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PURELY INFINITE $C^*$-ALGEBRAS ASSOCIATED TO ÉTALE GROUPOIDS

JONATHAN BROWN, LISA ORLOFF CLARK, AND ADAM SIERAKOWSKI

Abstract. Let $G$ be a Hausdorff, étale groupoid that is minimal and topologically principal. We show that $C^*_r(G)$ is purely infinite simple if and only if all the nonzero positive elements of $C_0(G(0))$ are infinite in $C^*_r(G)$. If $G$ is a Hausdorff, ample groupoid, then we show that $C^*_r(G)$ is purely infinite simple if and only if every nonzero projection in $C_0(G(0))$ is infinite in $C^*_r(G)$. We then show how this result applies to $k$-graph $C^*$-algebras. Finally, we investigate strongly purely infinite groupoid $C^*$-algebras.

1. Introduction

Purely infinite simple $C^*$-algebras were introduced by Cuntz in [7] where he showed that the $K_0$ group of such algebras can be computed within the algebra itself without resorting to the usual direct limit construction. The $K$-theory groups of $C^*$-algebras have long been known to be computable invariants and Cuntz’s result shows that this computation is easier when the $C^*$-algebra is purely infinite simple. Elliott initiated a program to find a suitably large class of $C^*$-algebras on which the $K$-theory groups provide a complete isomorphism invariant (see [8]). This program has achieved remarkable success, most notably in a theorem of Kirchberg and Phillips [11, 20] which states that every Kirchberg algebra satisfying the Universal Coefficient Theorem (UCT) is classified by the isomorphism classes of its ordered $K$-theory groups. A Kirchberg algebra is a separable, nuclear, purely infinite simple $C^*$-algebra.

The allure of classification via the Kirchberg-Phillips theorem has lead many authors to study when various constructions of $C^*$-algebras yield purely infinite simple algebras. Kumjian, Pask and Raeburn show that a graph $C^*$-algebra of a cofinal graph is purely infinite simple if and only if every vertex can be reached from a loop with an entrance [17, Theorem 3.9]. Carlsen and Thomsen show that if the $C^*$-algebra constructed from a locally injective surjection $\theta$ on a compact metric space of finite covering dimension is simple, then it is purely infinite simple if and only if $\theta$ is not a homeomorphism [5, Corollary 6.6]. Rørdam and Sierakowski [24] show that if a countable exact group $H$ acts by an essentially free action on the Cantor set $X$ and the type semigroup of clopen subsets of $X$ is almost unperforated, then $C_0(X) \rtimes_r H$ is purely infinite if and only if every clopen set $E$ in $X$ is paradoxical. The constructions in each of the above examples are special cases of groupoid $C^*$-algebras.

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In this paper we investigate purely infinite $C^*$-algebras associated to Hausdorff, étale groupoids. In sections 3 through 5 we restrict our attention to simple groupoid $C^*$-algebras. Characterising simplicity of groupoid $C^*$-algebras is known and we readily make use of the following theorem from [1, Theorem 5.1]:

**Theorem 1.1** (Brown-Clark-Farthing-Sims). *Let $G$ be a second-countable, locally compact, Hausdorff and étale groupoid. Then $C^*(G)$ is simple if and only if all of the following conditions are satisfied.*

1. $C^*(G) = C^*_r(G)$;
2. $G$ is topologically principal;
3. $G$ is minimal.

However, necessary and sufficient conditions on the groupoid for the associated algebra to be *purely infinite* simple are not known. Anantharaman-Delaroche showed that ‘locally contracting’ is a sufficient condition on the groupoid in [1, Proposition 2.4] but whether locally contracting is necessary remains an open question. Part of the difficulty in characterizing those groupoids that give rise to purely infinite simple $C^*$-algebras is relating arbitrary projections in the groupoid $C^*$-algebra to the groupoid itself.

In Section 3 we look at a necessary and sufficient conditions for ensuring pure infiniteness of groupoid $C^*$-algebras. We show that for a Hausdorff, étale, topologically principal, and minimal groupoid $G$ the $C^*$-algebra $C^*_r(G)$ is purely infinite simple if and only if it all the nonzero positive elements of $C_0(G^{(0)})$ are infinite in $C^*_r(G)$ and (see Theorem 3.3). In Section 4 we specialize to Hausdorff, ample groupoids. This is an important class of examples because every Kirchberg algebra in UCT is Morita equivalent to an algebra associated to an Hausdorff, ample groupoid (see [26]). We show in Theorem 4.1 for a Hausdorff, ample groupoid $G$, that is also topologically principal and minimal, the $C^*$-algebra $C^*_r(G)$ is purely infinite if and only if every nonzero projection in $C_0(G^{(0)})$ is infinite in $C^*_r(G)$.

Theorem 4.1 is a generalisation of [10] about partial actions. In Section 5 we demonstrate how Theorem 4.1 applies to $k$-graph $C^*$-algebras.

In Section 6 we turn our attention to the non-simple case. In [13], Kirchberg and Rørdam introduce three separate notions of purely infinite $C^*$-algebras: weakly purely infinite, purely infinite and strongly purely infinite. Of these notions, the last one appears to be the most useful in the classification theory of non-simple $C^*$-algebras. Indeed, Kirchberg showed in [12] that two separable, nuclear, strongly purely infinite $C^*$-algebras with the same primitive ideal space $X$ are isomorphic if and only if they are $KK_X$-equivalent. We provide a characterization of when groupoid $C^*$-algebras are strongly purely infinite in Proposition 6.3.

2. Preliminaries

### 2.1. Groupoids

A *groupoid* $G$ is a small category in which every morphism is invertible. The set of objects in $G$ is identified with the set of identity morphisms and both are denoted by $G^{(0)}$. We call $G^{(0)}$ the *unit space* of $G$. Each morphism $\gamma$ in the category has a range and source denoted $r(\gamma)$ and $s(\gamma)$ respectively and thus $r$ and $s$ define maps $G \to G^{(0)}$.

A *topological groupoid* is a groupoid with a topology in which composition is continuous and inversion is a homeomorphism. An *open bisection* in a topological groupoid $G$ is an open set $B$ such that both $r$ and $s$ restricted to $B$ are homeomorphisms; in particular
these restrictions are injective. An étale groupoid is a topological groupoid where \( s \) is a local homeomorphism. If a groupoid \( G \) is Hausdorff and étale, then the unit space \( G^{(0)} \) is open and closed in \( G \). If \( G \) is a locally compact, Hausdorff groupoid, then \( G \) is étale if and only if there is a basis for the topology on \( G \) consisting of open bisections with compact closure. A topological groupoid is called ample if it has a basis of compact open bisections. If \( G \) is locally compact, Hausdorff and étale groupoid, then we note that \( G \) is ample if and only if \( G^{(0)} \) is totally disconnected (see [3] Proposition 4.1).

