty$. Then $\{X_i, i \geq 1\}$ is stationary ergodic and*

$$\frac{1}{n}\sum_{j=1}^n X_j^2 \to_{a.s.} EX_1^2 = \sigma^2 \sum_{k=0}^\infty \psi_k^2,$$

where $X_j = \sum_{k=0}^\infty \psi_k \epsilon_{j-k}, j \geq 1$.

Appendix C: Strong approximation theorems

This appendix gives the following theorems, which comes from Major (1976) and Komlós, Major and Tusnaády (1975, 1976) (also see Csörgö and Révész, 1981 or Csörgö and Horváth, 1993).

Theorem C.1. Let $\{X_j, j \geq 1\}$ be iid $(0, \sigma^2)$. Then, without changing the distribution of $\{X_j, j \geq 1\}$, we can redefine $\{X_j, j \geq 1\}$ on a richer probability space together with a sequence of random variables $\{Y_j, j \geq 1\}$, where $\{Y_j, j \geq 1\}$ is iid $N(0, \sigma^2)$, such that

$$\max_{1 \leq m \leq n} \left| \sum_{j=1}^{m} X_j - \sum_{j=1}^{m} Y_j \right| = o_P\left(n^{1/2}\right).$$

If in addition $E|X_1|^p < \infty$, where $p > 2$, then

$$\max_{1 \leq m \leq n} \left| \sum_{j=1}^{m} X_j - \sum_{j=1}^{m} Y_j \right| = o\left(n^{1/p}\right), \quad a.s..$$

Appendix D: List of Publications

Part I. Related to this thesis

1. The invariance principle for linear processes with applications. Accepted by *Econometric Theory.* (Chapter 5 and part of Chapter 6 are from this paper)

2. Asymptotics for general fractionally integrated processes with applications to unit roots. Submitted to *Ann. Statist.* (Chapter 2 and part of Chapter 6 are mainly from this paper)

3. Asymptotics for moving average processes with dependent innovations. Accepted by *Stat. Probab. Letters.* (Chapter 4 is mainly from this paper)

4. Asymptotics for nonstationary frctionally integrated processes without prehistorical influence. *Manuscript in preparation.*

5. The law of the iterated logarithm for long memory processes. Submitted to *Stochastic Processes and their Application.* (we do not put this paper in thesis)

Part II. Selected publications of author in other projects

1. An exponential nonuniform Berry-Esseen bound for self normalized sums, *Ann. Probab.* (1999), 27(4), 2068-2088.

2. Berry-Esseen bound for studentized statistics, *Ann. Probab.* (2000), 28(1), 511-535.

3. Kolmogrov and Erdos test for self normalized sums, *Stat. Probab. letters,* (1999), 42, 323-326.

4. On Berry-Esseen rates for m-dependent U-statistics, *Stat. Probab. Letters,* (1999), 41, 123-130.

5. Bernstein type inequalities for degenerate U-statistics with applications, *Chin. Ann. Math.* (1998), 19B(2), 157-166.

6. Non-uniform Rates of Convergence for Double Arrays of Independent Random Variables with Applications, *Acta Math. Appl. Sinica,* (English Series), (1996) 12 109–112.

7. On the Maximal Inequality, *Stat. and Probab. Letters,* (1996) 31 85-89.

8. Probabilities of large Deviations for U-statistics, *J. of Nanjing University Math. Biquarterly,* (1996) 13 168-172.

9. On the Non-uniform Convergence Rates for U-statistics, *Science in China (Series A),* (1995) 25 253-261.

10. The Strong Law of U-Statistics with ϕ^*-mixing Samples, *Stat. and Probab. Letters,* (1995) 23 151-155.

Bibliography

[1] Abramowitz, M. and Stegun, I. (1970), *Handbook of Mathematical Functions*, Dover, New York.

[2] Ahn, S. K. (1993), Some tests for unit roots in autoregressive-interated-moving average models with deterministic trends, *Biometrika*, 80(4), 855-864.

[3] Akonom, J. and Gourieroux, C. (1987), A functional central limit theorem for fractional Processes, *Discussion Paper #8801*, CEPREMAP, Paris.

[4] Avram, F. and Taqqu, M. S. (1987), Noncentral limit theorems and Appell polynomials, *Ann. Probab.*, 15, 767-775.

[5] Baillie, R. T. (1996), Long memory processes and fractional integration in econometrics, *Journal of Econometrics*, 73, 5-59.

