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Stock Loans Valuation

Endah R M Putri

University of Wollongong
UNIVERSITY OF WOLLONGONG

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Stock Loans Valuation

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Endah R.M. Putri S.Si (ITS), M.T (ITS)

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Abstract

Stock loans have been recognized as a popular financial contract and considered as an alternative for investors to increase liquidity. Consequently, an accurate and efficient valuation of stock loans is necessary. Although there are some existing literatures on the valuation of stock loans, to the best of our knowledge, most are for perpetual loans. In reality, stock loan contracts mostly last for less than 10 years. It is, therefore, important to develop accurate and efficient evaluation methods for stock loans of finite maturity. This thesis explores a semi-analytic method for evaluating American puts in Zhu [2006] and proposes a modified semi-analytic method to evaluate stock loans in finite maturity. Three types of stock loans are formulated and solved as the corresponding American option problems under the Black-Scholes framework.

First, finite non-recourse stock loans with three different dividend distributions are formulated as American calls, and solved using Zhu’s semi-analytic method. Our results are compared with those in Dai and Xu [2011]. Then margin call stock loans are formulated as American Down-and-Out call options with rebate, and solved using the semi-analytic method. The dependency of the optimal exit price and the stock loan value on payback amount, and other parameters such as effective interest rate and volatility, are analyzed as well. Finally, Stock loans under a two-state regime switching economy are formulated as the corresponding Regime switching American Call options. A modified semi-analytic method is proposed to find the solutions of the optimal exit prices and stock loan
values. Furthermore, fair service fee are calculated as well for different scenarios. The results of thesis could be used for borrowers and lenders to consider their exit policy and decide the service fee fairly.
dedicated to

ibu bapak

mama tua bapak tua

my husband

my children
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Chapter 1

Introduction

A stock loan is a financial contract in which a borrower who owns stocks borrows some money from a lender by providing the stocks as collateral. In this way, the borrower may increase liquidity from the stocks without selling the stocks. The quality of the stocks determines the loan amount which can be lent. The borrower should pay an agreed service fee to the lender. This financial contract allows the borrower to use the loan for various objectives such as diversifying their assets or stocks. This contract can be used to hedge against market downturn or even get the unlimited upside potential from the future stock appreciation. Unlike the common loan, to establish a stock loan contract there is no income statement and credit report needed [104].

In the financial industry, stock loans have been immensely popular worldwide. The trading value globally exceeded approximately £1 trillion in 2007 [36]. In U.S. market, the trading value was still nearly US$ 250 billion in 2009, in spite of the short selling restriction imposed by U.S. authorities in 2008 [100]. The stock loan programme of the OCC (Option Clearing Corporation) recorded 84% increase of new loans in January 2011 with daily trading volume exceeding US$ 14 billion [2]. These indicate the importance of developing accurate and efficient valuation methods of this financial product [102].
In this thesis, a study of the valuation of the following three types of stock loans which are formulated as the corresponding American options is presented.

A standard stock loan is a non-recourse stock loan. This means that the borrower may just walk away if the price of the collateralized stock drops sharply and the borrower fails to pay back the loan. In this situation, the maximum loss for the borrower is only the service fee. However, if the stock price increases significantly, the borrower could exit, retrieve the stock and even receive a profit [104]. This type of stock loan provides greater benefit to the borrower and higher risk to the lender. The valuation of a standard stock loan with finite maturity forms the subject of investigation in Chapter 4.

A margin call stock loan is a stock loan with an additional feature—margin call. In this type of contract a margin call is issued when the stock price drops below the loan amount, the borrower is obliged to pay back a portion of the loan amount. The contract then continues as a non-recourse contract after the payback with a new loan amount. This new loan amount is the difference between the cumulative loan amount and the payback amount. Only one margin call is allowed in the life of a contract. A margin call stock loan provides more protection to the lender from risk due to the possibility that the borrower can walk away [34]. Finite maturity margin call stock loan is the subject of study in Chapter 5.

A stock loan under regime-switching economy is a standard stock loan which accommodates changes in economic conditions. The economic change is represented by the “market mode” (regime) which switches among states. In a financial market, changes occur in market parameters such as stock volatilities and market interest rates. Economic policies, natural disasters, threats of war, and so on may cause the change in stock volatilities [28, 41, 88, 99]. In a stock loan under regime-switching economy, the stock price would fluctuate between state regimes. This kind of stock loan can capture the dynamics of the market better
than the standard stock loans [112]. The valuation of finite maturity stock loan under regime-switching economy is studied in Chapter 6.

The above three chapters form the major part of the thesis, which are largely based on three journal papers corresponding to these chapters. To give a complete description of the topics under investigation, the review of literature of stock loans and the corresponding American option is presented in Chapter 2, and the fundamental mathematical background including stochastic process, Ito formula, and the Black-Scholes model are presented in Chapter 3, with Chapter 7 providing the general conclusion of the thesis work.
Chapter 2

Literature Review

The first model to study stock loans as an option problem was proposed by Xia and Zhou [104]. Other studies could be categorized as standard stock loans, non-standard stock loans, and stock loans with regime-switching. In this thesis the non-standard stock loans being discussed are margin call stock loans.

The literature review of stock loans and the corresponding American call options are presented in this chapter.

2.1 Standard Stock Loans and American Call Options

A similar trading structure between non-recourse or standard stock loans and American call options has been discussed by Xia and Zhou [104]. The borrower of a stock loan can be regarded as the holder of a call option and has right to exercise the option anytime before maturity. The borrower is allowed to walk away and forfeit the stock to the lender when the collateralized stock price drops significantly. This situation is analogous to the ‘do-nothing’ position of a call option holder. As a consequence, the loss of the holder is limited to only the premium. Conversely, if the stock price increases, the borrower can repay the
loan to exit the contract, retrieve the stock and make a profit. The borrower’s right to exit from the contract anytime before maturity can be regarded as an early exercise of the American call option. The valuation of standard stock loans resembles the perpetual American call option valuations [104].

Various extension of standard perpetual stock loans have been presented in the following studies. As the finite maturity of stock loans is more appropriate in practice, Dai and Xu [32] studied the finite maturity stock loans in different dividend distribution. An analysis of the existence of the optimal exit boundary and the properties were presented in the study.

Pascucci et al. [89] extended the stock loan valuation with certain dividend distribution in Dai and Xu [32]. The dividend distribution discussed in their study is the accumulative dividend yield which is paid at maturity. The stock loan is modeled as a modified Black-Scholes equation. The equation represents an irreducible two-dimensional time-dependent problem. A mathematical analysis of the stock loans was carried out and numerical examples were presented as the analytical solution is unavailable. Graselli and Gomez [45] assumed that the stock loans are traded in an incomplete market since the borrower as stock owner is actually unable to trade the stock on the market. Hamilton-Jacobi-Bellman (HJB) PDE was used in the valuation of the stock loans in an incomplete market. Both, infinite and finite maturity stock loans in an incomplete market were discussed. Numerical method was used to obtain the value of typical stock loans in finite maturity.

The valuation of stock loans in the aforementioned studies use the corresponding American call options model. There are many studies about American options in the literature. In 1973, Black and Scholes proposed an analytical solution of European options, known as the Black-Scholes formula which has become the most widely used model in financial markets [10].
While the European option valuation have a closed-form solution under the Black-Scholes framework, American option valuation still remains one of the most challenging problems in mathematical finance [62, 74, 96]. The difficulty lies in what financial terms, is known as the possible early exercise right. This right constitutes the American option pricing as a free boundary problem [83]. As the early exercise boundary is time dependent and forms a part of the solution, the American option becomes a highly non-linear problem. It is similar to other free boundary problems such as the Stefan problem in physics [52].

Analytic solutions can be used, as a benchmark, since they are undoubtedly accurate and do not need any degree of approximation. They are also mathematically elegant. But it seems impossible to obtain analytic solutions for American options where the early exercise may be optimal [103]. Some studies have been carried out however, to obtain the solution. A closed-form solution for a perpetual American option is obtained by Merton [85]. Recently, a breakthrough has been made by Zhu [114] which has a remarkable contribution to the theoretical side of option pricing. An exact, closed-form, and explicit solution of a non-dividend paying American put option is obtained in the form of an infinite Taylor series expansion. The series involves two infinite sums of infinite double-integral. However, a large amount of computing time is needed to reach the series convergence [115]. Apart from the analytical solution for the typical American options discussed by Zhu [114], there is no analytical solution for the more complex American options problem. Therefore, numerical and analytic approximation methods become the only solution for the market practitioner.

Extensive efforts have been made to find accurate and efficient approximation methods to obtain the value of the American options. The methods include numerical and analytical approximation. Each method has its advantages and limitations.
There are two basic subcategories of numerical methods. The first category is the methods which are based on the discretization of the Black-Scholes equation to both, a time variable and a stock variable. These methods include the finite difference method [15, 103], the finite element method [1] and the radial basis function method [54]. The second category is the methods which are based on risk-neutral valuation of each step. These include the binomial method [30], the Monte Carlo method [44] and the least square method [74]. Intensive computation is required in some of these methods to reach convergence and reasonable accuracy.

In order to reduce the computational intensity in the valuation process, analytical approximation methods have been developed. These methods still have some degree of computation but not as much as the fully numerical methods. Some of these methods include compound option approximation [40], integral equation approximation [21, 59, 67] and the transform method [23, 77, 96].

One of the most prominent studies of American option valuation using the transform method was conducted by Zhu [113]. In this study, a simple exact and elegant formula of an optimal exercise boundary of non-dividend American put options was obtained. A pseudo-steady-state approximation is used by considering the similarity of the free boundary problem of the American options with the Stefan problem in heat transfer. In the Stefan problem, the behavior of the slow moving of heat conduction interface compared with the conduction has motivated the use of approximation in pricing options. In a similar way, the optimal exercise price moves slowly in comparison with the option price. A simple formula to calculate the optimal exercise price was obtained in Laplace space. Subsequently, the option value was obtained directly. Zhu’s method is more appropriate for a valuation of American options with long-time-maturity [69]. The method is extended to the dividend paid American put options [119] where a
numerical Laplace inversion has been applied without losing the efficiency of the computation.

## 2.2 Margin Call Stock Loans and American Barrier Options

In margin call stock loans, if the stock price decreases below the loan amount, a margin call is issued and the contract is suspended. Only one margin call is allowed in the contract. The borrower has to pay back a pre-determined fraction of the loan due to the call. After the payback, the contract continues as a non-recourse contract with a reduced loan amount. The new loan amount is determined as the difference between the amount of the loan and the payback [34].

When the margin call is issued and the margin call stock loan continues as a non-recourse stock loan after the payback, the borrower actually get benefits. The borrower does not need to provide a new collateral for the non-recourse stock loan. The value of this non-recourse stock loan minus the payback is the incentive for the borrower. This value can be regarded as the stock loans refund value for the borrower due to the call [34].

The loan amount of a stock loan is considered as a barrier of an American down-and-out call option and the refund value for the borrower is similar to a rebate in an American down-and-out call. Therefore a margin call stock loan is modeled as an American down-and-out call option with rebate [34].

There are some studies of non-standard stock loans incorporating different features to provide more protection to the lender. Capped stock loans have been proposed by Liu and Xu [71]. The stock loans have a cap feature to limit the rise of the stock price so that the lender can avoid the large loss due to the
rise. These stock loans resemble the perpetual American capped call options. A more complicated type of non-standard stock loans which contains an automatic termination clause, cap and margin has been presented by Jiang et al. [61]. This type of stock loan corresponds to a generalized perpetual American option. The cap and the margin protect the lender from stock price fluctuations, both up and down. A different approach to stock loan valuation has been proposed by Yam et al. [105] for callable stock loans. Instead of resembling certain types of American barrier options, the stock loan valuation used Dynkin’s game to add an additional right of the lender to cancel the loan contract. The lender pays a pre-determined penalty to the borrower for the cancelation. All these studies have infinite maturity for the typical stock loans where, in practice, the stock loans have finite maturity.

In an American down-and-out call option, the option is terminated if the stock price falls and hits the pre-determined barrier. The American down-and-out option may have a rebate which is paid to the option holder at the option termination [68]. On the other hand, if the stock price increases then the option holder can exercise the option anytime before maturity and gain a profit [85].

In the following section, a more detailed literature review of barrier options is given. Barrier options as one class of exotic options have been attractive to the market [33]. The trading volume has doubled every year [55]. The variety of barrier options has also grown [20], including, for example: rainbow options, capped options, and roll up and down options. The types of standard barrier options include: up-and-out, down-and-out, down-and-in, and up-and-in [64, 68]. Closed form solutions for some types of European barrier options have been presented by some researchers. Merton [85] first proposed the closed-form solution of European down-and-out barrier options. Other studies about European up and out, down-and-in and up-and-in options can be seen in Benson and Daniel [9], Carr
Valuing the European barrier options through the PDE method and employing the symmetry properties of the Black-Scholes model have been performed by Buchen [16] for all barrier option types. A barrier option may pay a rebate when the barrier is hit. The payment can be at the hitting time or delayed until the expiry date. The analytical solution exists for this option even though the formula becomes more complicated [97].

The valuation of standard barrier options in the American style is more intricate than its European counterparts because of the presence of the free boundary. None of the existing studies obtained the analytical solution for the American barrier options except the following two studies. Haug [50] obtained analytic formulas for knock-in American options under certain conditions. Dai and Kwok [31] provided an integral representation of the knock-in American option and showed that an in-out barrier relationship could not be obtained.

Both numerical and analytical approximations, have been extensively developed to provide appropriate valuation of American barrier options. The most common numerical methods are the binomial and trinomial methods which might be viewed as certain types of finite difference methods [120]. Boyle and Lau [13] presented the valuation of American barrier options using the binomial method. This method is limited to a single barrier. Ritchken [92] proposed the trinomial method in the American barrier options which improved the limitation in [13].

However, those methods require many steps when the stock price is near the barrier. Because of this requirement, the methods need to have a large number of time steps and this results in slow convergence. Cheuk and Vorst [27] improved Ritchken’s work by incorporating a time dependent shift in the trinomial tree to solve the near-barrier problem. This still needs a large number of time steps.

The valuation of the American barrier option using finite difference has been
proposed by Boyle and Tian [14], Ndgo mo and Ntwiga [87], and Zvan et al. [120]. The convergence of the finite difference method can be reached in a relatively large number of time steps but this leads to huge computational time. On the other hand, analytical approximation methods can take less computational time so that the methods can be more efficient. Gao et al. [39] proposed a decomposition technique separating the value up-in and up-out American barrier options into the European value and the early exercise premium. Chang et al. [22] presented a modified quadratic approximation method based on Barone-Adesi-Whaley [5] which has more accurate result than the tree method. Lu and Rhodes [76] extended a semi-analytical valuation of American down-and-out call options without rebate. The method extended Zhu’s method [113] which was applied for a non-dividend American put option valuation. The method can obtain good results for an optimal exercise price and option value.

2.3 Stock Loans Under A Regime-Switching Economy and American Options Under A Regime-Switching Economy

In stock loans, the stocks owned by the borrower are collateralized without being withdrawn from the market. As the stock is still traded in the market, the stock price is volatile. The stock market volatility represents risk of the stock returns [37] and the economic condition [8, 94]. In the real market, the stock market volatility changes over time and this has been shown by Schwert [94], Scruggs and Nardari [95] and Shiller [98]. Therefore, the use of a constant volatility assumption in a standard Black-Scholes model is inappropriate because it does not fully reflect the market condition. And it is necessary to carry out a valuation of standard stock loans incorporating the economic changes.
The studies of stock loans valuation incorporating the economic changes are proposed by some researchers. Prager and Zhang [90] proposed a valuation of finite European stock loans with switching between the geometric Brownian motion and mean reverting model. Assuming that the stock price follows the Levy process, Liang et al. [70] proposed the perpetual stock loan valuation incorporating the Levy process and Wong [102] broadened the valuation by using the mean-reverting stochastic volatility model.

Since it is more commonly used by the market practitioners, the Black-Scholes model is used as the governing equation for the stock loan valuation. Zhang and Zhou [112] presented the valuation of perpetual stock loans under a regime-switching economy as a resemblance of the perpetual American call options under a regime-switching economy. The stock loan value process is governed by the Black-Scholes model.

Except for the European style stock loan in a regime-switching economy [90], the aforementioned stock loan studies which assume non-constant volatility have infinite maturity. Since in practice, the stock loan contract is valid only for a few years, the valuation of stock loans in finite maturity becomes necessary.

It has been known that the valuation of American options under the Black-Scholes framework with a constant volatility assumption can not capture the market randomness [4]. The volatility can be measured as historical volatility [12, 42] or implied volatility [82]. These two methods only estimate the volatility based on the available information. None of the methods provide a model for the volatility.

The need for a more appropriate volatility model has been shown in several empirical studies (see [29, 79, 84, 93]). In the Black-Scholes framework, the asset returns are assumed to follow a normal distribution, satisfy the independent and identically distributed assumption, and to be continuous. It has been proven
empirically, however, that these assumptions can not be satisfied. The presence of volatility smile and volatility term structure also showed that the Black Scholes model can not fully reflect the market randomness (see [33, 58]).

There have been some attempts to provide a better volatility model. Instead of treating volatility as a deterministic function or as a stochastic process with some additional randomness, it is more general and natural to treat it as an unknown process. The simplest volatility model as an unknown process was proposed by Avellaneda et al. [3]. Introducing this unknown volatility, however, can give an extra non-linearity to the PDE which is certainly more difficult to solve [80]. Avoiding this difficulty in solving the non-linear PDE and applying a more flexible framework [73], a regime-switching model may be useful.

As the market movement generally affects the stock price movement, it is appropriate to allow the stock market volatility to depend on the market movement. Some studies have suggested that a Markov chain regime-switching could describe the market movement better (see [4, 6, 26, 35, 111]). In a regime-switching model, the economic changes are triggered by the mood of consumers and investors in the market, shifts in economics policies, takeovers, the threat of war and so on. Goldfeld and Quandt [43] first introduced a regime-switching regression model as a Markov chain process. In his celebrated paper, Hamilton [49] has extended this model to the important case of dependent data specifically the parameters of an autoregression process. Since then, the number of studies about the Markov chain regime-switching generated by Hamilton [49] has been growing (see [4, 35, 66]).

The valuation of options with regime-switching in both the European and the American style have been discussed in the literature. Employing the lattice method and Monte Carlo simulation, Bollen [11] demonstrated that the valuation of European options with a two-regime economy has been satisfactorily practical for investors. Instead of using numerical solutions for European style option
valuation, it is more challenging to find a closed-form analytic solution of the
typical options. Some attempts to provide a closed-form analytic solution of
European style options with regime-switching include: the solution of European
options with two state random jumps of the risky asset volatility in double integral
form [86], a closed-form solution of European options with regime-switching under
the Black-Scholes model using the PDE method [18, 78, 116], mean-variance
hedging for European options where the drifts and volatility are driven by regime-
switching [81], and an explicit solution of European options with jumps volatility
based on some chosen “basis” options [51].

A more general model has been proposed by Guo [46] as the drifts, volatilities
and continuous dividend yields are assumed to be driven by the state of the
economy. Similar formulation has been obtained by Fuh et al. [38] by improving
the error in the preceding study. Numerical valuation of American style options
with regime-switching can be traced back to Bollen [11], Liu [72], and Yuen and
Yang [110]. The lattice and tree methods have been studied in those papers.
However, these numerical methods are time consuming and lead to the efficiency
problem.

An analytical solution for the option with regime-switching in the American
style has been obtained only for certain case such as the perpetual one. A modified
smooth fit technique for pricing perpetual American put options was used by
Guo [47] and a closed-form solution has been obtained. The analytic solution
for the valuation of American options with regime-switching in finite maturity is
not available and the numerical solution is not efficient. Therefore the analytic-
approximation method for obtaining the valuation may be considerable.

Buffington and Elliot [17] presented an analytic approximation of non-dividend
American options with interest rates and drifts which are driven by a two-state
regime-switching. The valuation was carried out based on the PDE method and
a probabilistic approach following Barone-Adesi-Whaley [5]. A set of partial differential equations to evaluate the American option with regime-switching was obtained but it still needs a numerical method to calculate the results.
Chapter 3
Mathematical Background

In this chapter, we provide basic knowledge needed for evaluating the stock loan problems.

