A cause of chaos

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Abstract

An abstract dynamical system consists of a collection S of points together with a transformation, or function, f which maps points of S into points of S. The points in S stand for all possible states of the system. The transformation is a “process of change” over one time unit, that changes each state x in S into another state f(x). Then we can interpret the equation f(x)=y as meaning that if the system is in state x, over the next time unit it will change into the state y. Alternatively, x is a “cause” of y, or x is an “antecedent” of y. There is no reason to consider the elapse of only one time unit – the transformation can be applied over and over again. Thus, if we start off in an initial state x, the next state is f(x), then the next state is f(f(x)), and so on. The sequence of states x, f(x), f(f(x)), f(f(f(x))), …. . is called the orbit of x, and it describes the evolution of the system from an initial state x. Now the actual state x of the system may not be known, but may only be approximated, by the state y. Then, if the orbits of x,y are very different, this would mean that the behaviour of the system cannot be predicted. This inability to predict is an intrinsic feature of chaotic systems. In this paper, chaotic behaviour is linked to a property that a dynamical system may have: given a state y, it may have more than one antecedent or, alternatively, any state may have more than one cause. A proliferation of possible causes may lead to chaos.

Keywords

dynamical system, function, chaos, choice, state, transitivity, sensitive dependence, initial state, one-to-one, onto, antecedent

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A CAUSE OF CHAOS

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Abstract

Chaos in both mathematical and physical systems has been studied intensively over the last decade, even longer, but what conceptual causes may it have? Here it is argued that a cause of chaotic behaviour is an element of "freedom", or "ambiguity", or even "choice" which is in the system as it evolves. In physical terms, this corresponds to any state having more than one "cause", or initial condition, which produces the given state. In lighter vein, it might also be said that chaos arises as a result of there being two sides to every question!

CHAOS, CHAOTIC BEHAVIOUR

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The term "chaos" was introduced by T. Li and J. Yorke [6]. Since then, a wide literature has developed concerning chaotic phenomena within biological, mathematical and physical contexts [4, 8]. Associated concepts such as fractals have been studied intensively, in large part due to the recognition by B. Mandelbrot [7] that such structures, often previously regarded as no more than "mathematical zoo exhibits", to adapt a comment of H. Steinhaus [10], are commonplace in nature. A purpose of this article is to provide a heuristic and especially a *conceptual* understanding of a reason for the onset of chaos, as distinct from an understanding which arises from a calculation of chaotic effects in specific systems.

Consider a general collection S of "states", and let f be a function which assigns, to each state x in S, a unique state $f(x)$ in S. The totality of points of S may be thought of as the set of all possible states of a system which is evolving in discrete time, and the value $f(x)$ may be thought of as the state of the system after one time unit, given that the preceding state was x. Let $f^{[1]}(x) = f(x)$, let

$$f^{[2]}(x) = f(f^{[1]}(x)), f^{[3]}(x) = f(f^{[2]}(x)),$$

and so on, so that for the values $t = 2, 3, 4, \ldots,$

$$f^{[t]}(x) = f(f^{[t-1]}(x)).$$

Then $f^{[t]}(x)$ is the state of the system after t time units, given that its initial state was x. Let us say that a state x of the system is *antecedent* to a state y if $f(x) = y$. If x is antecedent to y, x may be thought of as the "cause" of y; or alternatively, x may be thought of as one of possibly several choices which will cause the state y to be attained at the next stage. If each state has *at most one* antecedent, the function f is said to be *one-to-one*. If each state has *at least one* antecedent, then f is said to be *onto*.

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Any notion of chaos involves some form of complex behaviour. For present purposes, chaos in the system will mean that it has the properties of *transitivity* and of *sensitive dependence upon initial conditions.* A description of these concepts depends upon there being a notion of distance between any two states of the system. Transitivity means that any given state of the system may be "approximately" attained from any other given state of the system within a finite time. Sensitive dependence upon inital conditions means that there is some positive constant such that, given any state of the system, there is another state as close to it as desired so that, at some future time, these two states will have evolved into states whose distance apart is greater than the constant. Sensitive dependence embodies the idea that errors of prediction are intrinsic everywhere within the system and cannot be eliminated. Recent work [1, 11] has identified conditions under which sensitive dependence is a consequence of transitivity.

