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Abstract

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1. Main results

In this paper, we study self-similar solutions to the inverse mean curvature flow in Euclidean space. After a brief introduction, we present the definitions of the homothetic and translating solitons and discuss the one-dimensional examples. We prove that families of cycloids are the only translating solitons (Theorem 8), and we show how to construct translating surfaces via a tilted product of cycloids.

Next, we consider the rigidity of homothetic solitons. In the class of closed homothetic solitons of codimension one, we prove that round hyperspheres are rigid (Theorem 10). For the higher codimension case, we observe that any minimal submanifold of the standard hypersphere is an expander, so in light of Lawson's construction [1970] of minimal surfaces in $\mathbb{S}^3$, there exist compact embedded expanders for any genus in $\mathbb{R}^4$.

We conclude with an investigation of homothetic solitons with rotational symmetry. First, we construct new examples of complete expanders with rotational symmetry, called *infinite bottles* (see Figure 1), which are topological hypercylinders that interpolate between two concentric round hypercylinders (Theorem 14). Then, we show how the analysis in the proof of Theorem 14 can be used to construct other examples of complete expanders with rotational symmetry, including the examples of Huisken and Ilmanen [1997a].

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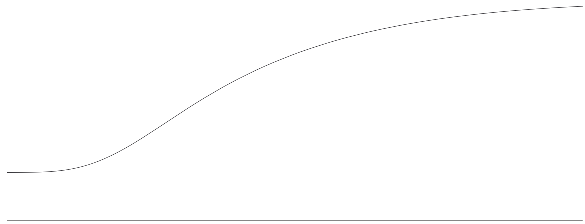

Figure 1. A numerical approximation of the part of a curve whose rotation about the horizontal axis is the self-expanding *infinite bottle* in $\mathbb{R}^3$.

2. Inverse mean curvature flow: history and applications

Round hyperspheres in Euclidean space expand under the inverse mean curvature flow (IMCF) with an exponentially increasing radius. This behavior is typical for the flow. Gerhardt [1990] and Urbas [1990] showed that compact, star-shaped initial hypersurfaces with strictly positive mean curvature converge under IMCF, after suitable rescaling, to a round sphere.

Strictly positive mean curvature is an essential condition. For the IMCF to be parabolic, the mean curvature must be strictly positive. Huisken and Ilmanen [2008] proved that smoothness at later times is characterized by the mean curvature remaining bounded strictly away from zero; see also Smoczyk [2000]. Within the class of strictly mean-convex surfaces, however, a solution to inverse mean curvature flow will, in general, become singular in finite time. For example, starting from a thin embedded torus with positive mean curvature in $\mathbb{R}^3$, the surface fattens up under IMCF and, after finite time, the mean curvature reaches zero at some points [Huisken and Ilmanen 2001, p. 364]. Thus, the classical description breaks down, and any appropriate weak definition of inverse mean curvature flow would need to allow for a change of topology.

Huisken and Ilmanen [2001] used a level-set approach and developed the notion of weak solutions for IMCF to overcome these problems. They showed existence for weak solutions and proved that Geroch's monotonicity [1973] for the Hawking mass carries over to the weak setting. This enabled them to prove the Riemannian Penrose inequality, which also gave an alternative proof for the Riemannian positive mass theorem. For a summary, we refer the reader to Huisken and Ilmanen [1997a; 1997b]. The work of Huisken and Ilmanen also shows that weak solutions become star-shaped and smooth outside some compact region and thus, by the results of Gerhardt [1990] and Urbas [1990], round in the limit. Using a different geometric evolution equation, Bray [2001] proved the most general form of the Riemannian

Penrose inequality. An overview of the different methods used by Huisken, Ilmanen, and Bray can be found in [Bray 2002]. An approach to solving the full Penrose inequality involving a generalized inverse mean curvature flow was proposed in [Bray et al. 2007]. To our knowledge, the full Penrose inequality is still an open problem.

Finally, let us mention some other applications and new developments in IMCF. Using IMCF, Bray and Neves [2004] proved the Poincaré conjecture for 3-manifolds with σ-invariant greater than that of $\mathbb{RP}^3$; see also [Akutagawa and Neves 2007]. Connections with p-harmonic functions and the weak formulation of inverse mean curvature flow are described in [Moser 2007], where a new proof for the existence of a proper weak solution is given, and in [Lee et al. 2011], where gradient bounds and nonexistence results are proved. Recently, Kwong and Miao [2014] discovered a monotone quantity for the IMCF, which they used to derive new geometric inequalities for star-shaped hypersurfaces with positive mean curvature.

3. Definitions and one-dimensional examples

Definition 1 (homothetic solitons of arbitrary codimension). A submanifold Σ^n of $\mathbb{R}^N$ with nonvanishing mean curvature vector field $\vec{H}$ is called a *homothetic soliton for the inverse mean curvature flow* if there exists a constant $C \in \mathbb{R} - \{0\}$ satisfying

$$-\frac{1}{|\vec{H}|^2}\vec{H} = CX^{\perp} \quad \text{on } \Sigma, \tag{1}$$

where the vector field $X^{\perp}$ denotes the normal component of X. Notice that, for any constant $\lambda \neq 0$, the rescaled immersion λX is a soliton with the same value of C.

Remark 2. On a homothetic soliton $\Sigma^n \subset \mathbb{R}^N$, we observe that the condition (1) implies

$$|\vec{H}|^2 = \langle \vec{H}, \vec{H}\rangle = \langle -C|\vec{H}|^2 X^{\perp}, \vec{H}\rangle = -C|\vec{H}|^2\langle X, \vec{H}\rangle.$$

Since the mean curvature vector field $\vec{H}$ is nonvanishing, this shows

$$-\langle \vec{H}, X\rangle = \frac{1}{C} \quad \text{or} \quad -\langle \triangle_g X, X\rangle = \frac{1}{C} \quad \text{or} \quad \triangle_g |X|^2 = 2\Big(n - \frac{1}{C}\Big),$$

where g denotes the induced metric on Σ.

