Purely infinite C*-algebras associated to etale groupoids

Jonathon Brown
United States Naval Academy

Les Clark
University of Otago

Adam Sierakowski
University of Wollongong, asierako@uow.edu.au

Publication Details
Purely infinite C*-algebras associated to etale groupoids

**Keywords**
algebras, infinite, groupoids, c, etale, associated, purely

**Disciplines**
Engineering | Science and Technology Studies

**Publication Details**

This journal article is available at Research Online: [http://ro.uow.edu.au/eispapers/3456](http://ro.uow.edu.au/eispapers/3456)
PURELY INFINITE C*-ALGEBRAS ASSOCIATED TO ÉTALE GROUPOIDS

JONATHAN BROWN, LISA ORLOFF CLARK, AND ADAM SIERAKOWSKI

Abstract. Let $G$ be a Hausdorff, étale groupoid that is minimal and topologically principal. We show that $C^*_r(G)$ is purely infinite simple if and only if all the nonzero positive elements of $C_0(G^{(0)})$ are infinite in $C^*_r(G)$. If $G$ is a Hausdorff, ample groupoid, then we show that $C^*_r(G)$ is purely infinite simple if and only if every nonzero projection in $C_0(G^{(0)})$ is infinite in $C^*_r(G)$. We then show how this result applies to $k$-graph $C^*$-algebras. Finally, we investigate strongly purely infinite groupoid $C^*$-algebras.

1. Introduction

Purely infinite simple $C^*$-algebras were introduced by Cuntz in [7] where he showed that the $K_0$ group of such algebras can be computed within the algebra itself without resorting to the usual direct limit construction. The $K$-theory groups of $C^*$-algebras have long been known to be computable invariants and Cuntz’s result shows that this computation is easier when the $C^*$-algebra is purely infinite simple. Elliott initiated a program to find a suitably large class of $C^*$-algebras on which the $K$-theory groups provide a complete isomorphism invariant (see [8]). This program has achieved remarkable success, most notably in a theorem of Kirchberg and Phillips [11, 20] which states that every Kirchberg algebra satisfying the Universal Coefficient Theorem (UCT) is classified by the isomorphism classes of its ordered $K$-theory groups. A Kirchberg algebra is a separable, nuclear, purely infinite simple $C^*$-algebra.

The allure of classification via the Kirchberg-Phillips theorem has led many authors to study when various constructions of $C^*$-algebras yield purely infinite simple algebras. Kumjian, Pask and Raeburn show that a graph $C^*$-algebra of a cofinal graph is purely infinite simple if and only if every vertex can be reached from a loop with an entrance [17, Theorem 3.9]. Carlsen and Thomsen show that if the $C^*$-algebra constructed from a locally injective surjection $\theta$ on a compact metric space of finite covering dimension is simple, then it is purely infinite simple if and only if $\theta$ is not a homeomorphism [5, Corollary 6.6]. Rørdam and Sierakowski [24] show that if a countable exact group $H$ acts by an essentially free action on the Cantor set $X$ and the type semigroup of clopen subsets of $X$ is almost unperforated, then $C_0(X) \rtimes_r H$ is purely infinite if and only if every clopen set $E$ in $X$ is paradoxical. The constructions in each of the above examples are special cases of groupoid $C^*$-algebras.
In this paper we investigate purely infinite $C^*$-algebras associated to Hausdorff, étale groupoids. In sections 3 through 5 we restrict our attention to simple groupoid $C^*$-algebras. Characterising simplicity of groupoid $C^*$-algebras is known and we readily make use of the following theorem from [4, Theorem 5.1]:

**Theorem 1.1** (Brown-Clark-Farthing-Sims). Let $G$ be a second-countable, locally compact, Hausdorff and étale groupoid. Then $C^*(G)$ is simple if and only if all of the following conditions are satisfied.

1. $C^*(G) = C^*_r(G)$;
2. $G$ is topologically principal;
3. $G$ is minimal.

However, necessary and sufficient conditions on the groupoid for the associated algebra to be purely infinite simple are not known. Anantharaman-Delaroche showed that ‘locally contracting’ is a sufficient condition on the groupoid in [1, Proposition 2.4] but whether locally contracting is necessary remains an open question. Part of the difficulty in characterizing those groupoids that give rise to purely infinite simple $C^*$-algebras is relating arbitrary projections in the groupoid $C^*$-algebra to the groupoid itself.

In Section 3 we look at a necessary and sufficient conditions for ensuring pure infiniteness of groupoid $C^*$-algebras. We show that for a Hausdorff, étale, topologically principal, and minimal groupoid $G$ the $C^*$-algebra $C^*_r(G)$ is purely infinite simple if and only if it all the nonzero positive elements of $C_0(G^{(0)})$ are infinite in $C^*_r(G)$ and (see Theorem 3.3). In Section 4 we specialize to Hausdorff, ample groupoids. This is an important class of examples because every Kirchberg algebra in UCT is Morita equivalent to an algebra associated to a Hausdorff, ample groupoid (see [26]). We show in Theorem 4.1 for a Hausdorff, ample groupoid $G$, that is also topologically principal and minimal, the $C^*$-algebra $C^*_r(G)$ is purely infinite if and only if every nonzero projection in $C_0(G^{(0)})$ is infinite in $C^*_r(G)$.

Theorem 4.1 is a generalisation of [10] about partial actions. In Section 5 we demonstrate how Theorem 4.1 applies to $k$-graph $C^*$-algebras.

In Section 6 we turn our attention to the non-simple case. In [13], Kirchberg and Rørdam introduce three separate notions of purely infinite $C^*$-algebras: weakly purely infinite, purely infinite and strongly purely infinite. Of these notions, the last one appears to be the most useful in the classification theory of non-simple $C^*$-algebras. Indeed, Kirchberg showed in [12] that two separable, nuclear, strongly purely infinite $C^*$-algebras with the same primitive ideal space $X$ are isomorphic if and only if they are $KK_X$-equivalent. We provide a characterization of when groupoid $C^*$-algebras are strongly purely infinite in Proposition 6.3.

2. **Preliminaries**

2.1. **Groupoids.** A groupoid $G$ is a small category in which every morphism is invertible. The set of objects in $G$ is identified with the set of identity morphisms and both are denoted by $G^{(0)}$. We call $G^{(0)}$ the unit space of $G$. Each morphism $γ$ in the category has a range and source denoted $r(γ)$ and $s(γ)$ respectively and thus $r$ and $s$ define maps $G → G^{(0)}$.