Intuitively, it seems more likely that chaos will occur if there are at least two distinct factors influencing the evolution of the system. One way of expressing this is to assume that the set S of states can be split into two subsets, a "left" set of states S_ℓ and a "right" set of states S_r; and further, to assume that each of the restrictions, of f to S_ℓ and of f to S_r, is a function which maps onto S. The assumptions imply that each state in S has at least two antecedents, one from S_ℓ and one from S_r. Thus, as f is applied to all the states in S, the values $f(x)$ of f cover S *at least twice*, whereas if f were one-to-one, S would be covered only once. The fact that the set of all possible states is covered at least twice by f is an indication that chaos may set in as successive iterates of f are taken, for essentially f is "working harder" to cover S twice when compared with a one-to-one function. Also, we may think of f as "stretching" each of S_ℓ and S_r so as to cover S, and as "folding" S upon itself because this covering occurs twice.

If $n_1, n_2, \ldots, n_k$ is any finite sequence of symbols, each of which is ℓ or r, let $S_{n_1 n_2 \ldots n_k}$ denote that set of states x for which x is in S_{n_1}, $f(x)$ is in S_{n_2}, $f^{[2]}(x)$ is in S_{n_3}, and so on until $f^{[k-1]}(x)$ is in S_{n_k}. Then, if k is given, the collection of all sets of states of the form $S_{n_1 n_2 \ldots n_k}$ splits S up into 2^k sets. Also, if x is in $S_{n_1 n_2 \ldots n_k}$, $f(x)$ will be in $S_{n_2 n_3 \ldots n_k}$ and, as every state in $S_{n_2 n_3 \ldots n_k}$ arises in this way, f maps $S_{n_1 n_2 \ldots n_k}$ onto $S_{n_2 n_3 \ldots n_k}$. This step may be carried out k times, and it follows that $f^{[k]}$ will map each set $S_{n_1 n_2 \ldots n_k}$ *onto* S. If $n_1, n_2, n_3, \ldots$ is any given infinite sequence of symbols, then the sets $S_{n_1 n_2 \ldots n_k}$ get smaller as k gets larger. Note also that the notion of distance between states leads to the concept of the diameter of a set of states; namely that the diameter is the greatest possible distance between two states in the set, assuming that this distance exists.

The extent to which the system as a whole is chaotic, and the extent to which different parts of the system are chaotic, are related to the behaviour, as k increases without limit, of the diameters of the sets $S_{n_1 n_2 \ldots n_k}$. A fully developed form of chaos occurs when for each sequence $n_1, n_2, n_3, \ldots$, the diameters of the sets of states of the form $S_{n_1 n_2 \ldots n_k}$ tend to 0 as k increases. For the time being, assume that this occurs. Then the system is chaotic in the sense that it must have transitivity and be sensitive to initial conditions. For, let x be in S, choose k to be appropriately large, and let x be in $S_{n_1 n_2 \ldots n_k}$. Here, "appropriately large" means that the diameter of $S_{n_1 n_2 \ldots n_k}$ will be as small as desired, so that any two states in $S_{n_1 n_2 \ldots n_k}$ will be "close". Let y be in S, and observe that because $f^{[k]}$ maps $S_{n_1 n_2 \ldots n_k}$ onto S, there is u in $S_{n_1 n_2 \ldots n_k}$ such that $f^{[k]}(u) = y$, and since the distance between x and u is at most the diameter of $S_{n_1 n_2 \ldots n_k}$, transitivity is illustrated. Again, because $f^{[k]}$ maps $S_{n_1 n_2 \ldots n_k}$ onto S, there are v, w in $S_{n_1 n_2 \ldots n_k}$ such that the distance between $f^{[k]}(v)$ and $f^{[k]}(w)$ is close to the diameter of S. However, the distance between $f^{[k]}(v)$ and $f^{[k]}(w)$

cannot be greater than the sum of the distances from $f^{[k]}(v)$ to $f^{[k]}(x)$ and from $f^{[k]}(x)$ to $f^{[k]}(w)$, since the sum of the lengths of two sides of a triangle is at least the length of the third side. This implies that the distance between $f^{[k]}(v)$ and $f^{[k]}(x)$, or that between $f^{[k]}(x)$ and $f^{[k]}(w)$, is close to half the diameter of S and thus illustrates sensitivity to initial conditions. In fact, in the latter argument, v and w may be chosen so that $f^{[k]}(v)$ and $f^{[k]}(w)$ respectively are equal to two preassigned states, a property which, in a weaker form, has been called blending and which is known, in some cases, to imply transitivity [2].