Proposition 3 (homothetic solitons of codimension one). *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a hypersurface with nowhere vanishing mean curvature vector field $\vec{H} = \triangle_g X$. Then, it becomes a homothetic soliton to the inverse mean curvature flow if and only if there exists a constant $C \in \mathbb{R} - \{0\}$ satisfying*

$$-\langle \vec{H}, X\rangle = \frac{1}{C} \quad \textit{or equivalently}, \quad -\langle \triangle_g X, X\rangle = \frac{1}{C}. \tag{2}$$

Proof. According to the observation in Remark 2, the vector equality in (1) implies the scalar equality in (2). To see that (2) implies (1), let N denote a unit normal vector, and let $H = -(\mathrm{div}_\Sigma \boldsymbol{N})$ be the corresponding scalar mean curvature. Then $\vec{H} = \triangle_g X = H\boldsymbol{N}$, and the condition (2) becomes

$$-\langle H\boldsymbol{N}, X\rangle = \frac{1}{C},$$

which implies

$$CX^\perp = \langle \boldsymbol{N}, CX\rangle \boldsymbol{N} = -\frac{1}{H}\boldsymbol{N} = -\frac{1}{H^2}\vec{H}. \qquad \square$$

Remark 4. A complete classification of the homothetic solitons for the inverse curve shortening flow in the plane was established by J. Urbas [1999]. If a plane curve $\mathcal{C}$ is a solution to (2), then its curvature function κ satisfies the Poisson equation

$$\triangle_{\mathcal{C}} \frac{1}{\kappa^2} = 2(C-1),$$

and this guarantees the existence of constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\kappa^2 = \frac{1}{(C-1)s^2 + \alpha_1 s + \alpha_2},$$

where s denotes an arc length parameter on the curve $\mathcal{C}$. It is a straightforward exercise to find explicit parametrizations of these homothetic solitons; for instance, see [Castro and Lerma 2016, Section 4]. Examples include circles, involutes of circles, classical logarithmic spirals, epicycloids, and hypocycloids.

Definition 5 (translators of arbitrary codimension). A submanifold $\Sigma^n \subset \mathbb{R}^N$ with nonvanishing mean curvature vector field $\vec{H}$ is called a *translator for the inverse mean curvature flow* if there exists a nonzero constant vector field $\boldsymbol{V}$ satisfying

$$(3) \qquad -\frac{1}{|\vec{H}|^2}\vec{H} = \boldsymbol{V}^\perp \quad \text{on } \Sigma,$$

where the vector field $\boldsymbol{V}^\perp$ denotes the normal component of $\boldsymbol{V}$. We say that $\boldsymbol{V}$ is the *velocity* of the translator Σ.

Proposition 6 (translators of codimension one). *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a hypersurface with nonvanishing mean curvature vector field $\vec{H} = \triangle_g X$, where g denotes the induced metric on Σ. Then Σ^n is a translator to the inverse mean curvature flow if and only if there exists a nonzero constant vector field $\boldsymbol{V}$ satisfying*

$$(4) \qquad \langle \boldsymbol{V}, \vec{H}\rangle = -1.$$

Proof. We first observe that the condition (3) implies the equality

$$-1 = \left\langle -\frac{1}{|\vec{H}|^2}\vec{H}, \vec{H}\right\rangle = \langle \boldsymbol{V}^\perp, \vec{H}\rangle = \langle \boldsymbol{V}, \vec{H}\rangle.$$

It remains to check that the scalar equality (4) implies the vectorial equality in (3). Let $\boldsymbol{N}$ denote a unit normal vector and $H = -(\mathrm{div}_\Sigma \boldsymbol{N})$ its scalar mean curvature, so that $\vec{H} = \triangle_g X = H\boldsymbol{N}$. Then the condition (4) becomes $-1 = \langle \boldsymbol{V}, \vec{H}\rangle = H\langle \boldsymbol{V}, \boldsymbol{N}\rangle$, which implies

$$\boldsymbol{V}^{\perp} = \langle \boldsymbol{V}, \boldsymbol{N}\rangle \boldsymbol{N} = -\frac{1}{H}\boldsymbol{N} = -\frac{1}{H^2}\vec{H}. \qquad \square$$

Corollary 7 (height function on translating hypersurfaces). *A submanifold Σ^n of $\mathbb{R}^{n+1}$ with nonvanishing mean curvature is a* ***translator to the inverse mean curvature flow*** *with velocity $\boldsymbol{V} = (0, \ldots, 0, 1)$ if and only if*

$$-1 = \triangle_\Sigma x_{n+1} \quad \text{on } \Sigma. \tag{5}$$

Now we prove that cycloids are the only one-dimensional translators in $\mathbb{R}^2$.

Theorem 8 (classification of translating curves in $\mathbb{R}^2$). *Any translating curves with unit speed for the inverse mean curvature flow in the Euclidean plane are congruent to cycloids generated by a circle of radius $\frac{1}{4}$.*

Proof. Let the connected curve $\mathcal{C}$ be a translator in the xy-plane with unit velocity $\boldsymbol{V} = (0, 1)$. Adopt the parametrization $X(s) = (x(s), y(s))$, where s denotes the arc length on $\mathcal{C}$, and introduce the tangential angle function $\theta(s)$ such that the tangent $dX/ds = (\cos\theta, \sin\theta)$ and the normal $N(s) = (-\sin\theta, \cos\theta)$. The translator condition reads

$$-\frac{1}{\kappa} = \cos\theta.$$

Now, we integrate

$$\left(\frac{dx}{d\theta}, \frac{dy}{d\theta}\right) = \left(\frac{ds}{d\theta}\frac{dx}{ds}, \frac{ds}{d\theta}\frac{dy}{ds}\right) = \left(\frac{1}{\kappa}\cos\theta, \frac{1}{\kappa}\sin\theta\right) = (-\cos^2\theta, -\cos\theta\sin\theta)$$

to recover, up to translation, the curve

$$(x, y) = \tfrac{1}{4}(-2\theta - \sin(2\theta), 1 + \cos(2\theta)).$$

After introducing the new variable $t = -\pi + 2\theta$, we have

$$(x, y) = \tfrac{1}{4}(-\pi - t + \sin t, 1 - \cos t).$$

Reflecting about the x-axis and then translating along the $(1, 0)$ direction, the translator is congruent to the cycloid represented by $\frac{1}{4}(t - \sin t, 1 - \cos t)$. Therefore, we conclude that $\mathcal{C}$ is congruent to the cycloid through the origin, generated by a circle of radius $\frac{1}{4}$. $\square$