3.1 Stochastic Process

The main point of valuation of an option is the knowledge of the current and future value of the underlying asset because this knowledge is essential for option valuation. The dynamics of an underlying asset of an option is assumed to be a stochastic process known as geometric Brownian motion which is modeled as in the following stochastic differential equation:

\[
\frac{dS}{S} = \mu dt + \sigma dW_t
\]  

(3.1)

where \(\frac{dS}{S}\) is the return on underlying asset, \(\mu\) is a deterministic constant mean of underlying asset return or drift, \(\sigma\) is a volatility and \(W_t\) is a Wiener process. The term \(dW_t\) is a normal distributed random variable with mean zero and variance \(dt\).
3.1.1 Ito’s Lemma

The geometric Brownian motion, which the underlying asset dynamic is assumed to follow, can be considered as a one dimension scaling limit of symmetric random walk. The value of an option can be obtained by direct simulation of the random walk in practical scale resulting in a large amount of data. To simplify the valuation of an option by using such a differential equation without dealing with huge data, mathematical manipulation of the random variable is necessary.

The following theorem, known as Ito’s lemma, is the tool to relate the small change in the random variable to a function of a random variable.

**Theorem 3.1 (Itô Lemma).** If $S$ is driven by the stochastic differential equation written in Equation (3.1) where $S$ is the value of the process, $\mu$ and $\sigma$ are adapted to the filtration $\mathcal{F}_t$, and $f \in C^{1,2}([0, \infty) \times \mathbb{R})$, then defining $f = f(S, t)$ implies that $f$ has a stochastic differential given by

$$df(S, t) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial S^2} \right\} dt + \sigma \frac{\partial f}{\partial S} dW_t \quad (3.2)$$

Ito’s lemma plays an important part in deriving the differential equations of financial derivatives such as options.

3.2 Arbitrage-free Principle

The arbitrage-free principle is one of the fundamental concept used in financial mathematics. The principle can also be stated as ‘there is no such thing as a free lunch’. This statement is interpreted in financial terms to mean that there are no opportunities to make a risk-free instantaneous profit. With this principle, the same assets will have the same price wherever they are traded.

The following illustrates how arbitrage cannot be done. Suppose that there
is an arbitrage opportunity in the stock markets which implies that the stock is priced differently in different markets. If Company ABC’s stock trades $7 per share on the New York Stock Exchange (NYSE) and $7.02 on Australian Stock Exchange (ASX), it is obvious that there is an arbitrage opportunity. The investor will buy the stocks on the NYSE and sell them directly on the ASX. A risk-free instant profit will be the price difference of $0.02 per share.

In reality, however, an arbitrage opportunity is found rarely and it lasts for a very short time due to the exposure of supply and demand in the market. To obtain significant profit, only a large company with a large trading volume can take an advantage. The transaction cost being deducted will reduce the profit. In this situation, if the presence of arbitrage opportunity is involved, the financial derivatives will be potentially mispriced. When the market is allowed to have an arbitrage-free-opportunity, the market will be able to reach an equilibrium of supply and demand so that the stock will be fairly priced.

### 3.3 Hedging

Hedging is a risk management strategy to allow investors to avoid loss in their portfolio by making an offsetting investments. This strategy can be formed by investing in two different securities with a highly negative correlation. When the investment decreases in value, the well-functioned hedging strategy can reduce the loss. Being opposite to the first, the second investment increases in value and the investor gains a reduced potential profit. In this way, hedging is not used to increase the profit but it can be used as insurance against the negative effect on profit to minimize risk. The investor could have a riskless position and that such a strategy is called a riskless hedging principle. This principle becomes the foundation of option pricing theory [68].
3.4 The Black-Scholes Model

The Black-Scholes model, the most widely-known financial derivatives pricing model, was first proposed by Black and Scholes [10] and Merton [85] in 1973. The Black-Scholes represents a continuous time option pricing problem in the form of a partial differential equation. In this section, the Black-Scholes partial differential equation is derived based on the riskless hedging principle without dividend payment and transaction cost.

The assumptions used in the Black-Scholes model are in the following [101]:

- the stock price follows the geometric Brownian motion
- the risk-free interest rate $r$ and the stock volatilities $\sigma$ are known and constant over time
- there is no transaction cost and taxes in the option trading
- the stock pays no-dividend
- there are arbitrage-free opportunities
- the trading of the stock is continuous
- short selling is permitted and the stock is divisible

Suppose that there is an option value of which is $V(S, t)$. The option depends only on stock price $S$ and time $t$. The stock price movement is assumed to follow geometric Brownian motion.

Using Ito’s lemma in Equation (3.2), $V(S, t)$ can be expressed as:

$$dV = \sigma S \frac{\partial V}{\partial S} dW_t + \left\{ \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right\} dt$$

(3.3)

Suppose that there is a risk-free portfolio with stock as the underlying asset. One can have a long position on one option $V$ with payoff $V(S, T) = \zeta(S)$ and a
short position in $\Delta$ amount of stock. The payoff function is $\zeta(S) = \max(S - K, 0)$ for call options and $\zeta(S) = \max(K - S, 0)$ for put options.

The value of the portfolio $\Pi$ at time $t$ is:

$$\Pi = V - \Delta S \quad (3.4)$$

where $\Delta$ is a constant. The change in the value of the portfolio over an infinitesimal time interval $dt$ is:

$$d\Pi = dV - \Delta dS \quad (3.5)$$

Combining Equations (3.1), (3.3), and (3.5), one obtains:

$$d\Pi = \sigma S \left\{ \frac{\partial V}{\partial S} - \Delta \right\} dW_t + \left\{ \mu S \frac{V}{S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right\} dt \quad (3.6)$$

To have riskless hedging of the portfolio, setting $\Delta = \frac{\partial V}{\partial S}$ will eliminate the stochastic part in Equation (3.5). The riskless hedging portfolio represents the minimum profit that the investor may have.

By imposing the arbitrage-free opportunity principle, portfolio $V$ must earn a risk-free interest rate $r$ and the growth in one time step $dt$ is:

$$d\Pi = r\Pi dt \quad (3.7)$$

Combining Equation (3.6) and (3.7) will result as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (3.8)$$

This equation is known as the Black-Scholes partial differential equation which represents the option price movement for both European and American options.
3.5 European Options

Following Equation (3.8) as the governing equation, standard European options can be solved by considering the conditions as follows [101]:

- final condition
  \[ C(S, T) = \max(S - K, 0) \]

- boundary conditions
  \[ C(0, t) = 0 \]
  \[ \lim_{x \to \infty} C(S, t) = S \]

Assuming that the interest rate \( r \) and volatility \( \sigma \) are constant in Equation (3.8), a closed-form solution is obtained:

\[ C(S, t) = S N(d_1) - Ke^{-r(T-t)} N(d_2) \]  \hspace{1cm} (3.9)

where \( N(x) \) is the standard normal cumulative distribution function as defined below:

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} \, dy \]  \hspace{1cm} (3.10)

and

\[ d_1 = \frac{\log \frac{S}{K} + \left( r + \frac{1}{2} \sigma^2 \right)(T-t)}{\sigma \sqrt{(T-t)}} \]  \hspace{1cm} (3.11)

\[ d_2 = \frac{\log \frac{S}{K} + \left( r - \frac{1}{2} \sigma^2 \right)(T-t)}{\sigma \sqrt{(T-t)}} \]  \hspace{1cm} (3.12)

If the option contract is extended into the case where the stock pays a continuous dividend at a fixed rate \( \delta \), the holder of the stock receives a dividend as an amount of \( \delta S dt \) in an infinitesimal time \( dt \). The Black-Scholes partial differential
equation incorporating the dividend payment is

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0
\]  (3.13)

and the value of the European call option with dividend payment is obtained as

\[
C(S, t) = e^{-\delta(T-t)}SN(z_1) - Ke^{-r(T-t)}N(z_2)
\]  (3.14)

where

\[
z_1 = \frac{\log S - (r - \delta + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{(T-t)}}
\]  (3.15)

\[
z_2 = \frac{\log S - (r - \delta - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{(T-t)}}
\]  (3.16)

### 3.6 American Options

#### 3.6.1 Optimal Exercise of American Options

American options have an additional feature where the option holder can exercise the options any time before maturity. When the stock price is optimal to exercise early, the American option problem becomes a free boundary problem. The optimal exercise price is time dependent and forms part of the solution [113].

An American call without dividend payment has the same value as its European counterpart. In this case the privilege of optimal exercise price does not exist, unless the American call option pays dividends. The value of an American option should be greater than its European counterpart. Otherwise there would be an arbitrage opportunity [101].

Suppose one has a dividend-paying American call option \( C(S, t) \) and the payoff function is \( max(S - K, 0) \) where \( S \) is the underlying asset price and \( K \) is the strike price. If the value of an American call option \( C(S, t) < max(S - K, 0) \), then the
call option holder can sell the stock at price $S$. At the same time, the call option holder can immediately exercise the option to use the right to buy the stock at price $K$. Then there is an instant profit which is calculated as $S - C - K > 0$. As the market is arbitrage-free, this situation cannot happen.

As a consequence of the principle, the constraint for an American option should be $C(S, t) > \max(S - K, 0)$ when early exercise is permitted. The holder of the American call option will find some values of $S$ which make the option value optimal before expiry. These values are called as the optimal exercise boundary.

The stock price $S$ which makes the option optimal at each time $t$, where $0 \leq S < \infty$ and $0 \leq t \leq T$, divides the pricing domain into two regions: the continuation region ($0 < S < S_f$) and the stopping region ($S_f < S < \infty$). The regions can be described as shown in Figure 3.1.

When the stock price $S = S_f(t)$, the option holder should exercise the option, otherwise the value will be too deep-in-the-money and the arbitrage-free principle cannot be satisfied. The optimal exercise boundary can be obtained when the profit is optimal and arbitrage-free.

As shown in Wilmott et al. [101], the optimal exercise boundary conditions
for the American call options are represented as

\[ V(S_f(t), t) = S_f(t) - K \]
\[ \frac{\partial V}{\partial S}(S_f(t), t) = 1 \]

These two conditions can be regarded as a tangential point where the value of the American call option meets the value of the payoff function. There is no more benefit to hold the option since it is already the same as the intrinsic value. Therefore the option can be exercised before maturity.

If a closed form solution of the optimal exercise price cannot be obtained in general, the optimal exercise price at \( t = T \) can be determined.

**Theorem 3.2.** [60] Suppose that the optimal exercise price for American options with dividend \( S = S_f(t) \) and \( 0 \leq t \leq T \), then

\[
S_f(T) = \begin{cases} 
\min \left\{ \frac{r}{\delta} K, K \right\}, & \text{(put options)} \\
\max \left\{ \frac{r}{\delta} K, K \right\}, & \text{(call options)}
\end{cases}
\]

(3.17)

where \( r \) is a risk free interest rate, \( \delta \) is dividend, and \( K \) is the strike price.

Since \( r \) could have negative value such as in the stock loans problem, Dai [32] proved that the theorem is still valid by using similar argument to Jiang [60]. In addition, Dai [32] showed that by using the continuous dependence of \( S_f \) and \( V \) on dividend \( \delta \) as \( T \to \infty \), the explicit solution of \( S_f \) as \( T \to \infty \) is identical to the optimal exercise boundary of American call options when \( r < \gamma \) and \( \delta > 0 \) where \( \gamma \) is the stock loan interest rate.

### 3.6.2 American Options Equations

The American options governing equation is the following equation (3.8) and the initial and boundary condition are the same as the conditions for European
options. The optimal exercise privilege (see Equation (3.17)) in American options makes the options become more complicated. There is no analytic solution available for American options except that of Zhu [114].

The complete PDE system of the American call options with dividend paid with the Black Scholes as the governing equation can be summarized as follows:

\[
\begin{align*}
&\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S(t) \frac{\partial V}{\partial S} - rV = 0 \\
&V(0, t) = 0 \\
&V(S, T) = \max(S - K, 0) \\
&V(S_f(t), t) = S_f(t) - K \\
&\frac{\partial V}{\partial S}(S_f(t), t) = 1
\end{align*}
\]

(3.18)

3.7 American Barrier Options

An American barrier option is an option where the payoff depends on whether the underlying assets breach the pre-set barrier during the lifetime of the option. The presence of a barrier reduces the holder’s opportunity to make a profit while the barrier gives more protection to the writer. To offset the holder smaller opportunity, the price of a barrier option is lower than the corresponding standard option thus making the barrier option more attractive.

Barrier options could be classified into two: ‘in’ and ‘out’ barrier options. The classification is based on how the options are ‘knocked’ and ‘alive’. An ‘in’ barrier option is alive when the underlying asset price knocks the barrier. Before the ‘in’ barrier option is knocked, the value of the option is nullified. On the contrary, an ‘out’ barrier option is terminated when the underlying asset price knocks the barrier. The value of the option is nullified as the barrier is knocked. These two
classes of barrier option can be divided into four main types of standard barrier options: \textit{up-and-out}, \textit{down-and-out}, \textit{up-and-in}, and \textit{down-and-in}. We focus on the American down-and-out call option with rebate since the margin call stock loan resembles this type of option.

An American down-and-out call option with rebate will be ‘knocked-out’ when the stock price $S$ falls significantly and touches the barrier $H$. In this situation, the option is terminated and the holder gets a rebate $R$ as the compensation. The amount of a rebate is determined in the beginning of the contract.

If the American down-and-out call option is not knocked out, the option can be exercised anytime before maturity as it reaches optimality. The holder will gain a profit from the early exercise. The optimal exercise privilege makes the valuation of this option more intriguing. Due to the presence of a barrier, the particular boundary condition for this option is $V(S = H, t) = R$ where $H$ is the predetermined barrier and $R$ is a rebate. This means that the option writer pays a rebate to the option holder when the option is knocked.

The governing equation for American down-and-out call options can be presented as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S(t)\frac{\partial V}{\partial S} - rV = 0$$

$$V(S, T) = \max(S - K, 0)$$

$$V(S_f(t), t) = S_f(t) - K$$

$$\frac{\partial V}{\partial S}(S_f(t), t) = 1$$

$$V(H, t) = R$$

(3.19)
3.8 American Options Under A Regime-Switching Economy

An American option under the standard Black-Scholes model assumes that the volatility is constant. The constant volatility cannot reflect the market dynamics precisely. Therefore a more appropriate model, such as a regime-switching model, is needed to better reflect the market environment. In a regime-switching model, the key parameters of an asset such as volatility is driven by the market regime. The market regime switches among a finite number of states which represent the change in financial markets.

Referring to Zhu et al. [116], we derive the American call option with regime-switching under the Black Scholes model. It is assumed that there are two regimes in the market which represent the economic condition: either in state of ’growth’ or ’recession’.

The derivation begins with the stock price dynamics incorporating the stochastic volatility. To accommodate an economic change $\varepsilon(t)$, the stock price dynamics in Equation (3.1) can be written as follows:

$$
\frac{dS}{S} = (\mu - \delta)dt + \sigma \varepsilon(t) dW_t ~ \tag{3.20}
$$

where $\varepsilon(t)$ is a stochastic process and represents the state of the economic cycle. $W_t$ is a standard Brownian motion and is independent of $\varepsilon(t)$.

The volatility $\sigma$ now is a stochastic process and has a different value for different states $\varepsilon(t)$. It is assumed that $\varepsilon(t)$ has finite state space $l, k \in I$ where $l \neq k$ and satisfies a Markov property. For a given present state, the future state is independent of the past state.

In mathematical formulation it can be written that for $t_0 < t_1 < t_2 < ... < t_n$
and $\varepsilon_0, ..., \varepsilon_n \in I$, 

$$P[\varepsilon(t + s) = l, \varepsilon(t_i) = \varepsilon_i] = P[\varepsilon(t + s) = l|\varepsilon(t) = k]$$  \hspace{1cm} (3.21)

It is assumed that there are two states representing two basic economic cycles $I = 1, 2$ then

$$\varepsilon(t) = \begin{cases} 
1, & \text{the economy is in a state of growth} \\
2, & \text{the economy is in a state of recession} 
\end{cases}$$

The value of an American call option in a state of growth $\varepsilon(t) = 1$ is $V_1$ and the volatility is $\sigma_{\varepsilon(t)} = \sigma_1$. When the condition is in a state of recession $\varepsilon(t) = 2$, the value of the option is represented by $V_2$ with volatility $\sigma_{\varepsilon(t)} = \sigma_2$.

Recalling the conditional probability $P[\varepsilon(t + s) = l|\varepsilon(t) = k], k, l \in I$, the transition probability from state $l$ to $k$, is denoted as $\lambda_{lk}$. A Poisson process with rate $\lambda$ is used to model the transition between two states with the following properties:

- $\varepsilon(0) = 1$
- The numbers of events occurring at disjointed time intervals are independent
- For a small time interval $\delta t$, the probability of a single event in the interval is given by:
  $$P[\varepsilon(t + \delta t) - \varepsilon(t) = 1] = \lambda \delta t + o(\delta t)$$  \hspace{1cm} (3.22)
  where $\lambda$ is the rate or intensity of the process.
- When more than a single event occurs in the same small time interval, the probability is negligible, in more formal term it is written as:
  $$P[\varepsilon(t + \delta t) - \varepsilon(t) \geq 2] = o(\delta t)$$  \hspace{1cm} (3.23)

Subsequently, the Poisson probability distribution with a rate $\lambda$ can be for-
mulated as follows:

\[ P[\xi(t+\delta t) - \xi(t) = k] = \frac{(\lambda t)_k e^{-\lambda t}}{k!} \]  
(3.24)

When the state changes from the state of growth to recession or vice versa, it is assumed that the transition between states can be illustrated as (Equation (3.22)): the jump from \( \xi(t) = 1 \) to \( \xi(t) = 2 \) occurs as a Poisson process with rate \( \lambda_{12} \) or conversely \( \xi(t) = 2 \) to \( \xi(t) = 1 \) occurs as a Poisson process with rate \( \lambda_{21} \).

The value of an option, \( V(S, t) \) depends on the current economic state and takes one of the two values \( V_1(S, t) \) or \( V_2(S, t) \). For example when \( \xi(t) = 1 \) then the value of the option is \( V_1(S, t) \). The probability of a change in state for an infinitesimal time \( dt \) is \( \lambda_{12} dt \) and the value of the option is:

\[
V(S(t+dt), t+dt) = \begin{cases} 
V_1(S(t+dt), t+dt), & \text{with probability } 1 - \lambda_{12} dt \\
V_2(S(t+dt), t+dt), & \text{with probability } \lambda_{12} dt 
\end{cases}
\]

and consequently the difference of the option value at interval \([t, t+dt]\) is written as follows

\[
dV = \begin{cases} 
V_1(S(t+dt), t+dt) - V_1(S(t), t), & \text{with probability } 1 - \lambda_{12} dt \\
V_2(S(t+dt), t+dt) - V_1(S(t), t), & \text{with probability } \lambda_{12} dt 
\end{cases}
\]
or

\[
dV = \begin{cases} 
dV_1, & \text{with probability } 1 - \lambda_{12} dt \\
dV_2 + V_2(S(t), t) - V_1(S(t), t), & \text{with probability } \lambda_{12} dt 
\end{cases}
\]

Suppose that the portfolio is constructed following the replicating strategy as in Section (3.4), the value of a riskless portfolio, consisting of an option and \( \Delta \) amount of stocks, is defined as

\[ \Pi = V - \Delta S \]  
(3.25)

where \( \Delta \) is a constant. The change in the value of the portfolio over infinitesimal
time interval $dt$ is:

$$d\Pi = dV - \Delta dS - \delta \Delta S dt$$

$$= \left\{ \begin{array}{ll}
        dV_1 - \Delta dS - \delta \Delta S dt, & \text{with probability } 1 - \lambda_{12} dt \\
        V_2 - V_1 + dV_2 - \Delta dS - \delta \Delta S dt, & \text{with probability } \lambda_{12} dt 
\end{array} \right.$$  

Taking the expected value of the portfolio and recalling the stochastic differential equation with regime-switching economy in Equation (3.20) results in:

$$E[d\Pi] = E[(dV_1 - \Delta dS - \delta \Delta S dt)(1 - \lambda_{12} dt) + ... + (V_2 - V_1 + dV_2 - \Delta dS - \delta \Delta S dt)\lambda_{12} dt]$$

$$= E[dV_1 - \Delta dS - \delta \Delta S dt + (V_2 - V_1 + dV_2 - dV_1)\lambda_{12} dt]$$

Applying Ito’s lemma for a function of the portfolio value $V_{1,2}(S, t)$ and referring to Equation (3.2) results in:

$$dV_{1,2}(S, t) = \left\{ \frac{\partial V_{1,2}}{\partial t} + \mu S \frac{\partial V_{1,2}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_{1,2}}{\partial S^2} \right\} dt + \sigma S \frac{\partial V_{1,2}}{\partial S} dW_t \quad (3.25)$$

Substituting Equation (3.25) from Equation (3.8) and taking $\Delta = \frac{\partial V_1}{\partial S}$ reduces the random part and the equation becomes completely deterministic which is known and is written as follows:

$$E[d\Pi] = \left\{ \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} \right\} dt + (V_2 - V_1)\lambda_{12} dt \quad (3.26)$$

It should be pointed out that the expected change in the portfolio in the left hand side of Equation (3.26) is no longer known with certainty. Therefore the free of arbitrage principle commonly used in the standard Black-Scholes model is no longer valid in this case. As an alternative, the derivation of the Black Scholes model with a regime-switching economy adopts a risk-neutral argument. It implies that the investors have the same risks and require no compensation for the risks. In the valuation, it is assumed that the risk associated with volatility is
not priced [84][86]. The expected return of the underlying assets is the risk-free
interest rate and this can be written as

$$E[d\Pi] = r\Pi dt = r\left(V_1 - S\frac{\partial V_1}{\partial S}\right) dt \quad (3.27)$$

Equation (3.27) shows that investors expect to gain maximum profit when they
invest their money in a risk-free environment.