When the diameters of the $S_{n_1 n_2 \ldots n_k}$ tend to 0 as k increases without limit, each state x corresponds uniquely to a sequence $n_1, n_2, \ldots$, which is given by requiring that $f^{[k-1]}(x)$ is in S_{n_k} for all $k = 1, 2, 3, \ldots$. Since such a sequence of symbols may be thought of as a sequence of random tosses of a coin, we get some insight into why there may be little to distinguish between randomness and chaos which arises deterministically, a problem discussed by R. May [8]. The binary expansion of a number is a special case of this situation, where the set S of states corresponds to the numbers in $[0, 1)$, and for each number x in $[0, 1)$ is assigned its sequence of binary digits, or symbols.

In general, however, for a given system, it may happen that the diameter of $S_{n_1 n_2 \ldots n_k}$ tends to 0 for some sequences $n_1, n_2, \ldots$ but not for others. Then, if $n_1, n_2, \ldots$ *is* a particular sequence such that the diameter of $S_{n_1 n_2 \ldots n_k}$ tends to 0 as k increases, the ideas above may be adapted to deduce that "chaos" occurs near any state, necessarily unique, which belongs to *all* of the sets S_{n_1}, $S_{n_1 n_2}$, $S_{n_1 n_2 n_3}$, ... A complicating point in the above discussion is that in some cases there may be a "small" set of exceptional states, each of which has only one antecedent, whereas we have assumed that each state has two antecedents; but the substance of the discussion nevertheless is not affected.

The preceding thoughts are the result, in part, of reflection upon the treatment by Devaney [3] of the function on $[-2, 2]$ given by $f(x) = 2|x| - 2$. While the discussion is primarily intended for continuous functions, it does apply to many functions which are discontinuous. Even so, note that if f is any continuous and one-to-one function on $[0, 1)$, then f is either increasing or decreasing, so chaos will not occur; but if a one-to-one function on $[0, 1)$ has even a single discontinuity, then chaotic behaviour may occur. Also, in general, chaos may occur even though the function is one-to-one and continuous, as shown by the Smale Horseshoe Map [5, 9] and the shift map on the set of all two-sided sequences of noughts and ones [5].

Whereas the occurrence of chaos often has been associated with the fact that the system is non-linear, the line of thought here suggests that, for systems described by a continuous function on an interval, chaotic behaviour should be associated rather with the fact that their evolutions are described by functions which are not one-to-one. When each state of a general system has at least two antecedents, we can think of the system as having a strong "element of freedom", in that each state of the system may have originated from different states. Then this element of freedom, or ambiguity, which almost one might call choice, is propagated geometrically through the system as time passes, and chaos is natural. Thus, it is reasonable to regard chaos as a normal state of affairs, or even as the usual one, so confirming a view expressed in John Milton's *Paradise Lost*, that in one part of the cosmos at least "Night And Chaos, ancestors of Nature, hold Eternal anarchy".

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References

[1] BANKS, J., BROOKS, J., CAIRNS, G., DAVIS, G. AND STACEY, P. (1992) On Devaney's definition of chaos. *Amer. Math. Month.* **99**, 332-334.

[2] CRANNELL, A. (1995) The role of transitivity in Devaney's definition of chaos, *Amer. Math. Month.* **102**, 788-793.

[3] DEVANEY, R. (1992) *A First Course in Chaotic Dynamical Systems*, Addison-Wesley.

[4] ECKMANN, J.-P. AND RUELLE, D. (1985) Ergodic theory of chaos and strange attractors, *Rev. Mod. Phys.* **57**, 617-656.

[5] GULICK, D. (1992) *Encounters with Chaos*, McGraw-Hill.

[6] LI, T. Y. AND YORKE, J. A. (1975) Period three implies chaos, *Amer. Math. Month.* **82**, 985-992.

[7] MANDELBROT, B. (1982) *The Fractal Geometry of Nature*, Freeman, San Francisco.

[8] MAY, R. M. (1995) Necessity and chance: deterministic chaos in ecology and evolution, *Bull. Amer. Math. Soc.* **32**, 291-308.

[9] SMALE, S. (1967) Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73**, 747-817.

[10] STEINHAUS, H. (1937) Introduction to *Mathematical Snapshots*, Stechert and Co., first edition.

[11] VELLEKOOP, M. AND BERGLUND, R. (1994) On intervals, transitivity = chaos, *Amer. Math. Month.* **101***, 353-355.*

(and it will occur when f is "stretching" on each of S_ℓ and S_r in the sense that there is a constant $C > 1$ such that if $x, y \in S_\ell$ or if $x, y \in S_r$, the distance between $f(x)$ and $f(y)$ is at least C times that between x and y).