University of Wollongong

Research Online

Faculty of Informatics - Papers (Archive)

Faculty of Engineering and Information Sciences

January 1998

Bose's Method of Differences Applied to Construct Bhaskar Rao Designs

Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

Follow this and additional works at: https://ro.uow.edu.au/infopapers

Part of the Physical Sciences and Mathematics Commons

Recommended Citation

Seberry, Jennifer: Bose's Method of Differences Applied to Construct Bhaskar Rao Designs 1998.
https://ro.uow.edu.au/infopapers/350

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

Bose's Method of Differences Applied to Construct Bhaskar Rao Designs

Abstract

In this paper we show that BIBD(v, b, r, k, λ), where v = pq or pq + 1, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs GBRD(p, b', r', k, λ; G) where G is a group of order q and vice versa. This gives many new GBRDs including a GBRD(9, 5, 5; Z5) and a GBRD(13, 7, 7; Z7).

Disciplines

Physical Sciences and Mathematics

Publication Details

This article was originally published as Seberry, J, Bose's Method of Differences Applied to Construct Bhaskar Rao Designs, Journal of Statistical Planning and Inference, 73, 1998, 215-224. Copyright Elsevier. Original journal available here. Paper originally presented at R. C. Bose Memorial Conference on Statistical Design and Related Combinatorics, Colorado Springs, 7-11 June, 1995.

This journal article is available at Research Online: https://ro.uow.edu.au/infopapers/350

Bose's Method of Differences Applied to Construct Bhaskar Rao Designs

Jennifer Seberry
Department of Computer Science
University of Wollongong
Wollongong, NSW, 2500
Australia

R.C.Bose Memorial Conference on Statistical design and Related Combinatorics

Abstract

In this paper we show that $BIBD(v, b, r, k, \lambda)$, where $v = pq$ or $pq + 1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p, b', r', k, \lambda; G)$ where G is a group of order q and vice versa.

This gives many new GBRDs including a $GBRD(9, 5, 5; Z_5)$ and a $GBRD(13, 7, 7; Z_7)$.

1 Introduction

Let $G = \{h_1 = e,\ h_2,\ \ldots\ ,\ h_g\}$ be a finite group of order g with identity e. Form the matrix W

$$W = h_1 A_1 + \ \ldots\ + \ h_g A_g$$

where $A_1,\ \ \ldots\ ,\ A_g$ are $v \times b$ (0,1)-matrices such that the Hadamard product $A_k * A_j = 0$ for any $k \neq j$. Let

$$W^+ = (h_1^{-1} A_1 + \ \ldots\ + \ h_g^{-1} A_g)^T$$

and

$$N = A_1 + A_2 + \ \ldots\ + \ A_g.$$

Then we say W is a *generalized Bhaskar Rao design* over G denoted by $GBRD(v, b, r, k, \lambda; G)$, or in abbreviated form $GBRD(v, k, \lambda; G)$, if N satisfies

$$NN^T = (r - \lambda)I \ + \ \lambda J :$$

that is, N is the incidence matrix of the $BIBD(v, k, \lambda)$ and

$$WW^+ = reI \ + \ (\lambda/g)(h_1 \ + \ \ldots\ + \ h_g)(J - I).$$

We say that the design W is based on the matrix N or that N is signed over the group G.

$GBRD$ have been studied by the author and others, for example see [7, 8, 9, 14, 16, 21, 22]. We now illustrate the construction of the $GBRD$ with some examples. For further examples the reader is referred to [20].

To check whether a set of blocks $(x_a^i,\ y_b^i,\ z_c^i)$ mod (v,g), each with k elements, are part of a $GBRD$ or a $BIBD$ we check all the differences: $(y-x)_{ba^{-1}}^i$, $(x-y)_{ab^{-1}}^i$, $(z-x)_{ca^{-1}}^i$, $(x-z)_{ac^{-1}}^i$, $(z-y)_{cb^{-1}}^i$ and $(y-z)_{bc^{-1}}^i$, where the subscripts such as ab^{-1} are from a group G of order $|G|=g$ and the subscripted elements such as $(x-y)^i$ come from a group V of order v.

