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Connection between trinomial trees and finite difference methods for option pricing with state-dependent switching rates

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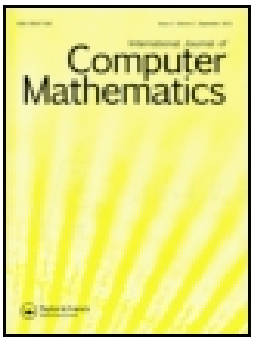

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Connection between trinomial trees and finite difference methods for option pricing with state-dependent switching rates

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Abstract

Tree approaches (binomial or trinomial trees) are very popularly used in finance industry to price financial derivatives. Such popularity stems from their simplicity and clear financial interpretation of the methodology. On the other hand, PDE (partial differential equation) approaches, with which standard numerical procedures such as the finite difference method, are characterized with the wealth of existing theory, algorithms and numerical software that can be applied to solve the problem. For a simple geometric Brownian motion model, the connection between these two approaches is studied, but it is lower-order equivalence. Moreover such a connection for a regime switching model is not so clear at all. This paper presents the high-order equivalence between the two for regime switching models. Moreover the convergence rates of trinomial trees for pricing options with state-dependent switching rates are first proved using the theory of the finite difference methods.

2010 Mathematics subject classification: 65C20, 65C40, 65M06, 91G20, 91G60

Keywords: Option pricing, trinomial tree methods, finite difference methods, regime switching models

1 Introduction

Markov regime switching models allow the model parameters (drift and volatility coefficients) to depend on a Markov chain which can reflect the information of the market environments and at the same time preserve the simplicity of the models. They are first

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introduced by Hamilton [16] and have had many applications in finance including equity options [2, 5, 7, 8, 11, 13, 14, 17, 20, 24, 26, 32, 35, 36, 12, 27, 18], bond prices [23] and interest rate derivatives [3, 25], portfolio selection [39], trading rules [10, 33, 34, 37, 38] , and others. There are many empirical studies on the Markov regime switching models (see e.g., [9], [15], [28], [4] and the references therein), which make the models popular and usable.

As a popular numerical solution approach, to actually produce values for these financial derivatives, trinomial tree methods (TTMs) are often used to price options. The first TTM is constructed by Boyle [6] for pricing options with single underlying asset using moments matching techniques. Later the approach is extended to the option pricing with two underlying assets. Tian [31] presents equal probability (1/3) trees with two different parameterizations for recombining trinomial tree and also another parameterization based on the idea of matching the first four moments. Rubinstein [29] explores that the trinomial tree can be constructed by viewing two steps of a binomial tree in combination as a single step of a trinomial tree.

Recently the trinomial tree methods are developed for the option pricing with regime-switching. Liu [24, 25] develops a linear tree for a regime-switching geometric Brownian motion model and extends it to a class of regime-switching mean-reverting models that have been frequently used for stochastic interest rates, energy and commodity prices. Liu and Zhao [26] develop a tree method for option pricing with two underlying assets under regime-switching models. Yuen and Yang [35, 36] construct an efficient trinomial tree method for option pricing in Markov regime-switching models and use the method to price Asian options and equity-indexed annuities. Ma and Zhu [27] prove the convergence rates of the trinomial tree of Yuen and Yang [35]. Liu and Zhao [26] propose a lattice method for option pricing with two underlying assets in the regime-switching model. Jiang, Liu and Nguyen [18] develop a recombining trinomial tree method for option pricing with state-dependent switching rates.

For a simple geometric Brownian motion model, the connection between these two approaches is studied (see e.g., [21], [1]), but it is lower-order equivalence. Moreover such a connection for a regime switching model is not so clear at all. This paper presents the high-order equivalence between the two for regime switching models. This paper presents the "bridge" between the two for the regime switching models. The main purpose of this paper is to explore this property for regime-switching option pricing models. We establish the high-order equivalence of the finite difference methods with the trinomial tree methods of [35] for regime switching models and the trinomial trees of [18] for state-dependent switching rates. The convergence rates for the TTMs can be established from the theory for the FDMs.

The remaining parts of the paper are arranged as follows. In Section 2, we study the relation of TTMs with FDMs for regime switching models; In Section 3, we explore the connection of the TTMs to the FDMs for state-dependent switching rates; In Section 4, we give numerical examples to verify the convergence rates of the TTMs and FDMs; Conclusions are given in the final section.

2 TTMs and FDMs for regime switching models

In the following, we describe the regime switching models and the trinomial method of Yuen and Yang [35] for pricing the European options.

Assume that the underlying asset price S_t follows a two-states regime switching model under risk-neutral measure:

$$\frac{dS(t)}{S(t)} = r(\alpha(t))\,dt + \sigma(\alpha(t))\,dW(t), \tag{1}$$

where $W(t)$ is a standard Brownian motion, $\alpha(t)$ is a continuous-time Markov chain with two states $(\alpha_1, \alpha_2, \ldots, \alpha_d)$. Assume also that at each state $\alpha(t) = \alpha_i$, $i \in \mathbb{D} = \{1, 2, \ldots, d\}$, the interest rate $r(\alpha_i) = r_i \geq 0$ and volatility $\sigma(\alpha_i) = \sigma_i$ for $i \in \mathbb{D}$ is constant. Let $\mathrm{A} = (a_{i\ell})_{i,\ell\in\mathbb{D}}$ be the generator matrix of the Markov chain process whose elements are constants satisfying $a_{i\ell} \geq 0$ for $i \neq \ell$ and $\sum_{\ell=1}^{d} a_{i\ell} = 0$ for $i \in \mathbb{D}$. Then from [32], the value of European option, $V(S,t,i)$, with maturity date T and payoff $f(S(T))$ satisfies the following PDEs

$$\begin{aligned} &\frac{\partial V(S,t,i)}{\partial t} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V(S,t,i)}{\partial S^2} + r_i S \frac{\partial V(S,t,i)}{\partial S} \\ &\quad - r_i V(S,t,i) + \sum_{\ell=1}^{d} a_{i\ell} V(S,t,\ell) = 0, \quad i \in \mathbb{D}, \end{aligned} \tag{2}$$

with terminal condition $V(S,T,i) = f(S)$, $i \in \mathbb{D}$. Here we assume the payoff function f is continuous.

Let $\Delta t = T/n$ be the time step-size. Then for all the regimes, the jump ratios of the lattice are taken as

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \tag{3}$$

where σ must satisfy

$$\sigma > \max_{i\in\mathbb{D}}\{\sigma_i\} \tag{4}$$

such that the risk-neutral probability measure exists. As suggested by Yuen and Yang [35], one possible value is

$$\sigma = \max_{i=\mathbb{D}}\{\sigma_i\} + (\sqrt{1.5} - 1)\widetilde{\sigma},$$

where $\widetilde{\sigma}$ is the arithmetic mean or the geometric mean of σ_i, $i \in \mathbb{D}$. For regime i, let π_u^i, π_m^i, π_d^i be the risk neutral probabilities corresponding to when the stock price increases, remains the same and decreases, respectively. Then the values of the probabilities are given by, for $i \in \mathbb{D}$, using first and second moments matching the original regime-switching model (1) (see [35]),

$$\pi_m^i = 1 - \frac{1}{\lambda_i^2}, \tag{5}$$

$$\pi_u^i = \frac{e^{r_i\Delta t} - e^{-\sigma\sqrt{\Delta t}} - \left(1 - 1/\lambda_i^2\right)\left(1 - e^{-\sigma\sqrt{\Delta t}}\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}, \tag{6}$$

$$\pi_d^i = \frac{e^{\sigma\sqrt{\Delta t}} - e^{r_i\Delta t} - \left(1 - 1/\lambda_i^2\right)\left(e^{\sigma\sqrt{\Delta t}} - 1\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}, \tag{7}$$

where $\lambda_i = \sigma/\sigma_i$.

Let $S_{j+1} = uS_j$ and $S_{j-1} = dS_j$ and denote $V^k(S_j, i)$ be the trinomial approximation of the European options for regime i at asset price S_j and time $t_k = k\Delta t$. Then for the trinomial trees (3) – (7), the trinomial value of European options for regime i can be recursively calculated by, for $k = 0, 1, \dots, n-1$,

$$\begin{aligned} &V^k(S_j, i) \qquad (8)\\ &= e^{-r_i \Delta t} \sum_{\ell=1}^{d} p_{i\ell} \left(\pi_u^i V^{k+1}(S_{j+1}, \ell) + \pi_m^i V^{k+1}(S_j, \ell) + \pi_d^i V^{k+1}(S_{j-1}, \ell) \right), \end{aligned}$$

with $V^n(S_j, i) = f(S_j)$ for $j = 0, \pm 1, \dots, \pm n$, in which $p_{i\ell}$ is the transition probability from regime state i to state ℓ for the time interval with length Δt. It is given by

$$(p_{i\ell})_{i,\ell \in \mathbb{D}} = e^{\mathrm{A}\Delta t} = \mathrm{I} + \sum_{l=1}^{\infty} (\Delta t)^l \mathrm{A}^l / l!, \tag{9}$$

where I is the identity matrix and A is the generator matrix of the Markov chain process.