Combining Equations (3.27) and (3.26) results in a modified Black-Scholes
PDE for $V_1(S,t)$ as follows:

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \delta)S \frac{\partial V_1}{\partial S} - rV_1 = \lambda_{12}(V_1 - V_2) \quad (3.28)$$

A similar procedure is used to obtain the PDE for $V_2(S,t)$:

$$\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - \delta)S \frac{\partial V_2}{\partial S} - rV_2 = \lambda_{21}(V_2 - V_1) \quad (3.29)$$

These modified Black-Scholes equations (3.28 and 3.29) are the governing equa-
tions for American call options incorporating the two-state regime-switching volatil-
ities.

The privilege of the optimal exercise right of the American call options in-
corporating the switching adds to the complexity of the valuation. Representing
the optimal exercise price in each state, there will be two free boundaries in the
valuation. The free boundaries are represented as

$$V_{1,2}(S_{f1,2}(t), t) = S_{f1,2} - K$$
$$\frac{\partial V_{1,2}}{\partial S}(S_{f1,2}(t), t) = 1$$

The boundary conditions of the options are $V_{1,2}(0, t) = 0$ and the terminal con-
ditions are written as $V_{1,2}(S, T) = \max(S - K, 0)$.

In summary the complete Black-Scholes equation system for American call
options under a two-state regime-switching economy can be presented as follows:

\[
\begin{aligned}
\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \delta) S \frac{\partial V_1}{\partial S} - r V_1 &= \lambda_{12}(V_1 - V_2) \\
V_1(0, t) &= 0 \\
V_1(S, T) &= max(S - K, T) \\
V_1(S_{f1}(t), t) &= S_{f1}(t) - K \\
\frac{\partial V_2}{\partial S}(S_{f1}(t), t) &= 1 \\
\frac{\partial V_2}{\partial t} + \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - \delta) S \frac{\partial V_2}{\partial S} - r V_2 &= \lambda_{21}(V_2 - V_1) \\
V_2(0, t) &= 0 \\
V_2(S, T) &= max(S - K, T) \\
V_2(S_{f2}(t), t) &= S_{f2}(t) - K \\
\frac{\partial V_2}{\partial S}(S_{f2}(t), t) &= 1
\end{aligned}
\]
Chapter 4

Semi-analytic Valuation of Stock Loans with Finite Maturity

4.1 Introduction

A stock loan is an interesting alternative for investors to increase liquidity from stock without selling the stock. The lender, a private company or a bank, offers loans with stock as collateral. The borrower gives custody of the stock to the lender without losing ownership. However, the lender has the right to take over the collateral if the borrower fails to pay the loan. The borrower may retrieve the stock any time before maturity or at maturity by repaying the loan and accumulated interest at a predetermined interest rate. If the stock price increases at anytime, the borrower is able to pay back the loan, retrieve the stock and get the unlimited upside potential profit. On the other hand, if the stock price decreases below the accumulated loan, the borrower can walk away by surrendering the stock (non-recourse loan). In this case the borrower only loses service fee paid at the beginning of the contract.

Xia and Zhou [104] initiated the study of a non-recourse or standard stock loan valuation as an option problem. They considered the standard stock loan as a perpetual American call option with time dependent strike price and possibly
negative interest rate. The valuation of the loan was carried out by using a probabilistic approach. Grasselli and Valez [45] extended the study to stock loans in incomplete market. They provided an explicit expression to calculate the service fee in infinite maturity and used numerical method for their valuation of stock loans of finite maturity. Stock loans with non-standard features have also been studied as option pricing problems recently, such as margin call stock loan [34], capped stock loan [71], and stock loan with automatic termination clause, cap and margin [61].

In practice, a stock loan has finite lifetime, but most research in this area is carried out assuming infinite maturity [34, 61, 71, 104, 118], only few studies that attempt to solve stock loan problems at finite maturity. Prager and Zhang [90] proposed a model of stock loan valuation under regime switching between Geometric Brownian Motion and Mean Reversion. However, the European style of the model drops an important feature of the stock loan, the borrower’s early exit possibility. Focusing on optimal exit strategy with various dividend distribution, Dai and Xu [32] presented a finite stock loan valuation, their results are calculated by using the binomial method. Grasselli and Valez [45] used finite difference method with PSOR (projected successive over relaxation) for their valuation of stock loans of finite maturity in incomplete market. Nevertheless, the existing valuation methods for finite maturity stock loans are purely numerical. In this study, we provide a semi-analytical evaluation of standard stock loans with finite maturity by using the method proposed in [113] for valuation of American options. Optimal exit boundaries and stock loan values for some different dividend distributions as discussed in Dai [32] are calculated more efficiently while maintaining the accuracy. This paper is organized as follows: Section 2 will focus on the connection of stock loans with American options. Section 3 presents the formulation of stock loan problems using the semi-analytic method in three dif-
ferent dividend distributions. Section 4 provides some numerical examples and Section 5 the conclusion.

4.2 Stock Loan and Its American Connection

The stock loan mechanism resembles that of American call option as discussed in [104]. The borrower who can exit from the contract anytime can be considered as the holder of a call option and the lender who is obligated to return the stock to the borrower can be seen as the writer of the option. When the stock price falls sharply below the loan, the borrower can surrender it to the lender and walk away. This feature is similar to an unexercised option that limits the loss of the holder to only the premium. If the stock price increases, the borrower can repay the loan to exit the contract and retrieve the stock to make a profit. This feature can be considered as an early exercise of a call option.

Due to the resemblance of the mechanisms of stock loans and American call options, the valuation of stock loan in finite maturity can be considered as an American style option problem with time dependent strike price. This motivated us to review the studies about American option pricing. Although extensive research on American option price and its early exercise possibility has been done, there is generally no closed-form solution for American options except in some special cases, such as non-dividend perpetual American put option obtained by Merton [85] and a non-dividend American put by Zhu [117]. The difficulty in the analysis of American options lies in the free boundary due to the the early exercise right. Existing solutions to American option pricing problems could be categorized into two basic types, numerical methods and analytical approximation methods. Numerical methods provide flexibility but require intensive computation for feasible results [15, 30, 74]. On the other hand, analytical approximation methods such as in [19, 25, 40, 56, 67, 77, 113] need less computation time than
fully numerical methods, but provide reasonable accurate results. Lauko and Sevcovic [69] provided a comparative study on some numerical and analytical approximation methods, and showed that Zhu’s method [113] is the most recommended in calculating the optimal exercise boundary of American option of long-term maturity. This method is particularly suitable for stock loan analysis, as the loans in general have longer maturity than options. Zhu’s solution is a simple closed-form approximation formula of non-dividend American put option by using Laplace transform. In this study, pseudo-steady state approximation is used to deal with the free boundary. An optimal exercise price formula was obtained explicitly whereby the option price can be calculated directly in the original time space. The simple exact approximation formula provides higher efficiency at reasonable accuracy. Zhu and Zhang [119] extended the method in [113] to pricing American puts with dividend. Lu and Rhodes applied this method in their study of American down-and-out call option [76]. In this study, we will apply Zhu’s approximation method [113] to obtain optimal exit price and stock loan value of a standard stock loan of finite maturity.

4.3 Stock Loan Formulation

Throughout this section we assume that the risk-neutral stock price process is described by the following stochastic differential equation (Geometric Brownian Motion)

$$dS = (r - \delta) S \, dt + \sigma S \, dW_t$$  \hspace{1cm} (4.1)

where $W_t$ is a Wiener process defined for $t \in [0, \infty)$, $r \geq 0$ is the risk-free interest rate, $\delta \geq 0$ the continuous dividend yield, and $\sigma$ the volatility of the stock.

It is assumed that at time 0, a client borrows amount $q$ at a predetermined interest rate $\gamma$ with one share of stock valued at $S_0$ as collateral, and pays a service fee, $c$. At any time $t \ (t \geq 0)$, the borrower owes the amount $qe^{\gamma t}$ (loan
principal plus accumulated interest).

By employing Ito’s lemma and arbitrage-free opportunity principle, we obtain the partial differential equation (PDE) for a finite maturity stock loan:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0$$ (4.2)

with boundary condition $V(0, t) = 0$, as when the stock price $S = 0$, the stock loan has no value. Equation (4.2) clearly resembles the Black-Scholes PDE in option pricing. However, in stock loan evaluation the terminal condition or the pay off function at maturity depends on the way the dividend is distributed to the borrower as discussed in Dai [32]. The early exit possibility in stock loans provides additional boundary conditions at $S = S_f(t)$ (optimal exit price, yet to be determined), which lead to a free (moving) boundary value problem similar to that of an American option pricing problem. In the following subsections, we will present the formulation and evaluation of finite stock loans with various dividend payment distributions to show how the optimal exit price and stock loan value are affected by the dividend payment method.

### 4.3.1 The Lender Collects Dividend Before Redemption

If the lender collects the dividend and pays it back to the borrower at maturity or early exit, the intrinsic value of stock loan at time $t$ is simply the difference between stock price $S$ and the accumulated loan amount $qe^{\alpha t}$, that is, $V(t) = \max(S - qe^{\alpha t}, 0)$. This is obviously similar to the intrinsic value of an American call option with a time dependent strike price.

The following partial differential equation system defines the stock loan problem in this section:
\[
\begin{align*}
\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S_1^2 \frac{\partial^2 V_1}{\partial S_1^2} + (r - \delta) S_1 \frac{\partial V_1}{\partial S_1} - rV_1 &= 0 \\
V_1(0, t) &= 0 \\
V_1(S_1, T) &= \max(S_1 - qe^{\gamma T}, 0) \\
V_1(S_{f1}(t), t) &= S_{f1}(t) - qe^{\gamma t} \\
\frac{\partial V_1}{\partial S_1}(S_{f1}(t), t) &= 1
\end{align*}
\]

(4.3)

where subscript 1 in \(V_1(S_1, t)\) and \(S_1\) indicates the stock loan value and the stock price in case 1, respectively. At maturity, the value of the stock loan is \(V_1(S_1, T) = \max(S_1 - qe^{\gamma T}, 0)\). The optimal exit boundary conditions are defined by the last two conditions in Equation (4.3). If \(S_1 \geq S_{f1}(t)\), the borrower should exit the loan contract.

Applying the following change of variables

\[
S_1 = X_1 qe^{\gamma t}, \quad V_1(S_1, t) = U_1(X_1, \tau) qe^{\gamma t}, \quad t = T - \frac{2\tau}{\sigma^2},
\]

we obtain a simpler differential equation system with initial and boundary conditions:

\[
\begin{align*}
- \frac{\partial U_1}{\partial \tau} + X_1^2 \frac{\partial^2 U_1}{\partial X_1^2} + (\alpha - \beta) X_1 \frac{\partial U_1}{\partial X_1} - \alpha U_1 &= 0 \\
U_1(0, \tau) &= 0 \\
U_1(X_1, 0) &= \max(X_1 - 1, 0) \\
U_1(X_{f1}(\tau), \tau) &= X_{f1}(\tau) - 1 \\
\frac{\partial U_1}{\partial X_1}(X_{f1}(\tau), \tau) &= 1
\end{align*}
\]

(4.4)

where \(\alpha = \frac{2(r - \gamma)}{\sigma^2}\), \(\beta = \frac{2\delta}{\sigma^2}\), and \(\tau\) is the dimensionless time to maturity.

Zhu’s approach [113] is applied to solve this differential equation system. For the easiness of reading, the details of the technique are repeated here. To make
the system more manageable, a function \( \tilde{V}_i(X, \tau) \) is introduced as

\[
\tilde{V}_i = \begin{cases} 
U_1 + 1 - X_1, & \text{if } 1 < X_1 \leq X_{f1}; \\
U_1, & \text{if } X_1 \leq 1;
\end{cases}
\]

Dropping all tildes in the new function, we obtain the following differential equation systems

\[
\begin{cases}
-\frac{\partial V_1}{\partial \tau} + X_1^2 \frac{\partial^2 V_1}{\partial X_1^2} + (\alpha - \beta)X_1 \frac{\partial V_1}{\partial X_1} - \alpha V_1 = \beta X_1 - \alpha, \\
V_1(X_1, 0) = 0, \\
V_1(X_{f1}(\tau), \tau) = 0, \\
\frac{\partial V_1}{\partial X_1}(X_{f1}(\tau), \tau) = 0.
\end{cases}
\]

\( \text{(4.5)} \)

\[
\begin{cases}
-\frac{\partial V_1}{\partial \tau} + X_1^2 \frac{\partial^2 V_1}{\partial X_1^2} + (\alpha - \beta)X_1 \frac{\partial V_1}{\partial X_1} - \alpha V_1 = 0, \\
V_1(X_1, 0) = 0, \\
V_1(0, \tau) = 0.
\end{cases}
\]

\( \text{(4.6)} \)

with the following continuity conditions

\[
\begin{cases}
\lim_{x \to 1^-} V_1 = \lim_{x \to 1^+} V_1 \\
\lim_{x \to 1^-} \frac{\partial V_1}{\partial X_1} = \lim_{x \to 1^+} \frac{\partial V_1}{\partial X_1} + 1
\end{cases}
\]

\( \text{(4.7)} \)

The boundary and initial conditions in Equations (4.5 - 4.6) are homogeneous, which are easier to be dealt with in the solution process.

We will apply Laplace transform with respect to the time variable \( \tau \) on the
system of equations (4.5 - 4.6) with following notations:

\[
\mathcal{L} V_1(X_1, \tau) = \int_0^\infty e^{-\nu \tau} V_1(X_1, \tau) d\tau = \hat{V}_1(X_1, p)
\]

\[
\mathcal{L} X_{f1}(\tau) = \int_0^\infty e^{-\nu \tau} X_{f1}(\tau) d\tau = \hat{X}_{f1}(p)
\]

We now have a system of ordinary differential equations (ODE) with continuity conditions in Laplace space:

\[
\begin{cases}
  X_1^2 \frac{d\hat{V}_1(X_1, p)}{dX_1^2} + (\alpha - \beta) X_1 \frac{d\hat{V}_1(X_1, p)}{dX_1} - (\alpha + p) \hat{V}_1(X_1, p) = \frac{\beta X_1 - \alpha}{p}, \\
  \hat{V}_1(p \hat{X}_{f1}, p) = 0, \\
  \frac{d\hat{V}_1}{dX_1}(p \hat{X}_{f1}, p) = 0.
\end{cases}
\]  

(4.8)

\[
\begin{cases}
  X_1^2 \frac{d\hat{V}_1(X_1, p)}{dX_1^2} + (\alpha - \beta) X_1 \frac{d\hat{V}_1(X_1, p)}{dX_1} - (\alpha + p) \hat{V}_1(X_1, p) = 0, \\
  \hat{V}_1(0, p) = 0.
\end{cases}
\]  

(4.9)

\[
\begin{cases}
  \hat{V}_1(1^-, p) = \hat{V}_1(1^+, p) \\
  \frac{d\hat{V}_1}{dX_1}(1^-, p) = \frac{d\hat{V}_1}{dX_1}(1^+, p) + \frac{1}{p}
\end{cases}
\]  

(4.10)

Since the behavior of the optimal exit boundary \( X_{f1} \) is similar to the moving boundary in Stefan problem, approximation is made on the moving boundary in the original time space following Zhu’s approach [113], that is, \( X_{f1}(\tau) \) is replaced in the Laplace space by \( p \hat{X}_{f1}(p) \).

The solution of the differential equations (4.8) and (4.9) in Laplace space is

\[
\hat{V}_1 = \begin{cases}
  C_1 X_1^{k_1} + C_2 X_1^{k_2} + \frac{p(\alpha - \beta X_1) + \alpha \beta (1 - X_1)}{p(p + \alpha)(p + \beta)}, & \text{if } 1 < X_1 \leq X_{f1} \\
  C_3 X_1^{k_3} + C_4 X_1^{k_2}, & \text{if } X_1 \leq 1
\end{cases}
\]  

(4.11)
where
\[
\begin{align*}
k_{1,2} &= \frac{1 + \beta - \alpha}{2} \pm \sqrt{\left(\frac{1 + \beta - \alpha}{2}\right)^2 + (p + \alpha)} \quad (4.12)
\end{align*}
\]

Note the value of \( C_4 \) is set to be zero using \( \frac{\partial \hat{V}_1}{\partial X_1}(0, p) = 0 \), following an argument similar to that in [32]. Nevertheless, \( \hat{V}_1(0, p) = 0 \) in Equation (5.3) is satisfied automatically with \( C_4 = 0 \).