For a subsets \( L, K \subseteq G \), denote \( LK = \{ \gamma : \gamma = \xi\zeta \text{ with } \xi \in L, \zeta \in K, s(\xi) = r(\zeta) \} \). With a slight abuse of notation, for \( u \in G^{(0)} \), we write \( uG \) and \( Gu \) for \( \{ u \} G \) and \( G \{ u \} \) respectively and denote by \( uGu \) the set

\[
\{ \gamma \in G : r(\gamma) = s(\gamma) = u \}.
\]

A topological groupoid \( G \) is topologically principal if the set \( \{ u \in G^{(0)} : uGu = \{ u \} \} \) is dense in \( G^{(0)} \), and minimal if \( G : u := \{ r(\gamma) : s(\gamma) = u \} \) is dense in \( G^{(0)} \) for all \( u \in G^{(0)} \). Recall, for a second countable, locally compact, Hausdorff, étale groupoid \( G \) the algebra \( C^*(G) \) is simple if and only if \( G \) is minimal, topologically principal, and \( C^*(G) = C^*_r(G) \).

2.2. Groupoid \( C^*\)-algebras. Let \( G \) be locally compact, Hausdorff étale groupoid and let \( C_c(G) \) denote the set of compactly supported continuous functions from \( G \) to \( \mathbb{C} \). Since every element \( \gamma \) of \( G \) has a neighbourhood \( B_\gamma \) such that \( r|_{B_\gamma} \) is injective, the set \( r^{-1}(u) \) is discrete for every \( u \in G^{(0)} \). Thus if \( f \in C_c(G) \) then \( \text{supp}(f) \cap r^{-1}(u) \) is finite for all \( u \in G^{(0)} \). With this, we are able to define a convolution and involution on \( C_c(G) \) such that for \( f, g \in C_c(G) \),

\[
f \ast g(\gamma) := \sum_{r(\eta) = r(\gamma)} f(\eta)g(\eta^{-1}\gamma) \quad \text{and} \quad f^*(\gamma) := \overline{f(\gamma^{-1})}.
\]

Under these operations, \( C_c(G) \) is a \(*\)-algebra. Next define for \( f \in C_c(G) \),

\[
\|f\|_I := \sup_{u \in G^{(0)}} \{ \max \{ \sum_{\gamma \in Gu} |f(\gamma)|, \sum_{\gamma \in uG} |f(\gamma)| \} \} \quad \text{and} \quad \|f\| := \sup \{ \|\pi(f)\| : \pi \text{ is a } \| \cdot \|_I \text{-decreasing representation} \}.
\]

Then \( C^*(G) \) is the completion of \( C_c(G) \) in the \( \| \cdot \| \)-norm.

Given a unit \( u \in G^{(0)} \), the regular representation \( \pi_u \) of \( C_c(G) \) on \( \ell^2(Gu) \) associated to \( u \) is characterized by

\[
\pi_u(f)\delta_\gamma = \sum_{s(\eta) = r(\gamma)} f(\eta)\delta_{\eta\gamma}.
\]

The reduced \( C^*\)-norm on \( C_c(G) \) is \( \|f\|_r = \sup \{ \|\pi_u(f)\| : u \in G^{(0)} \} \) and \( C^*_r(G) \) is the completion of \( C_c(G) \) in the \( \| \cdot \|_r \)-norm. Our attention will be focused on the reduced \( C^*\)-algebra and situations where the reduced and full algebras coincide. Also, we will often be assuming \( G \) is second-countable, implying \( C^*_r(G) \) is separable [21] Page 59.

Below we recall a few standard results (we heavily use) and their proofs to familiarise the reader with locally compact, Hausdorff étale groupoids.

**Lemma 2.1** (cf. [21]). Let \( G \) be a locally compact, Hausdorff and étale groupoid. Then

1. The extension map from \( C_c(G^{(0)}) \) into \( C_c(G) \) (where a function is defined to be zero on \( G - G^{(0)} \)) extends to an embedding of \( C_0(G^{(0)}) \) into \( C^*_r(G) \).
(2) The restriction map $E_0 : C_c(G) \to C_c(G^{(0)})$ extends to a conditional expectation $E : C_c^*(G) \to C_0(G^{(0)})$.

(3) The map $E$ from item (2) is faithful. That is, $E(b^*b) = 0$ implies $b = 0$ for $b \in C_c^*(G)$.

(4) For every closed invariant set $D \subseteq G^{(0)}$ we have the following commuting diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C_c^*(G|_U) & \overset{\iota_r}{\longrightarrow} & C_c^*(G) & \overset{\rho_r}{\longrightarrow} & C_c^*(G|_D) & \longrightarrow & 0 \\
E_U \downarrow & & E \downarrow & & E_D \downarrow & & & & \\
0 & \longrightarrow & C_0(U) & \overset{\iota_0}{\longrightarrow} & C_0(G^{(0)}) & \overset{\rho_0}{\longrightarrow} & C_0(D) & \longrightarrow & 0
\end{array}
\]

where $U = G^{(0)} - D$, $\iota_r$ and $\rho_r$ are determined on continuous functions by extension and restriction respectively. Moreover, $\text{image}(\iota_r) \subseteq \ker \rho_r$.

(5) The subalgebra $C_c(G^{(0)})$ contains an approximate unit for $C_c^*(G)$.

Proof. Since $G$ is Hausdorff and étale, $G^{(0)}$ is open and closed in $G$. Thus, the map $C_c(G^{(0)})$ into $C_c(G)$ is well defined. For $f, g \in C_c(G^{(0)})$, a quick computation gives

\[
f * g(\gamma) = \begin{cases} f(\gamma)g(\gamma), & \text{if } \gamma \in G^{(0)}; \\ 0, & \text{otherwise}, \end{cases}
\]

so the map from $C_c(G^{(0)})$ into $C_c(G)$ is a *-homomorphism. We claim the map is isometric, that is, we claim the reduced norm agrees with the infinity norm for functions in $C_c(G^{(0)})$. By evaluating at point masses in $\ell^2(Gu)$, one can show that $\|f\|_\infty \leq \|f\|$, for $f \in C_c(G)$. The reverse inequality can be verified for $f \in C_c(G^{(0)})$ and the claim follows. Thus the *-homomorphism from $C_c(G^{(0)})$ into $C_c(G) \subseteq C_c^*(G)$ extends by continuity to an isometric (hence injective) *-homomorphism from $C_0(G^{(0)})$ into $C_c^*(G)$.