A topological groupoid is a groupoid with a topology in which composition is continuous and inversion is a homeomorphism. An open bisection in a topological groupoid $G$ is an open set $B$ such that both $r$ and $s$ restricted to $B$ are homeomorphisms; in particular
these restrictions are injective. An étale groupoid is a topological groupoid where \( s \) is a local homeomorphism. If a groupoid \( G \) is Hausdorff and étale, then the unit space \( G^{(0)} \) is open and closed in \( G \). If \( G \) is a locally compact, Hausdorff groupoid, then \( G \) is étale if and only if there is a basis for the topology on \( G \) consisting of open bisections with compact closure. A topological groupoid is called ample if and only if \( G^{(0)} \) is totally disconnected (see [9, Proposition 4.1]).

For a subsets \( L, K \subseteq G \), denote \( LK = \{ \gamma : \gamma = \xi \zeta \mid \xi \in L, \zeta \in K, s(\xi) = r(\zeta) \} \). With a slight abuse of notation, for \( u \in G^{(0)} \), we write \( uG \) and \( Gu \) for \( \{u\}G \) and \( G\{u\} \) respectively and denote by \( uGu \) the set

\[
\{ \gamma \in G : r(\gamma) = s(\gamma) = u \}.
\]

A topological groupoid \( G \) is topologically principal if the set \( \{u \in G^{(0)} : uGu = \{u\}\} \) is dense in \( G^{(0)} \), and minimal if \( G : u := \{r(\gamma) : s(\gamma) = u\} \) is dense in \( G^{(0)} \) for all \( u \in G^{(0)} \). Recall, for a second countable, locally compact, Hausdorff, étale groupoid \( G \) the algebra \( C^*(G) \) is simple if and only if \( G \) is minimal, topologically principal, and \( C^*(G) = C^*_r(G) \).

### 2.2. Groupoid \( C^* \)-algebras.

Let \( G \) be locally compact, Hausdorff étale groupoid and let \( C_c(G) \) denote the set of compactly supported continuous functions from \( G \) to \( \mathbb{C} \). Since every element \( \gamma \) of \( G \) has a neighbourhood \( B_\gamma \) such that \( r|_{B_\gamma} \) is injective, the set \( r^{-1}(u) \) is discrete for every \( u \in G^{(0)} \). Thus if \( f \in C_c(G) \) then \( \text{supp}(f) \cap r^{-1}(u) \) is finite for all \( u \in G^{(0)} \). With this, we are able to define a convolution and involution on \( C_c(G) \) such that for \( f, g \in C_c(G) \),

\[
f * g(\gamma) := \sum_{r(\eta) = r(\gamma)} f(\eta)g(\eta^{-1}\gamma) \quad \text{and} \quad f^*(\gamma) := \overline{f(\gamma^{-1})}.
\]

Under these operations, \( C_c(G) \) is a \(*\)-algebra. Next define for \( f \in C_c(G) \),

\[
\|f\|_I := \sup_{u \in G^{(0)}} \left\{ \max\left\{ \sum_{\gamma \in Gu} |f(\gamma)|, \sum_{\gamma \in uG} |f(\gamma)| \right\} \right\} \quad \text{and} \quad \|f\| := \sup\{\|\pi(f)\| : \pi \text{ is a } \| \cdot \|_I\text{-decreasing representation}\}.
\]

Then \( C^*(G) \) is the completion of \( C_c(G) \) in the \( \| \cdot \| \)-norm.

Given a unit \( u \in G^{(0)} \), the regular representation \( \pi_u \) of \( C_c(G) \) on \( \ell^2(Gu) \) associated to \( u \) is characterized by

\[
\pi_u(f)\delta_\gamma = \sum_{s(\eta) = r(\gamma)} f(\eta)\delta_{u\eta}.
\]

The reduced \( C^* \)-norm on \( C_c(G) \) is \( \|f\|_r := \sup\{\|\pi_u(f)\| : u \in G^{(0)} \} \) and \( C^*_r(G) \) is the completion of \( C_c(G) \) in the \( \| \cdot \|_r \)-norm. Our attention will be focused on the reduced \( C^* \)-algebra and situations where the reduced and full algebras coincide. Also, we will often be assuming \( G \) is second-countable, implying \( C^*_r(G) \) is separable [21, Page 59].

Below we recall a few standard results (we heavily use) and their proofs to familiarise the reader with locally compact, Hausdorff étale groupoids.

**Lemma 2.1** (cf. [21]). Let \( G \) be a locally compact, Hausdorff and étale groupoid. Then

1. The extension map from \( C_c(G^{(0)}) \) into \( C_c(G) \) (where a function is defined to be zero on \( G - G^{(0)} \)) extends to an embedding of \( C_0(G^{(0)}) \) into \( C^*_r(G) \).
(2) The restriction map \( E_0 : C_c(G) \to C_c(G^{(0)}) \) extends to a conditional expectation \( E : C^*_r(G) \to C_0(G^{(0)}) \).

(3) The map \( E \) from item (2) is faithful. That is, \( E(b^*b) = 0 \) implies \( b = 0 \) for \( b \in C^*_r(G) \).

(4) For every closed invariant set \( D \subseteq G^{(0)} \) we have the following commuting diagram:

\[
\begin{array}{cccc}
0 & \xrightarrow{\iota_r} & C^*_r(G|U) & \xrightarrow{\rho_r} & C^*_r(G|D) & 0 \\
E_U & & E & & E_D & \\
0 & \xrightarrow{\iota_0} & C_0(U) & \xrightarrow{\rho_0} & C_0(D) & 0
\end{array}
\]

where \( U = G^{(0)} - D \), \( \iota_r \) and \( \rho_r \) are determined on continuous functions by extension and restriction respectively. Moreover, \( \text{image}(\iota_r) \subseteq \ker \rho_r \).

(5) The subalgebra \( C_c(G^{(0)}) \) contains an approximate unit for \( C^*_r(G) \).

Proof. Since \( G \) is Hausdorff and étale, \( G^{(0)} \) is open and closed in \( G \). Thus, the map \( C_c(G^{(0)}) \) into \( C_c(G) \) is well defined. For \( f, g \in C_c(G^{(0)}) \), a quick computation gives

\[
f \ast g(\gamma) = \begin{cases} f(\gamma)g(\gamma), & \text{if } \gamma \in G^{(0)}; \\ 0, & \text{otherwise,}
\end{cases}
\]

so the map from \( C_c(G^{(0)}) \) into \( C_c(G) \) is a *-homomorphism. We claim the map is isometric, that is, we claim the reduced norm agrees with the infinity norm for functions in \( C_c(G^{(0)}) \). By evaluating at point masses in \( \ell^2(GU) \), one can show that \( \|f\|_\infty \leq \|f\|_r \), for \( f \in C_c(G) \). The reverse inequality can be verified for \( f \in C_c(G^{(0)}) \) and the claim follows. Thus the *-homomorphism from \( C_c(G^{(0)}) \) into \( C_c(G) \subseteq C^*_r(G) \) extends by continuity to an isometric (hence injective) *-homomorphism from \( C_0(G^{(0)}) \) into \( C^*_r(G) \).