Applying the other boundary conditions, we obtain a set of algebraic equations for the constants \( C_1, C_2, \) and \( C_3 \), and the dimensionless exit price, \( \hat{X}_{f1} \),
\[
\begin{align*}
C_1(p\hat{X}_{f1})^{k_1} + C_2(p\hat{X}_{f1})^{k_2} + \frac{p(\alpha - \beta(p\hat{X}_{f1}) + \alpha\beta(1 - (p\hat{X}_{f1}))}{p(p + \alpha)(p + \beta)} &= 0 \\
C_1k_1(p\hat{X}_{f1})^{k_1-1} + C_2k_2(p\hat{X}_{f1})^{k_2-1} - \frac{\beta}{p(p + \beta)} &= 0 \\
C_1 + C_2 + \frac{(\alpha - \beta)}{(p + \alpha)(p + \beta)} &= C_3 \\
C_1k_1 + C_2k_2 - \frac{\beta}{p(p + \beta)} &= C_3k_1 - \frac{1}{p}
\end{align*}
\]
The values of constant \( C_1, C_2, \) and \( C_3 \) are determined as follows
\[
\begin{align*}
C_1 &= (p\hat{X}_{f1})^{k_2-k_1} \frac{k_2}{k_2-k_1}W \\
C_2 &= (p\hat{X}_{f1})^{1-k_2} \frac{\beta}{k_2p(p + \beta)} - \frac{k_1}{k_2-k_1}W \\
C_3 &= (1 - (p\hat{X}_{f1})^{k_2-k_1}) \frac{k_2}{k_2-k_1}W + \frac{(p\hat{X}_{f1})^{1-k_2}\beta + p}{k_1p(p + \beta)}
\end{align*}
\]

where
\[
W = \frac{\beta - \alpha}{(p + \alpha)(p + \beta)} + \frac{(p\hat{X}_{f1})^{1-k_2}\beta + p}{k_1p(p + \beta)} - \frac{(p\hat{X}_{f1})^{1-k_2}\beta}{k_2p(p + \beta)}
\]

By substituting the values of constant \( C_1, C_2, \) and \( C_3 \) in Equation (4.11), the value of stock loan in Laplace space is obtained as
CHAPTER 4. SEMI-ANALYTIC VALUATION OF STOCK LOANS WITH FINITE MATURITY

\[
\hat{V}_1 = \begin{cases} 
\frac{k_2 (p \hat{X}_f^1)^{k_2 - k_1}}{k_2 - k_1} W X_1^{k_1} + \frac{\beta (p \hat{X}_f^1)^{1 - k_2}}{k_2 p (p + \beta)} X_1^{k_2} + \frac{p (\alpha - \beta X_1) + \alpha \beta (1 - X_1)}{p (p + \alpha) (p + \beta)}, \\
\frac{k_2 (p \hat{X}_f^1)^{k_2 - k_1}}{k_2 - k_1} W + \frac{(p \hat{X}_f^1)^{1 - k_2} \beta + p}{k_1 p (p + \beta)} X_1^{k_1}, & \text{if } 1 < X_1 \leq X_f^1 \\
(1 - (p \hat{X}_f^1)^{k_2 - k_1}) \frac{k_2}{k_2 - k_1} W + \frac{(p \hat{X}_f^1)^{1 - k_2} \beta + p}{k_1 p (p + \beta)} X_1^{k_1}, & \text{if } X_1 \leq 1
\end{cases}
\]

(4.13)

More importantly, the equation for optimal exit price is obtained explicitly in Laplace space,

\[
\hat{X}_f^{k_2} \left\{ \frac{k_1 (\beta - \alpha) + (p + \alpha)}{k_1 (p + \alpha) (p + \beta)} \right\} + \hat{X}_f \left\{ \frac{\beta (1 - k_1)}{k_1 p^{k_2} (p + \beta)} \right\} = -\frac{\alpha}{p^{1 + k_2} (p + \alpha)}
\]

(4.14)

However, due to the highly non-linear feature of the equation, its solution could not be solved explicitly.

Let \( \lim_{\tau \to \infty} X_{f1}(\tau) = X_{f1\infty} \) be the optimal exit price for the perpetual stock loan. The explicit optimal exit formula discussed in Xia and Zhou [104] can be recovered from Equation (A-2) after the application of the final value theorem of Laplace transform

\[
X_{f1\infty} = \begin{cases} 
\frac{k_{10}}{k_{10} - 1}, & \text{if } r \geq \gamma \text{ and } \delta > 0 \text{ or } \delta = 0 \text{ and } r < \gamma - \frac{1}{2} \sigma^2 \\
+\infty, & \text{if } \delta = 0 \text{ and } \gamma - \frac{1}{2} \sigma^2 \leq r < \gamma
\end{cases}
\]

(4.15)

where \( k_{10} \) is the value of \( k_1 \) in Equation (4.12) with \( p = 0 \), and in the original variables

\[
k_{10} = \frac{-(r - \delta - \gamma - \frac{1}{2} \sigma^2) + \sqrt{(r - \delta - \gamma + \frac{1}{2} \sigma^2)^2 + 2 \sigma^2 \delta}}{\sigma^2}
\]

(4.16)
4.3.2 Dividend Reinvested Before Redemption

In this case, the dividend is reinvested immediately after being paid, and it will be returned as part of the collateral at redemption. The stock price movement is represented by

\[ e^{\delta t} S = S_0 \exp(rt + \sigma W_t - \frac{1}{2} \sigma^2 t) \]  

(4.17)

Therefore, the payoff function of this stock loan is

\[ \max(e^{\delta t} S - q e^{\gamma t}, 0), \quad t \in [0, T] \]

Let \( S_2 = e^{\delta t} S \) be the effective stock price for the collateral at time \( t \) for this stock loan problem, where \( S \), the stock price, follows Geometric Brownian Motion as in Equation (4.1). The stock loan PDE in Equation (4.2) becomes

\[
\begin{align*}
\frac{\partial V_2}{\partial t} + \frac{1}{2} \sigma^2 S_2^2 \frac{\partial^2 V_2}{\partial S_2^2} + r S_2 \frac{\partial V_2}{\partial S_2} - r V_2 &= 0 \\
V_2(0, t) &= 0 \\
V_2(S_2, T) &= \max(S_2 - q e^{\gamma T}, 0) \\
V_2(S_{f2}(t), t) &= S_{f2} - q e^{\gamma t} \\
\frac{\partial V_2}{\partial S_2}(S_{f2}(t), t) &= 1
\end{align*}
\]  

(4.18)

As pointed out by Dai [32], the stock loan in this case can be regarded as the one written on the non-dividend paying asset \( S_2 \). Similar change of variables as in Section (4.3.1) is used to reduce (4.18) to the following dimensionless PDE system:
\[
\begin{align*}
- \frac{\partial U_2}{\partial \tau} + X_2^2 \frac{\partial^2 U_1}{\partial X_2^2} + \alpha X_2 \frac{\partial U_2}{\partial X_2} - \alpha U_2 &= 0 \\
U_2(0, \tau) &= 0 \\
U_2(X_2, 0) &= \max(X_2(0) - 1, 0) \\
U_2(X_{f2}(\tau), \tau) &= X_{f2}(\tau) - 1 \\
\frac{\partial U_2}{\partial X_2}(X_{f2}(\tau), \tau) &= 1
\end{align*}
\]

(4.19)

where \( \alpha = \frac{2(r - \gamma)}{\sigma^2} \).

The PDE in (4.18) or (4.19) appears to be representing an American style call option without dividend payment, which should behave like a European option, as \( \delta \) does not appear explicitly. However, optimality still exists for values of \( \alpha < 0 \), that is, \( r < \gamma \), which is usually true for stock loans.

The solution of PDE system in (4.19) is obtained using the same procedure as that in Section (4.3.1). The value of the stock loan in Laplace space with dividend reinvested immediately is:

\[
\hat{V}_2 = \begin{cases} 
\frac{k_2(p \hat{X}_{f2})^{k_2-k_1}}{k_2-k_1} W X_2^{k_1} + \frac{\beta(p \hat{X}_{f2})^{1-k_2}}{k_2 p (p + \beta)} X_2^{k_2} + \frac{p(\alpha - \beta X_2) + \alpha \beta (1 - X_2)}{p(p + \alpha) (p + \beta)} & \text{if } 1 < X_2 \leq X_{f2} \\
\left\{ (1 - (p \hat{X}_{f2})^{k_2-k_1}) \frac{k_2}{k_2-k_1} W + \frac{(p \hat{X}_{f2})^{1-k_2} + p}{k_1 p (p + \beta)} \right\} X_2^{k_1} & \text{if } X_2 \leq 1
\end{cases}
\]

(4.20)

where

\[
W = \frac{-\alpha}{(p + \alpha) (p)} + \frac{(p \hat{X}_{f2})^{1-k_2} + p}{k_1 p^2}
\]

An explicit expression for the optimal exit price in Laplace space is obtained in this case:
\[ \hat{X}_{f_2} = \frac{1}{p} \left\{ \frac{\alpha k_1}{\alpha k_1 - (p + \alpha)} \right\} \frac{1}{k_2} \]  

(4.21)

where \( k_1 \) and \( k_2 \) are given by

\[ k_{1,2} = \frac{1 - \alpha}{2} \pm \sqrt{\left( \frac{1 - \alpha}{2} \right)^2 + (p + \alpha)} \]

It is worthy to point out that the optimal exit price in Laplace space in (4.21) can be inverted analytically to obtain an explicit formula in the original time space if \( r < \gamma - \frac{1}{2} \sigma^2 \), and therefore a close form analytical stock loan value can also be obtained for this case.

The optimal exit price for a perpetual stock loan is again recovered as

\[ X_{f_{2\infty}} = \begin{cases} \frac{\alpha}{\alpha + 1}, & \text{if } r < \gamma - \frac{1}{2} \sigma^2 \\ +\infty, & \text{if } \gamma - \frac{1}{2} \sigma^2 \leq r < \gamma \end{cases} \]  

(4.22)

### 4.3.3 Dividend Delivered to The Borrower

Since the dividend payment is continuous, during a time interval \( du \) the borrower is paid \( \delta S_u e^{r(t-u)} du \), \( 0 < u < t < T \). Therefore, at time \( t > 0 \), the borrower will receive cash in the amount

\[ I_t = \int_{u=0}^{u=t} \delta S_u e^{r(t-u)} du \]

The intrinsic value of stock loan is

\[ \max(S - qe^{r t}, 0) + I_t, \quad t \in [0, T] \]  

(4.23)
Let $H(S, t) = V_3(S, I, t) - I$, we obtain a modified PDE in terms of $H(S, t)$:

$$
\begin{align*}
\left\{ \frac{\partial H}{\partial t} + (r - \delta)S \frac{\partial H}{\partial S} + \frac{1}{2} \frac{\partial^2 H}{\partial S^2} \sigma^2 S^2 - rH \right\} + \delta S &= 0 \\
H(0, t) &= 0 \\
H(S, T) &= \max(S - qe^{\gamma T}, 0) \quad (4.24) \\
H(S_{f3}(t), t) &= S_{f3}(t) - qe^{\gamma t} \\
\frac{\partial H}{\partial S}(S_{f3}(t), t) &= 1
\end{align*}
$$

By using the same dimensionless variables in Section (4.3.1), the PDE system (4.24) is changed into the following

$$
\begin{align*}
\left\{ - \frac{\partial \tilde{H}}{\partial \tau} + (\alpha - \beta)X \frac{\partial \tilde{H}}{\partial X} + X^2 \frac{\partial^2 \tilde{H}}{\partial X^2} - \alpha \tilde{H} \right\} + \beta X(\tau) &= 0 \\
\tilde{H}(0, \tau) &= 0 \\
\tilde{H}(X_3, 0) &= \max(X_3 - 1, 0) \quad (4.25) \\
\tilde{H}(X_{f3}(\tau), \tau) &= X_{f3}(\tau) - 1 \\
\frac{\partial \tilde{H}}{\partial X}(X_{f3}(\tau), t) &= 1
\end{align*}
$$

where $\alpha = \frac{2(r - \gamma)}{\sigma^2}$, $\beta = \frac{2\delta}{\sigma^2}$.

The solution of PDE system in Equation (4.25) in Laplace space are obtained as follows

$$
\tilde{H} = \begin{cases} \\
\frac{k_2}{k_2 - k_1} (p\hat{X}_{f3})^{k_2-k_1} W X_3^{k_1} + \frac{k_2}{k_2 - k_1} W X_3^{k_2} + \frac{\alpha}{p(p + \alpha)}, & \text{if } 1 < X_3 \leq X_{f3} \\
\frac{k_2}{k_2 - k_1} W [(p\hat{X}_{f3})^{k_2-k_1} + 1] + \frac{1}{k_1(p + \beta)} X_3^{k_1} + \frac{\beta X_3}{p(p + \beta)}, & \text{if } X_3 \leq 1
\end{cases} \quad (4.26)
$$
where \( W = \frac{-\alpha}{p(p+\alpha)} + \frac{(p\hat{X}_{f3})^{1-k_2} + p}{k_1p^2} \)

We again obtained an explicit expression for the optimal exit price

\[
\hat{X}_{f3} = \frac{1}{p} \left\{ \frac{\alpha k_1(p+\beta)}{p[k_1(\alpha-\beta) - (p+\alpha)]} \right\}^{1/k_2}
\]

(4.27)

where

\[
k_{1,2} = \frac{1 + \beta - \alpha}{2} \pm \sqrt{\left(\frac{1 + \beta - \alpha}{2}\right)^2 + (p + \alpha)}
\]

(4.28)

Since \( \lim_{p \to 0} p\hat{X}_{f3} \) not exist, the value of perpetual optimal exit price, \( X_{f3\infty} \), does not exist either. Therefore, there is no optimality for a perpetual stock loan with this type of dividend distribution.

### 4.4 Numerical Results and Discussions

In this section, numerical results for the cases discussed in the previous section are presented to show the effects of different dividend payment methods on optimal exit price and stock loan value. The values of the fair service fee charged to the borrower for case 1 are also calculated.

We apply numerical inversion to transform the optimal exit price and stock loan value in Laplace space back to the original space. Stehfest algorithm is chosen due to its simplicity and ease of use as stated in [119]. The algorithm is written as

\[
\begin{align*}
X(t) & \approx \frac{[\ln^2 t]}{t} \sum_{n=1}^{N} V_n \hat{X}_f \left( \frac{n\ln^2 t}{t} \right) \\
V_n & = (-1)^{2+n} \sum_{k=\min(n,N/2)}^{n+1} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(n-k)!(2k-n)!}
\end{align*}
\]

(4.29)

where \( N \) is an even number, \( n \) is an integer, \( 1 \leq n \leq N \), and \( k \) is the greatest integer less than or equal to \( (n+1)/2 \). \( N \) being even is required for convergence acceleration, and \( N = 8 \) is used in our calculation because it gives better results.
in terms of accuracy and speed.

### 4.4.1 Optimal Exit Price

First we present a comparison of our results for the optimal exit price for Section (4.3.1) with those calculated by binomial tree method to verify the accuracy and efficiency of our semi-analytic method. Table 1 shows the results calculated on a PC with the following details: Intel(R)Core(TM)i7, CPU 860@2.80GHz 2.79 GHz, 4 GB RAM, Windows 7 enterprise service pack 1, 64 bit operating system.

**Table 4.1: Optimal Exit Price in Section (4.3.1) at \( t = 0 \)**

\( \gamma = 0.1, \delta = 0.03, \sigma = 0.4, r = 0.06, q = 0.7 \)

<table>
<thead>
<tr>
<th>( T ) (year)</th>
<th>Semi-analytic</th>
<th>Binomial(( N = 10,000 ))</th>
<th>Exact value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( S_f )</td>
<td>Computation time (sec)</td>
<td>( S_f )</td>
</tr>
<tr>
<td>1</td>
<td>1.197</td>
<td>0.21</td>
<td>1.163-1.168</td>
</tr>
<tr>
<td>5</td>
<td>1.527</td>
<td>1.02</td>
<td>1.524-1.538</td>
</tr>
<tr>
<td>20</td>
<td>1.825</td>
<td>3.77</td>
<td>1.839-1.872</td>
</tr>
<tr>
<td>perpetual</td>
<td>1.9658</td>
<td>17</td>
<td>1.98-2.06</td>
</tr>
</tbody>
</table>

As can be seen from the table, the semi-analytic method produces better results and uses less computing time than binomial tree method so it is indeed more accurate and efficient than the binomial method. Note that the results for the binomial method are obtained in an interval due to the natural oscillation of the method.

After validation of the semi-analytical method, we calculate the optimal exit price and stock loan value in two different parameter ranges, \( \gamma - \frac{1}{2} \sigma^2 \leq r < \gamma \) and \( r < \gamma - \frac{1}{2} \sigma^2 \). The reason for calculating the results in the two ranges is that the optimal exit prices for the two ranges behave differently. For the sake of comparison, we use the same parameters as those in Dai [32].
Dimensionless optimal exit prices for the dividend payment distributions discussed in Section (4.3) are shown in Figure 4.1 and 4.2 for the two parameter ranges mentioned above. Figure 4.1 displays the dimensionless optimal exit prices, for the cases in the first parameter range: $\gamma - \frac{1}{2}\sigma^2 \leq r < \gamma$. The value of $X_{f1}$ increases slowly, but $X_{f2}$ and $X_{f3}$ increase rather rapidly with respect to time to expiry ($\tau$). Although for perpetual stock loan, there is no optimality for case 2 and case 3 (see Section (4.3.2) and (4.3.3)) but in a finite stock loan contract, optimal exit price $X_{f2}$ and $X_{f3}$ exist. The optimal exit prices are monotonic with respect to the life of the contract $T$ (or $\tau$) in all three cases, and in case 1 the optimal exit price reaches the same limit as that of the perpetual optimal exercise price for the corresponding American call option.

![Figure 4.1: Dimensionless optimal exit price for $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$, and $T = 20$](image)

If the parameters are in the second range, $r < \gamma - \frac{1}{2}\sigma^2$, as shown in Figure 4.2, optimal exit prices $X_{f1}$ and $X_{f2}$ increase slowly with respect to $\tau$, but $X_{f3}$ increases sharply, and eventually, both $X_{f1}$ and $X_{f2}$ reach a limit respectively, as
CHAPTER 4. SEMI-ANALYTIC VALUATION OF STOCK LOANS WITH FINITE MATURITY

Figure 4.2: Dimensionless optimal exercise price for \( r - \gamma = -0.04, \delta = 0.03, \sigma = 0.15 \) and \( T = 20 \)

that of \( X_{f1} \) in parameter range 1.

The different behaviors of optimal exit prices in the two parameter ranges would mean different optimal exit strategies for the borrower. When the value of optimal exit price grows slowly and nearly constant, we can conclude that there is optimality and the borrower should exit as soon as optimality is reached. The optimal exit price \( X_{f3} \) is increasing in both ranges and this implies that the borrower should never exit.

4.4.2 Stock Loan Value and Fair Service Fee

Once the optimal stock price is obtained, the values of stock loans can be easily found from Equations (A-1), (4.20) and (4.26) for each case discussed in Section (4.3), respectively. The dimensionless stock loan values for the two different parameter ranges are presented in Figures 4.3 and 4.4. It is evident that the values of dimensionless stock loans \( U_1 < U_2 < U_3 \), just as \( X_{f1} < X_{f2} < X_{f3} \) for
both parameter ranges.

**Figure 4.3:** Dimensionless stock loan value for $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$, and $T = 5$

**Figure 4.4:** Dimensionless stock loan value for $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.15$, and $T = 5$
We now proceed to calculate the fair service fee that should be charged by the lender. At the beginning of the contract, the borrower obtains liquidity in the amount of \( q > 0 \) with one share of stock valued at \( S_0 \) as collateral, a service fee \( c \ (0 \leq c \leq q) \) is charged by the lender. The initial value of the stock loan at time \( t = 0 \) is \( V(0) = S_0 - (q - c) \), which could be used to determine a fair value of the service fee. In Table 4.2, we present the comparison of the results of our calculation of the fair service fee for a perpetual stock loan with those in [104]. The parameters used in this calculation are \( r = 0.05 \), \( \sigma = 0.15 \), \( \gamma = 0.07 \), \( \delta = 0 \), and \( S_0 = 100 \). The fair service fees are calculated for different loan to value ratios (LTV = \( q/S_0 \)). As can be seen from the table, our calculations are fairly accurate compared with the analytical results of [104]. It can also be seen that if the LTV is higher, the service fee needs to be higher due to the higher risk that the lender will have. A perpetual stock loan is marketable when the LTV is higher than \( q/S_{f\infty} = 1/X_{f\infty} = 0.4382 \) as reported in [104], whereas a corresponding 5 year finite stock loan contract is marketable for LTV greater than \( 1/X_f = 0.704 \). The calculated fair service fees for a 5 year stock loan contract for various loan to value ratios are listed in Table 4.3. The parameters used in the finite loan calculation are the same as those for the perpetual loan calculation.