Once again using that $G$ is Hausdorff and étale, we have that $G^{(0)}$ is open and closed in $G$ and hence $E_0$ is well defined. One may easily verify that $E_0$ is (a) positive (b) linear (c) idempotent, and (d) of norm one. Therefore $E_0$ extends by continuity to a map $E : C_c^*(G) \to C_0(G^{(0)})$ with the same properties (a)–(d). By [3, II.6.10.1] we conclude that $E$ is a conditional expectation.

Let $b \in C_c^*(G)$ such that $E(b^*b) = 0$. We need to show that $b = 0$. Let $V_\gamma : \mathbb{C} \to \ell^2(G_{s(\gamma)})$ be given by $c \mapsto c\delta_\gamma$. Then $V_\gamma^* \omega = \omega(\gamma)$. Since $\|b\|_r = \sup_{u \in G^{(0)}} \|\pi_u(b)\|$ and

\[
\|\pi_u(b)\delta_\gamma\|^2 = \langle \pi_u(b)\delta_\gamma, \pi_u(b)\delta_\gamma \rangle = \langle \pi_u(b^*b)\delta_\gamma, \delta_\gamma \rangle = V_\gamma^* \pi_u(b^*b)V_\gamma \delta_\gamma,
\]

it suffices to show that $V_\gamma^* \pi_u(b^*b)V_\gamma = 0$ for all $u \in G^{(0)}$ and $\gamma \in G$.

For $f \in C_c(G)$, $u \in G^{(0)}$, and $c \in \mathbb{C}$, we have

\[
(2.1) \quad V_u^* \pi_u(f)V_u c = V_u^* \pi_u(f)c\delta_u = V_u^*(\sum_{s(\eta)=u} f(\eta)c\delta_\eta) = f(u)c = E(f)(u)c.
\]

Thus by the continuity of $E$, for all $a \in C_c^*(G)$, $E(a)(u) = V_u^* \pi_u(a)V_u$ as operators on $\mathbb{C}$.

For every open bisection $B$ and $\gamma \in B$, pick a function $\phi_{\gamma,B} \in C_c(G)$ such that $\phi_{\gamma,B}(\gamma) = 1$, supp($\phi_{\gamma,B}$) $\subseteq B$, and $0 \leq \phi_{\gamma,B} \leq 1$. Now if $f \in C_c(G)$ and $B$ is an open bisection with

\[1\text{Recall: A conditional expectation }E : A \to B \text{ is a contractive, linear, completely positive map such that for every } b \in B, a \in A \text{ we have } E(b) = b, E(ba) = bE(a) \text{ and } E(ab) = E(a)b , \text{ see [3, II.6.10.1].}\]
γ \in B$, then
\[(E(\phi^*_{\gamma,B}f\phi_{\gamma,B}))(u) = \sum_{r(\xi)=r(\zeta)=u} \phi_{\gamma,B}(\xi^{-1})f(\xi^{-1}\zeta)\phi_{\gamma,B}(\zeta^{-1}),\]
which is zero unless \(\xi, \zeta \in B^{-1}\). Since \(r(\xi) = r(\zeta) = u\), we have that \(\xi = \zeta\) is the unique element of \(uB^{-1}\). So
\[(2.2) \quad (E(\phi^*_{\gamma,B}f\phi_{\gamma,B}))(u) = \phi_{\gamma,B}(B^{-1})f(s(\xi))\phi_{\gamma,B}(\zeta^{-1}) = E(f)(s(\xi)) \leq \|E(f)\|_{\infty}.\]
Now if \(a \in C^*_r(G)\) then \(\phi^*_{\gamma,B}a^*a\phi_{\gamma,B}\) is positive so \(E(\phi^*_{\gamma,B}a^*a\phi_{\gamma,B}) \geq 0\). Therefore by the continuity of \(E\) we can apply \((2.2)\) to obtain
\[0 \leq E(\phi^*_{\gamma,B}b^*b\phi_{\gamma,B}) \leq \|E(b^*b)\|_{\infty} = 0.\]
Thus \(E(\phi^*_{\gamma,B}b^*b\phi_{\gamma,B}) = 0\) for all open bisections \(B\) and \(\gamma \in B\).

For \(\gamma \in G\) pick an open bisection \(B\) such that \(\gamma \in B\). Notice for \(c \in \mathbb{C}\)
\[\pi_{s(\gamma)}(\phi_{\gamma,B})V_{s(\gamma)}c = \pi_{s(\gamma)}(\phi_{\gamma,B})c\delta_{s(\gamma)} = \sum_{s(\eta)=s(\gamma)} \phi_{\gamma,B}(\eta)c\delta_{\eta} = c\delta_{\gamma} = V_{\gamma}c.\]
Thus \(\pi_{s(\gamma)}(\phi_{\gamma,B})V_{s(\gamma)} = V_{\gamma}\) as operators. Now by equation \((2.1)\) and the above observation we get for all \(\gamma \in G\) that
\[V^*_{\gamma}\pi_a(b^*b)V_{\gamma} = V^*_{s(\gamma)}\pi_{s(\gamma)}(\phi^*_{\gamma,B}b^*b\phi_{\gamma,B})V_{s(\gamma)} = E(\phi^*_{\gamma,B}b^*b\phi_{\gamma,B}) = 0\]
as desired. Therefore \(b = 0\) and hence \(E\) is faithful.

4. The diagram commutes when restricting to continuous functions with compact support. Commutativity then passes to the respective completions by continuity. Since we know \(\rho_r(t_r(f)) = 0\) for all \(f \in C_c(G|U)\) we obtain image(\(t_r\)) \(\subseteq \ker \rho_r\) by continuity.

5. Let \(C\) be the set of compact sets in \(G^{(0)}\) ordered by inclusion. For each \(C \in C\) pick a function \(e_C\) in \(C_c(G^{(0)})\) such that \(0 \leq e_C \leq 1\) and \(e_C|_C \equiv 1\). Fix \(f \in C_c(G)\) vanishing outside a compact set \(K \subseteq G\). For \(C\) such that \(s(K) \subseteq C\), \(f \ast e_C = f\). It follows that \((e_C)\) is an approximate unit for \(C^*_r(G)\).

2.3. **Purely infinite simple \(C^*\)-algebras.** Given a \(C^*\)-algebra \(A\) we denote its positive elements by \(A^+\). If \(B\) is a subalgebra of \(A\) then \(B^+ \subseteq A^+\). In particular, if \(C_0(X)\) is an abelian subalgebra of \(A\) and \(f \in C_0(X)\) such that \(f(x) \geq 0\) for all \(x \in X\), then \(f \in A^+\).

