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Normal numbers without measure theory

Abstract

Any number can be expanded to the base 10, leading to a sequence of digits between 0 and 9 corresponding to the number. Also, any number can be expanded to the base 2, leading to a sequence of digits, each one being either 0 or 1, corresponding to the number. It is result due to Émile Borel in 1904 that “almost all” numbers have the property that, when expanded to the base 2, each of the digits 0 and 1 appears with an asymptotic frequency of $1/2$. That is, if we regard the sequence of digits in the expansion to the base 2 as a sequence of ‘heads’ and ‘tails’ resulting from a coin-tossing experiment, then, in the language of probability theory, the probability of getting heads (that is a 0) is $1/2$, and the probability of getting tails (that is a 1) is also $1/2$. Numbers with this property are called “simply normal numbers” to the base 2. Traditionally, the proof of Borel’s Theorem relies on a knowledge of measure theory, which generally lies outside the undergraduate curriculum. Here, a proof of Borel’s Theorem is presented which requires only an introductory knowledge of sequences and series, and a knowledge of how to integrate step functions on an interval. This makes it possible to discuss Borel’s theorem at the level of a first or second year course in mathematical analysis.

Keywords

Borel, normal number, simply normal number, Rademacher functions, set, Beppo Levi Theorem, measure zero, binary expansion, zeros, ones, base 2

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Normal Numbers Without Measure Theory

Rodney Nillsen

1. INTRODUCTION. Consider a given number x in $[0, 1)$. The binary expansion of this number produces a sequence of digits, each of which is zero or one. For $n = 1, 2, 3, \dots$ let $S_n(x)$ denote the number of ones which appear among the first n digits of the binary expansion of x . Then x is said to be *simply normal to base 2* if

$$\lim_{n \rightarrow \infty} \frac{S_n(x)}{n} = \frac{1}{2}.$$

Thus, a number is simply normal to base 2 if it has an “equal” number of zeros and ones in its binary expansion. A similar definition to the above may be made for a number to be simply normal to other bases. The following result was proved by Émile Borel in 1904.

Borel’s Theorem. *There is a subset \mathcal{Z} of $[0, 1)$ which has measure zero and is such that every number in $[0, 1)$ which is not in \mathcal{Z} is simply normal to base 2. That is, almost every number in $[0, 1)$ is simply normal to base 2.*

In [2], Marc Kac described a very elegant approach to proving Borel’s Theorem, using the *Rademacher functions*. More recently, in a paper in this *Monthly*, Goodman [1] has shown how Kac’s approach may be extended so as to obtain deeper results relating to normal numbers, including some of those obtained by Mendès France [4] using more difficult concepts and techniques. The approach of Kac is elementary, and is quite accessible to undergraduate students, except at one point, where it is necessary to invoke the Beppo Levi Theorem to interchange the order of summation and integration in a series of non-negative functions. The Beppo Levi Theorem is similarly invoked by Goodman [1], in his extension of Kac’s approach to other aspects of normal numbers.

The main aim of this note is to show how to avoid using Beppo Levi’s Theorem (and the associated background in measure and integration), in the approaches of Kac and Goodman, thus making results on normal numbers more accessible to undergraduate students. In fact, the intention is to make Borel’s Theorem completely accessible to the student who knows what the integral of a step function is and who is familiar with convergent sequences and series.

2. AVOIDING THE MEASURE THEORY. Well, measure theory cannot be completely avoided, since Borel’s Theorem requires the notion of a set of measure zero. If J is an interval, let $\mu(J)$ denote its length.

DEFINITION. A subset A of $[0, 1]$ is called a set of *measure zero* if for each $\varepsilon > 0$ there is a sequence (J_n) of intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} J_n \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(J_n) < \varepsilon.$$

This definition of a set of measure zero is quite accessible, and it is easy to show that any countable set has measure zero and that any interval of positive length does not have measure zero. It can also be shown at the undergraduate level that an uncountable set such as the Cantor set is a set of measure zero. If a statement holds for all x except for those x in some set of measure zero, we say that the statement holds for *almost all* x .

Now, let's look at Kac's approach as in [2]. For $n = 1, 2, 3, \dots$ the n^{th} Rademacher function r_n on $[0, 1]$ is given by

$$r_n(x) = \begin{cases} -1, & \text{if } x \in \left[\frac{(k-1)}{2^n}, \frac{k}{2^n} \right) \text{ and } k \text{ is odd;} \\ 1, & \text{if } x \in \left[\frac{(k-1)}{2^n}, \frac{k}{2^n} \right) \text{ and } k \text{ is even.} \end{cases}$$

Equivalently, $r_n(x) = -1$ if the n^{th} binary digit of x is 0, while $r_n(x) = 1$ if the n^{th} binary digit is 1 [2, p.3]. This relationship between $r_n(x)$ and the n^{th} binary digit of x implies that Borel's Theorem may be expressed equivalently in terms of the Rademacher functions as follows [2, pp.16-17]:

$$\lim_{n \rightarrow \infty} \frac{1}{n} (r_1(x) + r_2(x) + \dots + r_n(x)) = 0, \quad \text{for almost all } x \in [0, 1].$$

Then from this point, Kac's proof of Borel's Theorem is along the following lines.