**Table 4.2: Comparison of Fair Service Fees for A Perpetual Stock Loan**

<table>
<thead>
<tr>
<th>LTV ((q/S_0))</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c) [104]</td>
<td>0.701</td>
<td>3.998</td>
<td>9.027</td>
<td>15.177</td>
<td>22.098</td>
<td>29.572</td>
</tr>
<tr>
<td>(c) (Our Method)</td>
<td>0.700</td>
<td>3.997</td>
<td>9.025</td>
<td>15.174</td>
<td>22.095</td>
<td>29.569</td>
</tr>
</tbody>
</table>

**4.5 Conclusion**

In this paper, we formulated stock loans with finite maturity in three different dividend payment distributions as the corresponding American options problems,
Table 4.3: Fair Service Fee for A Finite Maturity Stock Loan \((T = 5 \text{ year})\)

<table>
<thead>
<tr>
<th>LTV ((q/S_0))</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>0.464</td>
<td>3.854</td>
<td>9.120</td>
</tr>
</tbody>
</table>

and solved the pricing problems using the semi-analytic method in Zhu(2006). It is shown that the semi-analytic method is more efficient and accurate than the binomial method. Optimal exit price and stock loan values are calculated for three different dividend distributions. These results are useful for the borrower to decide a strategy to exit from the stock loan contract, and could aid the lender in deciding the fair service fee.
Chapter 5

Finite Maturity Margin Call Stock Loan

5.1 Introduction

A stock loan is a financial contract that allows the borrower to obtain a loan with stocks as a collateral. The borrower may retrieve the stock at any time before maturity or at maturity by repaying the loan and accumulated interest at a predetermined interest rate. If the stock price increases at anytime, the borrower is able to pay back the loan, retrieve the stock and get the unlimited upside potential profit. For a non-recourse stock loan, if the stock price decreases below the accumulated loan, the borrower can walk away by surrendering the stock. In this case the borrower only loses the service fee paid at the beginning of the contract. A margin call stock loan contract provides more security for the lender than its non-recourse counterpart. When the stock price falls below the accumulated loan value, the lender issues a margin call. Once the margin call is issued, the borrower must pay back a pre-determined percentage of the loan, then the contract continues as a non-recourse loan with a reduced loan amount. Only one margin call is allowed in the life of a margin call stock loan contract.

Ekstrm [34] showed that a perpetual margin call stock loan resembles a per-
petual American down and out call option with rebate with a possibly negative interest rate and a time-dependent strike price. Explicit formulas for the value and optimal exit time of a perpetual margin call stock loan were obtained in their study. However, the assumption of infinite maturity is not practical as stock loans usually have only a number of years of maturity. In this work, we will formulate and solve the pricing problem of a finite maturity margin call stock as the corresponding American down-and-out option with rebate, negative interest rate and time dependent strike price.

In the early study about barrier options, Merton [85] in his fundamental paper in 1973, established a closed-form solution for valuing a barrier option in European style with the presence of a down-and-out barrier. Subsequently, barrier options in American style were investigated by other researchers. Similar to vanilla American options, analytic solutions are available only for limited cases, such as, a closed-form solution of a perpetual American up and out put option valuation using the variational inequality method [63], an integral representation of the knock-in American option pricing formula [31]. Both numerical and analytical approximation methods have been developed for the valuation of American barrier options. The main numerical methods in the literature are the tree method [13, 27, 92], and the finite difference method [14, 87, 120]. Since numerical methods are usually time consuming, analytical approximation methods are much preferred. Gao et al. [39] proposed a decomposition technique separating the value up-and-in and up-and-out American barrier options to the corresponding European value and an early exercise premium. Chang et al. [22] presented a modified quadratic approximation method based on the method in Barone-Adesi-Whaley [5] for American options. Lu and Rhodes [76] presented a valuation of American down-and-out call options using the analytical approximation method first proposed by Zhu [113]. This method is based on the Laplace transform with
pseudo-steady state approximation of the moving boundary, and it is particularly suitable for long term American option valuation as pointed out in [69]. Since stock loans usually have a longer life time than ordinary options, the method is suitable in stock loan evaluation as presented in [75]. Therefore, the approach in [76, 113] will form the base for the solution of our problem in this study.

This paper has four main sections. Section 1 provides an introduction, Section 2 discusses margin call stock loans as the corresponding American Barrier options with rebate. Numerical results are presented in Section 3 and conclusion is given in Section 4.

5.2 Margin Call Stock Loans as American Barrier Options

The mechanism of a margin call stock loan contract can be described as follows. At time 0, a client borrows amount $q$ at a predetermined interest rate $\gamma$ with one share of stock valued at $S_0$ as collateral. At any time $t$ ($t \geq 0$), the borrower may pay back the amount $qe^{\gamma t}$ (loan principal plus accumulated interest) to regain the collateral. However, the stock price drops below the accrued loan amount, the contract is suspended. Then the lender issues a margin call and forces the borrower to pay back a predetermined percentage $\Delta$ of the loan. After the call and payback, the contract then continues as a non-recourse loan with a reduced loan amount. It is assumed that only one margin call is allowed during the life of the loan.

As mentioned above, a stock loan contract with a margin call has two parts: a margin call phase and a non-recourse stock loan phase. The margin call phase resembles an American down-and-out call option and the non-recourse phase resembles a vanilla American call, both with negative interest rate and time dependent strike. We will concentrate on the margin call phase in this work, the
evaluation of non-recourse stock loans is presented in the paper by the authors [75].

5.2.1 Formulation of A Standard Margin Call Stock Loan

A standard margin call stock loan has the margin at the compounded loan size $qe^{\gamma t}$. When the stock price falls to or below the loan value at $t = t_q$ ($S \leq S = qe^{\gamma t_q}$), the lender issues a margin call, and forces the borrower to pay back the amount of $\Delta qe^{\gamma t_q}$. Let the value of the margin call stock loan be $V(S, t; q)$. This satisfies the following differential equation system under the Black-Scholes framework:

$$
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV &= 0 \\
V(S, T) &= \max(S - qe^{\gamma T}, 0) \\
V(S_f(t), t) &= S_f(t) - qe^{\gamma t} \\
\frac{\partial V}{\partial S}(S_f(t), t) &= 1 \\
V(S_{t_q}, t) &= \phi(t)
\end{align*}
$$

(5.1)

where $r \geq 0$ is the risk-free interest rate, $\delta \geq 0$ the continuous dividend yield, $\sigma$ the volatility of the stock, and $T$ the maturity of the loan.

The equation system (5.1) resembles one for American down-and-out options with time dependent rebate and strike. $S_f(t)$ is the optimal exit price, that is, the stock price, at or above which it is optimal for the borrower to exit the loan contract. $\phi(t)$ is the rebate or the stock loan value when the margin call phase is terminated, and it is calculated as the difference of the value of the non-recourse phase and the payback amount, that is, $\phi(t) = V_{nr}(S_{t_q}, t; (1 - \Delta)qe^{\gamma t_q}) - \Delta qe^{\gamma t_q}$ where $V_{nr}$ is the non-recourse stock loan value. The derivation of the rebate is detailed in Section (5.2.2).
Applying the following change of variables

\[ t = T - \frac{2\tau}{\sigma^2}, \quad S = X q e^{\gamma t}, \quad V(S, t) = \tilde{V}(X, \tau) q e^{\gamma t} \]

we obtain a simpler differential equation system with initial and boundary conditions:

\[
\begin{cases}
- \frac{\partial \tilde{V}}{\partial \tau} + X^2 \frac{\partial^2 \tilde{V}}{\partial X^2} + (\alpha - \beta)X \frac{\partial \tilde{V}}{\partial X} - \alpha \tilde{V} = 0 \\
\tilde{V}(X, 0) = \max(X - 1, 0) \\
\tilde{V}(X_f(\tau), \tau) = X_f(\tau) - 1 \\
\frac{\partial \tilde{V}}{\partial X}(X_f(\tau), \tau) = 1 \\
\tilde{V}(1, \tau) = (1 - \Delta)\tilde{V}_{nr} \left(\frac{1}{1 - \Delta}, \tau\right) - \Delta
\end{cases}
\]

(5.2)

where \( \alpha = \frac{2(r - \gamma)}{\sigma^2} \), and \( \beta = \frac{2\delta}{\sigma^2} \). \( \tilde{V}_{nr} \left(\frac{1}{1 - \Delta}, \tau\right) \) is the dimensionless initial loan value for the non-recourse phase.

Defining a new function

\[ U = \tilde{V} + 1 - X, \quad 1 \leq X \leq X_f \]

the PDE system (5.2) becomes,

\[
\begin{cases}
- \frac{\partial U}{\partial \tau} + X^2 \frac{\partial^2 U}{\partial X^2} + (\alpha - \beta)X \frac{\partial U}{\partial X} - \alpha U = \beta X - \alpha \\
U(X, 0) = 0 \\
U(X_f, \tau) = 0 \\
\frac{\partial U}{\partial X}(X_f, \tau) = 0 \\
U(1, \tau) = (1 - \Delta)U_{nr} \left(\frac{1}{1 - \Delta}, \tau\right)
\end{cases}
\]

(5.3)

The next step is to transform the equation system (5.3) into Laplace space,
we obtain an ODE system as follows

\[
\begin{align*}
X^2 \frac{d \bar{U}(X, p)}{dX^2} + (\alpha - \beta)X \frac{d \bar{U}(X, p)}{dX} - (\alpha + p)\bar{U}(X, p) &= \frac{\beta X - \alpha}{p} \\
\bar{U}(p, X_f, p) &= 0 \\
\frac{d \bar{U}}{dX}(p, X_f, p) &= 0 \\
\bar{U}(1, p) &= (1 - \Delta)\bar{U}_{nr}(\frac{1}{1 - \Delta}, p)
\end{align*}
\]

(5.4)

Applying the boundary conditions, we obtain an explicit equation for the optimal exit price in Laplace space:

\[
\bar{X}_f^{k_2} \frac{k_2 \alpha}{p^{1+k_1}(p + \alpha)} + \bar{X}_f^{1-k_1} \frac{(1-k_2)\beta}{p^{k_1}(p + \beta)} - \bar{X}_f^{k_2} \frac{k_1 \alpha}{p^{1+k_2}(p + \alpha)} - \bar{X}_f^{1-k_2} \frac{(1-k_1)\beta}{p^{k_2}(p + \beta)} = (k_1 - k_2) \left\{ \bar{R}(p) + \frac{\beta - \alpha}{(p + \alpha)(p + \beta)} \right\}
\]

(5.5)

where \( \bar{R}(p) = (1 - \Delta)\bar{U}_{nr}(\frac{1}{1 - \Delta}, p) \), and

\[
k_{1,2} = \frac{1 + \beta - \alpha}{2} \pm \sqrt{\frac{(1 + \beta - \alpha)^2}{2} + (p + \alpha)}
\]

First, we need to obtain the rebate using the solution of the non-recourse loan, which is given in our previous work [75]. Then Equation (5.5) can be solved for \( \bar{X}_f(p) \), the solution for the system (5.4) will follow:

\[
\bar{U} = C_1 X^{k_1} + C_2 X^{k_2} + \frac{p(\alpha - \beta X) + \alpha \beta (1 - X)}{p(p + \alpha)(p + \beta)}
\]

(5.6)

where

\[
C_1 = -\frac{1}{k_1 - k_2} (p\bar{X}_f)^{-k_1} \left( \frac{(k_2 - 1)\beta}{p + \beta} \bar{X}_f - \frac{\alpha k_2}{p(p + \alpha)} \right)
\]

\[
C_2 = \frac{1}{k_1 - k_2} (p\bar{X}_f)^{-k_2} \left( \frac{(k_1 - 1)\beta}{p + \beta} \bar{X}_f - \frac{\alpha k_1}{p(p + \alpha)} \right)
\]

Equation (5.5) is highly non-linear, it needs to be solved numerically for \( \bar{X}_f(p) \). Once the optimal exit price, \( \bar{X}_f(p) \), is obtained, the calculation of the value of
the stock loan is straight-forward using Equation (5.6) in the Laplace space. Numerical Laplace inversion is then used to obtain the values in original time space.

### 5.2.2 Derivation of The Rebate $R$

During the life time, $T$, of a margin call stock loan contract, if the stock price $S$ falls and touches the loan value at $t = t_q \ (S_{t_q} = q e^{\gamma t_q})$, the contract is temporarily halted. The borrower must pay a pre-determined $\Delta$ percent of the accumulated loan in order for the contract to continue. At the time of the call, the margin call phase of the stock loan is terminated. After the payment, the contract resumes as a non-recourse contract until maturity.

The connection between the two phases is time $t_q$ which is the end of a margin call stock loan phase and the beginning of a new non-recourse phase. At time $t_q$, we reset the clock for the non-recourse phase by introducing $t' = t - t_q$ and $T' = T - t_q$. The non-recourse phase begins with the initial stock value $S_{t_q}$ as collateral and a new loan size $q' = (1 - \Delta)q e^{\gamma t_q}$ after the payback.

The non-recourse phase is the same as the standard stock loan discussed in our previous work [75]. Let the value of the non-recourse phase be $V_{nr}(S, t'; q')$, it has the initial value $V_{nr}(S_{t_q}, 0; (1 - \Delta)q e^{\gamma t_q})$. Because the borrower pays back a fraction of the loan, $\Delta q e^{\gamma t_q}$, the value of the borrower’s holding at time $t_q$ is reduced by this amount. That is, the net value which the borrower holds at time $t_q$ is $V_{nr}(S_{t_q}, 0; (1 - \Delta)q e^{\gamma t_q}) - \Delta q e^{\gamma t_q}$, this amount can be considered as a rebate (terminal value of the margin call phase) to compensate the borrower due to the margin call. Therefore, the value of the margin call stock loan at $t_q$ is

$$V(S_{t_q}, t_q; q) = V_{nr}(S_{t_q}, 0; (1 - \Delta)q e^{\gamma t_q}) - \Delta q e^{\gamma t_q}.$$  

Note that the time to maturity for the non-recourse phase is
\[ \tau' = T' - t' = (T - t_q) - (t - t_q) = \tau \]

Therefore, using
\[ t = T - \frac{2\tau}{\sigma^2}, \quad S = Xqe^{\gamma t}, \quad V(S, t) = \tilde{V}(X, \tau) qe^{\gamma t} \]

the last equation could be transformed into
\[ \tilde{V}(1, \tau_q) = (1 - \Delta)\tilde{V}_{nr}(X'_q, \tau_q) - \Delta \]

where \( \tilde{V}_{nr} = V_{nr}/q' = V_{nr}/(1 - \Delta)qe^{\gamma t_q} \), and \( X'_q = S_{t_q}/q' = 1/(1 - \Delta) \).

Dropping the \( q \), we have
\[ \tilde{V}(1, \tau) = (1 - \Delta)\tilde{V}_{nr}(\frac{1}{1 - \Delta}, \tau) - \Delta \]

Using the results from [75], \( \tilde{V}_{nr}(\frac{1}{1 - \Delta}, \tau) = \tilde{U}_{nr}(\frac{1}{1 - \Delta}, \tau) + \frac{1}{1 - \Delta} - 1 \), therefore,
\[ \tilde{V}(1, \tau) = (1 - \Delta)\tilde{U}_{nr}(\frac{1}{1 - \Delta}, \tau) \]

Taking the Laplace transform of the last equation, we obtain
\[ \tilde{V}(1, p) = (1 - \Delta)\tilde{U}_{nr}(\frac{1}{1 - \Delta}, p) \]

Finally, we derived the rebate of the margin call stock loan in the Laplace space,
\[ \tilde{R}(p) = (1 - \Delta)\tilde{U}_{nr}(\frac{1}{1 - \Delta}, p). \]

The detailed solution of \( \tilde{U}_{nr} \) can be found from Equation (A-1) in Appendix A.
5.3 Numerical Results

In this section, we provide numerical examples demonstrate that numerical results can easily be obtained using our formulation. We also discuss the effects of margin calls on the value of stock loans and their optimal exit prices.

5.3.1 Method Validation

Before a meaningful discussion is carried out, we validate our method as follows:

We first compare the result, obtained using our formulation by letting maturity $T \to \infty$, with that obtained by the exact solution in [34] for a perpetual margin call stock loan. The following parameters are used in the calculations: $r = 0.06, \gamma = 0.1, \delta = 0$, and $\sigma = 0.15$. The result from our calculation, $x_f = 1.3103$, matches up to 4 decimal places with the exact result computed from the perpetual margin call formula of [34].

![Figure 5.1: Dimensionless optimal exit price](image)

We then make the payback $\Delta = 0$ in the margin call stock loan to degenerate the margin call to a standard non-recourse stock loan. Our results for the optimal exit price and stock loan value when $\Delta = 0$ (no payment), agree perfectly with the results for the standard non-recourse loan. The comparison of the optimal
exit prices is shown in Figure 5.1.

The validation processes help us verify that the tedious algebraic manipulations are correct.

### 5.3.2 Margin Call Stock Loans

For the rest of the calculation in the section, unless otherwise stated the following parameters are used: the risk free interest rate $r = 0.06$, loan interest rate $\gamma = 0.1$, dividend paid at maturity $\delta = 0.03$, volitility $\sigma = 0.4$, and maturity $T = 5$ years.

![Dimensionless Optimal Exit Price](image)

**Figure 5.2:** Dimensionless optimal exit price of standard margin call stock loan

Kwok [68] stated that the optimal exercise price of an American down-and-out call option with rebate is a decreasing function of the barrier, but an increasing function of the rebate. As margin call stock loans resemble American down-and-out call options with rebate, the optimal exit prices of margin call stock loans are observed having similar behaviors. The dimensionless optimal exit prices for different paybacks are shown in Figure 5.2. For reference, the dimensionless
optimal exit price for non-recourse loan is also plotted in the figure. As can be seen from Figure 5.2, an increase in \( \Delta \), the percentage of payback, leads to a decrease in the optimal exit price, as more payback means lower rebate.

![Figure 5.3: Dimensionless margin call stock loan value](image)

The values of margin call stock loans are less than that of non-recourse loan, and decrease as the percentage payback \( \Delta \) increases as shown in Figure 5.3. This makes financial sense, as a margin call is supposed to provide more security for the lender; higher payback provides lower margin call value to the borrower.

Figures 5.4 and 5.5 show the variation of the dimensionless optional exit price and the dimensionless margin call value at \( t = 0 \), respectively, for a fixed payback, \( \Delta \), with respect to the change in the effective interest rate \( \bar{r} = r - \gamma \). It is clear from the figures that both optimal price and loan value are decreasing functions of \( \bar{r} \). It means that optimal price and loan value will decrease if the loan interest is higher, or the risk free interest is lower.
We provide a comparison of the dimensionless optimal exit prices for different volatility rates for a fixed payback $\Delta$ in Figure 5.6. As can be seen from the figure, the optimal exit price increases with the volatility $\sigma$. This is in agreement with the results for standard non-recourse stock loans. As expected, the margin call stock loan value is higher for higher $\sigma$, as shown in Figure 5.7.
Figure 5.6: Dimensionless optimal exit price of margin call stock loan

Figure 5.7: Dimensionless margin call stock loan value
5.4 Conclusion

In this study, a finite maturity margin call stock loan is formulated as an American down-and-out call option with rebate. The rebate is calculated from the non-recourse phase of the margin call and subsequently is used in the solution process of the margin call phase (an American down-and-out call option with rebate and negative interest rate). The optimal exit price and margin call stock loan value formulas are obtained in the Laplace space. Numerical inversion is performed to obtain the corresponding values in the original space.

Numerical results are presented to show the dependency of the optimal exit price and the value of the stock loan on the payback and other parameters. The optimal exit price and margin call stock loan value are decreasing functions of the payback, increasing functions of the volatility, and decreasing functions of the effective interest rate.
6.1 Introduction

Stock loans are financial contracts in which the borrower can borrow money from the lender by collateralizing his stock. The lender holds the stock during the lifetime of the stock loan and has the right to take over the stock if the borrower fails to pay back the loan. The borrower may redeem the stock at anytime before maturity if he can repay the loan and the accumulated interest at a predetermined rate. If the stock price increases significantly, the borrower can gain the unlimited upside potential instead of paying back the loan and redeeming the stock. On the other hand, if the stock price decreases below the accumulated loan, the borrower can surrender the stock and walkaway.

Evaluation of standard stock loans as an American call option problem was initiated by Xia [104] under the classical Black-Scholes framework. The stock loans allow possibly negative interest rate that differentiate them from the plain American call options. The Black-Scholes model, which assumes a constant volatility, can not fully describe the stock market volatility [57]. The presence of a volatility smile (or skew) and a volatility term structure shows the discrepancy of the
Black-Scholes model in reflecting the market dynamics [58].

There are various studies which aim to provide models to better capture the market dynamics for stock loan valuation. Wong and Wong [102] presented a valuation of stock loans by using stochastic volatility model and Liang et al. [70] carried out an evaluation in which the pricing is assumed to follow a Levy process, both in infinite maturity.