If each non-zero-zero difference, that is differences other than 0_0, including the differences 0_1, $\cdots$, 0_{g-1} occurs the same number of times, λ, we will have a $BIBD(v,k,\lambda)$. If each non-zero difference, that is not including the differences 0_0, 0_1, $\cdots$, 0_{g-1}, occurs the same number of times, λ, we will have a $GBRD(v,k,\lambda;G)$.

We also use a distinguished element ∞ which has the properties that

$$\infty - a_i = \infty_i^{-1}, \text{ and } a_i - \infty = -\infty_i,$$

for any $a_i \in V$.

For a Bhaskar Rao design ∞_j and $-\infty_j$ must occur λ times. For a $BIBD$ it is merely necessary that the number of elements in the sets with ∞ is $k-1$.

Example 1 Consider the initial blocks

$$\begin{array}{ccccc} (1_1,\ 6_1,\ 0_2) & (2_1,\ 5_1,\ 0_2) & (3_1,\ 4_1,\ 0_2) & (1_2,\ 6_2,\ 0_3) & (2_2,\ 5_2,\ 0_3) \\ (3_2,\ 4_2,\ 0_3) & (1_3,\ 6_3,\ 0_1) & (2_3,\ 5_3,\ 0_1) & (3_3,\ 4_3,\ 0_1) & (0_1,\ 0_2,\ 0_3) \end{array}$$

mod $(7,3)$ which Bose gives as the initial blocks of a $BIBD(21,3,1)$. Without the last block of $(0_1,\ 0_2,\ 0_3)$ we have a $GBRD(7,3,9;Z_3)$.

Example 2 The subscripts from Bose's [4, p373, example (i)] $BIBD(9,3,1)$ give a $GBRD(3,3,9;Z_3)$ (the $GBRD$ is the 3×9 array):

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 1 \end{bmatrix}$$

Clearly, the differences from a single starting block also suffice to give all the differences, and hence we have a $GBRD(3,3,3;Z_3)$.

Example 3 Consider the initial sets

$$(0_0,\ 1_0,\ 4_1) \text{ twice},\quad (\infty,\ 0_0,\ 3_0),\quad (\infty,\ 0_1,\ 3_1)\ \bmod(5,2)$$

To develop the subscripts we note

$$0^{-1}=0,\quad 1^{-1}=1,\quad 0\cdot 1^{-1}=1\cdot 0^{-1}=1,\quad 0\cdot 0^{-1}=1\cdot 1^{-1}=0.$$

We now note that the set $(\infty,\ 0_0,\ 3_0)$ means the differences ∞_0 and $-\infty_0$ each occur twice. The set $(\infty,\ 0_1,\ 3_1)$ means the differences ∞_1 and $-\infty_1$ also each occur twice.

When we check the non-∞ differences we get

$$\begin{array}{lllll} & 0_0-1_0=4_0, & 0_0-4_1=1_1, & 1_0-4_1=2_1, & \text{twice}, \\ & 1_0-0_0=1_0, & 4_1-0_0=4_1, & 4_1-1_0=3_1, & \text{twice}, \\ \text{and} & 3_0-0_0=3_0, & 0_0-3_0=2_0, & 3_1-0_1=3_0, & 0_1-3_1=2_0. \end{array}$$

As we have each non-zero difference occurring twice we have a $GBRD(6,3,4;Z_2)$. Its incidence matrix is

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	1	1	1
0	1	.	.	0	0	1	.	.	0	0	.	0	.	.	1	.	1	.	.
0	0	1	.	.	0	0	1	.	.	.	0	.	0	.	.	1	.	1	.
.	0	0	1	.	.	0	0	1	.	.	.	0	.	0	.	.	1	.	1
.	.	0	0	1	.	.	0	0	1	0	.	.	0	.	1	.	.	1	.
1	.	.	0	0	1	.	.	0	0	.	0	.	.	0	.	1	.	.	1

where 0, 1 are elements of the group, Z_2, and $\cdot$ means the zero of the group ring so $\cdot - 1 = \cdot$, and $\cdot - 2 = \cdot$.