[6] Beran, J. (1994), *Statistics for Long Memory Processes*, Chapman and Hall, New York.

[7] Beran, J. (1995), Maximum likelihood estimation of the differencing parameter for invertible short and long memory *ARIMA* models, *J. R. Statist. Soc.B,* 57, 695-672.

[8] Beran, J., Bhansali, R. J. and Ocker, D. (1998), On unified model selection for stationary and nonstationary short and long memory autoregressive processes, *Biometrika,* 85, 921-934.

[9] Bhargava, A. (1986), On the theory of testing for unit roots in observed time series, *Review of Economic Studies*, 53, 369-384.

[10] Bierens, H. J. (1993), Higher-order sample autocorrelations and the unit root hypothesis, *Journal of Econometrics*, 57, 137-160.

[11] Billingsley, P. (1968), *Convergence of Probability Measures*, Wiley, New York.

[12] Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1987), *Regular Variation*, Cambridge Univ. Press, Cambridge, UK.

[13] Brockwell, P. J. and Davis, R. A. (1987), *Time Series: Theory and Methods*, Springer-Verlag, New York.

[14] Cappuccio, N. and Lubian, D. (1997), Spurious regressions between I(1) processes with long memory errors, *J. Time Ser. Anal.* 18(4), 341-354.

[15] Chan, N. H. and Wei, C. Z. (1988), Limiting distributions of least squares estimates of unstable autoregressive processes, *Ann. Statist.*, 16, 367-401.

[16] Chan, N. H. and Terrin, N. (1995), Inference for unstable long-memory processes with applications to fractional unit root autoregressions, *Ann. Statist.*, 23, 1662-1683.

[17] Chan, N. H. and Tsay, B. S. (1996), Asymptotic inference for non-invertible moving-average time series, *J. Time Ser. Anal.*, 17(1), 1-17.

[18] Chen, Z. G. (1990), An extension of Lai and Wei's law of the iterated logarithm with applications to time analysis and regression, *J. Multiv. Anal.*, 32, 55-69.

[19] Cheng, Y. W. and Lai, K. S. (1993), A fractional cointegration analysis of purchasing power parity, *J. Business and Econ. Stat.*, 11, 103-112.

[20] Chow, Y. S. (1965), Local convergence of martingales and the law of large numbers, *Ann. Math. Statist.*, 36, 552-558.

[21] Chow, Y. S. and Teicher, H. (1988), *Probability Theory*, 2nd ed., Springer-Verlag, New York.

[22] Chung, C. F. (1996), A generalized fractionally integrated autoregressive moving average process, *J. Time Ser. Anal.*, 17(2), 111-140.

[23] Csörgö, M. and Révész, P. (1981), *Strong approximations in probability and statistics*, Academic Press, New York.

[24] Csörgö, M. and Horváth, J. (1993), *Weighted Approximation in Probability and Statistics*, John Wiley and Sons.

[25] Dahlhaus, R. (1989), Efficient parameter estimation for self-similar processes, *Ann. Statist.*, 17, 1749-1766.

[26] Dahlhaus, R. (1995), Efficient location and regression estimation for long range dependent regression models, *Ann. Statist.*, 23, 1029-1047.

[27] Davidson, J. E. H. (1995), *Stochastic Limit Theory*, 2nd ed. Oxford University Press.

[28] Davidson, R. and Mackinnon, J. G. (1993), *Estimation and Inference in Econometrics*, Oxford University Press.

[29] Davydov, Yu. A. (1970), The invariance principle for stationary processes, *Theory Probab. Appl.*, 15, 487-498.

[30] Deo, R. S. (1997), Asymptotic Theory for certain regression models with long memory errors, *J. Time Ser. Anal.*, 18, 385-393.

[31] Dickey, D. A. and Fuller, W. A. (1979), Distribution of the estimators for autoregressive time series with a unit root, *J. Amer. Stat. Assoc.*, 74, 427-431.

[32] Dickey, D. A. and Fuller, W. A. (1981), Likelihood ratio tests for autoregressive time series with a unit root, *Econometrica*, 49, 1057-1072.

[33] Diebold, F. X. (1988), Random walk versus fractional integration: power comparisons of scalar and joint tests of the variance-time function, *Technical Report 41*, Federal Reserve Board, Washington, DC.

[34] Feller, W. (1971), *An Introduction to the Theory of Probability and its Applications*, 2nd ed., John Wiley, New York.

