Pricing American-style Parisian options

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Certification

I, LE Nhat Tan, declare that this thesis, submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

LE Nhat Tan
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Barrier options are the most common path-dependent options traded in financial markets. They are particularly attractive to investors, because not only are they cheaper than vanilla options but they also offer different choices of investment, which allow investors to bet their views on the movement of the underlying asset prices. The “one-touch” breaching barrier however is prone to market manipulations which can be made by influential agents in order to free them from their liabilities to the option holders. Aiming to prevent such market manipulations, Parisian options were introduced, with an extended trigger clause, which makes the knock-in or knock-out feature much harder to be activated. Pricing Parisian options has become an increasingly important problem from both financial and mathematical perspectives. Financially, the introduction of Parisian options, which makes the market fairer in the sense that it protects the holder of Parisian options from deliberate action taken by the writer, requires an efficient way to precisely evaluate the option prices. On the other hand, due to the presence of the newly-added trigger clause, the valuation of Parisian options becomes a three-dimensional problem, a challenging problem, which has hindered the application of various mathematical methods. In this thesis, we explore the integral equation method and the “moving window” technique to price different types of Parisian options under the Black-Scholes framework.

Firstly, we price an American-style down-and-out call, which can be considered as a special Parisian option with zero “option window”. Instead of using the probability theory as used in the literature, we use the continuous Fourier sine transform to solve the partial differential equation system governing the option price. As a way of validating our approach, we show that the “early exercise premium representation” for American-style down-and-out calls without rebate can be re-derived by using our approach. We then examine the case that time-dependent rebates are included in the contract of American-style down-and-out calls. As a result, a more general integral representation for the price of an American-style down-and-out call, with the presence of an extra term associated with the rebate, can be obtained. Our numerical method based on the newly-derived integral representation appears to be efficient.
in computing the price and the hedging parameters for American-style down-and-out calls with rebates. In addition, significant effects of rebates on the option prices and the optimal exercise boundaries are illustrated through selected numerical results.

Secondly, in Chapters 4 and 5 we price two different types of American-style Parisian knock-in call options: up-type and down-type, respectively. Usually, pricing an American-style option is much more difficult than pricing its European-style counterpart because of the appearance of the optimal exercise boundary in the former. Fortunately, the optimal exercise boundary associated with an American-style Parisian knock-in option only appears implicitly in its pricing partial differential equation systems, instead of explicitly as in the case of an American-style Parisian knock-out option. As a result, the “moving window” technique developed for pricing European-style Parisian knock-out call options can be adopted to price American-style Parisian knock-in options as well. In particular, we have obtained simple analytical solutions for American-style Parisian knock-in call options, which can be easily computed numerically.

Thirdly, in Chapters 6 and 7 we propose an integral equation approach for pricing two different types of American-style Parisian knock-out call options: up-type and down-type, respectively. The corresponding three-dimensional pricing problem is first reduced to solving a pair of coupled two-dimensional partial differential equations by applying the “moving window” technique. The newly-derived two-dimensional systems are then analytically solved separately as if they were not coupled. As a result, we can obtain integral representations for the option prices at any asset price, in terms of unknown quantities: the option prices at the asset barrier and the optimal exercise prices. These unknown quantities are in turn governed by a pair of coupled integral equations, which can be efficiently solved by using the Newton-Raphson iterative procedure. Consequently, the option prices and the hedging parameters can be obtained both accurately and efficiently. Numerical results are also examined in order to provide new insight into interesting features about prices of American-style Parisian knock-out call options and the behavior of the associated optimal exercise boundaries.

Finally, in the last chapter, we briefly summarize the main results achieved in this thesis and propose future research directions to extend these results.
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Chapter 1

Introduction

1.1 Options and option pricing problems

Since the establishment of the first exchange-traded options market, the Chicago Board of Options Exchange (CBOE), in 1973, that the option industry has developed rapidly. More specifically, the total option trading volumes in the six options exchanges in the US (CBOE, American Stock Exchange, Pacific Stock Exchange, Philadelphia Stock Exchange, Midwest Stock Exchange and New York Stock Exchange) increased more than one hundred times from 1973 to 1982 [80, p. 6]. Huge volumes of options, worth many billions of US dollars, have been now traded daily on both global exchanges and over-the-counter markets [47, 80]. As pointed out by Hunt and Kennedy [48], there have been two main factors behind this phenomenal growth.

The first factor is that options can fulfil two of the main demands from investors: hedging and speculating. Nowadays, many individuals and financial organizations are exposed to unavoidable financial risks associated with the movement of the world markets, which are beyond their control. For instance, companies are usually susceptible to the increase in the price of the raw materials; multinational corporations are exposed to the unfavourable changes in exchange rates; pension funds are exposed to high inflation rates and low interest rates. Suitable options can not only reduce effectively these risks but also help the option holders gain profits when the markets moves favourably. Alternatively, options can also be used for speculation. Typically, investors have their own views about the movement of markets. Depending on whether their views are right or wrong, they can earn high profits or suffer
great losses. By investing in options, investors may earn high profits but at a much cheaper cost, in comparison with the cost of investing directly in the underlying assets. For example, if investors believe the price of some stocks will go up for a certain period of time, then using suitable call options can produce a higher return for the investors than buying the stocks directly [47, p. 14].

The second factor is the parallel development of the financial mathematics, which provides powerful tools for pricing options. Since the introduction of the Black-Scholes formula [6], the option pricing theory has experienced rapid growth and become a major area in today’s quantitative finance research. To meet various needs of customers, variety of option types have been created by adding additional features to the financial contracts of plain vanilla options. Consequently, the corresponding pricing problems have become much more challenging and thus there are obvious needs for more research effort on how to determine the reasonable prices of newly-created options.

1.2 Literature review

There are many types of options in the option markets. In this thesis, we however limit ourselves to three common option types: vanilla options, barrier options and Parisian options, written on stocks. This section will present the literature review of the pricing of these options under the well-known Black-Scholes framework.

1.2.1 Vanilla options

A European vanilla call (put) option is a financial contract that gives the option holder (the buyer) a right, but no obligation, to buy (sell) a certain amount of a specified asset (the underlying asset) at a predetermined price (the exercise price) only on a certain date (the expiry date). The option writer (the seller), on the other hand, has an obligation to take part in the transaction if the option holder decides to exercise the option. It is clear that the option writer might suffer an arbitrarily large loss if the market goes unfavorably against the option writer. To compensate for this potential risk, the option writer should be paid up front some payment, which is called an option price. A fair option price, which does not allow the existence of any risk-free arbitrage opportunity, needs to be determined before the option can
be traded.

It is clear that the value of a European vanilla option depends heavily on the future random movement of the underlying stock price. For instance, if the stock price increases and ends up at a position that is far beyond the exercise price, then the call option holder will earn great profit while the holder of the corresponding put option will earn nothing. As a result, in this case, the call option has a great value, while the put option is worth almost nothing. A mathematical model that reasonably describes the behavior of the stock price is obviously needed to analytically price European vanilla options.

In 1973, Black and Scholes [6] and Merton [62] proposed such a model, which assumes that the underlying stock price follows a geometric Brownian motion with constant drift and volatility. This model, which is known as the Black-Scholes model (or Black-Scholes-Merton model), is perhaps the world’s most well-known option pricing model. An elegant formula for pricing European vanilla call, under the Black-Scholes framework, was derived by Black and Scholes [6], and Merton [62]. This formula has been widely used in global financial markets by traders and investors to calculate the theoretical price of European options, which has been demonstrated to be very close to the observed market prices. It is the introduction of this formula that has led to the development of pricing formulas for more complicated options, such as American vanilla options.

American vanilla options are very similar to their European option counterparts, except that they can be exercised at any time before and up to expiry. It is the additional flexibility of the early exercise right that makes American vanilla options become more valuable than their European option counterparts. On the other hand, this additional right has also caused much greater difficulties for pricing American-style options than pricing their European option counterparts [46, 51, 59, 73]. More precisely, this early exercise right has changed the pricing problem of American-style options into a so-called free boundary problem because the boundary of the pricing domain usually varies with time and needs to be determined as part of the solution. The valuation of American-style options therefore becomes a highly nonlinear problem and far more difficult to deal with, in comparison with European-style options.

In 2006, Zhu [81] made a great breakthrough by successfully showing the existence of an exact solution for the pricing problem of American vanilla options. A key idea behind the
author's approach is to reduce the original highly nonlinear pricing problem to a solvable linear one. As a result, an analytical pricing formula, in the form of a Taylor series expansion, for the price of American put options can be obtained. By evaluating this closed-form pricing formula, the price of the vanilla options can be achieved within any desired accuracy, a feature that none of the existing approximation methods can match. To evaluate the pricing formula however takes a relatively long time.

By contrast, approximation methods usually produce the prices of vanilla options faster with acceptable accuracy. In the literature, there are predominately two types of approximation methods for the valuation of an American-style option. They are numerical methods and analytical approximations. The numerical methods typically include: the finite difference method \[63, 73, 79, 84\], the binomial tree method \[25\], the moving boundary approach \[65\], the Monte Carlo simulation technique \[36\], and the least square approach \[59\]. On the other hand, the analytical approximations commonly are: the compound-option approximation method \[38\], the quadratic approximation method \[4, 60\], the randomization approach \[13\], the integral equation method \[20, 21, 23, 49, 52, 54, 61\], and the Laplace transform method \[82\]. Even though the resulting formulas obtained from these analytical approximation methods still require a certain degree of computation to numerically realize the solution at the end, the computational workload is reduced significantly, compared with the numerical methods.

Among the above approximation methods, the integral equation method has been a very useful tool for pricing American vanilla options. For instance, Kim \[54\] formulated the optimal exercise boundary of an American vanilla option as an integral equation, which shows clearly that the value of an American vanilla option is equal to that of the corresponding European option plus an early exercise premium. The quantity of this extra premium showing the difference between the values of an American vanilla option and its European counterpart however is not easy to achieve at all from other approximation methods. By using Gaussian quadrature, the integral equation can be cast into a system of algebraic equations, which can be easily solved using the Newton-Raphson iterative procedure \[52\]. The resulting numerical solution, which reveals interesting features about the price of an American call option and the behavior of the associated free boundary, fits well with the results gained from other methods,
such as the binomial method and finite difference method \[20\].

### 1.2.2 Barrier options

Theoretically, the holder of a vanilla option can earn large profit if the underlying stock price goes far beyond the exercise price, i.e., increases to infinity or decreases to zero. In practice, the stock price however usually varies only around the exercise price. In exchange for a cheaper price, some investors therefore are willing to lose the exercise right of the vanilla option in the event that the underlying stock price touches a certain price level, the asset barrier, which could be above or below the exercise price. To meet such a demand, barrier options were introduced in the option market.

A barrier option is an option whose payoff depends on whether or not the underlying asset price momentarily touches a pre-specified level, the barrier, during the life of the option. Barrier options can be classified as either knock-out options or knock-in options. A knock-out option is very similar to its vanilla counterpart, except that it will cease to exist when the underlying asset price reaches a predetermined constant barrier. A knock-in option, on the other hand, becomes the embedded vanilla option only if the knock-in feature is activated, i.e., the underlying asset price touches a predetermined constant barrier. It should be emphasized that the holder of the knock-in option does not have any exercise right to buy or sell the underlying stock until the knock-in feature is activated. Barrier options can be further categorized as either down-type options (i.e., down-and-in options or down-and-out options) or up-type options (i.e., up-and-in options or up-and-out options), depending on whether the barrier is set below or above the underlying price at inception, respectively. All these contracts exist in the form of puts and calls. It should be noted that the knock-out or knock-in feature will be activated immediately when the price of the underlying asset momentarily touches the asset barrier, no matter how briefly the breaching occurs. That is why this type of barrier options is also called “one-touch” barrier options.

Like the relationship between an American vanilla option and its European counterpart, the valuation problem of American barrier options in general is much more difficult than that of their European counterparts. While a simple closed-form solution of the latter has already been found by Merton \[62\], Rich \[68\], Rubinstein and Reiner \[70\], no such simple solution exists
for the former. The extra difficulty of pricing American barrier options, in comparison with their European counterparts, mainly stems from the early exercise right, which has changed the pricing problem of American barrier options into a so-called free boundary problem.

It should also be noted that the sum of the prices of a European down-and-in option and its European down-and-out option counterpart is equal to the price of the embedded European vanilla option (cf. [56, 78]). The value of a European down-and-out option can therefore be easily found once the value of its down-and-in option counterpart is available, or vice versa. Such “in-out parity” relation however does not hold in the case of American barrier options (cf. [17, 27]). Thus, the values of both American knock-out options and their knock-in counterparts are usually solved separately, using different pricing solution procedures. It is interesting to mention here that the level of complexity of pricing American-style barrier options varies a lot between knock-in or knock-out options.

For a knock-in American option, one does not need to deal directly with its optimal exercise boundary. This is because the holder does not have any exercise right until the “knock-in” feature is triggered, and once this happens, the optimal exercise boundary of the knock-in option is the same as that of the embedded American vanilla option, the calculation of which has been thoroughly studied in the literature (see Section 1.2.1). Dai and Kwok [27] successfully derived the pricing formulas of knock-in American options. These formulas take different analytical forms, depending on the relation between the asset barrier and the optimal exercise price of the embedded American vanilla option.

For an American knock-out option, one has to deal directly with the unknown optimal exercise boundary. Because of the risk of being knocked-out, the value of the knock-out option should be somewhat less than that of its embedded vanilla option. This in turn implies that the optimal exercise boundary of the knock-out option is lower than that of its embedded vanilla option. Until now, two main approaches for pricing an American-style knock-out option have been proposed. The first one is to use numerical methods such as the binomial tree method [7, 19, 33, 69] and the finite difference method (FDM) [8, 85, 86]. These lattice/grid-based methods are easy to be implemented. They however cannot handle the knock-out feature very well, especially for asset prices near the barrier, as pointed out in a number of articles [19, 33, 35]. As a result, the obtained results for the option prices and the hedging parameters
are not reliable in the region near the barrier. This issue indeed forms the main motivation for the second approach: the probabilistic approach developed by AitSahlia et al. \cite{3}, Detemple \cite{31}, Gao et al. \cite{35}, Kwok \cite{56}. This approach has been used to decompose the price of an American-style knock-out option without rebate into the sum of the price of its European counterpart and the early exercise premium associated with the early exercise right. Gao et al. \cite{35} claimed that the above probabilistic approach can be easily extended to price an American-style knock-out option with a rebate but they have not presented any results for this case. In Chapter \cite{3}, we adopt a different approach, an integral equation approach, for pricing American-style down-and-out calls with time-dependent rebates.

### 1.2.3 Parisian options

As discussed in Section 1.2.2, barrier options are common path-dependent options traded in financial markets. They provide a more flexible and cheaper way for hedging and speculating than vanilla options because the option buyers only pay a premium for scenarios they perceive as likely. The “one-touch” breaching barrier however is prone to market manipulations. For instance, an influential agent in the financial market, who has written a barrier option and has noticed the underlying asset price approaching the predetermined asset barrier, could try to push or pull the underlying price across the barrier, even momentarily. This will make the barrier option worthless so that the agent can eliminate its liabilities to the option holder.

To partially prevent such market manipulations, Parisian options were first introduced by Chesney et al. \cite{18}. Parisian options are very similar to their barrier option counterparts, except that it is much harder to activate their knock-in or knock-out features: the underlying asset has to continually stay above or below the asset barrier for a prescribed amount of time, which is called the “option window” \cite{18}. Such a requirement certainly makes the market fairer in the sense that it protects the holder of Parisian options from deliberate action taken by the writer. This extended trigger clause, the “option window”, can also be found in some derivative contracts, such as callable convertible bonds and executive warrants \cite{27}. The “option window” can also be useful in studying an optimal decision to invest in a project when delays are involved \cite{37}. It is also worthwhile to note that Parisian options can be a useful tool in corporate finance \cite{5}, credit risk and life insurance \cite{15,64}. 
Under the Black-Scholes model, the price of a Parisian option depends not only on the current asset price, the current time but also on the “excursion time”, which starts counting from 0 each time the underlying asset price touches the asset barrier from below (above) and stops counting when the underlying asset price touches the barrier from above (below). The pricing problem of Parisian options is therefore a three-dimensional (3-D) problem and it is no doubt much more complicated to solve than the two-dimensional (2-D) pricing problem of barrier options.

In the literature, many works have been devoted for pricing European-style Parisian options. More specifically, in [18, 29, 72], the price of a European-style Parisian option can be found after performing the inverse Laplace transform of the “Parisian stopping time”, which is the first time the length of the excursion reaches the predetermined option window. Numerically performing Laplace inversion however could be unstable and sensitive to round-off errors [16, 57]. Several researchers have also studied techniques to improve the accuracy of the inverse Laplace transforms that need to be performed in order to obtain the option price [5, 58]. An alternative way is to directly use the Laplace transform to obtain a recursive formula for the density of the “Parisian stopping time”. As a result, Dassios and Lim [28] obtained the option price without numerically performing Laplace inversion. This significantly increases the speed and accuracy of calculating the Parisian option price. Even more, a closed-form pricing formula for European-style Parisian options has also been found by Zhu and Chen [83]. A key idea behind their work is to reduce the 3-D pricing problem to a solvable 2-D one by using the “moving window” technique.

It is interesting to point out that the solution procedure for pricing a European-style Parisian knock-in option can be used to price its American-style counterpart. In fact, before the knock-in feature is activated, the two options are the same as both of them do not offer to their holders any exercise right to buy or sell the underlying stock. The difference between the two options appears only after the knock-in feature is activated, one becomes an American vanilla option, and the other becomes a European vanilla option. As a result, the valuation of an American-style Parisian knock-in option is very similar to that of its European-style counterpart. In particular, the “moving window” technique proposed by Zhu and Chen [83] could be applied to find analytical solutions for both American-style and European-style
1.2. LITERATURE REVIEW

Parisian knock-in options. In Chapters 4 and 5, this technique is applied to find simple analytical solutions for Parisian down-and-in calls and Parisian up-and-in calls, respectively.

Unlike knock-in cases, the valuation problem of American-style Parisian knock-out options is much more difficult than that of their European-style option counterparts because of the complexity of the determination of the optimal exercise price. For example, in the case of an American-style Parisian up-and-out call option, the complexity of determining the optimal exercise price is financially due to the conflict between the early exercise policy and the risk of losing the option when the asset price stays above the barrier. On one hand, the option holder has the incentive to wait for the asset price to further increase, hoping to gain more profit when exercising the option. On the other hand, the option holder also has to bear a higher risk of losing the option altogether if the asset price continue to stay above the asset barrier for “too long” and eventually the “knock-out” mechanism is triggered. The optimal exercise price now depends not only on the current time but also on the “excursion time”. In other words, the optimal exercise boundary is mathematically now a 3-D surface, instead of a 2-D curve as for the case of an American barrier option. Consequently, determining the 3-D optimal exercise boundary, which needs to be done in order to obtain the option value, becomes the primary source of difficulty for pricing an American-style Parisian up-and-out option.

Only few researchers have studied American-style Parisian knock-out options. Until now, two main approaches for pricing American-style Parisian knock-out options were proposed. The first one was to use numerical methods such as finite difference method, as was studied in detail by Haber et al. [39]. In their paper, a pair of two partial differential equation (PDE) systems governing the prices of Parisian options was established and then solved by using the explicit finite difference scheme. While this method is flexible and easy to implement, there are some limitations in their pricing systems, as pointed out in [83].

The second approach was to use analytical methods such as the probabilistic method [17]. More specifically, Chesney and Gauthier [17] first reduced the pricing problem of American-style Parisian options to finding the Laplace transform of the distribution of the “Parisian stopping time”. This Laplace transform was then obtained by using the Brownian meander and the Azema martingale. As a result, the option price can be computed by numerically
inverting the Laplace transform. As discussed in \cite{16, 57}, numerically performing Laplace inversion sometimes could be unstable and sensitive to round-off errors. In addition, the inverse Laplace transform techniques developed in \cite{5, 58} for European-style Parisian options have not been extended for American-style Parisian options. There is a need to find a new method that can eliminate drawbacks in the previous methods. This is the main motivation for Chapters 6 and 7, which propose a new approach, an integral equation approach, for pricing American-style Parisian knock-out options.

1.3 Structure of Thesis

In Chapter 2, we provide basic knowledge needed for pricing options, especially for pricing American-style Parisian options. We recall some stochastic processes, which can be used to model the asset price dynamics. We then re-derive the well-known Black-Scholes equation and Black-Scholes formula of European vanilla calls. We also present the derivations of the well-known integral representation of American vanilla calls and the closed form pricing formula for European down-and-out calls with rebates. Some of the techniques used in these derivations will be extended for pricing different types of Parisian options in the subsequent chapters.

In Chapter 3, an integral equation approach is adopted to price American-style down-and-out calls, which is a special type of Parisian options. Using this approach, the price of an American-style down-and-out call with time-dependent rebate can be decomposed into two components: the price of its European counterpart with the given rebate and an early exercise premium associated with the American-style early exercise right. Interesting numerical results are provided to illustrate the effects of time-dependent rebates on the prices of American-style down-and-out calls as well as their optimal exercise boundaries.

In Chapters 4 and 5, we price American-style Parisian up-and-in call options and American-style Parisian down-and-in call options, respectively. We adopt the “moving window” technique developed by Zhu and Chen \cite{83} for pricing European-style Parisian up-and-out call options to price these types of American-style Parisian knock-in options. As a result, we obtain simple analytical solutions for American-style Parisian knock-in call options, which can be easily computed numerically.

In Chapters 6 and 7, we propose an integral equation approach for pricing American-
style Parisian up-and-out call options and American-style Parisian down-and-out call options, respectively. The corresponding three-dimensional pricing problems are first reduced to two-dimensional ones by applying the “moving window” technique developed by Zhu and Chen[83]. The two-dimensional problems are then further simplified to solving pairs of two coupled integral equations, which govern two unknown quantities: the option price at the asset barrier and the optimal exercise price. As a result, once these two unknown quantities are efficiently solved by using the Newton-Raphson iterative procedure, the option price and the hedging parameters can be obtained both accurately and efficiently. Selected numerical results are also used to validate our approach as well as to illustrate interesting features about the option prices and the optimal exercise boundaries.

The thesis ends with some concluding remarks about the main results achieved in this thesis as well as some future research directions in Chapter 8.
Chapter 2

Theoretical background

In this chapter, we provide some theoretical background needed for pricing options. We start with the discussion on stochastic processes, which can be used to model unpredictable movements of the underlying asset prices. We then discuss on the Black-Scholes model, which is the simplest and most well-known model in option pricing theory. This model is the main option pricing model used in this thesis. In the subsequent sections, through re-deriving the well-known pricing formulas of some simple options (European vanilla calls, American vanilla calls and European down-and-out calls with rebates), we introduce some fundamental mathematical techniques used in option pricing theory. Some of these techniques will be extended for pricing more complicated exotic options, such as Parisian options, in the subsequent chapters.

2.1 Asset price dynamics and stochastic processes

By definition, the price of an option depends heavily on the price of its underlying asset. The knowledge about the value of the underlying asset therefore plays a critical role in determining the option price. In an efficient market, any information relevant to the determination of the underlying asset price is rapidly assimilated by rational traders and the asset price is then adjusted accordingly [26, p.54]. More precisely, the current and past information is assumed to be immediately incorporated into the current asset price and future changes in the asset price are caused by future information. As a result, the movements of the asset price are unpredictable simply because future information is by definition unpredictable. This is
consistent with the fact that in practice, no one can actually predict exactly the underlying asset price at some point in the future. Mathematically, the price of the underlying asset at a future time \( t \) can be considered as a random variable \( S_t \) and the asset price dynamics over a continuous time interval, i.e., a collection of \( \{S_t\}_{t \geq 0} \), can be modeled as a continuous-time stochastic process.

Bachelier made one of the first attempts to use continuous-time stochastic processes for pricing options. On March 19, 1900, Bachelier successfully defended his doctoral thesis, entitled “The Theory of Speculation”, which was credited with a high degree of originality. He introduced many important concepts which were developed and named after other mathematicians working at considerably later dates, such as Brownian motion, Markov property, Chapman-Kolmogorov equation \[30\]. These concepts play a critical role in our understanding of the random nature of asset price fluctuations. Bachelier also initiated the quest for a rigorous option valuation formula. He assumed that the asset price was normally distributed and that the asset price followed an arithmetic Brownian motion, defined by:

\[
dS = \mu dt + \sigma dW, \tag{2.1.1}
\]

where \( \mu \) and \( \sigma \) are constant drift and diffusion terms, respectively. Here \( W \) denotes a standard Brownian motion (Wiener process), which is a continuous-time stochastic process satisfying the following properties \[50, 74\]:

1. Continuity path: \( W(0) = 0 \), \( W(t) \) is a continuous function of \( t \).

2. Normal increments: for \( 0 \leq s < t \), \( W(t) - W(s) \) is normally distributed with mean 0 and variance \( t - s \), i.e., \( W(t) - W(s) \sim N(0, t - s) \).

3. Independence of increments: \( W(v) - W(u) \) is independent of \( W(t) - W(s) \), \( \forall v > u \geq t > s \).

Bachelier was able to derive a closed-form pricing formula by equating the option price to the expected difference between the stock price and the exercise price \[55\]. He also showed some intuition about how to use combinations of futures and options \[40\].

Although Bachelier’s work laid a strong foundation for the development of the theory of option pricing, Bachelier’s model has fundamental flaws: the model permits negative under-
lying asset prices as well as option values in excess the prices of the underlying assets, neither of which is possible. To remedy these drawbacks, Samuelson [71] used asset returns (returns obtained by investing in assets) to replace asset prices in the model of Bachelier. More specifically, asset prices were assumed to be log-normally distributed and that asset prices followed a geometric Brownian motion, defined by:

\[ dS = \mu S dt + \sigma S dW, \]  

(2.1.2)

where \( \mu \) and \( \sigma \) are constant expected rate and standard deviation (volatility) of the asset return, respectively. Here \( W \) denotes a standard Brownian motion. By using this model, Samuelson [71] derived a pricing formula for a European call option. Let \( C(S, \tau) \) be the price of a call option written on the underlying price \( S \) at time to expiry \( \tau \), \( E \) be the exercise price, then:

\[ C = Se^{(\mu - w)\tau}N(d_1) - Ee^{-w\tau}N(d_2), \]  

(2.1.3)

where \( w \) is the expected return on the call option and

\[ d_1 = \frac{\ln(S/E) + (\mu + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}. \]  

(2.1.4)

The formula (2.1.3) involves two parameters \( \mu \) and \( w \), which are difficult to determine due to the dependence on the risk preference of individual investors. As a result, this formula has not been widely used in the real world trading [40].

In 1973, Black and Scholes [6] and Merton [62] revolutionized the theory of option pricing by deriving an elegant pricing formula for European calls, which is independent of the risk preference of individual investors. More specifically, the formula is given by:

\[ C = SN(d_1) - Ee^{-r\tau}N(d_2), \]  

(2.1.5)

where the notations are the same as those used in the formula (2.1.4), except \( \mu \) is replaced by \( r \), which denotes the risk-free interest rate. It is interesting to point out that the Black-Scholes formula (2.1.5) can be derived from the Samuelson formula (2.1.3) by putting all investors in a risk-neutral world where both the expected returns on the underlying asset and the option
equals the risk-free interest rate \( r \). The Black-Scholes formula can quickly produce the price of a call option at inception once all the necessary inputs, which includes: the current asset price, the exercise price, the expiry date, the risk-free interest rate and the volatility, are provided. Almost all of these inputs, except the current volatility, can be obtained from the specification of the option contract (the exercise price, the expiry date) or from the current information in the market (the current asset price, the risk-free interest rate). To estimate the current volatility, one can use the historical volatility, which measures the variability of past prices. Alternatively, one can calculate the implied volatility by inserting the option market price in the Black-Scholes formula and then solving for the volatility.

The Black-Scholes model has been one of the most widely used models in practice. It has been used as the benchmark model for pricing options written on different underlying assets such as stocks, currencies, and futures. The introduction of the Black-Scholes formula has led to high growth of the option market, from a sparse and thinly traded option market to one of the largest and most active security markets \[47\]. In the next section, we provide more details on the derivation of the Black-Scholes formula \[2.1.5\].

### 2.2 The Black-Scholes model

In this section, the derivations of the Black-Scholes partial differential equation (PDE) and the Black-Scholes formula for European vanilla calls are given.

#### 2.2.1 The Black-Scholes equation

The assumptions used in the Black-Scholes model are given as follows \[78\]:

1. The underlying asset pays a continuous dividend yield \( \delta \), that is proportional to the asset’s price.

2. The risk-free interest rate \( r \) and the volatility rate \( \sigma \) are known and constant during the option life.

3. The underlying asset price \( S \) follows a geometric Brownian motion governed by:

\[
    dS = (\mu - \delta)Sdt + \sigma SdZ, \tag{2.2.6}
\]
where $\mu$ is the expected return on the underlying asset and $Z$ is a standard Brownian motion.

4. There are no risk-free arbitrage opportunities.

5. Security trading is continuous.

6. There are no transaction costs or taxes in option trading.

7. Short selling of securities is permitted and all securities are perfectly divisible.

It should be noted that Black and Scholes [6] and Merton [62] originally assumed that underlying assets pay no dividend. In practice however this is rarely the case. A typical extension of the Black-Scholes model is to assume underlying assets pay a continuous dividend yield $\delta$. This assumption certainly allows us to apply the model to price a wider range of options.

Under the above assumptions, the price of a European vanilla call then depends on the asset price $S$, the current time $t$, the exercise price $E$, in addition to other parameters such as the volatility rate $\sigma$, the risk-free interest rate $r$ and the expiry time $T$. Let $V(S,t)$ be the price the European call associated with the underlying price $S$ and time $t$. Because the value of $S$ is random, $V(S,t)$ is also random. The following well-known Ito’s lemma [47] plays a critical role in our understanding about the changes of $V$ over time.

**Lemma 2.2.1.** Suppose that the value of a random variable $S$ follows the following Ito process:

$$dS = a(S,t)dt + b(S,t)dZ,$$

where $Z$ is a standard Brownian motion and $a$, $b$ are given well-behaved functions of $S$ and $t$. Then any function of $S$ and $t$, say $G(S,t)$, follows the corresponding Ito process defined by:

$$dG = \left( a(S,t) \frac{\partial G}{\partial S}(S,t) + \frac{\partial G}{\partial t}(S,t) + \frac{1}{2} b^2(S,t) \frac{\partial^2 G}{\partial S^2}(S,t) \right) dt + b(S,t) \frac{\partial G}{\partial S}(S,t)dZ,$$

where $Z$ is the same standard Brownian motion used in (2.2.7).

By applying the Ito’s lemma with $a(S,t) = (\mu - \delta)S$ and $b(S,t) = \sigma S$, $V(S,t)$ satisfies the
following Ito process:

\[ dV = \left( (\mu - \delta)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dZ, \]  

(2.2.9)

where \( Z \) is the same standard Brownian motion used in (2.2.6). Therefore, both \( S \) and \( V \) are governed by the same underlying source of uncertainty. This allows us to construct a risk-free portfolio, which includes one unit of the call option and \( \frac{\partial V}{\partial S} \) unit of the underlying asset. By using the no arbitrage assumption, the Black-Scholes equation governing the price of the call option can be obtained as [56, 78]:

\[ \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0. \]  

(2.2.10)

It should be noted this important PDE will be used throughout this thesis for pricing different types of options under the Black-Scholes framework.

### 2.2.2 The governing PDE system

We now derive the governing PDE system of the price \( V \) of a European vanilla call option. First, the terminal condition is given by the payoff at expiry, i.e.,

\[ V(S, T) = \max(S - E, 0). \]  

(2.2.11)

When \( S \) approaches zero, the option is deeply out-of-money so that its value is worth almost nothing. That means:

\[ \lim_{S \to 0} V(S, t) = 0. \]  

(2.2.12)

On the other hand, if \( S \) approaches infinity at any time \( t \) then the option is deeply in-the-money and its value is virtually the same with the payoff received upon exercising the option immediately. Mathematically, this can be expressed as:

\[ V(S, t) \sim S - E \text{ as } S \to \infty. \]  

(2.2.13)
Equations (2.2.10)-(2.2.13) constitute the PDE system governing the value of a European vanilla call at any $S$, and any $t$. For convenience, they are summarized as:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0,
&V(S, T) = \max(S - E, 0),
&\lim_{S \to 0} V(S, t) = 0,
&V(S, t) \sim S - E \text{ as } S \to \infty
\end{aligned} \tag{2.2.14}$$

where $\mathcal{A}$ is defined on $t \in [0, T]$, $S \in [0, \infty)$.

To make the system $\mathcal{A}$ even simpler, we introduce the following dimensionless variables:

$$x = \ln \frac{S}{E}, \quad \tau = \frac{\sigma^2}{2} (T - t), \quad \gamma = \frac{2r}{\sigma^2}, \quad q = \frac{2\delta}{\sigma^2}, \quad k = \gamma - q - 1,$$

$$\alpha = -\frac{k}{2}, \quad \beta = -\frac{k^2}{4} - \gamma, \quad C(x, \tau) = E^{-1} e^{-\alpha x - \beta \tau} V(S, t). \tag{2.2.15}$$

System (2.2.14) now becomes a dimensionless system, which includes a standard heat equation together with the corresponding initial and boundary conditions:

$$\begin{aligned}
&\frac{\partial C}{\partial \tau}(x, \tau) = \frac{\partial^2 C}{\partial x^2}(x, \tau),
&C(x, 0) = e^{-\alpha x} \max(e^x - 1, 0),
&\lim_{x \to -\infty} C(x, \tau) = 0,
&C(x, \tau) \sim e^{(1-\alpha)x} - e^{-\alpha x} \text{ as } x \to +\infty.
\end{aligned} \tag{2.2.16}$$

2.2.3 Initial-value heat problem in the infinite domain

The pricing of European vanilla options is now reduced to solving the corresponding initial-value heat problem in the infinite domain. This standard problem has been studied in a number of text books [24, 32, 42, 53]. We recall here some useful results on this problem.
2.2. THE BLACK-SCHOLES MODEL

The general form of an initial-value heat problem in the infinite interval is [53, p.6]

\[
\begin{aligned}
\frac{\partial u}{\partial \tau}(x, \tau) &= a^2 \frac{\partial^2 u}{\partial x^2}(x, \tau), \quad -\infty < x < +\infty, \quad \tau > 0, \quad a > 0 \\
u(x, 0) &= f(x), \\
u(x, \tau) &\sim f(x) \text{ as } x \to \pm\infty,
\end{aligned}
\]

(2.2.17)

where \( f \) is an arbitrarily prescribed function. The solution of this problem is given by [32, p.43]:

\[
u(x, \tau) = \frac{1}{2a\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4a^2\tau}} d\xi.
\]

(2.2.18)

We now consider an example to illustrate how to solve an initial-value heat problem in the infinite domain. More specifically, we consider the following system:

\[
\begin{aligned}
\frac{\partial u}{\partial \tau}(x, \tau) &= 4 \frac{\partial^2 u}{\partial x^2}(x, \tau), \quad -\infty < x < +\infty, \quad \tau > 0, \\
u(x, 0) &= e^x, \\
u(x, \tau) &\sim e^x \text{ as } x \to \pm\infty,
\end{aligned}
\]

(2.2.19)

which has a unique solution \( u(x, \tau) = e^{x+4\tau} \).

Using the formula (2.2.18) with \( a = 2 \) and \( f(x) = e^x \), we easily obtain the solution of the system (2.2.19) as follows:

\[
u(x, \tau) = \frac{1}{4\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} e^z e^{-\frac{(x-\xi)^2}{16\tau}} d\xi = \frac{1}{4\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi+8\tau)^2}{16\tau}+4\tau} d\xi.
\]

Using the variable transformation: \( z = \frac{x - \xi + 8\tau}{4\sqrt{\tau}} \), the solution of the system (2.2.19) can be expressed as:

\[
u(x, \tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} e^{x+4\tau} dz = e^{x+4\tau}.
\]

Therefore, we obtain the unique solution of the system.

We are now ready to solve the system (2.2.16) to obtain the closed-form Black-Scholes formula for European vanilla options.
2.2.4 The Black-Scholes formula

Using the formula (2.2.18) with \( a = 1 \) and \( f(x) = \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0) \), we easily obtain the solution of the system (2.2.16) as follows:

\[
C(x, \tau) = \int_{-\infty}^{+\infty} \max(e^{(1-\alpha)\xi} - e^{-\alpha \xi}, 0) e^{-\frac{(x-\xi)^2}{4\tau}} d\xi
= \frac{1}{2\sqrt{\pi \tau}} \int_{0}^{+\infty} e^{-\frac{(x-\xi+2(1-\alpha)\tau)^2}{4\tau}+(1-\alpha)^2\tau+(1-\alpha)x} d\xi
- \frac{1}{2\sqrt{\pi \tau}} \int_{0}^{+\infty} e^{-\frac{(x-\xi-2\alpha\tau)^2}{4\tau}+\alpha^2\tau-\alpha x} d\xi.
\]

Using the variable transformations: \( u = \frac{x-\xi+2(1-\alpha)\tau}{\sqrt{2\tau}} \) and \( v = \frac{x-\xi-2\alpha\tau}{\sqrt{2\tau}} \) to the integrals \( I \) and \( J \), respectively, \( C(x, \tau) \) can be expressed as:

\[
C(x, \tau) = e^{(1-\alpha)^2\tau+(1-\alpha)x} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x+2(1-\alpha)\tau}{\sqrt{2\tau}}} e^{-\frac{u^2}{2}} du - e^{\alpha^2\tau-\alpha x} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-2\alpha\tau}{\sqrt{2\tau}}} e^{-\frac{v^2}{2}} dv
= e^{(1-\alpha)^2\tau+(1-\alpha)x} N\left(\frac{x+2(1-\alpha)\tau}{\sqrt{2\tau}}\right) - e^{\alpha^2\tau-\alpha x} N\left(\frac{x-2\alpha\tau}{\sqrt{2\tau}}\right),
\]

where \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \), the cumulative distribution function of the standard normal distribution.

Converting the solution to the original coordinate space \((S, t)\), with the notice that

\[
\alpha^2 + \beta = -\gamma, \quad (1-\alpha)^2 + \beta = -q,
\]

we obtain:

\[
V(S, t) = Ee^{\alpha x + \beta \tau} C(x, \tau) = M_1(S, T-t, E),
\]

where

\[
M_1(x, y, z) = xe^{-d_1} N(d_1(x, y, z)) - Ee^{-rT} N(d_2(x, y, z)),
\]

\[
d_1(x, y, z) = \frac{\ln(x/z) + (r - \delta + \sigma^2/2)y}{\sigma \sqrt{y}},
\]

\[
d_2(x, y, z) = d_1(x, y, z) - \sigma \sqrt{y}. \quad (2.2.20)
\]

We have now obtained the well-known Black-Scholes formula for European vanilla options.
2.3 The integral representation of American vanilla options

The integral representation of the price of an American vanilla option has been studied in a number of works \cite{20, 49, 52, 54}. Particularly, readers are referred to \cite{20} for a detailed review on this. In this section, we will recall the derivation of this integral representation by using the (incomplete) Fourier transform technique. The purpose of doing this is to familiarize readers with the solution procedure and necessary techniques that can be extended for pricing more complicated American-style options.

Under the Black-Scholes framework, the price of an American vanilla call depends on the underlying asset price $S$, the current time $t$ and other constant parameters: the exercise price $E$, the volatility rate $\sigma$, the risk-free interest rate $r$, the dividend rate $\delta$ and the expiry time $T$. Let $V(S, t)$ be the option price associated with the underlying asset price $S$ and time $t$. Typically, at each time $t$ during the life of the option, there exists an unknown value of the underlying asset referred to as the optimal exercise price denoted by $S_f(t)$, above which the option should be exercised immediately and its value is then equal to the payoff received by exercising the option. As a result, we only need to price the option when the asset price is below this optimal exercise boundary. The pricing domain $I$ of the option can be expressed mathematically as:

$$I = \{(S, \tau) | 0 \leq S \leq S_f(\tau), 0 \leq t \leq T\}.$$  

Using arguments similar to those used in Section \ref{sect:2.2.2}, it can be shown that $V(S, t)$ satisfies equations \eqref{eq:2.2.10}-\eqref{eq:2.2.12}. In addition, in order to determine the optimal exercise boundary, the following smooth pasting conditions are needed \cite{12}:

$$V(S_f(t), t) = S_f(t) - E, \quad \frac{\partial V}{\partial S}(S_f(t), t) = 1. \quad (2.3.21)$$

As a result, the PDE system governing the value of an American vanilla call at any $S$ and
any $t$ is given by:

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2 \partial^2 V}{2 \partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV &= 0, \\
V(S, T) &= \max(S - E, 0), \\
V(S_f(t), t) &= S_f(t) - E, \\
\frac{\partial V}{\partial S}(S_f(t), t) &= 1, \\
\lim_{S \to 0} V(S, t) &= 0,
\end{aligned}
\]  

(2.3.22)

where $\mathcal{A}$ is defined on $t \in [0, T]$, $S \in [0, \infty)$.