Once again using that \( G \) is Hausdorff and étale, we have that \( G^{(0)} \) is open and closed in \( G \) and hence \( E_0 \) is well defined. One may easily verify that \( E_0 \) is (a) positive (b) linear (c) idempotent, and (d) of norm one. Therefore \( E_0 \) extends by continuity to a map \( E : C^*_r(G) \to C_0(G^{(0)}) \) with the same properties (a)–(d). By [3] II.6.10.1] we conclude that \( E \) is a conditional expectation.

Let \( b \in C^*_r(G) \) such that \( E(b^*b) = 0 \). We need to show that \( b = 0 \). Let \( V_\gamma : \mathbb{C} \to \ell^2(G_{s(\gamma)}) \) be given by \( c \mapsto c\delta_\gamma \). Then \( V_\gamma^*\omega = \omega(\gamma) \). Since \( \|b\|_r = \sup_{u \in G^{(0)}} \|\pi_u(b)\| \) and

\[
\|\pi_u(b)\delta_\gamma\|^2 = \langle \pi_u(b)\delta_\gamma, \pi_u(b)\delta_\gamma \rangle = \langle \pi_u(b^*b)\delta_\gamma, \delta_\gamma \rangle = V_\gamma^*\pi_u(b^*b)V_\gamma \delta_\gamma,
\]

it suffices to show that \( V_\gamma^*\pi_u(b^*b)V_\gamma = 0 \) for all \( u \in G^{(0)} \) and \( \gamma \in G \).

For \( f \in C_s(G), \ u \in G^{(0)}, \) and \( c \in \mathbb{C}, \) we have

\[
(2.1) \quad V_u^*\pi_u(f)V_u^*c = V_u^*\pi_u(f)c\delta_u = V_u^*(\sum_{s(\eta) = u} f(\eta)c\delta_\eta) = f(u)c = E(f)(u)c.
\]

Thus by the continuity of \( E \), for all \( a \in C^*_r(G), \ E(a)(u) = V_u^*\pi_u(a)V_u \) as operators on \( \mathbb{C} \).

For every open bisection \( B \) and \( \gamma \in B, \) pick a function \( \phi_{\gamma,B} \in C_s(G) \) such that \( \phi_{\gamma,B}(\gamma) = 1, \) \( \text{supp}(\phi_{\gamma,B}) \subseteq B, \) and \( 0 \leq \phi_{\gamma,B} \leq 1. \) Now if \( f \in C_c(G) \) and \( B \) is an open bisection with

\footnote{Recall: A conditional expectation \( E : A \to B \) is a contractive, linear, completely positive map such that for every \( b \in B, a \in A \) we have \( E(b) = b, E(ba) = bE(a) \) and \( E(ab) = E(a)b, \) see [3] II.6.10.1.]
\( \gamma \in B \), then
\[
(E(\phi^*_B f \phi_B))(u) = \sum_{r(\zeta) = u} \phi_B(\xi^{-1}) f(\xi^{-1} \zeta) \phi_B(\zeta^{-1}),
\]
which is zero unless \( \xi, \zeta \in B^{-1} \). Since \( r(\zeta) = r(\xi) = u \), we have that \( \xi = \zeta \) is the unique element of \( uB^{-1} \). So
\[
(E(\phi^*_B f \phi_B))(u) = \phi_B(\xi^{-1}) f(\xi^{-1}) \xi \phi_B(\zeta^{-1}) \leq E(f)(s(\xi)) \leq \|E(f)\|_\infty.
\]
Now if \( a \in C^*_e(G) \) then \( \phi_B^* a^* a \phi_B^* \) is positive so \( E(\phi_B^* a^* a \phi_B^*) \geq 0 \).
Therefore by the continuity of \( E \) we can apply (2.2) to obtain
\[
0 \leq E(\phi_B^* b^* b \phi_B^*) \leq \|E(b^* b)\|_\infty = 0.
\]
Thus \( E(\phi_B^* b^* b \phi_B^*) = 0 \) for all open bisections \( B \) and \( \gamma \in B \).

For \( \gamma \in G \) pick an open bisection \( B \) such that \( \gamma \in B \). Notice for \( c \in \mathbb{C} \)
\[
\pi_s(\gamma)(\phi_B) V_s(\gamma) c = \pi_s(\gamma)(\phi_B) c \delta_s(\gamma) = \sum_{s(\eta) = s(\gamma)} \phi_B(\eta) c \delta_\eta = c \delta_\gamma = V_\gamma c.
\]
Thus \( \pi_s(\gamma)(\phi_B) V_s(\gamma) = V_\gamma \) as operators. Now by equation (2.1) and the above observation we get for all \( \gamma \in G \) that
\[
V_\gamma^* \pi_u(b^* b) V_\gamma = V_\gamma^* \pi_s(\gamma) (\phi_B b^* b \phi_B^*) V_s(\gamma) = E(\phi_B^* b^* b \phi_B^*) = 0
\]
as desired. Therefore \( b = 0 \) and hence \( E \) is faithful.

\[\square\]

### 2.3. Purely infinite simple C*-algebras.

Given a C*-algebra \( A \) we denote its positive elements by \( A^+ \). If \( B \) is a subalgebra of \( A \) then \( B^+ \subseteq A^+ \). In particular, if \( C_0(X) \) is an abelian subalgebra of \( A \) and \( f \in C_0(X) \) such that \( f(x) \geq 0 \) for all \( x \in X \), then \( f \in A^+ \).

For positive elements \( a \in M_n(A) \) and \( b \in M_m(A) \), \( a \) is Cuntz below \( b \), denoted \( a \lesssim b \), if there exists a sequence of elements \( x_k \in M_{m,n}(A) \) such that \( x_k^* b x_k \to a \) in norm. Notice that \( \lesssim \) is transitive: if \( a \lesssim b \) and \( b \lesssim c \) there exist sequences of element \( x_n \) and \( y_n \) such that \( x_n^* b y_n \to a \) and \( y_n^* c y_n \to b \) in norm, so \( x_n^* y_n^* c y_n x_n \to a \) in norm, that is \( a \lesssim c \). We say \( A \) is purely infinite if there are no characters on \( A \) and for all \( a, b \in A^+ \), \( a \lesssim b \) if and only if \( a \in A^bA \). A nonzero positive element \( a \in A \) is properly infinite if \( a \not\lesssim a \). By Theorem 4.16 \( A \) is purely infinite if and only if every nonzero positive element in \( A \) is properly infinite.