[35] Fox, R. and Taqqu, M. S. (1986), Large sample properties of parameter estimates for strongly dependent stationary Gaussian time series, *Ann. Statist.,* 14, 517-532.

[36] Fuller, W. A. (1996), *Introduction to Statistical Time Series*, second edition, John Wiley and Sons, INC.

[37] Giraitis, L. and Surgailis, D. (1990), A central limit theorem for quadratic forms in strongly dependent linear variables and its applications to the asymptotic normality of Whittle's estimate, *Probab. Theory and Related Fields,* 86, 87-104.

[38] Gorodetskii, V. V. (1977), On convergence to semi-stable Gaussian process, *Theory Probab. Appl.*, 22, 498-508.

[39] Granger, C. W. J. and Joyeux, A. (1980), An introduction to long memory time series models and fractional differencing, *J. Time Ser. Anal.,* 1, 15-29.

[40] Grenander, U. (1954), On the estimation of regression coefficients in the case of autocorrelated disturbance, *Ann. Math. Statist.,* 25, 252-272.

[41] Hall, A. R. (1989), Testing for a unit root in the presence of moving average errors, *Biometrika*, 76, 49-56.

[42] Hall, P. (1992), Convergence rates in the central limit theorem for means of autoregressive and moving average sequences, *Stoch. Process Appl.*, 43, 115-131.

[43] Hall, P. and Heyde, C. C. (1980), *Martingale Limit Theory and its Applications*, Academic Press, New York.

[44] Hamori, S. and Tokihisa, A. (1997), Testing for a unit root in the presence of a variance shift, *Economics Letters*, 57, 245-253.

[45] Hannan, E. J. (1976), The asymptotic distribution of series covariances, *Ann. Statist.*, 4, 396-399.

[46] Hannan, E. J. (1979), The central limit theorem for time series regression, *Stoch. Process Appl.*, 9, 281-289.

[47] Hannan, E. J. and Heyde, C. C. (1972), On limit theorems for quadratic functions of discrete time series, *Ann. Math. Statist.*, 43(6), 2056-2066.

[48] Hassler, U. (1993), Regression of spectral estimators with fractionally integrated time series, *J. Time Ser. Anal.*, 14(4), 369- 380.

[49] Hassler, U. (1994), The sample autocorrelation function of $I(1)$ process, *Statistical Papers*, 35, 1-16.

[50] Hassler, U. (1997), Sample autocorrelations of nonstationary fractionally integrated series, *Statistical Papers*, 38, 43-62.

[51] Hasza, D. P. (1980), The asymptotic distribution of the sample autocorrelations for an integrated ARMA process, *J. Amer. Stat. Assoc.*, 75, 349-352.

[52] Horvath, L. and Shao, Q. M. (1996), Darling-Erdős-type theorems for sums of Gaussian variables with long-range dependence, *Stoch. Process. Appl.*, 63, 117-137.

[53] Hosking, J. R. M. (1981), Fractional differencing, *Biometrika*, 68, 165-176.

[54] Hosking, J. R. M. (1984), Modeling persistence in hydrological time series using fractional differencing, *Water Resour. Res.*, 20, 1898-1907.

[55] Hosking, J. R. M. (1984), Asymptotic distributions of the sample mean, autocovariances of long memory time series, *Technical summary report 2752* (Mathematics research center, University of Wisconsin, Madison, WI).

[56] Hosking, J. R. M. (1996), Asymptotic distributions of the sample mean, autocovariances and autocorrelations of long memory time series, *Journal of Econometrics*, 73, 261-284.

[57] Jeganathan, P. (1999), On asymptotic inference in cointegrated time series with fractionally integrated errors, *Econometric Theory*, 15, 583-625.

[58] Komlós, J., Major, P. and Tusnády, G. (1975), An approximation of partial sums of independent R.V.'s and sample DF. I. *Z. Wahrsch. verw. Gebiete*, 32, 111-131.

[59] Komlós, J., Major, P. and Tusnády, G. (1976), An approximation of partial sums of independent R.V.'s and sample DF. II. *Z. Wahrsch. verw. Gebiete*, 34, 33-58.

[60] Kwiatkowski, D., Phillips, P. C. B., Schmidt, P. and Shin, Y. (1992), Testing the null hypothesis of stationary against the alternative of a unit root: How sure are we that economic time series have a unit root? *Journal of Econometrics*, 54, 159-178.

[61] Lai, T. L. and Stout, W. (1978), The Law of the iterated logarithm and upper-lower class tests for partial sums of stationary Gaussian sequences, *Ann. Probab.*, 6, 731-750.