To make the system $\mathcal{A}$ even simpler, we use the same dimensionless variables defined in (2.2.15), with an extra one: $x_f(\tau) = \ln \frac{S_f(\tau)}{E}$. Under this variable transformation, the system (2.3.22) now becomes a dimensionless system, which includes a standard heat equation together with the corresponding initial and boundary conditions:

\[
\begin{aligned}
\frac{\partial C}{\partial \tau}(x, \tau) &= \frac{\partial^2 C}{\partial x^2}(x, \tau), \\
C(x, 0) &= f(x), \\
C(x_f(\tau), \tau) &= g_1(x_f(\tau), \tau), \\
\frac{\partial C}{\partial x}(x_f(\tau), \tau) &= g_2(x_f(\tau), \tau), \\
\lim_{x \to -\infty} C(x, \tau) &= 0,
\end{aligned}
\]  

(2.3.23)

where $f, g_1, g_2$ are functions defined as:

\[
\begin{aligned}
f(x) &= \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0), \\
g_1(x, y) &= e^{(1-\alpha)x-\beta y} - e^{-\alpha x-\beta y}, \\
g_2(x, y) &= (1 - \alpha)e^{(1-\alpha)x-\beta y} + \alpha e^{-\alpha x-\beta y}.
\end{aligned}
\]  

(2.3.24)

As shown in [20], this system can be efficiently solved by using the Fourier transform. The Fourier transform of an arbitrarily continuous function $f(x)$, denoted by $\mathcal{F}\{f(x)\}$, is defined
as:
\[ F\{f(x)\} \equiv U(\omega) = \int_0^\infty f(x)e^{i\omega x}dx, \]
with the corresponding inversion:
\[ f(x) \equiv F^{-1}\{U(\omega)\} = \frac{2}{\pi} \int_0^\infty U(\omega)e^{-i\omega x}d\omega. \]

Here \( \omega \) is called the Fourier transform parameter. It should be noted that a continuous function \( f(x) \) is Fourier transformable if \( \int_{-\infty}^{+\infty} |f(x)|dx < \infty \). Readers are referred to [66] to obtain some properties and applications of the Fourier transform in solving partial differential equations.

To apply the Fourier transform to the system (2.3.23), its \( x \)-domain first needs to be extended to an infinite domain first by expressing the PDE as:
\[ H(x_f(\tau) - x)\frac{\partial C}{\partial \tau}(x, \tau) = H(x_f(\tau) - x)\frac{\partial^2 C}{\partial x^2}(x, \tau) \tag{2.3.25} \]
where \( H(x) \) is the Heaviside function, defined as:
\[ H(x) = \begin{cases} 
1, & \text{if } x > 0, \\
1/2, & \text{if } x = 0, \\
0, & \text{if } x < 0. 
\end{cases} \tag{2.3.26} \]

The reason for the appearance of the factor of 1/2 at the point of discontinuity is explained in [20]. The initial and boundary conditions remain unchanged.

For the purposes of the transform method, we assume that the function \( C(x, \tau) \) and its first derivatives with respect to \( x \) can be treated as zero when \( x \) tends to infinity. This assumption is subsequently justified by virtue of the facts that the solution obtained after applying the Fourier transform satisfies the system (2.3.23), and that the solution of the system (2.3.23) is unique.

We are now ready to apply the Fourier transform to the system (2.3.23) with respect to
2.3. THE INTEGRAL REPRESENTATION OF AMERICAN VANILLA OPTIONS

x. By definition, we have:

$$\mathcal{F}\left\{ H(x_f(\tau) - x)C(x, \tau) \right\} = \int_{-\infty}^{x_f(\tau)} C(x, \tau) e^{ix\omega} dx \equiv \hat{C}(\omega, \tau),$$

where for convenience, $\hat{C}(\omega, \tau)$ denotes the Fourier transform of $H(x_f(\tau) - x)C(x, \tau)$.

Direct calculation shows that:

$$\mathcal{F}\left\{ H(x_f(\tau) - x) \frac{\partial C}{\partial \tau}(x, \tau) \right\} = \frac{\partial \hat{C}}{\partial \tau}(\omega, \tau) - x'_f(\tau)g_1(x_f(\tau), \tau)e^{i\omega x_f(\tau)}, \quad (2.3.27)$$

and

$$\mathcal{F}\left\{ H(x_f(\tau) - x) \frac{\partial^2 C}{\partial x^2}(x, \tau) \right\} = g_2(x_f(\tau), \tau)e^{i\omega x_f(\tau)} - i\omega e^{i\omega x_f(\tau)}g_1(x_f(\tau), \tau) - \omega^2 \hat{C}(\omega, \tau). \quad (2.3.28)$$

Note that here the notation $x'_f(\tau)$ denotes the first derivative of $x_f$ with respect to $\tau$, and $g_1, g_2$ are predetermined functions defined in (2.3.24).

As a result of applying the Fourier transform with respect to $x$, the PDE (2.3.25) can be reduced to the following linear first-order ODE in the Fourier space:

$$\frac{\partial \hat{C}}{\partial \tau}(\omega, \tau) + \omega^2 \hat{C}(\omega, \tau) = G(\omega, \tau)$$

where

$$G(\omega, \tau) = e^{i\omega x_f(\tau)}g_2(x_f(\tau), \tau) - i\omega e^{i\omega x_f(\tau)}g_1(x_f(\tau), \tau) + x'_f(\tau)g_1(x_f(\tau), \tau)e^{i\omega x_f(\tau)}$$

with initial condition $\hat{C}(\omega, 0) = \int_{-\infty}^{x_f(\tau)} f(x)e^{ix\omega} dx$.

The solution of this initial-value ODE can be easily solved as:

$$\hat{C}(\omega, \tau) = \hat{C}(\omega, 0)e^{-\omega^2\tau} + \int_0^{\tau} e^{-\omega^2(\tau-\xi)}G(\omega, \xi)d\xi. \quad (2.3.29)$$

As $\hat{C}(\omega, \tau)$ denotes the Fourier transform of $H(x_f(\tau) - x)C(x, \tau)$, from (2.3.29), we can
now express the solution of the system (2.3.23) as follows:

$$H(xf(\tau) - x)C(x, \tau) = \mathcal{F}^{-1}\left\{\hat{C}(\omega, 0)e^{-\omega^2\tau}\right\} + \mathcal{F}^{-1}\left\{\int_0^\tau e^{-\omega^2(\tau - \xi)}G(\omega, \xi)d\xi\right\}. \quad (2.3.30)$$

Furthermore, as $S = Ee^x$, $Sf(\tau) = Ee^{xf(\tau)}$, and $V(S, t) = Ee^{\alpha x + \beta t}C(x, \tau)$, by multiplying $Ee^{\alpha x + \beta t}$ to both sides of (2.3.30), we can express the solution of the system (2.3.22) as follows:

$$H(\ln Sf(t) - \ln S)V(S, t) = Ee^{\alpha x + \beta \tau}F^{-1}\left\{\hat{C}(\omega, 0)e^{-\omega^2\tau}\right\}$$

$$+ Ee^{\alpha x + \beta \tau}F^{-1}\left\{\int_0^\tau e^{-\omega^2(\tau - \xi)}G(\omega, \xi)d\xi\right\}. \quad (2.3.31)$$

The first and second terms in the right hand side of (2.3.31) clearly need to be calculated explicitly in order to obtain an integral representation for the solution of the system (2.3.22).

**Compute the first term of (2.3.31).**

We first calculate the inverse Fourier transform of $e^{-\omega^2\tau}$ as:

$$\mathcal{F}^{-1}\left\{e^{-\omega^2\tau}\right\} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\omega^2\tau}e^{-i\omega u}d\omega = \frac{e^{-\frac{x^2}{4\tau}}}{2\sqrt{\pi}}.$$

Applying the Convolution theorem for the Fourier transform [66], we obtain:

$$\mathcal{F}^{-1}\left\{\hat{C}(\omega, 0)e^{-\omega^2\tau}\right\} = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{H(xf(0) - x)f(x)\right\}\mathcal{F}\left\{e^{\frac{x^2}{4\sqrt{\tau}}}\right\}\right\}$$

$$= \int_{-\infty}^{\infty} H(xf(0) - u)f(u)\frac{e^{-(x-u)^2}}{4\sqrt{\pi\tau}}du = \int_{-\infty}^{xf(0)} \max\left(e^{(1-\alpha)u} - e^{-\alpha u}, 0\right)\frac{e^{-(x-u)^2}}{4\sqrt{\pi\tau}}du$$

$$= \frac{1}{2\sqrt{\pi\tau}} \int_0^{xf(0)} e^{(1-\alpha)u} - \frac{(x-u)^2}{4\sqrt{\pi\tau}}du - \frac{1}{2\sqrt{\pi\tau}} \int_0^{xf(0)} e^{-\alpha u - \frac{(x-u)^2}{4\sqrt{\pi\tau}}}du$$

$$= \frac{e^{(1-\alpha)^2\tau + (1-\alpha)x}}{2\sqrt{\pi\tau}} \int_0^{xf(0)} e^{-(x-u+2(1-\alpha)\tau)^2}du - \frac{e^{\alpha^2\tau - \alpha x}}{2\sqrt{\pi\tau}} \int_0^{xf(0)} e^{-(x-u+2\alpha\tau)^2}du.$$

By using the variable transformations $v = \frac{x - u + 2(1-\alpha)\tau}{\sqrt{2\tau}}$ and $w = \frac{x - u - 2\alpha\tau}{\sqrt{2\tau}}$ to the
2.3. THE INTEGRAL REPRESENTATION OF AMERICAN VANILLA OPTIONS

last two integrals, respectively, we obtain:

\[
\mathcal{F}^{-1}\{\tilde{C}(\omega, 0)e^{-\omega^2\tau}\} = \frac{e^{(1-\alpha)^2\tau + (1-\alpha)x}}{\sqrt{2\pi}} \int_{x-x_f(0) + 2(1-\alpha)\tau}^{x} e^{-\frac{v^2}{2}} dv \\
- \frac{e^{2\alpha^2\tau - \alpha x}}{\sqrt{2\pi}} \int_{x-x_f(0) - 2\alpha\tau}^{x} e^{-\frac{w^2}{2}} dw
\]

Consequently, the first term of (2.3.31) can be expressed as:

\[
Ee^{\alpha x + \beta \tau} \mathcal{F}^{-1}\{\tilde{C}(\omega, 0)e^{-\omega^2\tau}\} = M_1(S, T - t, E) - M_1(S, T - t, S_f(T)),
\]

where \(M_1\) is a function of three variables defined in (2.2.20).

**Compute the second term of (2.3.31).**

\[
\mathcal{F}^{-1}\left\{ \int_0^\tau e^{-\omega^2(\tau - \xi)} G(\omega, \xi) d\xi \right\} = \int_0^\tau \mathcal{F}^{-1}\left\{ e^{-\omega^2(\tau - \xi)} G(\omega, \xi) d\xi \right\}
\]

\[
= \frac{1}{2\pi} \int_0^\tau \int_{-\infty}^{+\infty} e^{-\omega^2(\tau - \xi) - i\omega(x-x_f(\xi))} \left[ g_2(x_f(\xi), \xi) + x'_f(\xi)g_1(x_f(\xi), \xi) - i\omega g_1(x_f(\xi), \xi) \right] d\omega d\xi
\]

\[
= \frac{1}{2\pi} \int_0^\tau \int_{-\infty}^{+\infty} e^{-\omega^2(\tau - \xi) - i\omega(x-x_f(\xi))} d\omega d\xi
\]

\[
- \frac{1}{2\pi} \int_0^\tau i g_1(x_f(\xi), \xi) \int_{-\infty}^{+\infty} \omega e^{-\omega^2(\tau - \xi) - i\omega(x-x_f(\xi))} d\omega d\xi
\]

\[
= \int_0^\tau \left[ g_2(x_f(\xi), \xi) + x'_f(\xi)g_1(x_f(\xi), \xi) \right] e^{-\frac{(x-x_f(\xi))^2}{4(\tau-\xi)}} \frac{1}{2\sqrt{\pi(\tau-\xi)}} d\xi
\]

\[
- \int_0^\tau g_1(x_f(\xi), \xi) \frac{x-x_f(\xi)}{4\sqrt{\pi(\tau-\xi)^3}} e^{-\frac{(x-x_f(\xi))^2}{4(\tau-\xi)}} d\xi
\]

\[
= \int_0^\tau \frac{e^{-\frac{(x-x_f(\xi))^2}{4(\tau-\xi)}}}{2\sqrt{\pi(\tau-\xi)}} \left[ g_2(x_f(\xi), \xi) + g_1(x_f(\xi), \xi) \left( x'_f(\xi) - \frac{x-x_f(\xi)}{2(\tau-\xi)} \right) \right] d\xi
\]

\[
= J(x, \tau),
\]
where \( J \) is a function of two variables and is defined as:

\[
J(x, y) = \int_0^\tau e^{-\frac{(x-x_f(\xi))^2}{4(y-\xi)}} \left[ g_2(x_f(\xi), \xi) + g_1(x_f(\xi), \xi) \left( x_f'(\xi) - \frac{x - x_f(\xi)}{2(y-\xi)} \right) \right] d\xi. \tag{2.3.34}
\]

Here \( g_1, g_2 \) are defined in (2.3.24). By substituting \( g_1 \) and \( g_2 \) in the formula of \( J(x, \tau) \), we can express:

\[
J(x, \tau) = J_1(x, \tau; 1 - \alpha) - J_1(x, \tau; -\alpha),
\]

where

\[
J_1(x, \tau; 1 - \alpha) = \int_0^\tau e^{-\frac{(x-x_f(\xi))^2}{4(y-\xi)} + (1-\alpha)x_f(\xi) - \beta \xi} \frac{1}{2\sqrt{\pi(\tau - \xi)}} \left( 1 - \alpha + x_f'(\xi) - \frac{x - x_f(\xi)}{2(\tau - \xi)} \right) d\xi
\]

\[
= -\int_0^\tau \frac{\partial}{\partial \xi} \left( \frac{x - x_f(\xi) + 2(1 - \alpha)(\tau - \xi)}{\sqrt{2(\tau - \xi)}} \right) e^{-\frac{(x-x_f(\xi))^2+2(1-\alpha)(\tau-\xi)}{4(\tau-\xi)} + (1-\alpha)^2(\tau-\xi) - \beta \xi + (1-\alpha)x} d\xi
\]

\[
= -e^{-(1-\alpha)x+(1-\alpha)^2\tau} \lim_{\xi \to \tau} e^{-((1-\alpha)^2 + \beta)\xi} N \left( \frac{x - x_f(\xi) + 2(1 - \alpha)(\tau - \xi)}{\sqrt{2(\tau - \xi)}} \right) d\xi
\]

\[
+ e^{-(1-\alpha)x+(1-\alpha)^2\tau} N \left( \frac{x - x_f(0) + 2(1 - \alpha)\tau}{\sqrt{2\tau}} \right)
\]

\[
- e^{-(1-\alpha)x+(1-\alpha)^2\tau} \int_0^\tau ((1 - \alpha)^2 + \beta) e^{-((1-\alpha)^2 + \beta)\xi} N \left( \frac{x - x_f(\xi) + 2(1 - \alpha)(\tau - \xi)}{\sqrt{2(\tau - \xi)}} \right) d\xi.
\]

We can also obtain \( J_1(x, \tau; -\alpha) \) by simply replacing \( 1 - \alpha \) by \( \alpha \) in the above formula of \( J_1(x, \tau; 1 - \alpha) \). Therefore,

\[
J(x, \tau) = -e^{-(1-\alpha)x+(1-\alpha)^2\tau} \left[ e^{\gamma \tau} 1_{x=x_f(\tau)} - N \left( \frac{x - x_f(0) + 2(1 - \alpha)\tau}{\sqrt{2\tau}} \right) \right]
\]

\[
+ e^{-(1-\alpha)x+(1-\alpha)^2\tau} \int_0^\tau q e^{\xi} \frac{x - x_f(\xi) + 2(1 - \alpha)(\tau - \xi)}{\sqrt{2(\tau - \xi)}} d\xi
\]

\[
+ e^{-\alpha x + \alpha^2\tau} \left[ e^{\tau} 1_{x=x_f(\tau)} - N \left( \frac{x - x_f(0) - 2\alpha\tau}{\sqrt{2\tau}} \right) \right]
\]

\[
- e^{-\alpha x + \alpha^2\tau} \int_0^\tau \gamma e^{\xi} N \left( \frac{x - x_f(\xi) - 2\alpha(\tau - \xi)}{\sqrt{2(\tau - \xi)}} \right) d\xi,
\]
where

\[ 1_{x=x_f(\tau)}(x) = \begin{cases} 
 1 & \text{if } x = x_f(\tau), \\
 0 & \text{if } x \neq x_f(\tau).
\end{cases} \]

The second term of (2.3.31) can be now calculated explicitly as

\[ E e^{\alpha x + \beta \tau} \mathcal{F}^{-1} \left\{ \int_0^T e^{-\omega^2 (\tau - \xi)} G(\omega, \xi) d\xi \right\} = -(S - E)1_{S=S_f(t)} + M_1(S, T - t, S_f(T)) + \int_t^T Q_1(S, t, u, S_f(u)) du, \]  

(2.3.35)

where

\[ 1_{S=S_f(t)}(S) = \begin{cases} 
 1 & \text{if } S = S_f(t), \\
 0 & \text{if } S \neq S_f(t),
\end{cases} \]  

(2.3.36)

and \( Q_1 \) is a function of four variables defined by:

\[ Q_1(x, y, z, w) = x e^{-\delta(z-y)} N(d_1(x, z - y, w)) - E e^{-r(z-y)} N(d_2(x, z - y, w)). \]  

(2.3.37)

Here \( M_1, d_1, d_2 \) are functions already defined in (2.2.20).

Using results in (2.3.33) and (2.3.35), the solution of the system (2.3.22) can be expressed as:

\[ H(\ln S_f(t) - \ln S)V(S, t) = -(S - E)1_{S=S_f(t)} + M_1(S, T - t, E) + \int_t^T Q_1(S, t, u, S_f(u)) du. \]  

(2.3.38)

The expression (2.3.38) is the integral representation of the price of an American vanilla call. This integral representation expresses the value of an American vanilla call as the sum of the value of its European counterpart and the early exercise premium, which depends on the optimal exercise function \( S_f(t) \). To calculate the price of the American vanilla call, we need to determine \( S_f(t) \), which is governed by the following integral equation (derived by letting
S = S_f(t) in (2.3.38):

\[ S_f(t) - E = M_1(S_f(t), T - t, E) + \int_t^T Q_1(S_f(t), t, u, S_f(u))du, \]  

(2.3.39)

where functions \( M_1 \) and \( Q_1 \) are defined in (2.2.20) and (2.3.37), respectively.

The pricing problem of American vanilla calls is now reduced to solving an integral equation. In the next section, we will recall some concepts, techniques to solve integral equations.

### 2.4 Integral equations

In general, an integral equation is an equation in which the unknown function appears inside the integral sign. A typical form of an integral equation, with the unknown \( u(x) \), is of the form:

\[ u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x, t, u(t))dt, \]  

(2.4.40)

where \( K, \alpha, \beta, f \) are given functions and \( \lambda \) is a constant parameter. Here \( \alpha, \beta \) are called limits of the integration.

The most frequently used integral equations fall under two major classes, namely Fredholm and Volterra integral equations. They are classified based on whether the limits of integration are fixed constants or at least one limit is a variable. More precisely, a typical Fredholm integral equation is of the form:

\[ u(x) = f(x) + \lambda \int_a^b K(x, t, u(t))dt. \]

On the other hand, a typical Volterra integral equation is of the form:

\[ u(x) = f(x) + \lambda \int_a^x K(x, t, u(t))dt, \]

where \( a, b \) are constant.

Integral equations can also be categorized as linear or nonlinear integral equations, which depend on whether the function \( K \) in (2.4.40) is linear or not with respect to the unknown \( u(x) \), respectively. Many linear integral equations and some simple nonlinear integral equations can be solved analytically by using powerful techniques such as Adomian decomposition method.
variational iteration method [43–45], the successive approximation methods [77], the Laplace transform method [77]. For highly nonlinear integral equations, these analytical methods however are no longer valid and consequently numerical methods are used to find the solutions.

A common numerical method used to solve highly nonlinear integral equations is proceeded in two main steps. First, integrals appear in the equations are approximated by using quadrature rules, such as the Trapezoidal rule or the Simpson rule. As a result, the integral equations are reduced to systems of nonlinear algebraic equations. Second, the Newton-Raphson iterative method is used to efficiently solve the newly-derived systems. This solution technique is illustrated through the following selected examples.

Example 1. Considering the following nonlinear Fredhom equation:

\[ u(x) = \sin x - x^2(e - \sqrt{e}) - \frac{1}{2}\sqrt{e} + \int_{\pi/6}^{\pi/2} (x^2 + \sin t) \cos te^{u(t)} \, dt, \]

which has the exact solution \( y = \sin x \). We now solve numerically this integral equation and then compare the obtained result with the exact solution.

The integral equation can be rewritten as:

\[ u(x) = f(x) + \int_{\pi/6}^{\pi/2} K(x, t, u(t)) \, dt, \quad (2.4.41) \]

where \( f(x) = \sin x - x^2(e - \sqrt{e}) - \frac{1}{2}\sqrt{e} \) and \( K(x, t, u(t)) = (x^2 + \sin t) \cos te^{u(t)} \).

Let \( \Pi \) be a uniform partition of the interval \([\pi/6, \pi/2]\) such that:

\[ \Pi : \pi/6 = t_1 < t_2 < \ldots < t_n < t_{n+1} = \pi/2, \quad h = \frac{\pi/2 - \pi/6}{n}. \]

Using the composite Trapezoidal rule, we can approximate the integral in \((2.4.41)\) as:

\[ \int_{\pi/6}^{\pi/2} K(x, t, u(t)) \, dt \simeq \frac{h}{2} \left[ K(x, t_1, u(t_1)) + 2 \sum_{i=2}^{n} K(x, t_i, u(t_i)) + K(x, t_{n+1}, u(t_{n+1})) \right]. \]

The integral equation is now reduced to the following algebraic equation:

\[ u(x) = f(x) + \frac{h}{2} \left[ K(x, t_1, u(t_1)) + 2 \sum_{i=2}^{n} K(x, t_i, u(t_i)) + K(x, t_{n+1}, u(t_{n+1})) \right]. \quad (2.4.42) \]
Our aim now is to find an approximation of the solution of (2.4.41) at each discrete point \( \{t_j\}_{j=1}^{n+1} \), i.e., we find \( u(t_j) \) for each \( j \in \{1, \ldots, n+1\} \). To this end, for each \( j \in \{1, \ldots, n+1\} \), we let \( x = t_j \) in (2.4.42) to get the equation:

\[
 u(t_j) = f(t_j) + \frac{h}{2} \left[ K(t_j, t_1, u(t_1)) + 2 \sum_{i=2}^{n} K(t_j, t_i, u(t_i)) + K(t_j, t_{n+1}, u(t_{n+1})) \right].
\]

As a result, we obtain a system of \( (n+1) \) equations of \( (n+1) \) unknown values \( \{u(t_j)\}_{j=1}^{n+1} \).

Using the Newton-Raphson iterative procedure to solve this system, we can obtain the desired values of \( \{u(t_j)\}_{j=1}^{n+1} \). We now plot the obtained numerical result to compare with the exact solution, as illustrated in Figure 2.1(a).

![Graph](image)

(a) Solution

![Graph](image)

(b) Numerical error

Figure 2.1: Comparison between the numerical solution and the exact solution
Figure 2.1(b) illustrates the percentage error of the numerical result compared with the exact solution when the number of discrete points used is \( n = 25 \). It is clear from this figure that the error is very small so that our numerical scheme can produce results with high accuracy.

**Example 2.** Considering the following nonlinear Volterra equation:

\[
u(x) = 2x - 2xe^x + e^x - 1 + \int_0^x (x + t)e^{u(t)}dt, \quad 0 \leq x \leq 1,
\]

which has the exact solution \( y = x \).

The integral equation can be rewritten as:

\[
u(x) = f(x) + \int_0^x K(x, t, u(t))dt,
\]

where \( f(x) = 2x - 2xe^x + e^x - 1 \) and \( K(x, t, u(t)) = (x + t)e^{u(t)} \).

Let \( \Pi \) be a uniform partition of the interval \([0, 1]\) such that:

\[
\Pi : 0 = t_1 < t_2 < \ldots < t_n < t_{n+1} = 1, h = \frac{1}{n}.
\]

Our aim now is to find an approximation of the solution of (2.4.43) at each discrete point \( \{t_j\}_{j=1}^{n+1} \), i.e., we find \( u(t_j) \) for each \( j \in \{1, \ldots, n + 1\} \). To this end, substituting \( x = t_1 \) into the integral equation (2.4.43), we obtain: \( u(t_1) = f(t_1) \). That means \( u(t_1) \) is now known.

We then substitute \( x = t_2 \) into (2.4.43), we have the following equation:

\[
u(t_2) = f(t_2) + \int_0^{t_2} K(t_2, t, u(t))dt,
\]

Using the Trapezoidal rule, we can reduce this equation to the following algebraic equation of \( u(t_2) \):

\[
u(t_2) = f(t_2) + \frac{h}{2} \left[ K(t_2, t_1, u(t_1)) + K(t_2, t_2, u(t_2)) \right].
\]

This equation can be easily solved by the Newton-Raphson iteration procedure so that one can obtain the value of \( u(t_2) \). Repeating the above procedure until we can recursively find all
the values of $u(t_3), \ldots, u(t_{n+1})$.

Figure 2.2 presents a comparison between the obtained numerical result with the exact solution when the number of discrete points used is $n = 25$. The two results agree very well, with the error is less than 0.45%, as shown in Figure 2.2(b). One can even obtain more accurate results when using more discrete points.

Example 3. We now use the solution technique, presented in the above two examples, to solve the integral equation (2.3.39) that governs the optimal exercise price of an American
vanilla call option. For convenience, we recall the equation here:

\[ S_f(t) - E = M_1(S_f(t), T - t, E) + \int_t^T Q_1(S_f(t), t, u, S_f(u))du. \]

Let \( \Pi \) be a uniform partition of the interval \([0, T]\) such that:

\[ \Pi : T = t_1 > t_2 > \ldots > t_n > t_{n+1} = 0, h = \frac{T}{n}. \]

Our aim now is to find an approximation of the solution of (2.3.39) at each discrete point \( \{t_j\}_{j=1}^{n+1} \), i.e., we find \( S_f(t_j) \) for each \( j \in \{1, \ldots, n+1\} \). To this end, by taking the limit \( t \to T \) of the integral equation, we can obtain: \( S_f(t_1) = S_f(T) = \max(E, rE/\delta) \) (readers are referred to [20] to see the proof of this). In order to find \( S_f(t_2) \), we substitute \( t = t_2 \) into the integral equation (2.3.39) and obtain:

\[ S_f(t_2) - E = M_1(S_f(t_2), t_1 - t_2, E) + \int_{t_2}^{t_1} Q_1(S_f(t_2), t_2, u, S_f(u))du. \]

Applying the Trapezoidal rule to approximate the integral, we obtain the following algebraic equation:

\[ S_f(t_2) - E = M_1(S_f(t_2), t_1 - t_2, E) + \frac{h}{2} \left[ Q_1(S_f(t_2), t_2, t_1, S_f(t_1)) + Q_1(S_f(t_2), t_2, t_2, S_f(t_2)) \right]. \]

This equation can be easily solved by the Newton-Raphson iteration procedure so that one can obtain the value of \( S_f(t_2) \). Repeating the above procedure until we can recursively find all the values of \( S_f(t_3), \ldots, S_f(t_{n+1}) \).

Figure 2.3 illustrates the obtained numerical result for the optimal exercise price of an American vanilla call option, the parameters are set as: \( E = $100, T = 1\text{ (year)}, D = 2\%, r = 5\%, \sigma = 20\% \) and the number of discrete points used is \( n = 17 \). It should be noted that the numerical method used here is the same as that used in [52]. The accuracy and efficiency of this method were discussed in [52].

Once the optimal exercise price has been found, we can obtain straightforwardly the option price using the pricing formula (2.3.38). Figure 2.4 compares the prices of an American call and its European counterpart, the parameters are set as: \( E = $100, T = 1\text{ (year)}, S = $100, \)
Figure 2.3: Optimal exercise price of an American call $E = 100$, $T = 1\text{ (year)}$, $D = 2\%$, $r = 5\%$, $\sigma = 20\%$.

Figure 2.4: Option price of an American call with $E = 100$, $T = 1\text{ (year)}$, $S = 100$, $D = 7\%$, $r = 5\%$, $\sigma = 20\%$. The number of discrete points used here is $n = 17$. As one can expect that due to the early exercise premium, the American call price is always greater or at least equal to that of its European option counterpart (cf. Section 2.3). As can also be seen from Figure 2.4, the early exercise premium, which is the difference between the two option prices, increases with time to expiry. This is indeed reasonable as with a greater time to expiry, the holder of an American vanilla option has more chances to optimally exercise
the option and thus has greater advantage over that of the European option counterpart.

In Chapters 6 and 7, we show that the 3-D pricing problem of Parisian options can be reduced to solving a pair of coupled integral equations. Therefore, we also need to know how to solve a pair of coupled integral equations. We now consider an example to illustrate this.

**Example 4.** Considering the following pair of coupled integral equations on \([0, \pi/2]\):

\[
\begin{align*}
    u_1(x) &= 2x \cos x - \frac{5}{6} x^3 - \sin x + \int_0^x (x + t)(u_1(t) + u_2(t))dt, \\
    e^{u_1(x)}u_2(x) &= \frac{1}{2}e^x(\sin x + \cos x) + x^2(1 - e^x) - \frac{1}{2} + \int_0^x (x^2 + u_2(t))e^{u_1(t)}dt,
\end{align*}
\]

which has the exact solution \(u_1(x) = e^{-x}\) and \(u_2(x) = e^x\).

The pair of coupled integral equations can be rewritten as:

\[
\begin{align*}
    u_1(x) &= f_1(x) + \int_0^x K_1(x, t, u_1(t), u_2(t))dt, \\
    u_2(x) &= f_2(x) + \int_0^x K_2(x, t, u_1(t), u_2(t))dt. \quad (2.4.44)
\end{align*}
\]

where \(f_1(x) = 2x \cos x - \frac{5}{6} x^3 - \sin x, K_1(x, t, u_1(t), u_2(t)) = (x + t)(u_1(t) + u_2(t)), \)

\(f_2(x) = \frac{1}{2}e^x(\sin x + \cos x) + x^2(1 - e^x) - \frac{1}{2}, \) and \(K_2(x, t, u_1(t), u_2(t)) = (x^2 + u_2(t))e^{u_1(t)}\).

Let \(\Pi\) be a uniform partition of the interval \([0, \pi/2]\) such that:

\[\Pi : 0 = x_1 < x_2 < \ldots < x_n < x_{n+1} = \pi/2, h = \frac{\pi/2}{n}.\]

Our aim now is to find an approximation of the solution of \((2.4.44)\) at each discrete point \(\{t_j\}_{j=1}^{n+1}\), i.e., we find \(u_1(x_j), u_2(x_j)\) for each \(j \in \{1, \ldots, n+1\}\). To this end, we first substitute \(x = x_1\) into \((2.4.44)\) to obtain:

\[
\begin{align*}
    u_1(x_1) &= 2x_1 \cos x_1 - \frac{5}{6} x_1^3 - \sin x_1, \\
    u_2(x_1) &= \frac{1}{2}e^{x_1}((\sin x_1 + \cos x_1) + x_1^2(1 - e^{x_1}) - \frac{1}{2}).
\end{align*}
\]

Therefore, \(u_1(x_1)\) and \(u_2(x_1)\) are now known.

In order to solve \(u_1(x_2)\) and \(u_2(x_2)\), we substitute \(x = x_2\) into \((2.4.44)\) and approximate the corresponding integrals by using the Trapezoidal rule. As a result, we obtain a pair of
coupled algebraic equations as:

\[ u_1(x_2) = f_1(x_2) + \frac{h}{2} [K_1(x_2, x_1, u_1(x_1), u_2(x_1)) + K_1(x_2, x_2, u_1(x_2), u_2(x_2))] , \]

\[ u_2(x_2) = f_2(x_2) + \frac{h}{2} [K_2(x_2, x_1, u_1(x_1), u_2(x_1)) + K_2(x_2, x_2, u_1(x_2), u_2(x_2))] . \]

Using the Newton-Raphson iteration procedure, we can easily solve this system of algebraic equation to obtain \( u_1(x_2) \) and \( u_2(x_2) \). Repeat the solution procedure applied to find \( u_1(x_2) \) and \( u_2(x_2) \), we can recursively obtain \( u_1(x_3) \) and \( u_2(x_3) \), ..., \( u_1(x_{n+1}) \) and \( u_2(x_{n+1}) \).

Figure 2.5: Comparison between the exact solution and the numerical solution

Figure 2.5 compares the obtained numerical solution and the exact solution at 17 discrete points. As can be seen from the figure, the two solutions agree perfectly. This proves the validity of our solution technique, which will be used later to solve the optimal exercise prices of American Parisian knock-out options.

2.5 Pricing European barrier options

2.5.1 Formulation

Under the Black-Scholes framework, the price of a European down-and-out call usually depends on the asset price \( S \), the current time \( t \), the asset barrier \( \bar{S} \), the exercise price \( E \), in addition to other constant parameters such as the volatility rate \( \sigma \), the risk-free interest rate
r, the time-dependent rebate $R(t)$ and the expiry time $T$. Usually, we assume $\bar{S} < E$ because the holder often accepts the loss of his/her option only when the option is out-of-money.

By definition, if at any time the asset price stays below the barrier, the value of a European down-and-out call will then be the corresponding value of the rebate, which is already determined. Therefore, we only need to price the barrier option when $\bar{S} < S < \infty$. The pricing domain $I$ of the option can be expressed mathematically as:

$$I = \{(S, t) | \bar{S} \leq S \leq \infty, 0 \leq t \leq T \}.$$  

Let $V(S, t)$ be the price of a European down-and-out call written on the underlying asset with price $S$, at time $t$. Using arguments similar to those used in Section 2.2.2 it can be shown that $V(S, t)$ satisfies equations (2.2.10), (2.2.11), (2.2.13). In addition, the knock-out feature demands:

$$V(\bar{S}, t) = R(t), \quad (2.5.45)$$

where $R(t)$ is the amount of rebate paid to the option holder if the knock-out feature is activated at time $t$. As a result, the PDE system governing the value of a European down-and-out call at any $S$, and any $t$ is given by:

$$\mathcal{A} \begin{cases} 
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0, \\
V(S, T) = \max(S - E, 0), \\
V(S, t) \sim S - E \text{ as } S \to \infty, \\
V(\bar{S}, t) = R(t), 
\end{cases} \quad (2.5.46)$$

where $\mathcal{A}$ is defined on $t \in [0, T], \ S \in [\bar{S}, \infty]$.

To make the system $\mathcal{A}$ even simpler, we use the same dimensionless variables defined in (2.2.15), except:

$$x = \ln \frac{S}{\bar{S}}, \quad R(\tau) = \bar{S}^{-1}R(t), \quad C(x, \tau) = \bar{S}^{-1}e^{-\alpha x - \beta \tau}V(S, t).$$

System (2.5.46) now becomes a dimensionless system, which includes a standard heat equation.
in a semi-finite domain together with the corresponding initial and boundary conditions:

\[
\begin{cases}
\frac{\partial C}{\partial \tau}(x, \tau) = \frac{\partial^2 C}{\partial x^2}(x, \tau), \\
C(x, 0) = \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0), \\
C(x, \tau) \sim e^{(1-\alpha)x} - e^{-\alpha x} \text{ as } x \to +\infty, \\
C(0, \tau) = e^{-\beta \tau} R(\tau).
\end{cases}
\]  

(2.5.47)

2.5.2 Initial-value heat problem in a semi-infinite domain

The pricing of European down-and-out call options is now reduced to solving the corresponding initial-value heat problem in a semi-infinite domain. This standard problem has been studied in a number of text books [24, 32, 42, 53]. We recall here some useful results on this problem.

The general initial-value problem for the homogeneous diffusion equation in a semi-infinite interval is:

\[
\begin{cases}
\frac{\partial u}{\partial \tau}(x, \tau) = a^2 \frac{\partial^2 u}{\partial x^2}(x, \tau), \ 0 < x < \infty, \ \tau \geq 0, a > 0 \\
u(x, 0) = f(x), \\
u(x, \tau) \sim f(x) \text{ as } x \to \infty, \\
u(0, \tau) = g(\tau),
\end{cases}
\]  

(2.5.48)

where \(f, g\) are arbitrarily prescribed functions. The solution of this problem is given by:

\[
u(x, \tau) = \frac{1}{2a\sqrt{\pi \tau}} \int_0^{+\infty} f(\xi) \left( e^{-\frac{(x-\xi)^2}{4a^2\tau}} - e^{-\frac{(x+\xi)^2}{4a^2\tau}} \right) d\xi + \int_0^\tau g(\xi) \frac{xe^{-\frac{4a^2(\tau-\xi)}{4a^2\pi(\tau-\xi)^3}}} {2a\sqrt{\pi(\tau-\xi)^3}} d\xi.
\]  

(2.5.49)

We consider an example to illustrate how to solve an initial-value heat problem in a semi-
infinite domain. More specifically, we consider the following system:

\[
\begin{aligned}
\frac{\partial u}{\partial \tau} (x, \tau) &= 4 \frac{\partial^2 u}{\partial x^2} (x, \tau), \quad 0 < x < +\infty, \quad 0 \leq \tau \leq 1, \\
u(x, 0) &= e^x, \\
u(x, \tau) &\sim e^x \text{ as } x \to \infty, \\
u(0, \tau) &= e^{4\tau}.
\end{aligned}
\] (2.5.50)

It can be easily checked that \(u_1(x, \tau) = e^{x+4\tau}\) is the exact solution of the system (2.5.50).

We now check whether the solution obtained by using the formula (2.5.49) is the same with the exact solution. More precisely, using the formula (2.5.49) with \(a = 2\) and \(f(x) = e^x\), we easily obtain the solution of the system (2.5.50) as follows:

\[
u_2(x, \tau) = \frac{1}{4\sqrt{\pi\tau}} \int_0^\tau e^x \left( e^{-\frac{(x-6\tau)^2}{16\tau}} - e^{-\frac{(x+\xi)^2}{16\tau}} \right) d\xi + \int_0^\tau e^{4\xi} \frac{xe^{-\frac{x^2}{16(\tau-\xi)}}}{4\sqrt{\pi(\tau-\xi)^3}} d\xi
\]

\[
e^{x+4\tau} N \left( \frac{x+8\tau}{2\sqrt{2\tau}} \right) - e^{-x+4\tau} N \left( \frac{-x+8\tau}{2\sqrt{2\tau}} \right) + \int_0^\tau e^{4\xi} \frac{xe^{-\frac{x^2}{16(\tau-\xi)}}}{4\sqrt{\pi(\tau-\xi)^3}} d\xi.
\]

The solution \(u_2\) looks differently from the solution \(u_1\). However, they are actually the same but in different equivalent forms. We can plot the graphs of the two solutions to illustrate this.

![Figure 2.6: Compare values of \(u_1(1, \tau)\) and \(u_2(1, \tau)\)](image)
Figure (2.6) illustrates the perfect agreement between \( u_1(x, \tau) \) and \( u_2(x, \tau) \) when \( x \) is fixed at 1. We can do similarly for many values of \( x \) to convince ourselves that the two solutions are the same.

We are now ready to solve the system (2.5.47) to obtain the closed-form formula for European down-and-out call options with rebates.

### 2.5.3 A closed-form solution of European down-and-out calls options

Using the formula (2.5.49) with \( a = 1 \) and \( f(x) = \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0) \), we easily obtain the solution of the system (2.5.47) as follows:

\[
C(x, \tau) = \int_0^{+\infty} \max\left( e^{(1-\alpha)\xi} - \frac{E}{S} e^{-\alpha \xi} - \frac{e^{-(x-\xi)^2/4\tau}}{2\sqrt{\pi\tau}} - \frac{e^{-(x+\xi)^2/4\tau}}{2\sqrt{\pi\tau}} \right) d\xi
\]

\[
+ \int_0^\tau e^{-\beta \xi} R(\xi) \frac{x e^{-(x-\xi)^2/(4(\tau-\xi)^2)}}{2\sqrt{\pi(\tau-\xi)^3}} d\xi.
\]

By using the assumption \( \bar{S} < E \), we can simplify \( I \) as follows:

\[
I = \int_{\ln \frac{E}{\bar{S}}}^{+\infty} \left( e^{(1-\alpha)\xi} - \frac{E}{\bar{S}} e^{-\alpha \xi} \right) \left( \frac{e^{-(x-\xi)^2/4\tau}}{2\sqrt{\pi\tau}} - \frac{e^{-(x+\xi)^2/4\tau}}{2\sqrt{\pi\tau}} \right) d\xi
\]

\[
= K(x, \tau, 1-\alpha) - \frac{E}{\bar{S}} K(x, \tau, -\alpha) - K(-x, \tau, 1-\alpha) + \frac{E}{\bar{S}} K(-x, \tau, -\alpha),
\]

where \( K \) is a functions of three variables and is defined as:

\[
K(x, y, z) = \frac{1}{2\sqrt{\pi y}} \int_{\ln \frac{E}{\bar{S}}}^{+\infty} e^{x\xi - \frac{(x-\xi)^2}{4y}} d\xi = \frac{e^{x^2yz^2}}{2\sqrt{\pi y}} \int_{\ln \frac{E}{\bar{S}}}^{+\infty} e^{-\frac{(x-z/2y)^2}{4y}} d\xi
\]

\[
= e^{z^2yz^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln \frac{E}{\bar{S}}+2yz}{\sqrt{2y}}} e^{-\frac{u^2}{2}} du, \quad u = \frac{x - \xi + 2yz}{\sqrt{2y}}
\]

\[
= e^{z^2yz^2} N\left( \frac{x - \ln \frac{E}{\bar{S}} + 2yz}{\sqrt{2y}} \right).
\]

Converting the solution to the original coordinate space \((S, t)\), with the notice that

\[
\alpha^2 + \beta = -\gamma, \quad (1-\alpha)^2 + \beta = -q,
\]
we obtain:

\[ V(S, t) = M(S, T - t, E) + \left(\frac{S}{S}\right)^\alpha \int_t^T K(S, t, u)\,du. \]

where

\[
M(x, y, z) = M_1(x, y, z) - \left(\frac{x}{S}\right)^{2\alpha} M_1\left(\frac{S^2}{x}, y, z\right)
\]

\[
K(x, y, z) = \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{2\pi} \sqrt{(z - y)^3}} e^{-\frac{(\ln x - \ln \bar{S})^2}{2\sigma^2(z - y)} + \beta \sigma^2(z - y)R(z)}. \tag{2.5.51}
\]

Here the function \( M_1 \) is defined in \((2.2.20)\).