An alternative approach to capturing market movement more precisely is a regime switching model. Zhang and Zhou [112] extended the work in [104] to the valuation of stock loans under a regime-switching economy in infinite maturity. The switching between states is driven by the market volatility. Prager and Zhang [90] proposed the valuation of European stock loans under a regime-switching model in finite maturity. Rather than consider switching between the values of the stock loan parameters, they proposed switching which allows jumps between two models, mean-reverting and geometric Brownian motion.

Since valuation of stock loans in a regime-switching economy is carried out in a similar manner to the corresponding American call options, it is important to refer some studies about valuation of option under a regime-switching economy.

Regime-switching models for options valuation have been discussed extensively in the literature [24, 47, 48, 57, 107, 109, 110]. While European options with regime-switching have been solved analytically, [18, 78, 86, 116], there is no analytical solution for the American counterpart due to the presence of the free-boundary, except for Guo’s perpetual option [47].

Numerical and analytical approximation methods have been developed as alternatives since the analytic solution for the American options with regime-switching is extremely difficult or even impossible to obtain. Numerical methods to value American options with regime-switching include, among others, the penalty method [65], finite element method [53, 106], tree method [110].
So far the only analytical approximation method available in the literature is the valuation of American options with regime-switching in finite maturity by Buffington and Elliott [17]. Their valuation is based on risk-neutral pricing, and an extension of the quadratic approximation of Barone-Adesi-Whaley [5]. A set of partial differential equation for the option valuation was derived, but needs to be solved numerically.

In this work, we show our formulation of finite stock loans in a regime-switching economy as the corresponding American call option problem with negative interest rate and time dependent strikes. The analysis and solution of the regime-switching American option problem is also of interest to us. We will provide a semi-analytical approximation method for the solution of the option problem, and then apply the same procedure to the stock loan problem.

The new method is developed based on the semi-analytic method in Zhu [113] which proposed the use of a pseudo-steady state approximation in Laplace space to tackle the free boundary in finite maturity American options. We follow Buffington and Elliott [17] and Zhang and Zhou [112] in constructing the pricing domain to deal with coupled partial differential equations arising from the switching states. Lauko and Sevcovic [96] recommend Zhu’s method [113] for evaluating American options with long maturity. Stock loans generally have long maturity, and a modified form of the semi-analytic method is appropriate for stock loan valuation.

This paper is organized as follows. Section 2 presents the evaluation of American options under a regime-switching economy using the modified semi-analytic method. The equations to calculate optimal exercise price formulas and the options are obtained. Numerical results are presented to show the implementation of the proposed method with various parameters. Section 3, presents the valuation of stock loans formulated as American call options with a possibly negative
interest rate under a regime-switching economy. Numerical results for the stock loans under a regime-switching economy are also presented. These numerical results are then analyzed to give more insights from the financial point of view. Concluding remarks are given in Section 4.

6.2 American Call Options in A Two-State Regime-Switching Economy

In this section, we focus on American call options under a regime-switching economy. We assume that these options have two states, growth or recession which correspond to the volatilities $\sigma_1$ and $\sigma_2$ [116] respectively. The switching rate between states 1 and 2 is represented by $\lambda_{12}$ when switching from state 1 to state 2 and $\lambda_{21}$ when switching from state 2 to state 1.

We begin the formulation by setting up the pricing domain, following Buffington and Elliott [17]. Since the American options under a two-state regime-switching economy have two optimal exercise boundaries, the pricing domain is set up based on the position of the two free-boundaries as shown in Figure 6.1. The boundaries divide the domain into two parts: a common continuation region and a transition region.

For simplicity, it is assumed that $\sigma_2 < \sigma_1$ so that $S_{f2} < S_{f1}$. The stock price $S$ is in the common continuation region if $0 \leq S \leq S_{f2}$, and is in the transition region if $S_{f2} \leq S \leq S_{f1}$.
6.2.1 Common Continuation Region

In the common continuation region \((0 \leq S \leq S_f)\), there are two coupled equations for the value of American options \(V_1\) in state 1 and \(V_2\) in state 2:

\[
\begin{align*}
\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma_1^2S^2\frac{\partial^2 V_1}{\partial S^2} + (r - \delta)S\frac{\partial V_1}{\partial S} - rV_1 &= \lambda_{12}(V_1 - V_2) \\
\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma_2^2S^2\frac{\partial^2 V_2}{\partial S^2} + (r - \delta)S\frac{\partial V_2}{\partial S} - rV_2 &= \lambda_{21}(V_2 - V_1)
\end{align*}
\]

(6.1)

where \(r\) is the constant risk-free interest rate and \(\lambda_{12}\) and \(\lambda_{21}\) are the switching rates. Equation (6.1) is the governing system for option valuation under the Black-Scholes framework.

An option will be worthless when the stock price is zero. At the end of the contract, the value of an option should be equal to its payoff function. Therefore, we have

\[
V_{1,2}(0, t) = 0
\]

(6.2)

\[
V_{1,2}(S, T) = max(S - K, 0)
\]

(6.3)

where \(V_{1,2}\) is the option value in state 1 and 2, \(K\) is the strike price, and \(T\) is the maturity.
Unlike their European counterpart, American options can be exercised at any time before maturity. The early exercise possibility provides additional conditions at $S = S_f(t)$, the so called optimal exercise price, which needs to be solved as part of the solution. It makes the American option pricing problem a moving boundary problem.

The optimal exercise price $S_{f1}$ is reached in the transition region, which is examined in the next subsection. In the common continuation region, the only known conditions for option value $V_1$ are Equation (6.2) and (6.3).

The optimal exercise price $S_{f2}$ bounds the common continuation region. Accordingly, the optimal exercise boundary conditions for $V_2$ in the common continuation region are:

$$
\begin{align*}
V_2(S_{f2}, t) &= S_{f2}(t) - K \\
\frac{\partial V_2}{\partial S}(S_{f2}, t) &= 1
\end{align*}
$$

The following change of variables allows the simplification of Equation (6.1) and the associated conditions:

$$
S = \frac{X}{K}, \quad \tau = T - t, \quad U_{1,2} = \frac{V_{1,2}}{K}
$$

The resulting partial differential equation (PDE) system for $U_1$ and $U_2$ is as
follows
\[
\begin{cases}
  -\frac{\partial U_1}{\partial \tau} + \frac{1}{2}\sigma_1^2X^2\frac{\partial^2 U_1}{\partial X^2} + (r - \delta)X\frac{\partial U_1}{\partial X} - rU_1 = \lambda_{12}(U_1 - U_2) \\
  U_1(0, \tau) = 0 \\
  U_1(X, 0) = \max(X - 1, 0) \\
  -\frac{\partial U_2}{\partial \tau} + \frac{1}{2}\sigma_2^2X^2\frac{\partial^2 U_2}{\partial X^2} + (r - \delta)X\frac{\partial U_2}{\partial X} - rU_2 = \lambda_{21}(U_2 - U_1) \\
  U_2(0, \tau) = 0 \\
  U_2(X, 0) = \max(X - 1, 0) \\
  U_2(X_{f2}, \tau) = X_{f2} - 1 \\
  \frac{\partial U_2}{\partial X}(X_{f2}, \tau) = 1
\end{cases}
\] (6.5)

We follow the method in Zhu [113] to solve Equation (6.5), and the solution of the PDE system in Laplace space are (see Appendix B for detail):

\[
\bar{U}_1(X, p) = \begin{cases}
  A_1X^{k_1} + A_2X^{k_2} + A_3X^{k_3} + A_4X^{k_4} + \phi(X, p), & \text{if } 1 < X \leq X_{f2}; \\
  A_5X^{k_3} + A_6X^{k_4}, & \text{if } 0 \leq X \leq 1;
\end{cases}
\] (6.6)

\[
\bar{U}_2(X, p) = \begin{cases}
  B_1X^{k_1} + B_2X^{k_2} + B_3X^{k_3} + B_4X^{k_4} + \phi(X, p), & \text{if } 1 < X \leq X_{f2}; \\
  B_5X^{k_3} + B_6X^{k_4}, & \text{if } 0 \leq X \leq 1;
\end{cases}
\] (6.7)

where
\[
\phi(X, p) = \frac{1}{p + \delta}X - \frac{1}{p + r},
\]

\(k_1 < k_2 < 0 < k_3 < k_4\) are the roots of the quartic indicial equation for the ODE system, \(A_i\) and \(B_i\), \(i = 1, 2, 3 \cdots 6\), are some constants.

Note that \(A_i\) and \(B_i\) are related so only \(A_i\) appear in the final equations. The details for finding the quartic roots, and the constants are presented in
**APPENDICES**

### 6.2.2 Transition Region

The optimal exercise price $S_{f2}$ is reached in transition region ($S_{f2} \leq S \leq S_{f1}$). The value of $V_2$ in the transition region is just the payoff, that is, $V_2 = S - K$.

The PDE system for this region only contains $V_1$ as an unknown function.

$$
\begin{align*}
\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \delta) S \frac{\partial V_1}{\partial S} - rV_1 &= \lambda_{12} (V_1 - (S - K)) \\
V_1(S,T) &= \max(S - K, 0) \\
V_1(S_{f1}, t) &= S_{f1} - K \\
\frac{\partial V_1}{\partial S}(S_{f1}, t) &= 1
\end{align*}
$$

(6.8)

After a similar change of variables process, we obtain the PDE system for the transition region:

$$
\begin{align*}
\frac{\partial U_1}{\partial \tau} + \frac{1}{2} \sigma_1^2 X^2 \frac{\partial^2 U_1}{\partial X^2} + (r - \delta) X \frac{\partial U_1}{\partial X} - rU_1 &= \lambda_{12} (U_1 - (X - 1)) \\
U_1(X,0) &= X - 1 \\
U_1(X_{f1}, \tau) &= X_{f1} - 1 \\
\frac{\partial U_1}{\partial X}(X_{f1}, \tau) &= 1
\end{align*}
$$

(6.9)

Note that since in the transition region $S \geq K$ so the terminal condition $V_1(S,T) = \max(S - K, 0)$ is simply transformed into $U_1(X,0) = X - 1$. We follow a similar procedure to that in the previous subsection to solve Equation (6.9). The detailed process for finding the solution is presented in Appendix C.

The solution for the option in the transition region in Laplace space is:

$$
\bar{U}_1 = m_1 X^{\gamma_1} + m_2 X^{\gamma_2} + \Psi_1 X - \Psi_2
$$

(6.10)

where $\Psi_1 = \frac{p + \lambda_{12}}{p(p + \delta + \lambda_{12})}$; $\Psi_2 = \frac{p + \lambda_{12}}{p(p + r + \lambda_{12})}$; $m_{1,2}$ are the integration con-
sants, and \( \gamma_{1,2} \) are the solutions to the indicial equation:

\[
\frac{1}{2} \sigma_{1}^{2} \gamma^{2} + (r - \delta - \frac{1}{2} \sigma_{1}^{2}) \gamma - (p + r + \lambda_{12}) = 0.
\]

### 6.2.3 Optimal Exercise Price and Option Value

In the two previous subsections, solutions of for \( \bar{U}_1 \) and \( \bar{U}_2 \), in both the common continuation region and transition region, have been derived.

Since the pricing domain for \( V_1 \) (or \( U_1 \) in dimensionless variable) consists of two regions, it is important to assure the continuity and smoothness of \( U_1 \) at the boundary (intersection) \([47]\). These conditions are:

\[
\lim_{x \to X_{f2}^{-}} U_1 = \lim_{x \to X_{f2}^{+}} U_1 \quad (6.11)
\]

\[
\lim_{x \to X_{f2}^{-}} \frac{\partial U_1}{\partial X} = \lim_{x \to X_{f2}^{+}} \frac{\partial U_1}{\partial X}
\]

In the Laplace space, the continuity conditions for \( \bar{U}_1 \) can be written as follows:

\[
\bar{U}_1(p\bar{X}_{f2}^{-}, p) = \bar{U}_1(p\bar{X}_{f2}^{+}, p) \quad (6.12)
\]

\[
\frac{d\bar{U}_1}{dX}(p\bar{X}_{f2}^{-}, p) = \frac{d\bar{U}_1}{dX}(p\bar{X}_{f2}^{+}, p)
\]

Combining the continuity conditions in (6.12) with all other conditions in Equations (6.6), (6.7), and (6.10), we obtain a system of 10 equations in 10 unknowns consisting of 8 constants and 2 optimal exercise prices \( \bar{X}_{f1} \) and \( \bar{X}_{f2} \):

1) \( A_1 + A_2 + A_3 + A_4 + \frac{1}{p + \delta} - \frac{1}{p + r} = A_5 + A_6 \)

2) \( A_1 k_1 + A_2 k_2 + A_3 k_3 + A_4 k_4 + \frac{1}{p + \delta} = A_5 k_3 + A_6 k_4 \)

3) \( A_1 l_1 + A_2 l_2 + A_3 l_3 + A_4 l_4 + \frac{1}{p + \delta} - \frac{1}{p + r} = A_5 l_3 + A_6 l_4 \)

4) \( A_1 l_1 k_1 + A_2 l_2 k_2 + A_3 l_3 k_3 + A_4 l_4 k_4 + \frac{1}{p + \delta} = A_5 l_3 k_3 + A_6 l_4 k_4 \)
5) \( A_1 l_1(p\bar{X}_{f_2})^{k_1} + A_2 l_2(p\bar{X}_{f_2})^{k_2} + A_3 l_3(p\bar{X}_{f_2})^{k_3} + A_4 l_4(p\bar{X}_{f_2})^{k_4} + \ldots \)
\[
\frac{1}{p + \delta} (p\bar{X}_{f_2}) - \frac{1}{p + r} = \frac{1}{p} (p\bar{X}_{f_2}) - \frac{1}{p}
\]

6) \( A_1 l_1 k_1(p\bar{X}_{f_2})^{k_1} + A_2 l_2 k_2(p\bar{X}_{f_2})^{k_2} + A_3 l_3 k_3(p\bar{X}_{f_2})^{k_3} + A_4 l_4 k_4(p\bar{X}_{f_2})^{k_4} + \ldots \)
\[
\frac{1}{p + \delta} (p\bar{X}_{f_2}) = \frac{1}{p} (p\bar{X}_{f_2})
\]

7) \( m_1(p\bar{X}_{f_1})^{\gamma_1} + m_2(p\bar{X}_{f_1})^{\gamma_2} + \Psi_1(p\bar{X}_{f_1}) - \Psi_2 = \frac{1}{p} (p\bar{X}_{f_1}) - \frac{1}{p} \)

8) \( m_1 \gamma_1(p\bar{X}_{f_1})^{\gamma_1} + m_2 \gamma_1(p\bar{X}_{f_1})^{\gamma_2} + \Psi_1(p\bar{X}_{f_1}) = \frac{1}{p} (p\bar{X}_{f_1}) \)

9) \( A_1(p\bar{X}_{f_2})^{k_1} + A_2(p\bar{X}_{f_2})^{k_2} + A_3(p\bar{X}_{f_2})^{k_3} + A_4(p\bar{X}_{f_2})^{k_4} + \frac{1}{p + \delta} (p\bar{X}_{f_2}) - \frac{1}{p + r} = m_1(p\bar{X}_{f_2})^{\gamma_1} + m_2(p\bar{X}_{f_2})^{\gamma_2} + \Psi_1(p\bar{X}_{f_2}) - \Psi_2 \)

10) \( A_1 k_1(p\bar{X}_{f_2})^{k_1} + A_2 k_2(p\bar{X}_{f_2})^{k_2} + A_3 k_3(p\bar{X}_{f_2})^{k_3} + A_4 k_4(p\bar{X}_{f_2})^{k_4} + \frac{1}{p + \delta} (p\bar{X}_{f_2}) = m_1 \gamma_1(p\bar{X}_{f_2})^{\gamma_1} + m_2 \gamma_2(p\bar{X}_{f_2})^{\gamma_2} + \Psi_1(p\bar{X}_{f_2}) \)

It is difficult and time consuming to solve the 10 equations simultaneously due to non-linearity. Instead, we solve the equations ‘by parts’, that is, in blocks.

We first assume that the optimal exercise prices \( \bar{X}_{f_1} \) and \( \bar{X}_{f_2} \) are already known, obtain explicit expressions for \( A_i \)’s and \( m_1, m_2 \).

The first four listed above form the matrix equation below, from which we obtain the values of \( A_1, A_2, A_3 - A_5 \) and \( A_4 - A_6 \).

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
k_1 & k_2 & k_3 & k_4 \\
l_1 & l_2 & l_3 & l_4 \\
l_1 k_1 & l_2 k_2 & l_3 k_3 & l_4 k_4
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
A_3 - A_5 \\
A_4 - A_6
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{p + r} - \frac{1}{p + \delta} \\
-\frac{1}{p + \delta} \\
\frac{1}{p + r} - \frac{1}{p + \delta} \\
-\frac{1}{p + \delta}
\end{pmatrix}
\] (6.13)
Then, we use the next two equations to obtain $A_3$ and $A_4$. 

\[
\begin{pmatrix}
l_3 & l_4 \\
l_3k_3 & l_4k_4
\end{pmatrix} \times \begin{pmatrix}(p\bar{X}_{f_2})^{k_3} & 0 \\
0 & (p\bar{X}_{f_2})^{k_4}
\end{pmatrix} \times \begin{pmatrix}A_3 \\
A_4
\end{pmatrix} = (6.14)
\]

\[
\begin{pmatrix}-A_1(p\bar{X}_{f_2})^{k_1} - A_2(p\bar{X}_{f_2})^{k_2} + \left\{\frac{1}{p} - \frac{1}{p + \delta}\right\}(p\bar{X}_{f_2}) - \left\{\frac{1}{p} - \frac{1}{p + r}\right\} \\
-A_1l_1(p\bar{X}_{f_2})^{k_1} - A_2l_2(p\bar{X}_{f_2})^{k_2} + \left\{\frac{1}{p} - \frac{1}{p + \delta}\right\}(p\bar{X}_{f_2})
\end{pmatrix}
\]

Subsequently, we obtain $A_5$ and $A_6$.

The values of $m_1$ and $m_2$ are calculated from Equation 7) and 8) listed above:

\[
\begin{pmatrix}1 & 1 \\
g_1 & g_2
\end{pmatrix} \times \begin{pmatrix}(p\bar{X}_{f_1})^{g_1} & 0 \\
0 & (p\bar{X}_{f_1})^{g_2}
\end{pmatrix} \times \begin{pmatrix}m_1 \\
m_2
\end{pmatrix} = \begin{pmatrix}\Psi_1 p\bar{X}_{f_1} - \Psi_2 \\
\Psi_1 p\bar{X}_{f_1}
\end{pmatrix} (6.15)
\]

Since all the constants can be expressed explicitly, we now substitute them
into the final two equations to calculate the optimal exercise prices $\bar{X}_{f1}$ and $\bar{X}_{f2}$:

$$
\begin{pmatrix}
1 & 1 \\
3 & 4
\end{pmatrix}
\times
\begin{pmatrix}
(p\bar{X}_{f2})^3 & 0 \\
0 & (p\bar{X}_{f2})^4
\end{pmatrix}
\times
\begin{pmatrix}
A_3 \\
A_4
\end{pmatrix}
+ \\
\begin{pmatrix}
A_1(p\bar{X}_{f2})^1 + A_2(p\bar{X}_{f2})^2 + \left\{ \frac{1}{p+\delta} \right\}(p\bar{X}_{f2}) - \left\{ \frac{1}{p+r} \right\} \\
A_1k_1(p\bar{X}_{f2})^1 + A_2k_2(p\bar{X}_{f2})^2 + \left\{ \frac{1}{p+\delta} \right\}(p\bar{X}_{f2})
\end{pmatrix}

(6.16)
$$

Equation (6.16) allows us to obtain the optimal exercise prices in Laplace space, $\bar{X}_{f1}$ and $\bar{X}_{f2}$. It is then straightforward to calculate the dimensionless values of American call options $\bar{U}_1$ and $\bar{U}_2$ as all constants and the optimal exercise prices are known. Numerical Laplace inversion is needed to obtain the values in the original time space.

### 6.2.4 Numerical Examples

We now present numerical examples to verify the accuracy and efficiency of the modified semi-analytic method described above.