[62] Lai, T. L. and Stout, W. (1980), Limit theorems for sums of dependent random variables, *Z. Wahrsch. verw. Gebiete*, 51 1-14.

[63] Lai, T. L. and Wei, C.Z. (1982), A law of the iterated logrithm for double arrays of independent random variables with applications to regression and time series models, *Ann. Probab.*, 10, 320-335.

[64] Lee, D. and Schmidt, P. (1996), On the power of the KPSS test of stationarity against frationally-integrated alternatives, *Journal of Econometrics*, 73, 285-302.

[65] Li, W. K. and McLeod, A. I. (1986), Fractional time series modeling, *Biometrika,* 73(1), 217-221.

[66] Liu M. (1998), Asymptotics of Nonstationary fractional integrated series, *Econometric Theory,* 14, 641-662.

[67] Major, P. (1976), Approximation of partial sums of i.i.d. r.v.s when the summands have only two moments, *Z. Wahrash. verw. Gebiete,* 35, 213-220

[68] Mandelbrot, B. B. and Van Ness, J. W. (1968), Fractional Brownian motions, Fractional noises and applications, *SIAM Review,* 10, 423-437.

[69] Mandelbrot, B. B. M. (1983), *The fractal Geometry of Nature* , San Francisco, CA: Freeman.

[70] Marinucci, D. and Robinson, P. M. (1999), Alternative forms of fractional Brownian motion, *Journal of Statistical Planning and Inference,* 80, 13-35.

[71] Marinucci, D. and Robinson, P. M. (2000), Weak convergence of multivariate fractional processes, *Stoch. Process. Appl.,* 86, 103-120.

[72] Mcleish, D. L. (1975, a), A maximal inequality and dependent strong law, *Ann. Probab.*, 3, 829-839.

[73] Mcleish, D. L. (1975, b), Invariance principles for dependent variables, *Z. Wahrash. verw. Gebiete,* 32, 165-178.

[74] Mcleish, D. L. (1977), On the invariance principle for nonstationary mixingale, *Ann. Probab.*, 5, 616-621.

[75] Mcleod, A. I. (1998), Heperbolic Decay time series, *J. Time Ser. Anal.*, 4, 473-383.

[76] Merlevede, F. (1996), Central limit theorem for linear processes with values in a Hilbert space, *Stoch. Processes Appl.*, 65, 103-114.

[77] Mielniczuk, J. (1997), Long and short-range dependent sums of infinite-order moving averages and regression estimation, *Acta Sci. Math.*, 63, 301-316.

[78] Nelson, C. R. and Plloser, C. I. (1982), Trends and random walks in macroeconomic time series: some evidence and implications, *Journal of Monetary Economics,* 10, 139-62.

[79] Odaki, M. (1993), On the invertibility of fractionally differenced ARIMA processes, *Biometrika*, 80(3), 703-709.

[80] Oodaira, H. (1973), The log log law for certain dependent random sequences, Proc.Second Japan-USSR symp.Prob.Theory, *Lecture Notes in Math.*, 330, 355-369, Berlin: Springer Verlag.

[81] Parks, J. Y. and Phillips, P. C. B. (1988), Statistical inference in regressions with integrated processes: Part 1, *Econometric Theory*, 4, 468-497.

[82] Parks, J. Y. and Phillips, P. C. B. (1989), Statistical inference in regressions with integrated processes: Part 2, *Econometric Theory*, 5 95-131.

[83] Peligrad M. (1982), Invariance principles for mixing sequences of random variables, *Ann. Probab.*, 10, 968-981.

[84] Peligrad M. (1986), Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables, *Progress in Probab. and Statist.*, 11, 193-223.

[85] Peligrad M. (1998), Maximum of partial sums and an invariance principle for a class of weak dependent random variables, *Proceedings of the American Math. Society,* 126(4), 1181-1189.

[86] Peligrad M. and Utev, S. (1997), Central limit theorems for linear processes, *Ann. Probab.* 25 443-456.

[87] Phillips, P. C. B. (1987), Time series regression with a unit root, *Econometrica*, 55, 277-302.

[88] Phillips, P. C. B. (1991), To criticize the critics: an objective Bayesian analysis of stochastic trends, *Journal of Applied Econometrics*, 6, 334-364.

[89] Phillips, P. C. B. and Perron, P. (1988), Testing for a unit root in time series regression, *Biometrika*, 75, 335-346.