We have now obtained the closed-form pricing formula for European down-and-out call options with time-dependent rebates, under the Black-Scholes model. It should be noted that this pricing formula is the same as the one found by Kwok \[56, \text{p. 231-232}\].
Chapter 3

Pricing American-style down-and-out calls with rebates

3.1 Introduction

Barrier options are among the most common exotic options used in foreign exchange, interest rate and equity option markets. One of the reasons for the popularity of barrier options is that they provide a more flexible and cheaper way for hedging and speculating than their vanilla option counterparts. For instance, a corporation may wish to control its raw material prices by hedging against the risk that the prices might go “too high”, i.e., over a certain barrier. In such a situation, the use of vanilla options may involve over-hedging (i.e. providing protection against risks that need not be hedged) with a high cost. In contrast, the use of a suitable barrier option will not only reduce effectively all potential risks with a lower cost, but also gain possible profits from favorable movement of the underlying price. In addition, speculators also prefer barrier options to vanilla options because they can choose a wider range of barrier options that suit their views about the likely future movement of the asset price, again with a relatively lower cost. For example, if the price of the underlying asset fluctuates strongly and then hits a certain price level, knock-in options can lead to a higher return than their vanilla option counterparts as a result of a lower cost of establishing a position in the market. Similarly, speculators can reduce costs with a knock-out option if they believe that the price of the underlying asset will remain within a stable price range.

Compared to vanilla options, holders of barrier options however always face a higher
risk that their option contracts might be worthless as a result of the barrier options being knocked out or failing to be knocked in. To compensate for this potential risk, barrier options sometimes come with a rebate, which is an amount of cash paid to the option holder if the worst scenario takes place. The rebate can be set to be a constant or to vary with time. In this paper, we shall focus on the time-dependent rebate for two main reasons. First, the constant rebate is a special case of the time-dependent rebate. Therefore, the solution procedure that works for the latter should also work for the former. Second, it is financially more reasonable to assume the rebate to be a decreasing function of time because the rebate is set to partly compensate for the loss of the embedded option, the value of which also decreases with time.

In subsequent sections, we use an integral equation approach to price American-style down-and-out calls. The continuous Fourier sine transform (FST) method is used in our approach rather than the probability theory used previously in [3, 31, 35, 56]. The key idea behind our approach is to reduce the partial differential equation (PDE) governing the price of an American-style down-and-out call to an ordinary differential equation (ODE), the solution of which can be easily found (in the Fourier sine space) and analytically inverted into the original space. As a result, we can re-derive the “early exercise premium representation” for American-style down-and-out calls without rebate [31, 56]. More importantly, when time-dependent rebates are included in the contract of the barrier options, our approach results in a more general integral representation, with the presence of an extra term associated with the rebate. More precisely, we show that the price of an American-style down-and-out call with a time-dependent rebate can be decomposed into two components: the price of its European counterpart with the given rebate and an early exercise premium associated with the early exercise right.

This chapter is organized as follows. Section 3.2 presents the governing PDE system of an American-style down-and-out call with a time-dependent rebate. The analytical solution procedure is presented in Section 3.3 while the numerical implementation is discussed in Section 3.4. Section 3.5 presents some numerical results to demonstrate the effects of time-dependent rebates on the prices of American-style down-and-out calls as well as their optimal exercise boundaries. The chapter ends with some concluding remarks in Section 3.6.
3.2 The governing PDE system

Under the Black-Scholes framework, the price of an American-style down-and-out call depends on the underlying asset price $S$, the current time $t$, in addition to other constant parameters: the exercise price $E$, the volatility rate $\sigma$, the risk-free interest rate $r$, the dividend rate $\delta$, the expiry time $T$ and the time-dependent rebate $R(t)$. Let $V(S, t)$ be the option price associated with asset price $S$ and time $t$. Usually, we assume $\bar{S} < E$ because the holder often accepts the loss of his/her option only when the option is out-of-money.

We first specify the pricing domain of an American-style down-and-out call. By definition, if at any time $t$ the asset price stays below the barrier, the value of the option will then be the corresponding value of the rebate, which means:

$$V(\bar{S}, t) = R(t).$$

(3.2.1)

On the other hand, if the asset price is above the optimal exercise boundary, $S_f(t)$, then the option should be exercised immediately. In this case, the option value is equal to the payoff received by exercising the option. We therefore only need to price the barrier option when $\bar{S} < S < S_f(t)$. The pricing domain $I$ of the option can be expressed mathematically as:

$$I = \{(S, t)|\bar{S} \leq S \leq S_f(t), 0 \leq t \leq T\}.$$

Using arguments similar to those used in Sections 2.3 and 2.5.1 it can be shown that $V(S, t)$ satisfies the following system:

$$\begin{cases}
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0, \\
V(S, T) = \max(S - E, 0), \\
V(S_f(t), t) = S_f(t) - E, \\
\frac{\partial V}{\partial S}(S_f(t), t) = 1, \\
V(\bar{S}, t) = R(t),
\end{cases}$$

(3.2.2)

where the PDE in (3.2.2) is defined on $t \in [0, T), S \in (\bar{S}, S_f(t))$. 

One can easily realize that system (3.2.2) is very similar to the system (2.3.22), which governs the prices of American-style vanilla calls. The only difference between them lies in the last equation of (3.2.2), which provides a non-homogeneous boundary condition at \( S = \bar{S} > 0 \), instead of a homogeneous boundary condition at \( S = 0 \) as in the case of vanilla options. Such a simple difference however has caused a considerable difficulty in pricing the barrier options both numerically and analytically. More specifically, the discontinuity of the option payoff at the barrier has caused the “near-barrier” issue when using the lattice/grid-based methods [19, 33, 35]. The results obtained from these methods are thus not reliable in the region near the barrier. In addition, the non-homogeneous boundary condition at \( S = \bar{S} \) has also prevented the Fourier transform method developed by Chiarella et al. [20] for pricing American vanilla options to be easily extended to price American barrier options. More details on this are discussed in the next section.

It should also be mentioned that for most contracts of barrier options, rebates are not given at expiry, i.e., \( R(T) = 0 \). In addition, the earlier the knock-out feature is activated, the more loss the holder suffers and the greater the rebate should be paid to the holder. As a result, \( R(t) \) should be chosen as a monotonically decreasing function of \( t \). It should also be noted that under the Black-Scholes model, \( V(S, t) \) is a smooth function with respect to \( t \), for all values of \( S \). From (3.2.1), it is therefore necessary to assume \( R(t) \) to be a smooth function with respect to \( t \) in order to guarantee the existence and uniqueness of the solution of the PDE system (3.2.2).

### 3.3 Our analytical solution procedure

To derive a decomposition for \( V(S, t) \), we solve the pricing system (3.2.2) using the continuous FST. More specifically, the PDE system (3.2.2) is first reduced to a dimensionless heat equation in a finite time-dependent domain. Then, by using FST, the resulting heat equation can be further reduced to an initial value ODE, the solution of which can be easily obtained in the Fourier sine space and analytically converted back to the real space coordinate.
3.3.1 Applying the Fourier sine transform

We shall first non-dimensionalize all variables by introducing the dimensionless variables and parameters:

\[ x = \ln \frac{S}{\bar{S}}, \quad l = \frac{\sigma^2}{2} (T - t), \quad x_f(l) = \ln \frac{S_f(t)}{\bar{S}}, \]
\[ R(l) = \tilde{S}^{-1} R(t), \quad C(x, l) = \tilde{S}^{-1} e^{-\alpha x - \beta l} V(S, t), \]  
\[(3.3.3)\]

where \(\alpha, \beta\) are defined in (2.2.15).

System (3.2.2) now becomes a dimensionless system, which includes a standard heat equation together with the following corresponding initial and boundary conditions:

\[
\begin{align*}
\frac{\partial C}{\partial t}(x, l) &= \frac{\partial^2 C}{\partial x^2}(x, l), \\
C(x, 0) &= f(x), \\
B \begin{cases} \\
C(x_f(l), l) &= g_1(x_f(l), l), \\
\frac{\partial C}{\partial x}(x_f(l), l) &= g_2(x_f(l), l), \\
C(0, l) &= e^{-\beta l} R(l), 
\end{cases}
\end{align*}
\]
\[(3.3.4)\]

where \(f, g_1, g_2\) are functions defined as in (2.3.24). Here \(B\) is defined on \(l \in [0, T \sigma^2/2]\) and \(x \in [0, x_f(l)]\).

Although the PDE system (3.3.4) is somewhat simpler than (3.2.2), it is still difficult to be directly solved. In fact, it is a heat problem in a finite time-dependent domain. The existence and uniqueness of the solution of the heat problem in time-dependent domains have been studied in several published works [9–11, 20]. In particular, Chiarella et al. [20] have successfully solved a heat problem in a semi-infinite time-dependent domain by using the Fourier transform. It is however difficult to extend their method to solve (3.3.4) because the \(x\)-domain here is a finite time-dependent one. To the best of our knowledge, there has been no published work that uses an analytical method to simultaneously obtain the unknown pair \(C(x, l)\) and \(x_f(l)\) in (3.3.4). This is the focus of our work in this chapter. More precisely, we use the FST to formulate \(C(x, l)\) in terms of \(x_f(l)\), where \(x_f(l)\) is the solution of an explicit
integral equation.

For readers’ convenience, we recall here the definition of the FST of a function. More specifically, the FST of \( C(x, l) \) with respect to \( x \), denoted by \( \mathcal{F}_s\{C(x, l)\} \), is defined as:

\[
\mathcal{F}_s\{C(x, l)\} \equiv U(\omega, l) = \int_0^\infty C(x, l) \sin(\omega x) dx,
\]

with the corresponding inversion:

\[
\mathcal{F}_s^{-1}\{U(\omega, l)\} = \frac{2}{\pi} \int_0^\infty U(\omega, l) \sin(\omega x) d\omega.
\]

As we will use the continuous Fourier cosine transform (FCT) in our solution procedure later, we also recall here the definition of FCT and its inversion as:

\[
\mathcal{F}_c\{C(x, l)\} \equiv U(\omega, l) = \int_0^\infty C(x, l) \cos(\omega x) dx, \quad \mathcal{F}_c^{-1}\{U(\omega, l)\} = \frac{2}{\pi} \int_0^\infty U(\omega, l) \cos(\omega x) d\omega,
\]

respectively. To apply the FST, the \( x \)-domain of (3.3.4), which is a finite domain, first needs to be extended to a semi-infinite domain first by expressing the PDE as:

\[
H(x_f(l) - x) \frac{\partial C}{\partial l}(x, l) = H(x_f(l) - x) \frac{\partial^2 C}{\partial x^2}(x, l) \tag{3.3.5}
\]

where \( H(x) \) is the Heaviside function, defined as:

\[
H(x) = \begin{cases} 
1, & \text{if } x > 0, \\
1/2, & \text{if } x = 0, \\
0, & \text{if } x < 0.
\end{cases}
\tag{3.3.6}
\]

The reason for the appearance of the factor of \( 1/2 \) at the point of discontinuity is explained in [20]. The initial and boundary conditions remain unchanged.

By definition, we have:

\[
\mathcal{F}_s\left\{ H(x_f(l) - x) C(x, l) \right\} = \int_0^{x_f(l)} C(x, l) \sin(\omega x) dx \equiv \tilde{C}(\omega, l),
\]
where for convenience, we denote \( \hat{C}(\omega, l) \) as the FST of \( H(x_f(l) - x)C(x, l) \).

Direct calculation shows that:

\[
\mathcal{F}_s \left\{ H(x_f(l) - x) \frac{\partial C(x, l)}{\partial l} \right\} = \frac{\partial \hat{C}}{\partial l}(\omega, l) - x_f'(l)C(x_f(l), l) \sin(\omega x_f(l)) \tag{3.3.7}
\]

and

\[
\mathcal{F}_s \left\{ H(x_f(l) - x) \frac{\partial^2 C(x, l)}{\partial x^2} \right\} = \sin(\omega x_f(l)) \frac{\partial C(x_f(l), l)}{\partial x} - \omega \cos(\omega x_f(l)) C(x_f(l), l) \\
+ \omega C(0, l) - \omega^2 \hat{C}(\omega, l), \tag{3.3.8}
\]

and

\[
\mathcal{F}_c \left\{ H(x_f(l) - x) \frac{\partial^2 C(x, l)}{\partial x^2} \right\} = \cos(\omega x_f(l)) \frac{\partial C(x_f(l), l)}{\partial x} + \omega \sin(\omega x_f(l)) C(x_f(l), l) \\
- \frac{\partial C}{\partial x}(0, l) - \omega^2 \mathcal{F}_c \left\{ H(x_f(l) - x) C(x, l) \right\}. \tag{3.3.9}
\]

Note that here the notation \( x_f'(l) \) denotes the first derivative of \( x_f \) with respect to \( l \).

We now explain the reason of choosing FST over FCT in solving (3.3.4). From the formulas (3.3.8) and (3.3.9), it is clear that while \( \frac{\partial C}{\partial x}(0, l) \) vanishes from \( \mathcal{F}_s \left\{ H(x_f(l) - x) \frac{\partial^2 C(x, l)}{\partial x^2} \right\} \), it does appear in \( \mathcal{F}_c \left\{ H(x_f(l) - x) \frac{\partial^2 C(x, l)}{\partial x^2} \right\} \). Therefore, if the FCT is used to solve the system (3.3.4), the term \( \frac{\partial C}{\partial x}(0, l) \) must be eliminated during the solution procedure because it is also unknown. Since this complicates the solution procedure unnecessarily, to effectively solve the system (3.3.4), FST is a better choice than FCT.

As a result of applying the FST with respect to \( x \), the system (3.3.5) can be reduced to the following initial-value ODE in the Fourier sine space:

\[
\frac{\partial \hat{C}}{\partial l}(\omega, l) + \omega^2 \hat{C}(\omega, l) = G(\omega, l),
\]

where

\[
G(\omega, l) = \sin(\omega x_f(l)) \frac{\partial C(x_f(l), l)}{\partial x} - \omega \cos(\omega x_f(l)) C(x_f(l), l) + \omega C(0, l) \\
+ x_f'(l)C(x_f(l), l) \sin(\omega x_f(l)),
\]
with initial condition \( \hat{C}(\omega, 0) = \int_0^{x_f(l)} C(x, 0) \sin(\omega x) dx \). The solution of this initial-value ODE can be easily solved as:

\[
\hat{C}(\omega, l) = \hat{C}(\omega, 0) e^{-\omega^2 l} + \int_0^l e^{-\omega^2 (l - \xi)} G(\omega, \xi) d\xi.
\] (3.3.10)

### 3.3.2 Inverting the Fourier sine transform

As \( \hat{C}(\omega, l) \) denotes the FST of \( H(x_f(l) - x) C(x, l) \), from (3.3.10), we can now express the solution of the system (3.3.4) as follows:

\[
H(x_f(l) - x) C(x, l) = \hat{C}(\omega, 0) e^{-\omega^2 l} + \int_0^l e^{-\omega^2 (l - \xi)} G(\omega, \xi) d\xi.
\] (3.3.11)

Furthermore, as \( S = \bar{S} e^x \), \( S_f(t) = \bar{S} e^{x_f(t)} \) and \( V(S, t) = \bar{S} e^{\alpha x + \beta l} C(x, l) \), by multiplying \( \bar{S} e^{\alpha x + \beta l} \) to both sides of (3.3.11), we can express the solution of the system (3.2.2) as follows:

\[
H \left( \ln S_f(t) - \ln S \right) V(S, t) = \bar{S} e^{\alpha x + \beta l} \mathcal{F}_s^{-1} \{ \hat{C}(\omega, 0) e^{-\omega^2 l} \} + \bar{S} e^{\alpha x + \beta l} \mathcal{F}_s^{-1} \left\{ \int_0^l e^{-\omega^2 (l - \xi)} G(\omega, \xi) d\xi \right\}.
\] (3.3.12)

The first and second terms in the right hand side of (3.3.12) need to be calculated explicitly in order to obtain an integral representation for the solution of the system (3.2.2).

**Proposition 1.** The first term in the right hand side of (3.3.12) can be expressed as:

\[
\bar{S} e^{\alpha x + \beta l} \mathcal{F}_s^{-1} \{ \hat{C}(\omega, 0) e^{-\omega^2 l} \} = M(S, T - t, E) - M(S, T - t, S_f(T)),
\]

where \( M \) is defined as in (2.5.51).

**Proof.** We first calculate the inverse FCT of \( e^{-\omega^2 l} \) as follows:

\[
\mathcal{F}_c^{-1} \{ e^{-\omega^2 l} \} = \frac{2}{\pi} \int_0^{+\infty} e^{-\omega^2 l} \cos(\omega x) dx = \frac{e^{-\frac{x^2}{4l}}}{\sqrt{\pi l}}.
\]

Applying the Convolution theorem of the FST [66, chapter 3]

\[
\mathcal{F}_s^{-1} \{ \mathcal{F}_s(f) \mathcal{F}_c(g) \} = \frac{1}{2} \int_0^{+\infty} f(\zeta) [g(|x - \zeta|) - g(|x + \zeta|)] d\zeta,
\]
it follows:

\[
\mathcal{F}_s^{-1}\{\hat{C}(\omega, 0)e^{-\omega^2 t}\} = \mathcal{F}_s^{-1}\left\{ \mathcal{F}_s\left\{ H(x_f(0) - x)f(x) \right\} \mathcal{F}_c\left\{ \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{\pi l}} \right\} \right\}
\]

\[
= \frac{1}{2\sqrt{\pi l}} \int_{0}^{+\infty} H(x_f(0) - x)f(x) \left( e^{-\frac{(x-u)^2}{4}} - e^{-\frac{(x+u)^2}{4}} \right) du
\]

\[
= \frac{1}{2\sqrt{\pi l}} \int_{0}^{x_f(0)} \max\left( e^{(1-\alpha)u - \frac{E}{S}e^{-\alpha u}}, 0 \right) \left( e^{-\frac{(x-u)^2}{4}} - e^{-\frac{(x+u)^2}{4}} \right) du
\]

\[
= \frac{1}{2\sqrt{\pi l}} \int_{\ln \frac{E}{S}}^{x_f(0)} e^{(1-\alpha)u - \frac{(x-u)^2}{4}} du - \frac{E}{2S\sqrt{\pi l}} \int_{\ln \frac{E}{S}}^{x_f(0)} e^{-\alpha u - \frac{(x-u)^2}{4}} du
\]

\[
- \frac{1}{2\sqrt{\pi l}} \int_{\ln \frac{E}{S}}^{x_f(0)} e^{(1-\alpha)u - \frac{(x+u)^2}{4}} du + \frac{E}{2S\sqrt{\pi l}} \int_{\ln \frac{E}{S}}^{x_f(0)} e^{-\alpha u - \frac{(x+u)^2}{4}} du
\]

\[
= K(x, l, 1 - \alpha) - \frac{E}{S}K(x, l, -\alpha) - K(-x, l, 1 - \alpha) + \frac{E}{S}K(-x, l, -\alpha), \quad (3.3.13)
\]

where

\[
K(x, y, z) = \frac{1}{2\sqrt{\pi y}} \int_{\ln \frac{E}{S}}^{x_f(0)} e^{zv - \frac{(x-u)^2}{4y}} du = \frac{1}{2\sqrt{\pi y}} e^{xz^2/4y} \int_{\ln \frac{E}{S}}^{x_f(0)} e^{-\frac{1}{2}(x-u+2yz)^2} du
\]

\[
= \frac{1}{\sqrt{2\pi}} e^{xz^2/4y} \int_{\frac{v - \ln \frac{E}{S} + 2yz}{\sqrt{2y}}}^{\frac{x - x_f(0) + 2yz}{\sqrt{2y}}} e^{-\frac{u^2}{2}} du, \quad v = \frac{x - u + 2yz}{\sqrt{2y}}
\]

\[
= e^{xz^2/4y} \left[ N\left( \frac{x - \ln \frac{E}{S} + 2yz}{\sqrt{2y}} \right) - N\left( \frac{x - x_f(0) + 2yz}{\sqrt{2y}} \right) \right].
\]

Therefore, by converting back the dimensionless variables to the original variables, we obtain:

\[
\tilde{S}e^{\alpha x + \beta t} \mathcal{F}_s^{-1}\{\hat{C}(\omega, 0)e^{-\omega^2 t}\} = M(S, T - t, E) - M(S, T - t, S_f(T)),
\]

where \( M \) is defined as in (2.5.51).

**Proposition 2.** The second term in the right hand side of (3.3.12) can be expressed as:

\[
\tilde{S}e^{\alpha x + \beta t} \mathcal{F}_s^{-1}\left\{ \int_{0}^{t} e^{-\omega^2(l-\xi)} G(\omega, \xi) d\xi \right\}
\]

\[
= -(S - E) \mathcal{F}_s\{ T(S_f(t) - S) + M(S, T - t, S_f(T)) + \int_{t}^{T} Q(S, t, u, S_f(u)) du \}
\]
where $1_{S=S_f(t)}$ and $M$ are defined as in (2.3.36) and (2.5.51), respectively. Here,

$$Q(x, y, z, w) = Q_1(x, y, z, w) - \left(\frac{x}{S}\right)^{2\alpha} Q_1 \left(\frac{S^2}{x}, y, z, w\right) + \left(\frac{x}{S}\right)^\alpha K(x, y, z), \quad (3.3.14)$$

with $Q_1$ and $K$ are defined as in (2.3.37) and (2.5.51), respectively.

**Proof.** We first calculate $F_s^{-1}\left\{\int_0^l e^{-\omega^2(t-\xi)} G(\omega, \xi) d\xi\right\}$ as:

$$F_s^{-1}\left\{\int_0^l e^{-\omega^2(t-\xi)} G(\omega, \xi) d\xi\right\} = \frac{2}{\pi} \int_0^l \int_0^{+\infty} e^{-\omega^2(t-\xi)} G(\omega, \xi) \sin(\omega x) d\omega d\xi,$$

$$(\text{Fubini’s theorem})$$

$$= \frac{2}{\pi} \int_0^l \int_0^{+\infty} e^{-\omega^2(t-\xi)} G(\omega, \xi) \sin(\omega x) d\omega d\xi,$$

$$= \frac{2}{\pi} \int_0^l \left(g_2(x_f(\xi), \xi) + x_f(\xi) g_1(x_f(\xi), \xi)\right) \int_0^{+\infty} e^{-\omega^2(t-\xi)} \sin(\omega x_f(\xi)) \sin(\omega x) d\omega d\xi$$

$$- \frac{2}{\pi} \int_0^l g_1(x_f(\xi), \xi) \int_0^{+\infty} e^{-\omega^2(t-\xi)} \omega \cos(\omega x_f(\xi)) \sin(\omega x) d\omega d\xi$$

$$+ \frac{2}{\pi} \int_0^l e^{-\beta \xi} R(\xi) \int_0^{+\infty} e^{-\omega^2(t-\xi)} \sin(\omega x) d\omega d\xi.$$

By noting the following equality:

$$\int_0^{+\infty} e^{-ax^2} \cos(kx) dx = \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-\frac{k^2}{4a}}, \forall a > 0 \quad (3.3.15)$$

we can obtain:

$$F_s^{-1}\left\{\int_0^l e^{-\omega^2(t-\xi)} G(\omega, \xi) d\xi\right\}$$

$$= \int_0^l \frac{1}{2\sqrt{\pi(l-\xi)}} \left(g_2(x_f(\xi), \xi) + x_f(\xi) g_1(x_f(\xi), \xi)\right) \left(e^{-\frac{(x_f(\xi)-x)^2}{4(l-\xi)}} - e^{-\frac{(x_f(\xi)+x)^2}{4(l-\xi)}}\right) d\xi$$

$$- \int_0^l \frac{1}{4\sqrt{\pi(l-\xi)^3}} g_1(x_f(\xi), \xi) \left((x + x_f(\xi)) e^{-\frac{(x+x_f(\xi))^2}{4(l-\xi)}} + (x - x_f(\xi)) e^{-\frac{(x-x_f(\xi))^2}{4(l-\xi)}}\right) d\xi$$

$$+ \int_0^l \frac{x}{2\sqrt{\pi(l-\xi)^3}} e^{-\beta \xi} R(\xi) e^{-\frac{x^2}{4(l-\xi)}} d\xi$$

$$= P(x, l, 1 - \alpha) - \frac{E}{S} P(x, l, -\alpha) - P(-x, l, 1 - \alpha) + \frac{E}{S} P(-x, l, -\alpha)$$

$$+ \int_0^l \frac{x}{2\sqrt{\pi(l-\xi)^3}} e^{-\beta \xi} R(\xi) e^{-\frac{x^2}{4(l-\xi)}} d\xi,$$
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where

\[
P(x, y, z) = \int_{0}^{y} e^{-\frac{(z-x_f(\xi))^2}{2(y-\xi)}} e^{x+y(\xi-
\frac{z-x_f(\xi)}{2(y-\xi)})} d\xi
\]

\[
= -\frac{1}{\sqrt{2\pi}} \int_{0}^{y} e^{x+y(\xi-
\frac{z-x_f(\xi)+2z(y-\xi)}{\sqrt{2(y-\xi)}})} \frac{\partial}{\partial \xi} \left( \frac{x-x_f(\xi)+2z(y-\xi)}{\sqrt{2(y-\xi)}} \right) d\xi
\]

\[
= -e^{x+y\xi} \lim_{\xi \to y} e^{-(z^2+\beta)\xi} N \left( \frac{x-x_f(\xi)+2z(y-\xi)}{\sqrt{2(y-\xi)}} \right) + e^{x+y^2} N \left( \frac{x-x_f(0)+2zy}{\sqrt{2y}} \right)
\]

\[
- e^{x+y^2} \frac{z^2+\beta}{\sqrt{2(y-\xi)}} d\xi.
\]

Therefore, by converting back the dimensionless variables to the original variables, we obtain:

\[
\tilde{S} e^{\alpha x+\beta t} \mathcal{F}^{-1}_s \left\{ \int_{0}^{t} e^{-\omega^2(t-\xi)} G(\omega, \xi) d\xi \right\}
\]

\[
= -(S-E) \mathbf{1}_{S=S_f(t)}(S) + M(S, T-t, S_f(T)) + \int_{t}^{T} Q(S, t, u, S_f(u)) du.
\]

3.3.3 Integral representation

With Proposition 1 and Proposition 2, we can now obtain an integral representation for the price of an American-style down-and-out call at any asset price \(S\) and time \(t\), denoted by \(C^{do}(S, t; S_f^{do}(t))\), with a given time-dependent rebate \(R(t)\). Here \(S_f^{do}(t)\) denotes the associated optimal exercise price of the option. The below theorem is the main result of this chapter.

**Theorem 1.** The value of an American-style down-and-out call with a time-dependent rebate can be decomposed into two components, the value of its European counterpart with the given rebate, \(C^{do}(S, t; S_f^{do}(t))\), and the early exercise premium, \(EP(S, t; S_f^{do}(t))\), associated with the early exercise right:

\[
C^{do}_A(S, t; S_f^{do}(t)) = C^{do}_E(S, t) + EP(S, t; S_f^{do}(t)) \quad (3.3.16)
\]
where
\[
C_E^{do}(S, t) = M(S, T - t, E) + \int_t^T \left( \frac{S}{S} \right)^\alpha K(S, t, u)du
\]
and
\[
EP(S, t; S_f^{do}(t)) = \int_t^T \left[ Q_1(S, t, u, S_f^{do}(u)) - \left( \frac{S}{S} \right)^{2\alpha} Q_1 \left( \frac{S^2}{S}, t, u, S_f^{do}(u) \right) \right] du.
\]
Moreover, the optimal exercise boundary, \(S_f^{do}(t)\), is given by:
\[
S_f^{do}(t) - E = C_E^{do}(S_f^{do}(t), t) + EP(S_f^{do}(t), t; S_f^{do}(t)).
\]
(3.3.17)

Here \(M, K\) are defined as in (2.5.51), while \(Q_1\) is defined as in (2.3.37).

**Proof.** Substituting the result of Proposition 1 and Proposition 2 into (3.3.12), we obtain:
\[
H(\ln S_f^{do}(t) - \ln S)C_A^{do}(S, t; S_f^{do}(t)) = -(S - E).1_{S=S_f^{do}(t)}(S) + M(S, T - t, E) + \int_t^T \left( \frac{S}{S} \right)^\alpha K(S, t, u)du
\]
\[
+ \int_t^T \left[ Q_1(S, t, u, S_f^{do}(u)) - \left( \frac{S}{S} \right)^{2\alpha} Q_1 \left( \frac{S^2}{S}, t, u, S_f^{do}(u) \right) \right] du.
\]
(3.3.18)

For \(0 < S < S_f^{do}(t)\), as \(H(\ln S_f^{do}(t) - \ln S) = 1\) and \(1_{S=S_f^{do}(t)}(S) = 0\), we achieve:
\[
C_A^{do}(S, t; S_f^{do}(t)) = C_E^{do}(S, t) + EP(S, \tau; S_f^{do}(t))
\]
where
\[
C_E^{do}(S, t) = M(S, T - t, E) + \int_t^T \left( \frac{S}{S} \right)^\alpha K(S, t, u)du
\]
and
\[
EP(S, t; S_f^{do}(t)) = \int_t^T \left[ Q_1(S, t, u, S_f^{do}(u)) - \left( \frac{S}{S} \right)^{2\alpha} Q_1 \left( \frac{S^2}{S}, t, u, S_f^{do}(u) \right) \right] du.
\]
For $S = S_f^{do}(t)$, as $H(0) = \frac{1}{2}$, $C^d_A(S_f^{do}(t), t; S_f^{do}(t)) = S_f^{do}(t) - E$, and $1_{S=S_f^{do}(t)}(S) = \frac{1}{2}$, from (3.3.18), we obtain the governing equation for the optimal exercise boundary, $S_f^{do}(t)$, as follows:

$$S_f^{do}(t) - E = C^d_E(S_f^{do}(t), \tau) + EP(S_f^{do}(t), \tau; S_f^{do}(t)).$$

It should be noted that if the rebate is set to zero, the expression (3.3.16) will reduce to the “early exercise premium representation” (derived in [31, 56]) for American-style down-and-out calls without rebate. This clearly shows that our results can be viewed as a generalization of those in the literature. It is also worthwhile to note that once the optimal exercise boundary is determined by numerically solving the integral equation (3.3.17), we can easily obtain the option price by using (3.3.16). Hedging parameters $\Delta$, $\Gamma$, $\Theta$, $\text{Vega}$ and $\text{Rho}$, known as the Greeks, can also then be obtained by differentiating the formula (3.3.16) with respect to the relevant parameters.

### 3.3.4 Hedging parameters

As an illustrative example, we calculate explicitly the Delta below. Other hedging parameters can be calculated in a similar manner.

**Proposition 3.** The Delta of the option when $S \geq \bar{S}$, i.e., $\frac{\partial V_2}{\partial S}(S, u; t)$, can be calculated as:

$$\frac{\partial}{\partial S} C^d_A(S, t; S_f^{do}(t)) = \tilde{K}_1(S, t) + \tilde{M}(S, T - t, E) + \int_t^T L(S, t, u, S_f^{do}(u)) du, \quad \forall S < S_f^{do}(t),$$

where,

$$\tilde{K}_1(x, y) = \frac{\sqrt{2} \alpha x^{\alpha - 1}}{\sqrt{\pi} S^\alpha} \int_0^\infty R \left( y + \left( \frac{\ln x - \ln \bar{S}}{\sigma (\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T-y}}}) \right)^2 \right) \frac{\beta (\ln x - \ln \bar{S})^2}{2^{\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T-y}}}} \frac{1}{2 (\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T-y}})^2} d\eta,$$

$$+ \left( \frac{x}{S} \right)^\alpha \frac{\sqrt{2} \sigma}{\sqrt{\pi} x} \int_0^{\sqrt{T-y}} e^{-\frac{(\ln x - \ln \bar{S})^2}{2 \sigma^2 y^2}} + \frac{\beta}{2} \sigma^2 y^2 \left[ \beta R(y + v^2) + \frac{2}{\sigma^2} R'(y + v^2) \right] dv$$

$$- \frac{\sqrt{2} R(T)}{x \sigma \sqrt{\pi (T - y)}} \left( \frac{x}{S} \right)^\alpha e^{\frac{\beta^2}{2} (T-y) - \frac{(\ln x - \ln \bar{S})^2}{2 \sigma^2 (T-y)}}.$$
and

\[ \tilde{M}(x, y, z) = \tilde{M}_1(x, y, z) - \frac{2\alpha x^{2\alpha - 1}}{S^{2\alpha}} M_1 \left( \frac{S^2}{x}, y, z \right) + \left( \frac{x}{S} \right)^{2\alpha - 2} \tilde{M}_1 \left( \frac{S^2}{x}, y, z \right), \]

\[ L(x, y, z, w) = \tilde{Q}_1(x, y, z, w) - \frac{2\alpha x^{2\alpha - 1}}{S^{2\alpha}} Q_1 \left( \frac{S^2}{x}, y, z, w \right) + \left( \frac{x}{S} \right)^{2\alpha - 2} \tilde{Q}_1 \left( \frac{S^2}{x}, y, z, w \right). \]

Here

\[ \tilde{M}_1(x, y, z) = e^{-\delta y} \left[ N(d_1(x, y, z)) + \frac{\tilde{N}(d_1(x, y, z))}{\sigma \sqrt{y}} \left( 1 - \frac{E}{z} \right) \right], \]

\[ \tilde{Q}_1(x, y, z, w) = e^{-\delta(z-y)} \left[ \delta N(d_1(x, z - y, w)) + \frac{\tilde{N}(d_1(x, z - y, w))}{\sigma \sqrt{z - y}} \left( \delta - \frac{E_r}{w} \right) \right], \]

with

\[ \tilde{N}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \text{ and } d_1(x, y, z), \ d_2(x, y, z) \text{ are defined in } (2.2.20) \]

**Proof.** From the formula (3.3.16), we can derive that \( \forall S < S_f \),

\[ \frac{\partial V_2}{\partial S}(S, u; t) = \frac{\partial M_1}{\partial S}(S, T - t, E) - \frac{2\alpha S^{2\alpha - 1}}{S^{2\alpha}} M_1 \left( \frac{S^2}{S}, T - t, E \right) + \left( \frac{S}{S} \right)^{2\alpha - 2} \frac{\partial M_1}{\partial \frac{S^2}{S}} \left( \frac{S^2}{S}, T - t, E \right) + \frac{\partial K_1}{\partial S}(S, t) + \int_t^T \frac{\partial Q_1}{\partial S}(S, t, u, S_f^{do}(u)) du \]

\[ - \int_t^T \left[ \frac{2\alpha S^{2\alpha - 1}}{S^{2\alpha}} Q_1 \left( \frac{S^2}{S}, t, u, S_f^{do}(u) \right) + \left( \frac{S}{S} \right)^{2\alpha - 2} \frac{\partial Q_1}{\partial \frac{S^2}{S}} \left( \frac{S^2}{S}, t, u, S_f^{do}(u) \right) \right] du, \]

where \( K_1(x, y) = \int_y^T \left( \frac{x}{S} \right)^\alpha K(x, y, z) dz \), and \( M_1, K, Q_1 \) are defined as in (2.2.20), (2.5.51) and (2.3.37), respectively. Therefore, in order to prove the formula (3.3.19), we only need to show that

\[ \frac{\partial M_1}{\partial x}(x, y, z) = \tilde{M}_1(x, y, z), \quad \frac{\partial Q_1}{\partial x}(x, y, z, w) = \tilde{Q}_1(x, y, z, w), \quad \text{and} \quad \frac{\partial K_1}{\partial x}(x, y) = \tilde{K}_1(x, y). \]

where \( \tilde{K}_1, \tilde{M}_1, \tilde{Q}_1 \) are defined as in Proposition 3. Before proving these equalities, we notice that \( d_1(x, y, z) - d_2(x, y, z) = \sigma \sqrt{y} \). This implies

\[ xe^{-\delta y} \tilde{N}(d_1(x, y, z)) = ze^{-\gamma y} \tilde{N}(d_2(x, y, z)), \quad (3.3.20) \]
where $\tilde{N}(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$.

**Proof of $\frac{\partial}{\partial x} M_1(x, y, z) = \tilde{M}_1(x, y, z)$**. We have

$$
\frac{\partial}{\partial x} M_1(x, y, z) = \frac{\partial}{\partial x} \left[ xe^{-\delta y} N(d_1(x, y, z)) - E e^{-ry} N(d_2(x, y, z)) \right]
= e^{-\delta y} N(d_1(x, y, z)) + \frac{e^{-\delta y}}{\sigma \sqrt{y}} \tilde{N}(d_1(x, y, z)) - \frac{E e^{-ry}}{x \sigma \sqrt{y}} \tilde{N}(d_2(x, y, z)).
$$

Using the formula (3.3.20), it follows

$$
\frac{E e^{-ry}}{x \sigma \sqrt{y}} \tilde{N}(d_2(x, y, z)) = \frac{E e^{-\delta y}}{z \sigma \sqrt{y}} \tilde{N}(d_1(x, y, z)).
$$

Therefore,

$$
\frac{\partial}{\partial x} M_1(x, y, z) = e^{-\delta y} \left[ N(d_1(x, y, z)) + \frac{\tilde{N}(d_1(x, y, z))}{\sigma \sqrt{y}} \left(1 - \frac{E}{z}\right)\right] = \tilde{M}_1(x, y, z).
$$

**Proof of $\frac{\partial}{\partial x} Q_1(x, y, z) = \tilde{Q}_1(x, y, z)$**. We have

$$
\frac{\partial}{\partial x} Q_1(x, y, z) = \frac{\partial}{\partial x} \left[ x \delta e^{-\delta (z-y)} N(d_1(x, z-y, w)) - E e^{-r(z-y)} N(d_2(x, z-y, w)) \right]
= \delta e^{-\delta (z-y)} N(d_1(x, z-y, w)) + \frac{\delta e^{-\delta (z-y)}}{\sigma \sqrt{z-y}} \tilde{N}(d_1(x, z-y, w))
- \frac{E e^{-r(z-y)}}{x \sigma \sqrt{z-y}} \tilde{N}(d_2(x, z-y, w)).
$$

Using the formula (3.3.20), it follows

$$
e^{-r(z-y)} \tilde{N}(d_2(x, z-y, w)) = \frac{x e^{-\delta (z-y)}}{w} \tilde{N}(d_1(x, z-y, w)).
$$

Therefore,

$$
\frac{\partial}{\partial x} Q_1(x, y, z) = e^{-\delta (z-y)} \left[ \delta N(d_1(x, z-y, w)) + \frac{\tilde{N}(d_1(x, z-y, w))}{\sigma \sqrt{z-y}} \left(\delta - \frac{E r}{w}\right)\right] = \tilde{Q}_1(x, y, z).
$$

**Proof of $\frac{\partial}{\partial x} K_1(x, y; t) = \tilde{K}_1(x, y; t)$**. Note that $K_1(x, y) = \int_y^T \left(\frac{x}{S}\right)^{\frac{1}{2}} K(x, y, z) dz$ has remov-
able singularities at \((\bar{S}, y)\). In this case, one way to calculate \(\frac{\partial K_1}{\partial x}(x, y)\) when \(x\) approaches \(\bar{S}\) is to remove these singularities by using the following variable transformation:

\[
\xi = \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{z - y}}.
\]

As a result,

\[
K_1(x, y) = \int_{\frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}}}^{+\infty} \left( \frac{x}{S} \right) \alpha \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{\xi^2}{2}} R\left( y + \frac{(\ln x - \ln \bar{S})^2}{\sigma^2 \xi^2} \right) d\xi.
\]

By using the Leibniz integral rule, we can calculate the derivative of the above integral and obtain:

\[
\frac{\partial}{\partial x} K_1(x, y) = \int_{\frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}}}^{+\infty} \left( \frac{x}{S} \right) \alpha \frac{\sqrt{2}}{\sqrt{\pi}} \ln \frac{x - \ln \bar{S}}{x\xi^2} e^{-\frac{\xi^2}{2}} R\left( y + \frac{(\ln x - \ln \bar{S})^2}{\sigma^2 \xi^2} \right) d\xi
\]

\[
+ \left( \frac{x}{S} \right) \alpha \frac{\sqrt{2}}{\sqrt{\pi}} \int_{\frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}}}^{+\infty} \ln \frac{x - \ln \bar{S}}{x\xi^2} e^{-\frac{\xi^2}{2}} R\left( y + \frac{(\ln x - \ln \bar{S})^2}{\sigma^2 \xi^2} \right) d\xi.
\]

By using variable transformations \(\eta = \xi - \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}}\) and \(v = \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}}\) for the first and second integral in (3.3.21), we obtain the following expression for \(\frac{\partial}{\partial x} K_1(x, y)\):

\[
\tilde{K}_1(x, y) = \int_{0}^{+\infty} R\left( y + \frac{(\ln x - \ln \bar{S})^2}{\sigma (\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}})} \right) e^{\frac{\beta(\ln x - \ln \bar{S})^2}{2(\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}})^2} - \frac{1}{2}(\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}})^2} d\eta
\]

\[
+ \left( \frac{x}{S} \right) \alpha \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\sqrt{T - y}} e^{-\frac{(\ln x - \ln \bar{S})^2}{2\sigma^2 \xi^2}} + \frac{2\sigma^2 v^2}{2} \left[ \beta R(y + v^2) + \frac{2}{\sigma^2} R'(y + v^2) \right] dv
\]

\[
- \frac{\sqrt{2} R(T)}{\sigma \sqrt{T - y}} \left( \frac{x}{S} \right) \alpha \frac{\sqrt{2}}{\sqrt{\pi}} e^{\frac{\beta(\ln x - \ln \bar{S})^2}{2(\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}})^2} - \frac{1}{2}(\eta + \frac{\ln x - \ln \bar{S}}{\sigma \sqrt{T - y}})^2} d\eta
\]

\[
= \tilde{K}_1(x, y).
\]

This completes the proof of Proposition 3. 

\[\square\]
3.4 Numerical implementation

In this section, we solve numerically the integral equation (3.3.17) by using a combination of quadrature rules and the Newton-Raphson iteration procedure. Specifically, the equation (3.3.17) will first be reduced to a system of nonlinear algebraic equations by approximating the integrals contained in (3.3.17) using quadrature rules. The newly-established system of nonlinear algebraic equations can then be solved by using the Newton-Raphson iteration procedure to obtain the optimal exercise boundary at each discrete point in time. It should be noted that our numerical procedure here is very similar to the one proposed by Kallast and Kivinukk [52] for pricing American-style vanilla calls.