To change these sets into starting blocks for a $BIBD$ we would also need to have the difference 0_1 occur twice. This can be done but not in a straight forward way.

This clearly shows that Bhaskar Rao designs are not another representation of $BIBD$s. $\square$

Example 4 To illustrate how a design developed from blocks using differences can be shown to be a $BIBD$ or $GBRD$ we consider the initial blocks

$$(0_0,\ 1_0,\ 2_0)\ \ (0_0,\ 1_1,\ 2_1)\ \ (0_1,\ 2_0,\ 3_1)\ \ (0_0,\ 2_1,\ 4_1)\ \ (0_0,\ 2_0,\ 0_1)\ \ (0_0,\ 0_1,\ 4_1)\ \ \mathrm{mod}(5,-)$$

So the differences are

1_0	2_0	1_0	4_0	3_0	4_0
1_0	1_1	2_1	4_0	4_1	3_1
1_1	2_1	3_0	4_1	3_1	2_0
2_1	4_1	2_0	3_1	1_1	3_0
2_0	0_1	3_1	3_0	0_1	2_1
4_0	0_1	4_1	1_0	0_1	1_1

This means each of the differences $0_1, 1_0, 1_1, 2_0, 2_1, 3_0, 3_1, 4_0, 4_1$ occurs 4 times, that is 2 times corresponding to each difference where the direction it is taken in is considered. Hence we have, after developing the initial blocks modulo 5, exactly Bose's example (iv) [4, p370] for a $BIBD(10,3,2)$.

1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0
0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0
0	0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0
1	0	0	1	1	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	1	0	0	1	0	0	1	0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0	1	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	0	0
0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	1	1	0
0	0	0	0	0	1	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	1	1

It has GBRD type incidence matrix for the last three sets of

0		1	.	1	01	.	0	.	.	01	.	.	.	1
1	0	.	1	.	.	01	.	0	.	1	01	.	.	.
.	1	0	.	1	.	.	01	.	0	.	1	01	.	.
1	.	1	0	.	0	.	.	01	.	.	.	1	01	.
.	1	.	1	0	.	0	.	.	01	.	.	.	1	01

It is not a $GBRD$ as the difference 0_1 occurs 4 times. This is represented by the "01" in the above incidence matrix. However the first four blocks give a $GBRD(6,3,6;Z_2)$

$$(0_0,\ 1_0,\ 2_0)\ \ (0_0,\ 1_1,\ 2_1)\ \ (0_1,\ 2_0,\ 3_1)\ \ (0_0,\ 2_1,\ 4_1)\ \ \mathrm{mod}(5,Z_2),$$

which has incidence matrix:

$$\begin{array}{ccccc@{\quad}ccccc@{\quad}ccccc@{\quad}ccccc}
0 & 0 & 0 & . & . & 0 & 1 & 1 & . & . & 0 & . & 0 & 1 & . & 0 & . & 1 & . & 1 \\
. & 0 & 0 & 0 & . & . & 0 & 1 & 1 & . & . & 0 & . & 0 & 1 & 1 & 0 & . & 1 & . \\
. & . & 0 & 0 & 0 & . & . & 0 & 1 & 1 & 1 & . & 0 & . & 0 & . & 1 & 0 & . & 1 \\
0 & . & . & 0 & 0 & 1 & . & . & 0 & 1 & 0 & 1 & . & 0 & . & 1 & . & 1 & 0 & . \\
0 & 0 & . & . & 0 & 1 & 1 & . & . & 0 & . & 0 & 1 & . & 0 & . & 1 & . & 1 & 0
\end{array}$$

Example 5 David Glynn [12] has found the only $GW(v,k,G)$ known to the author where G is not an abelian group. Consider the multiplication table for S_3

	1	2	3	4	5	6	
1	1	2	3	4	5	6	$1 \leftrightarrow e$
2	2	3	1	5	6	4	$2 \leftrightarrow (123)(456)$
3	3	1	2	6	4	5	$3 \leftrightarrow (132)(465)$
4	4	6	5	1	3	2	$4 \leftrightarrow (14)(26)(35)$
5	5	4	6	2	1	3	$5 \leftrightarrow (15)(24)(36)$
6	6	5	4	3	2	1	$6 \leftrightarrow (16)(25)(34)$