A projection \( p \) in a C*-algebra \( A \) is infinite if it is Murray-von Neumann equivalent to a proper subprojection of itself, i.e., if there exists a partial isometry \( s \) such that \( s^* s = p \) but \( ss^* \not\leq p \). By Proposition 4.7 a C*-algebra \( A \) is purely infinite if every nonzero hereditary C*-subalgebra in every quotient of \( A \) contains an infinite projection. For simple C*-algebras the converse is also true, thus a simple C*-algebra is purely infinite precisely when every hereditary subalgebra contains an infinite projection.
3. Topologically Principal Groupoids and Positive Elements of $C_0(G^{(0)})$

In this section we consider, locally compact, Hausdorff and étale groupoids. We will show that we can determine when $C_\ast^r(G)$ is purely infinite simple by restricting our attention to elements of $C_0(G^{(0)})$ (see Theorems 4.4 and 5.5). Before we do that, we need the following technical lemmas.

**Lemma 3.1.** Let $G$ be a locally compact, Hausdorff and étale groupoid and $E : C_\ast^r(G) \to C_0(G^{(0)})$ be the faithful conditional expectation extending restriction. Suppose that $G$ is topologically principal. For every $\epsilon > 0$ and $c \in C_\ast^r(G)^+$, there exists $f \in C_0(G^{(0)})^+$ such that:

1. $\|f\| = 1$;
2. $\|fcf - fE(c)f\| < \epsilon$
3. $\|fE(c)f\| > \|E(c)\| - \epsilon$

**Proof.** Let $\epsilon > 0$. For $c = 0$ the result is trivial so let $c \in C_\ast^r(G)^+$ such that $c \neq 0$. Define

$$a := \frac{c}{\|E(c)\|}.$$ 

To find an appropriate $f$, we use the construction in the proof of [1] Proposition 2.4]; we include the details below. Find $b \in C_c(G) \cap C_\ast^r(G)^+$ so that $\|a - b\| < \frac{\epsilon}{2\|E(c)\|}$. Then

$$\|E(b)\| > 1 - \frac{\epsilon}{2\|E(c)\|},$$

because $E$ is linear and $\|E(a)\| = 1$. Now, let $K := \text{supp}(b - E(b))$, which is a compact subset of $G \setminus G^{(0)}$. Let

$$U := \{u \in G^{(0)} \mid E(b)(u) > 1 - \frac{\epsilon}{2\|E(c)\|}\}.$$ 

Since $G$ is topologically principal, [1 Lemma 2.3] implies that there exists a nonempty open set $V \subseteq U$ such that $VKV = \emptyset$. Using regularity, fix a nonempty open set $W$ such that $\overline{W} \subseteq V$. Using normality, select a positive (nonzero) real-valued function $f \in C_c(G^{(0)})$ such that $f|_W = 1$, $\text{supp}(f) \subseteq V$, and $0 \leq f(x) \leq 1$ for all $x \in G^{(0)}$.

Therefore, $f$ is positive in $C_\ast^r(G)$ and satisfies item [1].

To see that item [2] holds, a direct computation gives

$$fbf = fE(b)f.$$  

Since $\|a - b\| < \frac{\epsilon}{2\|E(c)\|}$, $\|f\| = 1$ and $E$ is norm decreasing we have

$$\|fE(a)f - fE(b)f\| < \frac{\epsilon}{2\|E(c)\|}.$$  

Combining equations (3.1) and (3.2) we get

$$\|faf - fE(a)f\| = \|faf - fbf + fbf - fE(b)f + fE(b)f - fE(a)f\| < \frac{\epsilon}{\|E(c)\|}.$$ 

Thus multiplying by $\|E(c)\|$ gives $\|fcf - fE(c)f\| < \epsilon$ as needed in [2].

To see item [3] notice that since $\text{supp} f \subseteq U$ we have

$$fE(b)f \geq (1 - \frac{\epsilon}{2\|E(c)\|})f^2.$$
Since $\|f\| = 1$, from the above equation and equation (3.2) we get
$$
\|fE(a)f\| > \|fE(b)f\| - \frac{\epsilon}{2\|E(c)\|} \geq 1 - \frac{\epsilon}{2\|E(c)\|} - \frac{\epsilon}{2\|E(c)\|} = 1 - \frac{\epsilon}{\|E(c)\|}.
$$
Multiplying by $\|E(c)\|$ we obtain $\|fE(c)f\| > \|E(c)\| - \epsilon$ as needed. □

**Lemma 3.2.** Let $G$ be a locally compact, Hausdorff and étale groupoid and $E : C_r(G) \to C_0(G(0))$ be the faithful conditional expectation extending restriction. Suppose that $G$ is topologically principal. For every nonzero $a \in C_r(G)^+$, there exists nonzero $h \in C_0(G(0))^+$ such that $h \prec a$.

**Proof.** Let $a \in C_r(G)^+$ such that $a \neq 0$. Since $E$ is faithful, $E(a)$ is nonzero. Applying Lemma 3.1 to $c := \frac{a}{\|E(a)\|}$ and $\epsilon = 1/4$ gives us an $f \in C_0(G(0))$ such that items (1), (2) and (3) of Lemma 3.1 hold. In particular $\|fE(c)f\| > \frac{\epsilon}{4}$.

Following [14, p. 640], for each $d \in C_0(G(0))^+$ we define the element
$$(d - 1/2)_+ := \phi_{1/2}(d) \in C_0(G(0))^+$$
where $\phi_{1/2}(t) = \max\{t - 1/2, 0\}$ for $t \in \mathbb{R}^+$. Notice that
$$\|\phi_{1/2}(d)\| = \max\{|d| - 1/2, 0\},$$
for each $d \in C_0(G(0))^+$.

Now let $h := (fE(c)f - 1/2)_+ \in C_0(G(0))^+$. Using item (2) of Lemma 3.1 and [13, Lemma 2.2], we can find $g \in C_r(G)$ so that $h = g^* f g$. Therefore $h \preceq a$. Finally, $h \neq 0$ since
$$\|h\| = \|(fE(c)f - 1/2)_+\| \geq \|fE(c)f\| - 1/2 \geq 1/4 > 0.$$  □

We are now in a position to prove the main result of this section.

**Theorem 3.3.** Let $G$ be a locally compact, Hausdorff and étale groupoid. Suppose that $G$ is minimal and topologically principal. Then $C_r(G)$ is purely infinite if and only if every nonzero positive element of $C_0(G(0))$ is infinite in $C_r(G)$.