For positive elements \(a \in M_n(A)\) and \(b \in M_m(A)\), \(a\) is Cuntz below \(b\), denoted \(a \preceq b\), if there exists a sequence of elements \(x_k \in M_{m,n}(A)\) such that \(x^*_kBx_k \rightarrow a\) in norm. Notice that \(\preceq\) is transitive: if \(a \preceq b\) and \(b \preceq c\) there exist sequences of element \(x_n\) and \(y_n\) such that \(x^*_nbx^n \rightarrow a\) and \(y^*_ncy_n \rightarrow b\) in norm, so \(x^*_ny_ny^*_nx_n \rightarrow a\) in norm, that is \(a \preceq c\). We say \(A\) is purely infinite if there are no characters on \(A\) and for all \(a, b \in A^+\), \(a \preceq b\) if and only if \(a \in A b A\) \(\text{[14 Definition 4.1]}\). A nonzero positive element \(a \in A\) is properly infinite if \(a \oplus a \not\preceq a\). By \(\text{[14 Theorem 4.16]}\) \(A\) is purely infinite if and only if every nonzero positive element in \(A\) is properly infinite.

A projection \(p\) in a \(C^*\)-algebra \(A\) is infinite if it is Murray-von Neumann equivalent to a proper subprojection of itself, i.e., if there exists a partial isometry \(s\) such that \(s^*s = p\) but \(ss^* \not\leq p\). By \(\text{[14 Proposition 4.7]}\) a \(C^*\)-algebra \(A\) is purely infinite if every nonzero hereditary \(C^*\)-subalgebra in every quotient of \(A\) contains an infinite projection. For simple \(C^*\)-algebras the converse is also true, thus a simple \(C^*\)-algebra is purely infinite precisely when every hereditary subalgebra contains an infinite projection.
3. TOPOLOGETICALLY PRINCIPAL GROUPOIDS AND POSITIVE ELEMENTS OF $C_0(G^{(0)})$

In this section we consider, locally compact, Hausdorff and étale groupoids. We will show that we can determine when $C^*_r(G)$ is purely infinite simple by restricting our attention to elements of $C_0(G^{(0)})$ (see Theorems 1.1 and 3.3). Before we do that, we need the following technical lemmas.

**Lemma 3.1.** Let $G$ be a locally compact, Hausdorff and étale groupoid and $E : C^*_r(G) \to C_0(G^{(0)})$ be the faithful conditional expectation extending restriction. Suppose that $G$ is topologically principal. For every $\epsilon > 0$ and $c \in C^*_r(G)^+$, there exists $f \in C_0(G^{(0)})^+$ such that:

1. $\|f\| = 1$;
2. $\|fcf - fE(c)f\| < \epsilon$;
3. $\|fE(c)f\| > \|E(c)\| - \epsilon$.

**Proof.** Let $\epsilon > 0$. For $c = 0$ the result is trivial so let $c \in C^*_r(G)^+$ such that $c \neq 0$. Define

$$a := \frac{c}{\|E(c)\|}.$$ 

To find an appropriate $f$, we use the construction in the proof of [1, Proposition 2.4]; we include the details below. Find $b \in C_c(G) \cap C^*_r(G)^+$ so that $\|a - b\| < \frac{\epsilon}{2\|E(c)\|}$. Then

$$\|E(b)\| > 1 - \frac{\epsilon}{2\|E(c)\|}$$

because $E$ is linear and $\|E(a)\| = 1$. Now, let $K := \text{supp}(b - E(b))$, which is a compact subset of $G \setminus G^{(0)}$. Let

$$U := \{u \in G^{(0)} \mid E(b)(u) > 1 - \frac{\epsilon}{2\|E(c)\|}\}.$$ 

Since $G$ is topologically principal, [1, Lemma 2.3] implies that there exists a nonempty open set $V \subseteq U$ such that $VKV = \emptyset$. Using regularity, fix a nonempty open set $W$ such that $\overline{W} \subseteq V$. Using normality, select a positive (nonzero) real-valued function $f \in C_c(G^{(0)})$ such that $f|_W = 1$, $\text{supp}(f) \subseteq V$, and $0 \leq f(x) \leq 1$ for all $x \in G^{(0)}$. Therefore, $f$ is positive in $C^*_r(G)$ and satisfies item [1].

To see that item [2] holds, a direct computation gives

$$fbf = fE(b)f.$$ 

Since $\|a - b\| < \frac{\epsilon}{2\|E(c)\|}$, $\|f\| = 1$ and $E$ is norm decreasing we have

$$\|fE(a)f - fE(b)f\| < \frac{\epsilon}{2\|E(c)\|}.$$ 

Combining equations (3.1) and (3.2) we get

$$\|faf - fE(a)f\| = \|bf - bbf + fbf - fE(b)f + fE(b)f - fE(a)f\| < \frac{\epsilon}{\|E(c)\|}.$$ 

Thus multiplying by $\|E(c)\|$ gives $\|fcf - fE(c)f\| < \epsilon$ as needed in [2].

To see item (3) notice that since $\text{supp} f \subseteq U$ we have

$$fE(b)f \geq (1 - \frac{\epsilon}{2\|E(c)\|})f^2.$$
Since $\|f\| = 1$, from the above equation and equation (3.2) we get
$$
\|fE(a)f\| > \|fE(b)f\| - \frac{\epsilon}{2\|E(c)\|} \geq 1 - \frac{\epsilon}{2\|E(c)\|} - \frac{\epsilon}{2\|E(c)\|} = 1 - \frac{\epsilon}{\|E(c)\|}.
$$
Multiplying by $\|E(c)\|$ we obtain $\|fE(c)f\| > \|E(c)\| - \epsilon$ as needed. \hfill \Box

Lemma 3.2. Let $G$ be a locally compact, Hausdorff and étale groupoid and $E : C^*_r(G) \to C_0(G(0))$ be the faithful conditional expectation extending restriction. Suppose that $G$ is topologically principal. For every nonzero $a \in C^*_r(G)^+$, there exists nonzero $h \in C_0(G(0))^+$ such that $h \not\preceq a$.

Proof. Let $a \in C^*_r(G)^+$ such that $a \neq 0$. Since $E$ is faithful, $E(a)$ is nonzero. Applying Lemma 3.1 to $c := \frac{a}{\|E(a)\|}$ and $\epsilon = 1/4$ gives us an $f \in C_0(G(0))$ such that items (1), (2) and (3) of Lemma 3.1 hold. In particular $\|fE(c)f\| > 3/4$.

Following [14, p. 640], for each $d \in C_0(G(0))^+$ we define the element
$$(d - 1/2)_+ := \phi_{1/2}(d) \in C_0(G(0))^+$$
where $\phi_{1/2}(t) = \max\{t - 1/2, 0\}$ for $t \in \mathbb{R}^+$. Notice that
$$\|\phi_{1/2}(d)\| = \max\{|d| - 1/2, 0\},$$
for each $d \in C_0(G(0))^+$.

