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Call four type 1(1,-1) matrices, x_1, x_2, x_3, x_4 ; of the same group of order m (odd) with the properties (i) $(X_i - I)^T = -(X_i - I)$, $i=1,2,3$, (ii) $X_4^T = X_4$ and the diagonal elements are positive, (iii) $X_i X_j = X_j X_i$ and (iv) $X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4mI_m$, best matrices. We use a computer to give, for the first time, all inequivalent best matrices of odd order $m \leq 31$. Inequivalent best matrices of order m , m odd, can be used to find inequivalent skew-Hadamard matrices of order $4m$. We use best matrices of order $1/4(s^2+3)$ to construct new orthogonal designs, including new $OD(2s^2+6; 1, 1, 2, 2, s^2, s^2)$.

Keywords

Circulant matrices, supplementary difference sets, orthogonal designs, Hadamard matrices, AMS Subject Classification: Primary 05B20, Secondary 05B30

Disciplines

Physical Sciences and Mathematics

Publication Details

This article was originally published as Georgiou, S, Koukouvinos, C and Seberry, J, On circulant best matrices and their applications, *Linear and Multilinear Algebra*, 48, 2001, 263-274. Copyright Taylor & Francis. Original journal available [here](#).

On circulant best matrices and their applications

S. Georgiou*, C. Koukouvinos* and Jennifer Seberry†

Abstract

Call four type 1 $(1, -1)$ matrices, X_1, X_2, X_3, X_4 , of the same group of order m (odd) with the properties (i) $(X_i - I)^T = -(X_i - I)$, $i = 1, 2, 3$, (ii) $X_4^T = X_4$ and the diagonal elements are positive, (iii) $X_i X_j = X_j X_i$ and (iv) $X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4m I_m$, *best matrices*. We use a computer to give, for the first time, all inequivalent best matrices of odd order $m \leq 31$. Inequivalent best matrices of order m , m odd, can be used to find inequivalent skew-Hadamard matrices of order $4m$. We use best matrices of order $\frac{1}{4}(s^2 + 3)$ to construct new orthogonal designs, including new $OD(2s^2 + 6; 1, 1, 2, 2, s^2, s^2)$.
AMS Subject Classification: Primary 05B20, Secondary 05B30

Key words and phrases: Circulant matrices, supplementary difference sets, orthogonal designs, Hadamard matrices.

1 Introduction and basic definitions

A $(1, -1)$ matrix of order n is called a Hadamard matrix if $HH^T = H^T H = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n . A $(1, -1)$ matrix A of order n is said to be of skew type if $A - I_n$ is skew-symmetric. If A is a skew type Hadamard matrix then A is said to be a skew-Hadamard matrix. Two $(1, -1)$ matrices A, B of order n are said to be amicable if $AB^T = BA^T$.

Let G be an additive abelian group of order n with elements g_1, g_2, \dots, g_n and X a subset of G . Define the type 1 $(1, -1)$ incidence matrix $M = (m_{ij})$ of order n of X to be

$$m_{ij} = \begin{cases} +1 & \text{if } g_j - g_i \in X \\ -1 & \text{otherwise} \end{cases}$$

and the type 2 $(1, -1)$ incidence matrix $N = (n_{ij})$ of order n of X to be

$$n_{ij} = \begin{cases} +1 & \text{if } g_j + g_i \in X \\ -1 & \text{otherwise} \end{cases}$$

In particular, if G is cyclic the matrices M and N are called circulant and back circulant respectively. In this case $m_{ij} = m_{1, j-i+1}$ and $n_{ij} = n_{1, i+j-1}$ respectively (indices should be reduced modulo n).

Definition 1 Let X_1, X_2, X_3, X_4 be four type 1 $(1, -1)$ matrices on the same group of order m (odd) with the properties

- (i) $(X_i - I)^T = -(X_i - I)$, $i = 1, 2, 3$
- (ii) $X_4^T = X_4$ and the diagonal elements are positive

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$$(iii) \quad X_i X_j = X_j X_i$$

$$(iv) \quad X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4m I_m$$

Call such matrices *best matrices* of order m .

Pre and post multiplying equation (iv) by e and e^T , respectively, where e is the $1 \times m$ matrix of all ones gives that $4m - 3 = s^2$ where s is odd integer.