**Proof.** The forward implication is trivial. To see the reverse, let $a \in C_r(G)^+$ such that $a \neq 0$. Using Lemma 3.2 we can find a nonzero $h \in C_0(G(0))^+$ such that $h \preceq a$. By assumption, we know $h$ is infinite. Since $C_r(G)$ is simple by [21, Proposition II.4.6], $h$ is properly infinite by [14, Proposition 3.14]. Thus $a$ is properly infinite by [14, Lemma 3.8], hence $C_r(G)$ is purely infinite. □

Recall that a Kirchberg algebra is a separable, nuclear, purely infinite simple $C^*$-algebra. We combine Theorem 3.3 with results from [2] [4] [21] to obtain the following characterization of groupoid Kirchberg algebras.

**Corollary 3.4.** Let $G$ be a second-countable, locally compact, Hausdorff and étale groupoid. Then $C^*(G)$ is a Kirchberg algebra if and only if $G$ is minimal, topologically principal, measure-wise amenable and every non-zero positive element of $C_0(G(0))$ is infinite in $C^*(G)$.

**Proof.** Suppose $C^*(G)$ is a Kirchberg algebra. Then $C^*(G)$ is simple by definition and so $C^*(G) = C_r(G)$, $G$ is minimal and $G$ topologically principal [4, Theorem 5.1]. Since $C^*(G)$ is nuclear, $C_r(G)$ is also nuclear hence $G$ is measure-wise amenable by [2, Corollary 6.2.14]. Finally, we apply Theorem 3.3 to see that every non-zero positive element of $C_0(G(0))$ is infinite in $C^*(G)$.
Conversely, suppose $G$ is minimal, topologically principal, measure-wise amenable and that every non-zero positive element of $C_0(G^{(0)})$ is infinite in $C^*(G)$. Then $C^*_r(G) = C^*(G)$ is nuclear by [2, Corollary 6.2.14], simple by [4, Theorem 5.1], separable because $G$ is second countable [21, Remark (iii) page 59] and purely infinite by Theorem 3.3. □

4. $C^*$-algebras associated to ample groupoids

In this section, we will restrict our attention to ample groupoids. Although this might seem a very restrictive class of groupoids, it actually includes a lot of important examples. Again, every Kirchberg algebra in UCT is Morita equivalent to a $C^*$-algebra associated to a Hausdorff, ample groupoid (see [26]). The ample case is far more manageable than the general case. In particular there is a large number of projections in the associated algebra. Let $G$ be a locally compact, Hausdorff and étale groupoid. If $G$ is ample, then the complex Steinberg algebra associated to $G$ is

$$A(G) := \text{span}\{\chi_B : B \text{ is a compact open bisection} \} \subseteq C_c(G)$$

where $\chi_B$ denotes the characteristic function of $B$, is dense in $C^*_r(G)$ see [6, Proposition 4.2] (see also [27]). A quick computation shows that $\chi_B * \chi_D = \chi_{BD}$ and $\chi_B^* = \chi_{B^{-1}}$, so that if $B \subseteq G^{(0)}$ is compact open, then $\chi_B$ is a projection.

**Theorem 4.1.** Let $G$ be a second countable, Hausdorff, ample groupoid. Suppose that $G$ is topologically principal, minimal and that $\mathcal{B}$ is a basis of $G^{(0)}$ consisting of compact open sets. Then $C^*_r(G)$ is purely infinite if and only if every nonzero projection $p$ in $C_0(G^{(0)})$ with $\text{supp}(p) \in \mathcal{B}$ is infinite in $C^*_r(G)$.

**Proof.** The forward implication is trivial. To see the reverse, suppose every nonzero projection $p$ of $C_0(G^{(0)})$ with $\text{supp}(p) = U$ for some $U \in \mathcal{B}$ is infinite in $C^*_r(G)$. By Theorem 3.3 it suffices to show that every positive element in $C_0(G^{(0)})^+$ is infinite. Let $a \in C^*_r(G^{(0)})^+$ be a nonzero element. We show that $a$ is properly infinite. We claim there is a nonzero projection $p \in C_0(G^{(0)})^+$ with $\text{supp}(p) \subseteq U$ for some $U \in \mathcal{B}$ such that $p \lesssim a$. To see this, first note that characteristic functions of the form $\chi_V$ are projections in $C_0(G^{(0)})$ for every compact open set $V \subseteq G^{(0)}$. Since $\mathcal{B}$ is a basis of compact open sets, there exists a compact open set $U_0 \in \mathcal{B}$ and a nonzero $s \in \mathbb{R}^+$ such that $\chi_{U_0}(x) \leq sa(x)$ for every $x \in G^{(0)}$. Then $p := \chi_{U_0} \leq sa$. Applying [14, Proposition 2.7] we get that $p \lesssim sa$ and so $p \lesssim a$ as claimed. Since $p$ is infinite by assumption and $C^*_r(G)$ is simple, $p$ is properly infinite by [14, Proposition 3.14]. Hence $a$ is properly infinite by [14, Lemma 3.8]. □

**Corollary 4.2.** Let $G$ be a second countable, Hausdorff, ample groupoid. Suppose that $G$ is topologically principal and minimal. Then $C^*_r(G)$ is purely infinite if and only if there exists a point $x \in G^{(0)}$ and a neighbourhood basis $\mathcal{D}$ at $x$ consisting of compact open sets so that every nonzero projection $q$ in $C_0(G^{(0)})$ with $\text{supp}(q) \in \mathcal{D}$ is infinite in $C^*_r(G)$.

**Proof.** Again, the forward direction is trivial. For the reverse implication, suppose there exist a point $x \in G^{(0)}$ and neighbourhood basis $\mathcal{D}$ of $x$ consisting of compact open sets such that that every nonzero projection $q$ in $C_0(G^{(0)})$ with $\text{supp}(q) \in \mathcal{D}$ is infinite in $C^*_r(G)$. Let $\mathcal{B}$ be a basis of $G^{(0)}$ consisting of compact open sets and suppose $p := \chi_U$ is a nonzero projection in $C_0(G^{(0)})$ with $U \in \mathcal{B}$. By Theorem 4.1, it suffices to show that $p$
is infinite. Since $G$ is minimal and ample, there exists a compact open bisection $B$ such that $x \in s(B)$ and $r(B) \cap U \neq \emptyset$. By shrinking $B$, we may assume that $r(B) \subseteq U$. Since $s(B)$ is an compact open neighbourhood of $x$, there exists a $V \in D$ such that $V \subseteq s(B)$. By shrinking $B$ again, we may assume that $s(B) = V$. Thus,

$$
\chi_V = \chi_{B}^* \chi_{r(B)} \chi_{B}.
$$

That is, $\chi_V \preceq \chi_{r(B)}$. Hence, $\chi_{r(B)}$ is properly infinite by [13, Lemma 3.8]. Finally, since $\chi_V = \chi_{r(B)} + \chi_{U - r(B)}$, $\chi_V$ is infinite.