#### Example 1:

In this example, we compare our results of the optimal exercise prices of perpetual American puts under a regime-switching economy with those obtained by
the analytic method in [47]. The optimal exercise prices for the standard perpetual American puts are also presented. For the sake of comparison, the same parameters are used as those in [47].

The formulas to evaluate American put options can be obtained in a similar way to those for the call counterparts in the previous section. Alternatively, the call formulas can be used to obtain the put results by using Put-Call symmetry. Therefore, we will not present detailed formulation here.

As can be seen from Table 6.1, it is evident that our results are very close to the analytical results of [47]. This indicates that our method is accurate for evaluating perpetual American options. The optimal exercise prices under a regime-switching economy $S_{f1}$ and $S_{f2}$ are indeed between the results for the standard options (non-switching) as predicted by the theoretical analysis of Yi [108].

**Table 6.1: Optimal Exercise Price of Perpetual American Puts**

<table>
<thead>
<tr>
<th>Classical</th>
<th>Regime-Switching</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balck-Scholes</td>
<td>Analytic in [47]</td>
</tr>
<tr>
<td>$S_{f1}$</td>
<td>$S_{f1}$</td>
</tr>
<tr>
<td>$S_{f2}$</td>
<td>$S_{f2}$</td>
</tr>
<tr>
<td>0.34483</td>
<td>0.96774</td>
</tr>
<tr>
<td>0.61156</td>
<td>0.61163</td>
</tr>
</tbody>
</table>

**Example 2:**

We now compare the values of one-year American puts under a regime-switching economy in dimensional variables with those of [106]. The same parameters are used for the sake of comparison. As for Example 1, the formulation will not be presented due to the similarity with the call counterpart.

The results in Table 6.2 show a maximum discrepancy of approximately 2% between our results and those from the Crank Nicholson (CN12800) and Lattice
Table 6.2: American Put Value with Regime-Switching at $t = 0$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>4.5257</td>
<td>2.3083</td>
<td>4.5257</td>
<td>2.3083</td>
<td>4.4568</td>
<td>2.2928</td>
</tr>
</tbody>
</table>

Method (LM51200) in [106]. The Crank-Nicholson method is considered to give ‘true’ values. However, our time savings far outweigh the difference in values, and to produce the ‘true’ values, 12800 and 51200 time steps are needed in the Crank-Nicholson method and the Lattice method, respectively. In addition, our method produces more accurate results as the contract term becomes longer, and it has a maximum relative error of 0.01% for the perpetual case in example 1.

In summary, our method provides a more efficient valuation for American options. This method is also accurate, especially for longer term contracts.

Example 3:

In this example, we present our calculation of the optimal exercise prices and the values of American call options in a regime-switching economy in finite maturity. The parameters used in the calculations are: $r = 0.06$, $\delta = 0.08$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $\lambda_{12} = 1.375968919$, $\lambda_{21} = 1.031976689$, $K = $100, and $T = 1$.

The dimensionless optimal exercise prices of one year American call options under a regime-switching economy are illustrated together with option values calculated under the classical Black-Scholes model in Figure 6.2. As expected, we find that the optimal exercise prices with regime-switching are bounded between those for the standard American calls.

The values of a one year American call option under a regime-switching economy for various stock price $S$ are presented, and compared with those for stan-
Figure 6.2: Dimensionless optimal exercise prices

Table 6.3: American Call Option Values at $t = 0$

<table>
<thead>
<tr>
<th>$S$</th>
<th>Our results</th>
<th>BS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$V_1$</td>
<td>$V_2$</td>
</tr>
<tr>
<td>80</td>
<td>2.264831</td>
<td>1.47024</td>
</tr>
<tr>
<td>100</td>
<td>9.351436</td>
<td>7.78078</td>
</tr>
<tr>
<td>120</td>
<td>22.50430</td>
<td>20.71355</td>
</tr>
</tbody>
</table>

standard American calls in Table 6.3. As expected the values of American call under a regime-switching economy are between those of standard American calls, that is, $V^{BS}_2 < V_2 < V_1 < V^{BS}_1$ for $\sigma_2 < \sigma_1$.

6.3 Stock Loans in A Two-State Regime-Switching Economy

In this section, we formulate finite maturity stock loans under a regime-switching economy as finite American call options with a negative interest rate under a regime-switching economy, and then solve the problem using the method devel-
Let $r$ be the risk-free interest rate, $\delta$ the continuous dividend rate, $\gamma$ the loan interest rate, $q$ the initial loan amount, $\sigma_{1,2}$ the volatilities and $T$ the maturity. The stock loan PDE systems are derived by following the same procedure as that in [75] for the common continuation region and the transition region. Assuming $\sigma_1 > \sigma_2$, we obtain, in the common continuation region,

$$
\begin{align*}
\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \gamma - \delta) S \frac{\partial V_1}{\partial S} - (r - \gamma) V_1 &= \lambda_{12} (V_1 - V_2) \\
V_1(0, t) &= 0 \\
V_1(S, T) &= \max(S - qe^{\gamma T}, 0) \\
V_1(S_{f2}(t), t) &= S_{f2}(t) - qe^{\gamma t} \\
\frac{\partial V_1}{\partial S}(S_{f2}(t), t) &= 1
\end{align*}
$$

(6.17)

and in the transition region,

$$
\begin{align*}
\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \gamma - \delta) S \frac{\partial V_1}{\partial S} - (r - \gamma) V_1 &= \lambda_{12} (V_1 - (S - qe^{\gamma t})) \\
V_1(S, T) &= \max(S - qe^{\gamma T}, 0) \\
V_1(S_{f1}(t), t) &= S_{f1}(t) - qe^{\gamma t} \\
\frac{\partial V_1}{\partial S}(S_{f1}(t), t) &= 1
\end{align*}
$$

(6.18)

After performing the following change of variables:

$$
S = X qe^{\gamma t}, \ V_{1,2}(S, t) = U_{1,2}(X, \tau) qe^{\gamma t}, \ t = T - \tau,
$$
we obtain the stock loan PDE systems under a regime-switching economy, which are the same as those for the corresponding American option problem in Section 2, except \( r \) being replaced by \( \bar{r} = r - \gamma \). However, it is possible that \( \bar{r} \) could have a negative value. The rationale is that, in practice, the loan interest rate \( \gamma \) is usually higher than the risk-free interest rate \( r \). Stock loans evaluated as American call option problems could reach optimality even for dividend \( \delta = 0\% \) because of the presence of the loan interest rate \( \gamma \). It has been shown in Dai [32] and Lu and Putri [75], that for stock loans with \( \delta = 0\% \), the optimal exit price exists if \( r < \gamma \).

The equations in Laplace space for the stock loans in a regime switching economy have similar forms to Equations (6.13 - 6.16). The same procedures are followed to obtain the values of the optimal exit prices and the stock loans in the Laplace space. Numerical inversion is then used to invert the values back to the original time space, and from there it is straightforward to compute the service fees.

### 6.3.1 Numerical Examples

We now present numerical examples to show the performance of the proposed method in evaluating stock loans under a regime-switching economy. We present the calculation of optimal exit prices, stock loan values and service fees.

**Comparison Study for A Perpetual Stock Loan**

In this section, we present a comparison of our method and the analytic method in Zhang and Zhou [112] for a regime-switching perpetual American stock loan. The reason that we compare our results for a perpetual loan is due to the lack of results for a finite stock loan. We have to split our study into two parts due to the available data in [112] for the sake of comparison.

1. **Optimal exit prices:**
For comparison, we used the same parameters as those in [112]: the risk-free interest rate \( \bar{r} = -0.03 \), the variance for state 1 \( \sigma_1^2 = 0.04 \), the switching rates \( \lambda_{12} = \lambda_{21} = 2 \), and the initial loan amount \( q = 1 \).

Table 6.4 shows the dimensionless optimal exit prices for various values of \( \sigma_2^2 \) with a fixed \( \sigma_1^2 \) value.

### Table 6.4: Optimal Exit Price of A Perpetual Stock Loan (\( \sigma_1^2 = 0.04 \))

<table>
<thead>
<tr>
<th>( \sigma_2^2 )</th>
<th>( X_{f1} )</th>
<th>( X_{f2} )</th>
<th>( X_{f1} )</th>
<th>( X_{f2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>1.589</td>
<td>1.423</td>
<td>1.589</td>
<td>1.423</td>
</tr>
<tr>
<td>0.005</td>
<td>1.664</td>
<td>1.525</td>
<td>1.663</td>
<td>1.524</td>
</tr>
<tr>
<td>0.01</td>
<td>1.770</td>
<td>1.653</td>
<td>1.769</td>
<td>1.652</td>
</tr>
<tr>
<td>0.02</td>
<td>2.034</td>
<td>1.946</td>
<td>2.033</td>
<td>1.944</td>
</tr>
<tr>
<td>0.03</td>
<td>2.422</td>
<td>2.378</td>
<td>2.420</td>
<td>2.372</td>
</tr>
</tbody>
</table>

2. Stock loan values:

The values of a perpetual American stock loan under a regime-switching economy are calculated and compared with those obtained in [112] with the same parameters: the risk-free interest rate \( r = 0.05 \), the loan interest rate \( \gamma = 0.15 \), the volatility for state 1 \( \sigma_1 = 0.4 \), the volatility for state 2 \( \sigma_2 = 0.15 \), switching rates \( \lambda_{12} = \lambda_{21} = 4 \), and the initial loan amount \( q = 5 \).

As can be seen from Table 6.4 and Table 6.5, there is a good agreement between our results and those obtained by the analytical method in [112]. This indicates that our method achieved sufficient accuracy.

### Numerical Results for Finite Stock Loans

Having validated the accuracy of our method in the previous section, we present our calculation of the optimal exit prices and values of a finite maturity (\( T = 5 \) year) stock loan under a regime-switching economy in dimensionless variables.
The service fee is presented in dimensional variables to facilitate analysis of its relationship with the Loan To Value (LTV) ratio.

1. Optimal exit prices

Figure 6.3 presents a comparison of the optimal exit prices under a regime-switching economy and those under the standard Black-Scholes framework. The parameters used are: $r = -0.03$, $\delta = 0$, $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.02$, $\lambda_{12} = \lambda_{21} = 2$, $q = 1$, and $T = 5$. As expected, the curves of the optimal exit prices for the higher volatility are above the ones for the lower volatility, and the ones for regime-switching are bounded by the ones for the standard Black-Scholes model.

We then present the results of the optimal exit prices for different $\bar{r}$, with all other parameters the same as those in the previous figure. Figure 6.4 illustrates the variation of the optimal exit prices with $\bar{r}$. As can be seen when $\bar{r}$ becomes lower (more negative), the optimal exit price also decreases. This makes financial sense, since $\bar{r} = r - \gamma$, a more negative $\bar{r}$ means a higher relative loan interest rate compared to the risk-free interest rate, and one needs to exit sooner because of the larger accumulated loan.

Figure 6.5 depicts the optimal exit prices for different values of dividend $\delta$, all other parameters are the same as in Figure 6.3. The dividend rate affects the
Figure 6.3: Dimensionless optimal exit price between (RS: Regime-Switching; BS: Black-Sholes)

Figure 6.4: Dimensionless optimal exit price with regime-switching for different $\bar{r}$
Figure 6.5: Dimensionless optimal exit price with regime-switching for different $\delta$

Stock loan value through its effect on the collateralized stock. The stock price is expected to drop by the dividend amount on the ex-dividend date so a higher amount of dividend paid will lead to a lower optimal exit price and a lower value of the stock loans. An early exit from a stock loan contract, requires consideration of the cash received from the dividend so that optimal profit can be obtained from the optimal stock price plus the cash dividend.

2. Stock loan values

Table 6.6 presents the value of a dimensionless finite stock loan for various dimensionless stock price $X$ for the stock loans under a regime-switching economy and the Black-Scholes model. One can see from the table that the non-switching and switching stock loan values satisfy

$$U_{2BS} < U_2 < U_1 < U_{1BS},$$

which is similar to the relationship of the optimal exit prices, $X_{2BS} < X_{f2} < X_{f1} < X_{1BS}$. The reason $U_{1BS}$ is higher than $U_1$, is that the economy might change from a state of growth to recession during the life of the loan contract,
whereas, \( U_2 \) is higher than \( U_2^{BS} \), due to the possibility of the economy moving to a growth state. Higher collateralized stock prices imply that the stock loans have higher value.

**Table 6.6: Dimensionless Finite Stock Loan Value at \( t = 0 \)**

<table>
<thead>
<tr>
<th>( X )</th>
<th>( U_1^{BS} )</th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( U_2^{BS} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.869e-3</td>
<td>0.263e-3</td>
<td>0.222e-3</td>
<td>0.016e-3</td>
</tr>
<tr>
<td>0.6</td>
<td>0.012</td>
<td>0.006</td>
<td>0.006</td>
<td>0.002</td>
</tr>
<tr>
<td>0.8</td>
<td>0.051</td>
<td>0.035</td>
<td>0.033</td>
<td>0.019</td>
</tr>
<tr>
<td>1.0</td>
<td>0.129</td>
<td>0.104</td>
<td>0.101</td>
<td>0.080</td>
</tr>
<tr>
<td>1.2</td>
<td>0.248</td>
<td>0.220</td>
<td>0.216</td>
<td>0.205</td>
</tr>
</tbody>
</table>

### 3. Calculation of the service fee

The fair service fee for a finite stock loan under a regime-switching economy is computed in a similar way to that for the standard stock loan [75]. The service fee is determined at the beginning of the contract based on the initial value of the stock loan, and it can be calculated as \( c_i = V_i^0 - (S_0 - q) \), where \( c_i, \ (i = 1, 2) \) is the fair service fee depending on the state of the economy when the contract is made, \( S_0 \) is the initial value of the collateral stock, \( q \) is the amount of loan, and \( V_i^0, \ (i = 1, 2) \) is the initial value of the stock loan for volatility rate \( \sigma_1 \) and \( \sigma_2 \), respectively.

With the same quality of stock \( S \) and the same loan amount \( q \), the value of stock loans is affected by whether the valuation considers the economic changes. Thus, the fair service fee is also dependent on the state of the economy, given \( \sigma_1 > \sigma_2, \ c_2^{BS} < c_2 < c_1 < c_1^{BS} \). For the stock loans with switching from state 1 to 2 and vice versa, \( V_1 \) is more valuable than \( V_2 \) since the corresponding volatility \( \sigma_1 \) is greater than \( \sigma_2 \). This implies the service fee \( c_1 \) is more expensive than \( c_2 \). Under the regime-switching model, the calculated service fee is fairer since the influence of change in the economy on the stock loan value is taken into consideration.
The dependency of the service fee on the Loan-To-Value (LTV) ratios, which is defined as \( \text{LTV} = \left\{ \frac{q}{S_0} \right\} \), is shown in Table 6.7. The stock loan values used in the calculation are from Table 6.6. The service fee is presented in dimensional variables for different LTV ratios. With fixed \( S_0 \), different LTV imply different loan amount \( q \) in the calculation. The lender will charge a service fee based on the loan amount for the collateralized stock. A larger loan amount requires a higher service fee as the lender bears more risk. A stock loan contract which considers a regime-switching economy is marketable for any \( \text{LTV} > \left\{ \frac{q}{S_{f1}} \right\} = 0.740 \) if the economy is in state 1 and \( \text{LTV} > \left\{ \frac{q}{S_{f2}} \right\} = 0.687 \) if the economy is in state 2.

Table 6.7: Fair Service Fee for A Finite Maturity Stock Loan with \( S_0 = 1 \)

<table>
<thead>
<tr>
<th>LTV</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>0.056</td>
<td>0.063</td>
<td>0.104</td>
<td>0.167</td>
<td>0.243</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0.052</td>
<td>0.059</td>
<td>0.101</td>
<td>0.165</td>
<td>0.242</td>
</tr>
</tbody>
</table>

6.4 Conclusion

The new solution procedure for American call options under a regime-switching economy is presented first, followed by the valuation of stock loans under a regime-switching economy. Under the two-state regime-switching economy, there are two optimal boundaries, which divide the pricing domain into two parts: the common continuation region and the transition region. PDE systems for each region are set up and solved using the new method. Analytic approximation formulas to calculate optimal exercise price and American call option value are obtained in Laplace space. Numerical inversion is carried out to obtain the values in the original domain.

Numerical examples are presented in finite and infinite maturity for both American options and stock loans with regime-switching. The results show that the modified semi-analytic method is accurate and efficient. The results satisfy theoretical analysis in [108] that the switching values should be between the non-switching ones. For stock loans valuation, the optimal exit price, stock loan value, and service fee under a regime-switching economy are calculated. The dependency of these on other parameters are discussed. The results can be used to assist the borrower and lender to reach a decision about the exit policy and fair service fee charged by considering economic change.
Chapter 7

Concluding Remarks

In this thesis, there are three types of stock loans being studied as American options problem in finite maturity. The three types are non-recourse stock loans in three different dividend payment distributions, margin call stock loan, and non-recourse stock loans under a regime-switching economy. The stock loans are evaluated as the corresponding American options with possibly negative interest rate and time dependent strike price.

In Chapter 4, the valuations of non-recourse finite maturity stock loans in three different payment dividend distributions has been presented. The non-recourse stock loans are examined as American call options problems and solved using the semi-analytic method [113]. We obtained formulas to calculate the optimal exit price and stock loans value in finite maturity for the three dividend payment distributions. The numerical examples showed that the method is more efficient than the binomial method. The results could be used by the borrower and the lender to decide exit policy and the amount of service fee charged.

In Chapter 5, a new approach to evaluate margin call stock loans in finite maturity has been proposed. The margin call stock loan is solved as a pricing problem of American down and out call options with rebate in finite maturity using the semi-analytic method. The formulas to calculate the optimal exit price
and the margin call stock loan value in finite maturity are obtained. Numerical results are presented to show the dependency of the optimal exit price on payback and other parameters.

In Chapter 6, we proposed a new method to evaluate the stock loans under a regime-switching economy in finite maturity. The stock loan under a regime-switching economy in finite maturity is formulated as American call options problem under a regime-switching economy. It is assumed that the stock loans have a two-state regime-switching which describes the economic condition whether in the state of growth or recession. The new method is developed by combining the works in Guo [47] for perpetual American put options with regime-switching, Buffington and Elliott [17] for finite maturity American put options with regime-switching, Zhu [113] for finite maturity non-switching American put option and Zhang and Zhou [112] for perpetual stock loans with regime-switching. The formulas to calculate the optimal exercise price and American call option value under a regime-switching economy in finite maturity have been obtained. The valuation of stock loans under a regime-switching economy is carried out by following the solution process for the corresponding American call options. Some numerical examples are presented in finite and infinite maturity for both American options and stock loans with regime-switching. The results showed that the modified semi-analytic method is efficient and accurate to value the American options and stock loans under a regime-switching economy in finite maturity. The results could be useful in making early exit decision and the service fee charged.

The analytical approximation procedure used in this thesis for calculating optimal exit price and stock loan value is relatively robust to the grid size as the calculation in one particular time requires no discretization. It certainly costs less computational time than numerical methods. However, the Black-Scholes model which is used in all stock loans cases discussed in this thesis are more
appropriate for the stock price which has a tendency to follow a random-walk. If
the stock price tends to return to some average value over time then the Mean-
Reverting model can be used. Therefore, in the future it would be interesting
to apply another model such as Mean-Reverting model for the first two types
of stock loans or stock loan with regime-switching between Mean-Reverting and
Geometric Brownian Motion. The constant interest rate assumed in all stock
loans cases could also be replaced by the stochastic interest rate.
Appendices
Appendix A: Solution of A Non-Recourse Stock Loan

As the rebate is related to the initial value of a non-recourse stock loan, we recall the stock loan contract evaluation from [75].