In this paper we consider circulant best matrices, so condition (iii) is trivially satisfied. Hence, multiplying on the left by e^T (the $1 \times m$ vector of one's) and on the right by e both sides of (iv) we conclude that circulant (or type 1) best matrices can only exist for orders m of which $4m = 1^2 + 1^2 + 1^2 + a^2$, where a is the sum of the elements of the first row of the symmetric matrix X_4 and a is an odd integer.

An *orthogonal design* of order n and type (s_1, s_2, \dots, s_u) ($s_i > 0$), denoted $OD(n; s_1, s_2, \dots, s_u)$, on the commuting variables x_1, x_2, \dots, x_u , is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$ such that

$$AA^T = \left(\sum_{i=1}^u s_i x_i^2 \right) I_n$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [1], where this was first defined, it was mentioned that

$$A^T A = \left(\sum_{i=1}^u s_i x_i^2 \right) I_n$$

and so our alternative description of A applies equally well to the columns of A . It was also shown in [1] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$. For more details and constructions of orthogonal designs the reader can consult the book of Geramita and Seberry [2].

In section 2 we describe briefly the method of construction, in section 3 we give all inequivalent circulant best matrices of odd order $m \leq 31$, and in section 4 we use best matrices to construct some new orthogonal designs and families of Hadamard matrices.

2 Method of construction

In order to describe our construction for best matrices, we need a few more definitions. Let n be a positive integer.

Definition 2 Four subsets S_0, S_1, S_2, S_3 of $\{1, 2, \dots, n-1\}$ are called $4-(n; n_0, n_1, n_2, n_3; \lambda)$ *supplementary difference sets (sds)* modulo n if $|S_k| = n_k$ for $k = 0, 1, 2, 3$ and for each $m \in \{1, 2, \dots, n-1\}$ we have $\lambda_0(m) + \dots + \lambda_3(m) = \lambda$, where $\lambda_k(m)$ is the number of solutions (i, j) of the congruence $i - j \equiv m \pmod{n}$ with $i, j \in S_k$.

Suppose that S_k are $4-(n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo n having the following additional properties:

$$n + \lambda = n_0 + n_1 + n_2 + n_3 \tag{1}$$

$$i \in S_k \iff n - i \notin S_k, \quad k = 0, 1, 2 \tag{2}$$

$$i \in S_t \iff n - i \in S_t, \quad t = 3 \tag{3}$$

where in (2) and (3) it is assumed that $i \in \{1, 2, \dots, n-1\}$.

Let $a_k = (a_{k0}, a_{k1}, \dots, a_{k_{n-1}})$, $k = 0, 1, 2, 3$, be the row vector defined by

$$a_{k_i} = \begin{cases} -1 & \text{if } i \in S_k \\ 1 & \text{otherwise} \end{cases}$$

Furthermore let A_k , $k = 0, 1, 2, 3$ be the circulant matrices with first row a_k . Then it can be easily verified that A_0, A_1, A_2, A_3 are four matrices of order n as described in definition 1.

Let r be an integer relatively prime to n , and set

$$S'_k = \{r_i \pmod{n} : i \in S_k\} \subset \{1, 2, \dots, n-1\}$$

for $k = 0, 1, 2, 3$. These sets are also $4 - (n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo n satisfying the conditions (1), (2), (3). We shall say that such quadruples S_0, S_1, S_2, S_3 and S'_0, S'_1, S'_2, S'_3 are equivalent.

We now give a brief description of the method of computation used to find the necessary sds's. The numbers n_i are easy to determine (see [6]). We first generate a number of subsets of size n_i of $\{1, 2, \dots, n\}$ having the required symmetry properties (2) or (3), and at the same time compute the corresponding set of differences. We store the multiplicities of these differences in a file, say f_i , saving only sets of differences with different multiplicities. After creating these files for each of the sizes n_0, \dots, n_3 , we try to match the items in the four files to produce an sds. This is done by examining items in two files only, say f_0 and f_1 and creating a new file in which we record the pairs which produce different total multiplicities of the differences. The procedure is repeated with the remaining two files f_2 and f_3 . Finally the resulting two files are examined in order to find a perfect match.