(I) Any product of distinct Rademacher functions has integral zero. This property allows a direct calculation which shows that

$$\int_0^1 \left(\frac{r_1(x) + r_2(x) + \dots + r_n(x)}{n} \right)^4 dx = \frac{3n-2}{n^3},$$

and we deduce that

$$\sum_{n=1}^{\infty} \left(\int_0^1 \left(\frac{r_1(x) + r_2(x) + \dots + r_n(x)}{n} \right)^4 dx \right) < \infty. \quad (1)$$

(II) Beppo Levi's Theorem means we can change the order of summation and integration in (1), to deduce that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \left(\frac{r_1(x) + r_2(x) + \dots + r_n(x)}{n} \right)^4 \right) dx < \infty.$$

It follows that

$$\sum_{n=1}^{\infty} \left(\frac{r_1(x) + r_2(x) + \dots + r_n(x)}{n} \right)^4 < \infty, \quad \text{for almost all } x \in [0, 1]. \quad (2)$$

(III) Since the n^{th} term of a convergent series has limit zero, we deduce from (2) that

$$\lim_{n \rightarrow \infty} \frac{r_1(x) + r_2(x) + \dots + r_n(x)}{n} = 0, \quad \text{for almost all } x \in [0, 1]. \quad (3)$$

Then, as noted above, Borel's Theorem follows.

Step (I) above can be carried out at the undergraduate level, but steps (II) and (III) require a preliminary course in measure and integration theory. We now show how to replace steps (II) and (III) with a direct argument which leads to the conclusion (3).

Lemma. *Let (a_n) be a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Then there is a sequence (b_n) of positive numbers such that*

$$\lim_{n \rightarrow \infty} b_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n b_n < \infty.$$

Proof. As $\sum_{n=1}^{\infty} a_n < \infty$, there is a sequence $k_1 < k_2 < k_3 < \dots$ such that $k_1 = 0$ and

$$\sum_{n=k_j+1}^{\infty} a_n < 2^{-j}, \quad \text{for all } j = 2, 3, \dots$$

We define the sequence (b_n) as follows: if $n \in \mathbb{N}$, there is a unique $j \in \mathbb{N}$ with $n \in \{k_j + 1, k_{j+1}, \dots, k_{j+1}\}$, in which case we put $b_n = j$. Now, for all $j = 2, 3, 4, \dots$,

$$\sum_{n=k_j+1}^{k_{j+1}} a_n b_n = \sum_{n=k_j+1}^{k_{j+1}} j a_n \leq j \sum_{n=k_j+1}^{\infty} a_n < j 2^{-j}.$$

Hence,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{k_2} a_n + \sum_{j=2}^{\infty} \left(\sum_{n=k_j+1}^{k_{j+1}} a_n b_n \right) < \sum_{n=1}^{k_2} a_n + \sum_{j=2}^{\infty} j 2^{-j} < \infty.$$

□

Theorem. *Let (ϕ_n) be a sequence of real or complex valued step functions on $[0, 1)$ such that*

$$\sum_{n=1}^{\infty} \left(\int_0^1 |\phi_n(x)| \, dx \right) < \infty.$$

Then,

$$\lim_{n \rightarrow \infty} \phi_n(x) = 0, \quad \text{for almost all } x \in [0, 1).$$

Proof. By the Lemma, there is a sequence (b_n) of positive numbers such that

$$\lim_{n \rightarrow \infty} b_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \left(\int_0^1 |\phi_n(x)| \, dx \right) < \infty. \quad (4)$$

Now let \mathcal{Z} denote the set of all points x in $[0, 1)$ such that the sequence $(\phi_n(x))$ does *not* converge to 0, and let $x \in \mathcal{Z}$. Then, by the definition of a convergent sequence, there is some $\eta > 0$ such that $|\phi_n(x)| > \eta$ for an infinite number of $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} b_n = \infty$, $b_n^{-1} < \eta$ for all sufficiently large n . It follows that $|\phi_n(x)| > b_n^{-1}$ for an infinite number of n .