Let $x = \log S$. Then $V(S, t, i) = V(e^x, t, i) \equiv \widehat{V}(x, t, i)$ for $i \in \mathbb{D}$. Then the PDEs (2) can be rewritten as

$$\begin{aligned} &\frac{\partial \widehat{V}(x,t,i)}{\partial t} + \frac{\sigma_i^2}{2} \frac{\partial^2 \widehat{V}(x,t,i)}{\partial x^2} + \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{\partial \widehat{V}(x,t,i)}{\partial x} \qquad (10)\\ &-r_i \widehat{V}(x,t,i) + \sum_{\ell=1}^{d} a_{i\ell} \widehat{V}(x,t,\ell) = 0, \quad \text{for } i \in \mathbb{D}, \end{aligned}$$

with terminal condition $\widehat{V}(x, T, i) = f(e^x)$, $i \in \mathbb{D}$.

Denote $\Delta x \equiv x_{j+1} - x_j$, for $j = 0, \pm 1, \dots, \pm(n-1)$ and let $\widehat{V}_j^k(i) \approx \widehat{V}(x_j, t_k, i)$. Then the explicit finite difference method (FDM) for solving (10) is given by

$$\begin{aligned} &\frac{\widehat{V}_j^{k+1}(i) - \widehat{V}_j^k(i)}{\Delta t} + \frac{\sigma_i^2}{2} \frac{\widehat{V}_{j+1}^{k+1}(i) - 2\widehat{V}_j^{k+1}(i) + \widehat{V}_{j-1}^{k+1}(i)}{(\Delta x)^2} + \left(r_i - \frac{\sigma_i^2}{2} \right) \frac{\widehat{V}_{j+1}^{k+1}(i) - \widehat{V}_{j-1}^{k+1}(i)}{2\Delta x}\\ &+ \left[\left[\left(\frac{r_i^2}{2} - \frac{\sigma^2 \sigma_i^2}{24} \right) - \frac{\sigma}{6} \left(r_i - \frac{\sigma_i^2}{2} \right) \right] \Delta t \right] \frac{\widehat{V}_{j+1}^{k+1}(i) - \widehat{V}_{j-1}^{k+1}(i)}{2\Delta x}\\ &-r_i \widehat{V}_j^k(i) + \sum_{\ell=1}^{d} a_{i\ell} \widehat{V}_j^{k+1}(\ell) = 0, \quad k = 0, 1, \dots, n-1;\ i \in \mathbb{D}, \qquad (11) \end{aligned}$$

with $\widehat{V}_j^n(i) = f(e^{x_j})$ for $j = 0, \pm 1, \dots, \pm n$; $i \in \mathbb{D}$.

Since it contains a perturbation term $\left[\left[\left(\frac{r_i^2}{2} - \frac{\sigma^2 \sigma_i^2}{24} \right) - \frac{\sigma}{6} \left(r_i - \frac{\sigma_i^2}{2} \right) \right] \Delta t \right] \frac{\widehat{V}_{j+1}^{k+1}(i) - \widehat{V}_{j-1}^{k+1}(i)}{2\Delta x}$, the scheme (11) is not a standard explicit FDM. We name the scheme (11) as perturbed FDM. The perturbed FDM (11) has the same convergence rate as the standard one as will be shown in Theorem 2.3. The motivation of constructing the perturbed FDM (11) is to set the second-order equivalence between the TTM and FDM, which means that the difference between the formulas of FDM and TTM is proportional to $(\Delta t)^2$ (see Theorem 2.1). The second-order equivalence is necessary to obtain the first-order difference between the solutions of FDM and TTM and to make the approximations of FDM and TTM to the price of options are both the first-order in time (see Theorem 2.3). For the option pricing with geometric Brownian motion (GBM) model, Ahn and Song [1] proves the first-order equivalence between the standard explicit FDM and the TTM. In fact the equivalence between the standard explicit FDM and the TTM is of 3/2 order. However the 3/2-order equivalence is not enough to keep the first-order difference for solutions between the FDM and TTM. The second-order equivalence studied in this section covers the GBM model in that the regime-switching model is reduced to the GBM model when there is no regime-switching occurrence (i.e., $a_{i\ell} = 0$ for $i, \ell \in \mathbb{D}$).

Theorem 2.1 *Let $x_j = \log S_j$ for $j = 0, \pm 1, \ldots, \pm n$. Then for European option pricing with regime switching model* (1)*, under condition*

$$\left| \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) + \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2} \right| < \frac{\sigma_i^2}{2\sigma^2}, \quad i \in \mathbb{D}, \tag{12}$$

the explicit perturbed FDM (11) *is equivalent to the TTM* (8) *by neglecting the high-order term $O\left((\Delta t)^2\right)$. Here and throughout the paper $O\left((\Delta t)^2\right)$ denotes a term that is proportional to $(\Delta t)^2$.*

Proof Since $\Delta x \equiv x_{j+1} - x_j = \log S_{j+1} - \log S_j = \log(S_{j+1}/S_j) = \log u = \sigma\sqrt{\Delta t}$ and $\widehat{V}_j^k(i) \approx \widehat{V}(x_j, t_k, i) = V(S_j, t_k, i)$, $j = 0, \pm 1, \ldots, \pm k$; $k = 0, 1, \ldots, n$; $i \in \mathbb{D}$, then the FDM (11) gives the following recursive formula

$$\begin{aligned} \widehat{V}_j^k(i) \;=\; & \frac{1}{1 + r_i\Delta t}\Bigg\{\Big[\frac{\sigma_i^2}{2\sigma^2} + \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) + \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\widehat{V}_{j+1}^{k+1}(i) \\ & + \Big(1 - \frac{\sigma_i^2}{\sigma^2}\Big)\widehat{V}_j^{k+1}(i) \\ & + \Big[\frac{\sigma_i^2}{2\sigma^2} - \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) - \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\widehat{V}_{j-1}^{k+1}(i) \\ & + \Delta t \sum_{\ell=1}^{d} a_{i\ell}\widehat{V}_j^{k+1}(\ell)\Bigg\}, \quad i \in \mathbb{D}, \end{aligned} \tag{13}$$

with $\widehat{V}_j^n(i) = f(e^{x_j}) = f(S_j)$ for $j = 0, \pm 1, \ldots, \pm n$; $i \in \mathbb{D}$.

Rewrite (6) and (7) into the following forms, for $i \in \mathbb{D}$,

$$\pi_u^i = \frac{e^{r_i\Delta t} - 1 + (1-d)\sigma_i^2/\sigma^2}{u - d}, \tag{14}$$

$$\pi_d^i = \frac{1 - e^{r_i\Delta t} + (u-1)\sigma_i^2/\sigma^2}{u - d}. \tag{15}$$

Using the following expansions

$$e^{r_i\Delta t} = 1 + r_i\Delta t + \frac{r_i^2}{2}(\Delta t)^2 + O\left((\Delta t)^3\right), \tag{16}$$

$$u = e^{\sigma\sqrt{\Delta t}} = 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t + \frac{\sigma^3}{6}(\Delta t)^{3/2} + \frac{\sigma^4}{4!}(\Delta t)^2 + O\left((\Delta t)^{5/2}\right), \tag{17}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t - \frac{\sigma^3}{6}(\Delta t)^{3/2} + \frac{\sigma^4}{4!}(\Delta t)^2 + O\left((\Delta t)^{5/2}\right), \tag{18}$$

we write (14) into the following form

$$\pi_u^i = \frac{r_i\Delta t + \frac{r_i^2}{2}(\Delta t)^2 + \left[\sigma\sqrt{\Delta t} - \frac{\sigma^2}{2}\Delta t + \frac{\sigma^3}{6}(\Delta t)^{3/2} - \frac{\sigma^4}{4!}(\Delta t)^2\right]\sigma_i^2/\sigma^2 + O((\Delta t)^{5/2})}{2\sigma\sqrt{\Delta t} + \frac{\sigma^3}{3}(\Delta t)^{3/2} + O((\Delta t)^{5/2})}.$$

Then we have

$$\begin{aligned} 2\sigma\sqrt{\Delta t}\,\pi_u^i &= r_i\Delta t + \frac{r_i^2}{2}(\Delta t)^2 + \left[\sigma\sqrt{\Delta t} - \frac{\sigma^2}{2}\Delta t\right]\sigma_i^2/\sigma^2 \\ &+ \left[\frac{\sigma^3}{6}(\Delta t)^{3/2} - \frac{\sigma^4}{4!}(\Delta t)^2\right]\sigma_i^2/\sigma^2 + O((\Delta t)^{5/2}) \\ &- \left[\frac{\sigma^3}{3}(\Delta t)^{3/2} + O((\Delta t)^{5/2})\right]\pi_u^i. \end{aligned}$$

This leads to

$$\begin{aligned} \pi_u^i &= \frac{\sigma_i^2}{2\sigma^2} + \frac{\sqrt{\Delta t}}{2\sigma}\left(r_i - \frac{\sigma_i^2}{2}\right) + \frac{r_i^2}{4\sigma}(\Delta t)^{3/2} + \frac{\sigma_i^2}{12}\Delta t - \frac{\sigma\sigma_i^2}{48}(\Delta t)^{3/2} + O((\Delta t)^2) \\ &- \left[\frac{\sigma^2}{6}\Delta t + O((\Delta t)^2)\right]\pi_u^i. \end{aligned} \tag{19}$$