5. AN APPLICATION: $k$-GRAPH $C^*$-ALGEBRAS

In this section, we apply Theorem 4.1 to $C^*$-algebras associated to $k$-graphs. We assume the reader is familiar with the basic definitions and constructions of $k$-graphs and their $C^*$-algebras found in [16], but we recall a few facts here. Let $\Lambda$ be a $k$-graph. Then the associated $C^*$-algebra $C^*(\Lambda)$ is the universal $C^*$-algebra generated by a Cuntz-Krieger $\Lambda$-family $\{s_\lambda : \lambda \in \Lambda\}$. To keep things clean, we will restrict our attention to row-finite $k$-graphs with no sources but similar results hold in the more general setting. We think our results will be useful in this setting because necessary and sufficient conditions on $\Lambda$ for $C^*(\Lambda)$ to be purely infinite simple are not known.

Following [16] we recall how $C^*(\Lambda)$ can be realised as the $C^*$-algebra of a second countable, Hausdorff, ample groupoid $G_\Lambda$ as follows. Let $\Lambda^\infty$ denote the infinite path space of $\Lambda$ and $\Lambda^\infty(v)$ be the set of infinite paths with range $v$. Define

$$
G_\Lambda := \{(x, n, y) \in \Lambda^\infty \times \mathbb{N}^k \times \Lambda^\infty : \sigma^l(x) = \sigma^m(y), n = l - m\}
$$

where $\sigma$ is the shift map. We view $(x, n, y)$ as a morphism with source $y$ and range $x$. Composition is given by $(x, n, y)(y, m, w) = (x, n + m, w)$. The unit space $G_\Lambda^{(0)}$ is identified with $\Lambda^\infty$. For $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ we define

$$
Z(\lambda, \mu) := \{(\lambda z, d(\lambda) - d(\mu), \mu z) : z \in \Lambda^\infty(s(\lambda))\}.
$$

The (countable) collection of all such $Z(\lambda, \mu)$ generate a topology under which $G_\Lambda$ is a second countable, Hausdorff, ample groupoid by [16, Proposition 2.8]. Further, the relative topology on the unit space $\Lambda^\infty$ has a basis of compact cylinder sets

$$
Z(\lambda) := \{\lambda x \in \Lambda^\infty : x \in \Lambda^\infty(s(\lambda))\}
$$

by identifying $Z(\lambda, \lambda)$ and $Z(\lambda)$ from [16, Lemma 2.6 and Proposition 2.8]. Note that $G_\Lambda$ is amenable by [16, Theorem 5.5] and hence $C^*_r(G_\Lambda) = C^*(G_\Lambda)$. It was shown in [16] that

$$
C^*(\Lambda) \cong C^*(G_\Lambda).
$$

More specifically, by [16, Corollary 3.5(i)], there is a (unique) isomorphism $\phi : C^*(\Lambda) \rightarrow C^*(G_\Lambda)$ such that $\phi(s_\lambda) = \chi_{Z(\lambda, s(\lambda))}$. Note that

$$
\phi(s_\mu s_\mu^*) = \chi_{Z(\mu, s(\mu))} \chi_{Z(\mu, s(\mu))} = \chi_{Z(\mu, s(\mu))} \chi_{Z(\mu, s(\mu))} = \chi_{Z(\mu)}.
$$

With all of this theory in place, along with the simplicity results of [23] and [4], the following is an immediate corollary of Theorem 4.1 and Corollary 4.2.

**Corollary 5.1.** Suppose $\Lambda$ is a row-finite $k$-graph with no sources such that $\Lambda$ is aperiodic and cofinal in the sense of [23]. Then

1. For $\mu \in \Lambda$, $s_\mu s_\mu^*$ is infinite if and only if $s_\mu$ is.
2. $C^*(\Lambda)$ is purely infinite simple if and only if $s_v$ is infinite for every $v \in \Lambda^0$. 
(3) $C^*(\Lambda)$ is purely infinite simple if and only if there exists $x \in \Lambda^\infty$ such that $s_v$ is infinite for every vertex $v$ on $x$.

**Proof.** For [11], we use a trick used in [25, Lemma 8.13]. Recall that infiniteness is preserved under von Neumann equivalence, hence $s_\mu s_\mu^*$ is infinite if and only if $s_\mu^* s_\mu = s_{s(\mu)}$ is infinite.

For (2), we apply Theorem 4.4 to the second countable, Hausdorff, ample groupoid $G_\Lambda$; first we check the remaining hypotheses of Theorem 4.4. Since $\Lambda$ is cofinal and aperiodic, $C^*(G_\Lambda) \cong C^*(\Lambda)$ is simple by [23, Theorem 3.1]. Thus $C^*(G_\Lambda) = C^*_r(G_\Lambda)$ is simple and hence $G_\Lambda$ is topologically principal and minimal by [4, Theorem 5.1].

We have that the collection of cylinder sets of the form $Z(\mu)$ form a basis $B$ of $G^{(0)}$ consisting of compact open sets. Now we apply Theorem 4.4 to see that $C^*(G_\Lambda)$ is purely infinite if and only if each $\chi_{Z(\mu)}$ is infinite. Let $\phi : C^*(\Lambda) \to C^*(G_\Lambda)$ be the isomorphism characterized by $s_\mu \mapsto \chi_{Z(\mu)}$. Since $\phi$ is an isomorphism, this gives $\chi_{Z(\mu)}$ is infinite if and only if $\phi^{-1}(\chi_{Z(\mu)}) = s_\mu s_\mu^*$ is infinite. Finally, $s_\mu s_\mu^*$ if and only if $s_{s(\mu)}$ is infinite by (1).

For (3), given an infinite path $x$, the collection of compact open sets of the form $Z(x(0, n))$ for $n \in \mathbb{N}^k$ form a neighbourhood base at $x$. Now proceed as in the proof of (2) replacing Theorem 4.4 with Corollary 4.2 and $\mu$ with $x(0, n)$. \hfill $\square$

6. THE NON-SIMPLE CASE

Let $A$ be a $C^*$-algebra. A pair of positive elements $(a_1, a_2) \in A \times A$ has the matrix diagonalization property in $A$ in the sense of [15, Definition 3.3] if for every positive matrix $(a_{ij})$ with $b_{ij} \in A$ and every $\epsilon_1, \epsilon_2, \delta > 0$ there exists $d_1, d_2 \in A$ with

$$||d_i^* a_i d_i - a_i|| < \epsilon_i \quad \text{and} \quad ||d_i^* b_{ij} d_j|| < \delta.$$ 

A subset $\mathcal{F}$ of $A^+$ is a filling family for $A$, in the sense of [15, Definition 3.10], if for every hereditary $C^*$-subalgebra $H$ of $A$ and every primitive ideal $I$ of $A$ with $H \not\subseteq I$ there exist $f \in \mathcal{F}$ and $z \in A$ with $z^* z \in H$ and $z z^* = f \not\in I$.