The results that we found applying this algorithm are presented in the next section.

3 The inequivalent supplementary difference sets

In this section we give for the first time all inequivalent supplementary difference sets which satisfy the condition (1), (2), (3) for all odd $m \leq 31$.

Table 1.

All inequivalent circulant best matrices of odd order $m \leq 31$

$m = 3$; $4 - (3; 1, 1, 1, 0; 0)$

$$S_0 = \{1\}, \quad S_1 = \{1\}, \quad S_2 = \{1\}, \quad S_3 = \emptyset$$

$m = 7$; $4 - (7; 3, 3, 3, 6; 8)$

$$S_0 = \{1, 3, 5\}, \quad S_1 = \{1, 2, 3\}, \quad S_2 = \{1, 4, 5\}, \quad S_3 = \{1, 2, 3, 4, 5, 6\}$$

$m = 13$; $4 - (13; 6, 6, 6, 10; 15)$

$$1. \quad \begin{aligned} S_0 &= \{1, 3, 5, 6, 9, 11\}, \quad S_1 = \{2, 6, 8, 9, 10, 12\}, \\ S_2 &= \{1, 4, 5, 6, 10, 11\}, \quad S_3 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \end{aligned}$$

2. $S_0 = \{2, 6, 89, 10, 12\}$, $S_1 = \{1, 4, 5, 6, 10, 11\}$,
 $S_2 = \{1, 2, 3, 4, 6, 8\}$, $S_3 = \{1, 2, 3, 4, 6, 7, 9, 10, 11, 12\}$

Table 1. (continued)

$\mathbf{m} = 21$; $4 - (21; 10, 10, 10, 6; 15)$

1. $S_0 = \{1, 3, 6, 10, 12, 13, 14, 16, 17, 19\}$, $S_1 = \{1, 2, 3, 4, 7, 8, 10, 12, 15, 16\}$,
 $S_2 = \{1, 2, 5, 11, 12, 13, 14, 15, 17, 18\}$, $S_3 = \{4, 6, 10, 11, 15, 17\}$,
2. $S_0 = \{1, 3, 4, 5, 6, 8, 10, 12, 14, 19\}$, $S_1 = \{1, 2, 3, 4, 7, 9, 10, 13, 15, 16\}$,
 $S_2 = \{3, 4, 6, 7, 8, 9, 11, 16, 19, 20\}$, $S_3 = \{2, 8, 9, 12, 13, 19\}$,
3. $S_0 = \{1, 3, 4, 6, 10, 12, 13, 14, 16, 19\}$, $S_1 = \{2, 3, 4, 7, 8, 9, 10, 15, 16, 20\}$,
 $S_2 = \{1, 2, 3, 4, 5, 7, 8, 11, 12, 15\}$, $S_3 = \{1, 3, 8, 13, 18, 20\}$,
4. $S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}$, $S_1 = \{4, 5, 10, 12, 13, 14, 15, 18, 19, 20\}$,
 $S_2 = \{1, 2, 3, 4, 5, 6, 9, 11, 13, 14\}$, $S_3 = \{1, 5, 9, 12, 16, 20\}$,
5. $S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}$, $S_1 = \{1, 2, 3, 4, 8, 11, 12, 14, 15, 16\}$,
 $S_2 = \{1, 6, 11, 12, 13, 14, 16, 17, 18, 19\}$, $S_3 = \{2, 6, 8, 13, 15, 19\}$,
6. $S_0 = \{1, 3, 7, 10, 12, 13, 15, 16, 17, 19\}$, $S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}$,
 $S_2 = \{1, 3, 4, 5, 6, 11, 12, 13, 14, 19\}$, $S_3 = \{1, 5, 9, 12, 16, 20\}$,
7. $S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}$, $S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}$,
 $S_2 = \{1, 7, 8, 9, 11, 15, 16, 17, 18, 19\}$, $S_3 = \{2, 4, 9, 12, 17, 19\}$,