[90] Phillips, P. C. B. and Solo, V. (1992), Asymptotics for linear processes, *Ann. Statist.*, 20(2), 971-1001.

[91] Phillips, P. C. B. and Xiao, Z. (1998), A primer on unit root testing, manuscript.

[92] Rao, M. M. (1984), *Probability Theory with Applications,* A series of monographs and textbooks, Ed. by Birnbaum Z.W. and Lukacs, E. Academic Press, INC. New York.

[93] Robinson, P. M. (1994), Time series with strong dependence, In *Advance in Econometrics, Sixth World Congress* (C.A. Sims, ed.), 1, 47-95, Cambridge Univ. Press.

[94] Robinson, P. M. (1997), Large sample inference for nonparametric regression with dependent errors, *Ann. Statist.* 25, 2054-2084.

[95] Robinson, P. M. and Hidalgo, F. J. (1997), Time series regression with long range dependence, *Ann. Statist.* 25, 77-104.

[96] Rodin, W. (1986), Real and Complex Analysis, New-York: McGraw-Hill.

[97] Said, S. E. and Dickey, D. A. (1984), Testing for unit roots in autoregressive moving average models of unkown order, *Biometrika*, 71, 599-607.

[98] Samorodnitsky, G. and Taqqu, M. S. (1994), *Stable Non-Gaussian Random processes*, Chapman and Hall, New York.

[99] Shao, Q. M. (1988), A moment inequality and its applications, *Acta Math. Sinica*, 31, 736-747. [In Chinese.]

[100] Shao, Q. M. (1995, a), Maximal inequalities for partial sums of ρ-mixing sequences, *Ann. Probab.*, 23, 948-965.

[101] Shao, Q. M. (1995, b), Strong Approximation Theorems for independent ranom variables and their applications, *J. Multiv. Anal.*, 52, 107-130.

[102] Silveira, G. (1991), Contributions to strong approximations in time series with applications in nonparametric statistics and functional central limit theorems, Ph.D. thesis, University of London.

[103] Sims, C. A., Stock, J. H. and Watson, M. W. (1990), Inference in linear time series models with some unit roots, *Econometrica*, 58, 113-144.

[104] Sowell, F. B. (1990), The fractional unit root distribution, *Econometrica*, 58(2), 495-505.

[105] Sowell, F. B. (1992, a), Modeling long-run behavior with the fractional differenced model to the monetary aggregates, *J. Monetary Econ.*, 29, 277-302.

[106] Sowell, F. B. (1992, b), Maximum likelihood estimation of stationary univariate fractinally integrated time series models, *Journal of Econometrics*, 53, 165-188.

[107] Stadtmüller, U. and Trautner, R. (1985), Asymptotic behaviour of discrete linear processes, *J. Time Ser. Anal.*, 6(2), 97-108.

[108] Stout, W. F. (1974), *Almost Sure Convergence*, Academic Press, New York, London.

[109] Strassen, V. (1964), An invariance principle for the law of the iterated logarithm. *Z. Wahrach. Verw. Gebiete*, 3 211-226.

[110] Tanaka, K. (1996), *Time Series Analysis: Nonstationary and Noninertible Distribution Theory,* John Wiley and Sons, INC.

[111] Tanaka, K. (1999), The Nonstationary fractional unit root, *Econometric Theory,* 15, 549-582.

[112] Taqqu, M. S. (1975), Weak convergence to fractional Brownian motion and to the Rosenblatt process, *Z. Wahrsch. Verw. Gebiete*, 31, 287-302.

[113] Taqqu, M. S. (1977), Law of the iterated logarithm for sums of nonlinear functions of Gaussian variables that exhibit a long range dependence, *Z. Wahrsch. Verw. Gebiete*, 40, 203-238.

[114] Truong-van, B. (1995), Invariance principles for semi-stationary sequence of linear processes and applications to ARMA process, *Stoch. Processes Appl.*, 58, 155-172.

[115] Yajima, Y. (1988), On estimation of a regression model with long-memory stationary errors, *Ann. Statist.*, 16, 791-807.

[116] Yajima, Y. (1991), Asymptotic properties of the LSE in a regression model with long-memory stationary errors, *Ann. Statist.*, 19, 791-807.

[117] Yokoyama, R. (1995), On the central limit theorem and law of the iterated logarithm for stationary processes with applications to linear processes, *Stoch. Processes Appl.*, 59, 343-351.