Example 9 (tilted cycloid products: one-parameter family of translators with the same speed in $\mathbb{R}^3$). We can use cycloids (one-dimensional translators in $\mathbb{R}^2$) to construct a one-parameter family of two-dimensional translators with velocity $(0, 0, 1)$ in $\mathbb{R}^3$. Let $(\alpha(s), \beta(s))$ denote a unit speed patch of the translating curve

$\mathcal{C}$ with velocity $(0,1)$ in the $\alpha\beta$-plane, so that $\beta''(s) = -1$ on the translator $\mathcal{C}$. For each constant $\mu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we introduce orthonormal vectors

$$\boldsymbol{v}_1 = (\cos\mu, 0, -\sin\mu), \quad \boldsymbol{v}_2 = (0,1,0), \quad \boldsymbol{v}_3 = (\sin\mu, 0, \cos\mu),$$

and associate the product surface $\Sigma_\mu = \mathbb{R} \times \dfrac{1}{\cos\mu}\mathcal{C}$ defined by the patch

$$\boldsymbol{X}(s,h) = h\boldsymbol{v}_1 + \frac{\alpha(s)}{\cos\mu}\boldsymbol{v}_2 + \frac{\beta(s)}{\cos\mu}\boldsymbol{v}_3.$$

A straightforward computation yields

$$\langle \triangle_{\Sigma_\mu} X, (0,0,1)\rangle = \left\langle \frac{1}{\cos\mu}(\alpha''(s)\boldsymbol{v}_2 + \beta''(s)\boldsymbol{v}_3), (0,0,1)\right\rangle = \beta''(s) = -1,$$

which guarantees that Σ_μ becomes a translator with velocity $(0,0,1)$ in $\mathbb{R}^3$.

4. Rigidity of hyperspheres and spherical expanders

We first prove that hyperspheres, as homothetic solitons to the inverse mean curvature flow, are exceptionally rigid. This is a higher-dimensional generalization of Andrews' result [2003, Theorem 1.7] that circles centered at the origin are the only compact homothetic solitons in $\mathbb{R}^2$. We then explain that the moduli space of spherical expanders of higher codimension is large. Hereafter, we assume $n \geq 2$.

Theorem 10 (uniqueness of spheres as compact solitons). *Let Σ^n be a homothetic soliton hypersurface for the inverse mean curvature flow in $\mathbb{R}^{n+1}$. If Σ is closed, then it is a round hypersphere (centered at the origin).*

Proof. Since Σ is a compact hypersurface with nonvanishing mean curvature vector, there exists an inward pointing unit normal vector field $\boldsymbol{N}$ along Σ. Then $\vec{H} = \triangle_g X = H\boldsymbol{N}$, where the scalar mean curvature $H = -\operatorname{div}_\Sigma \boldsymbol{N}$ is positive. Since Σ is a homothetic soliton, we have

$$\frac{1}{C} = -\langle X, \vec{H}\rangle = -H\langle X, \boldsymbol{N}\rangle, \tag{6}$$

for some constant $C \neq 0$. The Hsiung–Minkowski formula [Hsiung 1956] gives

$$0 = \int_\Sigma \left(1 + \frac{1}{n}\langle X, \vec{H}\rangle\right) d\Sigma = \left(1 - \frac{1}{nC}\right)\int_\Sigma 1\, d\Sigma.$$

It follows that $C = 1/n$. Let $\kappa_1, \dots, \kappa_n$ be principal curvature functions on Σ. In terms of

$$\sigma_2 = \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n}\kappa_i\kappa_j = \frac{H^2}{n^2} - \frac{1}{n^2(n-1)}\sum_{1\leq i<j\leq n}(\kappa_i - \kappa_j)^2,$$

we have the classical symmetric means inequality

$$\frac{H^2}{n^2}-\sigma_2=\frac{1}{n^2(n-1)}\sum_{1\le i<j\le n}(\kappa_i-\kappa_j)^2\ge 0.$$

Applying the Hsiung–Minkowski formula again, we obtain the integral identity

$$0=\int_\Sigma\Big(\frac{H}{n}+\frac{\sigma_2}{H}\langle X,\vec{H}\rangle\Big)\,d\Sigma=\int_\Sigma\Big(\frac{H}{n}-\frac{n\sigma_2}{H}\Big)\,d\Sigma=\int_\Sigma\frac{n}{H}\Big(\frac{H^2}{n^2}-\sigma_2\Big)\,d\Sigma.$$

Hence, $H^2/n^2-\sigma_2$ vanishes on Σ, which implies that $\kappa_1=\cdots=\kappa_n$ on Σ. Since Σ^n is a closed umbilic hypersurface in Euclidean space, it is a hypersphere. It follows from (6) that this hypersphere is centered at the origin. □

Lemma 11. *A minimal submanifold of the hypersphere $\mathbb{S}^{q\ge 2}$ is an expander for the inverse mean curvature flow in $\mathbb{R}^{q+1}$.*

Proof. Let $\Sigma^{p\ge 1}$ be a minimal submanifold of the hypersphere $\mathbb{S}^q\subset\mathbb{R}^{q+1}$, and let X denote the position vector field in $\mathbb{R}^{q+1}$. On the one hand, since X is already normal to the hypersphere $\mathbb{S}^q\subset\mathbb{R}^{q+1}$, we observe the equality

$$X^\perp:=X^{\perp(\Sigma\subset\mathbb{R}^{q+1})}=X.$$

On the other hand, according to the minimality of Σ^p in $\mathbb{S}^q$, we obtain

$$\triangle_g X+pX=0, \tag{7}$$

where g denotes the induced metric on Σ^p. Thus, we have

$$\vec{H}:=\vec{H}_{\Sigma\subset\mathbb{R}^{q+1}}(X)=\triangle_g X=-pX\quad\text{and}\quad |\vec{H}|=p|X|=p. \tag{8}$$

Combining the four equalities on Σ and taking $C=\dfrac{1}{p}>0$, we get

$$-\frac{1}{|\vec{H}|^2}\vec{H}=CX^\perp,$$

which indicates that Σ is an expander for the inverse mean curvature flow. □

Theorem 12. *For any integer $g\ge 1$, there exists at least one two-dimensional compact embedded expander of genus g in $\mathbb{R}^4$.*

Proof. For any integer g, Lawson [1970] showed that there exists a compact embedded minimal surface Σ of genus g in $\mathbb{S}^3$. Lemma 11 shows that Σ becomes an expander to the inverse mean curvature flow in $\mathbb{R}^4$. □

Remark 13. Castro and Lerma [2016] proved that the converse of Lemma 11 holds.