$\mathbf{m} = 31$; $4 - (31; 15, 15, 15, 10; 24)$

1. $S_0 = \{1, 5, 6, 8, 10, 13, 14, 16, 19, 20, 22, 24, 27, 28, 29\}$,
 $S_1 = \{2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 16, 20, 22, 23, 30\}$,
 $S_2 = \{1, 3, 4, 5, 12, 16, 17, 18, 20, 21, 22, 23, 24, 25, 29\}$,
 $S_3 = \{3, 6, 8, 12, 13, 18, 19, 23, 25, 28\}$,
2. $S_0 = \{1, 4, 7, 10, 12, 14, 15, 18, 20, 22, 23, 25, 26, 28, 29\}$,
 $S_1 = \{1, 2, 6, 12, 13, 14, 16, 20, 21, 22, 23, 24, 26, 27, 28\}$,
 $S_2 = \{5, 6, 7, 9, 10, 11, 12, 14, 15, 18, 23, 27, 28, 29, 30\}$,
 $S_3 = \{2, 3, 10, 12, 14, 17, 19, 21, 28, 29\}$,

4 Constructions using best matrices

Theorem 1 *Suppose there exist best matrices of order $t = \frac{1}{4}(s^2 + 3)$, then there exists an $OD(8t; 1, 1, 2, 2, s^2, s^2)$.*

Proof. Suppose $I + X, I + X_2, I + X_3$ and X_4 are the circulant best matrices of order t . Let a, b, c, d, e, f be commuting variables. Define

$$X = \frac{1}{2}(X_2 + X_3); Y = \frac{1}{2}(X_2 - X_3)$$

So

$$X^T = -X, Y^T = -Y, XX^T + YY^T = \frac{1}{2}(X_2X_2^T + X_3X_3^T).$$

Now define

$$\begin{aligned}
A_1 &= aI + bX_1 \\
A_2 &= dX_4 \\
A_3 &= cI - dX_1 \\
A_4 &= bX_4 \\
A_5 &= eI + bX + dY \\
A_6 &= fI - dX + bY \\
A_7 &= eI - bX - dY \\
A_8 &= fI + dX - bY
\end{aligned}$$

It is straightforward to check, using the properties of best matrices, that A_1, A_2, \dots, A_8 satisfy the additive property and

$$\sum_{i=1}^8 A_i A_i^T = (a^2 + c^2 + 2e^2 + 2f^2 + (4t-3)b^2 + (4t-3)d^2)I_t.$$

We now check if the matrices form an amicable set. First we see

$$\begin{aligned}
A_1 A_2^T &= adX_4 + bdX_1X_4 \\
A_2 A_1^T &= adX_4 - bdX_1X_4 \\
A_3 A_4^T &= cbX_4 - bdX_1X_4 \\
A_4 A_3^T &= cbX_4 + bdX_1X_4
\end{aligned}$$

So

$$A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T = 0.$$

Then we have

$$\begin{aligned}
A_5 A_6^T &= efI + bfX + dfY + edX + bdX^2 + d^2XY - ebY - b^2XY - dbY^2 \\
A_6 A_5^T &= efI - bfX - dfY - edX + bdX^2 + d^2XY + ebY - b^2XY - dbY^2 \\
A_7 A_8^T &= efI - bfX - dfY - edX + bdX^2 + d^2XY + ebY - b^2XY - dbY^2 \\
A_8 A_7^T &= efI + bfX + dfY + edX + bdX^2 + d^2XY - ebY - b^2XY - dbY^2
\end{aligned}$$

So

$$A_5 A_6^T - A_6 A_5^T + A_7 A_8^T - A_8 A_7^T = 0.$$

Hence $A_1 \dots A_8$ are amicable set of circulant matrices satisfying the additive property. Hence we may use them in Kharaghani array [3] to form $OD(8t; 1, 1, 2, 2, 4t-3, 4t-3)$. \square

Remark 1 We note there is no construction known which gives $OD(8t; 1, 1, 2, 2, 4t-3, 4t-3)$.

Hence we have $OD(56; 1, 1, 2, 2, 25, 25)$, $OD(104; 1, 1, 2, 2, 49, 49)$, $OD(168; 1, 1, 2, 2, 81, 81)$ and $OD(248; 1, 1, 2, 2, 121, 121)$ for the first time.

Theorem 2 Suppose there are best matrices of order m then there exists an $OD(4m; 1, 1, 1, 4m-3)$.