By Proposition 3.13 and Lemma 3.12 of [15], if $A^+$ contains a filling family $\mathcal{F}$ that is closed under $\epsilon$-cut-downs and every pair of elements $(a_1, a_2) \in \mathcal{F} \times \mathcal{F}$ has the matrix diagonalization property, then $A$ is strongly purely infinite. In this section we provide a characterization of when the reduced groupoid $C^*$-algebra is strongly purely infinite (Proposition 6.3). In our proof of Proposition 6.3 we will use results from [4] to describe ideals of reduced groupoid $C^*$-algebras. First we need the following lemma. Recall that a subset $D \subseteq G^{(0)}$ is said to be invariant if $G \cdot D := \{r(\gamma) : s(\gamma) \in D\} \subseteq D$.

**Lemma 6.1.** Let $G$ be a second countable, locally compact, Hausdorff and étale groupoid such that $C^*(G) = C^*_r(G)$. Then the following properties are equivalent:

1. For every closed invariant set $D \subseteq G^{(0)}$

$$C^*(G|_D) = C^*_r(G|_D).$$

2. For every closed invariant set $D \subseteq G^{(0)}$ the sequence

$$0 \longrightarrow C^*_r(G|_{G^{(0)} - D}) \overset{\iota_r}{\longrightarrow} C^*_r(G) \overset{\rho_r}{\longrightarrow} C^*_r(G|_D) \longrightarrow 0$$

is exact where $\iota_r$ and $\rho_r$ are determined on continuous functions by extension and restriction respectively.
Remark 6.2. In [22, Remark 4.10], Renault mentions that if $G|_D$ is amenable for every closed invariant set $D \subseteq G^{(0)}$, then item (2) of Lemma 6.1 follows. Thus Lemma 6.1 is a strengthening of Renault’s comment.

Proof. Fix a closed invariant set $D \subseteq G^{(0)}$ and let $U = G^{(0)} - D$. Consider the following diagram:

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\iota_r} & C^*_r(G|_U) & \xrightarrow{\iota} & C^*_r(G) & \xrightarrow{\rho} & C^*_r(G|_D) & \xrightarrow{\rho_r} & 0 \\
\pi_U & & \downarrow \pi & & \downarrow \pi_D & & \downarrow \pi_D & & \\
0 & \xrightarrow{\iota_r} & C^*_r(G|_U) & \xrightarrow{\iota} & C^*_r(G) & \xrightarrow{\rho_r} & C^*_r(G|_D) & \xrightarrow{\rho_r} & 0 \\
\end{array}
\]

where $\pi_U, \pi$ and $\pi_D$ are the respective quotient maps, and $\iota, \iota_r$ and $\rho, \rho_r$ extend extension and restriction respectively. Since all of the maps involved are continuous, the diagram commutes. We also have that the top row of (6.1) is exact by [18, Lemma 2.10].

(6.2) $\Rightarrow (\mathbf{1})$: We show the surjective map $\pi_D$ is injective. Fix any $a \in C^*_r(G|_D)$ with $\pi_D(a) = 0$. Find $b \in C^*(G)$ with $\rho(b) = a$. From $\pi_D(\rho(b)) = \rho_r(\pi(b)) = 0$, exactness of (6.1), surjectivity of $\pi_U$, and $\iota \circ \pi_U = \pi \circ \iota$ we obtain

$\pi(b) \in \ker \rho_r = \iota_r(C^*_r(G|_U)) = \iota_r \circ \pi_U(C^*(G|_U)) = \pi \circ \iota(C^*(G|_U))$.

Find $c \in C^*(G|_U)$ with $\pi(b) = \pi \circ \iota(c)$. As $\pi$ is an isomorphism by assumption we obtain that $b = \iota(c)$. Hence $a = \rho(b) = \rho \circ \iota(c) = 0$, and $C^*(G|_D) = C^*_r(G|_D)$.

(\mathbf{1}) $\Rightarrow (\mathbf{2})$: By assumption the maps $\pi$ and $\pi_D$ are isomorphisms. Using the commutative diagram (6.1) and the exactness of the top line of that diagram, the exactness of the bottom line follows from a easy diagram chase. \[ \square \]

Let $G$ be a second countable, locally compact, Hausdorff and étale groupoid and $D$ be a closed invariant set of $G^{(0)}$ and define $U = G^{(0)} - D$. Recall from Lemma 2.1(4) we have the commuting diagram:

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\iota_r} & C^*_r(G|_U) & \xrightarrow{\iota} & C^*_r(G) & \xrightarrow{\rho} & C^*_r(G|_D) & \xrightarrow{\rho_r} & 0 \\
E_U & & \downarrow E & & \downarrow E & & \downarrow E_D & & \\
0 & \xrightarrow{\iota_r} & C_0(U) & \xrightarrow{\iota} & C_0(G^{(0)}) & \xrightarrow{\rho_0} & C_0(D) & \xrightarrow{\rho_0} & 0 \\
\end{array}
\]

Notice that the bottom row in (6.2) is exact. We will use this diagram several times. We also use the notation Ideal[$S$] for the ideal in $C^*_r(G)$ generated by $S \subseteq C^*_r(G)$.

**Proposition 6.3.** Let $G$ be a second countable, locally compact, Hausdorff and étale groupoid such that $C^*(G) = C^*_r(G)$. Then the following properties are equivalent:

1. The $C^*$-algebra $C^*_r(G)$ is strongly purely infinite, and for every ideal $I$ in $C^*_r(G)$, $I = \text{Ideal}[I \cap C_0(G^{(0)})]$.

2. For every closed invariant set $D \subseteq G^{(0)}$, $G|_D$ is topologically principal; the sequence

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\iota_r} & C^*_r(G|_U) & \xrightarrow{\iota} & C^*_r(G) & \xrightarrow{\rho_r} & C^*_r(G|_D) & \xrightarrow{\rho_r} & 0 \\
\end{array}
\]

is exact where $U = G^{(0)} - D$, $\iota_r$ and $\rho_r$ are determined on continuous functions by extension and restriction respectively; and for every pair of elements $a, b$ in $C_0(G^{(0)})^+$ the 2-tuple $(a, b)$ has the matrix diagonalization property in $C^*_r(G)$. 