Define, for $n = 1, 2, 3, \dots$,

$$\mathcal{A}_n = \left\{ y : y \in [0, 1) \text{ and } |\phi_n(y)| > b_n^{-1} \right\}.$$

We have

$$\begin{aligned}
x \in \mathcal{Z} &\implies |\phi_n(x)| > b_n^{-1} \text{ for an infinite number of } n, \\
&\implies x \in \mathcal{A}_n \text{ for an infinite number of } n, \\
&\implies x \in \bigcup_{k=n}^{\infty} \mathcal{A}_k, \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Hence,

$$\mathcal{Z} \subseteq \bigcup_{k=n}^{\infty} \mathcal{A}_k, \text{ for all } n \in \mathbb{N}. \quad (5)$$

Now note that because each function ϕ_n is a step function, the set \mathcal{A}_n is a finite union of intervals. Then, \mathcal{A}_n may be expressed as a finite union of *disjoint* intervals, J_1, J_2, \dots, J_r say, and let us put $\mu(\mathcal{A}_n) = \sum_{j=1}^r \mu(J_j)$. (It is easy to see that this definition of $\mu(\mathcal{A}_n)$ is independent of the manner in which \mathcal{A}_n is expressed as such a finite disjoint union.) Also, it follows from the definition of \mathcal{A}_n that

$$b_n |\phi_n(x)| \geq 1, \text{ for all } x \in \mathcal{A}_n.$$

Consequently,

$$\begin{aligned}
\mu(\mathcal{A}_n) &= \int_{\mathcal{A}_n} 1 \, dx \\
&\leq \int_{\mathcal{A}_n} b_n |\phi_n(x)| \, dx \\
&\leq b_n \int_0^1 |\phi_n(x)| \, dx.
\end{aligned}$$

It now follows from (4) that

$$\sum_{n=1}^{\infty} \mu(\mathcal{A}_n) \leq \sum_{n=1}^{\infty} \left(b_n \int_0^1 |\phi_n(x)| \, dx \right) < \infty,$$

so that

$$\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(\mathcal{A}_k) = 0. \quad (6)$$

Now each set \mathcal{A}_n is finite union of disjoint intervals and $\mu(\mathcal{A}_n)$ is the sum of the lengths of these intervals. So, it follows from (5) and (6) that \mathcal{Z} is a set of measure zero. But as \mathcal{Z} is by definition the set of all points x in $[0, 1)$ such that the sequence $(\phi_n(x))$ does not converge to 0, we have

$$\lim_{n \rightarrow \infty} \phi_n(x) = 0, \text{ for almost all } x.$$

□

This Theorem implies Borel's Theorem, as we can see by taking ϕ_n to be the step function $n^{-1}(r_1 + r_2 + \dots + r_n)$ and using (1) to deduce from the Theorem that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(r_1(x) + r_2(x) + \dots + r_n(x) \right) = 0, \text{ for almost all } x.$$

3. CONCLUDING REMARKS. Let $|A|$ denote the number of elements in a finite set A . Then, a number x in $[0, 1)$ is *normal to base 2*, as distinct from *simply normal to base 2*, if it has the following

property: if $d_1(x), d_2(x), \dots$ denotes the sequence of zeros and ones in the binary expansion of x , and if b_1, b_2, \dots, b_r is a finite sequence of zeros and ones, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ j : 1 \leq j \leq n \text{ and } d_j(x) = b_1, d_{j+1}(x) = b_2, \dots, d_{j+r-1}(x) = b_r \right\} \right| = \frac{1}{2^r}.$$

It is known that *almost every number in $[0, 1)$ is normal to base 2*, a result which is known as the *Normal Numbers Theorem for base 2* [1]. It was Mendès France [4] who made a connection between the numbers normal to base 2 and the *Walsh functions*, which are formed by taking products of the Rademacher functions. In [1], Goodman shows that the argument which Kac used on the Rademacher functions to prove Borel's Theorem can be used in a like manner on the Walsh functions, to deduce the Normal Numbers Theorem for base 2. Goodman's argument uses the Beppo Levi Theorem. But, just as the approach in this paper shows how to avoid measure theory in Kac's approach to Borel's Theorem, so too this approach avoids the use of measure theory in Goodman's approach to the Normal Numbers Theorem to base 2. In this sense, the Normal Numbers Theorem to base 2 is as accessible to students as Borel's Theorem. For bases other than 2, Goodman uses complex valued functions that correspond to the Rademacher functions, but the Theorem of Section 2 still applies, and the discussion in [1] for more general bases thus may proceed independently of measure theory.

The Theorem in Section 2 is stated in terms of step functions to emphasise that the present approach to Borel's Theorem requires a knowledge of integration extending only to step functions. In [6, p.345] Weyl proves a similar result to the Theorem of Section 2, but where the functions are continuous. Weyl's proof uses measure theory, and does in fact apply to step functions, in which case his argument simplifies somewhat, becoming independent of measure theory and providing a different proof of the Theorem in Section 2. However, the proof of the Theorem in Section 2 can be modified so as to apply to continuous functions, without using measure theory. Then, for students who know about integrals of continuous functions, and who know that a 2π -periodic continuous function may be uniformly approximated by trigonometric polynomials, Weyl's criterion for uniformly distributed sequences may be proved. This means that measure theory may be avoided in discussing results on the uniform distribution of sequences where Beppo Levi's Theorem is routinely invoked, such as in [3, pp.32-33]. Note also that the distinction between the concept of a set of measure zero and the general theory of measure is discussed by F. Riesz in [5], especially pp.363-365. Riesz also outlines a proof of Borel's Theorem [5, pp.369-370], which the interested reader may care to compare with both Kac's approach and the ideas presented here.

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