3.4.1 The optimal exercise boundary just prior to expiry

As input to our numerical technique, the value of the optimal exercise boundary just prior to expiry, i.e. at $\tau = 0^+$. is needed. This value will be given in the following corollary.

**Corollary 3.4.1.** The optimal exercise price of an American-style down-and-out call just prior to expiry, $S_{f}^{do}(T^-)$, is given by:

$$S_{f}^{do}(T^-) = \max(E, \frac{rE}{\delta}).$$

**Proof.** To simplify the notation, we replace $S_{f}^{do}(t)$ by $S_{f}(t)$ in (3.3.17). By rearranging the integral equation (3.3.17), we obtain:

$$\frac{S_{f}(t)}{E} = \frac{T_{1}(t, S_{f}(t))}{T_{2}(t, S_{f}(t))},$$

(3.4.22)

where

$$T_{1}(t, S_{f}(t)) = 1 - e^{-r(T-t)}N\left(d_{2}(S_{f}(t), T - t, E)\right)$$

$$+ \left(\frac{S_{f}(t)}{S}\right)^{2\alpha} e^{-r(T-t)}N\left(d_{2}\left(\frac{\bar{S}^{2}}{S_{f}(t)}, T - t, E\right)\right)$$

$$- \int_{t}^{T} re^{-r(u-t)}N\left(d_{2}(S_{f}(t), u - t, S_{f}(u))\right) dv$$

$$+ \left(\frac{S_{f}(t)}{S}\right)^{2\alpha} \int_{t}^{T} re^{-r(u-t)}N\left(d_{2}\left(\frac{\bar{S}^{2}}{S_{f}(t)}, u - t, S_{f}(u)\right)\right) dv,$$
and

\[ T_2(t, S_f(t)) = 1 - e^{-\delta(T-t)} N \left( d_1 \left( S_f(t), T - t, E \right) \right) + \left( \frac{S_f(t)}{S} \right)^{2a-2} e^{-\delta(T-t)} N \left( d_1 \left( \frac{S^2}{S_f(t)}, T - t, E \right) \right) \]

\[ - \int_t^T \delta e^{-\delta(u-t)} N \left( d_1 \left( S_f(t), u - t, S_f(u) \right) \right) \, dv \]

\[ + \left( \frac{S_f(t)}{S} \right)^{2a-2} \int_t^T \delta e^{-\delta(u-t)} N \left( d_1 \left( \frac{S^2}{S_f(t)}, u - t, S_f(u) \right) \right) \, dv \]

\[ - \frac{S_f(t)^{a-1}}{S^a \sigma \sqrt{2\pi}} \int_t^T \ln S_f(t) - \ln \bar{S} \, dv + \int_t^T \ln S_f(t) - \ln \bar{S} \, dv. \]

Before proceeding further, we note that \( S_f(T^-) \geq E \) as the option should be exercised only when it is in-the-money or at-the-money. Consider the first case where \( S_f(T^-) = E \). Taking the limit of equation \([3.4.22]\) as \( t \) tends to \( T^- \), we obtain \( \lim_{t \to T^-} \frac{S_f(t)}{E} = 1 \) and thus \( S_f(T^-) = E \) is a possible solution for \( S_f(T^-) \).

Now we consider the second case where \( S_f(T^-) > E \). In this case, we have

\[ \lim_{t \to T^-} T_1(t, S_f(t)) = \lim_{t \to T^-} T_2(t, S_f(t)) = 0. \]

The limit of equation \([3.4.22]\) therefore is an indeterminate form, which can be resolved by using L'Hospital’s rule. Before applying L’Hospital’s rule, “redundant” terms in \( T_1 \) and \( T_2 \) should be eliminated. First, we have:

\[ \lim_{t \to T^-} \frac{1 - e^{-r(T-t)} N \left( d_2 \left( S_f(t), T - t, E \right) \right)}{T - t} = \lim_{t \to T^-} \frac{1 - e^{-r(T-t)}}{T - t} = r. \]

As a result, \( 1 - e^{-r(T-t)} N \left( d_2 \left( S_f(t), T - t, E \right) \right) \sim T - t \) as \( t \to T^- \), where the notation “\( \sim \)” shows the equivalence of two infinitesimal functions of \( u \). In addition, as \( t \to T^- \), we have:

\[ \lim_{T-t \to 0^+} \frac{\int_{-\infty}^{\frac{1}{\sqrt{T-t}}} e^{-\frac{1}{T-t}} \, dt}{T - t} = \lim_{T-t \to 0^+} \frac{e^{-\frac{1}{T-t}}}{T - t} = 0. \]

The term \( \left( \frac{S_f(t)}{S} \right)^{2a} e^{-r(T-t)} N \left( d_2 \left( \frac{S^2}{S_f(t)}, T - t, E \right) \right) \) thus decays to 0, as \( t \to T^- \), at a
faster rate than the term $1 - e^{-r(T-t)}N\left(d_2(S_f(t), T-t, E)\right)$. We also have:

$$
\lim_{u \to t} N\left(d_2\left(\frac{\bar{S}^2}{S_f(t)}, u-t, S_f(u)\right)\right) = 0,
$$

$$
\lim_{u \to t} N\left(d_2\left(\frac{S_f(t)}{S_f(t)}, u-t, S_f(u)\right)\right) = \frac{1}{2}.
$$

The terms $\left(\frac{S_f(t)}{S}\right)^{2a} \int_t^T r e^{-r(u-t)} N\left(d_2\left(\frac{\bar{S}^2}{S_f(t)}, u-t, S_f(u)\right)\right) dv$ therefore decays to 0, as $t \to T^-$, at a faster rate than the term $\int_t^T r e^{-r(u-t)} N\left(d_2\left(\frac{S_f(t)}{S_f(t)}, u-t, S_f(u)\right)\right) dv$.

From the above results, we conclude that as $t \to T^-$

$$
T_1 \sim T_3 = 1 - e^{-r(T-t)}N\left(d_2(S_f(t), T-t, E)\right) - \int_t^T r e^{-r(u-t)}N\left(d_2\left(\frac{S_f(t)}{S_f(t)}, u-t, S_f(u)\right)\right) dv.
$$

Similarly, we have:

$$
T_2 \sim T_4 = 1 - e^{-\delta(T-t)}N\left(d_1\left(S_f(t), T-t, E\right)\right) - \int_t^T \delta e^{-\delta(u-t)}N\left(d_1\left(S_f(t), u-t, S_f(u)\right)\right) dv.
$$

Similar to the case in Chiarella et al. [20], it can be shown that $\lim_{t \to T^-} \frac{T_3}{T_4} = \frac{r}{\delta}$. As a result,

$$
\lim_{t \to T^-} \frac{S_f(t)}{E} = \lim_{t \to T^-} \frac{T_1(t, S_f(t))}{T_2(t, S_f(t))} = \lim_{t \to T^-} \frac{T_3}{T_4} = \frac{r}{\delta}.
$$

Combining the results of the above three cases, we have:

$$
S_f(T^-) = \max(E, \frac{rE}{\delta}).
$$

This completes our proof. \qed

One can see from the above corollary that just prior to expiry, the optimal exercise price of an American-style down-and-out call with a time-dependent rebate is the same as that of its embedded vanilla option. In other words, just prior to expiry, the asset barrier and the rebate have no effect on the optimal exercise price of the barrier option. This is not surprising as the rebate is set to zero just prior to expiry, in order to eliminate a possible singularity in the PDE system governing the option price, as discussed in Section 3.2.
3.4.2 Numerical procedure

We now describe our numerical procedure to approximate the optimal exercise price and the option price. To simplify the notation, we replace $S(t)$ by $S_f(t)$ in (3.3.17). Let II be a uniform partition of the interval $[0,T]$ such that:

$$\Pi : T = t_1 > t_2 > \ldots > t_n > t_{n+1} = 0, h = \frac{T}{n}.$$ 

Our aim now is to find the optimal exercise price at each of these $n+1$ discrete points. We first approximate the integral equation (3.3.17), at each discrete point $x_i$, by an algebraic equation. As a result, we have a system of $n+1$ non-linear algebraic equations of $n+1$ unknown variables $\{S_f(x_i)\}_{i=1}^{n+1}$. Solving these algebraic equations, by using the Newton-Raphson method, will give us numerical values of the optimal exercise prices at the above discrete points. We can then interpolate these values to obtain the whole optimal exercise boundary on the interval $[0,T]$.

The first algebraic equation, $S_f(t_1) = \max(E, \frac{rE}{\delta})$, comes from the result of Corollary 3.4.1. Other algebraic equations are derived by substituting $t = t_j, j \geq 2$ into the integral equation (3.3.17) as follows:

$$S_f(t_j) - E = C_E^{do}(S_f(t_j), t_j) + EP(S_f(t_j), t_j; S_f(t_j)). \quad (3.4.24)$$

The first four integrals in $EP(S_f(t_j), t_j; S_f(t_j))$, which are integrals of smooth functions on finite domains, can be approximated by using simple numerical integration rules such as composite Trapezoidal rule. The last integrand in $C_E^{do}(S_f(t_j), t_j)$ however has a singularity at $u = t_j$. Therefore, a composite Trapezoidal rule would not provide a good approximation for the integral. One way to approximate this integral is first to transform it into an integral on a semi-infinite domain by using the following variable transformation:

$$v = \frac{\ln S_f(t_j) - \ln \bar{S}}{\sigma \sqrt{2(u-t_j)}} - \frac{\ln S_f(t_j) - \ln \bar{S}}{\sigma \sqrt{2(T-t_j)}}.$$
The integral then becomes:

\[
\frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-\left( v + \frac{\ln S_f(t_j) - \ln \bar{S}}{\sigma \sqrt{2(T-t_j)}} \right)^2} \frac{\beta(\ln S_f(t_j) - \ln \bar{S})^2}{4 \left( v + \frac{\ln S_f(t_j) - \ln \bar{S}}{\sigma \sqrt{2(T-t_j)}} \right)^2} \right) R \left( t_j + \frac{(\ln S_f(t_j) - \ln \bar{S})^2}{2\sigma^2(v + \frac{\ln S_f(t_j) - \ln \bar{S}}{\sigma \sqrt{2(T-t_j)}})^2} \right) dv.
\]

The Gauss-Laguerre rule is finally applied to evaluate the integral. As a result, at each \( x_i \), (3.4.24) can be reduced to an algebraic equation. We now have a system of nonlinear algebraic equations, which can be solved by using the Newton-Raphson iteration procedure to obtain the optimal exercise boundary at each discrete point in time (cf. Section 2.4).

Our next steps to obtain the option price are very similar to the ones proposed by Kallast and Kivinukk [52] for pricing American-style vanilla calls. Therefore, we will not discuss the details here.

### 3.5 Numerical results

In this section, we first validate our numerical scheme for an American-style down-and-out call (without rebate) by comparing our results with those in [85]. The accuracy and efficiency of our method are then examined. Finally, the effects of rebates on the prices of American-style down-and-out calls and their optimal exercise boundaries are clearly illustrated through numerical examples.

#### 3.5.1 Validation of our numerical scheme

We shall validate our numerical scheme before it is applied to examine the effects of rebates on the prices of American-style down-and-out calls and their optimal exercise boundaries. A reliable way to do so is to compare the results obtained by the integral equation (IE) approach with those obtained by other published schemes. Due to the lack of available published results for none-zero rebate case, we will use our formulation to do the calculation for the case without rebate and compare our results with those obtained through an implicit finite difference method (FDM) proposed by Zhu et al. [85]. All of our experiments were performed using Matlab R2014b on an Intel Core i7, 3.40 GHZ machine. All of our numerical results are obtained using 37 Gauss-Legendre points and 37 Gauss-Laguerre points with the
3.5. NUMERICAL RESULTS

tolerance $10^{-5}$.

Table 3.1 presents a comparison of the prices of the American-style down-and-out calls without rebate calculated by using the IE method with those reported in [85]. All parameters are kept the same as those used in [85]. From Table 3.1 it is clear that our analytic results agree well with those reported in [85]. More specifically, the point-wise relative errors are less than 0.02%. The agreement of these two sets of results proves the validity of our numerical scheme.

Table 3.1: Option values calculated by the IEM and FDM methods, versus various asset barriers

$E = \$100, \sigma = 20\%, r = 10\%, D = 5\%, T - t = 1(year)$.

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<th></th>
<th>S = $85$</th>
<th></th>
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3.5.2 The accuracy and efficiency of our numerical scheme

This section is devoted to study the level of accuracy and efficiency of the IE method. To this end, we first calculate several values of the option and its Delta when $S$ is chosen closer to a fixed barrier $\bar{S} = \$90$, at $\$100, \$95, \$91, \$90.5, \$90.1$. These values are calculated with the uniform time-step points, denoted by $m$, being consecutively doubled from 5 to 80. Here, other parameters are set as: $E = \$100, \sigma = 20\%, r = 10\%, D = 5\%, T - t = 1(year)$, with a specific rebate function: $R_1(\tau) = 6\tau$. We then also calculate the relative error $\epsilon_a$ of the IE method, defined by:

$$
\epsilon_a = \frac{\text{current approximation} - \text{previous approximation}}{\text{current approximation}}.
$$
It is well-known that for an iterative method, like the Newton-Raphson iteration procedure used in our calculation, if \( \epsilon_a < 0.5 \times 10^{-n} \) then the current approximation is correct to at least \( n \) significant digits, where \( n \) is an integer number.

Table 3.2: Prices and Deltas of an American down-and-out call option at different \( S \). The parameters are set as: \( R_1(\tau) = 6\tau \), \( E = $100 \), \( \sigma = 20\% \), \( r = 10\% \), \( D = 5\% \), \( T - t = 1(\text{year}) \). The IE method is used with 37 Gauss–Legendre points and 37 Gauss–Laguerre points, with tolerance \( 10^{-5} \).

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<td>0.40 E-08</td>
<td>0.30</td>
<td>0.312095789</td>
<td>0.09 E-06</td>
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<td>6.190033605</td>
<td>0.22 E-08</td>
<td>0.785</td>
<td>0.312095762</td>
<td>0.42 E-07</td>
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<td>0.299920798</td>
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<td>10</td>
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<td>0.33 E-09</td>
<td>2.75</td>
<td>0.299920646</td>
<td>0.43 E-07</td>
</tr>
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</table>

Table 3.2 presents the values of the option and its Delta, the CPU time which is measured in seconds and the relative errors associated with the number of discrete points being used. Two main points can be observed from this table. First, the IE method appears to be very efficient in computing the option price and the Delta. More specifically, as can be seen from Table 3.2 within 1 second, one can obtain the option price correct to at least 7 significant digits. In addition, if results for both the option prices and the Deltas correct to 6 digits are
required, only about 0.07 seconds is needed to do the calculation. This level of efficiency is very important for market practitioners. Second, the accuracy and efficiency of the IE method appears to be independent from $S$. More specifically, from Table 3.2 it seems that no matter whether $S$ is far away or close to the barrier $\bar{S} = 90$, to achieve 6 significant digits of accuracy, one only needs to use 5 uniform time-steps in $\tau$-direction and then obtain the results within 0.07 seconds. The “near-barrier” issue [19, 33, 35] faced by grid-based methods is completely eliminated from our formulation because the option price or the hedging parameters can be obtained straightforwardly after the optimal exercise boundary is identified by solving its governing integral equation, which is independent from $S$.

We have also compared the IE method with the FDM in terms of numerical efficiency. Since adding a rebate term would only worsen the efficiency by a tiny bit because the computational effort for a non-zero but given analytical function $R(\tau)$ is hardly different from that for the value of zero, we could compare the efficiency of these two methods without the rebate term. By using the fully implicit FDM method, Rhodes [67] computed the prices of an American down-and-out call without rebate for a range of barrier values with parameters: $S = 100$, $E = $100, $\sigma = 40\%$, $r = 10\%$, $D = 5\%$, $T - t = 1$ (year). The minimum consumed CPU time reported for the entire range of barrier values was 0.84(s). We apply the IE method to the same problem and find that the consumed CPU times for the same range of barrier values are all around 0.05(s) to 0.07(s). Although Rhodes’ numerical experiments were conducted on a different machine, but of roughly the same order of “floops”, a 10-fold reduction of CPU time with the IE method demonstrates that the IE method is superior to the FDM as far as numerical efficiency is concerned.

### 3.5.3 Effects of rebates on the optimal exercise price

In this section, we examine the effect of rebates on the optimal exercise price. To this end, we first choose four different rebate functions as functions of $\tau$: $R_1(\tau), R_2(\tau), R_3(\tau)$ and $R_4(\tau)$, where $\tau = T - t$. Of course, one can choose for each of $R_1(\tau), R_2(\tau), R_3(\tau), R_4(\tau)$ any given function that is smoothly increasing with $\tau$ and becomes zero at expiry. In this paper, as a simple illustrative example, we choose: $R_1(\tau) = 6\tau$, $R_2(\tau) = 3\tau$, $R_3(\tau) = 10\tau^2$, and $R_4(\tau) = 0$. It is clear from our choice that $R_1(\tau) > R_2(\tau) > R_4(\tau), \forall \tau$. In addition,
$R_2(\tau) > R_3(\tau), \forall 0 < \tau < 0.3$, but $R_2(\tau) < R_3(\tau), \forall 0.3 < \tau < T$.

Figure 3.1 compares the optimal exercise boundaries of an American-style down-and-out call associated with the rebate functions: $R_1(\tau), R_2(\tau), R_4(\tau)$, respectively (other parameters remain unchanged). We make the following interesting observations. First, it is clear from Figure 3.1 that the optimal exercise boundary associated with a larger rebate function is higher than those associated with smaller rebate functions. This suggests that the optimal exercise price increases as the rebate increases. This is indeed expected as a greater rebate means more compensation for the loss of the holder in the event the knock-out feature is activated. Consequently, the option holder would prefer to choose a higher asset price to optimally exercise the option. Second, it can be seen that the effect of the rebates on the optimal exercise boundary is gradually diminished when time is close to expiry, i.e, $\tau$ approaches 0. This is not surprising because our rebate functions are decreasing functions of $\tau$ and become zero at expiry.

![Figure 3.1: Optimal exercise boundaries of American-style down-and-out calls associated with: $R_1, R_2, R_4$. Parameters are $E = \$100, \bar{S} = \$90, \sigma = 20\%, r = 5\%, D = 10\%, T - t = 1(\text{year})$.](image)

The behavior of the optimal exercise boundaries become even more interesting when we compare those associated with $R_2(\tau)$ and $R_3(\tau)$, which satisfy: $R_2(\tau) > R_3(\tau), \forall \tau < 0.3$, but $R_2(\tau) < R_3(\tau), \forall \tau > 0.3$. The American-style down-and-out call associated with $R_2(\tau)$ will give more compensation than that associated with $R_3(\tau)$ if $\tau^* < 0.3$, but will give less if $\tau^* > 0.3$, where $\tau^*$ denotes the time when the knock-out feature is activated. Therefore, if the time to expiry $\tau$ is “long” enough, say $\tau = 1$, the latter would give more benefit to the
holder than the former. As a result, as can be seen from Figure 3.2, when the time to expiry increases, the optimal exercise boundary associated with the latter starts at a lower position, but later increases at a faster rate and eventually will be larger than that associated with the former. In particular, when \( \tau \) is about 0.6, the optimal exercise boundary associated with \( R_3 \) starts to dominate that associated with \( R_2 \).

![Figure 3.2: Optimal exercise boundaries of American-style down-and-out calls associated with: \( R_2 \) and \( R_3 \). Parameters are \( E = \$100 \), \( \bar{S} = \$90 \), \( \sigma = 20\% \), \( r = 5\% \), \( D = 10\% \), \( T - t = 1(\text{year}) \).](image)

It should also be noted that if we decrease \( \bar{S} \), all of the effects of rebates on the optimal exercise boundary appear to diminish very quickly. This of course also depends on how large the rebate is. More precisely, for larger rebates, it might take a smaller \( \bar{S} \) to diminish the effect. As shown in Figure 3.3, when \( \bar{S} = 60 \), all four optimal exercise boundaries merge into one. This phenomenon can be explained as follows. As \( \bar{S} \) decreases, the chance that the option will be knocked out is smaller, and hence the effect of the rebate on the optimal exercise boundary \( S_f \) is also smaller. In particular, when \( \bar{S} \to 0 \), the behavior of \( S_f \) can be proved to be the same with that of the optimal exercise boundary of the vanilla option counterpart, and therefore in this case \( S_f \) does not depend on the rebate at all.

### 3.5.4 Effects of rebates on the option price

Like the optimal exercise price, the option price is expected to be an increasing function of \( \tau \) because the larger \( \tau \) is, the more chance the holder can optimally exercise the option and
3.5. NUMERICAL RESULTS

Figure 3.3: Optimal exercise boundaries of American-style down-and-out calls associated with: $R_1, R_2, R_3, R_4$. Parameters are $E = \$100, \bar{S} = \$90, \sigma = 20\%, r = 5\%, D = 10\%, T - t = 1(\text{year})$.

then gain profits. Figure 3.4 illustrates the changes in the option price when we increase the amount of rebate. It is clear from Figure 3.4 that the option price associated with a greater rebate function is always larger than those associated with smaller rebate functions (other parameters remain unchanged). This is indeed expected as a larger rebate always brings more benefits to the option holder and thereby increases the option price.

Figure 3.4: Prices of American-style down-and-out calls associated with: $R_1, R_2, R_4$. Parameters are $E = \$100, \bar{S} = \$90, \sigma = 20\%, r = 5\%, D = 10\%, T - t = 1(\text{year})$. 
We now also compare the values of an American-style down-and-out call associated with rebate functions $R_2(\tau), R_3(\tau)$, respectively. When other parameters are fixed, these values can be considered as functions of $\tau$ (see Figure 3.5). As discussed in Section 3.5.3, if $\tau$ is “long” enough, say $\tau = 1$, the option associated with $R_3(\tau)$ would give more benefit to the holder than that associated with $R_2(\tau)$. Consequently, the value of the latter is expected to start at a lower position, but later will increase at a faster rate and eventually will be larger than the former. This expectation is clearly demonstrated in Figure 3.5. In particular, when $\tau$ approaches 1, the option price associated with $R_3$ becomes significantly larger than that associated with $R_2$.

![Figure 3.5: Prices of American-style down-and-out calls associated with: $R_2, R_3$. Parameters are: $E = $100, $\bar{S} = $90, $\sigma = 20\%$, $r = 5\%$, $D = 10\%$, $T - t = 1(\text{year})$.](image)

It is also expected that the price of an American-style down-and-out call without rebate will increase if the asset price $S$ increases. This is because the farther $S$ is away from the asset barrier, the less likely that the option will be knocked out as well as the more likely the option is in-the-money and can be optimally exercised. In other words, the holder of an American-style down-and-out call without rebate will benefit more when the asset price increases. Consequently, the option price should be a monotonically increasing function of asset price, as shown by Curve: $R_4 = 0$ in Figure 3.6.

With a rebate, the price of an American-style down-and-out call however might not always monotonically increase with asset price. For instance, one can easily observe from two Curves: $R_1 = 6\tau$ and $R_2 = 3\tau$ in Figure 3.6 that the value of the option decreases with asset price in
the neighborhood of the barrier. This interesting phenomenon occurs because in this case the asset barrier \( \bar{S} = 75 \) is much lower than the exercise price \( E = 100 \) and thus when \( S \) is close to \( \bar{S} \), the option is deeply out-of-money and its value mainly results from the rebate premium, which is a strictly decreasing function of \( S \) (the farther \( S \) is away from the asset barrier, the less likely the knock-out is triggered and consequently the less amount of money is needed to compensate for the risk of the option holder). One can expect that the lower the asset barrier is, the more likely this phenomenon will happen.

To further study the effects of the rebates across different barrier values, we plot the option prices when \( \bar{S} = \{75, 95\} \) for both \( R_1(\tau) \) and \( R_4(\tau) \) on the same figure, Figure 3.7. One can also observe from Figure 3.7 that Curve \( R_4 = 0, \bar{S} = 75 \) stays above Curve \( R_4 = 0, \bar{S} = 95 \). This illustrates the fact that the price of an American-style down-and-out call without rebate is a monotonically decreasing function of asset barriers because a higher asset barrier always increases the risk of losing the option and thus reduces the value of the option. With a rebate, this risk however is partly compensated for and consequently the price of an American-style down-and-out call with a higher asset barrier might be greater than that with a smaller asset barrier, at some asset prices (as shown in the two Curves \( R_1 = 6\tau, \bar{S} = 75 \) and \( R_1 = 6\tau, \bar{S} = 95 \) in Figure 3.7). Of course, the greater the rebate is, the more likely this phenomenon will happen.
3.6 Conclusion

This chapter applies an integral equation approach to price American-style down-and-out calls. A key step of our approach is to use the Fourier sine transform to reduce the original 2-D governing PDE system into a simple ODE, the solution of which is readily obtainable (in the Fourier sine space) and can be analytically converted to the original space. As a result, we can re-derive the “early exercise premium representation” for American-style down-and-out calls without rebate. More importantly, our approach has led to a slightly more general integral representation for an American-style down-and-out call with a time-dependent rebate, which is decomposed into two components: the price of its European counterpart with the given rebate and an early exercise premium associated with the American-style early exercise right. We have also demonstrated the validity and efficiency of our approach in computing the price and the Delta of the option. In particular, it has strong advantages over the FDM. In addition, through numerical examples, the effects of time-dependent rebates on the prices of American-style down-and-out calls and their optimal exercise boundaries have been carefully examined in this chapter.
Chapter 4

Pricing American-style Parisian up-and-in options

4.1 Introduction

Barrier options are common path-dependent options traded in financial markets. One of reasons for the popularity of barrier options is that they provide a more flexible and cheaper way for hedging and speculating than vanilla options because the holders of barrier options only pay a premium for scenarios they perceive as likely. The “one-touch” breaching barrier however may have an undesirable feature of suddenly losing all proceeds (knock-out options) or suddenly receiving the embedded options (knock-in options) if the price of the underlying asset momentarily touches the asset barrier, no matter how briefly the breaching occurs. This opens up the possibility that market practitioners may deliberately manipulate the underlying asset price to force the cancelation or activation of the option. To prevent such an attempt, Parisian options were introduced, with a unique feature that the underlying asset price has to continually stay above or below the asset barrier for a prescribed amount of time before the knock-out or knock-in feature is activated. This extended trigger clause can also be found in some derivative contracts, such as callable convertible bonds and executive warrants. It is also worthwhile to note that Parisian options can be a useful tool in corporate finance.

Like the relationship between an American vanilla option and its European counterpart, the valuation problem of an American-style Parisian option, in general, is much more difficult
than that of its European-style counterpart. While a closed-form solution of the latter has already been found by Zhu and Chen \[83\], a closed-form solution of the former only exists for the perpetual knock-in case \[17\]. The extra difficulty of pricing an American-style Parisian knock-out option, in comparison with its European-style counterpart, mainly stems from the complexity of the determination of the optimal exercise boundary, which has hindered the application of various mathematical methods (cf. Chapters \[6\] and \[7\]).

Fortunately, the above difficulty disappears in the valuation of an American-style Parisian knock-in option because its optimal exercise boundary can be easily determined. More precisely, by definition, the option holder does not have any exercise right to buy or sell the underlying stock until the knock-in feature is activated; and once the “knock-in” feature is activated, the exercise right associated with the Parisian option is the same as that associated with the embedded vanilla option. In other words, the optimal exercise boundary of the Parisian option, which only exists after the activation of the knock-in feature, can be easily identified as it is equal to that of the embedded vanilla option, the calculation of which is well-known in the literature (cf. Section \[1.2.1\]).

In addition, an American-style Parisian knock-in option is very similar to its European-style option counterpart, except that once the asset price touches the barrier, the former immediately becomes its embedded American vanilla option while the latter immediately becomes its European vanilla option counterpart. This suggests that American-style Parisian knock-in options and their European-style counterparts can be priced by using the same solution procedure.

This chapter aims to derive an explicit analytical solution for both American-style and European-style Parisian up-and-in call options, using the “moving window” technique proposed by Zhu and Chen \[83\]. With a growing demand for trading exotic options in today’s finance industry, our solution procedures may lead to the development of pricing formulas for other exotic derivatives, such as the Edokko options introduced by Fujita and Miura \[34\], which are generalizations of both Parisian and delayed barrier options.

The chapter is organized as follows. In Section \[4.2\] we introduce the PDE systems governing the price of a Parisian up-and-in call. The solution procedure is presented in Section \[4.3\] while Section \[4.4\] provides numerical examples to illustrate the results obtained from our
pricing formula. Some concluding remarks are given in Section 4.5.

4.2 Formulation

By definition, the knock-in feature of a Parisian up-and-in call is activated if the underlying asset price has continually stayed above the barrier $\bar{S}$ for a prescribed time period $\bar{J}$, the “option window”. Once the knock-in feature is activated, the value of the Parisian option then equals that of its embedded vanilla option, the calculation of which is well-known in the literature. What we actually need to solve is therefore the value of the Parisian option before the knock-in feature is activated. Under the Black-Scholes model, this value depends on the underlying asset price $S$, the current time $t$ and the “excursion time” $J$, in addition to other parameters such as the volatility rate $\sigma$, the risk-free interest rate $r$, the exercise price $E$, the expiry time $T$, the barrier $\bar{S}$ and the “option window” $\bar{J}$.

It is interesting to point out that if $\bar{S}$ and $\bar{J}$ take some extreme values, the Parisian up-and-in call will become worthless or degenerate to either a “one-touch” barrier option or a vanilla option. For example, if $\bar{J}$ approaches zero, the option will be immediately “knocked in” once the underlying asset price touches the barrier from the below, which is the same as the specification of a “one-touch” barrier call option with up-and-in feature. Similarly, it can be deduced that if $\bar{J}$ is greater than the option life, $T$, or $\bar{S}$ approaches infinity, the option values nothing as the knock-in feature can never be activated.

We now focus on non-degenerated cases. Based on financial arguments similar to those used in [83], the pricing domains of those non-degenerated cases can be elegantly reduced as:

$$I : \{0 \leq S \leq \bar{S}, \ 0 \leq t \leq T - \bar{J}, \ J = 0\},$$

$$II : \{\bar{S} \leq S < \infty, \ J \leq t \leq J + T - \bar{J}, \ 0 \leq J \leq \bar{J}\}.$$

Let $V_1(S, t)$ and $V_2(S, t, J)$ denote the option prices in the region $I$ and $II$, respectively. Following the arguments of Haber et al. [39] and Zhu and Chen [83], we can show that $V_1$ and $V_2$ should satisfy the following two PDE systems $A_1$ and $A_2$ defined in domain $I$ and domain
II, respectively,

\[ \begin{align*}
\mathcal{A}_1 & \quad \begin{cases} 
\frac{\partial V_1}{\partial t} + LV_1 = 0, \\
V_1(S,T - \bar{J}) = 0, \\
V_1(0,t) = 0, \\
V_1(\bar{S},t) = V_2(\bar{S},t,0),
\end{cases} \\
\mathcal{A}_2 & \quad \begin{cases} 
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + LV_2 = 0, \\
V_2(S,t,\bar{J}) = C(S,t), \\
V_2(S,t,J) \sim S as S \to +\infty, \\
V_2(\bar{S},t,J) = V_2(\bar{S},t,0),
\end{cases}
\end{align*} \tag{4.2.1} \]

with the Delta condition: \( \frac{\partial V_1}{\partial S}(\bar{S},t) = \frac{\partial V_2}{\partial S}(\bar{S},t,0). \) \( \tag{4.2.2} \)

Here \( C = C_A \) (the price of the embedded American vanilla option) if the Parisian option is of American-style, or \( C = C_E \) (the price of the embedded European vanilla option) if the Parisian option is of European-style, and operator \( \mathbb{L} \) is defined as:

\[ \mathbb{L} = \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - D)S \frac{\partial}{\partial S} - rI, \] \( \tag{4.2.3} \)

with \( I \) being the identity operator. It should be mentioned that the Delta condition is required here to ensure the continuity of the Delta across the barrier.

There are several remarks about the systems \( \mathcal{A}_1, \mathcal{A}_2 \). First, we point out that the option will become worthless if the asset price still remains below or at the asset barrier when \( t \) reaches \( T - \bar{J} \), because then there is not enough time left for \( J \) to reach \( \bar{J} \). Therefore, \( V_1(S,t) = 0 \), for all \( t \geq T - \bar{J}, \ S \leq \bar{S} \). This fact explains the “terminal condition” in \( \mathcal{A}_1 \) at \( t = T - \bar{J} \). Second, the “terminal condition”, with respect to \( J \), in \( \mathcal{A}_2 \) corresponds to the “knock-in” feature that the option price is equal to that of the embedded call, denoted by \( C_A(S,t) \) or \( C_E(S,t) \), at the time \( t \) the option is activated. Third, we have the inhomogeneous boundary condition in \( \mathcal{A}_2 \) when \( S \) approaches infinity because in this case the knock-in feature will surely be triggered and thereby the knock-in option price would be the same as its embedded option price, which is then equivalent with the asset price \( S \). Finally, the last equation in \( \mathcal{A}_2 \) holds only for \( 0 \leq J < \bar{J} \), i.e., before the “knock in” feature is triggered.

The above coupled PDE systems resemble those in \([83]\), so the “moving window” technique can be adopted to obtain the solution for our problem. In the next section, we shall discuss
the solution procedure.

4.3 Solution procedure

Following the method presented in [83] with the same notations, the 3-D PDE systems (4.2.1-4.2.2) can be further simplified to the following two 2-D PDE systems:

\[
\begin{align*}
A_3: & \quad \begin{cases}
\frac{\partial V_1}{\partial t} + LV_1 = 0, \\
V_1(S, T - \tilde{J}) = 0, \\
V_1(0, t) = 0, \\
V_1(\bar{S}, t) = W(t),
\end{cases} \\
A_4: & \quad \begin{cases}
\frac{\partial V_2}{\partial t'} + LV_2 = 0, \\
V_2(S, J; t) = C(S, t + \tilde{J}), \\
V_2(S, l'; t) \sim S \text{ as } S \to +\infty, \\
V_2(\bar{S}, l'; t) = W(t + l'),
\end{cases}
\end{align*}
\]

with the Delta condition: \( \frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, 0; t). \) \hfill (4.3.5)

Here \( A_3 \) is defined on \( t \in [0, T - \tilde{J}], \; S \in [0, \bar{S}] \); \( A_4 \) is defined on \( t' \in [0, \bar{J}], \; S \in [\bar{S}, \infty) \), with the parameter \( t \in [0, T - \tilde{J}] \). The unknown function \( W(t) = V_2(\bar{S}, 0; t) \), which provides the coupling between \( A_3 \) and \( A_4 \), needs to be solved as part of the solution.

It should be pointed out that the derivation procedure to obtain \( A_4 \) of \hfill (4.3.4) from \( A_2 \) of \hfill (4.2.1) is very similar to the one presented in [83] already. To make it easier for readers, a brief explanation is provided here. The 3-D governing equation contained in \( A_2 \) of \hfill (4.2.1) is reduced to a 2-D one by replacing \( \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} \) with \( \sqrt{2} \frac{\partial V_2}{\partial l} \). By denoting \( \frac{l}{\sqrt{2}} \) as \( l' \), we obtain the governing equation contained in \( A_2 \) of \hfill (4.3.4). On the other hand, by realizing the fact that the solution \( V_2(S, l'; t) \) in the 45° rotated coordinate system \( (l', t) \) is equal to \( V_2(S, t + l', l') \) in the coordinate system \( (t, J) \), we have \( V_2(S, \bar{J}; t) = V_2(S, t + \bar{J}, \bar{J}) = C(S, t + \bar{J}) \) when \( l' = \bar{J} \). For more details, interested readers are referred to [83].

To solve the newly established pricing systems \hfill (4.3.4)-(4.3.5) effectively, we shall first non-dimensionlize all variables by introducing the following variables:

\[
\begin{align*}
x = \ln \frac{S}{\bar{S}}, \; \tau = (T - \bar{J} - t)\frac{\sigma^2}{2}, \; \tilde{l} = \frac{\sigma^2}{2}(\bar{J} - l'), \; j' = \frac{\sigma^2}{2}, \; T' = \frac{\sigma^2 T}{2}, \; W'(\tau) = \bar{S}^{-1}W(t), \\
V_1'(x, \tau) = \bar{S}^{-1}V_1(S, t), V_2'(x, \tilde{l}, \tau) = \bar{S}^{-1}V_2(S, l'; t), \; C'(x, \tau) = \bar{S}^{-1}C(S, t + \bar{J}).
\end{align*}
\]
With all primes and tildes dropped from now on, $A_3$ and $A_4$ in (4.3.4-4.3.5) are transformed to $B_1$ and $B_2$, respectively, as follows:

\[
\begin{align*}
B_1 \quad & \quad \left\{ \begin{array}{l}
\frac{\partial V_1}{\partial \tau} = K V_1, \\
V_1(x, 0) = 0, \\
\lim_{x \to -\infty} V_1(x, \tau) = 0, \\
V_1(0, \tau) = W(\tau),
\end{array} \right. \\
B_2 \quad & \quad \left\{ \begin{array}{l}
\frac{\partial V_2}{\partial l} = K V_2, \\
V_2(x, 0; \tau) = C(x, \tau), \\
V_2(x, l; \tau) \sim e^{x} \text{ as } x \to +\infty, \\
V_2(0, l; \tau) = W(\tau - J + l),
\end{array} \right. 
\tag{4.3.7}
\end{align*}
\]

with the Delta condition: \( \frac{\partial V_1}{\partial x}(0, \tau) = \frac{\partial V_2}{\partial x}(0, J; \tau). \)  \( \tag{4.3.8} \)

Here $B_1$ is defined on $\tau \in [0, T - J]$, $x \in (-\infty, 0]$; $B_2$ is defined on $l \in [0, J]$, $x \in [0, \infty)$, with the parameter $\tau \in [0, T - J]$. Operator $K$ is defined as:

\[
K = \frac{\partial^2}{\partial x^2} + k \frac{\partial}{\partial x} - \gamma I, 
\tag{4.3.9}
\]

with $\gamma = \frac{2r}{\sigma^2}$, $q = \frac{2D}{\sigma^2}$, $k = \gamma - q - 1$. Note that $S$, $C$, $W$, $V_1$ and $V_2$ are non-dimensionalized by the barrier $\bar{S}$ here, but not by the exercise price $E$ as in [83]. As a result, the $x$-domains in $B_1$ and $B_2$ are semi-infinite.

By applying the Laplace transform technique, the solution of $B_1$ can be easily found as:

\[
V_1(x, \tau) = \int_{0}^{\tau} W(s) g_1(x, \tau - s) ds, \quad \forall x \leq 0, 
\tag{4.3.10}
\]

where

\[
g_1(x, \tau) = -\frac{x}{2\sqrt{\pi} \tau^2} e^{\alpha x + \beta \tau - \frac{x^2}{4\tau}}, \quad \forall x \leq 0. 
\tag{4.3.11}
\]

Here

\[
\alpha = \frac{k}{2}, \beta = -\frac{k^2}{4} - \gamma, 
\tag{4.3.12}
\]

Since the PDE in $B_2$ is linear, its solution can be found by superposition of the solutions of
the following two systems:

\[
B_3 \begin{cases}
\frac{\partial V_2}{\partial l} = KV_2, \\
V_2(x, 0; \tau) = 0, \\
\lim_{x \to +\infty} V_2(x, l; \tau) = 0, \\
V_2(0, l; \tau) = W(\tau - \bar{J} + l),
\end{cases}
\]

\[
B_4 \begin{cases}
\frac{\partial V_2}{\partial l} = KV_2, \\
V_2(x, 0; \tau) = C(x, \tau), \\
V_2(x, l; \tau) \sim e^x \text{ as } x \to +\infty, \\
V_2(0, l; \tau) = 0.
\end{cases}
\]

System $B_3$ is very similar to that of $B_1$ so its solution can be easily found as:

\[
V_2^{(1)}(x, l; \tau) = \int_0^l W(\tau - \bar{J} + s)g_2(x, l - s)ds, \quad \forall x \geq 0,
\]

where

\[
g_2(x, \tau) = \frac{x}{2\sqrt{\pi} \tau^{3/2}} e^{\alpha x + \beta \tau - \frac{x^2}{4\tau}}, \quad \forall x \geq 0,
\]

with $\alpha, \beta$ are defined in (4.3.12).