Proof. Let x_1, x_2, x_3 and x_4 be four commuting variables. Write $I + B_1, I + B_2, I + B_3$ and B_4 for the best matrices of order m . Further write $A_1 = x_1 I + x_4 B_1, A_2 = x_2 I + x_4 B_2, A_3 = x_3 I + x_4 B_3$ and $A_4 = x_4 B_4$ for the four circulant (or type 1) matrices of order m satisfying

$$A_1 A_1^T + A_2 A_2^T + A_3 A_3^T + A_4 A_4^T = (x_1^2 + x_2^2 + x_3^2 + (4m - 3)x_4^2)I_m.$$

Let $R = r_{ij}$, where $r_{ij} = 1$ for $i + j = m + 1$ and 0 otherwise. Then using the Goethals-Seidel array

$$\begin{bmatrix} A_1 & A_2 R & A_3 R & A_4 R \\ -A_2 R & A_1 & A_4^T R & -A_3^T R \\ -A_3 R & -A_4^T R & A_1 & A_2^T R \\ -A_4 R & A_3^T R & -A_2^T R & A_1 \end{bmatrix},$$

is the required $OD(4m; 1, 1, 1, 4m - 3)$. \square

Corollary 2 *Let m be the order of best matrices. Then an $OD(4m; 1, 1, 1, 4m - 3)$ exists.*

Corollary 3 *Let $m \in \{3, 7, 13, 21, 31\}$. Then an $OD(4m; 1, 1, 1, 4m - 3)$ exists.*

Corollary 4 *Let m be the order of best matrices. Then there exist up to 8 inequivalent skew-Hadamard, and Hadamard, matrices of order $4m$.*

Proof. Let X_1, X_2, X_3, X_4 be best matrices of order m . Then choosing $A_1 = X_1, A_2 = I \pm (X_2 - I), A_3 = I \pm (X_3 - I)$ and $A_4 = \pm X_4$, in the Goethals-Seidel array gives the required result, (Note choosing $A_2 = \pm I + (X_2 - I)$, and $A_3 = \pm I + (X_3 - I)$ is an alternative choice.) \square

We have constructed the Hadamard matrices of order 28 made, using as A_1, A_2, A_3 and A_4 , the first rows given below in the Goethals-Seidel array

$$\begin{array}{cccc} 1 & 1 & 1-1 & 1-1-1; & 1 & 1 & 1-1 & 1-1-1; & 1 & 1 & 1-1 & 1-1-1; & 1-1-1-1-1-1-1-1 \\ 1 & 1 & 1-1 & 1-1-1; & 1 & 1 & 1-1 & 1-1-1; & -1 & 1 & 1-1 & 1-1-1; & 1-1-1-1-1-1-1-1 \\ 1 & 1 & 1-1 & 1-1-1; & -1 & 1 & 1-1 & 1-1-1; & 1 & 1 & 1-1 & 1-1-1; & 1-1-1-1-1-1-1-1 \\ 1 & 1 & 1-1 & 1-1-1; & -1 & 1 & 1-1 & 1-1-1; & -1 & 1 & 1-1 & 1-1-1; & 1-1-1-1-1-1-1-1 \end{array}$$

We believe that the four Hadamard matrices thus produced are H-inequivalent and inequivalent skew-Hadamard matrices.

Corollary 5 *Suppose there are best matrices of order m and an Hadamard matrix, H , of order $4m/3$, then there is an Hadamard matrix of order $4m(4m - 3)/3$.*

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$.

Write J for the $4m/3 - 1 \times 4m/3 - 1$ matrix of all ones. Normalize the Hadamard matrix, H , of order $4m/3$ so that its first row and column is all ones, then discard the first row and column to obtain the core of the Hadamard matrix, B , of order $4m/3 - 1$, which satisfies $BJ = -J$ and $BB^T = 4m/3 I_{4m/3-1} - J_{4m/3-1}$. Then replacing the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by J, J, J and B , which satisfy

$$3JJ^T + (4m - 3)BB^T = (4m - 3)J + 4m(4m - 3)/3I - (4m - 3)J = 4m(4m - 3)/3I,$$

gives the required matrix. \square

Example 1 We have found best matrices of orders $m = 3$ and 21 . These give Hadamard matrices of orders 36 and 2268 . These orders are not new, but, since Kimura [4, 5] has found some 487 inequivalent Hadamard matrices of order 28 which can be used in the corollary for $m = 21$ we may have constructed new, inequivalent, Hadamard matrices of order 2268 . Since the variables can also be replaced by $J, \pm J, \pm J$ and $\pm B$ there is further potential for inequivalent Hadamard matrices. \square

Corollary 6 *Suppose there are best matrices of order m and a symmetric Hadamard matrix of order h*

1. $h = 4(m + 1)/3$;
2. $h = 4(m + 2)/3$;
3. $h = 4(m + 3)/3$,

then there is an Hadamard matrix of order $4m(h - 1)$.