5. Expanders with rotational symmetry

In this section, we investigate homothetic solitons in $\mathbb{R}^{n+1}$ with rotational symmetry about a line through the origin. To a profile curve $\mathcal{C}$ parametrized by $(r(t), h(t))$ for $t \in I$ in the half-plane $\{(r, h) \mid r > 0, h \in \mathbb{R}\}$, we associate the rotational hypersurface in $\mathbb{R}^{n+1}$ defined by

$$\Sigma^n = \left\{ X = (r(t)\boldsymbol{p}, h(t)) \in \mathbb{R}^{n+1} \mid (r(t), h(t)) \in \mathcal{C},\ \boldsymbol{p} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \right\}.$$

The rotational hypersurface Σ satisfies the homothetic soliton (2) if and only if the profile curve $(r(t), h(t))$ satisfies the ODE

$$-\left(\frac{\dot{r}\ddot{h} - \dot{h}\ddot{r}}{(\dot{r}^2 + \dot{h}^2)^{3/2}} + \frac{n-1}{(\dot{r}^2 + \dot{h}^2)^{1/2}} \cdot \frac{\dot{h}}{r} \right) \frac{-\dot{h}r + \dot{r}h}{(\dot{r}^2 + \dot{h}^2)^{1/2}} = \frac{1}{C} \tag{9}$$

for some constant $C > 0$. We observe:

i. As long as the quantity $r\dot{h} - h\dot{r}$ is nonzero, we may write (9) as
$$\frac{\dot{r}\ddot{h} - \dot{h}\ddot{r}}{\dot{r}^2 + \dot{h}^2} = -\frac{(n-1)}{r}\dot{h} + \frac{\dot{r}^2 + \dot{h}^2}{C(r\dot{h} - h\dot{r})}.$$

ii. The ODE (9) is invariant under the dilation $(r, h) \mapsto (\lambda r, \lambda h)$, unlike the profile curve equation for shrinkers or expanders for the mean curvature flow.

iii. Spheres are expanders. The half-circle $(r(t), h(t)) = (R\cos t, R\sin t)$ with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ having the origin as its center obeys the ODE (9) with $C = 1/n$.

iv. Cylinders become expanders. The lines $r(t) = \text{constant}$ are solutions to the ODE (9) when $C = 1/(n-1)$.

v. We outline a way to deduce the ODE (9) using the homothetic soliton equation
$$\triangle_g |X|^2 = 2\left(n - \frac{1}{C}\right).$$

We observe that Σ is a homothetic soliton with rotational symmetry if and only if

$$2\left(n - \frac{1}{C}\right) = \triangle_g(r^2 + h^2) = \frac{1}{r^{n-1}\left(\dot{r}^2 + \dot{h}^2\right)^{1/2}} \frac{d}{dt}\left(\frac{r^{n-1}}{\left(\dot{r}^2 + \dot{h}^2\right)^{1/2}} \frac{d}{dt}(r^2 + h^2) \right), \tag{10}$$

which is equivalent to (9).

5.1. ***Construction of expanding infinite bottles.*** Writing the profile curve $\mathcal{C}$ as a graph $(r(h), h)$, we have the second-order nonlinear differential equation

$$\frac{r''}{1 + r'^2} = \frac{n-1}{r} - \frac{1 + r'^2}{C(r - hr')}. \tag{11}$$

When $C = 1/(n-1)$, this equation becomes

$$\frac{r''}{1+r'^2} = (n-1)\left(\frac{1}{r} - \frac{1+r'^2}{r-hr'}\right). \tag{12}$$

Observe that $r(h) =$ constant is a solution to (12), which corresponds to a round hypercylinder expander. Moreover, if $r(h)$ is a solution to (12) with $r'(a) = 0$ for some $a \in \mathbb{R}$, then $r(h) \equiv r(a)$. Consequently, any nonconstant solution to (12) must be strictly monotone.

In this section, we construct new examples of entire solutions to (12), which correspond to hypercylinder expanders that interpolate between two concentric round hypercylinders.

Theorem 14 (construction of infinite bottles). *Let r_0, h_0, and r'_0 be constants satisfying $r_0 > 0$, $h_0 < 0$, and $r'_0 \in (0, -h_0/r_0)$, and let $r(h)$ be the unique solution to (12) satisfying the initial conditions $r(h_0) = r_0$ and $r'(h_0) = r'_0$. Then $r(h)$ is an entire solution, and there are constants $0 < r_{\text{bot}} < r_{\text{top}} < \infty$ such that $r(h)$ interpolates between r_{bot} and r_{top}. More precisely, $r(h)$ is strictly increasing, $\lim_{h\to-\infty} r(h) = r_{\text{bot}}$, $\lim_{h\to\infty} r(h) = r_{\text{top}}$, and there exists a point $h_1 \in (h_0, 0)$ such that $r''(h_1) = 0$ and $r''(h)$ has the same sign as $(h_1 - h)$ when $h \neq h_1$.*

Proof. We separate the proof into two parts. First, we show that the solution is entire and increasing, and there is a unique point where the concavity changes sign. Second, we establish estimates that bound the solution between two positive constants. We note that the rotation of the profile curve about the h-axis has the appearance of an infinite bottle, which interpolates between two concentric cylinders.