By using the variable transform $V_2(x, l; \tau) = e^{\alpha x + \beta \tau}u(x, l; \tau)$, $B_4$ can be transferred to a standard heat problem in a semi-infinite domain, whose solution can be found in [41]. As a result, the solution of $B_4$ can be obtained as:

\[
V_2^{(2)}(x, l; \tau) = \int_0^{+\infty} F(x, l, z, \tau)dz,
\]

where

\[
F(x, l, z, \tau) = \frac{1}{2\sqrt{\pi l}} e^{\alpha(x-z)+\beta l} \left[ e^{-\frac{(x-z)^2}{4l}} - e^{-\frac{(x+z)^2}{4l}} \right] C(z, \tau).
\]

We now can obtain the solution of $B_2$ in (4.3.7) as:

\[
V_2(x, l; \tau) = \int_0^l W(\tau - \bar{J} + s)g_2(x, l - s)ds + \int_0^{+\infty} F(x, l, z, \tau)dz.
\]

Applying the connectivity condition (4.3.8) to (4.3.10) and (4.3.15), we obtain an integral
equation governing $W(\tau)$:

$$
\int_0^\tau W(s) \frac{\partial g_1}{\partial x} (x, \tau - s) \, ds \bigg|_{x=0} = \int_0^{+\infty} \frac{\partial F}{\partial x} (0, \bar{J}, z, \tau) \, dz + \int_0^\tau W(\tau - \bar{J} + s) \frac{\partial g_2}{\partial x} (x, \bar{J} - s) \, ds \bigg|_{x=0},
$$

(4.3.16)

where

$$
\frac{\partial F}{\partial x} (0, \bar{J}, z, \tau) = \frac{z C(z, \tau)}{2 \sqrt{\pi \bar{J}^3}} e^{-\alpha z + \beta \bar{J} - \frac{z^2}{4 \bar{J}}}.
$$

(4.3.17)

Now, taking a simple coordinate transform, $\xi = \tau - \bar{J} + s$, in the last integral on the right-hand side of the equation (4.3.16) leads to:

$$
\int_0^\tau W(s) \frac{\partial g_1}{\partial x} (x, \tau - s) \, ds \bigg|_{x=0} = \int_0^{+\infty} \frac{\partial F}{\partial x} (0, \bar{J}, z, \tau) \, dz + \int_{\bar{J}}^\tau W(\xi) \frac{\partial g_2}{\partial x} (x, \xi) \, d\xi \bigg|_{x=0},
$$

(4.3.18)

It can be observed that the left-hand side of (4.3.18) contains the information of $W(s)$, with $s \in [0, \tau]$, while its right-hand side integral involves the value of $W(\xi)$, $\xi \in [\tau - \bar{J}, \tau]$, which coincides with the projection of the “slide” (a plane is of 45° angle to both of the plane $t = 0$, and $J = 0$) passing through $(\bar{S}, \tau, 0)$ on the plane $J = 0$. As in [83], we also name such a projection a “window”.

We now solve the integral equation (4.3.18) for $\tau \in [0, \bar{J}]$ to obtain the solution for $W_1(\tau)$, the value of $W$ in the first window. Note that $W(\xi) = 0, \forall \xi \in [-\bar{J}, 0]$ because $V_1(S, t) = 0$, for all $t \geq T - \bar{J}, S \leq \bar{S}$ (as already explained in Section 2). Therefore, for $\tau \in [0, \bar{J}]$, we can rewrite (4.3.18) as follows:

$$
\int_0^\tau W_1(s) \left( \frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x} \right) (x, \tau - s) \, ds \bigg|_{x=0} = \int_0^{+\infty} \frac{\partial F}{\partial x} (0, \bar{J}, z, \tau) \, dz.
$$

(4.3.19)

Clearly, the left hand-side of the last equation is a convolution integral involving the unknown function $W_1$. Taking the Laplace transform of equation (4.3.19) with respect to $\tau$, we obtain:

$$
\mathcal{L} [W_1(\tau)] \mathcal{L} \left[ \left( \frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x} \right) (x, \tau) \right] \bigg|_{x=0} = \mathcal{L} \left[ \int_0^{+\infty} \frac{\partial F}{\partial x} (0, \bar{J}, z, \tau) \, dz \right],
$$

where

$$
\mathcal{L} \left[ \left( \frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x} \right) (x, \tau) \right] \bigg|_{x=0} = 2 \sqrt{p - \beta}
$$
with \( p \) being the Laplace parameter \( [83] \). Thus,

\[
\mathcal{L}[W_1(\tau)] = \frac{1}{2\sqrt{p - \beta}} \mathcal{L} \left[ \int_0^{+\infty} \frac{\partial F}{\partial x}(0, \bar{J}, z, \tau) \, dz \right].
\] (4.3.20)

Taking the inverse Laplace transform on both sides of (4.3.20) yields:

\[
W_1(\tau) = \int_0^{+\infty} \int_0^{\tau} e^{\beta(\tau-s)} \frac{\partial F}{\partial x}(0, \bar{J}, z, s) \, ds \, dz.
\] (4.3.21)

Similar to the case in \( [83] \), for a state point \((S, \tau, J)\), one can evaluate \( W \) forwards, window by window, until the value at the required time \( \tau \) is found. The determination of \( W_{n+1} \), assuming \( W_n \) is known for \( n \geq 1 \), is however slightly different from that of \( W_1 \). More specifically, the coupled 2-D PDE systems governing the option price in the \((n+1)\)th window can be expressed as:

\[
C_1 \begin{cases} 
\frac{\partial V_1}{\partial t} + \mathbb{L} V_1 = 0, \\
V_1(S, T - (n + 1)J) = \bar{S} f_n(\ln \frac{S}{\bar{S}}), \\
V_1(0, t) = 0, \\
V_1(\bar{S}, t) = W(t),
\end{cases}
\]

\[
C_2 \begin{cases} 
\frac{\partial V_2}{\partial \ell'} + \mathbb{L} V_2 = 0, \\
V_2(S, \ell'; t) = C(S, t + \bar{J}), \\
V_2(S, \ell'; t) \sim S \text{ as } S \to +\infty, \\
V_2(\bar{S}, \ell'; t) = W(t + \ell'),
\end{cases}
\] (4.3.22)

with the Delta condition:

\[
\frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, 0; t),
\] (4.3.23)

where

\[
f_n(x) = \sum_{i=1}^{n} \int_{(i-1)J}^{iJ} W_i(s) g_1(x, nJ - s) \, ds.
\] (4.3.24)

Here, \( C_1 \) is defined on \( t \in [T - (n + 2)J, T - (n + 1)J] \), \( S \in [0, \bar{S}] \); \( C_2 \) is defined on the domain \( \ell' \in [0, J] \), \( S \in [\bar{S}, \infty) \), and parameter \( t \in [T - (n + 2)J, T - (n + 1)J] \); operator \( \mathbb{L} \) is defined in (4.2.3).

It should be noted that the systems (4.3.22, 4.3.23) are very similar to those of (4.3.4, 4.3.5), except the inhomogeneous initial condition in \( C_1 \): \( \bar{S} f_n(\ln \frac{S}{\bar{S}}) > 0, \forall S < \bar{S} \). To non-dimensionalize the systems (4.3.22, 4.3.23), we use the same variables introduced in (4.3.6), except that \( \tau \) and \( W'(\tau) \) are replaced by \( \tilde{\tau} = (T - (n + 1)J - t)\sigma^2/2 = \tau - nJ' \) and \( U(\tilde{\tau}) \),
respectively. Dropping all primes from now on, $C_1$ and $C_2$ in (4.3.22, 4.3.23) are transformed to $C_3$ and $C_4$, respectively, as follows:

\[
\begin{align*}
C_3: & \quad \begin{cases} 
\frac{\partial V_1}{\partial \tilde{\tau}} = KV_1, \\
V_1(x, 0) = f_n(x), \\
\lim_{x \to -\infty} V_1(x, \tilde{\tau}) = 0, \\
V_1(0, \tilde{\tau}) = U(\tilde{\tau}),
\end{cases} \\
C_4: & \quad \begin{cases} 
\frac{\partial V_2}{\partial l} = KV_2, \\
V_2(x, 0; \tilde{\tau}) = C(x, \tilde{\tau}), \\
V_2(x, l; \tilde{\tau}) \sim e^x \text{ as } x \to +\infty, \\
V_2(0, l; \tilde{\tau}) = U(\tilde{\tau} - \bar{J} + l),
\end{cases}
\end{align*}
\]

(4.3.25)

Delta condition: \[
\frac{\partial V_1}{\partial x}(0, \tilde{\tau}) = \frac{\partial V_2}{\partial x}(0, \bar{J}; \tilde{\tau}),
\]

(4.3.26)

where \(f_n(x)\) is defined in (4.3.24); \(C_3\) is defined on \(\tilde{\tau} \in [0, \bar{J}], x \in (-\infty, 0)\); \(C_4\) is defined on \(l \in [0, \bar{J}], x \in [0, \infty)\), with parameter \(\tilde{\tau} \in [0, \bar{J}]\); operator \(K\) is defined in (4.3.9).

Applying the same method used to solve the system \(B_2\) in (4.3.7), the solution of \(C_3\) can be easily solved as:

\[
V_1(x, \tilde{\tau}) = \int_{-\infty}^{0} G(x, \tilde{\tau}, z) dz + \int_{0}^{\tilde{\tau}} U(s) g_1(x, \tilde{\tau} - s) ds,
\]

(4.3.27)

where

\[
G(x, \tilde{\tau}, z) = \frac{1}{2\sqrt{\pi \tilde{\tau}}} e^{\alpha(x-z)+\beta \tilde{\tau}} \left[ e^{-\frac{(x-z)^2}{4\tilde{\tau}}} - e^{-\frac{(x+z)^2}{4\tilde{\tau}}} \right] f_n(z).
\]

(4.3.28)

Consequently, the corresponding integral equation governing \(U(\tilde{\tau})\) is

\[
\int_{-\infty}^{0} \frac{\partial G}{\partial x}(0, \tilde{\tau}, z) dz + \int_{0}^{\tilde{\tau}} U(s) \frac{\partial g_1}{\partial x}(x, \tilde{\tau} - s) ds \bigg|_{x=0} = \int_{0}^{+\infty} \frac{\partial F}{\partial x}(0, \bar{J}, z, \tilde{\tau}) dz + \int_{0}^{\bar{J}} U(\tilde{\tau} - \bar{J} + s) \frac{\partial g_2}{\partial x}(x, \bar{J} - s) ds \bigg|_{x=0},
\]

(4.3.29)

where

\[
\frac{\partial G}{\partial x}(0, \tilde{\tau}, z) = \frac{zf_n(z)}{2\sqrt{\pi \tilde{\tau}^3}} e^{-\alpha z + \beta \tilde{\tau} - \frac{z^2}{4\tilde{\tau}}},
\]

(4.3.30)

and \(\frac{\partial F}{\partial x}(0, \bar{J}, z, \tilde{\tau})\) is defined as in (4.3.17). Now, taking a simple coordinate transform,
\( \xi = \tilde{\tau} - \tilde{J} + s \), in the integral on the right-hand side of the equation (4.3.29) leads to:

\[
\int_{-\infty}^{0} \frac{\partial G}{\partial x}(0, \tilde{\tau}, z) \, dz + \int_{0}^{\tilde{\tau}} U(s) \frac{\partial g_{1}}{\partial x}(x, \tilde{\tau} - s) \, ds \bigg|_{x=0} = \int_{0}^{+\infty} \frac{\partial F}{\partial x}(0, \tilde{J}, z, \tilde{\tau}) \, dz + \int_{\tilde{\tau}-\tilde{J}}^{\tilde{\tau}} U(\xi) \frac{\partial g_{2}}{\partial x}(x, \tilde{\tau} - \xi) \, d\xi \bigg|_{x=0}. \tag{4.3.31}
\]

Let \( U_0(\xi) = W_n(\xi + n\tilde{J}), \forall \xi \in [-\tilde{J}, 0] \), the equation (4.3.31) can be written as:

\[
\int_{0}^{\tilde{\tau}} U(s) \left( \frac{\partial g_{1}}{\partial x} - \frac{\partial g_{2}}{\partial x} \right)(x, \tilde{\tau} - s) \, ds \bigg|_{x=0} = \int_{0}^{+\infty} \frac{\partial F}{\partial x}(0, \tilde{J}, z, \tilde{\tau}) \, dz \bigg|_{x=0} = \int_{0}^{+\infty} \frac{\partial \tilde{F}}{\partial x}(0, \tilde{J}, z, \tilde{\tau}) \, dz - \int_{-\infty}^{0} \frac{\partial G}{\partial x}(0, \tilde{\tau}, z) \, dz + \int_{\tilde{\tau}-\tilde{J}}^{0} U_0(\xi) \frac{\partial g_{2}}{\partial x}(x, \tilde{\tau} - \xi) \, d\xi \bigg|_{x=0}. \tag{4.3.32}
\]

Taking the Laplace transform on both sides of (4.3.32) with respect to \( \tilde{\tau} \), we obtain:

\[
\mathcal{L}[U(\tilde{\tau})] \mathcal{L} \left[ \left( \frac{\partial g_{1}}{\partial x} - \frac{\partial g_{2}}{\partial x} \right)(x, \tilde{\tau}) \right] \bigg|_{x=0} = \mathcal{L} \left[ \int_{0}^{+\infty} \frac{\partial \tilde{F}}{\partial x}(0, \tilde{J}, z, \tilde{\tau}) \, dz \right] - \mathcal{L} \left[ \int_{-\infty}^{0} \frac{\partial G}{\partial x}(0, \tilde{\tau}, z) \, dz \right] + \mathcal{L} \left[ \int_{\tilde{\tau}-\tilde{J}}^{0} U_0(\xi) \frac{\partial g_{2}}{\partial x}(x, \tilde{\tau} - \xi) \, d\xi \bigg|_{x=0} \right].
\]

Therefore,

\[
\mathcal{L}[U(\tilde{\tau})] = \frac{1}{2\sqrt{p-\beta}} \mathcal{L} \left[ \int_{0}^{+\infty} \frac{\partial \tilde{F}}{\partial x}(0, \tilde{J}, z, \tilde{\tau}) \, dz \right] \tag{4.3.33}
\]

\[\quad + \frac{1}{2\sqrt{p-\beta}} \left( \mathcal{L} \left[ \int_{0}^{\tilde{\tau}-\tilde{J}} U_0(\xi) \frac{\partial g_{2}}{\partial x}(x, \tilde{\tau} - \xi) \, d\xi \bigg|_{x=0} \right] - \mathcal{L} \left[ \int_{-\infty}^{0} \frac{\partial G}{\partial x}(0, \tilde{\tau}, z) \, dz \right] \right).\]

By taking the inverse Laplace transform on both sides of (4.3.33), we can obtain the solution of (4.3.29) as follows:

\[
U(\tilde{\tau}) = \int_{0}^{+\infty} \int_{0}^{\tilde{\tau}} e^{\beta(\tilde{\tau}-s)} \frac{\partial E}{\partial x}(0, \tilde{J}, z, s) \, dsdz + \int_{0}^{\tilde{\tau}} e^{-\alpha z+\beta \tilde{\tau}-\frac{\tilde{J}^2}{2\tilde{s}}} f_{n}(z)dz + \frac{U_{0}(0)}{2} e^{\beta \tilde{\tau}} \]
\[\quad - \frac{e^{\beta \tilde{J}}}{2\pi \sqrt{\tilde{s}}} \int_{0}^{\tilde{\tau}} e^{\beta(\tilde{s}-s)} U_{0}(s-\tilde{J})ds - \frac{1}{\pi} \int_{0}^{\tilde{\tau}} e^{\beta(\tilde{s}-s)} \int_{\sqrt{\tilde{s}}}^{\sqrt{\tilde{s}}} e^{\beta \tilde{s}^2} \left[ (-\beta)U_{0}(s-\tilde{s}^2) + U'_{0}(s-\tilde{s}^2) \right]dt ds,
\]

where \( U_{0}(\tilde{\tau}) = W_{n}(\tilde{\tau} + n\tilde{J}), \forall \tilde{\tau} \in [-\tilde{J}, 0] \).

Note that the inverse Laplace of the first term on the right hand side of (4.3.33) is the same as that in the calculation of \( W_{1} \), while the inverse Laplace of the last two terms on the right hand side of (4.3.33) were also carried out analytically, the detailed calculation can be seen
Thus, we have obtained an analytical formula for Parisian up-and-in calls. This formula can be used for the valuation of American-style and European-style Parisian up-and-in calls, once \( C \) is substituted by \( C_A \) and \( C_E \) in the above formulas of \( V_1 \), \( V_2 \), and \( W \), respectively. It should not be too difficult to calculate \( C_A \) or \( C_E \) because the valuations of European-style vanilla options and American-style vanilla options have been thoroughly studied in the literature [35, 36, 49, 54, 81].

### 4.4 Numerical example and discussion

In this section, we provide some graphs to illustrate the results obtained from our analytical pricing formula as well as reveal some interesting features of a Parisian up-and-in call.

It should be noted that the calculation procedure for an American-style Parisian up-and-in call option is similar to that for a European-style Parisian up-and-out call as presented in [83], except that we have replaced the value of the European vanilla option by the numerical value of its American counterpart, which can be obtained by using the highly efficient integral equation method ([54]). Once the value of the American vanilla option is found, the integrals in our analytical formula can be computed by using quadrature rules (Gauss-Laguerre, Gauss-Legendre, Gauss-Jacobi rules) in a very similar way as that in [83]. Therefore, the computation cost of our formula should not be too much different with that in [83].

Figures 4.1(a) and 4.1(b) present comparisons of the values of Parisian up-and-in calls for various \( J \) values with those of their embedded vanilla calls. The parameters used in our calculations are \( E = \$10 \), \( \bar{S} = \$18 \), \( T - t = 0.8 \) (year), \( \bar{J} = 0.2 \) (year), \( \sigma = 30\% \), \( r = 5\% \), \( D = 10\% \). As can be seen clearly from both the figures that the values of the Parisian options
4.4. NUMERICAL EXAMPLE AND DISCUSSION

Figure 4.1: Comparison between the prices of a Parisian up-and-in call at various $J$ with that of its embedded vanilla option; parameters are: $E = $10, $S = $18, $T - t = 0.8$ (year), $\bar{J} = 0.2$ (year), $\sigma = 30\%$, $r = 5\%$, $D = 10\%$.

are always less than those of their embedded vanilla options. This is indeed expected as the holders of the Parisian calls have to wait until the knock-in feature is activated to obtain the same exercise right as the holders of the embedded vanilla options. This waiting period, with the risk that the knock-in feature may never be activated, would definitely devalue the Parisian calls, in comparison with their embedded vanilla counterparts.

Figures 4.1(a) and 4.1(b) also reveal some interesting properties of the Parisian up-and-in calls with respect to changes in $S$ and $J$. One can observe that when $J$ is fixed, the Parisian call prices are increasing functions of asset price. In fact, when the asset price increases,
the knock-in feature is more likely to be activated and thus the values of the Parisian calls increase and finally approach the values of their embedded vanilla options. Similarly, with a fixed value of $S$, the knock-in feature is more likely to be activated when $J$ gets closer to $\bar{J}$. As a result, the Parisian option prices increase when $J$ increases.

![Figure 4.2: Differences between the prices of American-style and European-style Parisian up-and-in calls at various $J$, with parameters: $E = $10, $T - t = 0.8$ (year), $\bar{S} = $18, $\bar{J} = 0.2$ (year), $\sigma = 30\%$, $r = 5\%$, $D = 10\%$.](image)

It is also expected that the price of an American-style Parisian up-and-in call should be higher than that of its European-style option counterpart because upon the activation of the knock-in feature, the holder of the former can buy the underlying asset before and up to expiry, while that of the latter can only buy the underlying asset at expiry. This expectation is clearly illustrated in Figure 4.2 which shows the differences between the prices of the former at various $J$ and those of the latter. It is clear from this figure that the differences between the two sets of prices become larger when $S$ becomes larger or $J$ gets closer to $\bar{J}$. This is because the knock-in feature is then more likely to be activated and the values of the Parisian options approaches those of their embedded options. As a result, the differences between the values of the Parisian options approach those of their embedded vanilla options, which become larger when $S$ increases.
4.5 Conclusion

In this chapter, we have derived a simple analytical formula for Parisian up-and-in calls by using the “moving window” technique proposed in [83]. Unlike the “knock-out” cases, the valuation of American-style Parisian up-and-in calls is very similar to that of its European counterpart and both can be handled with the same solution procedure. As a result, we are able to derive a pricing formula that can be used to evaluate both American-style and European-style Parisian up-and-in calls. We have also provided examples to illustrate some interesting features of a Parisian up-and-in call.
Chapter 5

Pricing American-style Parisian down-and-in options

5.1 Introduction

Continuing on the topic of pricing Parisian knock-in options, this chapter discusses the pricing problem of another type of these options, the down-type options. The difference between the up-type and down-type options lies on the knock-in condition, i.e., the condition activates the knock-in feature. More specifically, the knock-in feature of a Parisian down-and-in option is activated only if the underlying asset price has continually stayed below the barrier $\bar{S}$ for a prescribed time period $\bar{J}$. This knock-in condition of the down-type option clearly contrasts with that of its up-type option counterpart.

It should be noted that there is no clear relation between the pricing formulas of Parisian up-and-in options and Parisian down-and-in options. In other words, knowing an analytical pricing formula of the former does not allow us to straightforwardly obtain that of the latter. Finding an analytical formula for the latter is therefore not a trivial task, even though that of the former is already derived in Chapter 4. In this chapter, we show that the “moving window” technique can be used to derive an analytical solution for American-style Parisian down-and-in calls.

This chapter is organized as follows. In Section 5.2, we introduce the PDE systems governing the price of an American-style Parisian down-and-in call option. The solution procedure is presented in Section 5.3 while Section 5.4 provides some selected graphs to illustrate the
implementation of our formulas. This chapter ends with some concluding remarks given in Section 5.5.

5.2 The PDE systems

Similar to its Parisian up-and-in option counterpart, for some extreme values of \( \bar{S} \) and \( \bar{J} \), an American-style Parisian down-and-in call option becomes worthless or degenerates to either a one-touch barrier option or a vanilla option. For other non-degenerate cases, the price of an American-style Parisian down-and-in call option is however not trivial. Under the Black-Scholes model, it depends on the underlying asset price \( S \), the current time \( t \) and the excursion time \( J \), in addition to other constant parameters such as the volatility rate \( \sigma \), the risk-free interest rate \( r \), the exercise price \( E \), the expiry time \( T \), the barrier \( \bar{S} \), and the “option window” \( \bar{J} \).

Using financial arguments similar to those used in [83], the pricing domains of non-degenerated cases can be elegantly reduced as:

\[
I : \{ \bar{S} \leq S < \infty, \ 0 \leq t \leq T - \bar{J}, \ J = 0 \}, \\
II : \{ 0 \leq S < \bar{S}, \ J \leq t \leq J + T - \bar{J}, \ 0 \leq J \leq \bar{J} \}.
\]

Let \( V_1(S,t) \) and \( V_2(S,t,J) \) denote the option prices in the region \( I \) and \( II \), respectively. Based on the definition of the option, the continuity conditions of both the option price and the “Delta” of the option, it can be shown that the option price should satisfy [39, 83]:

\[
\begin{align*}
A_1 & : \begin{cases}
\frac{\partial V_1}{\partial t} + \mathbb{L} V_1 = 0, \\
V_1(S,T - \bar{J}) = 0, \\
\lim_{S \to \infty} V_1(S,t) = 0, \\
V_1(\bar{S},t) = V_2(\bar{S},t,0),
\end{cases} \\
A_2 & : \begin{cases}
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathbb{L} V_2 = 0, \\
V_2(S,t,\bar{J}) = C_A(S,t) , \\
V_2(0,t,J) = 0 , \\
V_2(\bar{S},t,J) = V_2(\bar{S},t,0),
\end{cases}
\end{align*}
\]

(5.2.1)

connectivity condition : \( \frac{\partial V_1}{\partial S}(\bar{S},t) = \frac{\partial V_2}{\partial S}(\bar{S},t,0) \),

(5.2.2)

where \( A_1 \) and \( A_2 \) defined in domain \( I \) and domain \( II \), respectively. Here, operator \( \mathbb{L} \) is
defined as in (4.2.3). It should be noted that the last equation in \( A_2 \) of (5.2.1) holds only for 
\[ 0 \leq J < \bar{J}, \] i.e., the “knock in” feature has not been triggered.

One can observe that the above PDE systems are quite similar to those governing an 
American-style Parisian up-and-in call option. There are however two key differences. Firstly, 
it is obvious that the pricing domains of a Parisian down-and-in option are reversed from those 
of its up-and-in counterpart. Secondly, instead of an inhomogeneous boundary condition as 
in \( A_2 \) of (4.2.1), we have the homogeneous boundary condition in \( A_1 \) of (5.2.1) when \( S \) approaches infinity, because in this case the knock-in feature can never be activated.

Albeit different, the above coupled PDE systems can be solved by adopting the “moving 
window” technique developed in \[83\]. In the next section, we shall briefly discuss the solution 
procedure.

### 5.3 Solution of the coupled PDE systems

Following the method presented in \[83\] with the same notations, the 3-D PDE systems (5.2.2) 
can be further simplified to the following two 2-D PDE systems:

\[
A_3 \begin{cases}
\frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\
V_1(S,T - \bar{J}) = 0, \\
\lim_{S \to \infty} V_1(S,t) = 0, \\
V_1(\bar{S},t) = W(t),
\end{cases}
A_4 \begin{cases}
\frac{\partial V_2}{\partial l'} + \mathbb{L}V_2 = 0, \\
V_2(S,\bar{J}; t) = C_A(S,t + \bar{J}), \\
V_2(0,l'; t) = 0, \\
V_2(\bar{S},l'; t) = W(t + l'),
\end{cases}
\]

connectivity condition: \( \frac{\partial V_1}{\partial S}(\bar{S},t) = \frac{\partial V_2}{\partial S}(\bar{S},0;t) \),

where \( W(t) = V_2(\bar{S},0;t) \). This unknown function \( W(t) \) that provides the coupling between 
the two systems also needs to be solved as part of the solution. Here \( A_3 \) is defined on 
\( t \in [0,T - \bar{J}], S \in [\bar{S},\infty) \) and \( A_4 \) is defined on \( t \in [0,T - \bar{J}], l' \in [0,\bar{J}], S \in [0,\bar{S}] \).

To solve the newly established pricing systems (5.3.3-5.3.4) effectively, we shall first non-
dimensionalize all variables, using dimensionless variables defined in (4.3.6). With all primes
and tildes dropped from now on, the dimensionless PDE systems read:

\[
\begin{align*}
B_1 & \begin{cases}
\frac{\partial V_1}{\partial \tau} = KV_1, \\
V_1(x, 0) = 0, \\
\lim_{x \to \infty} V_1(x, \tau) = 0, \\
V_1(0, \tau) = W(\tau),
\end{cases} & \quad \begin{cases}
\frac{\partial V_2}{\partial l} = KV_2, \\
V_2(x, 0; \tau) = C_A(x, \tau), \\
\lim_{x \to -\infty} V_2(x, l; \tau) = 0, \\
V_2(0, l; \tau) = W(\tau - \bar{J} + l),
\end{cases}
\end{align*}
\]

(5.3.5)

connectivity condition: \[
\frac{\partial V_1}{\partial x}(0, \tau) = \frac{\partial V_2}{\partial x}(0, \bar{J}; \tau),
\]

(5.3.6)

where \(B_1\) is defined on \(\tau \in [0, T - \bar{J}]\), \(x \in [0, \infty)\), \(B_2\) is defined on \(\tau \in [0, T - \bar{J}], l \in [0, \bar{J}], x \in (-\infty, 0]\), and operator \(K\) is defined as in (4.3.9).

Using methods similar to those used to solve the systems (4.3.7, 4.3.8), we obtain

\[
V_1(x, \tau) = \int_0^\tau W(s)g_2(x, \tau - s)ds, \quad \forall x \geq 0
\]

(5.3.7)

and

\[
V_2(x, l; \tau) = \int_{-\infty}^0 F(x, l, z, \tau)dz + \int_0^l W(\tau - \bar{J} + s)g_1(x, l - s)ds, \quad \forall x \leq 0,
\]

(5.3.8)

where functions \(g_1, g_2, F\) are defined as in (4.3.11), (4.3.13), and (4.3.14), respectively. Now, applying the connectivity condition (5.3.6) to (5.3.7) and (5.3.8), we obtain the integral equation governing \(W(\tau)\) as

\[
\int_0^\tau W(s) \frac{\partial g_2}{\partial x}(x, \tau - s)ds \bigg|_{x=0} = \int_{-\infty}^0 \frac{\partial F}{\partial x}(0, \bar{J}, z, \tau)dz + \int_0^{\bar{J}} W(\tau - \bar{J} + s) \frac{\partial g_1}{\partial x}(x, \bar{J} - s)ds \bigg|_{x=0},
\]

(5.3.9)

which can be written as below after a new variable transform \(\xi = \tau - \bar{J} + s\) is introduced

\[
\int_0^\tau W(s) \frac{\partial g_2}{\partial x}(x, \tau - s)ds \bigg|_{x=0} = \int_{-\infty}^0 \frac{\partial F}{\partial x}(0, \bar{J}, z, \tau)dz + \int_{\tau - \bar{J}}^\tau W(\xi) \frac{\partial g_1}{\partial x}(x, \tau - \xi)d\xi \bigg|_{x=0}.
\]

(5.3.10)

Here \(\frac{\partial F}{\partial x}(0, \bar{J}, z, \tau)\) are defined as in (4.3.17).

One can easily observe that the equation (5.3.10) is very similar to (4.3.18). Therefore,
using the method used to solve (4.3.18), we can obtain the solution of (5.3.10) for \( \tau \in [0, \bar{J}] \), i.e., the value of \( W \) in the first window, denoted by \( W_1(\tau) \) as follows:

\[
W_1(\tau) = - \int_{-\infty}^{0} \int_{0}^{\tau} \frac{e^{\beta(\tau-s)}}{2\sqrt{\pi(\tau-s)}} \frac{\partial F}{\partial x}(0, \bar{J}, z, s) ds dz
\]

As can be expected, the formula (5.3.11) is very similar to (4.3.21). They share the same integrand and differ only in the sign and the limits of the integrations.

We now determine the value of \( W \) in the \((n+1)\)-th window, denoted by \( W_{n+1} \). In the new coordinate system \( \tilde{\tau} = \tau - n\bar{J} \), solving \( W_{n+1}(\tau) \) with the known option price on the \( n \)th window is equivalent to determining \( U(\tilde{\tau}) \) from the following PDE systems:

\[
B_3 \begin{cases} \\
\frac{\partial V_1}{\partial \tilde{\tau}} = KV_1, \\
V_1(x, 0) = f_n(x), \\
\lim_{x \to \infty} V_1(x, \tilde{\tau}) = 0, \\
V_1(0, \tilde{\tau}) = U(\tilde{\tau}), \\
c\end{cases}
\]

\[
B_4 \begin{cases} \\
\frac{\partial V_2}{\partial l} = KV_2, \\
V_2(x, 0; \tilde{\tau}) = C_A(x, \tilde{\tau}), \\
\lim_{x \to -\infty} V_2(x, l; \tilde{\tau}) = 0, \\
V_2(0, l; \tilde{\tau}) = U(\tilde{\tau} - \bar{J} + l), \\
c\end{cases}
\]

connectivity condition : \( \frac{\partial V_1}{\partial x}(0, \tilde{\tau}) = \frac{\partial V_2}{\partial x}(0, \bar{J}; \tilde{\tau}) \).

Here \( f_n(x) = V_1(x, n\bar{J}) = \sum_{i=1}^{n} \int_{(i-1)\bar{J}}^{i\bar{J}} W_i(s)g_2(x, n\bar{J} - s) ds, \) \( K \) is defined as in (4.3.9), \( B_3 \) is defined on \( \tilde{\tau} \in [0, \bar{J}], \) \( x \in [0, \infty), \) and \( B_4 \) is defined on \( \tilde{\tau} \in [0, \bar{J}], \) \( l \in [0, \bar{J}], \) \( x \in (-\infty, 0] \).

The inhomogeneous initial condition of the system \( B_3 \) makes its solution procedure more complicated than that of \( B_1 \). However, using the Laplace transform technique and the Green function method, we have managed to derive its solution as

\[
V_1(x, \tilde{\tau}) = \int_{0}^{+\infty} G(x, \tilde{\tau}, z) dz + \int_{0}^{\tilde{\tau}} U(s)g_2(x, \tilde{\tau} - s) ds,
\]

where \( G(x, \tilde{\tau}, z) \) is defined as in (4.3.28). As a result, the corresponding integral equation
governing $U(\bar{\tau})$ is given by:

\[
\int_0^{+\infty} \frac{\partial G}{\partial x}(0, \bar{\tau}, z) dz + \int_0^{\bar{\tau}} U(s) \frac{\partial g_2}{\partial x}(x, \bar{\tau} - s) ds \bigg|_{x=0} = \int_{-\infty}^{0} \frac{\partial F}{\partial x}(0,\bar{J},z,\bar{\tau}) dz + \int_0^{\bar{J}} U(\bar{\tau} - \bar{J} + s) \frac{\partial g_1}{\partial x}(x, \bar{J} - s) ds \bigg|_{x=0},
\]

(5.3.15)

where $\frac{\partial G}{\partial x}(0, \bar{\tau}, z)$ and $\frac{\partial F}{\partial x}(0, \bar{J}, z, \bar{\tau})$ are defined as in (4.3.30) and (4.3.17), respectively. Using the method similar to that used to solve the equation (4.3.29), we can obtain the solution of (5.3.15) as:

\[
U(\bar{\tau}) = -\int_{-\infty}^{0} \int_{0}^{\bar{\tau}} \frac{e^{\beta(\tau-s)}}{2\sqrt{\pi(\tau-s)}} \frac{\partial F}{\partial x}(0, \bar{J}, z, s) ds dz + \int_0^{+\infty} \frac{e^{-\alpha z + \beta \tau - \frac{\beta^2}{4}}}{2\sqrt{\pi \tau}} f_n(z) dz + \frac{U_0(0)}{2} e^{\beta \bar{\tau}}
\]

\[-\frac{e^{\beta J}}{2\pi \sqrt{J}} \int_{0}^{\bar{\tau}} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} U_0(s - \bar{J}) ds - \frac{1}{\pi} \int_0^{\bar{\tau}} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} \int_{\sqrt{\tau}}^{\sqrt{J}} e^{\beta t^2} \left[ (-\beta) U_0(s - t^2) + U_0'(s - t^2) \right] dt ds,
\]

where $U_0(\bar{\tau}) = W_n(\bar{\tau} + n\bar{J}), \forall \bar{\tau} \in [-\bar{J}, 0]$.

Consequently, the analytical formula for $W_{n+1}(\tau)$ is

\[
W_{n+1}(\tau) = -\int_{-\infty}^{0} \int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{2\sqrt{\pi(\tau-s)}} \frac{\partial F}{\partial x}(0, \bar{J}, z, s) ds dz + \int_0^{+\infty} \frac{e^{-\alpha z + \beta(\tau-n\bar{J}) - \frac{\beta^2}{4(\tau-n\bar{J})}}}{2\sqrt{\pi(\tau-n\bar{J})}} f_n(z) dz
\]

\[+ \frac{W_n(n\bar{J})}{2} e^{\beta(\tau-n\bar{J})} - \frac{e^{\beta J}}{2\pi \sqrt{J}} \int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} W_n(s - \bar{J}) ds \] (5.3.16)

\[-\frac{1}{\pi} \int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} \int_{\sqrt{s-nJ}}^{\sqrt{J}} e^{\beta t^2} \left[ (-\beta) W_n(s - t^2) + W_n'(s - t^2) \right] dt ds.
\]

One can easily observe that the formula (5.3.16) is very similar to (4.3.31). More specifically, the last three terms in the two formulas are identical, while the first two terms share the same integrands and differ only in the sign and the limits of the integrations. It should also be remarked here that from the above solution for an American-style Parisian down-and-in call option, we can immediately derive an analytical pricing formula for its European-style option counterpart by replacing the value of the American vanilla call option in the formulas (5.3.11) and (5.3.16) by the value of the European vanilla call. Moreover, using American-style Parisian put-call symmetry as established by Chesney and Gauthier [17], closed-form pricing formulas for Parisian knock-in put options can be obtained straightforwardly from
those of the call counterparts.

5.4 Numerical example and discussion

In this section, we provide some graphs to illustrate some interesting features of a Parisian down-and-in call.

![Comparison graphs](image)

(a) American-style options

(b) European-style options

Figure 5.1: Comparison between the prices of a Parisian down-and-in call at various \( J \) with that of its embedded vanilla option; parameters are: \( E = $10, \bar{S} = $12, T - t = 0.8 \) (year), \( \bar{J} = 0.2 \) (year), \( \sigma = 30\% \), \( r = 5\% \), \( D = 10\% \).

Figures 5.1(a) and 5.1(b) present comparisons of the values of Parisian down-and-in calls at various \( J \) values with those of their embedded vanilla calls, for American-style and European-
style cases, respectively. The parameters used in our calculations are $E = $10, $\bar{S} = $12, $T - t = 0.8$ (year), $\bar{J} = 0.2$ (year), $\sigma = 30\%$, $r = 5\%$, $D = 10\%$. Similar to the “up-and-in” case, the values of the Parisian down-and-in calls at various $J$ values are always less than those of their embedded vanilla options. This is reasonable as the holders of the Parisian down-and-in calls also face an extra risk that the knock-in feature may never be activated. This risk becomes lower if the asset price decreases below the barrier and $J$ gets closer to $\bar{J}$. In this case, the values of the Parisian calls will approach the values of their embedded vanilla options. By contrast, if the knock-in feature is less likely to be activated, the prices of the Parisian options become much less than those of their embedded vanilla options. For instance, if the asset price increases beyond the barrier, the values of the Parisian options decrease to zero while those of the embedded vanilla options increase to infinity, and thus the difference between them become larger with an increase in the asset price.

It is obvious that the price of an American-style Parisian up-and-in call is always higher than or at least equal to that of its European-style option counterpart, because the holder of the former has more exercise right than that of the latter. As a result, the difference between the value of the former and that of the latter is always greater or at least equal to zero, as shown clearly in Figure 5.2. It is also clear from this figure that the difference associated with
a greater $J$ is larger than those associated with smaller $J$. This is because the closer $J$ gets to $\bar{J}$, the more likely the holder of the American-style Parisian call can buy the underlying asset before expiry, a right that the holder of the European-style option counterpart will not have until expiry. As a result, the American-style Parisian call is more valuable than its European counterpart when $J$ increases to $\bar{J}$.

5.5 Conclusion

In this chapter, a simple analytical solution for American-style Parisian down-and-in call options is derived. Similar to Chapter 4 we apply the “moving window” technique developed in [83] to simplify the pricing domain, and consequently reduce the 3-D pricing problem of American-style Parisian down-and-in call options to a 2-D one. The derived analytical formula can be easily realized numerically using quadrature rules. We have also provided some selected graphs to illustrate some interesting features of a Parisian down-and-in call.
Chapter 6

Pricing American-style Parisian up-and-out call options

6.1 Introduction

Parisian options are a natural extension of “one-touch” barrier options to prevent such an attempt to manipulate the asset price so as to unfairly force the activation of the knock-in or knock-out feature. The asset price has to stay continually above (or below) the asset barrier for a prescribed amount of time in order to activate the knock-out or knock-in feature of Parisian options. Such an requirement certainly makes it harder and more costly for those who wish to manipulate the asset price. However, the introduction of the “excursion time” also causes much more difficulty in pricing Parisian option as the valuation problem is now a three-dimensional problem, instead of a two-dimensional one as is the case for barrier options.

Interestingly, the level of difficulty in pricing Parisian options varies a lot between knock-in and knock-out options. As discussed in Section 4.1, one can price an American-style Parisian knock-in option and its European counterpart by the same solution procedure. In fact, in Chapters 4 and 5, the “moving window” technique proposed by Zhu and Chen [83] is applied to find analytical solutions for both types of Parisian knock-in options: up-type and down-type, respectively.

Unlike knock-in cases, the valuation problem of American-style Parisian knock-out options is much more difficult than that of their European-style counterparts. In fact, a closed-form analytical solution of European-style Parisian up-and-out options has already been found by
Zhu and Chen [83], whereas finding a closed-form solution of American-style Parisian knock-out options is still a very challenging task. The main difficulty of pricing an American-style knock-out Parisian option, in comparison with its European counterpart, is that the optimal exercise boundary needs to be determined in order to obtain the option value. This optimal exercise boundary depends not only on the current time, but also on how long the asset price has stayed continually above the barrier (the “excursion time”). In other words, the optimal exercise boundary is now a 3-D surface, instead of a 2-D curve as is the case for a “one touch” barrier option.

In the literature, two main approaches for pricing American-style Parisian knock-out options were proposed. The first one was to use numerical methods such as finite difference method, as was studied in detailed by Haber et al. [39]. While this method is flexible and easy to implement, there are some deficiencies in their pricing systems, as already pointed out in [83]. The second approach was to use analytical approximation methods such as the probability method [17]. By using the theory of Brownian excursions, Chesney and Gauthier [17] derived formulas for the prices of American-style Parisian knock-out options, which involve the inverse Laplace transforms of the “Parisian stopping time”, which is the first time the length of the excursion reaches the predetermined option window. The inverse Laplace transforms need to be performed in order to obtain the option prices. As discussed in [16, 57], numerically performing Laplace inversion sometimes could be unstable and sensitive to round-off errors. In addition, the inverse Laplace transform techniques developed by Bernard et al. [5], Labart and Lelong [58] for European-style Parisian options have not been extended for American-style Parisian options. Therefore, there is a need to find a new method to obtain the option price without numerically performing Laplace inversion. This is the main motivation for our work on American-style Parisian knock-out options.