Now let $h := (fE(c)f - 1/2)_+ \in C_0(G(0))^+$. Using item (2) of Lemma 3.1 and [13, Lemma 2.2], we can find $g \in C^*_r(G)$ so that $h = g^*fcfg$. Therefore $h \not\preceq a$. Finally, $h \neq 0$ since
$$\|h\| = \|(fE(c)f - 1/2)_+\| \geq \|fE(c)f\| - 1/2 \geq 1/4 > 0.$$ We are now in a position to prove the main result of this section.

Theorem 3.3. Let $G$ be a locally compact, Hausdorff and étale groupoid. Suppose that $G$ is minimal and topologically principal. Then $C^*_r(G)$ is purely infinite if and only if every nonzero positive element of $C_0(G(0))$ is infinite in $C^*_r(G)$.

Proof. The forward implication is trivial. To see the reverse, let $a \in C^*_r(G)^+$ such that $a \neq 0$. Using Lemma 3.2 we can find a nonzero $h \in C_0(G(0))^+$ such that $h \not\preceq a$. By assumption, we know $h$ is infinite. Since $C^*_r(G)$ is simple by [21, Proposition II.4.6], $h$ is properly infinite by [14, Proposition 3.14]. Thus $a$ is properly infinite by [14, Lemma 3.8], hence $C^*_r(G)$ is purely infinite. \hfill \Box

Recall that a Kirchberg algebra is a separable, nuclear, purely infinite simple $C^*$-algebra. We combine Theorem 3.3 with results from [2, 4, 21] to obtain the following characterization of groupoid Kirchberg algebras.

Corollary 3.4. Let $G$ be a second-countable, locally compact, Hausdorff and étale groupoid. Then $C^*_r(G)$ is a Kirchberg algebra if and only if $G$ is minimal, topologically principal, measure-wise amenable and every non-zero positive element of $C_0(G(0))$ is infinite in $C^*_r(G)$.

Proof. Suppose $C^*_r(G)$ is a Kirchberg algebra. Then $C^*_r(G)$ is simple by definition and so $C^*_r(G) = C^*_r(G), G$ is minimal and $G$ topologically principal [4, Theorem 5.1]. Since $C^*_r(G)$ is nuclear, $C^*_r(G)$ is also nuclear hence $G$ is measure-wise amenable by [2, Corollary 6.2.14]. Finally, we apply Theorem 3.3 to see that every non-zero positive element of $C_0(G(0))$ is infinite in $C^*_r(G)$. 

Conversely, suppose $G$ is minimal, topologically principal, measure-wise amenable and that every non-zero positive element of $C_0(G(0))$ is infinite in $C^*(G)$. Then $C^*_r(G) = C^*(G)$ is nuclear by [2, Corollary 6.2.14], simple by [3, Theorem 5.1], separable because $G$ is second countable [21, Remark (iii) page 59] and purely infinite by Theorem 3.3. \qed

4. $C^*$-algebras associated to ample groupoids

In this section, we will restrict our attention to ample groupoids. Although this might seem a very restrictive class of groupoids, it actually includes a lot of important examples. Again, every Kirchberg algebra in UCT is Morita equivalent to a $C^*$-algebra associated to a Hausdorff, ample groupoid (see [20]). The ample case is far more manageable than the general case. In particular there is a large number of projections in the associated algebra. Let $G$ be a locally compact, Hausdorff and étale groupoid. If $G$ is ample, then the complex Steinberg algebra associated to $G$ is

$$A(G) := \text{span}\{\chi_B : B \text{ is a compact open bisection} \} \subseteq C_c(G)$$

where $\chi_B$ denotes the characteristic function of $B$, is dense in $C^*_r(G)$ see [6, Proposition 4.2] (see also [27]). A quick computation shows that $\chi_B \star \chi_D = \chi_{BD}$ and $\chi_B^* = \chi_{B^{-1}}$, so that if $B \subseteq G(0)$ is compact open, then $\chi_B$ is a projection.

**Theorem 4.1.** Let $G$ be a second countable, Hausdorff, ample groupoid. Suppose that $G$ is topologically principal, minimal and that $\mathcal{B}$ is a basis of $G(0)$ consisting of compact open sets. Then $C^*_r(G)$ is purely infinite if and only if every nonzero projection $p$ in $C_0(G(0))$ with $\text{supp}(p) \in \mathcal{B}$ is infinite in $C^*_r(G)$.

**Proof.** The forward implication is trivial. To see the reverse, suppose every nonzero projection $p$ of $C_0(G(0))$ with $\text{supp}(p) = U$ for some $U \in \mathcal{B}$ is infinite in $C^*_r(G)$. By Theorem 3.3 it suffices to show that every positive element in $C_0(G(0))^+$ is infinite. Let $a \in C^*_r(G(0))^+$ be a nonzero element. We show that $a$ is properly infinite. We claim there is a nonzero projection $p \in C_0(G(0))^+$ with $\text{supp}(p) \subseteq U$ for some $U \in \mathcal{B}$ such that $p \lesssim a$. To see this, first note that characteristic functions of the form $\chi_V$ are projections in $C_0(G(0))$ for every compact open set $V \subseteq G(0)$. Since $\mathcal{B}$ is a basis of compact open sets, there exists a compact open set $U_0 \in \mathcal{B}$ and a nonzero $s \in \mathbb{R}^+$ such that $\chi_{U_0}(x) \leq sa(x)$ for every $x \in G(0)$. Then $p := \chi_{U_0} \leq sa$. Applying [14, Proposition 2.7] we get that $p \lesssim sa$ and so $p \lesssim a$ as claimed. Since $p$ is infinite by assumption and $C^*_r(G)$ is simple, $p$ is properly infinite by [14, Proposition 3.14]. Hence $a$ is properly infinite by [14, Lemma 3.8]. \qed

In the next corollary, we show how we can use the minimality of $G$ to strengthen our result.

**Corollary 4.2.** Let $G$ be a second countable, Hausdorff, ample groupoid. Suppose that $G$ is topologically principal and minimal. Then $C^*_r(G)$ is purely infinite if and only if there exists a point $x \in G(0)$ and a neighbourhood basis $\mathcal{D}$ of $x$ consisting of compact open sets so that every nonzero projection $q$ in $C_0(G(0))$ with $\text{supp}(q) \in \mathcal{D}$ is infinite in $C^*_r(G)$.

**Proof.** Again, the forward direction is trivial. For the reverse implication, suppose there exist a point $x \in G(0)$ and neighbourhood basis $\mathcal{D}$ of $x$ consisting of compact open sets such that that every nonzero projection $q$ in $C_0(G(0))$ with $\text{supp}(q) \in \mathcal{D}$ is infinite in $C^*_r(G)$. Let $\mathcal{B}$ be a basis of $G(0)$ consisting of compact open sets and suppose $p := \chi_U$ is a nonzero projection in $C_0(G(0))$ with $U \in \mathcal{B}$. By Theorem 4.1, it suffices to show that $p$
That is, $\chi_V \preceq \chi_{r(B)}$. Hence, $\chi_{r(B)}$ is properly infinite by [14, Lemma 3.8]. Finally, since $\chi_V = \chi_{r(B)} + \chi_{U - r(B)}$, $\chi_V$ is infinite.