Part 1: Existence of expanding infinite bottles.

Notice that the condition $r'(h_0) = r'_0 > 0$ shows that r is a nonconstant solution and guarantees that $r'(h) > 0$. Also, observe that the assumption $r'_0 \in (0, -h_0/r_0)$ coupled with the defining initial conditions for $r(h)$ shows that $h + r'r$ is negative at $h = h_0$. In fact, by assumption, the terms r', $-h - r'r$, r, and $r - hr'$ are all positive at $h = h_0$. So, writing (12) as

$$r'' = (n-1)(1+r'^2)\frac{r'(-h-r'r)}{r(r-hr')}, \tag{13}$$

we see that $r''(h_0) > 0$.

In the following lemma, we show that the concavity of $r(h)$ changes sign exactly once when $r(h)$ is a maximally extended solution.

Lemma 15 (existence of a unique inflection point). *Let $r : (h_{\min}, h_{\max}) \to \mathbb{R}^+$ be a maximally extended solution. Then there exists a point $h_1 \in (h_0, 0)$ such that $r''(h_1) = 0$. Furthermore, $r''(h)$ has the same sign as $(h_1 - h)$ when $h \neq h_1$.*

Proof. Step A. We claim that there exists a point $h_1 \in (h_0, 0)$ such that $r''(h_1) = 0$. We first treat the case where $h_{\max} \le 0$. In this case, proving the claim is equivalent to showing there is a point $h_1 \in (h_0, h_{\max})$ such that $r''(h_1) = 0$. Suppose to the contrary that

$$r''(h) > 0 \quad \text{for all } h \in (h_0, h_{\max}).$$

As $h_{\max} \le 0$ and both r and r' are positive, we have $(r - hr') > 0$ for $h \in (h_0, h_{\max})$. In fact, since $(d/dh)(r - hr') = -hr'' > 0$, we see that $(r - hr') > r_0 - h_0 r_0'$. Using (13) and the positivity of the functions r, r', $(r - hr')$, and r'', we arrive at the inequality $(-h - rr') > 0$, which leads to the estimate

$$0 < r'(h) < -\frac{h}{r} < -\frac{h_0}{r_0} \quad \text{for all } h \in (h_0, h_{\max}).$$

Now, returning to (13), we have the estimate

$$0 \le r''(h) = (n-1)(1 + r'^2)\frac{r'(-h - r'r)}{r(r - hr')} \le (n-1)\left(1 + \left(\frac{h_0}{r_0}\right)^2\right)\frac{(-h_0/r_0)(-h_0)}{r_0(r_0 - h_0 r_0')}$$

for $h \in (h_0, h_{\max})$. These estimates contradict the finiteness of the maximal endpoint $h_{\max}$, and we conclude that the claim is true in the case where $h_{\max} \le 0$.

It still remains to prove the claim in the case where $h_{\max} > 0$. However, in this case the solution $r(h)$ is defined when $h = 0$, and (12) implies

$$r''(0) = -(n-1)\frac{r'(0)^2}{r(0)}(1 + r'(0)^2) < 0.$$

It follows that there exists a point $h_1 \in (h_0, 0)$ such that $r''(h_1) = 0$.

Step B. We claim that $r''(h)$ has the same sign as $h_1 - h$. Taking a derivative of (11), we have

$$\frac{r'''}{1 + r'^2} = \frac{2r'(r'')^2}{(1 + r'^2)^2} - \frac{n-1}{r^2}r' - \frac{2r'r''}{C(r - hr')} - \frac{1 + r'^2}{C(r - hr')^2}hr''.$$

At the point h_1, we obtain

$$\frac{r'''(h_1)}{1 + r'(h_1)^2} = -(n-1)\frac{r'(h_1)}{r(h_1)^2} < 0,$$

which shows that $r''(h)$ has the same sign as $h_1 - h$ in a neighborhood of h_1. In fact, at any point $\bar{h}$ where $r''(\bar{h}) = 0$, we have $r'''(\bar{h}) < 0$. This property tells us that the sign of r'' can only change from positive to negative, and consequently r'' vanishes at most once. Thus, $r''(h)$ has the same sign as $h_1 - h$ for all $h \in (h_{\min}, h_{\max})$. □

Next, we prove that the profile curves corresponding to the infinite bottles come from entire graphs.

Lemma 16 (existence of entire solutions). *We have $h_{\min} = -\infty$ and $h_{\max} = \infty$.*

Proof. Step A. We claim that $h_{\max} = \infty$. First, we show that $h_{\max} > 0$. To see this, notice that $0 \le r'(h) \le r'(h_1)$, $r(h) \ge r_0$, and $r - hr' \ge r_0$ whenever $h_1 \le h \le 0$. It follows from (12) that the solution $r(h)$ can be extended past $h \le 0$. Thus, $h_{\max} > 0$. Next, we show that $h_{\max} = \infty$. Since $h_1 < 0$, we have $(d/dh)(r - hr') = -hr'' \ge 0$ when $h \ge 0$ so that $(r - hr') \ge r(0)$ when $h \ge 0$. We also have $0 \le r'(h) \le r'(h_1)$ and $r(h) \ge r_0$ when $h \ge 0$. As before, it follows from (12) that the solution $r(h)$ can be extended past any finite point.