In this chapter, we propose an integral equation method for pricing American-style Parisian up-and-out call options. Using similar financial arguments used in [83], the pricing domain can be simplified, and consequently the 3-D pricing problem can be reduced to a 2-D one. The Fourier sine transform is then applied to derive a pair of coupled integral equations, which can be further simplified to a system of nonlinear algebraic equations. Consequently, the option prices and the hedging parameters can be straightforwardly obtained after efficiently solving
the algebraic equation system using the Newton-Raphson iterative procedure. Our numerical results not only show the validity of our proposed approach but also reveal interesting features of the option prices and the optimal exercise boundaries.

The chapter is organized as follows. In Section 6.2, we introduce the PDE systems governing the prices of American-style Parisian up-and-out call options. The solution procedure is presented in Section 6.3, while the numerical implementation is discussed in Section 6.4. Section 6.5 presents selected numerical results to demonstrate interesting properties of the prices of American-style Parisian up-and-out calls as well as their optimal exercise boundaries. The chapter ends with some concluding remarks given in Section 6.6.

6.2 Formulation

An American Parisian up-and-out call option is very similar to its one-touch option counterpart, except that the knock-out feature is activated only when the asset price \( S \) continually stays above a predetermined constant barrier \( \bar{S} \) for a prescribed time period \( \bar{J} \), the “option window”. Therefore, for pricing Parisian up-and-out call options with the PDE approach, we need a state variable \( J \) to measure the “excursion time”, i.e., the total time the asset price has spent continually above the barrier. More specifically, when \( S > \bar{S} \), the value of \( J \) starts to accumulate at the same rate as the passing time \( t \), and when \( S \leq \bar{S} \), \( J \) is reset to zero, and remains zero until the asset price is beyond \( \bar{S} \) again. Mathematically, we can describe the movement of \( J \) as follows:

\[
\begin{align*}
J &= 0, \quad dJ = 0, \quad S \leq \bar{S}, \\
\frac{dJ}{dt} &= 1, \quad S > \bar{S}.
\end{align*}
\]

One can expect that if \( \bar{J} \) takes some extreme values, an American-style Parisian up-and-out call option will degenerate to either a one-touch barrier option or a vanilla option. For instance, a Parisian option degenerates to a barrier option when \( \bar{J} = 0 \), as the option will be immediately “knocked out” once the asset price touches the barrier. On the other hand, if \( \bar{J} \) is greater than the option life, denoted by \( T \), the knock-out feature can never be activated, and thus the Parisian option would be identical to its embedded vanilla option.

Another important parameter in the pricing of an American-style Parisian up-and-out
call option is the asset barrier, which can have a great effect on the early exercise policy of the Parisian option. Typically, an American-style Parisian up-and-out call option has to be exercised at a lower optimal exercise price than that of its embedded vanilla option in order to avoid the worst scenario, i.e, the Parisian option is knocked out. More precisely, the lower the asset barrier is, the more likely the Parisian option will be knocked out, and thereby the lower the optimal exercise price of the Parisian option is in comparison with that of its embedded option. In other words, the lower the asset barrier is, the more apparent the effect of the knock-out feature on the early exercise policy of the Parisian option.

The formulation for the pricing of an American style Parisian up-and-out call depends on the position of the asset barrier surface relative to the optimal exercise price curve of the embedded call. Let $S^V_f(t)$ denote the optimal exercise price of the embedded vanilla option at time $t$, we will show the formulations for the following cases.

**Case 1:** $\bar{S} > S^V_f(0)$. In this case, the barrier surface is completely above the optimal exercise boundary of the Parisian option so that the Parisian option could always be optimally exercised before the asset price reaches the barrier. As a result, the Parisian option degenerates to its embedded option as the knock-out feature has no effect on the optimal exercise policy.

**Case 2:** $S < S^V_f(T)$. If $\bar{S} < S^V_f(T)$, the barrier surface is completely below the optimal exercise boundary of the embedded vanilla option. It can then be deduced that the barrier surface is also completely below the optimal exercise boundary of the Parisian option. This is because if there exists $t_0 \in [0, T]$ such that the optimal exercise price of the Parisian option at $t_0$, denoted by $S_f(t_0)$, satisfies $S_f(t_0) \leq \bar{S}$, then by using arguments similar to those used for the case $\bar{S} > S^V_f(0)$, we can obtain $S^V_f(t) = S_f(t) \leq \bar{S}, \forall t \geq t_0$, which clearly contradicts with the assumption $\bar{S} < S^V_f(T)$.

The pricing domain of an American Parisian up-and-out call option in this case can be divided into two main parts: $D_1$ and $D_2$, where the asset price $S$ is below or above the asset barrier $\bar{S}$, respectively. When $S > \bar{S}$, $J$ starts to accumulate and so the option price is governed by a 3-D PDE system. On the other hand, $J$ is reset to zero and remains zero when $S \leq \bar{S}$ and consequently, the option price is governed by a 2-D PDE system only.
Mathematically, $D_1$ and $D_2$ can be expressed as:

$$D_1 = \{(S,t)|0 \leq S \leq \bar{S}, 0 \leq t \leq T\},$$

$$D_2 = \{(S,t,J)|\bar{S} \leq S \leq S_f(t,J), 0 \leq t \leq T, 0 \leq J \leq \bar{J}\},$$

where $S_f(t,J)$ denotes the optimal exercise price of the Parisian option at current time $t$, and the excursion time $J$. By using financial arguments similar to those used in [83], $D_1$ and $D_2$ can even be simplified more. More specifically, the subdomain

$$D_3 = \{(S,t,J)|\bar{S} \leq S \leq S_f(t,J), t < J, 0 \leq J \leq \bar{J}\}$$

of $D_2$ can be cut off because in this region the elapsed time is always less than the barrier time, a case that can never happen. Moreover, in the domain

$$D_4 = \{(S,t,J)|0 \leq S \leq S_f(t,J), t > T - \bar{J} + J, 0 \leq J \leq \bar{J}\},$$

Figure 6.1: Pricing domain of the case: $\bar{S} < S_f^V(T)$
there is not enough time left for \( J \) to reach \( \bar{J} \) and so the option can never be knocked out. As a result, the Parisian option degenerates to its embedded vanilla option in this region. That leaves the actual pricing domains as:

\[
I = \{(S,t)|0 \leq S \leq \bar{S}, 0 \leq t \leq T - \bar{J}\},
\]

\[
II = \{(S,t,J)|\bar{S} \leq S \leq S_f(t,J), J \leq t \leq T - \bar{J} + J, 0 \leq J \leq \bar{J}\}.
\]

(6.2.1)

Under the Black-Scholes model, the price of an American-style Parisian up-and-out call option depends on the underlying price \( S \), the current time \( t \) and the excursion time \( J \), in addition to other constant parameters such as the asset barrier \( \bar{S} \), the “option window” \( \bar{J} \), the volatility rate \( \sigma \), risk-free interest rate \( r \), the dividend yield rate \( \delta \) and the expiry time \( T \). Let \( V_1(S,t) \) and \( V_2(S,t,J) \) denote the prices of an American Parisian up-and-out call option in the region \( I \) and \( II \), respectively and \( C_A(S,t) \) be the price of the embedded American vanilla call option at time \( t \). It can be shown that \( V_1(S,t) \) should satisfy the classical Black-Scholes equation:

\[
\frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0,
\]

(6.2.2)

where \( \mathbb{L} = \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - \delta)S \frac{\partial}{\partial S} - rI \), with \( I \) being the identity operator. The terminal condition for \( V_1 \) is given by the value of the embedded American vanilla call option, i.e.,

\[
V_1(S,T - \bar{J}) = C_A(S,T - \bar{J}).
\]

(6.2.3)

This is because the Parisian call is identical to its embedded vanilla call in the region \( D_4 \). In addition, the fact that a call option becomes worthless when \( S \) approaches zero gives:

\[
\lim_{S \to 0} V_1(S,t) = 0,
\]

(6.2.4)

whereas the continuity of the option price across the barrier \( \bar{S} \) demands:

\[
\lim_{S \to \bar{S}} V_1(S,t) = \lim_{S \to \bar{S}} V_2(S,t,0).
\]

(6.2.5)

On the other hand, in the region \( II \), the excursion time \( J \) accumulates at the same rate as
the passing time $t$. As a result, $V_2(S, t, J)$ is governed by a modified Black-Scholes equation (cf. 39):

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + LV_2 = 0,$$

(6.2.6)

with the operator $L$ is defined earlier.

By definition, when $J = \bar{J}$, the option is knocked out and becomes worthless. Therefore, just before $J$ reaches $\bar{J}$, the option holder should immediately exercise the option if it is still in-the-money, i.e.,

$$\lim_{J \to \bar{J}} V_2(S, t, J) = \max(S - E, 0).$$

(6.2.7)

Also, similar to the case of American vanilla options, the two “smooth pasting” conditions for determining the optimal exercise boundary surface of an American-style Parisian up-and-out call option are:

$$V_2(S_f(t, J), t, J) = S_f(t, J) - E, \quad \frac{\partial V_2}{\partial S}(S_f(t, J), t, J) = 1.$$

(6.2.8)

Moreover, the boundary condition at $S = \bar{S}$ is specified by the so-called “reset condition”, i.e.,

$$\lim_{S \to \bar{S}} V_2(S, t, J) = \lim_{S \to \bar{S}} V_2(S, t, 0),$$

(6.2.9)

which indicates that $J$ is reset to zero every time the underlying $S$ falls back to the barrier $\bar{S}$ from above.

To ensure no arbitrage opportunity, we should explicitly demand that the Delta of the option be continuous across the barrier $S = \bar{S}$, i.e.,

$$\lim_{S \to \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \to \bar{S}} \frac{\partial V_2}{\partial S}(S, t, 0).$$

(6.2.10)

From now on, the condition (6.2.10) is called the Delta condition.

Equations (6.2.2)-(6.2.10) constitute a pair of coupled PDE systems governing the value of an American-style Parisian up-and-out call option in the domain $I$ and $II$. To summarize,
the PDE systems can be written as follows:

\[ \begin{align*}
A_1 & \quad \begin{cases} 
\frac{\partial V_1}{\partial t} + \mathcal{L}V_1 = 0, \\
V_1(S, T - \bar{J}) = C_A(S, T - \bar{J}), \\
\lim_{S \to 0} V_1(S, t) = 0, \\
\lim_{S \to \bar{S}} V_1(S, t) = \lim_{S \to \bar{S}} V_2(S, t, 0) 
\end{cases} \\
A_2 & \quad \begin{cases} 
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathcal{L}V_2 = 0, \\
\lim_{J \to \bar{J}} V_2(S, t, J) = \max(S - E, 0), \\
V_2(S_f(t, J), t, J) = S_f(t, J) - E, \\
\frac{\partial V_2}{\partial S}(S_f(t, J), t, J) = 1, \\
\lim_{S \to \bar{S}} V_2(S, t, J) = \lim_{S \to \bar{S}} V_2(S, t, 0), 
\end{cases}
\end{align*} \]

\[ (6.2.11) \]

\[ \Delta \text{ condition: } \lim_{S \to \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \to \bar{S}} \frac{\partial V_2}{\partial S}(S, t, 0), \quad (6.2.12) \]

where \( A_1 \) and \( A_2 \) are defined on the region \( I \) and \( II \), respectively.

**Case 3:** \( S_f^A(T) \leq \bar{S} \leq S_f^A(0) \). In this case, the barrier surface intersects with the optimal exercise price curve of the embedded option, i.e., there exists \( 0 \leq t^* \leq T \) such that:

\[ S_f^A(t^*) = \bar{S}. \quad (6.2.13) \]

Because \( S_f^A(t) \) is a decreasing function of \( t \) and \( S_f(t, J) \leq S_f^A(t), \forall t \leq T, \ J < \bar{J} \), from \( (6.2.13) \), it follows:

\[ S_f(t, J) \leq S_f^A(t) \leq S_f^A(t^*) = \bar{S} \quad \forall t \geq t^*, J < \bar{J}. \]

This implies that when \( t \) already reaches \( t^* \), no matter how close \( J \) to \( \bar{J} \), it is always optimal to exercise the Parisian option if \( S \geq \bar{S} \). It can be now shown that the Parisian option will degenerate to its embedded option in a region that satisfies: \( t^* - t < \bar{J} - J \). In fact, in this region, before \( J \) reaches \( \bar{J} \), \( t \) already reaches \( t^* \). Before the knock-out feature is activated, the Parisian option could therefore always be optimally exercised if the asset price still consecutively stays above the barrier. In other words, the knock-out feature has no effect on the optimal exercise policy of the Parisian option in the concerned region. As a result, the Parisian option should be identical with its embedded vanilla option in this region.
The above argument leaves the actual pricing domains as:

\[ III = \{(S,t)|0 \leq S \leq \bar{S}, 0 \leq t \leq t^* - \bar{J}\}, \]

\[ IV = \{(S,t,J)|\bar{S} \leq S \leq S_f(t, J), J \leq t \leq t^* - \bar{J} + J, 0 \leq J \leq \bar{J}\}. \] (6.2.14)

It is clear that these domains are smaller than those in (6.2.1) because \( t^* \leq T \). In addition, the domains in (6.2.14) heavily depend on \( \bar{S} \). More precisely, the greater \( \bar{S} \) is, the smaller \( t^* \) is, and consequently the smaller the domains in (6.2.14) are.

Following arguments similar to those used for the case \( \bar{S} < S_f^V(T) \), we can establish the properly closed PDE systems governing the price of an American-style Parisian up-and-out

Figure 6.2: Pricing domain of the case: \( S_f^A(T) \leq \bar{S} \leq S_f^A(0) \)
call option in the case \( S_f^V(T) \leq \bar{S} \leq S_f^V(0) \) as follows:

\[
\begin{align*}
A_3 \left\{ 
\frac{\partial V_1}{\partial t} + \mathcal{L}V_1 &= 0, \\
V_1(S, t^* - \bar{J}) &= C_A(S, t^* - \bar{J}), \\
\lim_{S \to 0} V_1(S, t) &= 0, \\
\lim_{S \to \bar{S}} V_1(S, t) &= \lim_{S \to \bar{S}} V_2(S, t, 0) 
\right. \\
A_4 \left\{ 
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathcal{L}V_2 &= 0, \\
\lim_{J \to \bar{J}} V_2(S, t, J) &= S - E, \\
V_2(S_f(t, J), t, J) &= S_f(t, J) - E, \\
\frac{\partial V_2}{\partial S}(S_f(t, J), t, J) &= 1, \\
\lim_{S \to \bar{S}} V_2(S, t, J) &= \lim_{S \to \bar{S}} V_2(S, t, 0), \\
\left. \right\}
\end{align*}
\]

Delta condition: \( \lim_{S \to \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \to \bar{S}} \frac{\partial V_2}{\partial S}(S, t, 0) \), (6.2.16)

where \( A_3 \) and \( A_4 \) are defined on the region III and IV, respectively.

**The general PDE systems.**

It should be noted that the two PDE systems (6.2.11, 6.2.12) and (6.2.15, 6.2.16) are derived from two different financial arguments. However, one can easily observed that the two PDE systems are very similar. Let

\[
t_f = \begin{cases} 
T & \text{if } \bar{S} < S_f^V(T), \\
t^* & \text{if } S_f^V(T) \leq \bar{S} \leq S_f^V(0),
\end{cases}
\] (6.2.17)

we can combine the two PDE systems (6.2.11, 6.2.12) and (6.2.15, 6.2.16) to obtain the general PDE systems governing the prices of American-style Parisian up-and-out call options for all non-degenerate cases as follows:

\[
\begin{align*}
B_1 \left\{ 
\frac{\partial V_1}{\partial t} + \mathcal{L}V_1 &= 0, \\
V_1(S, t_f - \bar{J}) &= C_A(S, t_f - \bar{J}), \\
\lim_{S \to 0} V_1(S, t) &= 0, \\
\lim_{S \to \bar{S}} V_1(S, t) &= \lim_{S \to \bar{S}} V_2(S, t, 0) 
\right. \\
B_2 \left\{ 
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathcal{L}V_2 &= 0, \\
\lim_{J \to \bar{J}} V_2(S, t, J) &= \max(S - E, 0), \\
V_2(S_f(t, J), t, J) &= S_f(t, J) - E, \\
\frac{\partial V_2}{\partial S}(S_f(t, J), t, J) &= 1, \\
\lim_{S \to \bar{S}} V_2(S, t, J) &= \lim_{S \to \bar{S}} V_2(S, t, 0), \\
\left. \right\}
\end{align*}
\] (6.2.18)
Delta condition: \[
\lim_{S \to \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \to \bar{S}} \frac{\partial V_2}{\partial S}(S, t, 0),
\]
(6.2.19)
where \(B_1\) is defined on \(S \in [0, \bar{S}]\), \(t \in [0, t_f - \bar{J}]\), \(B_2\) is defined on \(S \in [\bar{S}, S_f(t, J)]\), \(J \in [0, \bar{J}]\), \(t \in [J, t_f - \bar{J} + J]\).

It should be noted that the initial conditions in \(B_1\) and \(B_2\) are derived from the corresponding initial conditions in the two cases: \(\bar{S} < S_f^V(T)\) and \(S_f^V(T) \leq \bar{S} \leq S_f^V(0)\) because

\[
V_1(S, t_f - \bar{J}) = \begin{cases} 
C_A(S, T - \bar{J}) & \text{if } \bar{S} < S_f^V(T), \\
C_A(S, t^* - \bar{J}) & \text{if } S_f^V(T) \leq \bar{S} \leq S_f^V(0), 
\end{cases}
\]

and

\[
V_2(S, t, J) = \begin{cases} 
\max(S - E, 0) & \text{if } \bar{S} < S_f^V(T), \\
S - E & \text{if } S_f^V(T) \leq \bar{S} \leq S_f^V(0). 
\end{cases}
\]

The boundary conditions in \(B_1\) and \(B_2\) are the same with those in the two cases \(\bar{S} < S_f^V(T)\) and \(S_f^V(T) \leq \bar{S} \leq S_f^V(0)\).

Before leaving this section, we should remark that \((6.2.18, 6.2.19)\) is a set of two coupled PDE systems, with the solution of one system being used as the boundary condition for the other. In our approach, we first need to compute the optimal exercise price of the embedded vanilla option in order to determine the initial conditions for the system \((6.2.18, 6.2.19)\). This step actually does not cause much difficulty because the pricing of vanilla options is well studied in the literature. In particular, one can use the highly efficient integral equation method \([20, 54]\) to obtain the optimal exercise price of a vanilla option. On the other hand, the primary source of difficulty for solving \((6.2.18, 6.2.19)\) lies on the presence of the optimal exercise boundary surface, \(S_f(t, J)\), which is a 3-D surface, instead of a 2-D curve as is the case for a “one-touch” barrier option. However, we have managed to develop an approach, which is discussed in the next section, to efficiently resolve this difficulty.

6.3 Our solution procedure

The solution of \((6.2.18, 6.2.19)\) would give rise to \(V_1(S, t), V_2(S, t, J)\) and \(S_f(t, J)\) of the Parisian option. However, the process of finding the solution is not simple. First the 3-
D problem in the coupled systems is reduced to a 2-D one by using the “moving window” technique in [83]. The coupled systems are then transformed into two coupled dimensionless heat problems: one is in a semi-infinite domain and the other is in a finite time-dependent domain. These systems are first solved separately, as if they were not coupled. The solution of the 2-D heat problem in a semi-infinite domain can be easily found in [24, 32, 42, 53]. The 2-D heat problem in the time-dependent domain is solved using the continuous Fourier sine transform. The solutions together with the continuity condition at the interface of the two domains form a pair of coupled integral equations. Once these integral equations are solved by using the Newton-Raphson iteration procedure, we can easily obtain the option price.

6.3.1 The dimensionless heat systems

Following the method of Zhu and Chen [83], the 3-D system in (6.2.18-6.2.19) can be reduced to a 2-D system by replacing the sum of the partial derivatives of $V$, $\partial^2 V/\partial t + \partial^2 V/\partial J$, by its directional derivative $\sqrt{2}\partial V/\partial v$, in the direction of $(\sqrt{2}, \sqrt{2})$. After a further change of variable by $u = \frac{v}{\sqrt{2}}$, the PDE systems in (6.2.18-6.2.19) is transformed to the following 2-D PDE systems:

$$
\begin{align*}
B_3 & \left\{ \begin{array}{l}
\frac{\partial V_1}{\partial t} + LV_1 = 0, \\
V_1(S, T_f) = C_A(S, T_f), \\
V_1(0, t) = 0, \\
V_1(\bar{S}, t) = W(t),
\end{array} \right. \\
B_4 & \left\{ \begin{array}{l}
\frac{\partial V_2}{\partial u} + LV_2 = 0, \\
\lim_{u \to J} V_2(S, u; t) = \max(S - E, 0), \\
V_2(S_f(u; t), u; t) = S_f(u; t) - E, \\
\frac{\partial V_2}{\partial S}(S_f(u; t), u; t) = 1, \\
\lim_{S \to \bar{S}} V_2(S, u; t) = W(t + u),
\end{array} \right.
\end{align*}
$$

Delta condition: $\frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, 0; t)$, (6.3.20)

where $W(t) = V_2(\bar{S}, 0; t)$, $T_f = t_f - J$, $B_3$ is defined on $\{(S, t)|0 \leq S \leq \bar{S}, 0 \leq t \leq T_f\}$ and $B_4$ is defined on $\{(S, J)|0 \leq S \leq S_f(t, J), 0 \leq t \leq T_f, 0 \leq u \leq J\}$. Here $C_A(S, t)$ denotes the price of the embedded American vanilla call at the underlying $S$ and time $t$.

It should be noted that the notation $t$ in the system $B_4$ serves as a parameter, instead of a variable as in the system $B_3$. In addition, the solution $V_2(S, u; t)$ in the new coordinate
system is equal to the solution $V_2(S, t + u, u)$ in the original pricing domain. For more details, interested readers are referred to [83].

In order to make the newly-derived 2-D PDE system even simpler, we shall now non-dimensionalize the PDE systems by introducing dimensionless variables:

$$x = \ln \frac{S}{\bar{S}}, \quad \tau = \frac{\sigma^2}{2}(T_f - t), \quad l = (\bar{J} - u)\frac{\sigma^2}{2};$$

constants

$$L = \frac{\sigma^2}{2} \bar{J}, \quad \gamma = \frac{2r}{\sigma^2}, \quad q = \frac{2\delta}{\sigma^2}, \quad k = \gamma - q - 1, \quad \alpha = -\frac{k}{2}, \quad \beta = -\alpha^2 - \gamma;$$

and unknown functions $C_1(x, \tau)$, $C_2(x, l; \tau)$, $U(\tau)$ and $x_f(l; \tau)$ defined by

$$C_1(x, \tau) = \bar{S}^{-1} e^{-\alpha x - \beta \tau} V_1(S, t), \quad U(\tau) = \bar{S}^{-1} W(t),$$

$$C_2(x, l; \tau) = \bar{S}^{-1} e^{-\alpha x - \beta l} V_2(S, u; t), \quad x_f(l; \tau) = \ln \frac{S_f(u; t)}{S}.$$
6.3. OUR SOLUTION PROCEDURE

The solution procedure in Section 6.3.1 consists of finding the integral representations of the option prices. The solution procedure is as follows:

\[ g_3(x_f(l; \tau), l; \tau) = (1 - \alpha)e^{(1-\alpha)x_f(l; \tau)} - \beta l + \alpha E \bar{S} e^{-\alpha x_f(l; \tau)} - \beta l. \]  

(6.3.23)

It can be seen clearly that once \( U(\tau) \) is found, the PDE systems in (6.3.21–6.3.22) will be uncoupled so each of the system could be solved separately. However, it is not straightforward to find \( U(\tau) \). In this work, we will solve the systems in an “inverse” way. First, we solve the two PDE systems as if they were not coupled to find the integral representations of \( C_1 \) and \( C_2 \) in terms of unknown functions \( U(\tau) \) and \( x_f(l; \tau) \). Then we use one boundary condition at the free boundary and the Delta condition at the barrier to establish the equations to find \( U(\tau) \) and \( x_f(l, \tau) \).

6.3.2 Integral representations of the option prices

The system I is clearly a classical heat problem in a semi-finite domain and has been well-studied in a number of text books [24, 32, 42, 53]. Therefore, the integral representation of \( V_1(S, t) \) in terms of the unknown function \( W(t) \) can easily be found as (cf. Appendix A.1):

\[ V_1(S, t) = \int_0^S \frac{C_A(u, T_f)}{u \sigma \sqrt{2\pi(T_f - t)}} \left( \frac{S}{u} \right)^{\alpha} e^{\frac{\beta S^2}{2}(T_f - t)} \left( e^{-(\ln S - \ln u)^2} \frac{1}{2\sigma \sqrt{2\pi(T_f - t)}} - e^{-(\ln S + \ln u - 2\ln \bar{S})^2} \frac{1}{2\sigma \sqrt{2\pi(T_f - t)}} \right) du + \int_T^T \frac{W(u)}{(u - t)^{\beta}} e^{\frac{\beta S^2}{2}(u - t) - \frac{(\ln S - \ln u)^2}{2\sigma^2(u - t)^2}} dz. \]

(6.3.24)

On the other hand, the system II, which is a heat problem in a finite time-dependent domain, is much more complicated to solve. Fortunately, this system resembles the system (3.3.4), which has already been solved in Chapter 3 using the Fourier sine transform. As a result, we can easily deduce the integral representation of the solution of the system II as follows:

\[ \bar{S} e^{\alpha x + \beta l} H(x_f(l; \tau) - x) C_2(x, l; \tau) = -(\bar{S} e^x - E)1_{x=x_f(l; \tau)}(x) + M(x, l) + \int_0^l Q(x, l, \xi, x_f(\xi; \tau); \tau) d\xi. \]

(6.3.25)
where

\[
1_{x=x_f(l;\tau)}(x) = \begin{cases} 
\frac{1}{2} & \text{if } x = x_f(l;\tau), \\
0 & \text{if } x \neq x_f(l;\tau), 
\end{cases}
\]

and

\[
\mathcal{M}(x, l) = \tilde{S}e^{x-q(l)}N \left( \frac{x - \max(0, \ln \frac{x}{\tilde{S}}) + 2(1-\alpha)l}{\sqrt{2l}} \right) 
- Ee^{-\gamma l}N \left( \frac{x - \max(0, \ln \frac{x}{\tilde{S}}) - 2\alpha l}{\sqrt{2l}} \right) 
- \tilde{S}e^{-q(l-\xi)}N \left( \frac{-x - \max(0, \ln \frac{x}{\tilde{S}}) + 2(1-\alpha)l}{\sqrt{2l}} \right) 
- Ee^{2\alpha x-\gamma l}N \left( \frac{-x - \max(0, \ln \frac{x}{\tilde{S}}) - 2\alpha l}{\sqrt{2l}} \right)
\]  \hspace{1cm} (6.3.26)

and

\[
Q(x, l, \xi, x_f(\xi;\tau); \tau) = q\tilde{S}e^{x-q(l-\xi)}N \left( \frac{x - x_f(\xi;\tau) + 2(1-\alpha)(l-\xi)}{\sqrt{2(l-\xi)}} \right) 
- E\gamma e^{-\gamma(l-\xi)}N \left( \frac{x - x_f(\xi;\tau) - 2\alpha(l-\xi)}{\sqrt{2(l-\xi)}} \right) 
- e^{2\alpha x} \left[ q\tilde{S}e^{-q(l-\xi)-x}N \left( \frac{-x - x_f(\xi;\tau) + 2(1-\alpha)(l-\xi)}{\sqrt{2(l-\xi)}} \right) 
- E\gamma e^{-\gamma(l-\xi)}N \left( \frac{-x - x_f(\xi;\tau) - 2\alpha(l-\xi)}{\sqrt{2(l-\xi)}} \right) \right] 
+ \frac{x\tilde{S}U(\tau+\xi-L)}{2\sqrt{\pi(l-\xi)^3}} e^{\beta(l-\xi)+\alpha x - \frac{x^2}{4(l-\xi)}}.
\]  \hspace{1cm} (6.3.27)

Transfer (6.3.25), (6.3.26), (6.3.27) back to the original variable \(S\) and \(t\), we obtain the following relation between \(V_2(S, u; t)\) and \(S_f(u; t)\):

\[
H(S_f(u; t) - S)V_2(S, u; t) = -(S - E)1_{S=S_f(u; t)}(S) + M \left( S, \bar{J} - u, \max(E, S) \right) 
+ \int_u^\bar{J} Q(S, u, v, S_f(v; t); t) dv,
\]  \hspace{1cm} (6.3.28)
where $1_{S=S_f(u,t)}$, $M$ are defined as in (2.3.36) and (2.5.51), respectively. Here

$$Q(x, y, z, w; t) = Q_1(x, y, z, w) - \left(\frac{x}{S}\right)^{2\alpha} Q_1\left(\frac{S^2}{x}, y, z, w\right) + \left(\frac{x}{S}\right)^\alpha K(x, y, z; t),$$

with

$$K(x, y, z; t) = \frac{(\ln x - \ln \bar{S})W(z + t)}{\sigma\sqrt{2\pi}(z - y)^3} e^{-\frac{(\ln x - \ln \bar{S})^2 + \beta^2}{2}(z - y)}.$$

and $Q_1$ is defined as in (2.3.37).

### 6.3.3 Coupled integral equations

In order to solve for $S_f(u; t)$ and $W(t)$, we need to derive a pair of coupled integral equations that govern their values. The first one can be easily derived by substituting $S = S_f(u; t)$ into (6.3.28). As a result, we obtain an integral governing the values of $S_f(u; t)$ and $W(t)$ as follows:

$$S_f(u; t) - E = M \left(S_f(u; t), \bar{J} - u, \max(E, \bar{S})\right) + \int_u^J Q(S_f(u; t), u, v, S_f(v; t); t) dv. \quad (6.3.31)$$

The second integral equation governing the values of $S_f(u; t)$ and $W(t)$ can be derived from the Delta condition (6.3.29). To this end, we first need to calculate the Delta ($\Delta$) of the option when $S$ is below or above $\bar{S}$, i.e., $\frac{\partial V_1}{\partial S}(S, t)$ and $\frac{\partial V_2}{\partial S}(S, u; t)$, respectively.

**Proposition 4.** The Delta of the option when $S$ is below $\bar{S}$ can be calculated as:

$$\frac{\partial V_1}{\partial S}(S, t) = \int_0^S C_A(u, T_f) S^{\alpha-1} e^{\frac{\beta S^2}{2}(T_f - t)} \left[e^{-\frac{(\ln \bar{S} - \ln u)^2}{2\sigma^2(T_f - t)}} \left(\frac{\ln S - \ln u}{\sigma^2(T_f - t)} - \frac{\ln S + \ln u - 2\ln \bar{S}}{\sigma^2(T_f - t)}\right)\right] du$$

$$+ \sqrt{\frac{2}{\pi}} \frac{\alpha S^{\alpha-1}}{\sqrt{S}} \int_{\ln \frac{\ln S - \ln \bar{S}}{\sqrt{T_f - t}}}^{+\infty} W\left(t + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2v^2}\right) e^{\frac{\beta}{2}(\ln \bar{S} - \ln S)^2 - \frac{v^2}{2}} dv$$

$$- \frac{\sigma \sqrt{2}}{\sqrt{\pi}} \frac{S^\alpha}{\sqrt{S}} \int_0^{\sqrt{T_f - t}} e^{-\frac{(\ln \bar{S} - \ln S)^2}{2\sigma^2u^2} + \frac{\beta^2}{2}u^2} \left[\beta W(t + u^2) + \frac{2}{\sigma^2} W'(t + u^2)\right] du$$

$$+ \left(\frac{S}{\bar{S}}\right)^\alpha \frac{\sqrt{2W(T_f)}}{\bar{S} \sigma \sqrt{\pi(T_f - t)}} e^{\frac{\beta^2}{2}(T_f - t) - \frac{(\ln \bar{S} - \ln S)^2}{2\sigma^2(T_f - t)}}. \quad (6.3.32)$$
6.3. OUR SOLUTION PROCEDURE

Proof. From the formula (6.3.24), we obtain:

$$\frac{\partial V_1}{\partial S}(S,t) = \frac{\partial I_1}{\partial S}(S,t) + \frac{\partial I_2}{\partial S}(S,t),$$

where

$$I_1(S,t) = \int_0^S \frac{C_A(u,T_f)}{u\sigma \sqrt{2\pi (T_f - t)}} \left( \frac{S}{u} \right)^\alpha e^{\frac{\beta^2}{2}(T_f - t)} \left[ e^{-\frac{(\ln S - \ln u)^2}{2\sigma^2(T_f - t)}} - e^{-\frac{(\ln S - \ln u)^2}{2\sigma^2(T_f - t)}} \right] du,$$

$$I_2(S,t) = \left( \frac{S}{\bar{S}} \right)^\alpha \int_t^{T_f} \frac{W(T_f - u + t)}{\sigma \sqrt{2\pi}} \left( \frac{\ln \bar{S} - \ln S}{\sqrt{T_f - u}} \right) e^{\frac{\beta^2}{2}(T_f - u)} \frac{e^{-\frac{(\ln S - \ln u - 2\ln \bar{S})^2}{2\sigma^2(T_f - u)^2}}}{\sigma^2(T_f - u)} dz.$$

By using the Leibniz Integral Rule, $\frac{\partial I_1}{\partial S}(S,t)$ can be obtained straightforwardly as follows:

$$\frac{\partial I_1}{\partial S}(S,t) = \int_0^S \frac{C_A(u,T_f)}{u^{\alpha+1}\sigma \sqrt{2\pi (T_f - t)}} \left( \frac{S}{u} \right)^{\alpha-1} e^{\frac{\beta^2}{2}(T_f - t)} \left[ e^{-\frac{(\ln S - \ln u)^2}{2\sigma^2(T_f - t)}} \left( \alpha - \ln S - \ln u \right) - e^{-\frac{(\ln S + \ln u - 2\ln \bar{S})^2}{2\sigma^2(T_f - u)^2}} \left( \alpha - \ln S + \ln u - 2\ln \bar{S} \right) \right] du.$$

The calculation of $\frac{\partial I_2}{\partial S}(S,t)$, on the other hand, is much more complicated because of removable singularities of $I_2(S,t)$ at $S = \bar{S}$. We first use the following variable transform to remove these singularities:

$$v = \frac{\ln \bar{S} - \ln S}{\sigma \sqrt{T_f - u}}.$$

As a result, $I_2(S,t)$ can be expressed as

$$I_2(S,t) = \int_{\ln \bar{S} - \ln S}^{+\infty} \left( \frac{S}{\bar{S}} \right)^{\alpha} \sqrt{\frac{v}{2\pi}} W \left( t + \left( \frac{\ln \bar{S} - \ln S}{\sigma v} \right)^2 \right) e^{-\frac{v^2}{2\sigma^2}} \frac{e^{-\frac{(\ln S - \ln S)^2}{2\sigma^2(T_f - u)^2}}}{\sigma^2(T_f - u)^2} dv.$$
By using the Leibniz Integral Rule again, \( \frac{\partial I_2}{\partial S}(S,t) \) can be obtained as follows:

\[
\frac{\partial I_2}{\partial S}(S,t) = \frac{\sqrt{2} \alpha S^{\alpha-1}}{\sqrt{\pi} \sigma^\alpha} \int_{\ln \bar{S} - \ln S}^{+\infty} \frac{W\left(t + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2}\right)e^{\frac{(\ln \bar{S} - \ln S)^2}{2\sigma^2}} - e^{2\bar{S}W(t)}}{\sigma^2 v^2} dv
\]

\[
-\frac{\sqrt{2}}{\sqrt{\pi} \sigma^\alpha} \int_{\ln \bar{S} - \ln S}^{+\infty} \frac{\ln \bar{S} - \ln S}{\sigma^2 v^2} e^{\frac{(\ln \bar{S} - \ln S)^2}{2\sigma^2}} dv
\]

\[
\left[ \beta W\left(t + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2}\right) + \frac{2}{\sigma^2} W'\left(t + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2}\right) \right] dv
\]

\[
+\left( \frac{S}{\bar{S}} \right)^\alpha \frac{\sqrt{2}W(T_f)}{S\sigma \sqrt{\pi(T_f - t)}} e^{\frac{\sigma^2}{2}(T_f - t)} \frac{(\ln \bar{S} - \ln S)^2}{2\sigma^2(T_f - t)}.
\]

To simplify the second integral in the above formula, we let \( u = \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2} \). As a result, we can express the second integral in the above formula as

\[
-\frac{\sigma \sqrt{2}}{S\sqrt{\pi} \bar{S}^{\alpha}} \int_0^{\sqrt{T_f - t}} e^{-\frac{(\ln \bar{S} - \ln S)^2}{2\sigma^2 v^2} + \frac{\beta^2 \sigma^2}{4} u} \left[ \beta W(t + u^2) + \frac{2}{\sigma^2} W'(t + u^2) \right] d\xi.
\]

Now we can obtain the formula for \( \frac{\partial V_1}{\partial S}(S,t) \) as stated in Proposition 4. This completes our proof for Proposition 4.

By substituting \( S = \bar{S} \) into the formula (6.3.32), we can obtain the value of the Delta of the option when \( S \) approaches \( \bar{S} \) from below as follows:

\[
\lim_{S \to \bar{S}} \frac{\partial V_1}{\partial S}(S,t) = \int_0^\bar{S} \frac{\sqrt{2}CA(u, T_f) \bar{S}^{\alpha-1}(\ln u - \ln \bar{S})}{u^{\alpha+1} - \ln\bar{S}} e^{\frac{\sigma^2}{2}(T_f - t)} \frac{\beta^2 \sigma^2}{2\sigma^2(T_f - t)} du + \frac{\alpha}{S} W(t)
\]

\[
+\frac{\sqrt{2}W(T_f)}{S\sigma \sqrt{\pi(T_f - t)}} e^{\frac{\sigma^2}{2}(T_f - t)} \int_0^{\sqrt{T_f - t}} e^{\frac{\beta^2 \sigma^2}{4} u} \left[ \beta W(t + u^2) + \frac{2}{\sigma^2} W'(t + u^2) \right] du.
\]

(6.3.33)

**Proposition 5.** The Delta of the option when \( S \) is above \( \bar{S} \) can be calculated as follows:

\[
\frac{\partial}{\partial S} V_2(S, u; t) = \tilde{K}_1(S, u; t) + M(S, \bar{J} - u, \max(E, \bar{S})) + \int_u^{\bar{J}} L(S, u, v, S_f(v; t)) dv, \quad \forall S < S_f(u; t),
\]

(6.3.34)
where

\[\tilde{K}_1(x, y; t) = -\frac{\sqrt{2}W(t + \tilde{J})}{x\sqrt{\pi}(\tilde{J} - y)} \left(\frac{x}{\tilde{S}}\right)^{\alpha} e^{\frac{\tilde{S}^2}{\tilde{S}}(\tilde{J} - y) - \frac{\ln x - \ln \tilde{S}}{\alpha^2(\tilde{J} - y)}} + \frac{\sqrt{2}\alpha x^{\alpha-1}}{\sqrt{\pi}S^\alpha} \int_0^{+\infty} W\left(t + y + \left(\frac{\ln x - \ln \tilde{S}}{\sigma + \sqrt{\tilde{J} - y}}\right)^2\right) e^{\frac{\beta(\ln x - \ln \tilde{S})^2}{2(\sigma + \sqrt{\tilde{J} - y})^2} - \frac{1}{2}(\eta + \sqrt{\tilde{J} - y})^2} d\eta,\]

+ \left(\frac{x}{\tilde{S}}\right)^{\alpha} \frac{\sqrt{2}\sigma}{\sqrt{\pi}x} \int_0^{\sqrt{\tilde{J} - y}} e^{-\frac{(\ln x - \ln \tilde{S})^2}{2\sigma^2x^2} + \frac{2\sigma^2v^2}{S}} \beta W(t + y + v^2) + \frac{2}{\sigma^2} W'(t + y + v^2) dv.

Here \(\tilde{M}\) and \(L\) are defined as in Proposition 3 in Section 3.3.4.

Proof. The calculation of \(\frac{\partial}{\partial S} V_2(S, u; t)\) is very similar to that of the Delta of the American down-and-out call options. Interested readers are referred to the proof of Proposition 3 in Section 3.3.4.

Based on the Proposition 3 we now can calculate the Delta of the option when \(S\) approaches \(\tilde{S}\) from above, i.e., \(\lim_{S \to \tilde{S}} \frac{\partial V_2}{\partial S}(S, 0; t)\), by letting \(S = \tilde{S}, u = 0\) in the formula (6.3.34).