Recursively using (19), we arrive at

$$\pi_u^i = \frac{\sigma_i^2}{2\sigma^2} + \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) + \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2} + O((\Delta t)^2). \tag{20}$$

Similarly to the derivation of (20), we obtain that

$$\pi_d^i = \frac{\sigma_i^2}{2\sigma^2} - \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) - \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2} + O((\Delta t)^2). \tag{21}$$

Moreover, it follows from (9) that for $i \in \mathbb{D}$,

$$p_{ii} = 1 + a_{ii}\Delta t + O\left((\Delta t)^2\right), \tag{22}$$

$$p_{i\ell} = a_{i\ell}\Delta t + O\left((\Delta t)^2\right), \quad i \neq \ell. \tag{23}$$

Furthermore, using the following Taylor expansion

$$e^{-r_i\Delta t} = 1 - r_i\Delta t + O((\Delta t)^2),$$

we derive that

$$(1 + r_i\Delta t)e^{-r_i\Delta t} = 1 - r_i^2(\Delta t)^2 + O((\Delta t)^2) = 1 + O((\Delta t)^2).$$

So we have

$$e^{-r_i\Delta t} = \frac{1}{1 + r_i\Delta t} + O((\Delta t)^2). \tag{24}$$

Using (5), (20), (21), (22), (23), and (24), we obtain from (8) that

$$\begin{aligned} V^k(S_j, i) &= \frac{1}{1 + r_i\Delta t}\bigg\{\Big[\frac{\sigma_i^2}{2\sigma^2} + \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) \\ &+ \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]V^{k+1}(S_{j+1}, i) \\ &+ \Big(1 - \frac{\sigma_i^2}{\sigma^2}\Big)V^{k+1}(S_j, i) + \Big[\frac{\sigma_i^2}{2\sigma^2} - \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) \\ &- \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]V^{k+1}(S_{j-1}, i) \\ &+ \Delta t\sum_{\ell=1}^{d} a_{i\ell}V^{k+1}(S_j, \ell)\bigg\} + O((\Delta t)^2), \quad i \in \mathbb{D}, \end{aligned} \tag{25}$$

with $V^n(S_j, i) = f(S_j)$, (25) is equivalently written as

$$\begin{aligned}
\widehat{V}^k(x_j,i) &= \frac{1}{1+r_i\Delta t}\bigg\{\Big[\frac{\sigma_i^2}{2\sigma^2}+\frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\\
&+ \Big[\Big(\frac{r_i^2}{4\sigma}-\frac{\sigma\sigma_i^2}{48}\Big)-\frac{\sigma}{12}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\widehat{V}^{k+1}(x_{j+1},i)\\
&+ \Big(1-\frac{\sigma_i^2}{\sigma^2}\Big)\widehat{V}^{k+1}(x_j,i)+\Big[\frac{\sigma_i^2}{2\sigma^2}-\frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\\
&- \Big[\Big(\frac{r_i^2}{4\sigma}-\frac{\sigma\sigma_i^2}{48}\Big)-\frac{\sigma}{12}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\widehat{V}^{k+1}(x_{j-1},i)\\
&+ \Delta t\sum_{\ell=1}^{d}a_{i\ell}\widehat{V}^{k+1}(x_j,\ell)\bigg\}+O((\Delta t)^2), \quad i\in\mathbb{D}, \qquad (26)
\end{aligned}$$

with $\widehat{V}^n(x_j, i) = f(e^{x_j})$. Therefore (13) is equivalent to (26) by neglecting the high-order term $O((\Delta t)^2)$. Moreover, to ensure the probabilities π_u^i in (20) and π_d^i in (21) are non-negative, it requires that (12) holds true. Thus the proof of the Theorem is complete. □

Theorem 2.2 *Set $\Delta x = \sigma\sqrt{\Delta t}$. Then under condition (12), TTM (8) is stable.*

Proof From the theory for FDM (see [19]), we know that both condition

$$\frac{\sigma^2\Delta t}{(\Delta x)^2} \le 1, \qquad (27)$$

and (12) are required to guarantee the stability of the FDM (11). Thus, the same conditions are expected to be imposed for TTM (8) as they are equivalent (see Theorem 2.1). In fact condition (27) is always satisfied as we have already set $\Delta x = \sigma\sqrt{\Delta t}$. □

The convergence rates for the TTMs have been proved by Ma and Zhu [27]. The convergence rates for the TTMs can also be established from the theory of the convergence rates for FDMs. In the following theorem, we prove the convergence rates of the TTMs using the theory of the convergence rates for FDMs. Denote the errors of FDM (11) and TTM (8) with notation $\widehat{V}^k(x_j,i) = V^k(S_j,i)$ for pricing the European option under GBM regime-switching model (1) which satisfies the PDE (10) respectively by

$$\epsilon_j^k(i) := \widehat{V}(x_j, t_k, i) - \widehat{V}_j^k(i),$$

and

$$\eta_j^k(i) := \widehat{V}(x_j, t_k, i) - \widehat{V}^k(x_j, i),$$

for $j = 0, \pm1, \ldots, \pm k$, $k = 0, 1, \ldots, n-1$ and $i \in \mathbb{D}$. Define the discrete infinity norm at time t_k by

$$\|\nu^k(i)\|_\infty := \max_{-k\le j\le k} |\nu_j^k(i)|, \quad i \in \mathbb{D}.$$

Then we present the following results for the convergence rates of FDMs and TTMs.

Theorem 2.3 *Assume the payoff function f is continuous. Then under the set-up $\Delta x = \bar{\sigma}\sqrt{\Delta t}$ and condition (12), the convergence rates of the FDM (11) at time t_k are estimated by*

$$\|\epsilon^k(i)\|_\infty = |O(\Delta t)|, \quad k = 0, 1, \ldots, n-1;\ i \in \mathbb{D}, \qquad (28)$$

and the convergence rates of the TTM (8) *are given by*

$$\|\eta^k(i)\|_\infty = |O(\Delta t)|, \quad k = 0, 1, \ldots, n-1;\ i \in \mathbb{D}. \tag{29}$$

Proof Define local truncation error by

$$\begin{aligned}
&\frac{\widehat{V}(x_j, t_{k+1}, i) - \widehat{V}(x_j, t_k, i)}{\Delta t} + \frac{\sigma_i^2}{2}\frac{\widehat{V}(x_{j+1}, t_{k+1}, i) - 2\widehat{V}(x_j, t_{k+1}, i) + \widehat{V}(x_{j-1}, t_{k+1}, i)}{(\Delta x)^2} \\
&+ \left(r_i - \frac{\sigma_i^2}{2}\right)\frac{\widehat{V}(x_{j+1}, t_{k+1}, i) - \widehat{V}(x_{j-1}, t_{k+1}, i)}{2\Delta x} \\
&+ \left[\left[\left(\frac{r_i^2}{2} - \frac{\sigma^2\sigma_i^2}{24}\right) - \frac{\sigma}{6}\left(r_i - \frac{\sigma_i^2}{2}\right)\right]\Delta t\right]\frac{\widehat{V}(x_{j+1}, t_{k+1}, i) - \widehat{V}(x_{j-1}, t_{k+1}, i)}{2\Delta x} \\
&- r_i\widehat{V}(x_j, t_k, i) + \sum_{\ell=1}^{d} a_{i\ell}\widehat{V}(x_j, t_{k+1}, \ell) = \mathcal{T}_j^k(i),
\end{aligned} \tag{30}$$

with $\widehat{V}(x_j, t_n, i) = f(e^{x_j})$, $i \in \mathbb{D}$. Since the PDEs (10) are a kind of linear parabolic PDEs with constant coefficients, from the PDE theory in [22], we know that the solutions to the PDEs (10) have high-order smoothness. Therefore we can conduct the Taylor expansions for $\widehat{V}$ at point (x_j, t_{k+1}), and then utilize PDE (10) and expansion

$$\begin{aligned}
&\left[\left[\left(\frac{r_i^2}{2} - \frac{\sigma^2\sigma_i^2}{24}\right) - \frac{\sigma}{6}\left(r_i - \frac{\sigma_i^2}{2}\right)\right]\Delta t\right]\frac{\widehat{V}(x_{j+1}, t_{k+1}, i) - \widehat{V}(x_{j-1}, t_{k+1}, i)}{2\Delta x} \\
&= \left[\left[\left(\frac{r_i^2}{2} - \frac{\sigma^2\sigma_i^2}{24}\right) - \frac{\sigma}{6}\left(r_i - \frac{\sigma_i^2}{2}\right)\right]\Delta t\right]\left[\frac{\partial\widehat{V}(x_j, t_{k+1}, i)}{\partial x} + O((\Delta x)^2)\right] = O(\Delta t),
\end{aligned}$$

to get

$$|\mathcal{T}_j^k(i)| = |O(\Delta t) + O((\Delta x)^2)| = |O(\Delta t)|. \tag{31}$$