Step B. We claim that $h_{\min} = -\infty$. Suppose to the contrary that $h_{\min} > -\infty$. Then at least one of the functions r', $1/r$, or $1/(r - hr')$ must blow up at the finite point $h = h_{\min}$. Since $r'' > 0$ on $(h_{\min}, h_1)$, the positive function r' is increasing, and we have $r'(h) \le r'(h_0) = r_0'$ for all $h \in (h_{\min}, h_0)$. So, the function r' does not blow up at $h_{\min}$. If the function $1/r$ is bounded above on $(h_{\min}, h_0)$, then the inequality $0 < r(h) < r(h) - hr'(h)$ (when $h \le 0$) guarantees that $1/(r - hr')$ is also bounded above on $(h_{\min}, h_0)$, in which case, the solution can be extended prior to $h_{\min}$. Therefore, the function $1/r$ must blow up at $h = h_{\min}$. In other words,

$$\lim_{h \to {h_{\min}}^+} r(h) = 0.$$

Observing this and using $0 < r'(h) < r_0'$ on $(h_{\min}, h_0)$, we can find a sufficiently small $\delta > 0$ so that $r'(h)r(h) \ge -h_0/2$ for all $h \in (h_{\min}, h_{\min} + \delta]$. Also, the inequality $(d/dh)(r - hr') = -hr'' > 0$ guarantees that

$$0 < r(h) - hr'(h) \le \epsilon_1 := r(h_{\min} + \delta) - (h_{\min} + \delta) r'(h_{\min} + \delta).$$

It follows from these estimates and (12) that

$$\frac{d}{dh}(\arctan r') = \frac{r''}{1 + r'^2} = (n-1)\frac{-(h + r'r)}{r - hr'} \cdot \frac{r'}{r} \ge \epsilon_2 \frac{d}{dh}(\ln r),$$

where

$$\epsilon_2 = \frac{(n-1)(-h_0/2)}{\epsilon_1} > 0$$

is a constant. Hence, the function $F(h) := \arctan(dr/dh) - \epsilon_2 \ln r(h)$ is increasing on $(h_{\min}, h_{\min} + \delta]$. Thus, we have the estimate

$$\epsilon_2 \ln r(h) \ge -F(h_{\min} + \delta) + \arctan r' > -F(h_{\min} + \delta).$$

Taking the limit as $h \to {h_{\min}}^+$ and using $\lim_{h \to {h_{\min}}^+} r(h) = 0$ leads to a contradiction. We conclude that $h_{\min} = -\infty$. $\square$

So far, we have proved the existence of an entire bottle solution $r(h)$ to (12). In the next part of the proof we will establish estimates that squeeze the ends of the infinite bottles between two cylinders.

Part 2: Squeezing infinite bottles by two hypercylinders.

To establish upper and lower bounds for the solution $r(h)$, we study the profile curve $\mathcal{C}$ by writing it as a graph over the axis of rotation: $(r, h(r))$. Then, we have the second-order nonlinear differential equation

$$\frac{h''}{1+h'^2} = -\frac{(n-1)}{r}h' + \frac{1+h'^2}{C(rh'-h)}, \tag{14}$$

or equivalently,

$$\frac{h''}{1+h'^2} = \frac{(n-1)hh' + \frac{1}{C}r}{r(rh'-h)} + \left(\frac{1}{C} - (n-1)\right)\frac{h'^2}{(rh'-h)}. \tag{15}$$

Throughout this section, we take $C = 1/(n-1)$, so that (14) takes the form

$$\frac{h''}{1+h'^2} = -(n-1)\left(\frac{h'}{r} - \frac{1+h'^2}{rh'-h}\right) = \frac{n-1}{r}\cdot\frac{r+hh'}{rh'-h}. \tag{16}$$

Now, let $h(r)$ be a maximally extended solution to (16) defined on $(r_{\text{bot}}, r_{\text{top}})$. Lemma 15 tells us that there is a point $r_1 \in (r_{\text{bot}}, r_{\text{top}})$ such that $h'(r) > 0$ and $h''(r) > 0$ for all $r \in (r_1, r_{\text{top}})$ and that $r_1 h'(r_1) - h(r_1) > 0$.

Lemma 17 (existence of the outside cylinder barrier). *We have*

$$r_{\text{top}} < \infty, \qquad \lim_{r\to r_{\text{top}}^-} h'(r) = \infty, \quad \textit{and} \quad \lim_{r\to r_{\text{top}}^-} h(r) = \infty.$$

Proof. We introduce the angle functions $\theta, \phi : (r_1, r_{\text{top}}) \to \left(0, \frac{\pi}{2}\right]$, defined by

$$\theta(r) = \arctan\frac{dh}{dr} \quad \text{and} \quad \phi(r) = \arctan\frac{h}{r},$$

to rewrite the profile curve (16) as

$$\frac{d\theta}{dr} = \frac{n-1}{r\cdot\tan(\theta-\phi)}. \tag{17}$$

Combining this and $0 < \tan(\theta-\phi) \le \tan\theta$, we have $d\theta/dr \ge (n-1)/(r\cdot\tan\theta)$, which implies

$$\frac{d}{dr}\left(\frac{\tan\theta}{r^{n-1}}\right) \ge \frac{n-1}{r^n\tan\theta} \ge 0.$$

This tells us that the continuous function $(\tan\theta)/r^{n-1}$ is increasing for $r > r_1$. Set $\theta_1 = \theta(r_1)$. According to the estimate

$$\frac{d}{dr}\left(h - \frac{\tan\theta_1}{nr_1^{n-1}}r^n\right) = \tan\theta - \frac{\tan\theta_1}{r_1^{n-1}}r^{n-1} = \left(\frac{\tan\theta}{r^{n-1}} - \frac{\tan\theta_1}{r_1^{n-1}}\right)r^{n-1} \ge 0,$$

we see that the function

$$h - \frac{\tan\theta_1}{nr_1^{n-1}}r^n$$

is increasing. In particular, we have the height estimate

$$h \geq h_1 + \frac{\tan\theta_1}{n{r_1}^{n-1}}(r^n - r_1^n).$$

Observe that

$$\frac{1}{\tan(\theta-\phi)} = \frac{1+\tan\theta\tan\phi}{\tan\theta-\tan\phi} \geq \tan\phi.$$

Combining this with (17), we have

$$\frac{1}{n-1}\frac{d\theta}{dr} \geq \frac{\tan\phi}{r} = \frac{h}{r^2} \geq \frac{1}{r^2}\left(h_1 + \frac{\tan\theta_1}{n{r_1}^{n-1}}(r^n - r_1^n)\right),$$

which implies

$$\frac{d}{dr}\left(\frac{\theta}{n-1} + \left(h_1 - \frac{\tan\theta_1}{n}r_1\right)\frac{1}{r} - \frac{\tan\theta_1}{n(n-1)r_1^{n-1}}r^{n-1}\right) \geq 0.$$