More specifically,

\[\lim_{S \to \tilde{S}} \frac{\partial V_2}{\partial S}(S, 0; t) = \tilde{K}_1(\tilde{S}, 0; t) + \tilde{M}(\tilde{S}, \tilde{J}, \max(E, \tilde{S})) + \int_0^{\tilde{J}} L(\tilde{S}, 0, v, S_f(v; t)) dv, \quad (6.3.35)\]

where \(\tilde{K}_1, \tilde{M}, L\) are defined as in the Proposition 3. Consequently, from (6.3.33) and (6.3.35), using the Delta condition (6.3.29), we can derive the second integral equation that governs the values of \(S_f(u; t)\) and \(W(t)\) as follows:

\[
\int_0^{\tilde{S}} \sqrt{2C_A(u, T_f)} \tilde{S}^{\alpha-1}(\ln u - \ln \tilde{S}) e^{\frac{\tilde{S}^2}{\tilde{S}}(T_f - t)} - \frac{\ln \tilde{S} - \ln u}{\alpha^2(T_f - t)} du + \alpha \frac{\sqrt{2}}{S} W(t) + \frac{\sqrt{2}}{S\sqrt{\pi}(T_f - t)} e^{\frac{\beta^2}{\sigma^2}(T_f - t)} - \frac{\sqrt{2}}{S\sqrt{\pi}} \int_0^{\sqrt{T_f - t}} e^{\frac{\beta^2}{\sigma^2}u} \left[\beta W(t + u^2) + \frac{2}{\sigma^2} W'(t + u^2)\right] du = \tilde{K}_1(\tilde{S}, 0; t) + \tilde{M}(\tilde{S}, \tilde{J}, \max(E, \tilde{S})) + \int_0^{\tilde{J}} L(\tilde{S}, 0, v, S_f(v; t)) dv. \quad (6.3.36)
\]

It should be noted that once \(S_f(u; t)\) and \(W(t)\) are determined by numerically solving the coupled integral equations (6.3.31)-(6.3.36), we can easily obtain the value of an American Parisian up-and-out call when \(S\) is below or above \(\tilde{S}\) by using formulas: (6.3.24) and (6.3.28), respectively. Similarly, the hedging parameters Delta, Gamma, Theta, Vega and Rho, known
as the Greeks, can also then be obtained by differentiating the formulas (6.3.24) and (6.3.28) with respect to the relevant parameters. For instance, the value of \( \Delta \) associated with \( S \) is below or above \( \bar{S} \) can be easily calculated by using formulas: (6.3.32) and (6.3.34), respectively. It is therefore very important to build an efficient numerical scheme in order to solve the coupled integral equations (6.3.31)-(6.3.36). This is the focus of the next section.

6.4 Numerical implementation

We now describe a numerical procedure to compute \( W(t) \) and and \( S_f(u; t) \). Our first important step is to determine the actual pricing domains of an American-style Parisian up-and-out call option (cf. Section 6.2). To this end, one need to solve for the optimal exercise boundary of its embedded vanilla option. This can be easily done with high accuracy and efficiency using the integral equation method [20, 54]. As a result, one can easily determine \( t_f \) defined in Section 6.2 as follows:

\[
t_f = \begin{cases} 
T & \text{if } \bar{S} < S_f^V(T), \\
\ast & \text{if } S_f^V(T) \leq \bar{S} \leq S_f^V(0), 
\end{cases}
\]

where \( \ast \) is the unique solution of the algebraic equation: \( S_f^V(t) = \bar{S} \). Here \( S_f^V(t) \) denotes the optimal exercise boundary of the embedded vanilla option at time \( t \). We now can express the actual pricing domains as follows:

\[
III = \{(S, t)|0 \leq S \leq \bar{S}, 0 \leq t \leq T_f\}, \\
IV = \{(S, t, J)|\bar{S} \leq S \leq S_f(t, J), 0 \leq t \leq T_f, 0 \leq u \leq \bar{J}\},
\]

(6.4.37)

where \( T_f = t_f - \bar{J} \).

Without losing generality, we can assume that \( T_f = \frac{p}{q}\bar{J} \), where \( p, q \) are positive integer numbers. With this assumption, we can choose the same uniform step size for both \( u \)-axis and \( t \)-axis. More specifically, we can choose uniform partitions for the intervals \([0, \bar{J}]\) and \([0, T_f]\) such that:

\[
\Pi_1 : \bar{J} = u_1 > u_2 > \ldots > u_n > u_{n+1} = 0, u_i = \bar{J} - (i - 1)h, \forall i, h = \frac{\bar{J}}{n}, n = qk.
\]
\[ \Pi_2: T_f = t_1 > t_2 > \ldots > t_m > t_{m+1} = 0, t_j = T_f - (j - 1)h, \forall j, h = \frac{J}{n}, m = pk. \]

Here \( k \) is some integer number.

At each \( j \in \{1, 2, \ldots, m+1\} \), we aim to find \( W(t_j) \) and \( S_f(u_i; t_j), \forall i \in \{1, 2, \ldots, n+1\} \).

We first start at \( j = 1 \). Here it should be noticed that \( W(t_1) = C_A(S, T_f) \) (cf. Section 6.2).

In addition, \( S_f(u_1; t_1) \) can be determined from the following corollary:

**Corollary 6.4.1.** Taking the limit as \( u \) tends to \( J \) in the integral equation (6.3.31), the optimal exercise price of an American Parisian up-and-out call option at any time \( t \) and just before \( u \) reaches \( J \) is given by

\[
\lim_{u \to J} S_f(u; t) = \max(E, \bar{S}, \frac{r}{\delta} E).
\]

**Proof.** See Appendix [A.2]

As a result, \( S_f(u_1; t_1) = \max(E, \bar{S}, \frac{r}{\delta} E) \). We can also obtain values of \( S_f(u_i; t_1), \) for \( i > 1 \) by using a recursive scheme. More specifically, in order to obtain the value of \( S_f(u_2; t_1), \) we solve numerically the following integral equation:

\[
S_f(u_2; t_1) - E = M(S_f(u_2; t_1), \bar{J} - u_2, \max(E, \bar{S})) + \int_{u_2}^{\bar{J}} Q(S_f(u_2; t_1), u_2, v, S_f(v; t); t_1)dv.
\]

(6.4.38)

The equation (6.4.38) can be converted to an algebraic equation by computing the integral terms contained in the equation using suitable quadrature rules. Most of the integrals are of smooth functions on finite domains and can be evaluated using Trapezoidal rule. The only integrand that needs special attention is the last term of \( Q(S_f(u_2; t_1), u_2, v, S_f(v; t); t_1) \), which is:

\[
\left( \frac{S_f(u_2; t_1)}{S} \right)^\alpha K(S_f(u_2; t_1), u_2, v; t_1).
\]

This integrand has a singularity at \( v = u_2 \). One way to approximate this integral is first to transform it into an integral on a semi-infinite domain by using the following variable
transformation:
\[ w = \frac{\ln S_f(u_2; t_1) - \ln \bar{S}}{\sigma \sqrt{2(v - u_2)}} - \frac{\ln S_f(u_2; t_1) - \ln \bar{S}}{\sigma \sqrt{2(\bar{J} - u_2)}}. \]

The Gauss-Laguerre rule is then applied, which is an efficient way to evaluate integrals on semi-infinite domains.

The obtained algebraic equation can then be efficiently solved using the Newton-Raphson procedure in a similar way as in \[52\]. As a result, we can obtain the value of \( S_f(u_2; t_1) \). The values of \( S_f(u_i; t_1), i > 2 \) can then be solved recursively.

Now we find the values of \( W(t_2) \) and \( S_f(u_i; t_2), 1 \leq i \leq n + 1 \). As \( W(t) = C_A(S, t) \), \( \forall t \geq T_f \) (cf. Section \[6.2\], we can easily obtain \( S_f(u_i; t_2), 1 \leq i \leq n \) by proceeding a similar procedure for \( S_f(u_i; t_1), 1 \leq i \leq n \). However, it is more complicated to solve for \( S_f(u_{n+1}; t_2) \) because its governing integral equation

\[
S_f(u_{n+1}; t_2) - E = M \left( S_f(u_{n+1}; t_2), \bar{J} - u_{n+1}, \max(E, \bar{S}) \right)
+ \int_{u_{n+1}}^{\bar{J}} Q(S_f(u_{n+1}; t_2), u_{n+1}, v, S_f(v; t); t_2) dv,
\]

involves the unknown value \( W(t_2) \), which is in turn governed by the equation \[6.3.36\]. We use the following iterative procedure to solve for \( S_f(u_{n+1}; t_2) \) and \( W(t_2) \). We first use \( W(t_1) \) as the first approximation of \( W(t_2) \), i.e., \( W(1)(t_2) = W(t_1) \), for solving \[6.4.39\] to obtain the first approximation of \( S_f(u_{n+1}; t_2) \), denoted by \( S_f^{(1)}(u_{n+1}; t_2) \). Next, the values of \( S_f(u_i; t_2), 1 \leq i \leq n \) and \( S_f^{(1)}(u_{n+1}; t_2) \) are used as inputs for solving \[6.3.36\], using Trapezoidal rule and the Newton-Raphson iteration procedure. As a result, we can obtain the second approximation of \( W(t_2) \), denoted by \( W(2)(t_2) \). The value \( W(2)(t_2) \) is then used to obtain the second approximation \( S_f(u_{n+1}; t_2) \), denoted by \( S_f^{(2)}(u_{n+1}; t_2) \). Repeating this iterative procedure until the \( k \)-th step, when

\[
\left| \frac{S_f^{(k)}(u_{n+1}; t_2) - S_f^{(k-1)}(u_{n+1}; t_2)}{S_f^{(k)}(u_{n+1}; t_2)} \right| < \text{tol}, \quad \left| \frac{W^{(k)}(t_2) - W^{(k-1)}(t_2)}{W^{(k)}(t_2)} \right| < \text{tol},
\]

where \( \text{tol} \) is a pre-set value of tolerance. In our calculation, we use \( \text{tol} = 10e^{-5} \).

Proceeding in a similar manner as above, the values of \( W(t_j) \) and \( S_f(u_i; t_j) \) at each pair \((t_j, u_i)\) can be obtained. The values of the option, namely \( V_1(S, t) \) and \( V_2(S, u; t) \), can then
be calculated by numerical integration of (6.3.24) and (6.3.28). In addition, the Delta values can be calculated from (6.3.32) and (6.3.34).

6.5 Numerical examples and discussions

In this section, we will validate our method and provide numerical examples to illustrate some interesting properties of the price and the optimal exercise boundary of an American-style Parisian up-and-out call option.

6.5.1 Validation of our IEM

We validate our approach by comparing our numerical results with those calculated using a commonly used numerical scheme, namely, the Crank Nicolson (C-N) scheme.

Table 6.1: Comparison of prices of an American-style Parisian up-and-out call option calculated from the IEM and FDM, with parameters set as \( E = $10 \), \( r = 5\% \), \( \delta = 10\% \), \( S = $9 \), \( \bar{J} = 0.2 \text{ (year)} \), \( T - t = 1 \text{ (year)} \).

<table>
<thead>
<tr>
<th>( S )</th>
<th>( J )</th>
<th>Price (C-N)</th>
<th>CPU(s)</th>
<th>Price (IEM)</th>
<th>CPU(s)</th>
<th>( \epsilon_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>0.07123</td>
<td>104.89</td>
<td>0.07144</td>
<td>74.61</td>
<td>0.29%</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0.17228</td>
<td>111.76</td>
<td>0.17291</td>
<td>75.46</td>
<td>0.36%</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0.33149</td>
<td>112.97</td>
<td>0.33196</td>
<td>81.66</td>
<td>0.14%</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.44137</td>
<td>104.89</td>
<td>0.44812</td>
<td>82.23</td>
<td>1.52%</td>
</tr>
<tr>
<td>11</td>
<td>0.1</td>
<td>1.06151</td>
<td>111.76</td>
<td>1.06332</td>
<td>84.87</td>
<td>0.17%</td>
</tr>
</tbody>
</table>

Shown in Table 6.1 are the prices of the Parisian option associated with the pairs \((S, J)\) chosen among two ranges of values: \( S \in \{7, 8, 9\} \) (\( S \leq \bar{S} \)) with \( J = 0 \) and \( S \in \{10, 11\} \) (\( S > \bar{S} \)) with \( J = 0.1 \). A fine grid of \( \Delta t = \Delta J = 1/2400 \), \( \Delta \ln S = 1/1200 \) is used in the calculation of results by the C-N scheme. The results of the IEM are calculated by using the same uniform discrete steps in both \( t \)-direction and \( J \)-direction, \( \Delta t = \Delta J = 1/200 \). All of our experiments were performed using Matlab R2012b on an Intel Core i7, 3.40 GHZ machine. As can be clearly seen from Table 6.1, the two sets of the results match between 2 to 3 significant digits, with their relative difference being less than 1.52\%. This provides a valuable validation for our method.

We also plot the absolute differences between two sets of option prices calculated by the IEM and C-N scheme in Figures 6.3(a) and 6.3(b). Different grids are used for the results
6.5. NUMERICAL EXAMPLES AND DISCUSSIONS

Figure 6.3: Absolute difference between prices of an American-style Parisian up-and-out call option as a function of \(S\), for various \(J\) values with fixed parameters \(E = \$10\), \(\bar{S} = \$9.5\), \(T - t = 1\) (year), \(\bar{J} = 0.2\) (year), \(\sigma = 30\%\), \(r = 5\%\), \(\delta = 10\%\).

for the C-N scheme, \(\Delta t = \Delta J = 1/2400\), \(\Delta \ln S = 1/1200\) in (a) and \(\Delta t = \Delta J = 1/4800\), \(\Delta \ln S = 1/2400\) in (b), whereas the same grid \(\Delta t = \Delta J = 1/200\) is used for the IEM in both the figures. One can observe that the absolute difference is fairly small for most of the asset prices except near the asset barrier, where the difference is reduced significantly when the grid for the C-N method becomes finer, whereas those used in the IEM are kept the same. In other words, the results of the C-N scheme are getting closer to those of the IEM as the number of grid points of the C-N scheme increases. This indicates that the IEM is not only
more accurate, but also more efficient than the C-N method, as to obtain results at the same level of accuracy near the asset barrier the later method would need much longer time.

6.5.2 The option price

In this section, we show the general trend of the price of an American-style Parisian up-and-out call option.

![Graph showing option prices](image)

Figure 6.4: Prices of an American-style Parisian up-and-out call option (as a function of $S$) produced using the IEM and C-N scheme, with other parameters: $E = $10, $T - t = 1$ (year), $J = 0.2$ (year), $\sigma = 30\%$, $r = 5\%$, $\delta = 10\%$.

As shown in Figures 6.4(a) and 6.4(b), the price of the Parisian option is always smaller...
than that of its embedded vanilla option and it decreases when \( J \) increases. This is indeed reasonable as compared with the vanilla option, the holder of the Parisian option always faces an extra risk that the option contract might be knocked out and this risk becomes even greater when \( J \) gets closer to \( \bar{J} \).

It can be further observed from Figures 6.4(a) and 6.4(b) that the price curve of the Parisian option is not smooth when the underlying price is at \( \bar{S} \), with the exception of the case for \( J = 0 \). This is indeed as a result of the Delta condition (6.2.19), which ensures the continuity of the Delta of the option only at \( J = 0 \), but not at other non-zero \( J \) values. Moreover, in a small neighborhood to the right of \( \bar{S} \), the Delta of the option decreases and even becomes negative as \( J \) increases towards \( \bar{J} \). This can be seen clearly if one compares the price curve associated with \( J = 0 \) and those associated with \( J = 0.1 \) and \( J = 0.15 \), in both the figures. More precisely, when \( S \) just slightly increases over the barrier, the price curve \( J = 0.1 \) experiences a slower increase rate than the price curve \( J = 0 \), whereas the price curve \( J = 0.15 \) in Figure 6.4(a) even decreases. This interesting phenomenon occurs due to the effect of the “knock-out” feature. The greater \( J \) is, the more likely the Parisian option is to be knocked out and thus the less value the Parisian option becomes. However, when \( S \) continues to increase and gets closer to the optimal exercise price, while \( J < \bar{J} \), the Parisian option price will increase as the option is more likely be optimally exercised before the knock-out feature is activated. In other words, the effect of the “knock-out” feature becomes less when \( S \) approaches the optimal exercise price.

In addition, by comparing Figures 6.4(a) and 6.4(b), one can observe that Parisian up-and-out option price increases when the barrier \( \bar{S} \) increases. More specifically, the value of the Parisian option increases towards the value of the plain American option when \( \bar{S} \) increases. This can be explained as that an increase in \( \bar{S} \) would reduce the danger of being “knocked out” and also increase the chance for the option to be optimally exercised.

6.5.3 The optimal exercise price

In this subsection, we point out some interesting properties of the optimal exercise price of an American-style Parisian up-and-out call option.

Figures 6.5(a) and 6.5(b) show the optimal exercise price of an American-style Parisian
Figure 6.5: Optimal exercise prices of an Parisian American up-and-out call option and its embedded option at different J values. Parameters are $E = $10, $T = 1$ (year), $\bar{J} = 0.2$ (year), $\sigma = 30\%$, $r = 5\%$, $\delta = 10\%$.

up-and-out call option as a function of time to expiry, at different J values for $\bar{S} = $9.5 and $\bar{S} = $10.95, respectively. For reference, the optimal exercise price of the corresponding American vanilla option is also plotted. The parameters used in the calculation are: $E = $10, $T = 1$ (year), $\bar{J} = 0.2$ (year), $\sigma = 30\%$, $r = 5\%$, $\delta = 10\%$. It is clear from both the figures that the optimal exercise price of the vanilla option is always greater than or at least equal to that of the Parisian option. This can be explained as follows. When $t_f - t \geq \bar{J} - J$, there is possibility that the Parisian option may be knocked-out. To avoid this risk, the holder of the
Parisian option has to exercise the option at a lower optimal exercise price than that of its embedded vanilla option. On the other hand, when $t_f - t < \bar{J} - J$, the Parisian option can never be knocked out so that its optimal exercise price is identical to that of the associated vanilla option (see Section 6.2). It should be noted here that for case $\bar{S} = $9.5, $t_f = 1$ (year) and for the case $\bar{S} = $10.95, $t_f = 0.98$ (year), where $t_f$ is defined in (6.2.17).

From Figures 6.5(a) and 6.5(b), one can also observe that the optimal exercise price of an American-style Parisian up-and-out call option is a decreasing function with respect to $J$. In other words, the closer $J$ is to $\bar{J}$, the lower the optimal exercise price curve of the Parisian option becomes. This is indeed expected because the risk of being “knocked out” for the option becomes greater when $J$ gets closer to $\bar{J}$. On the other hand, this risk is reduced if $\bar{S}$ increases (other parameters remain unchanged). That is why the optimal exercise price of the Parisian option increases toward that of its embedded option when $\bar{S}$ increases from $9.5$ to $10.95$, as shown in both the figures.

6.6 Conclusion

This chapter applies an integral equation approach to solve the three dimensional pricing problem of American-style Parisian up-and-out call options. The key idea behind our approach is to convert this pricing problem to solving two coupled integral equations. As a result, the option price, the optimal exercise price and the hedging parameters can be straightforwardly computed after solving these coupled integral equations using the Newton-Raphson iteration procedure. We have validated our proposed approach by showing that our obtained results agree well with the reference values obtained from adopting the standard Crank-Nicolson scheme. The main advantage of this approach is that not only the optimal exercise price and the option price can be found, but also the option Greeks can be obtained easily from the integral representation of the option price. Through selected numerical examples, interesting properties of the price and the optimal exercise boundary of American-style Parisian up-and-out call options have also been clearly revealed.
Chapter 7

Pricing American-style Parisian down-and-out call options

7.1 Introduction

Continuing on the topic of pricing Parisian knock-out call options, this chapter discusses the pricing problem of another type of these options, the down-type options. The main difference between the up-type and down-type options lies on the knock-out condition, i.e., the condition activates the knock-out features. More specifically, the knock-out feature of a Parisian down-and-out call option is activated only if the underlying asset price $S$ has continually stayed below the barrier $\bar{S}$ for a prescribed time period $\bar{J}$. The knock-out condition of the down-type option clearly contrasts with that of its up-type option counterpart. It is this difference that has led to two key differences between the pricing problems of the former and the latter.

One key difference is that the pricing domains of a Parisian down-and-out call option are reversed from those of its up-type option counterpart. In particular, when $S \leq \bar{S}$, the excursion time $J$ accumulates at the same rate with the passing time $t$ and has a great effect on the price of the Parisian down-and-out call option. As a result, the pricing domain below the barrier of the down-type option becomes a 3-D one, instead of a 2-D one as for the up-type option. By contrast, while the pricing domain above the barrier of the up-type option is a 3-D one, that of the down-type option is only a 2-D one, because $J$ is reset to zero and remains zero until the asset price stays below $\bar{S}$ again.
The other key difference is that the optimal exercise boundary of an American-style Parisian down-and-out call option is a 2-D curve, but not a 3-D surface as for its up-type option counterpart. This results from two main factors. First, its optimal exercise boundary must stay above the exercise price because the call option should be exercised only when it is in-the-money, i.e., when \( S > \bar{S} \). Second, for a down-type call option, the asset barrier is typically assumed to be less than the exercise price because the option holder usually accepts to lose the option only when the option is out-of-money. As a result, the optimal exercise boundary must stay above the barrier and thus it is a 2-D curve.

Albeit different, the pricing problems of the up-type and down-type options can be solved by the same solution procedure. More precisely, using the “moving window” technique, the 3-D pricing problem of an American-style Parisian down-and-out call option can be reduced to a 2-D one, which can be then converted into solving a pair of coupled integral equations. As a result, once these integral equations have been solved using the Newton-Raphson iterative procedure, the option price, the optimal exercise price and the hedging parameters can be obtained easily.

The chapter is organized as follows. In Section 7.2 we introduce the PDE systems governing the price of an American-style Parisian down-and-out call option. The solution procedure is presented in Section 7.3 while the numerical implementation is discussed in Section 7.4. Section 7.5 presents selected numerical results to demonstrate interesting properties of the option price as well as its optimal exercise boundary. The chapter ends with some concluding remarks given in Section 7.6.

### 7.2 Formulation

One can easily observe that if \( \bar{J} \) takes some extreme values, i.e., \( \bar{J} \) approaches zero or \( \bar{J} \) is greater than the option life, an American-style Parisian down-and-out call option will then degenerate to either a one-touch barrier option or a vanilla option, respectively. For any other non-degenerate cases, the pricing domain of an American-style Parisian down-and-out call option can be divided into two main parts

\[
D_1 = \{(S,t,J)|0 \leq S \leq \bar{S}, 0 \leq t \leq T, 0 \leq J \leq \bar{J}\},
\]
and

\[ D_2 = \{(S, t) | \bar{S} \leq S \leq S_f(t), 0 \leq t \leq T\}, \]

where \( S_f(t) \) denotes the optimal exercise price at time \( t \).

Using financial arguments similar to those used in [83], \( D_1 \) and \( D_2 \) can be even simplified more. In fact, the subdomain \( D_3 = \{(S, t, J) | 0 \leq S \leq \bar{S}, t < J, 0 \leq J \leq \bar{J}\} \) of \( D_1 \) can be cut off because in this region the elapsed time is always less than the barrier time, a case that can never happen. Moreover, in the domain:

\[ D_4 = \{(S, t, J) | 0 \leq S \leq S_f(t), t > T - \bar{J} + J, 0 \leq J \leq \bar{J}\}, \]

there is not enough time left for \( J \) to reach \( \bar{J} \) and so the option can never be knocked out. Therefore, the Parisian option degenerates to its embedded vanilla option in this region. As

Figure 7.1: Pricing domain of American-style Parisian down-and-out call options
a result, the pricing domains of those non-degenerated cases can be elegantly reduced as:

\[
I = \{(S, t, J) | 0 \leq S \leq \bar{S}, J \leq t \leq T^* + J, 0 \leq J \leq \bar{J}\},
\]

\[
II = \{(S, t) | \bar{S} \leq S \leq S_f(t), 0 \leq t \leq T^*\},
\]

where \(T^* = T - \bar{J}\).

Under the Black-Scholes framework, the price of an American-style Parisian down-and-out call option depends on the underlying price \(S\), the current time \(t\) and the excursion time \(J\), in addition to other constant parameters such as the asset barrier \(\bar{S}\), the “option window” \(\bar{J}\), the volatility rate \(\sigma\), risk-free interest rate \(r\) and the expiry time \(T\). Let \(V_1(S, t, J)\) and \(V_2(S, t)\) denote the prices of an American-style Parisian down-and-out call option in the region \(I\) and \(II\), respectively. We now establish the coupled PDE systems governing \(V_1(S, t, J)\) and \(V_2(S, t)\). In the region \(I\), it can be shown that \(V_1(S, t, J)\) is governed by a modified Black-Scholes equation (cf. [39]):

\[
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial J} + LV_1 = 0,
\]

(7.2.1)

where \(L = \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - \delta)S \frac{\partial}{\partial S} - rI\), with \(I\) being the identity operator. Also, by definition, when \(J\) reaches \(\bar{J}\), the option is knocked out and becomes worthless. That means:

\[
V_1(S, t, \bar{J}) = 0.
\]

(7.2.2)

In addition, the fact that a call option becomes worthless when the underlying price approaches zero gives:

\[
V_1(0, t, J) = 0.
\]

(7.2.3)

Moreover, the boundary condition at \(S = \bar{S}\) is specified by the so-called “reset condition”, i.e.,

\[
V_1(\bar{S}, t, J) = V_1(\bar{S}, t, 0),
\]

(7.2.4)

which indicates that \(J\) is reset to zero every time \(S\) approaches \(\bar{S}\) from below.

To ensure no arbitrage opportunity, we should explicitly demand that the option Delta be
continuous across the barrier $S = \bar{S}$, i.e.,

$$\frac{\partial V_1}{\partial S}(\bar{S}, t, 0) = \frac{\partial V_2}{\partial S}(\bar{S}, t). \quad (7.2.5)$$

From now on, the condition (7.2.5) is called the Delta condition.

On the other hand, under the Black-Scholes model, $V_2(S, t)$ should satisfy the classical Black-Scholes equation:

$$\frac{\partial V_2}{\partial t} + \mathcal{L}V_2 = 0, \quad (7.2.6)$$

with the operator $\mathcal{L}$ is defined earlier. The terminal condition for $V_2$ is given by:

$$V_2(S, T^*) = C_A(S, T^*), \quad (7.2.7)$$

where $C_A(S, T^*)$ denotes the price of the embedded American vanilla call option at the asset price $S$ and time $t = T^*$. This is because the Parisian call is identical to its embedded vanilla call in the region $D_1$.

In addition, the continuity of the option price across the barrier $\bar{S}$ demands:

$$V_2(\bar{S}, t) = V_1(\bar{S}, t, 0). \quad (7.2.8)$$

Also, the two necessary conditions for determining the optimal exercise boundary are given by:

$$V_2(S_f(t), t) = S_f(t) - E, \quad \frac{\partial V_2}{\partial S}(S_f(t), t) = 1. \quad (7.2.9)$$

Equations (7.2.1)-(7.2.9) constitute a pair of coupled PDE systems governing the value of an American-style Parisian down-and-out call option at any underlying price $S$, any excursion time $J$, and any time $t$ before the expiration $T$. To summarize, the PDE systems can be
written as follows:

\[
\mathcal{A}_1 \begin{cases}
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial J} + \mathcal{L}V_1 = 0,
\lim_{J \to \bar{J}} V_1(S, t, J) = 0,
V_1(0, t, J) = 0,
V_1(\bar{S}, t, J) = V_1(\bar{S}, t, 0),
\end{cases}
\]

\[
\mathcal{A}_2 \begin{cases}
\frac{\partial V_2}{\partial t} + \mathcal{L}V_2 = 0,
V_2(S, T^*) = C_A(S, T^*),
V_2(S_f(t), t) = S_f(t) - E,
\frac{\partial V_2}{\partial S}(S_f(t), t) = 1,
V_2(\bar{S}, t) = V_1(\bar{S}, t, 0),
\end{cases}
\]

Delta condition: \( \frac{\partial V_1}{\partial S}(\bar{S}, t, 0) = \frac{\partial V_2}{\partial S}(\bar{S}, t) \).

where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are defined on the region \( I \) and \( II \), respectively.

One can observe that the governing PDE systems (7.2.10-7.2.11) of the value of an American-style Parisian down-and-out call option are quite similar to those (6.2.18–6.2.19) of its up-type option counterpart; both of them consist of a 2-D system coupled with a 3-D one. More precisely, the former systems are a bit easier to handle with than the latter systems because their unknown free boundary is only a 2-D curve, but not a 3-D surface as for the latter ones. In the next section, we show that the solution procedure used to solve the latter systems can be adopted to solve the former ones.

### 7.3 Solution procedure

We now present in details the solution procedure for pricing an American-style Parisian down-and-out call option. The 3-D coupled systems (7.2.10-7.2.11) are first reduced to 2-D ones by using the “moving window” technique developed by Zhu and Chen [83]. These 2-D coupled systems are then further cast into a pair of coupled integral equations, which governs the optimal exercise price and the value of the option at the barrier. Once these integral equations have been solved using the Newton-Raphson iteration procedure, we can straightforwardly obtain the option price and the hedging parameters. It is clear that this solution procedure is similar to that used for pricing the up-type option counterpart.
7.3. SOLUTION PROCEDURE

7.3.1 The dimensionless heat systems

Following the method of [83], the 3-D systems in (7.2.10-7.2.11) can be reduced to 2-D systems by replacing the sum of the partial derivatives of $V_1$, $\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial J}$, by its directional derivative $\sqrt{2} \frac{\partial V_1}{\partial v}$, in the direction of ($\sqrt{2}/2$, $\sqrt{2}/2$). After a further change of variable by $u = \frac{v}{\sqrt{2}}$, the PDE systems in (7.2.10-7.2.11) is transformed to the following 2-D PDE systems:

\[ B_1 \begin{cases} \frac{\partial V_1}{\partial u} + LV_1 = 0, \\ \lim_{u \to J} V_1(S, u; t) = 0, \\ V_1(0, u; t) = 0, \\ V_1(\bar{S}, u; t) = W(t + u), \end{cases} \]

\[ B_2 \begin{cases} \frac{\partial V_2}{\partial t} + LV_2 = 0, \\ V_2(S, T^*) = C_A(S, T^*), \\ V_2(S_f(t), t) = S_f(t) - E, \\ \frac{\partial V_2}{\partial S}(S_f(t), t) = 1, \\ V_2(\bar{S}, t) = W(t). \end{cases} \]

Delta condition: $\frac{\partial V_1}{\partial S}(\bar{S}, 0; t) = \frac{\partial V_2}{\partial S}(\bar{S}, t)$, \hspace{1cm} \hspace{1cm} \hspace{1cm} (7.3.12)

where $W(t) = V_1(\bar{S}, 0; t)$, $B_1$ is defined on $S \in [0, \bar{S}]$, $t \in [0, T^*]$, $u \in [0, \bar{J}]$ and $B_2$ is defined on $S \in [\bar{S}, S_f(t)]$, $t \in [0, T^*]$. Recall here that $C_A(S, T^*)$ denotes the price of the embedded American vanilla call at the underlying price $S$ and time $t = T^*$, where $T^* = T - \bar{J}$.

In order to make the newly-derived 2-D PDE systems simpler, we shall now non-dimensionalize the PDE systems by introducing dimensionless variables:

\[ x = \ln \frac{S}{\bar{S}}, \quad \tau = \frac{\sigma^2}{2} (T^* - t), \quad l = (\bar{J} - u) \frac{\sigma^2}{2}; \]

constants

\[ L = \frac{\bar{J} \sigma^2}{2}, \quad \gamma = \frac{2r}{\sigma^2}, \quad q = \frac{2\delta}{\sigma^2}, \quad k = \gamma - q - 1, \quad \alpha = -\frac{k}{2}, \quad \beta = -\alpha^2 - \gamma; \]
and unknown functions $C_1(x, l; \tau), C_2(x, \tau), U(\tau)$ and $x_f(l; \tau)$ defined by:

$$C_1(x, l; \tau) = S^{-1} e^{-\alpha x - \beta l} V_1(S, u; t), \quad U(\tau) = S^{-1} W(t)$$

$$C_2(x, \tau) = S^{-1} e^{-\alpha x - \beta \tau} V_2(S, t), \quad x_f(\tau) = \ln \frac{S_f(t)}{S}.$$ 

Under this change of variables, the systems become dimensionless, and are given by:

$$\begin{align*}
\mathcal{A}_1 \quad & \begin{cases}
\frac{\partial C_1}{\partial l}(x, l; \tau) = \frac{\partial^2 C_1}{\partial x^2}(x, l; \tau), \\
C_1(x, 0; \tau) = 0, \\
\lim_{x \to -\infty} C_1(x, l; \tau) = 0, \\
C_1(0, l; \tau) = e^{-\beta l} U(\tau + l - L),
\end{cases} \\
\mathcal{A}_2 \quad & \begin{cases}
\frac{\partial C_2}{\partial \tau}(x, \tau) = \frac{\partial^2 C_2}{\partial x^2}(x, \tau), \\
C_2(x, 0) = g(x), \\
C_2(x_f(\tau), \tau) = g_2(x_f(\tau), \tau), \\
\frac{\partial C_2}{\partial x}(x_f(\tau), \tau) = g_3(x_f(\tau), \tau),
\end{cases}
\end{align*}$$

(7.3.13)

$$\text{Delta condition : } e^{\beta x} \frac{\partial C_1}{\partial x}(0, L; \tau) = e^{-\beta \tau} \frac{\partial C_2}{\partial x}(0, \tau),$$

(7.3.14)

where $U(\tau) = e^{\beta L} C_1(0, L; \tau), \mathcal{A}_1$ is defined on $x \in (-\infty, 0], \tau \in [0, \frac{\sigma^2}{2} T^*], l \in [0, L], \mathcal{A}_2$ is defined on $x \in [0, x_f(\tau)], \tau \in [0, \frac{\sigma^2}{2} T^*]$. Here datum $g(x), g_1(\tau), g_2(x_f(\tau), \tau)$ and $g_3(x_f(\tau), \tau)$ are given by:

$$g(x) = S^{-1} e^{-\alpha x} C_A(Se^x, T^*),$$

$$g_1(\tau) = e^{-\beta \tau} U(\tau),$$

$$g_2(x_f(\tau), \tau) = e^{(1-\alpha)x_f(\tau) - \beta \tau} - \frac{E}{S} e^{-\alpha x_f(\tau) - \beta \tau},$$

$$g_3(x_f(\tau), \tau) = (1 - \alpha) e^{(1-\alpha)x_f(\tau) - \beta \tau} + \alpha \frac{E}{S} e^{-\alpha x_f(\tau) - \beta \tau}.$$ 

(7.3.15)

From (7.3.13), (7.3.14), it can be observed that once $U(\tau)$ and $x_f(\tau)$ are found, $C_1$ and $C_2$ are no longer coupled, and the corresponding solutions $C_1$ and $C_2$ can be obtained straightforwardly. The determination of $U(\tau)$ and $x_f(\tau)$ is therefore a key step of solving (7.3.13), (7.3.14).

To this end, we shall first find the integral representations of $C_1$ and $C_2$, in terms of the unknown functions $U(\tau)$ and $x_f(\tau)$. This can be achieved by solving $\mathcal{A}_1$ and $\mathcal{A}_2$ in (7.3.13), (7.3.14).
separately, as if they were not coupled.

7.3.2 Integral representations of the option prices

The system $A_1$ in (7.3.13) is very similar to the system $I$ in (6.3.21), except that the non-homogeneous initial condition of the latter is replaced by the homogenous one of the former. As a result, the integral representation of the solution $V_1(S, u; t)$ of the former can be easily deduced and is simpler than that (6.3.24) of the latter:

$$V_1(S, u; t) = \frac{(\ln S - \ln S)}{\sigma^2 (v-u)} \int_u^t W(t+v) e^{\frac{\sigma^2 (v-u)}{2\sigma^2 (v-u)} \left( \ln S - \frac{\ln \bar{S}}{2} \right)^2} dv. \quad (7.3.16)$$

On the other hand, the system $A_2$, which is a heat problem in a finite time-dependent domain, is much more complicated to solve. Fortunately, this system resembles the system (3.3.4), which has already been solved in Chapter 3 using the Fourier sine transform. As a result, we can easily deduce the integral representation of the solution of the system $A_2$ as follows:

$$\bar{S}e^{\alpha x + \beta \tau} H(x_f(\tau) - x)C_2(x, \tau) = \bar{S}e^{\alpha x + \beta \tau} \int_0^{x_f(0)} \frac{g(u)}{2\sqrt{\pi \tau}} \left( e^{-\frac{(x-u)^2}{4\tau}} - e^{-\frac{(x+u)^2}{4\tau}} \right) du \quad (7.3.17)$$

$$-(\bar{S}e^x - E)1_{x=x_f(\tau)}(x) + \mathcal{M}(x, \tau) + \int_0^\tau Q(x, \tau, \xi, x_f(\xi))d\xi,$$

where

$$1_{x=x_f(\tau)}(x) = \begin{cases} 1 & \text{if } x = x_f(\tau), \\ 0 & \text{if } x \neq x_f(\tau), \end{cases}$$

and

$$\mathcal{M}(x, \tau) = \bar{S}e^{-q\tau} N \left( \frac{x - x_f(0) + 2(1-\alpha)\tau}{\sqrt{2\tau}} \right) - E e^{-\gamma \tau} N \left( \frac{x - x_f(0) - 2\alpha \tau}{\sqrt{2\tau}} \right)$$

$$- e^{2\alpha x} \left[ \bar{S}e^{-q\tau - x} N \left( \frac{-x - x_f(0) + 2(1-\alpha)\tau}{\sqrt{2\tau}} \right) - E e^{-\gamma \tau} N \left( \frac{-x - x_f(0) - 2\alpha \tau}{\sqrt{2\tau}} \right) \right]. \quad (7.3.18)$$
and

\[ Q(x, \xi, x_f(\xi)) = qS\text{e}^{-q(\tau-\xi)}N \left( \frac{x-x_f(\xi) + 2(1-\alpha)(\tau-\xi)}{\sqrt{2(\tau-\xi)}} \right) \]

\[ - E\gamma e^{-\gamma(\tau-\xi)}N \left( \frac{x-x_f(\xi) - 2\alpha(\tau-\xi)}{\sqrt{2(\tau-\xi)}} \right) \]

\[ - e^{2\alpha x} \left[ qS\text{e}^{-q(\tau-\xi)-x}N \left( \frac{-x-x_f(\xi) + 2(1-\alpha)(\tau-\xi)}{\sqrt{2(\tau-\xi)}} \right) \right] \]

\[ + \frac{xSU(\xi)}{2\sqrt{\pi(\tau-\xi)^3}}e^{\beta(\tau-\xi)+\alpha x-\frac{x^2}{4(\tau-\xi)}}. \] (7.3.19)

Transfer (7.3.17), (7.3.18), (7.3.19) back to the original variable \( S \) and \( t \), we obtain the following formula for \( V_2(S, t) \) in terms of \( S_f(t) \) and \( W(t) \):

\[ H(S_f(t) - S)V_2(S, t) = -(S - E)1_{S=S_f(t)}(S) + R(S, t) + M(S, T^* - t, S_f(T^*)) + \int_t^{T^*} Q(S, t, v, S_f(v))dv, \] (7.3.20)

where

\[ R(x, y) = \int_S^{S_f(T^*)} \left( \frac{x}{y} \right) \alpha e^{\frac{2}{\sigma^2(2\tau_0^2)}C_A(v, T^*)} \left( e^{\frac{(\ln x - \ln y)^2}{2\sigma^2(2\tau_0^2)}} - e^{\frac{(\ln x + \ln y - 2\ln S_f)^2}{2\sigma^2(2\tau_0^2)}} \right) dv, \] (7.3.21)

\( 1_{S=S_f(t)} \), \( M \) and \( Q \) are defined as in (2.3.36), (2.5.51), and (3.3.14), respectively.

Compared with the integral representation (6.3.28), there is a new term \( R(S, t) \) appearing in the integral representation (7.3.20) of \( V_2(S, t) \). This is indeed the result of the non-homogenous terminal condition (7.2.27) of the system \( A_2 \) in (7.2.10).

### 7.3.3 Coupled integral equations

In order to solve for \( S_f(t) \) and \( W(t) \), we need to derive a pair of coupled integral equations that governs their values. The first one can be easily derived by substituting \( S = S_f(t) \) into (7.3.20). As a result, we obtain an integral governing the values of \( S_f(t) \) and \( W(t) \) as follows:

\[ S_f(t) - E = M(S_f(t), T^* - t, S_f(T^*)) + R(S_f(t), t) + \int_t^{T^*} Q(S_f(t), t, v, S_f(v))dv. \] (7.3.22)
The second integral equation governing the values of \( S_f(t) \) and \( W(t) \) can be derived from the Delta condition (7.3.12). To this end, we first need to calculate the Delta \((\Delta)\) of the option when \( S \) is below or above \( \bar{S} \), i.e., \( \frac{\partial V_1}{\partial S}(S, u; t) \) and \( \frac{\partial V_2}{\partial S}(S, t) \), respectively.