Write (30) into a recursive form

$$\begin{aligned}
\widehat{V}(x_j, t_k, i) \;=\; \frac{1}{1 + r_i\Delta t}\Bigg\{&\Big[\frac{\sigma_i^2}{2\sigma^2} + \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) \\
&+\Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\widehat{V}(x_{j+1}, t_{k+1}, i) \\
&+\Big(1 - \frac{\sigma_i^2}{\sigma^2}\Big)\widehat{V}(x_j, t_{k+1}, i) + \Big[\frac{\sigma_i^2}{2\sigma^2} - \frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i - \frac{\sigma_i^2}{2}\Big) \\
&-\Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma\sigma_i^2}{48}\Big) - \frac{\sigma}{12}\Big(r_i - \frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\widehat{V}(x_{j-1}, t_{k+1}, i) \\
&+\Delta t\sum_{\ell=1}^{d} a_{i\ell}\widehat{V}(x_j, t_{k+1}, \ell) + \mathcal{T}_j^k(i)\Delta t\Bigg\},
\end{aligned} \tag{32}$$

with $\widehat{V}(x_j, t_n, i) = f(e^{x_j}) = f(S_j)$ for $j = 0, \pm 1, \ldots, \pm n$; $i \in \mathbb{D}$. Subtracting (13) from (32) gives that

$$\begin{aligned}\epsilon_j^k(i) &= \frac{1}{1+r_i\Delta t}\Bigg\{\Big[\frac{\sigma_i^2}{2\sigma^2}+\frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\\ &\quad +\Big[\Big(\frac{r_i^2}{4\sigma}-\frac{\sigma\sigma_i^2}{48}\Big)-\frac{\sigma}{12}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\epsilon_{j+1}^{k+1}(i)\\ &\quad +\Big(1-\frac{\sigma_i^2}{\sigma^2}\Big)\epsilon_j^{k+1}(i)+\Big[\frac{\sigma_i^2}{2\sigma^2}-\frac{\sqrt{\Delta t}}{2\sigma}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\\ &\quad -\Big[\Big(\frac{r_i^2}{4\sigma}-\frac{\sigma\sigma_i^2}{48}\Big)-\frac{\sigma}{12}\Big(r_i-\frac{\sigma_i^2}{2}\Big)\Big](\Delta t)^{3/2}\Big]\epsilon_{j-1}^{k+1}(i)\\ &\quad +\Delta t\sum_{\ell=1}^{d}a_{i\ell}\epsilon_j^{k+1}(\ell)+\mathcal{T}_j^k(i)\Delta t\Bigg\},\quad i\in\mathbb{D}. \end{aligned}\tag{33}$$

Using (12), the truncation error estimation (31), and $a_{i\ell}\ge 0$ for $i\ne\ell$ and $a_{ii}\le 0$, we obtain that

$$\begin{aligned}\|\epsilon^k(i)\|_\infty &\le \frac{1}{1+r_i\Delta t}\left[(1-a_{ii}\Delta t)\|\epsilon^{k+1}(i)\|_\infty+\sum_{\ell=1,\ell\ne i}^{d}a_{i\ell}\Delta t\|\epsilon^{k+1}(\ell)\|_\infty\right]\\ &\quad +|O((\Delta t)^2)|,\quad i\in\mathbb{D}.\end{aligned}\tag{34}$$

Denote vector $\mathbf{\Psi}_k=\big(\|\epsilon^k(1)\|_\infty,\dots,\|\epsilon^k(d)\|_\infty\big)'$. Then (34) can be written into a vector form

$$\mathbf{\Psi}_k\le\mathbf{D}\mathbf{\Psi}_{k+1}+|O((\Delta t)^2)|\mathbf{1},\tag{35}$$

where $\mathbf{1}=(1,\dots,1)'$ is a d-dimension vector and $\mathbf{D}$ is a $d\times d$ matrix

$$\mathbf{D}=\begin{bmatrix}1-a_{11}\Delta t & a_{12}\Delta t & \cdots & \cdots & a_{1d}\Delta t\\ a_{21}\Delta t & 1-a_{22}\Delta t & a_{23}\Delta t & \cdots & a_{2d}\Delta t\\ & \ddots & \ddots & \ddots & \\ a_{d-1,1}\Delta t & \cdots & a_{d-1,d-2}\Delta t & 1-a_{d-1,d-1}\Delta t & a_{d-1,d}\Delta t\\ a_{d1}\Delta t & \cdots & \cdots & a_{d,d-1}\Delta t & 1-a_{dd}\Delta t\end{bmatrix}.$$

Since each element of matrix $\mathbf{D}$ is nonnegative, iterating of inequality (35) gives that

$$\mathbf{\Psi}_k\le\mathbf{D}^{n-k}\mathbf{\Psi}_n+\left[\mathbf{I}+\sum_{m=1}^{n-k-1}\mathbf{D}^m\right]|O((\Delta t)^2)|\mathbf{1},\tag{36}$$

where $\mathbf{I}$ is a $d\times d$ identity matrix and

$$\begin{aligned}&\mathbf{I}+\sum_{m=1}^{n-k-1}\mathbf{D}^m\\ &=\begin{bmatrix}(n-k)-\frac{(n-k)(n-k-1)}{2}a_{11}\Delta t & \cdots & \cdots & \frac{(n-k)(n-k-1)}{2}a_{1d}\Delta t\\ \vdots & \vdots & \vdots & \vdots\\ \frac{(n-k)(n-k-1)}{2}a_{d1}\Delta t & \cdots & \cdots & (n-k)-\frac{(n-k)(n-k-1)}{2}a_{dd}\Delta t\end{bmatrix}\\ &\quad+\begin{bmatrix}|O((\Delta t)^2)| & \cdots & \cdots & |O((\Delta t)^2)|\\ \vdots & \vdots & \vdots & \vdots\\ |O((\Delta t)^2)| & \cdots & \cdots & |O((\Delta t)^2)|\end{bmatrix}.\end{aligned}$$

Since at the terminal time $t_n \equiv T$ the FDM value equals the true option value, we know that $\mathbf{\Psi}_n$ is a zero vector. Therefore (28) follows from (36) using $\Delta t = T/n$.

Now we prove (29). To this end, we write the error of the TTM into the following form

$$\begin{aligned} \eta_j^k(i) &= [\widehat{V}(x_j, t_k, i) - \widehat{V}_j^k(i)] + [\widehat{V}_j^k(i) - \widehat{V}^k(x_j, i)] \\ &= \epsilon_j^k(i) + \chi_j^k(i), \end{aligned} \tag{37}$$

where $\chi_j^k(i) := \widehat{V}_j^k(i) - \widehat{V}^k(x_j, i)$. Subtracting (26) from (13) gives that

$$\begin{aligned} \chi_j^k(i) &= \frac{1}{1 + r_i \Delta t} \Bigg\{ \Big[\frac{\sigma_i^2}{2\sigma^2} + \frac{\sqrt{\Delta t}}{2\sigma} \Big(r_i - \frac{\sigma_i^2}{2} \Big) \\ &\quad + \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma \sigma_i^2}{48} \Big) - \frac{\sigma}{12} \Big(r_i - \frac{\sigma_i^2}{2} \Big) \Big] (\Delta t)^{3/2} \Big] \chi_{j+1}^{k+1}(i) \\ &\quad + \Big(1 - \frac{\sigma_i^2}{\sigma^2} \Big) \chi_j^{k+1}(i) + \Big[\frac{\sigma_i^2}{2\sigma^2} - \frac{\sqrt{\Delta t}}{2\sigma} \Big(r_i - \frac{\sigma_i^2}{2} \Big) \\ &\quad - \Big[\Big(\frac{r_i^2}{4\sigma} - \frac{\sigma \sigma_i^2}{48} \Big) - \frac{\sigma}{12} \Big(r_i - \frac{\sigma_i^2}{2} \Big) \Big] (\Delta t)^{3/2} \Big] \chi_{j-1}^{k+1}(i) \\ &\quad + \Delta t \sum_{\ell=1}^{d} a_{i\ell} \chi_j^{k+1}(\ell) \Bigg\} + O((\Delta t)^2), \quad i \in \mathbb{D}. \end{aligned} \tag{38}$$

Since (38) has the same structure as (33), it can follow the lines (34) - (36) to obtain that

$$\|\chi^k(i)\|_\infty = |O(\Delta t)|, \quad k = 0, 1, \ldots, n-1; \ i \in \mathbb{D}. \tag{39}$$

Therefore, from (37), using triangle inequality and estimations (28) and (39), we obtain that

$$\|\eta^k(i)\|_\infty \le \|\epsilon^k(i)\|_\infty + \|\chi^k(i)\|_\infty = |O(\Delta t)|.$$

Thus the proof of this theorem is complete. □

3 TTMs and FDMs for state-dependent switching rates

From the previous sections, we prove the equivalence between a TTM and an explicit FDM for regime switching models with the jump-rate independent of the state. In this section, we explore the relation between the explicit FDM and TTM of [18] for the option pricing with state-dependent switching rates.