Then the circulant matrix

$$\begin{bmatrix}
0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 \\
0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 \\
4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 \\
0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 \\
6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 \\
5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 \\
6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 \\
1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 \\
1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 \\
0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 \\
4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 \\
1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 \\
5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0
\end{bmatrix}$$

is a generalized weighing matrix $GW(13,9;S_3)$.

In set notation this can be written as

$$(1_5\ 2_1\ 3_4\ 6_1\ 7_1\ 8_6\ 9_5\ 10_6\ 12_4)\ \ \mathrm{mod}\ (13,S_3)$$

There are two inequivalent circulant $GW(13,9;Z_2)$ (see Seberry and Wehrhahn [23]) and a total of eight inequivalent $GW(13,9;Z_2)$ (Ohmori [15]) but the existence of $GW(13,9;G)$ for $G=Z_3$ or Z_6 is not yet resolved.

Another interesting possibility occurs if there are parallel classes associated with a subdesign. For example the $GBRD(13,9,6;S_3)$ given by developing the above block mod (13) can be embedded in an $SBIBD(91,10,1)$ which also has an $SBIBD(13,4,1)$ embedded (see the construction in [19]). $\square$

In this paper we show that $BIBD(v,b,r,k,\lambda)$, where $v = pq$ or $pq+1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p,b',r',k,\lambda;G)$ where G is a group of order q.

Theorem 1 *Let G be a group. A $GBRD(v,k,\lambda;G)$, where $\mid G \mid$ is λ, is equivalent to a $BIBD(v\lambda,k,1)$ when both are written using starting blocks developed* $\mathrm{mod}(\lambda)$. *(The most common case has G a cyclic group but it is possible to develop a theory with other groups.)*

Proof. Suppose we have initial blocks

$$(x_{a_i}^{(i)}\ y_{b_i}^{(i)}\ \cdots\ z_{c_i}^{(i)})$$

for the $GBRD$. Then the equivalent starting blocks for the $BIBD$ are:

$$(x_{a_i}^{(i)},\ \cdots\ z_{c_i}^{(i)})\ (x_{a_i+1}^{(i)},\ \cdots\ z_{c_i+1}^{(i)})\ (x_{a_i+k-1}^{(i)},\ \cdots\ z_{c_i+k-1}^{(i)})\ (0_0,\ 0_1,\ \cdots\ 0_{k-1})\ \ \mathrm{mod}(v,\lambda).$$

The differences of the type $(x_{a_i}^{(i)} - y_{a_i}^{(i)})$, $a_i = 0,\ \cdots,\ \lambda$ yield the product elements of type $(x^{(i)} - y^{(i)})_{a_i a_i^{-1}})$ which must occur once each for the $GBRD$ and once each arising from each a_i for the $BIBD$.

The differences of type $(x_{a_i}^{(i)} - y_{b_i}^{(i)})$, $a_i = 0,\ \cdots,\ \lambda$, $a_i \neq b_i$, yield the product elements of type $(x^{(i)} - y^{(i)})_{a_i b_i^{-1}}$ which must occur equally often for the $GBRD$ and hence occur equally often for the $BIBD$. The block with zeros completes the differences for the $BIBD$.

□

See the case $k = 3$ in the next section for an example of how this theorem works.

When we have a $BIBD((k-1)v+1,k,k-1)$ developed mod $(v,k-1)$ with a special element ∞ we seek to form a $GBRD(v,k,(k-1)^2;G)$ where G is a group of order $k-1$. Again the converse, that the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [19]).