5. AN APPLICATION: $k$-GRAPH $C^*$-ALGEBRAS

In this section, we apply Theorem 4.11 to $C^*$-algebras associated to $k$-graphs. We assume the reader is familiar with the basic definitions and constructions of $k$-graphs and their $C^*$-algebras found in [16], but we recall a few facts here. Let $\Lambda$ be a $k$-graph. Then the associated $C^*$-algebra $C^*(\Lambda)$ is the universal $C^*$-algebra generated by a Cuntz-Krieger $\Lambda$-family $\{s_\lambda : \lambda \in \Lambda\}$. To keep things clean, we will restrict our attention to row-finite $k$-graphs with no sources but similar results hold in the more general setting. We think our results will be useful in this setting because necessary and sufficient conditions on $\Lambda$ for $C^*(\Lambda)$ to be purely infinite simple are not known.

Following [16] we recall how $C^*(\Lambda)$ can be realised as the $C^*$-algebra of a second countable, Hausdorff, ample groupoid $G_\Lambda$ as follows. Let $\Lambda^\infty$ denote the infinite path space of $\Lambda$ and $\Lambda^\infty(v)$ be the set of infinite paths with range $v$. Define

$$G_\Lambda := \{(x, n, y) \in \Lambda^\infty \times \mathbb{N}^k \times \Lambda^\infty : \sigma^l(x) = \sigma^m(y), n = l - m\}$$

where $\sigma$ is the shift map. We view $(x, n, y)$ as a morphism with source $y$ and range $x$. Composition is given by $(x, n, y)(y, m, w) = (x, n + m, w)$. The unit space $G_\Lambda^{(0)}$ is identified with $\Lambda^\infty$. For $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ we define

$$Z(\lambda, \mu) := \{(\lambda z, d(\lambda) - d(\mu), \mu z) : z \in \Lambda^\infty(s(\lambda))\}.$$

The (countable) collection of all such $Z(\lambda, \mu)$ generate a topology under which $G_\Lambda$ is a second countable, Hausdorff, ample groupoid by [16, Proposition 2.8]. Further, the relative topology on the unit space $\Lambda^\infty$ has a basis of compact cylinder sets

$$Z(\lambda) := \{\lambda x \in \Lambda^\infty : x \in \Lambda^\infty(s(\lambda))\}$$

by identifying $Z(\lambda, \lambda)$ and $Z(\lambda)$ from [16, Lemma 2.6 and Proposition 2.8]. Note that $G_\Lambda$ is amenable by [16, Theorem 5.5] and hence $C^*_r(G_\Lambda) = C^*(G_\Lambda)$. It was shown in [16] that $C^*(\Lambda) \cong C^*(G_\Lambda)$.

More specifically, by [16, Corollary 3.5(i)], there is a (unique) isomorphism $\phi : C^*(\Lambda) \to C^*(G_\Lambda)$ such that $\phi(s_\lambda) = \chi_{Z(\lambda, s(\lambda))}$. Note that

$$\phi(s_\mu s_\mu^*) = \chi_{Z(\mu, s(\mu))} \chi_{Z(\mu, s(\mu))} = \chi_{Z(\mu, s(\mu))} \chi_{Z(\mu, s(\mu))} = \chi_{Z(\mu, s(\mu))} = \chi_{Z(\mu)}.$$

With all of this theory in place, along with the simplicity results of [23] and [4], the following is an immediate corollary of Theorem 4.11 and Corollary 4.2.

**Corollary 5.1.** Suppose $\Lambda$ is a row-finite $k$-graph with no sources such that $\Lambda$ is aperiodic and cofinal in the sense of [23]. Then

1. For $\mu \in \Lambda$, $s_\mu s_\mu^*$ is infinite if and only if $s(\mu)$ is.
2. $C^*(\Lambda)$ is purely infinite simple if and only if $s_v$ is infinite for every $v \in \Lambda^0$. 
(3) $C^*(\Lambda)$ is purely infinite simple if and only if there exists $x \in \Lambda^\infty$ such that $s_v$ is infinite for every vertex $v$ on $x$.

Proof. For (1), we use a trick used in [25, Lemma 8.13]. Recall that infiniteness is preserved under von Neumann equivalence, hence $s_\mu s_\mu^*$ is infinite if and only if $s_\mu^* s_\mu = s_{s(\mu)}$ is infinite. For (2), we apply Theorem 4.1 to the second countable, Hausdorff, ample groupoid $G_\Lambda$; first we check the remaining hypotheses of Theorem 4.1. Since $\Lambda$ is cofinal and aperiodic, $C^*(\Lambda) \cong C^*(\Lambda)$ is simple by [23, Theorem 3.1]. Thus $C^*(\Lambda) = C^*_r(\Lambda)$ is simple and hence $G_\Lambda$ is topologically principal and minimal by [4, Theorem 5.1].

We have that the collection of cylinder sets of the form $Z(\mu)$ form a basis $B$ of $G_\Lambda^{(0)}$ consisting of compact open sets. Now we apply Theorem 4.1 to see that $C^*(\Lambda)$ is purely infinite if and only if each $\chi_{Z(\mu)}$ is infinite. Let $\phi : C^*(\Lambda) \to C^*(\Lambda)$ be the isomorphism characterized by $s_\mu \mapsto \chi_{Z(\mu)}$. Since $\phi$ is an isomorphism, this gives $\chi_{Z(\mu)}$ is infinite if and only if $\phi^{-1}(\chi_{Z(\mu)}) = s_\mu s_\mu^*$ is infinite. Finally, $s_\mu s_\mu^*$ if and only if $s_{s(\mu)}$ is infinite by (1).

For (3), given an infinite path $x$, the collection of compact open sets of the form $Z(x(0,n))$ for $n \in \mathbb{N}^k$ form a neighbourhood base at $x$. Now proceed as in the proof of (2) replacing Theorem 4.1 with Corollary 4.2 and $\mu$ with $x(0,n)$.