We use a Poisson process $\{N(t),\, t \ge 0\}$ with regime-dependent intensity $\lambda_{\alpha(t)}$ to model the random jump times for the asset price. That is, if the current regime is $\alpha(t) = i$, then the time until the next jump is given by an exponential random variable with mean $1/\lambda_i$. Hence $N(t)$ counts the total number of jumps in the asset price up to time t. For each $i \in \mathbb{D}$, let Z_k^i, $k \ge 1$ be a sequence of independent identically distributed (iid) random variables with the common density function $f_i(z)$, that specifies the jump sizes when the regime is i. Note that here we consider a very general model setup allowing different jump distributions $f_i(\cdot)$ for different regimes i.

The model (see [18]) is given by $S(t) = \exp(X(t))$ with $X(t)$ satisfying that

$$\begin{cases} dX(t) = [b_{\alpha(t)} - \lambda_{\alpha(t)} \kappa_{\alpha(t)}]dt + \sigma_{\alpha(t)} dW(t) + dJ(t), \\ X(0) = \ln S(0), \end{cases} \tag{40}$$

where for each $i \in \mathbb{D}$, $b_i = r_i - \sigma_i^2/2$ and $\kappa_i := E[e^{Z_1^i} - 1]$ denotes the mean percentage change in the risky asset price due to jump when the regime is i. $J(t)$ represents the cumulative jumps by time t, given by

$$J(t) = \sum_{k=1}^{N(t)} Z_k^{\alpha(\tau_k)}, \tag{41}$$

where τ_k denotes the kth jump time of the process $N(\cdot)$.

In the following we describe the TTM of [18]. Let l_i be the number of upward moves of X_{k+1}. Note that $l_i \in \mathbb{N}^+$ and l_i is independent of X_k. By matching the mean and variance implied by the trinomial tree to that implied by the SDE (40), the nodes $(X_{k+1}, \alpha(t_{k+1}))$ at $(k+1)$th time level, emanating from nodes $(X_k, \alpha(t_k)) = (x, i)$, $i \in \mathbb{D}$, at kth time level, are given by, for $\ell \in \mathbb{D}$,

$$(X_{k+1}, \alpha(t_{k+1})) = \begin{cases} (x + l_i\bar{\sigma}\sqrt{\Delta t}, \ell) & \text{with prob. } p_{i\ell}\left[(1-\lambda_i\Delta t)\pi_u^i + \lambda_i\Delta t\, d\mathcal{N}_i(l_i)\right], \\ (x, \ell) & \text{with prob. } p_{i\ell}\left[(1-\lambda_i\Delta t)\pi_m^i + \lambda_i\Delta t\, d\mathcal{N}_i(0)\right], \\ (x - l_i\bar{\sigma}\sqrt{\Delta t}, \ell) & \text{with prob. } p_{i\ell}\left[(1-\lambda_i\Delta t)\pi_d^i + \lambda_i\Delta t\, d\mathcal{N}_i(-l_i)\right], \\ (x + l\bar{\sigma}\sqrt{\Delta t}, \ell) & \text{with prob. } p_{i\ell}\lambda_i\Delta t\, d\mathcal{N}_i(l)],\ l \neq -l_i, 0, l_i, \end{cases} \tag{42}$$

where

$$\pi_u^i = \frac{\sigma_i^2 + (b_i - \lambda_i\kappa_i)l_i\bar{\sigma}\sqrt{\Delta t} + (b_i - \lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}, \tag{43}$$

$$\pi_m^i = 1 - \frac{\sigma_i^2 + (b_i - \lambda_i\kappa_i)^2\Delta t}{(l_i\bar{\sigma})^2}, \tag{44}$$

$$\pi_d^i = \frac{\sigma_i^2 - (b_i - \lambda_i\kappa_i)l_i\bar{\sigma}\sqrt{\Delta t} + (b_i - \lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}, \tag{45}$$

and

$$\begin{aligned} d\mathcal{N}_i(l) &:= P\{Z_k^i = l\bar{\sigma}\sqrt{\Delta t}\} \\ &= \mathcal{N}_i((l+0.5)\bar{\sigma}\sqrt{\Delta t}) - \mathcal{N}_i((l-0.5)\bar{\sigma}\sqrt{\Delta t}), \quad l = 0, \pm 1, \pm 2, \ldots, \pm M, \end{aligned} \tag{46}$$

where $\mathcal{N}_i(x) = \int_{-\infty}^{x} f_i(z)dz$ is the cumulative distribution function of Z_k^i, M is a sufficiently large positive integer such that the probability $d\mathcal{N}_i(l) = P\{Z_k^i = l\bar{\sigma}\sqrt{\Delta t}\}$ is extremely small.

The trinomial value of European options with maturity T for regime $i \in \mathbb{D}$ can be recursively calculated by

$$\begin{aligned} \widehat{V}^k(x,i) = {} & e^{-r_i\Delta t}\sum_{\ell=1}^{d} p_{i\ell}\Big[[\pi_u^i(1-\lambda_i\Delta t) + \lambda_i\Delta t\, d\mathcal{N}_i(l_i)]\widehat{V}^{k+1}(x + l_i\bar{\sigma}\sqrt{\Delta t}, \ell) \\ & + [\pi_m^i(1-\lambda_i\Delta t) + \lambda_i\Delta t\, d\mathcal{N}_i(0)]\widehat{V}^{k+1}(x, \ell) \\ & + [\pi_d^i(1-\lambda_i\Delta t) + \lambda_i\Delta t\, d\mathcal{N}_i(-l_i)]\widehat{V}^{k+1}(x - l_i\bar{\sigma}\sqrt{\Delta t}, \ell) \\ & + \sum_{l \neq -l_i, 0, l_i} \lambda_i\Delta t\, d\mathcal{N}_i(l)\widehat{V}^{k+1}(x_j + l\bar{\sigma}\sqrt{\Delta t}, \ell)\Big], \end{aligned} \tag{47}$$

with $\widehat{V}^n(x,i) = f(e^x)$ (payoff function) for $k = 0, 1, \ldots, n-1$ and $\bar{\sigma}$ satisfying

$$2\sigma_i/\sqrt{3} < l_i\bar{\sigma} \leq 2\sigma_i, \quad i \in \mathbb{D}.$$

(see [18]).

In the current studying, we explore the relation of trinomial tree method (47) with the finite difference method. Denote $x_j \equiv j\bar{\sigma}\sqrt{\Delta t}$. Then the trinomial value of European options at nodes $x = x_j$ for regime $i \in \mathbb{D}$ can be written as, for $k = 0, 1, \ldots, n-1$,

$$
\begin{aligned}
\widehat{V}^k(x_j, i) &= e^{-r_i\Delta t}\sum_{\ell=1}^{d} p_{i\ell}\Big[[\pi_u^i(1-\lambda_i\Delta t) + \lambda_i\Delta t\, d\mathcal{N}_i(l_i)]\widehat{V}^{k+1}(x_{j+l_i}, \ell) \\
&\quad + [\pi_m^i(1-\lambda_i\Delta t) + \lambda_i\Delta t\, d\mathcal{N}_i(0)]\widehat{V}^{k+1}(x_j, \ell) \\
&\quad + [\pi_d^i(1-\lambda_i\Delta t) + \lambda_i\Delta t\, d\mathcal{N}_i(-l_i)]\widehat{V}^{k+1}(x_{j-l_i}, \ell) \\
&\quad + \sum_{l\neq -l_i, 0, l_i} \lambda_i\Delta t\, d\mathcal{N}_i(l)\widehat{V}^{k+1}(x_{j+l}, \ell)\Big] \\
&= e^{-r_i\Delta t}\sum_{\ell=1}^{d} p_{i\ell}\Big[(1-\lambda_i\Delta t)\left(\pi_u^i\widehat{V}^{k+1}(x_{j+l_i}, \ell) + \pi_m^i\widehat{V}^{k+1}(x_j, \ell) + \pi_d^i\widehat{V}^{k+1}(x_{j-l_i}, \ell)\right) \\
&\quad + \sum_{l=-M}^{M} \lambda_i\Delta t\, d\mathcal{N}_i(l)\widehat{V}^{k+1}(x_{j+l}, \ell)\Big] \qquad (48)
\end{aligned}
$$

with $\widehat{V}^n(x_j, i) = f(e^{x_j})$ (payoff function) for $j = 0, \pm 1, \ldots, \pm n$; $i \in \mathbb{D}$ and M is a sufficiently large positive integer.

Following [12], the value of European option $\widehat{V}(x, t, i)$ with maturity date T and payoff $f(e^x)$ satisfies the following partial integro-differential equations (PIDEs)

$$
\begin{aligned}
&\frac{\partial \widehat{V}(x,t,i)}{\partial t} + \frac{\sigma_i^2}{2}\frac{\partial^2 \widehat{V}(x,t,i)}{\partial x^2} + (b_i - \lambda_i\kappa_i)\frac{\partial \widehat{V}(x,t,i)}{\partial x} - (r_i + \lambda_i)\widehat{V}(x,t,i) \\
&\quad + \sum_{\ell=1}^{d} a_{i\ell}\widehat{V}(x,t,\ell) + \lambda_i\int_{-\infty}^{\infty}\widehat{V}(x+y,t,i)\, d\mathcal{N}_i(y) = 0, \qquad (49)
\end{aligned}
$$

with terminal condition $\widehat{V}(x, T, i) = f(e^x)$, $i \in \mathbb{D}$.