Theorem 2 *Let G be a group. If the initial blocks*

$$(x_{a_1}^{(1)},\ x_{a_2}^{(2)},\ \ldots,\ x_{a_k}^{(k)})\ \ (y_{b_1}^{(1)},\ y_{b_2}^{(2)},\ \ldots,\ y_{b_k}^{(k)})\ldots(z_{c_1}^{(1)},\ z_{c_2}^{(2)},\ \ldots,\ z_{c_k}^{(k)})$$

give a $GBRD(v,k,k-1;G)$ where $\mid G \mid = k-1$, then the initial blocks

$$(x_{a_1+i}^{(1)},\ x_{a_2+i}^{(2)},\ldots,x_{a_k+i}^{(k)})\ (y_{b_1+i}^{(1)},\ y_{b_2+i}^{(2)},\ldots y_{b_k+i}^{(k)})\ldots(z_{c_1+i}^{(1)},\ z_{c_2+i}^{(2)},\ldots,z_{c_k+i}^{(k)})\ (\infty,\ 0_1,\ 0_2,\ \ldots,\ 0_{k-1})$$

$i = 0,\ \cdots,\ k-2$, *give a $BIBD((k-1)v+1,k,1)$, where $i = 0,1,\ldots,k-1$. The converse is also true. (The most common case has G a cyclic group but it is possible to develop a theory with other groups.)*

Proof. The proof follows by noticing that the mixed non-zero differences occur the same number of times. The extra block makes sure the zero elements occur the appropriate number of times. □

See the case $k = 4$ in the next section for an example of how this theorem works.

Comment. In some sense Bose's theorem implies that the subscripts are from a cyclic group. However the formulation in terms of $GBRD$ indicates clearly that any group will suffice.

The construction with non-cyclic groups can be visualized by replacing the 0 element of the $GBRD$ by a $(k-1)\times(k-1)$ zero matrix and the other elements by their right (left) regular matrix representation. Finally, the extra $k-1$ blocks corresponding to $(\infty\ 0_1\ 0_2\ldots 0_{k-1})$ are added.

2 Examples and Constructions

We now give some examples of these methods. The $GBRD$ are new.

k=3: To illustrate

$$(0_1, 1_0, 4_0), (0_1, 2_0, 3_0) \bmod(5, Z_3)$$

give a $GBRD(5, 3, 3; Z_3)$. The blocks

$$\begin{array}{llll} (0_1, 1_0, 4_0) & (0_1, 2_0, 3_0) & (0_0, 0_2, 0_3) & \bmod (5, 3) \\ (0_2, 1_1, 4_1) & (0_2, 2_1, 3_1) & & \\ (0_0, 1_2, 4_2) & (0_0, 2_2, 3_2) & & \end{array}$$

give a $BIBD(15, 3, 1)$. That the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [19]).

In Lemma 3 this method is applied to $BIBD(3v, 3, 1)$ and $GBRD(v, 3, 3; Z_3)$ for all odd v.

k=4: The initial blocks for the $BIBD(28, 4, 1)$, $i = 0,\ 1,\ 2$, are

$$[(21)_i, (01)_i, (12)_{i+1}, (10)_{i+1}],\ \ [(20)_i, (02)_i, (22)_{i+1}, (00)_{i+1}] \ \text{ and } \ [\infty, (00)_0, (00)_1, (00)_2].$$

The first two blocks, with $i = 0$, give a $GBRD(9, 4, 3; Z_3)$. The elements work modulo (3,3) and suffixes modulo 3.

k=5: The initial blocks of a $BIBD(45, 5, 1)$ are

$$\begin{array}{rl} & [(01)_0, (02)_0, (10)_2, (20)_2, (00)_1], \\ & [(21)_0, (12)_0, (22)_2, (11)_2, (00)_1] \\ \text{and} & [(00)_0, (00)_1, (00)_2, (00)_3, (00)_4]. \end{array}$$

The first two blocks give a $GBRD(9, 5, 5; Z_5)$ which is new. The elements work modulo (3,3) and suffixes modulo 5.