Proof. (1) $\Rightarrow$ (2): Fix a closed invariant set $D \subseteq G^{(0)}$ and $U = G^{(0)} - D$. For this $D$ and $U$ we have a commuting diagram as in (6.2). Define $I := \ker \rho_r \subseteq C^*_r(G)$. Using the diagram, $\rho_0(E(I)) = E_0(\rho_r(I)) = 0$, implying that $E(I) \subseteq \iota_0(C_0(U))$. Since $E(b) = b$ for $b \in C_0(G^{(0)}), I \cap C_0(G^{(0)}) \subseteq E(I)$. Using assumption (1) we have $I = \text{Ideal}[I \cap C_0(G^{(0)})]$. Hence

$$\ker \rho_r = I = \text{Ideal}[I \cap C_0(G^{(0)})] \subseteq \text{Ideal}[E(I)] \subseteq \text{Ideal}[\iota_0(C_0(U))] \subseteq \iota_r(C^*_r(G)_U));$$

that is $\ker \rho_r \subseteq \text{image}(\iota_r)$. Thus (6.3) is exact.

We know that each $G|_D$ is topologically principal by [4] Remark 5.10] provided that $C^*(G|_D) = C^*_r(G|_D)$. The latter follows from Lemma 6.1 since (6.3) is exact.

Since $C^*_r(G)$ is strongly purely infinite, Lemma 5.8 in [13] implies that every pair $(a, b)$ of positive elements in $C_0(G^{(0)})$ has the matrix diagonalization property in $C^*_r(G)$.

(2) $\Rightarrow$ (1): Since we assumed that $G|_D$ is topologically principal for all closed invariant $D \subseteq G^{(0)}$, by the proof of Corollary 5.9 in [4] (see also [4] Remark 5.10)), we know $I = \text{Ideal}[I \cap C_0(G^{(0)})]$ for every ideal $I$ in $C^*_r(G) = C^*(G)$ provided that $C^*(G|_D) = C^*_r(G|_D)$ for every closed invariant set $D \subseteq G^{(0)}$. But this follows from Lemma 6.1 since $C^*(G) = C^*_r(G)$ and (6.3) is exact, which are assumed in (2). Hence (2) implies $I = \text{Ideal}[I \cap C_0(G^{(0)})]$.

We prove $C^*_r(G)$ is strongly purely infinite. Define $F := C_0(G^{(0)})^+ \subseteq C^*_r(G)$. By functional calculus we know

$$f(a) \in F, \quad \text{for } f \in C_0(\mathbb{R})^+, \quad a \in F.$$

In particular $F$ is closed under $\varepsilon$-cut-downs, i.e., for each $a \in F$, and $\varepsilon \in (0, \|a\|)$ we have $(a - \varepsilon)_+ \in F$. By [2] each pair $(a, b)$ with $a, b \in F$ has the matrix diagonalization property (of [13] Definition 3.3]). Now by Lemma 3.12 of [13] we know that $F$ has the matrix diagonalization property of [13] Definition 3.10(ii)]. If follows from Proposition 3.13 of [13] that $C^*_r(G)$ is strongly purely infinite provided that $F$ is a filling family for $C^*_r(G)$, which we now show.

Fix any hereditary $C^*$-subalgebra $H$ of $C^*_r(G)$ and any ideal $I$ of $C^*_r(G)$ with $H \nsubseteq I$. We know $I = \text{Ideal}[I \cap C_0(G^{(0)})]$ , hence $I = \iota_r(C^*_r(G)_U))$ for some open invariant set $U \subseteq G^{(0)}$. Let $D = G^{(0)} - U$ and consider the corresponding commuting diagram (6.2).

Select $d \in H^+$, $d \notin I$. Define $c := \rho_r(d)$. As $d \notin I = \ker \rho_r$ by exactness in (2), we know $\rho_r(d) \neq 0$. Since $E_D$ is faithful and $c$ positive,

$$\epsilon := \frac{1}{4} \|E_D(c)\| > 0.$$

By (2) the groupoid $G|_D$ is topologically principal, hence Lemma 6.11 gives $f \in C_0(D)^+$ such that

$$\|f\| = 1, \quad \|fcf - fE_D(c)\| < \epsilon, \quad \|E_D(c)f\| > \|E_D(c)\| - \epsilon.$$

Recall [13] Lemma 2.2: For $x, y$ positive and $\delta > 0$ with $\|x - y\| < \delta$ there exist a contraction $a$ with $a^*xa = (y - \delta)_+$. Use this to find a contraction $a \in C^*_r(G|_D)$ such that

$$h := a^* f e a = (f E_D(c)f - \epsilon)_+ \in C_0(D)^+.$$

Notice that

$$\|h\| \geq \|f E_D(c)f\| - \epsilon > \|E_D(c)\| - 2\epsilon > 0.$$

Using that $\rho_r$ restricts to the map $C_0(G^{(0)}) \rightarrow C_0(D)$, select $b \in C_0(G^{(0)})^+$ such that $\rho_r(b) = h$. Also as $\rho_r$ is surjective find $w \in C^*_r(G)$ such that $\rho_r(w) = f a$. Since $\rho_r(b -
$w^*dw = h - a^* f c a = 0$ we have $b = w^*dw + v$ for some $v \in I$. Let $\{e_\lambda\}$ denote an approximate unit of $I = C^*_r(G_U)$ with $e_\lambda \in C_0(U)$ (see Lemma 2.1). Let $1$ be the unit of the unitisation of $C^*_r(G)$. Then $(1 - e_\lambda)v(1 - e_\lambda) \to 0$. For suitable $\lambda_0$ and $\epsilon := 1 - e_{\lambda_0}$ we get

$$\|ew^*dwe - ebe\| = \|eve\| < \epsilon.$$ 

Use Lemma 2.2 of [13] to find a contraction $u \in C^*_r(G)$ such that

$$g := u^* ew^*dweu = (ebe - \epsilon) + C_0(G^{(0)})^+ = \mathcal{F}.$$ 

Since $be_{\lambda_0} + e_{\lambda_0}b - e_{\lambda_0}be_{\lambda_0} \in C_0(U) \subseteq \ker \rho_r$ we obtain that $\rho_r(ebe) = \rho_r(b) = h$. Moreover by functional calculus we know $(h - \epsilon)_+ = (fE_D(c)f - 2\epsilon)_+$. We conclude

$$\|\rho_r(g)\| = \|(h - \epsilon)_+\| \geq \|fE_D(c)f\| - 2\epsilon > \|E_D(c)\| - 3\epsilon > 0,$$ 

eff ensuring $g \not\in I$. Finally with $z := u^* ew^*d^{1/2} \in C^*_r(G)$ we obtain $g = zz^*$ and $z^*z \in H$. By definition $\mathcal{F}$ is a filling family for $C^*_r(G)$ completing the proof. \hfill $\square$

**References**


Jonathan Brown, Mathematics Department, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506-2602, USA.

E-mail address: brownjh@math.kansas.edu

Lisa Orloff Clark, Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin Dunedin 9054, New Zealand.

E-mail address: lclark@maths.otago.ac.nz

Adam Sierakowski, School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia

E-mail address:asierakowski@uow.edu.au