Denote $\Delta x = x_{j+1} - x_j$ and $\widehat{V}_j^k(i) \approx \widehat{V}(x_j, t_k, i)$. Then the explicit perturbed FDM for solving PIDE (49) is given by, for $k = 0, 1, \ldots, n-1$,

$$
\begin{aligned}
&\frac{\widehat{V}_j^{k+1}(i) - \widehat{V}_j^k(i)}{\Delta t} + \frac{\sigma_i^2}{2}\frac{\widehat{V}_{j+l_i}^{k+1}(i) - 2\widehat{V}_j^{k+1}(i) + \widehat{V}_{j-l_i}^{k+1}(i)}{(l_i\Delta x)^2} \qquad (50) \\
&\quad + \left(\frac{(b_i - \lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\right)\frac{\widehat{V}_{j+l_i}^{k+1}(i) - 2\widehat{V}_j^{k+1}(i) + \widehat{V}_{j-l_i}^{k+1}(i)}{(l_i\Delta x)^2} \\
&\quad + (b_i - \lambda_i\kappa_i)\frac{\widehat{V}_{j+l_i}^{k+1}(i) - \widehat{V}_{j-l_i}^{k+1}(i)}{2l_i\Delta x} - (r_i + \lambda_i)\widehat{V}_j^k(i) \\
&\quad + \sum_{\ell=1}^{d} a_{i\ell}\widehat{V}_j^{k+1}(\ell) + \lambda_i\sum_{l=-M}^{M} V_{j+l}^{k+1}(i)\, d\mathcal{N}_i(l) = 0,
\end{aligned}
$$

with $\widehat{V}_j^n(i) = f(e^{x_j})$ for $j = 0, \pm 1, \ldots, \pm n$; $i \in \mathbb{D}$. Note that we have used the composite mid-point quadrature rules to discretize the integrals in PIDEs (49) based on the mesh nodes $y = x_l$, $l = 0, \pm 1, \pm 2, \ldots, \pm M$.

Theorem 3.1 *For the European option pricing with state-dependent switching rate model* (40), *the explicit perturbed FDM* (50) *is equivalent to the TTM* (48) *by neglecting high-order term* $O((\Delta t)^2)$ *under condition*

$$(b_i - \lambda_i\kappa_i)\ell_i\bar{\sigma}\sqrt{\Delta t} \le \sigma_i^2 + (b_i - \lambda_i\kappa_i)^2\Delta t \le (\ell_i\bar{\sigma})^2, \quad i \in \mathbb{D}. \tag{51}$$

Proof Using the fact $\Delta x = \bar{\sigma}\sqrt{\Delta t}$, we write the perturbed FDM (50) into the following recursive form

$$\begin{aligned}
\widehat{V}_j^k(i) &= \frac{1}{1+(r_i+\lambda_i)\Delta t}\Bigg\{\Big[\frac{\sigma_i^2}{2(l_i\bar{\sigma})^2} + \frac{\sqrt{\Delta t}}{2l_i\bar{\sigma}}\Big(b_i - \lambda_i\kappa_i\Big) + \frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\Big]\widehat{V}_{j+l_i}^{k+1}(i) \\
&+ \Big[1 - \frac{\sigma_i^2}{(l_i\bar{\sigma})^2} - \frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{(l_i\bar{\sigma})^2}\Big]\widehat{V}_j^{k+1}(i) \\
&+ \Big[\frac{\sigma_i^2}{2(l_i\bar{\sigma})^2} - \frac{\sqrt{\Delta t}}{2l_i\bar{\sigma}}\Big(b_i - \lambda_i\kappa_i\Big) + \frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\Big]\widehat{V}_{j-l_i}^{k+1}(i) \\
&+ \Delta t\sum_{\ell=1}^{d} a_{i\ell}\widehat{V}_j^{k+1}(\ell) + \lambda_i\Delta t\sum_{l=-M}^{M}\widehat{V}_{j+l}^{k+1}(i)\, d\mathcal{N}_i(l)\Bigg\}, \quad i \in \mathbb{D}.
\end{aligned} \tag{52}$$

with $\widehat{V}_j^n(i) = f(e^{x_j})$ for $j = 0, \pm 1, \ldots, \pm n$; $i \in \mathbb{D}$.

Using (22), (23) and the relation $\sum_{\ell=1}^{d} a_{i\ell} = 0$ for $i \in \mathbb{D}$, the TTM (48) is rewritten as

$$\begin{aligned}
\widehat{V}^k(x_j,i) &= e^{-r_i\Delta t}(1-\lambda_i\Delta t)\Bigg[\pi_u^i\widehat{V}^{k+1}(x_{j+l_i},i) + \pi_m^i\widehat{V}^{k+1}(x_j,i) + \pi_d^i\widehat{V}^{k+1}(x_{j-l_i},i) \\
&\quad +\Delta t\sum_{\ell=1}^{d} a_{i\ell}\Big[\pi_u^i\widehat{V}^{k+1}(x_{j+l_i},\ell) + \pi_m^i\widehat{V}^{k+1}(x_j,\ell) + \pi_d^i\widehat{V}^{k+1}(x_{j-l_i},\ell)\Big]\Bigg] \\
&\quad +e^{-r_i\Delta t}\lambda_i\Delta t\sum_{l=-M}^{M}\widehat{V}^{k+1}(x_{j+l},i)\, d\mathcal{N}_i(l) + O((\Delta t)^2).
\end{aligned} \tag{53}$$

The Taylor's expansion gives that

$$e^{-r_i\Delta t} = 1 - r_i\Delta t + O((\Delta t)^2).$$

Therefore we have

$$\begin{aligned}
e^{-r_i\Delta t}(1-\lambda_i\Delta t)\left[1+(r_i+\lambda_i)\Delta t\right] &= 1 - r_i^2(\Delta t)^2 - \lambda_i(r_i+\lambda_i)(1-r_i\Delta t)(\Delta t)^2 + O((\Delta t)^2) \\
&= 1 + O((\Delta t)^2).
\end{aligned}$$

So

$$e^{-r_i\Delta t}(1-\lambda_i\Delta t) = \frac{1}{1+(r_i+\lambda_i)\Delta t} + O((\Delta t)^2). \tag{54}$$

Finally inserting (43) – (45) and (54) into (53) gives that for $i \in \mathbb{D}$,

$$
\begin{aligned}
\widehat{V}^k(x_j,i) \;=\;& \frac{1}{1+(r_i+\lambda_i)\Delta t}\Bigg\{\Big[\frac{\sigma_i^2}{2(l_i\bar\sigma)^2}+\frac{\sqrt{\Delta t}}{2l_i\bar\sigma}\Big(b_i-\lambda_i\kappa_i\Big)+\frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar\sigma)^2}\Big]\widehat{V}^{k+1}(x_{j+l_i},i)\\
&+\Big[1-\frac{\sigma_i^2}{(l_i\bar\sigma)^2}-\frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{(l_i\bar\sigma)^2}\Big]\widehat{V}^{k+1}(x_j,i)\\
&+\Big[\frac{\sigma_i^2}{2(l_i\bar\sigma)^2}-\frac{\sqrt{\Delta t}}{2l_i\bar\sigma}\Big(b_i-\lambda_i\kappa_i\Big)+\frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar\sigma)^2}\Big]\widehat{V}^{k+1}(x_{j-l_i},i)\\
&+\Delta t\sum_{\ell=1}^{d}a_{i\ell}\widehat{V}^{k+1}(x_j,\ell)+\lambda_i\Delta t\sum_{l=-M}^{M}\widehat{V}^{k+1}(x_{j+l},i)\,d\mathcal{N}_i(l)\Bigg\}+O((\Delta t)^2),
\end{aligned}
\tag{55}
$$

with $\widehat{V}^n(x_j,i)=f(e^{x_j})$ for $j=0,\pm1,\ldots,\pm n$; $i\in\mathbb{D}$. To ensure the probabilities (43) - (45) are nonnegative, it requires that (51) holds true. Comparing (52) with (55) completes the proof of this Theorem. □

Theorem 3.2 *Set $\Delta x=\bar\sigma\sqrt{\Delta t}$. Then under condition* (51)*, the TTM* (48) *is stable.*

Proof From the theory for FDM (see [19]), we know that both

$$\frac{\bar\sigma^2\Delta t}{(l_i\Delta x)^2}\le 1, \tag{56}$$

and condition (51) are required to guarantee the stability of the FDM (50) and thus the stability of the TTM (48) as they are equivalent (see Theorem 3.1). In fact condition (56) is always satisfied as we have already set $\Delta x=\bar\sigma\sqrt{\Delta t}$ and $l_i\in\mathbb{N}^+$. □

Now we present the convergence rates of FDMs and TTMs for pricing the European option under state-dependent switching rate model (40).