Therefore, the function

$$F(r) = \frac{\theta}{n-1} + \left(h_1 - \frac{\tan\theta_1}{n}r_1\right)\frac{1}{r} - \frac{\tan\theta_1}{n(n-1)r_1^{n-1}}r^{n-1}$$

is increasing, and for all $r \in (r_1, r_{\text{top}})$, we have

$$\frac{\theta}{n-1} \geq F(r_1) - \left(h_1 - \frac{\tan\theta_1}{n}r_1\right)\frac{1}{r} + \frac{\tan\theta_1}{n(n-1)r_1^{n-1}}r^{n-1}.$$

Since the left-hand side is bounded above, and the right-hand side becomes arbitrarily large as r goes to ∞, we conclude that $r_{\text{top}} < \infty$. It then follows that the increasing, concave up function $h(r)$ satisfies $\lim_{r\to r_{\text{top}}^-} h'(r) = \infty$. If $h(r)$ has a finite limit as r approaches r_{top}, then by the uniqueness of the cylinder $r(h) \equiv r_{\text{top}}$, we get a contradiction. Therefore, we also have $\lim_{r\to r_{\text{top}}^-} h(r) = \infty$. □

Next, we prove the following lemma, which shows that a solution with $h < 0$, $h' > 0$, and $h'' < 0$ cannot approach the axis of rotation.

Lemma 18 (existence of the inside cylinder barrier). *We have*

$$r_{\text{bot}} > 0, \qquad \lim_{r\to r_{\text{bot}}^+} h'(r) = \infty, \qquad \textit{and} \qquad \lim_{r\to r_{\text{bot}}^+} h(r) = -\infty.$$

Proof. We first observe that $h - rh' < 0$ and $hh' < 0$. We introduce three well-defined functions $\theta : (r_{\text{bot}}, r_0] \to \left(0, \frac{\pi}{2}\right]$ and $\Psi_1, \Psi_2 : (r_{\text{bot}}, r_0] \to \mathbb{R}$ defined by

$$\theta(r) = \arctan\frac{dh}{dr}, \qquad \Psi_1(r) = \frac{-hh'}{rh'-h}, \qquad \text{and} \qquad \Psi_2(r) = \frac{r+hh'}{hh'},$$

and we rewrite the profile curve (16) as

$$\frac{d\theta}{dr} = -\frac{n-1}{r}\Psi_1\Psi_2. \tag{18}$$

Using the estimate

$$\frac{d\Psi_1}{dr} = \frac{-r(h')^3+h((h')^2+hh'')}{(h-rh')^2} \le 0,$$

we see that Ψ_1 is decreasing on $(r_{\text{bot}}, r_0]$, and setting $\epsilon_1 = \Psi_1(r_0)$, we have

$$\Psi_1(r) \ge \epsilon_1 > 0. \tag{19}$$

Observing $(hh')' = h'^2 + h''h > 0$ and defining a constant $\epsilon_2 = -h(r_0)h'(r_0) > 0$, we have the estimate $hh' \le -\epsilon_2$ for all $r \in (r_{\text{bot}}, r_0]$. It follows that

$$\Psi_2(r) = 1 + \frac{r}{hh'} \ge 1 - \frac{r}{\epsilon_2}. \tag{20}$$

Combining (18), (19), and (20), we have

$$\frac{d}{dr}\Big(\frac{\theta}{(n-1)\epsilon_1} + \ln r - \frac{r}{\epsilon_2}\Big) \le 0.$$

Therefore, the function $\Psi(r) = \dfrac{\theta}{(n-1)\epsilon_1} + \ln r - \dfrac{r}{\epsilon_2}$ is decreasing, and for all $r \in (r_{\text{bot}}, r_0]$, we have

$$\frac{\theta}{(n-1)\epsilon_1} \ge -\ln r + \frac{r}{\epsilon_2} + \Psi(r_0).$$

Since the left-hand side is bounded above, and the right-hand side becomes arbitrarily large as r goes to 0, we conclude that $r_{\text{bot}} > 0$. It then follows that the increasing, concave down function $h(r)$ satisfies $\lim_{r\to r_{\text{bot}}^+} h'(r) = \infty$. If $h(r)$ has a finite limit as r approaches r_{bot}, then by comparison with the cylinder $r(h) \equiv r_{\text{bot}}$, we get a contradiction. Therefore, we also have $\lim_{r\to r_{\text{bot}}^+} h(r) = -\infty$.

This completes the proof of both the lemma and Theorem 14. □

5.2. *Other examples of complete solitons.* Huisken and Ilmanen [1997a] used a phase-plane analysis to exhibit complete, rotationally symmetric expanders for the inverse mean curvature flow which are topological hyperplanes. For each $C > 1/n$, they showed there exists a half-entire solution to (11) which intersects the h-axis perpendicularly, and they provided numeric descriptions of these profile curves. For $C > 1/n$ and $C \ne 1/(n-1)$, they also indicated the existence of entire solutions to (11) which are symmetric about the r-axis and correspond to topological hypercylinders. (We note that the rotational expander constructed in Theorem 14 is nonsymmetric in the sense that its profile curve is not symmetric about the r-axis.) In this section, we explain how the techniques from Section 5.1 can be used to recover the examples and numeric pictures presented in [Huisken and Ilmanen 1997a].

Hyperplane expanders. We begin by considering the initial value problem where we shoot perpendicularly to the axis of rotation. For $C > 0$, let $h(r)$ be a solution to (14) with $h(0) = h_0 < 0$ and $h'(0) = 0$. This singular shooting problem is well-defined (see [Baouendi and Goulaouic 1976] and [Drugan 2015]), and the solution satisfies $h''(0) = -1/(nCh_0) > 0$. Differentiating (14) and analyzing the equation for $h'''(r)$ shows that, under the above conditions, we have $h''(r) > 0$ and $h'(r) > 0$, for $r > 0$, as long as the solution is defined. The global behavior of the solution ultimately depends on the value of C.