k=6: Denniston [10] has given a $BIBD(66, 6, 1)$ design. This design is rewritten in a convenient way as follows:

$$\begin{array}{ll} (0_3,\ 2_3,\ 3_0,\ 5_2,\ 6_2,\ 10_0), & (0_3,\ 1_4,\ 4_0,\ 6_0,\ 7_1,\ 11_2), \\ (0_4,\ 1_4,\ 3_2,\ 9_2,\ 11_1,\ 12_2), & (0_3,\ 3_4,\ 5_1,\ 7_0,\ 8_2,\ 12_1), \\ (0_4,\ 1_0,\ 3_4,\ 7_2,\ 9_0,\ 10_0), & (2_3,\ 4_4,\ 5_3,\ 6_3,\ 10_4,\ 12_4), \\ (0_4,\ 1_1,\ 3_1,\ 4_1,\ 8_0,\ 9_4), & (0_3,\ 2_0,\ 4_1,\ 5_0,\ 6_3,\ 9_1), \\ (0_3,\ 1_2,\ 2_1,\ 5_3,\ 6_1,\ 12_2), & (0_3,\ 2_2,\ 8_1,\ 9_4,\ 10_2,\ 11_0), \\ (\infty,\ 0_0,\ 0_1,\ 0_2,\ 0_3,\ 0_4), & \end{array}$$

all blocks developed $\bmod(13, 5)$. We note that the first 10 of these blocks also give a $GBRD(13, 6, 25; Z_5)$. If we knew how to sign the blocks $\{2\ 5\ 6\ 7\ 8\ 11\}$ and $\{1\ 3\ 4\ 9\ 10\ 12\}$, which develop mod(13) to give a $BIBD(13, 6, 5)$, we would have a $GBRD(13, 6, 5; Z_5)$.

k=7: The following three sets of initial blocks from Bhat-Nayak and Kane [1, 2] give initial blocks for a $BIBD(91, 7, 1)$:

Solution 1:

$$\begin{array}{rl} & [6_0, 7_0, 11_1, 2_1, 8_5, 5_5, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

Solution 2:

$$\begin{array}{rl} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_3, 3_3, 9_4, 4_4, 12_6, 1_6, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

Solution 3:

$$\begin{array}{rl} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

In each case the first two blocks mod $(13, 7)$ are initial blocks for a $GBRD(13, 7, 7; Z_7)$. These are new.

One typical result illustrates many possible theorems arising from constructions for $BIBD$ using Bose's method of differences.

Lemma 3 *The starting blocks*

$$(1_0,\ 2t_0,\ 0_1)\ \ (2_0,\ (2t-1)_0,\ 0_1)\ \ \ldots\ \ (t_0,\ (t+1)_0,\ 0_1)\ \ (0_0,\ 0_1,\ 0_2)\ \ \mathrm{mod}\ (2t+1, 3)$$

were used by Bose [4, p373, equation 4.22] *to form a* $BIBD(3(2t+1), 3, 1)$*. The starting blocks*

$$(1_0,\ 2t_0,\ 0_1)\ \ (2_0,\ (2t-1)_0,\ 0_1)\ \ \ldots\ \ (t_0,\ (t+1)_0,\ 0_1)\ \ \mathrm{mod}\ (2t+1, Z_3)$$

form a $GBRD(2t+1, 3, 3; Z_3)$.

Proof. A simple check shows all the differences for the $GBRD$, $\{1,\ \cdots,\ 2t\}$, are obtained with each subscript 0, 1, 2. This immediately gives the $GBRD(2t+1, 3, 3; Z_3)$.

It remains to notice that the starting blocks for the $BIBD$ are the same as for the $GBRD$ with the exception of the last. This means we have the required number of pairs both with the subscripts i and i, $i = 0,\ 1,\ 2$ and with cross subscripts i and j, $i \neq j$, $i,\ j = 0,\ 1,\ 2$. This gives the result after noting the block $(0_0,\ 0_1,\ 0_2)$ guarantees the remaining differences which are needed.

Acknowledgement The author wishes to thank the referee for carefully reading the paper and improving the exposition.

References

[1] V. N. Bhat-Nayak and V.D. Kane, A technique for obtaining non-isomorphic balanced in complete block designs, *Utilitas Mathematica*, **26,** (1984), 235-247.