Theorem 3.3 *Assume the payoff function f is continuous. Then under the set-up $\Delta x=\bar\sigma\sqrt{\Delta t}$ and* (51)*, the convergence rates of the FDM* (50) *at time t_k for pricing European option under state-dependent switching rate model* (40) *are estimated by*

$$\|\epsilon^k(i)\|_\infty=|O(\Delta t)|,\quad k=0,1,\ldots,n-1;\ i\in\mathbb{D}, \tag{57}$$

and the convergence rates of the TTM (48) *are given by*

$$\|\eta^k(i)\|_\infty=|O(\Delta t)|,\quad k=0,1,\ldots,n-1;\ i\in\mathbb{D}. \tag{58}$$

Proof We follow the proof for Theorem 2.3 to establish the results of convergence rates in this theorem. Define local truncation error by

$$
\begin{aligned}
&\frac{\widehat{V}(x_j,t_{k+1},i)-\widehat{V}(x_j,t_k,i)}{\Delta t}+\frac{\sigma_i^2}{2}\frac{\widehat{V}(x_{j+l_i},t_{k+1},i)-2\widehat{V}(x_j,t_{k+1},i)+\widehat{V}(x_{j-l_i},t_{k+1},i)}{(l_i\Delta x)^2}\\
&+\left(\frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar\sigma)^2}\right)\frac{\widehat{V}(x_{j+l_i},t_{k+1},i)-2\widehat{V}(x_j,t_{k+1},i)+\widehat{V}(x_{j-l_i},t_{k+1},i)}{(l_i\Delta x)^2}\\
&+(b_i-\lambda_i\kappa_i)\frac{\widehat{V}(x_{j+l_i},t_{k+1},i)-\widehat{V}(x_{j-l_i},t_{k+1},i)}{2l_i\Delta x}-(r_i+\lambda_i)\widehat{V}(x_j,t_k,i)\\
&+\sum_{\ell=1}^{d}a_{i\ell}\widehat{V}(x_j,t_{k+1},\ell)+\lambda_i\sum_{l=-M}^{M}\widehat{V}(x_{j+l},t_{k+1},i)d\mathcal{N}_i(l)=\mathcal{T}_j^k(i),
\end{aligned}
\tag{59}
$$

with $\widehat{V}(x_j, t_n, i) = f(e^{x_j})$, $i \in \mathbb{D}$. Now we estimate the truncation error. Using Taylor expansions for $\widehat{V}$ at point (x_j, t_{k+1}), we obtain that

$$\begin{aligned}
&\left(\frac{(b_i - \lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\right)\frac{\widehat{V}(x_{j+l_i}, t_{k+1}, i) - 2\widehat{V}(x_j, t_{k+1}, i) + \widehat{V}(x_{j-l_i}, t_{k+1}, i)}{(l_i\Delta x)^2} \\
&= \left(\frac{(b_i - \lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\right)\left[\frac{\partial^2\widehat{V}(x_j, t_{k+1}, i)}{\partial x^2} + O((\Delta x)^2)\right] = O(\Delta t). \qquad (60)
\end{aligned}$$

Moreover the Taylor expansions give that

$$\widehat{V}(x_j + y, t_{k+1}, i) = \widehat{V}(x_j + l\Delta x, t_{k+1}, i) + \frac{\partial\widehat{V}}{\partial y}(x_j + l\Delta x, t_{k+1}, i)(y - l\Delta x) + O((y - l\Delta x)^2),$$

and

$$f_i(y) = f_i(l\Delta x) + f_i'(l\Delta x)(y - l\Delta x) + O((y - l\Delta x)^2).$$

Noting that $x_j + l\Delta x = x_{j+l}$ and $\Delta x = \bar{\sigma}\sqrt{\Delta t}$, we thus have

$$\begin{aligned}
&\int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x} \widehat{V}(x_j + y, t_{k+1}, i)d\mathcal{N}_i(y) \qquad (61)\\
&= \widehat{V}(x_{j+l}, t_{k+1}, i)d\mathcal{N}_i(l) \\
&\quad + \int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}\left[\frac{\partial\widehat{V}}{\partial y}(x_{j+l}, t_{k+1}, i)(y - l\Delta x) + O((y - l\Delta x)^2)\right]d\mathcal{N}_i(y).
\end{aligned}$$

Furthermore we derive that

$$\begin{aligned}
&\int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}\left[\frac{\partial\widehat{V}}{\partial y}(x_{j+l}, t_{k+1}, i)(y - l\Delta x) + O((y - l\Delta x)^2)\right]d\mathcal{N}_i(y) \\
&= \int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}\left[\frac{\partial\widehat{V}}{\partial y}(x_{j+l}, t_{k+1}, i)(y - l\Delta x) + O((y - l\Delta x)^2)\right]f_i(y)dy \\
&= \int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}\left[\frac{\partial\widehat{V}}{\partial y}(x_{j+l}, t_{k+1}, i)f_i(l\Delta x)(y - l\Delta x) + O((y - l\Delta x)^2)\right]dy \\
&= \frac{\partial\widehat{V}}{\partial y}(x_{j+l}, t_{k+1}, i)f_i(l\Delta x)\int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}(y - l\Delta x)dy + \int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}O((y - l\Delta x)^2)dy \\
&= \int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}O((y - l\Delta x)^2)dy. \qquad (62)
\end{aligned}$$

Therefore, using (60), (61) and (62), carrying out the Taylor expansions for the other terms in (59), and utilizing the PIDEs (49), we arrive at

$$\begin{aligned}
\mathcal{T}_j^k(i) &= O(\Delta t) + O((\Delta x)^2) + \lambda_i\sum_{l=-M}^{M}\int_{(l-0.5)\Delta x}^{(l+0.5)\Delta x}O((y - l\Delta x)^2)dy \\
&= O(\Delta t) + O((\Delta x)^2) = O(\Delta t), \qquad (63)
\end{aligned}$$

where we have used $\Delta x = \bar{\sigma}\sqrt{\Delta t}$. Write (59) into a recursive form

$$\begin{aligned}
&\widehat{V}(x_j, t_k, i) \\
&= \frac{1}{1+(r_i+\lambda_i)\Delta t}\Bigg\{\Big[\frac{\sigma_i^2}{2(l_i\bar{\sigma})^2} + \frac{\sqrt{\Delta t}}{2l_i\bar{\sigma}}\Big(b_i - \lambda_i\kappa_i\Big) + \frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\Big]\widehat{V}(x_{j+l_i}, t_{k+1}, i) \\
&+\Big[1 - \frac{\sigma_i^2}{(l_i\bar{\sigma})^2} - \frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{(l_i\bar{\sigma})^2}\Big]\widehat{V}(x_j, t_{k+1}, i) \\
&+\Big[\frac{\sigma_i^2}{2(l_i\bar{\sigma})^2} - \frac{\sqrt{\Delta t}}{2l_i\bar{\sigma}}\Big(b_i - \lambda_i\kappa_i\Big) + \frac{(b_i-\lambda_i\kappa_i)^2\Delta t}{2(l_i\bar{\sigma})^2}\Big]\widehat{V}(x_{j-l_i}, t_{k+1}, i) \\
&+\Delta t\sum_{\ell=1}^{d} a_{i\ell}\widehat{V}_j^{k+1}(\ell) + \lambda_i\Delta t\sum_{l=-M}^{M}\widehat{V}(x_{j+l}, t_{k+1}, i)\, d\mathcal{N}_i(l) + \mathcal{T}_j^k(i)\Delta t\Bigg\}, \quad i \in \mathbb{D}, \qquad (64)
\end{aligned}$$

with $\widehat{V}(x_j, t_n, i) = f(e^{x_j})$ for $j = 0, \pm 1, \ldots, \pm n$; $i \in \mathbb{D}$. Subtracting (52) from (64), using the truncation error estimation (63), and inheriting the same notations as Theorem 2.3, we derive that

$$\begin{aligned}
\|\epsilon^k(i)\|_\infty \;\le\; & \frac{1}{1+(r_i+\lambda_i)\Delta t}\left[(1 - a_{ii}\Delta t + \lambda_i\Delta t)\|\epsilon^{k+1}(i)\|_\infty + \sum_{\ell=1,\ell\neq i}^{d} a_{i\ell}\Delta t\|\epsilon^{k+1}(\ell)\|_\infty\right] \\
& + |O((\Delta t)^2)|, \quad i \in \mathbb{D}. \qquad (65)
\end{aligned}$$

The rest of the proof is similar to that for Theorem 2.3. Thus the proof of this theorem is complete. □

4 Numerical examples

In this section we use numerical examples to verify the convergence rates of the TTMs and FDMs. The convergence rates are calculated by the commonly used formula ([27]):

$$\text{Rate} = \log\left|\frac{\text{Error with number of time step } N_1}{\text{Error with number of time step } N_2}\right| \Big/ \log\left(\frac{N_2}{N_1}\right).$$

Example 4.1 *In this example we assume that the stock price follows the regime-switching GBM model* (1) *with initial price* $S_0 = 100$. *The generator for the regime-switching process is taken as*

$$A = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \quad or \quad B = \begin{pmatrix} -2/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}.$$

The interest rates are $r_1 = 4\%$ *and* $r_2 = 6\%$, *and the volatilities* $\sigma_1 = 0.25$ *and* $\sigma_2 = 0.35$, *respectively for regime 1 and 2. We compute the price of the European call option with maturity date* $T = 1$ *year and strike price* $K = 100$.