When $h(r)$ is a solution to the above shooting problem, the graph $(r, h(r))$ is part of a profile curve $\mathcal{C}$, which corresponds to a rotational expander for the inverse mean curvature flow. Applying the techniques from the proof of Theorem 14 to the profile curve $\mathcal{C}$ leads to a description of the global behavior of this expander, which ultimately depends on the value of $C > 1/n$. In terms of the profile curve $\mathcal{C}$ written as a graph over the h-axis, we have the following result.

Theorem 19. *For $C > 1/n$ and $h_0 < 0$, there exists a half-entire solution $r(h)$ to (11) that is defined for $h > h_0$, and such that the curve $(h, r(h))$ intersects the h-axis perpendicularly when $h = h_0$. The solution $r(h)$ has three types of behavior, depending on the value of C*:

(1) *If $C = 1/(n-1)$, then $r' > 0$, $r'' < 0$, and there exists $0 < r_{\text{top}} < \infty$ such that $\lim_{h\to\infty} r(h) = r_{\text{top}}$.*

(2) *If $C > 1/(n-1)$, then $r' > 0$, $r'' < 0$, and $\lim_{h\to\infty} r(h) = \infty$.*

(3) *If $1/n < C < 1/(n-1)$, then there exists a point h_1 such that $r''(h)$ has the same sign as $(h - h_1)$, and $\lim_{h\to\infty} r(h) = 0$.*

Proof. When $C = 1/(n-1)$, the convexity of $h(r)$ along with the analysis from Lemma 17 shows that there is a point $r_{\text{top}} < \infty$ such that $\lim_{r\to r_{\text{top}}^-} h'(r) = \infty$ and $\lim_{r\to r_{\text{top}}^-} h(r) = \infty$. Written as a graph over the h-axis, this shows that there is a solution $r(h)$ to (11), defined for $h > h_0$, which intersects the h-axis perpendicularly at h_0 and satisfies $r' > 0$, $r'' < 0$, and $\lim_{h\to\infty} r(h) = r_{\text{top}}$.

Next, when $C > 1/(n-1)$, we claim that the solution $h(r)$ must exist for all $r > 0$. To see this, suppose to the contrary that h' increases to ∞ at a point $r_{\text{top}} < \infty$. Then, since $C > 1/(n-1)$, (14) forces $h \geq \epsilon r h'$ when r is close to r_{top}, for some $\epsilon > 0$. However, integrating this inequality shows that h' does not blow up at a finite point; hence the solution exists for all $r > 0$. Therefore, the solution $h(r)$ exists for all $r > 0$, and using $h'' > 0$ and $h' > 0$, we have $\lim_{r\to\infty} h(r) = \infty$. Written as a graph over the h-axis, this shows that there is a solution $r(h)$ to (11), defined for $h > h_0$, which intersects the h-axis perpendicularly at h_0 and satisfies $r' > 0$, $r'' < 0$, and $\lim_{h\to\infty} r(h) = \infty$.

Finally, when $1/n < C < 1/(n-1)$, the factor $\frac{1}{C} - (n-1)$ in (15) is positive and the analysis in Lemma 17 can be used to show that $h(r)$ does not exist for all

$r > 0$. Moreover, using the positivity of $\frac{1}{C} - (n-1)$ and integrating (14), we arrive at an inequality that provides an upper bound for h. In terms of the profile curve written as a graph over the h-axis, this says that the solution $r(h)$ achieves a global maximum at a finite point. Reading (9) in polar coordinates, we can show that $r(h)$ is defined for $h > h_0$. This forces the concavity of $r(h)$ to change sign at a finite point, and as in the proof of Lemma 15, it follows that there is a point h_1 such that $r''(h)$ has the same sign as $(h - h_1)$. Then, an argument similar to the one in the previous paragraph shows that $r(h)$ is not bounded below by a positive constant, and we conclude that $\lim_{h\to\infty} r(h) = 0$. □

We remark that when $1/n < C < 1/(n-1)$, the analogue of Lemma 17 holds, but as we saw in the proof of the previous theorem, the analogue of Lemma 18 is not true. Similarly, if $C > 1/(n-1)$, then the analogue of Lemma 18 holds, but the analogue of Lemma 17 does not.

Hypercylinder expanders. We finish this section with a result on the construction of rotational expanders that are topological hypercylinders.

Theorem 20. *For $C > 1/n$ and $r_0 > 0$, there is a unique solution $r(h)$ to (11) that is symmetric about the r-axis and satisfies the initial condition $r(0) = r_0$, $r'(0) = 0$. The solution $r(h)$ has three types of behavior, depending on the value of C:*

(1) *If $C = 1/(n-1)$, then $r(h) \equiv r_0$ (which corresponds to the round hypercylinder).*

(2) *If $C > 1/(n-1)$, then $r(h)$ has a global minimum at $h = 0$, and there exists a point $h_1 > 0$ such that $r''(h)$ has the same sign as $(h_1 - |h|)$. Also, $\lim_{h\to\infty} r(h) = \infty$.*

(3) *If $1/n < C < 1/(n-1)$, then $r(h)$ has a global maximum at $h = 0$, and there exists a point $h_1 > 0$ such that $r''(h)$ has the same sign as $(|h| - h_1)$. Also, $\lim_{h\to\infty} r(h) = 0$.*

Proof. It follows from (11) that the condition $r'(0) = 0$ forces the solution to be constant when $C = 1/(n-1)$, to have a global minimum at $h = 0$ when $C > 1/(n-1)$, and to have a global maximum at $h = 0$ when $1/n < C < 1/(n-1)$. To see that there is a finite point $h_1 > 0$ where the concavity of $r(h)$ changes sign when $C > 1/(n-1)$, we first observe that $r(h)$ is increasing when $h > 0$, and consequently, it is defined for all $h > 0$. An analysis of (14) shows that a positive solution $h(r)$ cannot satisfy $h''(r) < 0$ and $h'(r) > 0$ for all $r > 0$ when $C > 1/(n-1)$; hence, there is a finite point $h_1 > 0$ where the concavity of $r(h)$ changes sign. When $1/n < C < 1/(n-1)$, the analysis in the proof of Theorem 19 can be used to show that the concavity of $r(h)$ changes sign at a finite point $h_1 > 0$. The proofs of the remaining properties are similar to the proofs given for Theorems 14 and 19. □

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