6. The non-simple case

Let $A$ be a $C^*$-algebra. A pair of positive elements $(a_1, a_2) \in A \times A$ has the matrix diagonalization property in $A$ in the sense of [15, Definition 3.3.] if for every positive matrix $\left(\begin{array}{ll} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)$ with $b_{ij} \in A$ and every $\epsilon_1, \epsilon_2, \delta > 0$ there exists $d_1, d_2 \in A$ with

$$\|d_1^* a_i d_1 - a_i\| < \epsilon_1 \text{ and } \|d_2^* b_{ij} d_2\| < \delta.$$ 

A subset $\mathcal{F}$ of $A^+$ is a filling family for $A$, in the sense of [15, Definition 3.10], if for every hereditary $C^*$-subalgebra $H$ of $A$ and every primitive ideal $I$ of $A$ with $H \nsubseteq I$ there exist $f \in \mathcal{F}$ and $z \in A$ with $z^* z \in H$ and $z z^* = f \notin I$.

By Proposition 3.13 and Lemma 3.12 of [15], if $A^+$ contains a filling family $\mathcal{F}$ that is closed under $\epsilon$-cut-downs and every pair of elements $(a_1, a_2) \in \mathcal{F} \times \mathcal{F}$ has the matrix diagonalization property, then $A$ is strongly purely infinite. In this section we provide a characterization of when the reduced groupoid $C^*$-algebra is strongly purely infinite (Proposition 6.3). In our proof of Proposition 6.3 we will use results from [4] to describe ideals of reduced groupoid $C^*$-algebras. First we need the following lemma. Recall that a subset $D \subseteq G^{(0)}$ is said to be invariant if $G \cdot D := \{r(\gamma) : s(\gamma) \in D\} \subseteq D$.

Lemma 6.1. Let $G$ be a second countable, locally compact, Hausdorff and étale groupoid such that $C^*(G) = C^*_r(G)$. Then the following properties are equivalent:

1. For every closed invariant set $D \subseteq G^{(0)}$

$$C^*(G|_D) = C^*_r(G|_D).$$

2. For every closed invariant set $D \subseteq G^{(0)}$ the sequence

$$0 \longrightarrow C^*_r(G|_{G^{(0)} \setminus D}) \overset{\iota_r}{\longrightarrow} C^*_r(G) \overset{\rho_r}{\longrightarrow} C^*_r(G|_D) \longrightarrow 0$$

is exact where $\iota_r$ and $\rho_r$ are determined on continuous functions by extension and restriction respectively.
Remark 6.2. In [22, Remark 4.10], Renault mentions that if \( G \) is amenable for every closed invariant set \( D \subseteq G(0) \), then item (2) of Lemma 6.1 follows. Thus Lemma 6.1 is a strengthening of Renault’s comment.

Proof. Fix a closed invariant set \( D \subseteq G(0) \) and let \( U = G(0) - D \). Consider the following diagram:

\[
\begin{array}{ccc}
0 & \to & C^*(G_U) \to & C^*(G) \to & C^*(G_D) & \to & 0 \\
\pi_U & \downarrow & \iota & \downarrow & \pi & \downarrow & \pi_D \\
0 & \to & C^*_r(G_U) \to & C^*_r(G) \to & C^*_r(G_D) & \to & 0
\end{array}
\]

where \( \pi_U, \pi \) and \( \pi_D \) are the respective quotient maps, and \( \iota, \iota_r \) and \( \rho, \rho_r \) extend extension and restriction respectively. Since all of the maps involved are continuous, the diagram commutes. We also have that the top row of (6.1) is exact by [18, Lemma 2.10].

\( 2 \Rightarrow 1 \): We show the surjective map \( \pi_D \) is injective. Fix any \( a \in C^*_r(G_D) \) with \( \pi_D(a) = 0 \). Find \( b \in C^*(G) \) with \( \rho(b) = a \). From \( \pi_D(\rho(b)) = \rho_r(\pi(b)) = 0 \), exactness of (6.1), surjectivity of \( \pi_U \), and \( \iota \circ \pi_U = \pi \circ \iota_r \) we obtain

\[
\pi(b) \in \ker \rho_r = \iota_r(C^*_r(G_U)) = \iota_r \circ \pi_U(C^*(G_U)) = \pi \circ \iota(C^*(G_U)).
\]

Find \( c \in C^*(G_U) \) with \( \pi(b) = \pi(\iota(c)) \). As \( \pi \) is an isomorphism by assumption we obtain that \( b = \iota(c) \). Hence \( a = \rho(b) = \rho(\iota(c)) = 0 \), and \( C^*(G_D) = C^*_r(G_D) \).

\( 1 \Rightarrow 2 \): By assumption the maps \( \pi \) and \( \pi_D \) are isomorphisms. Using the commutative diagram (6.1) and the exactness of the top line of that diagram, the exactness of the bottom line follows from a easy diagram chase. \( \square \)

Let \( G \) be a second countable, locally compact, Hausdorff and \( \acute{e} \)tale groupoid and \( D \) be a closed invariant set of \( G(0) \) and define \( U = G(0) - D \). Recall from Lemma 2.1(4) we have the commuting diagram:

\[
\begin{array}{ccc}
0 & \to & C^*_r(G_U) \to & C^*_r(G) \to & C^*_r(G_D) & \to & 0 \\
E_U & \downarrow & E & \downarrow & E_D \\
0 & \to & C_0(U) \to & C_0(G(0)) \to & C_0(D) & \to & 0
\end{array}
\]

Notice that the bottom row in (6.2) is exact. We will use this diagram several times. We also use the notation \( \text{Ideal}[S] \) for the ideal in \( C^*_r(G) \) generated by \( S \subseteq C^*_r(G) \).

Proposition 6.3. Let \( G \) be a second countable, locally compact, Hausdorff and \( \acute{e} \)tale groupoid such that \( C^*(G) = C^*_r(G) \). Then the following properties are equivalent:

1. The \( C^* \)-algebra \( C^*_r(G) \) is strongly purely infinite, and for every ideal \( I \) in \( C^*_r(G) \), \( I = \text{Ideal}[I \cap C_0(G^{(0)})] \).

2. For every closed invariant set \( D \subseteq G(0) \), \( G|_D \) is topologically principal; the sequence

\[
\begin{array}{ccc}
0 & \to & C^*_r(G_U) \to & C^*_r(G) \to & C^*_r(G_D) & \to & 0 \\
\iota_r & \downarrow & \rho_r \\
E_U & \downarrow & E_D
\end{array}
\]

is exact where \( U = G(0) - D \), \( \iota_r \) and \( \rho_r \) are determined on continuous functions by extension and restriction respectively; and for every pair of elements \( a, b \) in \( C_0(G^{(0)})^+ \) the 2-tuple \((a, b)\) has the matrix diagonalization property in \( C^*_r(G) \).
Proof. (1) $\Rightarrow$ (2): Fix a closed invariant set $D \subseteq G^{(0)}$ and $U = G^{(0)} - D$. For this $D$ and $U$ we have a commuting diagram as in (6.2). Define $I := \ker \rho_r \subseteq C^*_r(G)$. Using the diagram, $\rho_0(E(I)) = E_D(\rho_r(I)) = 0$, implying that $E(I) \subseteq \iota_0(C_0(U))$. Since $E(b) = b$ for $b \in C_0(G^{(0)})$, $I \cap C_0(G^{(0)}) \subseteq E(I)$. Using assumption (1) we have $I = \text{Ideal}[I \cap C_0(G^{(0)})]$. Hence

$$\ker \rho_r = I = \text{Ideal}[I \cap C_0(G^{(0)})] \subseteq \text{Ideal}[E(I)] \subseteq \text{Ideal}[\iota_0(C_0(U))] \subseteq \iota_r(C^*_r(G)_U);$$

that is $\ker \rho_r \subseteq \text{image}(\iota_r)$. Thus (6.3) is exact.