[2] V. N. Bhat-Nayak and V.D. Kane, Construction and irreducibility of quasi-multiple designs, *Ars Combinatoria*, **18,** (1984), 113-129.

[3] R. C. Bose, On the construction of balanced incomplete block-designs, *Annals of Eugenics*, **9,** (1939), 358-399.

[4] R. C. Bose, A note on the resolvability of balanced incomplete block designs, *Sankyā*, **6,** (1942), 105-110.

[5] R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.*, 13(1963), 389-418.

[6] C. Colbourn, Invited Lecture, *20th Conference on Combinatorial Mathematics and Combinatorial Computing*, Auckland, December, 1994.

[7] W. de Launey, D.G. Sarvate and Jennifer Seberry, Generalized Bhaskar Rao designs of block size three over the group Z_4, *Ars Combinatoria*, **19A,** (1985), 273-286.

[8] Warwick de Launey and Jennifer Seberry, On Bhaskar Rao designs of block size four, *Combinatorics and Applications*, Proceedings of the Seminar on Combinatorics and its Applications in honour of Professor S.S.Shrikhande, ed K.S.Vijayan and N.M.Singhi, Indian Statistical Institute, Calcutta, (1984) 311-316.

[9] Warwick de Launey and Jennifer Seberry, Generalized Bhaskar Rao designs with block size four, *Congressus Numerantium*, **41,** (1984), 229-294.

[10] R. F. H. Denniston, A Steiner system with a maximal arc, *Ars Combinatoria*, **Vol. 9,** (1980), 247-248.

[11] J. Doyen, Constructions of disjoint Steiner triple systems, *Proceedings of the A.M.S.*, **32,** (1972), 409-416.

[12] D. Glynn, *Finite Projective Planes and Related Combinatorial Systems*, PhD. Thesis, University of Adelaide, Adelaide, 1978.

[13] M. Hall Jr, *Combinatorial Theory*, Blaisdell Publishing Company, Waltham, Mass. 1967.

[14] Clement Lam and Jennifer Seberry, Generalized Bhaskar Rao designs, *J.Statistical Planning and Inference*, **10,** (1984), 83-95.

[15] H. Ohmori, Classification of weighing matrices of order 13 and weight 9, *Discrete Math.,* **116,** (1993), pp. 55-78.

[16] William Palmer, Generalized Bhaskar Rao designs with two association classes, *Australas. J. Combinatorics,* **1,** (1990), pp. 161-180.

[17] C. R. Rao, A study of BIB designs with replications 11 to 15, *Sankhyā Ser A*, **23,** (1961), 117-127.

[18] C. R. Rao and P. N. Dandawate (Private Communication).

[19] Jennifer Seberry, Some remarks on generalized Hadamard matrices and theorems of Rajkundlia on $SBIBD$s, *Combinatorial Mathematics VI*, Lecture Notes in Mathematics, Vol. **748,** Springer-Verlag, Berlin- Heidelberg-New York, (1979), 378-386.

[20] Jennifer Seberry, Some families of partially balanced incomplete block designs, eds E. J. Billington, S. Oates-Williams and A. P. Street, *Combinatorial Mathematics IX*, Lecture Notes in Mathematics, Vol. **952,** Springer-Verlag, Berlin- Heidelberg-New York, (1982), 154-164.

[21] Jennifer Seberry, Regular group divisible designs and Bhaskar Rao designs with block size three, *J. Statistical Planning and Inference*, **10,** (1984), 69-82.

[22] Jennifer Seberry, Generalized Bhaskar Rao designs with block size three, *J.Statistical Planning and Inference*, **11,** (1985), 373-380.

[23] Jennifer Seberry and K. Wehrhahn, A class of codes generated by circulant weighing matrices , *Combinatorial Mathematics*: Proceedings of the international Conference, Canberra, August, 1977, Vol. **686**, in Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, (1978), 282-289.

[24] Jennifer Wallis, Part IV of W. D. Wallis, A. P. Street and J. S. Wallis, *Combinatorics: Room Squares, Sum free Sets, Hadamard Matrices*, Vol. **292**, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin- Heidelberg-New York, 1972, 273-490.