Let \( \tilde{V}(X, \tau) \) be the dimensionless stock loan value, define the following function

\[
U = \begin{cases} 
\tilde{V} + 1 - X, & 1 < X \leq X_f; \\
\tilde{V}, & 0 \leq X < 1.
\end{cases}
\]

The PDE system for the non-recourse stock loan is transformed into an ordinary differential equation (ODE) system in Laplace space, and the solution of \( \bar{U} \) in the Laplace space is presented below:

\[
\bar{U} = \begin{cases} 
\frac{k_2(p\bar{X}_f)^{k_2-k_1}}{k_2-k_1} W X^{k_1} + \frac{\beta(p\bar{X}_f)^{1-k_2}}{k_2 p(p + \beta)} X^{k_2} + \frac{p(\alpha - \beta p\bar{X}_f)) + \alpha \beta (1 - p\bar{X}_f))}{p(p + \alpha)(p + \beta)}, & \text{if } 1 < X \leq X_f \\
\left\{ (1 - (p\bar{X}_f)^{k_2-k_1}) \frac{k_2}{k_2-k_1} W + \frac{(p\bar{X}_f)^{1-k_2} \beta + p}{k_1 p(p + \beta)} \right\} X^{k_1}, & \text{if } X \leq 1
\end{cases}
\]

(A-1)
where \( k_1 \) and \( k_2 \) are the solutions of the following equation

\[
\frac{1}{2} \sigma^2 k^2 + (r - \delta - \frac{1}{2} \sigma^2)k - (p + r) = 0
\]

\[
W = \frac{\beta - \alpha}{(p + \alpha)(p + \beta)} + \frac{(p \bar{X}_f)^{1-k_2} \beta + p}{k_1 p (p + \beta)} - \frac{(p \bar{X}_f)^{1-k_2} \beta}{k_2 p (p + \beta)}
\]

In order to compute the value of \( \bar{U}(X, p) \), we need to obtain the optimal exit price \( \bar{X}_f(p) \) in the Laplace space from the following equation.

\[
\bar{X}_f^{k_2} \left\{ \frac{k_1 (\beta - \alpha) + (p + \alpha)}{k_1 (p + \alpha) (p + \beta)} \right\} + \bar{X}_f \left\{ \frac{\beta (1 - k_1)}{k_1 p^{k_2} (p + \beta)} \right\} = -\frac{\alpha}{p^{1+k_2} (p + \alpha)} \quad (A-2)
\]
Appendix B: Solution Procedure for Common Continuation Region

In this appendix the PDE systems in Equation (6.1) are solved by following Zhu [113].

\[
\begin{align*}
-\frac{\partial U_1}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_1}{\partial X^2} + (r - \delta)X \frac{\partial U_1}{\partial X} - rU_1 &= \lambda_{12}(U_1 - U_2) \\
U_1(0, \tau) &= 0 \\
U_1(X, 0) &= \max(X - 1, 0) \\
-\frac{\partial U_2}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_2}{\partial X^2} + (r - \delta)X \frac{\partial U_2}{\partial X} - rU_2 &= \lambda_{21}(U_2 - U_1) \\
U_2(0, \tau) &= 0 \\
U_2(X, 0) &= \max(X - 1, 0) \\
U_2(X_{f2}, \tau) &= X_{f2} - 1 \\
\frac{\partial U_2}{\partial X}(X_{f2}, \tau) &= 1
\end{align*}
\]

(B-1)

Employing two ranges in the common continuation region, \(1 < X \leq X_{f2}\) and \(X \leq 1\), in the PDE system of Equation (B-1), the PDEs can be represented as shown in Equation (B-2) and (B-4).
The PDE system for \( U_1 \),

\[
\begin{cases}
-\frac{\partial U_1}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_1}{\partial X^2} + (r - \delta)X \frac{\partial U_1}{\partial X} - rU_1 = \lambda_{12}(U_1 - U_2), & \text{if } 1 < X \leq X_{f2}; \\
U_1(0, \tau) = 0 \\
U_1(X, 0) = X - 1 \\
-\frac{\partial U_1}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_1}{\partial X^2} + (r - \delta)X \frac{\partial U_1}{\partial X} - rU_1 = \lambda_{12}(U_1 - U_2), & \text{if } X \leq 1; \\
U_1(0, \tau) = 0 \\
U_1(X, 0) = 0
\end{cases}
\]  

(B-2)

and the continuity conditions to ensure that \( U_1 \) is continuous at \( X = 1 \) are as follows:

\[
\begin{align*}
\lim_{x \to 1^-} U_1 &= \lim_{x \to 1^+} U_1 \\
\lim_{x \to 1^-} \frac{\partial U_1}{\partial X} &= \lim_{x \to 1^+} \frac{\partial U_1}{\partial X}
\end{align*}
\]  

(B-3)

The PDE system for \( U_2 \),

\[
\begin{cases}
-\frac{\partial U_2}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_2}{\partial X^2} + (r - \delta)X \frac{\partial U_2}{\partial X} - rU_2 = \lambda_{21}(U_2 - U_1), & \text{if } 1 < X \leq X_{f2}; \\
U_2(0, \tau) = 0 \\
U_2(X, 0) = X - 1 \\
U_2(X_{f2}, \tau) = X_{f2} - 1 \\
\frac{\partial U_2}{\partial X}(X_{f2}, \tau) = 1 \\
-\frac{\partial U_2}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_2}{\partial X^2} + (r - \delta)X \frac{\partial U_2}{\partial X} - rU_2 = \lambda_{21}(U_2 - U_1), & \text{if } X \leq 1; \\
U_2(0, \tau) = 0 \\
U_2(X, 0) = 0
\end{cases}
\]  

(B-4)

and the continuity conditions to ensure that \( U_2 \) is continuous at \( X = 1 \) are as
follows:

\[
\lim_{x \to 1^-} U_2 = \lim_{x \to 1^+} U_2 \tag{B-5}
\]

\[
\lim_{x \to 1^-} \frac{\partial U_2}{\partial X} = \lim_{x \to 1^+} \frac{\partial U_2}{\partial X}
\]

Subsequently, the non-dimensional PDEs are transformed to the Laplace space to obtain \( \bar{U}_1 \):

\[
\begin{align*}
\frac{1}{2} \sigma^2 X^2 \frac{d^2 \bar{U}_1}{dX^2} + (r - \delta)X \frac{d\bar{U}_1}{dX} - (p + r + \lambda_{12})\bar{U}_1 + \lambda_{12} \bar{U}_2 &= -(X - 1), \quad \text{if } 1 < X \leq X_{f1}; \\
\bar{U}_1(0, p) &= 0 \\
\frac{1}{2} \sigma^2 X^2 \frac{d^2 \bar{U}_1}{dX^2} + (r - \delta)X \frac{d\bar{U}_1}{dX} - (p + r + \lambda_{12})\bar{U}_1 + \lambda_{12} \bar{U}_2 &= 0, \quad \text{if } X \leq 1; \\
\bar{U}_1(0, p) &= 0
\end{align*}
\]  
\tag{B-6}

and the continuity conditions in Laplace space are

\[
\begin{align*}
\bar{U}_1(1^-, p) &= \bar{U}_1(1^+, p) \\
\frac{d\bar{U}_1}{dX}(1^-, p) &= \frac{d\bar{U}_1}{dX}(1^+, p)
\end{align*}
\]  
\tag{B-7}

and for \( \bar{U}_2 \)

\[
\begin{align*}
\frac{1}{2} \sigma^2 X^2 \frac{d^2 \bar{U}_2}{dX^2} + (r - \delta)X \frac{d\bar{U}_2}{dX} - (p + r + \lambda_{21})\bar{U}_2 + \lambda_{21} \bar{U}_1 &= -(X - 1), \quad \text{if } 1 < X \leq X_{f2}; \\
\bar{U}_2(0, p) &= 0 \\
\bar{U}_2(pX_{f2}, p) &= \bar{X}_{f2} - \frac{1}{p} \\
\frac{d\bar{U}_2}{dX}(pX_{f2}, p) &= \frac{1}{p} \\
\frac{1}{2} \sigma^2 X^2 \frac{d^2 \bar{U}_2}{dX^2} + (r - \delta)X \frac{d\bar{U}_2}{dX} - (p + r + \lambda_{21})\bar{U}_2 + \lambda_{21} \bar{U}_1 &= 0, \quad \text{if } X \leq 1; \\
\bar{U}_2(0, p) &= 0
\end{align*}
\]  
\tag{B-8}
and the continuity conditions are

\[
\begin{align*}
\bar{U}_2(1^-, p) &= \bar{U}_2(1^+, p) \\
\frac{d\bar{U}_2}{dX}(1^-, p) &= \frac{d\bar{U}_2}{dX}(1^+, p)
\end{align*}
\]  

(B-9)

Equations (B-6-B-9) are ODE systems which are easier to be solved. In performing the Laplace transform, the pseudo-steady-state approximation proposed by Zhu [113] is applied to the free boundary.

The Solutions to the ODE systems are in the following form:

\[
\bar{U}_1(X, p) = \begin{cases} 
A_1X^{k_1} + A_2X^{k_2} + A_3X^{k_3} + A_4X^{k_4} + AA(X, p), & \text{if } 1 < X \leq X_{f2}; \\
A_5X^{k_1} + A_6X^{k_2} + A_7X^{k_3} + A_8X^{k_4}, & \text{if } X \leq 1;
\end{cases}
\]  

(B-10)

\[
\bar{U}_2(X, p) = \begin{cases} 
B_1X^{k_1} + B_2X^{k_2} + B_3X^{k_3} + B_4X^{k_4} + BB(X, p), & \text{if } 1 < X \leq X_{f2}; \\
B_5X^{k_1} + B_6X^{k_2} + B_7X^{k_3} + B_8X^{k_4}, & \text{if } X \leq 1;
\end{cases}
\]  

(B-11)

where \(AA(X, p)\) and \(BB(X, p)\) are linear in terms of \(X\). The constants \(A_i\) and \(B_i\) are related by the following equation:

\[
\left\{ \frac{1}{2}\sigma_1^2k_i(k_i - 1) + (r - \delta)k_i - (p + r + \lambda_{12}) \right\} A_i + \lambda_{12}B_i = 0
\]

or

\[
\left\{ \frac{1}{2}\sigma_2^2k_i(k_i - 1) + (r - \delta)k_i - (p + r + \lambda_{21}) \right\} B_i + \lambda_{21}A_i = 0
\]

where \(k_i\) is the solutions to the quartic indicial equation shown in Appendix ??.

The fact that \(A_i\) are related to \(B_i\) allows us to solve only for the \(A_i\)'s in the final equations. Further, it can be proved that \(k_1 < k_2 < 0 < k_3 < k_4\).

Recall the boundary condition \(\bar{U}_{1,2}(0, p) = 0\), it is apparent that the solutions
in the region $X < 1$ can not involve negative powers of $X$. This means that $A_5, A_6$ and $B_5, B_6$ should be zero. Renaming the constants $A_7, A_8$ and $B_7, B_8$ as $A_5, A_6$ and $B_5, B_6$, the final solutions of $\bar{U}_1$ and $\bar{U}_2$ in the Laplace space are given as

$$\bar{U}_1(X, p) = \begin{cases} 
A_1X^{k_1} + A_2X^{k_2} + A_3X^{k_3} + A_4X^{k_4} + AA(X, p), & \text{if } 1 < X \leq X_{f2}; \\
A_5X^{k_3} + A_6X^{k_4}, & \text{if } X \leq 1;
\end{cases}$$

$$\bar{U}_2(X, p) = \begin{cases} 
B_1X^{k_1} + B_2X^{k_2} + B_3X^{k_3} + B_4X^{k_4} + BB(X, p), & \text{if } 1 < X \leq X_{f2}; \\
B_5X^{k_3} + B_6X^{k_4}, & \text{if } X \leq 1;
\end{cases}$$

where $AA(X, p) = BB(X, p) = \frac{1}{p + \delta}X - \frac{1}{p + r}$
Appendix C: Solution for Transition Region

In this appendix, we present the steps to solve the PDE system in transition region. We start with equation system for transition region (6.9).

\[
\begin{align*}
-\frac{\partial U_1}{\partial \tau} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 U_1}{\partial X^2} + (r - \delta) X \frac{\partial U_1}{\partial X} - r U_1 &= \lambda_{12} (U_1 - (X - 1)) \\
U_1(X, 0) &= X - 1 \\
U_1(X_{f1}, \tau) &= X_{f1} - 1 \\
\frac{\partial U_1}{\partial X}(X_{f1}, \tau) &= 1
\end{align*}
\]

(C-1)

The transformed system in the Laplace space is:

\[
\begin{align*}
\frac{1}{2} \sigma^2 X^2 \frac{d^2 \bar{U}_1}{dX^2} + (r - \delta) X \frac{d\bar{U}_1}{dX} - (p + r + \lambda_{12}) \bar{U}_1 &= -(X - 1) - \frac{\lambda_{12}(X - 1)}{p} \\
\bar{U}_1(p\bar{X}_{f1}, p) &= \bar{X}_{f1} - \frac{1}{p} \\
\frac{d\bar{U}_1}{dX}(p\bar{X}_{f1}, p) &= \frac{1}{p}
\end{align*}
\]

(C-2)

The equation (C-2) is an ordinary differential equation which is easier now to solve. Its solution has a general form of

\[
\bar{U}_1 = m_1 X^{\gamma_1} + m_2 X^{\gamma_2} + M(X, p)
\]

(C-3)

where \(m_{1,2}\) are integration constants, \(\gamma_{1,2}\) are the solutions of the indicial equation

\[
\frac{1}{2} \sigma^1 \gamma^2 + (r - \delta - \frac{1}{2} \sigma^1) \gamma - (p + r + \lambda_{12}) = 0,
\]
and $M(X, p)$ is the particular solution of Equation (C-2),

$$
M(X, p) = \left\{ \frac{p + \lambda_{12}}{p(p + \delta + \lambda_{12})} \right\} X - \left\{ \frac{p + \lambda_{12}}{p(p + r + \lambda_{12})} \right\}
$$

$$
\Psi_1 = \left\{ \frac{p + \lambda_{12}}{p(p + \delta + \lambda_{12})} \right\}, \quad \Psi_2 = \left\{ \frac{p + \lambda_{12}}{p(p + r + \lambda_{12})} \right\}
$$
Appendix D: Roots of Equation

Roots of Quartic Equation

The solutions in the Laplace space in the common continuation region are expected to have the following form:

\[
\tilde{U}_1(X, p) = \begin{cases} 
A_1X^{k_1} + A_2X^{k_2} + A_3X^{k_3} + A_4X^{k_4} + AA(X, p), & \text{if } 1 < X \leq X_{f_2}; \\
A_5X^{k_1} + A_6X^{k_2} + A_7X^{k_3} + A_8X^{k_4}, & \text{if } X \leq 1;
\end{cases} 
\]

\[\text{(D-1)}\]

\[
\tilde{U}_2(X, p) = \begin{cases} 
B_1X^{k_1} + B_2X^{k_2} + B_3X^{k_3} + B_4X^{k_4} + BB(X, p), & \text{if } 1 < X \leq X_{f_2}; \\
B_5X^{k_1} + B_6X^{k_2} + B_7X^{k_3} + B_8X^{k_4}, & \text{if } X \leq 1;
\end{cases}
\]

\[\text{(D-2)}\]

Where \(AA(X, p)\) and \(BB(X, p)\) are the particular solutions of the corresponding equations, they are linear in terms of \(X\).

Substituting the solutions in (D-1) and (D-2) and their derivatives into Equation (B-6) and (B-8), we obtain the relation between constants \(A_i\) and \(B_i\):

\[
\left\{ \frac{1}{2} \sigma_i^2 k_i (k_i - 1) + (r - \delta) k_i - (p + r + \lambda_{12}) \right\} A_i + \lambda_{12} B_i = 0 \quad \text{(D-3)}
\]

\[
\left\{ \frac{1}{2} \sigma_i^2 k_i (k_i - 1) + (r - \delta) k_i - (p + r + \lambda_{21}) \right\} B_i + \lambda_{21} A_i = 0 \quad \text{(D-4)}
\]
The last two equations are equivalent. Let \( l(k_i) = \frac{1}{2}\sigma_1^2k_i(k_i - 1) + (r - \delta)k_i - (p + r + \lambda_{12}) \), the relation between \( A_i \) and \( B_i \) can be expressed as follows

\[
B_i = -A_i \frac{l(k_i)}{\lambda_{12}}, \quad i = 1...4 \tag{D-5}
\]

\( A_i \) and \( B_i \) for \( i = 5, 6, 7, 8 \) can be expressed in a similar way to (D-5):

\[
B_i = -A_i \frac{l(k_{i-4})}{\lambda_{12}}, \quad i = 5...8 \tag{D-6}
\]

Note that for simplicity we use \( l_i = l(k_i), \quad i = 1, 2, 3, 4 \) in the main section of this paper.

In common continuation region, the indicial equation for the corresponding differential equations is a quartic equation as written below:

\[
F(k) = l(k)g(k) - \lambda_{12}\lambda_{21} \tag{D-7}
\]

where

\[
l(k) = \frac{1}{2}\sigma_1^2k^2 + (r - \delta - \frac{1}{2}\sigma_1^2)k - (p + r + \lambda_{12}) \quad \text{and} \quad g(k) = \frac{1}{2}\sigma_2^2k^2 + (r - \delta - \frac{1}{2}\sigma_2^2)k - (p + r + \lambda_{21}).
\]

Equation (D-7) has two quadratic functions \( l(k) \) and \( g(k) \). Let \( kk_1 \) and \( kk_2 \) be the roots of quadratic equation:

\[
l(kk) = \frac{1}{2}\sigma_1^2kk^2 + (r - \delta - \frac{1}{2}\sigma_1^2)kk - (p + r + \lambda_{12}) = 0 \tag{D-8}
\]

then \( l(\infty) = l_1(-\infty) = \infty \) and \( l(0) = -(p + r + \lambda_{12}) < 0 \).

Therefore it can be concluded that \( l(kk) \) has two real roots \( kk_1 < 0 \) and \( kk_2 > 0 \). The value of \( F(kk_1) \) and \( F(kk_2) \) equal \(-\lambda_{12}\lambda_{21} < 0 \) since \( \lambda_{12} > 0 \) and \( \lambda_{21} > 0 \). Note that \( F(0) = (p+r)(p+r+\lambda_{12}+\lambda_{21}) > 0 \) and \( F(\infty) = F(-\infty) = \infty \). Therefore, we can conclude that the quartic equation \( F(k) \) has four real roots \( k_1, k_2, k_3, k_4 \).
Roots of Quadratic Equation

For transition region, the indicial equation can be written as follows:

\[
\frac{1}{2}\sigma_1^2 \gamma^2 + (r - \delta - \frac{1}{2}\sigma_1^2)\gamma - (p + r + \lambda_{12}) = 0
\]

\[
\gamma_{1,2} = \frac{1}{2} - \frac{r - \delta}{\sigma_1^2} \pm \sqrt{\left\{\frac{1}{2} + \frac{r - \delta}{\sigma_1^2}\right\}^2 + \frac{2}{\sigma_1^4}\{p + r + \lambda_{12}\}}
\]
Appendix E: Initial Guessing

To calculate the dimensionless optimal exercise price in Laplace space from Equation (6.16), a pair of initial guesses are required for $\bar{X}_{f1}$ and $\bar{X}_{f2}$. The following adjusted optimal exercise prices $X_{f1,2(\tau)}$ will be used as initial guesses for states 1 and 2:

$$X_{f1,2}(\tau) = X_{f1,2}^\infty - (X_{f1,2}^\infty - X_{1,2}^o) e^{-\varepsilon \tau} \quad (E-1)$$

where $X_{f1,2}^\infty$ is the optimal exercise price at $T \rightarrow \infty$, $X_{1,2}^o$ is the optimal exercise price at $\tau = 0$ and $\varepsilon$ is the adjustment factor. The two entities $X_{f1,2}^\infty$ and $X_{1,2}^o$ are obtained analytically and they are monotonic increasing in $\tau$ for call and monotonic decreasing in $\tau$ for put.

To use the adjusted optimal exercise prices $X_{f1,2}(\tau)$ as initial guesses, one has to transform the values to Laplace space where the initial guesses can be written as $\bar{X}_{f1,2}(p)$. The adjustment factor $\varepsilon$ ranges between $0.00001 - 0.4$ for to obtain good results. Using the initial guesses, we can speed up the computation of the optimal exercise prices of finite American options under regime-switching economy.
Bibliography