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$.

Normalize the symmetric Hadamard matrix of order h so that its first row and column is all ones, then discard the first row and column to obtain the symmetric core of the symmetric Hadamard matrix, B , which satisfies $BJ = -J$ and $BB^T = hI_{h-1} - J_{h-1}$. Write $K = J - 2I$. Then

$$KJ^T = JK^T; \quad KB^T = BK^T; \quad JB^T = BJ^T.$$

Then replacing the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by

1. J, J, K and B ;
2. J, K, K and B ;
3. K, K, K and B

which satisfy

$$2JJ^T + KK^T + (4m - 3)BB^T = 2(h - 1)J + (h - 5)J + 4I + h(4m - 3)I - (4m - 3)J = 4m(h - 1)I;$$

$$JJ^T + 2KK^T + (4m - 3)BB^T = (h - 1)J + 2(h - 5)J + 8I + h(4m - 3)I - (4m - 3)J = 4m(h - 1)I;$$

$$3KK^T + (4m - 3)BB^T = 3(h - 5)J + 12I + h(4m - 3)I - (4m - 3)J = 4m(h - 1)I,$$

respectively giving the required matrices. \square

Example 2 From above we have four sequences of lengths $m = 3, 7, 13, 21$ and 31 which are the first rows for best matrices. Then using Corollary 5 and the best matrices of orders 3 and 21 we obtain Hadamard matrices of order 36 and 2268 . Using Corollary 6 we obtain Hadamard matrices of orders $84 = 4 \cdot 21$, $308 = 4 \cdot 77$, $988 = 4 \cdot 247 = 4 \cdot 13 \cdot 19$, $2604 = 4 \cdot 851 = 4 \cdot 21 \cdot 31$ and $5332 = 4 \cdot 31 \cdot 43$. None of these orders are new but there are possibly inequivalent Hadamard matrices. \square

Corollary 7 *Suppose there are best matrices of order m , a back-circulant $SBIBD(v, k, \lambda)$ and an Hadamard matrix with circulant core, B , of order*

1. $v = 4(k - \lambda) + 4m/3 - 1;$
2. $v = (8k - 8\lambda + 4m)/3 - 1;$
3. $v = 4(k - \lambda + m)/3 - 1;$

then there is an Hadamard matrix of order $4mv$.

Proof. Form the $OD(4m; 1, 1, 1, 4m - 3)$ as before.

As before B satisfies $BJ = -J$ and $BB^T = (v + 1)I_v - J_v$. Let A be the ± 1 incidence matrix of the $SBIBD(v, k, \lambda)$ then $AJ = (2k - v)J$ and $AA^T = 4(k - \lambda)I + (v - 4(k - \lambda))J$. We note $AB^T = BA^T$ as A is back-circulant and B is circulant. We now replace the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by (1) A, A, A and B , (2) A, A, J and B , and (3) A, J, J and B , respectively, which satisfy

$$3AA^T + (4m - 3)BB^T = 12(k - \lambda)I + 3(v - 4(k - \lambda))J + (4m - 3)(v + 1)I - (4m - 3)J = 4mvI,$$

$$2AA^T + JJ^T + (4m - 3)BB^T = 8(k - \lambda)I + 2(v - 4(k - \lambda))J + vJ + (4m - 3)(v + 1)I - (4m - 3)J = 4mvI,$$

$$AA^T + 2JJ^T + (4m - 3)BB^T = 4(k - \lambda)I + (v - 4(k - \lambda))J + 2vJ + (4m - 3)(v + 1)I - (4m - 3)J = 4mvI,$$

gives the required matrices. \square

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