From Table 1, we observe that the convergence rates for both TTMs and FDMs are about 1. *Moreover the absolute difference between the option values computed by TTMs and FDMs, which is denoted by "Diff" in the tables, is decreasing in* N *with rate* 1. *All these results are consistent with the theoretical findings.*

Table 1: Comparisons of TTMs and FDMs for pricing European call option under the regime-switching GBM model (1) for Example 4.1.

Regime 1 (generator A)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
20	12.6281680	0.129140	1.019162	12.653997	0.103506	1.014277	0.025829	1.027650
40	12.6935901	0.063718	1.026857	12.706259	0.051243	1.024525	0.012669	1.013392
80	12.7260368	0.031272	1.049163	12.732313	0.025190	1.048028	0.006276	1.006576
160	12.7421964	0.015112	1.100461	12.745320	0.012182	1.099897	0.003124	1.003284
320	12.7502606	0.007048	1.222843	12.751819	0.005684	1.222573	0.001558	1.001616
640	12.7542888	0.003020	1.585106	12.755067	0.002436	1.585053	0.000778	1.000658
1280	12.7563019	0.001006	–	12.756691	0.000812	–	0.000389	1.000771
2560	12.7573083	–	–	12.757503	–	–	0.000194	–
Regime 2 (generator A)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
20	15.756030	0.009180	0.891050	15.725206	0.039771	1.001321	0.030824	1.024717
40	15.760260	0.004950	0.967322	15.745110	0.019867	1.017986	0.015150	1.012110
80	15.762679	0.002532	1.020373	15.755167	0.009811	1.044745	0.007512	1.006005
160	15.763962	0.001248	1.086336	15.760222	0.004755	1.098253	0.003740	1.002965
320	15.764622	0.000588	1.215618	15.762756	0.002221	1.221750	0.001866	1.001518
640	15.764957	0.000253	1.580978	15.764025	0.000952	1.584641	0.000932	1.000790
1280	15.765126	0.000085	–	15.764660	0.000318	–	0.000466	1.000068
2560	15.765210	–	–	15.764977	–	–	0.000233	–
Regime 1 (generator B)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
20	12.9232455	0.136280	1.019197	12.9582194	0.101570	1.012814	0.034974	1.026628
40	12.9922863	0.067239	1.026891	13.0094535	0.050336	1.023844	0.017167	1.012917
80	13.0265269	0.032999	1.049189	13.0350340	0.024755	1.047699	0.008507	1.006365
160	13.0435794	0.015946	1.100472	13.0478142	0.011975	1.099735	0.004235	1.003160
320	13.0520889	0.007437	1.222859	13.0542017	0.005588	1.222493	0.002113	1.001575
640	13.0563396	0.003186	1.585196	13.0573948	0.002395	1.585013	0.001055	1.000786
1280	13.0584639	0.001062	–	13.0589912	0.000798	–	0.000527	1.000393
2560	13.0595258	–	–	13.0597894	–	–	0.000264	–
Regime 2 (generator B)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
20	16.0245607	0.019079	0.961531	16.0045053	0.038982	0.999113	0.020055	1.024673
40	16.0338422	0.009797	0.999109	16.0239845	0.019503	1.016951	0.009858	1.012103
80	16.0387377	0.004902	1.035548	16.0338500	0.009638	1.044245	0.004888	1.005994
160	16.0412481	0.002391	1.093713	16.0388144	0.004673	1.098007	0.002434	1.002983
320	16.0425189	0.001120	1.219495	16.0413045	0.002183	1.221628	0.001214	1.001488
640	16.0431581	0.000481	1.583517	16.0425516	0.000936	1.584581	0.000607	1.000743
1280	16.0434787	0.000161	–	16.0431756	0.000312	–	0.000303	1.000371
2560	16.0436392	–	–	16.0434877	–	–	0.000152	–

Example 4.2 *In this example we assume that the stock price follows the state-dependent switching rate model* (40) *with initial price* $S_0 = 100$. *The generator for the regime-switching process is taken as*

$$A = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \quad or \quad B = \begin{pmatrix} -2/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}.$$

The interest rates are $r_1 = r_2 = 5\%$, *and the volatilities* $\sigma_1 = 0.15$ *and* $\sigma_2 = 0.25$, *respectively for regime 1 and 2. We compute the price of the European call option with maturity date* $T = 1$ *year and strike price* $K = 100$. *The density function* $f_i(z)$ *is a mixture of two normal densities:*

$$f_i(z) = p_i \frac{1}{\sqrt{2\pi} b_{1,i}} e^{-\frac{(z-a_{1,i})^2}{2b_{1,i}^2}} + (1-p_i)\frac{1}{\sqrt{2\pi} b_{2,i}} e^{-\frac{(z-a_{2,i})^2}{2b_{2,i}^2}},$$

where $a_{1,i} > 0$, $a_{2,i} < 0$, *and* $0 < p_i < 1$. *In this case, we have*

$$\kappa_i = p_i e^{a_{1,i}+\frac{1}{2}b_{1,i}^2} + (1-p_i)e^{a_{2,i}+\frac{1}{2}b_{2,i}^2} - 1.$$

The parameters for the jump density functions are taken as: $a_{1,1} = a_{1,2} = 0.3753$, $a_{2,1} = a_{2,2} = -0.5503$, $b_{1,1} = b_{1,2} = 0.18$, $b_{2,1} = b_{2,2} = 0.6944$, $p_1 = p_2 = 0.3445$. *In addition we choose the intensities* $\lambda_1 = 5$, $\lambda_2 = 2$, *and* $l_1 = 1$, $l_2 = 2$.

From Table 2, we observe that the convergence rates for both TTMs and FDMs are about 1. *Moreover the absolute difference between the option values computed by TTMs and FDMs is decreasing in* N *with rate* 1. *All these results are consistent with the theoretical findings.*

5 Conclusions

In this paper we have established the equivalence of the explicit finite difference methods with the trinomial tree methods developed by Yuen and Yang [35] for the option pricing with regime switching models and the trinomial tree methods proposed by Jiang, Liu and Nguyen [18] for the option pricing with state-dependent switching rates. The second-order equivalence, which means that the difference between these two schemes is proportional to $(\Delta t)^2$), is derived. The convergence rates of the trinomial tree methods are established from the theory of the finite difference methods, as a result of the establishment of such equivalences. If there is a dividend yield associated with the underlying asset, the derivation essentially remains the same. The extension of the studies to American option pricing is much more challenging, which will be our future target.

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Table 2: Comparisons of TTMs and FDMs for pricing European call option under the state-dependent switching rate model (40) for Example 4.2.

Regime 1 (generator A)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
40	48.828522	0.547479	1.040940	48.764898	0.609152	1.042406	0.063624	1.008551
80	49.109920	0.266081	1.094467	49.078296	0.295754	1.095384	0.031624	1.004834
160	49.251393	0.124608	1.218237	49.235634	0.138416	1.219092	0.015759	1.005315
320	49.322444	0.053557	1.581455	49.314593	0.059457	1.582175	0.007851	1.004745
640	49.358105	0.017896	–	49.354193	0.019857	–	0.003912	1.003840
1280	49.376001	–	–	49.374050	–	–	0.001951	–
Regime 2 (generator A)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
40	38.351528	0.308790	1.038353	38.323374	0.336149	1.042463	0.028155	1.044384
80	38.509974	0.150345	1.093822	38.496323	0.163199	1.096502	0.013651	1.031759
160	38.589879	0.070439	1.218379	38.583203	0.076320	1.220468	0.006677	1.027259
320	38.630046	0.030272	1.582155	38.626770	0.032752	1.583627	0.003276	1.022055
640	38.650208	0.010110	–	38.648595	0.010928	–	0.001613	1.018977
1280	38.660319	–	–	38.659523	–	–	0.000796	–
Regime 1 (generator B)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
80	48.883737	0.244198	1.039294	48.171957	0.933018	1.026927	0.711780	0.975342
160	49.009117	0.118819	1.089773	48.647092	0.457883	1.089472	0.362025	0.989694
320	49.072110	0.055825	1.213060	48.889800	0.215174	1.212674	0.182310	0.991063
640	49.103855	0.024080	1.574763	49.012133	0.092841	1.581156	0.091722	0.998577
1280	49.119852	0.008084	–	49.073946	0.031029	–	0.045906	0.999496
2560	49.127935	–	–	49.104974	–	–	0.022961	–
Regime 2 (generator B)								
N	TTM			FDM			Diff	Rate
	Value	Error	Rate	Value	Error	Rate		
80	38.100803	0.131783	1.031288	37.663516	0.554898	1.026127	0.437287	0.976957
160	38.168108	0.064478	1.081751	37.945944	0.272469	1.082305	0.222164	0.983117
320	38.202123	0.030463	1.204417	38.089733	0.128680	1.211322	0.112389	0.991510
640	38.219366	0.013219	1.565188	38.162840	0.055573	1.577816	0.056526	0.997032
1280	38.228118	0.004467	–	38.199797	0.018616	–	0.028321	0.998834
2560	38.232586	–	–	38.218414	–	–	0.014172	–

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