**Proposition 6.** The Delta of the option when \( S < \bar{S} \), i.e., \( \frac{\partial V_1}{\partial S}(S, u; t) \) can be calculated as:

\[
\frac{\partial}{\partial S} V_1(S, u; t) = \frac{\sqrt{2}W(t + \bar{\nu})}{S\sigma\sqrt{\pi(J - u)}} \left( \frac{S}{\bar{S}} \right)^{\alpha} e^{\frac{\sigma^2}{2}(J - u) - \frac{(\ln S - \ln \bar{S})^2}{\sigma^2(J - u)}} \tag{7.3.23}
\]

\[+ \frac{\sqrt{2}\alpha S^{\alpha - 1}}{\sqrt{\pi}S^\alpha} \int_0^{\infty} e^{-\frac{1}{2}(\eta + \ln \bar{S} - \ln S)^2} \frac{2(\eta + \ln \bar{S} - \ln S)^2}{\sigma^2\sqrt{J - u}} \right\} W \left( t + u + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2(\eta + \ln \bar{S} - \ln S)^2} \right) d\eta \]

\[- \left( \frac{S}{\bar{S}} \right)^{\alpha} \frac{\sqrt{2}\sigma}{\sqrt{\pi}S} \int_0^{\infty} e^{-\frac{1}{2}(\eta + \ln \bar{S} - \ln S)^2} \frac{2(\eta + \ln \bar{S} - \ln S)^2}{\sigma^2\sqrt{J - u}} \right\} \left[ \beta W(t + u + v^2) + \frac{2}{\sigma^2} W'(t + u + v^2) \right] dv.
\]

**Proof.** Note that \( V_1(S, u; t) \) has removable singularities at \((\bar{S}, u)\). In this case, in order to calculate \( \frac{\partial V_1}{\partial S}(S, u; t) \) when \( S \) closes to \( \bar{S} \), we first need to remove these singularities by using the following variable transformation:

\[\xi = \frac{\ln S - \ln \bar{S}}{\sigma\sqrt{v - u}}.\]

As a result,

\[V_1(S, u; t) = \int_{\ln \bar{S} - \ln S}^{\infty} \left( \frac{S}{\bar{S}} \right)^{\alpha} \sqrt{\frac{2}{\pi}} e^{-\frac{\xi^2}{2} + \frac{\beta}{\sigma^2\sqrt{\xi}} \left( \frac{\ln \bar{S} - \ln S}{\xi} \right)^{2}} W \left( t + u + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2\xi^2} \right) d\xi.\]

By using the Leibniz integral rule, we can obtain:

\[
\frac{\partial}{\partial S} V_1(S, u; t) = \frac{\sqrt{2}W(t + \bar{\nu})}{S\sigma\sqrt{\pi(J - u)}} \left( \frac{S}{\bar{S}} \right)^{\alpha} e^{\frac{\sigma^2}{2}(J - u) - \frac{(\ln S - \ln \bar{S})^2}{\sigma^2(J - u)}} \tag{7.3.24}
\]

\[+ \left( \frac{S}{\bar{S}} \right)^{\alpha} \frac{\sqrt{2}\sigma}{\sqrt{\pi}S} \int_{\ln \bar{S} - \ln S}^{\infty} \ln \bar{S} - \ln S \frac{e^{-\frac{\xi^2}{2} + \frac{\beta}{\sigma^2\sqrt{\xi}} \left( \frac{\ln \bar{S} - \ln S}{\xi} \right)^{2}}}{S\xi^2} \right\} \left[ \beta W(t + u + v^2) + \frac{2}{\sigma^2} W'(t + u + v^2) \right] d\xi \]

\[+ \int_{\ln \bar{S} - \ln S}^{\infty} \frac{2\alpha S^{\alpha - 1}}{\sqrt{\pi}S^\alpha} e^{-\frac{\xi^2}{2} + \frac{\beta}{\sigma^2\sqrt{\xi}} \left( \frac{\ln \bar{S} - \ln S}{\xi} \right)^{2}} W \left( t + u + \frac{(\ln \bar{S} - \ln S)^2}{\sigma^2\xi^2} \right) d\xi.\]
By using variable transformations \( \eta = \xi - \frac{\ln \bar{S} - \ln S}{\sigma \sqrt{J - u}} \) and \( v = \frac{\ln \bar{S} - \ln S}{\sigma \xi} \) for the first and second integral in (7.3.24), we obtain the formula (7.3.23).

From the formula (7.3.23), we can obtain \( \frac{\partial V_1}{\partial S}(\bar{S}, 0; t) \) straightforwardly as follows:

\[
\frac{\partial V_1}{\partial S}(\bar{S}, 0; t) = \frac{\alpha}{S} W(t) + \frac{\sqrt{2} W(t + J)}{S \sigma \sqrt{\pi J}} e^{\frac{\beta^2 x^2}{2}} J - \frac{\sigma \sqrt{2}}{S \sqrt{\pi}} \int_0^J e^{\frac{\beta^2 x^2}{2} u^2} \left[ \beta W(t + u^2) + \frac{2}{\sigma^2} W'(t + u^2) \right] du. \tag{7.3.25}
\]

The calculation of \( \frac{\partial}{\partial S} V_2(S, t) \), on the other hand, is very similar to that of the Delta of the American down-and-out call options, which is already calculated in Section 3.3.4. As a result, one can easily deduce the following Proposition:

**Proposition 7.** The Delta of the option when \( S \geq \bar{S} \), i.e., \( \frac{\partial V_2}{\partial S}(S, t) \), can be calculated as:

\[
\frac{\partial}{\partial S} V_2(S, t) = \tilde{R}(S, t) + \tilde{K}_1(S, t) + \tilde{M}(S, T^* - t, S_f(T^*)) + \int_t^{T^*} L(S, t, v, S_f(v)) dv, \quad \forall S < S_f(t),
\]

where

\[
\tilde{R}(x, y) = \int_S^{S_f(T^*)} \frac{C_A(u, T^*)}{u^{\alpha + 1} \sigma \sqrt{2 \pi (T^* - y)}} x^{\alpha - 1} e^{\frac{\beta^2 x^2 (T^* - y)}{2}} e^{-\frac{(\ln x - \ln u)^2}{2 \sigma^2 (T^* - y)}} \left( \alpha - \frac{\ln x - \ln u}{\sigma^2 (T^* - y)} \right) du - \int_S^{S_f(T^*)} \frac{C_A(u, T^*)}{u^{\alpha + 1} \sigma \sqrt{2 \pi (T^* - y)}} x^{\alpha - 1} e^{\frac{\beta^2 x^2 (T^* - y)}{2}} e^{-\frac{(\ln x + \ln u - 2 \ln \bar{S})^2}{2 \sigma^2 (T^* - y)}} \left( \alpha - \frac{\ln x + \ln u - 2 \ln \bar{S}}{\sigma^2 (T^* - y)} \right) du,
\]

\[
\tilde{K}_1(x, y) = \frac{\sqrt{2} e^{\alpha x^2}}{\sqrt{\pi} \sigma x^\alpha} \int_0^{+\infty} W \left( y + \left( \frac{\ln x - \ln \bar{S}}{\sigma (\eta + \ln x - \ln \bar{S})} \right) x^{\alpha - 1} e^{\frac{\beta^2 x^2 (T^* - y)}{2}} e^{-\frac{(\ln x - \ln \bar{S})^2}{2 \sigma^2 (T^* - y)}} \right) du, \tag{7.3.26}
\]

\[
+ \left( \frac{x}{S} \right)^\alpha \frac{\sqrt{2} e^{\alpha x^2}}{\sqrt{\pi} \sigma x^\alpha} \int_0^{T^* - y} e^{-\frac{(\ln x - \ln \bar{S})^2}{2 \sigma^2 (T^* - y)}} e^{\frac{\beta^2 x^2 (T^* - y)}{2}} \left[ \beta W(y + v^2) + \frac{2}{\sigma^2} W'(y + v^2) \right] dv,
\]

\[
+ \left( \frac{x}{S} \right)^\alpha \frac{\sqrt{2} e^{\alpha x^2}}{\sqrt{\pi} \sigma x^\alpha} \int_0^{\sqrt{2} W(T^*)} e^{-\frac{(\ln x - \ln \bar{S})^2}{2 \sigma^2 (T^* - y)}} e^{\frac{\beta^2 x^2 (T^* - y)}{2}} \left[ \beta W(y + v^2) + \frac{2}{\sigma^2} W'(y + v^2) \right] dv.
\]

and \( \tilde{M} \) and \( L \) are defined as in Proposition [3 in Section 3.3.4.

Based on the Proposition 4 we now can calculate \( \frac{\partial V_2}{\partial S}(\bar{S}, t) \) as:

\[
\frac{\partial}{\partial S} V_2(\bar{S}, t) = \tilde{R}(\bar{S}, t) + \tilde{K}_1(\bar{S}, t) + \tilde{M}(\bar{S}, T^* - t, S_f(T^*)) + \int_t^{T^*} L(\bar{S}, t, v, S_f(v)) dv \tag{7.3.27}
\]
where \( \tilde{R}, \tilde{K}_1, \tilde{M}, L \) are defined as in the Proposition\(^7\) As a result, from (7.3.25) and (7.3.27), using the Delta condition (7.3.12), we can derive the second integral equation that governs the values of \( S_f(t) \) and \( W(t) \) as follows:

\[
\alpha \bar{S} W(t) + \frac{\sqrt{2} W(t + \tilde{J})}{S \sigma \sqrt{\pi}} e^{\beta \frac{\sigma^2}{2} \tilde{J}} - \frac{\sigma \sqrt{2}}{S \sqrt{\pi}} \int_0^{\sqrt{T}} e^{\beta \frac{\sigma^2}{2} u^2} \left[ \beta W(t + u^2) + \frac{2}{\sigma^2} W'(t + u^2) \right] du = \tilde{R}(\bar{S}, t) + \tilde{K}_1(\bar{S}, t) + \tilde{M}(\bar{S}, T^* - t, S_f(T^*)) + \int_t^{T^*} L(\bar{S}, t, v, S_f(v)) dv. \tag{7.3.28}
\]

We have now obtained the coupled integral equations governing the values of the optimal exercise boundary \( S_f(t) \) and the option price \( W(t) \) at the barrier. It can be easily observed that the coupled integral equations (7.3.22)-(7.3.28) are quite different with the corresponding ones (6.2.18–6.2.19) of the up-type option counterpart. First, there are new terms, such as \( \tilde{R}(\bar{S}, t) \) and \( \tilde{K}_1(\bar{S}, t) \), appearing in the coupled integral equations (7.3.22)-(7.3.28). Second, the optimal exercise price \( S_f(t) \) is now a function of only \( t \), but not both \( t \) and \( J \), as for the case of the up-type option counterpart. These differences make the numerical procedure of solving (7.3.22)-(7.3.28) become a bit different from that of solving (6.2.18–6.2.19), presented in Section 6.3.

### 7.4 Numerical procedure

We now describe our proposed numerical procedure to compute \( W(t) \) and \( S_f(t) \). Without losing generality, we can assume that \( T^* = \frac{p}{q} \tilde{J} \), where \( p, q \) are positive integer numbers. With this assumption, we can choose a uniform partition for the interval \([0, T^*]\) satisfying:

\[
\Pi : T^* = t_1 > t_2 > \ldots > t_m > t_{m+1} = 0, t_j = T^* - (j - 1)h, \forall j, h = \frac{J}{n}, n = qk, m = pk.
\]

Here \( k \) is some integer number. It should be noted here we do not need to discretize the \( J \)-axis, as we did in solving (6.2.18–6.2.19).

At each discrete time point \( t_j, j \in \{1, 2, \ldots, m + 1\} \), we aim to find only two unknown values of \( W(t_j) \) and \( S_f(t_j) \), but not \((n + 2)\) unknown values as in solving (6.2.18–6.2.19). This is why solving (7.3.22)-(7.3.28) is a bit easier than solving (6.2.18–6.2.19).

We first start at \( j = 1 \). Here it should be noticed that \( W(t_1) = C_A(\bar{S}, T^*) \) (cf. Section
In addition, $S_f(t_1)$ can be determined from the following Corollary:

**Corollary 7.4.1.** The optimal exercise price of an American-style Parisian down-and-out call option just before $t$ reaches $T^*$ is equal to that of its embedded vanilla option.

**Proof.** In order to prove Corollary 7.4.1 we need to take the limit as $t$ tends to $T^*$ of both sides of the integral equation (7.3.22). It is clear that

$$\lim_{t \to T^*} \int_t^{T^*} Q(S_f(t), t, v, S_f(v)) dv = 0.$$ 

In addition, we have:

$$\lim_{t \to T^*} N( d_1 (S_f(t), T^* - t, S_f(T^*)) ) = \lim_{t \to T^*} N( d_2 (S_f(t), T^* - t, S_f(T^*)) ) = \frac{1}{2},$$

$$\lim_{t \to T^*} N\left( d_1 \left( \frac{S_f^2(t)}{S_f(t)} T^* - t, S_f(T^*) \right) \right) = \lim_{t \to T^*} N\left( d_2 \left( \frac{S_f^2}{S_f(t)} T^* - t, S_f(T^*) \right) \right) = 0.$$ 

Therefore, it can be easily proved that

$$\lim_{t \to T^*} M( S_f(t), T^* - t, S_f(T^*) ) = \frac{1}{2} (S_f(T^*) - E).$$ 

The term $R(S_f(t), t)$ can be expressed as: \( R(S_f(t), t) = I_1(S_f(t), t) - I_2(S_f(t), t) \), where

$$I_1(S_f(t), t) = \int_{S_f(T^*)}^{S_f(T^*)} \left( \frac{S_f(t)}{v} \right)^\alpha \frac{e^{\beta \sigma^2 (T^* - t)} C_A(v, T^*)}{v \sigma \sqrt{2\pi (T^* - t)}} e^{-\frac{(\ln S_f(t) - \ln v)^2}{2\sigma^2(T^* - t)}} dv,$$

$$I_2(S_f(t), t) = \int_{S_f(T^*)}^{S_f(T^*)} \left( \frac{S_f(t)}{v} \right)^\alpha \frac{e^{\beta \sigma^2 (T^* - t)} C_A(v, T^*)}{v \sigma \sqrt{2\pi (T^* - t)}} e^{-\frac{(\ln S_f(t) - \ln v)^2}{2\sigma^2(T^* - t)}} dv.$$ 

Using a variable transformation $u = \frac{\ln S_f(t) - \ln v}{\sigma \sqrt{2(T^* - t)}}$, $I_1(S_f(t), t)$ can be transformed to

$$I_1(S_f(t), t) = \int_{\frac{\ln S_f(t) - \ln S_f(T^*)}{\sigma \sqrt{2(T^* - t)}}}^{\frac{\ln S_f(t) - \ln S_f(T^*)}{\sigma \sqrt{2(T^* - t)}}} e^{-u^2} \frac{C_A(S_f(t)e^{-\sigma u\sqrt{2(T^* - t)}}, T^*)e^{\sigma u\sqrt{2(T^* - t)} + \beta \sigma^2/2(T^* - t)}}{\sqrt{\pi}} du.$$ 

Therefore,

$$\lim_{t \to T^*} I_1(S_f(t), t) = C_A(S_f(T^*), T^*) \int_0^{+\infty} e^{-u^2} \frac{1}{\sqrt{\pi}} du = \frac{1}{2} C_A(S_f(T^*), T^*).$$
Similarly, we can prove that \( \lim_{t \to T^*} I_2(S_f(t), t) = 0 \). Now, by taking the limit as \( t \) tends to \( T^* \) of both sides of the integral equation (7.3.22), we obtain:

\[
S_f(T^*) - E = C_A(S_f(T^*), T^*).
\]

This implies that \( S_f(T^*) = S_f^V(T^*) \), where \( S_f^V(t) \) denotes the optimal exercise price of the embedded vanilla option at time \( t \). This completes our proof.

From Corollary 7.4.1, we obtain \( S_f(t_1) = S_f^V(T^*) \), which can be efficiently computed using the integral equation method [20, 54]. Now for solving \( S_f(t_2) \) and \( W(t_2) \), we use the following iterative procedure. We first use \( W(t_1) \) as the first approximation of \( W(t_2) \), i.e., \( W^{(1)}(t_2) = W(t_1) \). Then \( W^{(1)}(t_2) \) is used as an input for solving the governing integral equation of \( S_f(t_2) \), which is:

\[
S_f(t_2) - E = M(S_f(t_2), T^* - t_2, S_f(T^*)) + R(S_f(t_2), t_2) + \int_{t_2}^{T^*} Q(S_f(t_2), t_2, v, S_f(v))dv.
\]  

(7.4.29)

One can easily observe that the equation (7.4.29) is very similar to the equation (3.4.24), except that there is an extra term \( R(S_f(t_2), t_2) \). Fortunately, this term can be easily evaluated using the Gauss-Legendre rule. Therefore, the equation (7.4.29) can be converted to an algebraic equation, which can be efficiently solved using Newton-Raphson iterative procedure (see Section 3.4.2). Therefore, one can obtain the first approximation of \( S_f(t_2) \), denoted by \( S_f^{(1)}(t_2) \), as a solution of (7.4.29). Next, \( S_f^{(1)}(t_2) \) is used as an input for solving (7.3.28), using Trapezoidal rule and the Newton-Raphson iteration procedure. As a result, we can obtain the second approximation of \( W(t_2) \), denoted by \( W^{(2)}(t_2) \). The value \( W^{(2)}(t_2) \) is then used to obtain the second approximation of \( S_f(t_2) \), denoted by \( S_f^{(2)}(t_2) \). Repeating this iterative scheme until step \( k \)-th if the following conditions are hold:

\[
\left| \frac{S_f^{(k)}(t_2) - S_f^{(k-1)}(t_2)}{S_f^{(k)}(t_2)} \right| < \text{tol}, \quad \left| \frac{W^{(k)}(t_2) - W^{(k-1)}(t_2)}{W^{(k)}(t_2)} \right| < \text{tol},
\]

(7.4.30)

where \( \text{tol} \) is a given tolerance, say \( 10e^{-5} \).

Proceeding in a similar manner as for \( W(t_2) \) and \( S_f(t_2) \), the values of \( W(t_j) \) and \( S_f(t_j) \)
at each $j$ can be obtained. Once all the necessary values of $W(t)$ and $S_f(t)$ are determined, we can compute the values of $V_1(S,u;t)$ and $V_2(S,t)$ straightforwardly, using the formulas (7.3.16) and (7.3.20). Similarly, we can obtain the values of the Delta of the option via formulas (7.3.23) and (7.3.26).

7.5 Numerical examples and discussions

In this section, we provide selected examples to illustrate some interesting properties of the price and the optimal exercise boundary of an American-style Parisian down-and-out call option.

7.5.1 The optimal exercise price

Depicted in Figures 7.2(a) and 7.2(b) is the comparison between the optimal exercise price of an American-style Parisian down-and-out call option, as a function of time to expiry, and those of its embedded vanilla option and its American down-and-out call counterpart, when $\bar{S} = \$95$ and $\bar{S} = \$75$, respectively. Here parameters are set as: $E = \$100$, $T = 1$ (year), $\bar{J} = 1/15$ (year), $\sigma = 40\%$, $r = 6\%$, $\delta = 10\%$. It is clear from both the figures that the optimal exercise price of the Parisian option always lies between those of the other two options. This is indeed the result of the fact that the risk inherent in the contract of Parisian option is higher than that of its embedded option, but lower than that of its barrier option counterpart. More specifically, compared with the vanilla option, the Parisian option have an extra risk that they may be knocked out, i.e., become worthless before expiry. However, if compared with the barrier option, the risk of being “knocked out” of the Parisian option is still lower than, as its knock-out feature is harder to be activated.

By comparing Figures 7.2(a) and 7.2(b), one can also observe that the optimal exercise price of the Parisian option increases when the barrier $\bar{S}$ decreases. More specifically, when $\bar{S}$ decreases from $\$95$ to $\$75$, the optimal exercise price of the Parisian option increases towards that of its embedded vanilla option, which does not depend on $\bar{S}$. This can be explained as a decrease in $\bar{S}$ would not only reduce the likelihood of the “knocked out” event, but also reduce the loss for the option holder if the option is knocked out as the option is then deeply out-of-money. In other words, the knock-out feature has less effect on the Parisian option
Figure 7.2: Comparison between the optimal exercise boundaries of an American-style Parisian down-and-out call and its embedded vanilla option. Parameters are $E = 100$, $J = 1/15\text{(year)}$, $\sigma = 40\%$, $r = 6\%$, $D = 10\%$, $T - t = 1\text{(year)}$.
when $\bar{S}$ decreases and thus the difference between the optimal exercise prices of the Parisian option and the vanilla option becomes smaller in this case.

### 7.5.2 The option price

Depicted in Figure 7.3 is the comparison between the price of an American-style Parisian down-and-out call option, as a function of time to expiry $\tau$, and that of its embedded vanilla option. Here parameters are set as: $E = $100, $S = \bar{S} = $95, $T = 1 (year)$, $\bar{J} = 1/15 (year)$, $\sigma = 40\%$, $r = 6\%$, $\delta = 10\%$. It is clear from the figure that like its embedded vanilla option, the price of the Parisian option increases when time to expiry increases. This is because with a larger $\tau$, the option holder has more chances to optimally exercise the option and then gains profit. Figure 7.3 also demonstrates the fact that the price of the Parisian option is always less than or at most equal to that of the vanilla option, due to the extra risk of being “knocked out”. In particular, when $\tau < \bar{J}$, i.e., there is not enough time left for $J$ to reach $\bar{J}$, this risk disappears so that the two prices are equal.

![Figure 7.3: Comparison between prices of an American-style down-and-out call with its embedded option. Here $E = $100, $S = \bar{S} = $95, $\bar{J} = 1/15 (year)$, $\sigma = 40\%$, $r = 6\%$, $\delta = 10\%$, $T - t = 1 (year)$.](image)

On the other hand, Figures 7.4(a) and 7.4(b) compare the prices of an American-style Parisian down-and-out call option at different fixed $J$, as functions of $S$, with those of its embedded vanilla option and its American down-and-out call counterpart, when $\bar{S} = $95
Figure 7.4: Comparison between prices of an American-style Parisian down-and-out call, at different $J$, with that of its embedded option. Parameters are $E = $100, $\tilde{J} = 1/15$ (year), $\sigma = 40\%$, $r = 6\%$, $D = 10\%$, $T - t = 1$ (year).

and $\tilde{S} = $75, respectively. Like its optimal exercise price, the price of the Parisian option always lies between those of its embedded vanilla option and its barrier option counterpart. In addition, it is clear from both the figures that the price of the Parisian option is an increasing function of the underlying asset price. This is indeed reasonable as an increase in $S$ always brings more benefits to the option holder: reducing the risk of losing the option and increasing the chance of optimally exercising the option. By contrast, when $S$ decreases below the barrier, the Parisian option becomes out-of-money. Even worse, $J$ then starts to
accumulate at the same rate of the passing time and the risk of losing the option increases, which devaluates the option. This is clearly demonstrated in both Figures 7.4(a) and 7.4(b): the price of the Parisian option associated with a higher $J$ is lower than that associated with a lower $J$. It can be further observed from both the figures that only the option price curve associated with $J = 0$ passes the barrier smoothly, but not those associated with non-zero $J$ values. This is because the Delta condition (7.2.11) ensures the continuity of the Delta of the option only at $J = 0$, but not at other non-zero $J$ values.

7.6 Conclusion

This chapter adopts an integral equation approach for pricing American-style Parisian down-and-out call options. Using the “moving window” technique and the Fourier sine transform, this pricing problem can be reduced to solving a pair of coupled integral equations. As a result, once these integral equations have been solved using the Newton-Raphson iterative procedure, the option prices, the optimal exercise prices and the hedging parameters can be obtained easily. Through selected numerical examples, interesting properties of the price and the optimal exercise boundary of American-style Parisian down-and-out call options have also been revealed clearly.
In this thesis, we consider the pricing problems of different types of Parisian options under the Black-Scholes framework.

We start with an American-style down-and-out call, which is a “special” Parisian option with zero “option window”. More specifically, this option immediately becomes worthless if the underlying asset touches the asset barrier. It is this risk of being knocked out that has devaluated the price of the option, in comparison with its vanilla option counterpart. Instead of using the probability theory as used in the literature, we use the continuous Fourier sine transform to derive the “early exercise premium representation” of the option price. An advantage of this approach is that it can be easily extended to examine the significant effects of a time-dependent rebate, which may be included in the option contract, on the option price and the optimal exercise boundary. In addition, our proposed numerical method is efficient in computing the price and the hedging parameters for American-style down-and-out calls.

For a typical Parisian option, its price depends not only on the current asset price, the current time, but also the “excursion time”, which measures how long the underlying asset has continually stayed above (or below) the barrier. The corresponding pricing problem becomes a 3-D one, rather than a 2-D one as for the case of “one-touch” barrier options. More precisely, the Parisian option price is governed by a 3-D PDE system coupled with a 2-D one. Solving these coupled PDE systems is a great challenge as the complexities inherent in the coupled systems have hindered the application of various mathematical methods. Fortunately, using the “moving window” technique developed recently by Zhu and Chen [83], we are able to...
reduce the 3-D problem to a simpler 2-D one. This is the key step that has paved the way for our approaches for pricing both American-style Parisian knock-in and knock-out options.

For Parisian knock-in options, because the pricing problem of American-style options is very similar to that of their European-style option counterparts, we can solve the former by adopting a solution procedure similar to that used for solving the latter. As a result, we are able to derive simple pricing formulas for both up-type and down-type Parisian knock-in calls. We have also provided some selected graphs to illustrate results obtained from our pricing formulas as well as to reveal some interesting features of these Parisian knock-in calls.

Unlike knock-in cases, the valuation problem of an American-style Parisian knock-out option is much more difficult than that of its European-style option counterpart because of the extra challenging task of determining the optimal exercise boundary, which needs to be done in order to obtain the option value. This is especially true for an American-style Parisian up-and-out call option because its optimal exercise boundary is a 3-D surface, instead of a 2-D curve as for the case of a barrier option. Albeit difficult, we have managed to convert the corresponding 3-D pricing problem to solving a pair of two coupled integral equations, which can be solved efficiently using the Newton-Raphson iterative procedure. As a result, integral representations for the option prices and the hedging parameters can be easily obtained. Our proposed approach has been validated by showing that our obtained results agree well with the reference values obtained from adopting the standard Crank-Nicolson scheme. Interesting properties of the prices and the optimal exercise prices of American-style Parisian knock-out call options have also been revealed and discussed clearly through selected numerical examples.

Before closing the conclusion part, we remark that our approaches, which are based on the integral equation approach and the “moving window” technique, for pricing Parisian options could be extended to price Parisian options. These two types of options are very similar to each other and they differ only in the way the “excursion time” is measured. For Parisian options, the excursion time starts counting from zero each time the underlying asset price crosses the asset barrier from below (above) and then stops counting and is reset to zero when the underlying asset price crosses the barrier from above (below). For Parisian options, the excursion time accumulates all the time the underlying asset has stayed below (above) the barrier, without resetting to zero when the underlying asset price crosses the barrier from
above (below). We anticipate that the pricing problem of Parisian options could also be converted to solving a pair of coupled integral equations. These coupled integral equations, which could be more complicated than those of the Parisian option counterparts, may still be solved efficiently by using the Newton-Raphson iterative procedure. Therefore, extending our approaches to price Parisian options could be a promising research direction. Another interesting research direction is to extend our approaches, which are based on the well-known Black-Scholes model, to more complex option pricing models, such as jump-diffusion models with local or stochastic volatility. In fact, recently, some published works have been devoted to develop an integral equation approach for pricing American vanilla options under jump diffusion processes [14, 22]. These works can be served as good starting points to study how to extend our approaches to price Parisian options under jump-diffusion models.
Appendix A

Proofs of some propositions

A.1 Solving a classical heat problem in a semi-finite domain

The solution of \( I \) of 6.3.21 can be found by splitting the linear problem into two problems as follows:

\[
\begin{align*}
B_1: & \quad \begin{cases}
\frac{\partial C_1(x, \tau)}{\partial \tau} = \frac{\partial^2 C_1(x, \tau)}{\partial x^2}, \\
C_1(x, 0) = 0, \\
\lim_{x \to -\infty} C_1(x, \tau) = 0, \\
C_1(0, \tau) = e^{-\beta \tau} U(\tau),
\end{cases} \\
B_2: & \quad \begin{cases}
\frac{\partial C_1(x, \tau)}{\partial \tau} = \frac{\partial^2 C_1(x, \tau)}{\partial x^2}, \\
C_1(x, 0) = C_A(x, 0), \\
\lim_{x \to -\infty} C_1(x, \tau) = 0, \\
C_1(0, \tau) = 0,
\end{cases}
\end{align*}
\]

We will solve \( B_1 \) first. Taking the Laplace transform with respect to \( \tau \) to each term of \( B_1 \), we have:

\[
\mathcal{L}\{ C_1(x, \tau) \} = v(x, p), \quad \mathcal{L}\{ \frac{\partial C_1}{\partial \tau} (x, \tau) \} = pv(x, p), \quad \mathcal{L}\{ \frac{\partial^2 C_1}{\partial x^2} (x, \tau) \} = \frac{\partial^2 v}{\partial x^2} (x, p)
\]

The PDE of \( B_1 \) becomes \( \frac{\partial^2 v}{\partial x^2} (x, p) - pv(x, p) = 0 \). We also have:

\[
\lim_{x \to -\infty} C_1(x, \tau) = 0 \iff \lim_{x \to -\infty} v(x, p) = 0,
\]

\[
C(0, \tau) = e^{-\beta \tau} U(\tau) \iff v(0, p) = V(p) = \mathcal{L}\{ e^{-\beta \tau} U(\tau) \}
\]
The system $B_1$ now becomes

$$
B_1 \begin{cases}
\frac{\partial^2 v}{\partial x^2}(x, p) - pv(x, p) = 0, \\
\lim_{x \to -\infty} v(x, p) = 0, \\
v(0, p) = V(p),
\end{cases}
$$

(A.1.2)

The general solution of the above ODE is $v(x, p) = C_1 e^{\sqrt{px}} + C_2 e^{-\sqrt{px}}$. Since $\lim_{x \to -\infty} v(x, p) = 0$, we must have $C_2 = 0$. Therefore, the solution of the system has the form: $v(x, p) = C_1 e^{\sqrt{px}}$. Moreover, using the condition $v(0, p) = V(p)$, we obtain $C_1 = V(p)$ and thereby $v(x, p) = V(p)e^{\sqrt{px}}$. Taking the Inverse Laplace transform of $e^{\sqrt{px}}$, we have

$$
\mathcal{L}^{-1}\{e^{\sqrt{px}}\} = -\frac{x}{2\sqrt{\pi \tau^3}}e^{-\frac{x^2}{4\tau}} = g_1(x, \tau).
$$

Therefore, using the Convolution theorem, we can obtain the solution of $B_1$ as follows:

$$
C_1^i(x, \tau) = \mathcal{L}^{-1}\{v(x, p)\} = \mathcal{L}^{-1}\{\mathcal{L}\{e^{-\beta \tau U(\tau)}\} \mathcal{L}\{g_1(x, \tau)\}\} = g_1(x, \tau) * e^{\beta \tau U(\tau)} = \int_0^\tau e^{-\beta s}U(s)g_1(x, \tau-s)ds.
$$

Now we solve $B_2$ by using the method of Images. First, we consider the following standard Heat problems

$$
\begin{cases}
\frac{\partial w}{\partial \tau}(x, \tau) = \frac{\partial^2 w}{\partial x^2}(x, \tau), \\
w(x, 0) = \begin{cases} C_A(x, 0), & -\infty < x \leq 0 \\
-C_A(-x, 0), & 0 \leq x < \infty \end{cases}
\end{cases}
$$

(A.1.3)

It is well-known that the solution of this standard Heat system is

$$
w(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{+\infty} C_A(s, 0)e^{-\frac{(x-s)^2}{4\tau}}ds
$$

$$
= \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{0} C_A(s, 0)e^{-\frac{(x-s)^2}{4\tau}}ds - \frac{1}{2\sqrt{\pi \tau}} \int_{0}^{+\infty} C_A(-s, 0)e^{-\frac{(x-s)^2}{4\tau}}ds
$$

$$
= \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{0} C_A(s, 0)e^{-\frac{(x-s)^2}{4\tau}}ds - \frac{1}{2\sqrt{\pi \tau}} \int_{0}^{+\infty} C_A(s, 0)e^{-\frac{(x+s)^2}{4\tau}}ds
$$

$$
= \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{0} C_A(s, 0)\left[e^{-\frac{(x-s)^2}{4\tau}} - e^{-\frac{(x+s)^2}{4\tau}}\right]ds, \forall -\infty < x < +\infty
$$
The function $C_2^2(x, \tau) = w(x, \tau), \forall -\infty < x \geq 0$ is the solution of $B_2$. Now, we can obtain the solution of $A_1$ as

$$C_1(x, \tau) = \int_{-\infty}^{0} \frac{C_A(z, 0)}{2\sqrt{\pi \tau}} \left[ e^{-\frac{(x-z)^2}{4\tau}} - e^{-\frac{(x+z)^2}{4\tau}} \right] dz - \int_{0}^{\tau} \frac{e^{-\beta z} U(z)x}{2\sqrt{\pi (\tau-z)^3}} e^{-\frac{x^2}{4(\tau-z)}} dz \tag{A.2.4}$$

Having found $C_1(x, \tau)$, it is necessary to recover $V_1(S, \tau)$

$$V_1(S, \tau) = \int_{0}^{S} \frac{C_A(u, 0)}{u\sigma \sqrt{2\pi \tau}} \left( \frac{S}{u} \right)^{\alpha} e^{\beta \frac{S^2}{2\tau}} \left[ e^{-\frac{(\ln S - \ln u)^2}{2\sigma^2\tau}} - e^{-\frac{(\ln S + \ln u - 2\ln \bar{S})^2}{2\sigma^2\tau}} \right] du,$$

$$+ \frac{(\ln \bar{S} - \ln S)}{\sigma \sqrt{2\pi}} \left( \frac{S}{\bar{S}} \right)^{\alpha} \int_{0}^{\tau} \frac{W(z)}{\sqrt{(\tau-z)^3}} e^{\beta \frac{\bar{S}^2}{2(\tau-z)}} \frac{(\ln S - \ln \bar{S})^2}{2\sigma^2(\tau-z)} dz. \tag{A.2.4}$$

### A.2 Optimal exercise price at expiry

**Proof.** By rearranging the integral equation [6.3.31], we obtain:

$$\frac{S_f(u; t)}{E} = \frac{T_1(u, S_f(u; t); t)}{T_2(u, S_f(u; t); t)} \tag{A.2.4}$$

where

$$T_1(u, S_f(u; t); t) = 1 - e^{-r(J-u)} N \left( d_2(S_f(u; t), \bar{J} - u, \max(\bar{S}, E)) \right) \tag{A.2.4}$$

$$+ \left( \frac{S_f(u; t)}{S} \right)^{2\alpha} e^{-r(J-u)} N \left( d_2 \left( \frac{S^2}{S_f(u; t)}, \bar{J} - u, \max(\bar{S}, E) \right) \right) \tag{A.2.4}$$

$$- \int_{u}^{J} e^{-r(v-u)} N \left( d_2 \left( S_f(u; t), v - u, S_f(v; t) \right) \right) dv \tag{A.2.4}$$

$$+ \left( \frac{S_f(u; t)}{S} \right)^{2\alpha} \int_{u}^{J} e^{-r(v-u)} N \left( d_2 \left( \frac{S^2}{S_f(u; t)}, v - u, S_f(v; t) \right) \right) dv, \tag{A.2.4}$$
A.2. OPTIMAL EXERCISE PRICE AT EXPIRY

and

\[
T_2(u; S_f(u; t); t) = 1 - e^{-\delta(J-u)}N\left(d_1\left(S_f(u; t), \bar{J} - u, \max(\bar{S}, E)\right)\right) \\
+ \left(\frac{S_f(u; t)}{S}\right)^{2\alpha-2} e^{-\delta(\bar{J}-u)}N\left(d_1\left(\frac{\bar{S}^2}{S_f(u; t)}, \bar{J} - u, \max(\bar{S}, E)\right)\right) \\
- \int_u^\bar{J} \delta e^{-\delta(v-u)}N\left(d_1\left(S_f(u; t), v - u, S_f(v; t)\right)\right) dv \\
+ \left(\frac{S_f(u; t)}{S}\right)^{2\alpha-2} \int_u^\bar{J} \delta e^{-\delta(v-u)}N\left(d_1\left(\frac{\bar{S}^2}{S_f(u; t)}, v - u, S_f(v; t)\right)\right) dv \\
- \frac{S_f(u; t)^{\alpha-1}}{S^{\alpha} \sqrt{2\pi}} \int_u^\bar{J} \frac{\ln S_f(u; t) - \ln \bar{S}}{\sqrt{(v-u)^3}} W(v+t) e^{-\frac{(\ln S_f(u; t) - \ln \bar{S})^2}{2\sigma^2(v-u)} + \beta^2(v-u)} dv.
\]

Before proceeding further, we note that \(S_f(\bar{J}^-; t) \geq \max(\bar{S}, E)\) because of two main reasons. First, the option should be exercised only when it is in-the-money or at-the-money. Second, if the option is in-the-money or at-the-money, it should be exercised before it is knocked out.

We first consider the case: \(S_f(\bar{J}^-; t) = E\). Taking the limit of equation (A.2.4) as \(u\) tends to \(\bar{J}^-\), we obtain \(\lim_{u \to \bar{J}^-} \frac{S_f(u; t)}{E} = \frac{1}{2} = 1\) and thus \(S_f(\bar{J}^-; t) = E\) is a possible solution for \(S_f(\bar{J}^-; t)\). We also consider the second case: \(S_f(\bar{J}^-; t) = \bar{S}\). Taking the limit of (A.2.4) as \(u\) tends to \(\bar{J}^-\), we obtain \(\lim_{u \to \bar{J}^-} \frac{S_f(u; \tau)}{E} = \frac{1}{1 - \frac{\bar{S} - E}{\bar{S}}} = \frac{1}{\frac{S - E}{S}}\). Therefore, \(S_f(\bar{J}^-; t) = \bar{S}\) is a possible solution for \(S_f(\bar{J}^-; t)\).

Finally, we consider the case: \(S_f(\bar{J}^-; t) > \max(\bar{S}, E)\). Because

\[
\lim_{u \to \bar{J}^-} T_1(u; S_f(u; t); t) = \lim_{u \to \bar{J}^-} T_2(u; S_f(u; t); t) = 0,
\]

the limit of equation (A.2.4) is an indeterminate form, which can be resolved by using L’Hospital’s rule. However, before applying L’Hospital’s rule, we should eliminate “redundant terms” in \(T_1\) and \(T_2\). First, we have:

\[
\lim_{u \to \bar{J}^-} \frac{1 - e^{-r(J-u)}N\left(d_2(S_f(u; t), J - u, \max(\bar{S}, E))\right)}{J - u} = \lim_{u \to \bar{J}^-} \frac{1 - e^{-r(J-u)}}{J - u} = r.
\]

Thus \(1 - e^{-r(J-u)}N\left(d_2(S_f(u; t), J - u, \max(\bar{S}, E))\right) \sim J - u\) as \(u \to \bar{J}^-\), where the notation
Moreover, we have:

\[ N \left( d_2 \left( \frac{\bar{S}^2}{S_f(u; t)}, \bar{J} - u, \max(\bar{S}, E) \right) \right) \sim N \left( -\frac{1}{\sqrt{\bar{J} - u}} \right) \sim \int_{-\infty}^{-\frac{1}{\sqrt{\bar{J} - u}}} e^{-\frac{t^2}{2}} dt, \]

\[ \lim_{\bar{J} - u \to 0^+} \frac{\int_{-\infty}^{-\frac{1}{\sqrt{\bar{J} - u}}} e^{-\frac{t^2}{2}} dt}{\bar{J} - u} = \lim_{\bar{J} - u \to 0^+} \frac{e^{-\frac{1}{\bar{J} - u}}}{\bar{J} - u} = 0. \]

Therefore, the term \( \left( \frac{S_f(u; t)}{\bar{S}} \right)^{2\alpha} e^{-r(\bar{J} - u)} N \left( d_2 \left( \frac{\bar{S}^2}{S_f(u; t)}, \bar{J} - u, \max(\bar{S}, E) \right) \right) \) decays to 0, as \( u \to \bar{J}^- \), at a faster rate than the term \( 1 - e^{-r(\bar{J} - u)} N \left( d_2(S_f(u; t), \bar{J} - u, \max(\bar{S}, E)) \right) \).

Moreover, we have:

\[ \lim_{v \to u} N \left( d_2 \left( \frac{\bar{S}^2}{S_f(u; t)}, v - u, S_f(v; t) \right) \right) = 0, \]

\[ \lim_{v \to u} N \left( d_2(S_f(u; t), v - u, S_f(v; t)) \right) = \frac{1}{2}. \]

Therefore, the terms \( \left( \frac{S_f(u; t)}{\bar{S}} \right)^{2\alpha} \int_{u}^{\bar{J}} r e^{-r(v-u)} N \left( d_2 \left( \frac{\bar{S}^2}{S_f(u; t)}, v - u, S_f(v; t) \right) \right) dv \) decays to 0, as \( u \to \bar{J}^- \), at a faster rate than the term \( \int_{u}^{\bar{J}} r e^{-r(v-u)} N \left( d_2(S_f(u; t), v - u, S_f(v; t)) \right) dv \).

From the above results, we conclude that as \( u \to \bar{J}^- \)

\[ T_1 \sim T_3 = 1 - e^{-r(\bar{J} - u)} N \left( d_2(S_f(u; t), \bar{J} - u, \max(\bar{S}, E)) \right) \]

\[ - \int_{u}^{\bar{J}} r e^{-r(v-u)} N \left( d_2(S_f(u; t), v - u, S_f(v; t)) \right) dv. \]

Similarly, we have

\[ T_2 \sim T_4 = 1 - e^{-\delta(\bar{J} - u)} N \left( d_1 \left( S_f(u; t), \bar{J} - u, \max(\bar{S}, E) \right) \right) \]

\[ - \int_{u}^{\bar{J}} \delta e^{-\delta(v-u)} N \left( d_1(S_f(u; t), v - u, S_f(v; t)) \right) dv. \]

Similar to the case in Chiarella et al. [20], it can be shown that \( \lim_{u \to \bar{J}^-} \frac{T_3}{T_4} = \frac{r}{\delta} \). Therefore,

\[ \lim_{u \to \bar{J}^-} \frac{S_f(u; t)}{E} = \lim_{u \to \bar{J}^-} \frac{T_1(u; S_f(u; t); t)}{T_2(u; S_f(u; t); t)} = \lim_{u \to \bar{J}^-} \frac{T_3}{T_4} = \frac{r}{\delta}. \] (A.2.5)
Combining the results of the above three cases, we have

\[ S_f(J^-; t) = \max(E, \bar{S}, \frac{rE}{\delta}). \]
Bibliography


Publication list of the author