We know that each $G|_D$ is topologically principal by [4] Remark 5.10] provided that $C^*(G|_D) = C^*_r(G|_D)$. The latter follows from Lemma 6.1 since (6.3) is exact.

Since $C^*_r(G)$ is strongly purely infinite, Lemma 5.8 in [15] implies that every pair $(a, b)$ of positive elements in $C_0(G^{(0)})$ has the matrix diagonalization property in $C^*_r(G)$.

(2) $\Rightarrow$ (1): Since we assumed that $G|_D$ is topologically principal for all closed invariant $D \subseteq G^{(0)}$, by the proof of Corollary 5.9 in [4] (see also [4] Remark 5.10]), we know $I = \text{Ideal}[I \cap C_0(G^{(0)})]$ for every ideal $I$ in $C^*_r(G) = C^*(G)$ provided that $C^*(G|_D) = C^*_r(G|_D)$ for every closed invariant set $D \subseteq G^{(0)}$. But this follows from Lemma 6.1 since $C^*(G) = C^*_r(G)$ and (6.3) is exact, which are assumed in (2). Hence (2) implies $I = \text{Ideal}[I \cap C_0(G^{(0)})]$.

We prove $C^*_r(G)$ is strongly purely infinite. Define $\mathcal{F} := C_0(G^{(0)})^+ \subseteq C^*_r(G)$. By functional calculus we know

$$f(a) \in \mathcal{F}, \quad \text{for } f \in C_0(\mathbb{R})^+, \quad a \in \mathcal{F}.$$ 

In particular $\mathcal{F}$ is closed under $\varepsilon$-cut-downs, i.e., for each $a \in \mathcal{F}$, and $\varepsilon \in (0, \|a\|)$ we have $(a - \varepsilon)_+ \in \mathcal{F}$. By (2) each pair $(a, b)$ with $a, b \in \mathcal{F}$ has the matrix diagonalization property (of [15] Definition 3.3). Now by Lemma 3.12 of [15] we know that $\mathcal{F}$ has the matrix diagonalization property of [15] Definition 3.10(ii)]. It follows from Proposition 3.13 of [15] that $C^*_r(G)$ is strongly purely infinite provided that $\mathcal{F}$ is a filling family for $C^*_r(G)$, which we now show.

Fix any hereditary $C^*$-subalgebra $H$ of $C^*_r(G)$ and any ideal $I$ of $C^*_r(G)$ with $H \nsubseteq I$. We know $I = \text{Ideal}[I \cap C_0(G^{(0)})]$, hence $I = \iota_r(C^*_r(G|_U))$ for some open invariant set $U \subseteq G^{(0)}$. Let $D = G^{(0)} - U$ and consider the corresponding commuting diagram (6.2).

Select $d \in H^+, \quad d \notin I$. Define $c := \rho_r(d)$. As $d \notin I = \ker \rho_r$ by exactness in (2), we know $\rho_r(d) \neq 0$. Since $E_D$ is faithful and $c$ positive,

$$\epsilon := \frac{1}{4} \|E_D(c)\| > 0.$$ 

By (2) the groupoid $G|_D$ is topologically principal, hence Lemma 6.1 gives $f \in C_0(D)^+$ such that

$$\|f\| = 1, \quad \|f cf - f E_D(c)f\| < \epsilon, \quad \|f E_D(c)f\| > \|E_D(c)\| - \epsilon.$$ 

Recall [13] Lemma 2.2: For $x, y$ positive and $\delta > 0$ with $\|x - y\| < \delta$ there exist a contraction $a$ with $a^*x a = (y - \delta)_+$. Use this to find a contraction $a \in C^*_r(G|_D)$ such that

$$h := a^* f c a = (f E_D(c)f - \epsilon)_+ \in C_0(D)^+.$$ 

Notice that

$$\|h\| \geq \|f E_D(c)f\| - \epsilon > \|E_D(c)\| - 2\epsilon > 0.$$ 

Using that $\rho_r$ restricts to the map $C_0(G^{(0)}) \to C_0(D)$, select $b \in C_0(G^{(0)})^+$ such that $\rho_r(b) = h$. Also as $\rho_r$ is surjective find $w \in C^*_r(G)$ such that $\rho_r(w) = f a$. Since $\rho_r(b -
\(w^*dw) = h - a^*fca = 0\) we have \(b = w^*dw + v\) for some \(v \in I\). Let \(\{e_\lambda\}\) denote an approximate unit of \(I = C^*_r(G_U)\) with \(e_\lambda \in C_0(U)\) (see Lemma \[2.1\]). Let 1 be the unit of the unitisation of \(C^*_r(G)\). Then \((1 - e_\lambda)v(1 - e_\lambda) \to 0\). For suitable \(\lambda_0\) and \(e := 1 - e_{\lambda_0}\) we get

\[\|ew^*dwe - ebe\| = \|eve\| < \varepsilon.\]

Use Lemma 2.2 of \[13\] to find a contraction \(u \in C^*_r(G)\) such that

\[g := u^*ew^*dweu = (ebe - \varepsilon) + C_0(G^{(0)}) = \mathcal{F}.\]

Since \(be_{\lambda_0} + e_{\lambda_0}b - e_{\lambda_0}be_{\lambda_0} \in C_0(U) \subseteq \ker \rho_r\) we obtain that \(\rho_r(ebe) = \rho_r(b) = h\). Moreover by functional calculus we know \((h - \varepsilon)_+ = (fE_D(c)f - 2\varepsilon)_+\). We conclude

\[\|\rho_r(g)\| = \|(h - \varepsilon)_+\| = \|(fE_D(c)f - 2\varepsilon)_+\| \geq \|fE_D(c)f\| - 2\varepsilon > \|E_D(c)\| - 3\varepsilon > 0,\]

ensuring \(g \notin I\). Finally with \(z := u^*ew^*d^{1/2} \in C^*_r(G)\) we obtain \(g = zz^*\) and \(z^*z \in H\).

By definition \(\mathcal{F}\) is a filling family for \(C^*_r(G)\) completing the proof. \(\square\)

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