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PRODUCTS AND POWERS, POWERS AND EXPONENTIATIONS, ...

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ABSTRACT. The Horadam recurrence relation $w_{n+1}(a, b; p, q) = pw_n(a, b; p, q) - qw_{n-1}(a, b; p, q)$ (with $w_0 = a$ and $w_1 = b$) has inspired consideration of the recurrence $z_n(a, b; p, q) = z^n_0(a, b; p, q)z^n_{n-1}$ (with $z_0 = a$ and $z_1 = b$). This paper defines a natural sequence of such recurrence relations of which $w_n$ and $z_n$ are the first and second.

1. The functions $w_n(a, b; p, q)$ and $z_n(a, b; p, q)$

The Horadam functions (Horadam [6] p 161) and the functions $z_n(a, b; p, q)$ (Bunder [2] p 279 and Larcombe, Bagdasar and Fennesey [8]) are given by:

Definition 1.1. $w_0(a, b; p, q) = a, \ w_1(a, b; p, q) = b, \ w_{n+1}(a, b; p, q) = pw_n(a, b; p, q) - qw_{n-1}(a, b; p, q)$.

Definition 1.2. $z_0(a, b; p, q) = a, \ z_1(a, b; p, q) = b, \ z_{n+1}(a, b; p, q) = (z_n(a, b; p, q))^p \cdot (z_{n-1}(a, b; p, q))^q, \ w_n(a, b; p, q)$ will usually be written as $w_n$ and $z_n(a, b; p, q)$ as $z_n$.

2. A sequence of functions starting with $w_n$ and $z_n$

The Horadam recurrence of Definition 1.1 involves the sum of two products (i.e. repeated additions) $pw_n$ and $(-q)w_{n-1}$. The recurrence in Definition 1.2 involves the product of two powers (i.e. repeated multiplications) $z^n_0$ and $z^n_{n-1}$. Taking this to the next level, the recurrence would involve the exponentiation of repeated exponentiations $t_n^{w_n}$ and $t_{n-1}^{w_{n-1}}$. Where there are $p \ t_n s$ and $q \ t_{n-1} s$. There are of course two different exponentiations, but we will consider only one.

The first aim of this paper is to generate a natural infinite sequence of such functions $<w_n, z_n, t_n, ...>$ and the second to see whether $t_n$ and later functions can be defined in simple terms or in terms of functions coming earlier in the sequence, just as $z_n$ can be defined in terms of $w_n$. Bunder [2] and Larcombe, Bagdasar and Fennesey [5] show that:

$$z_n = a^{w_n(1, 0; p, q)}b^{w_n(0, 1; p, q)}.$$

The first aim can be achieved by using the following function due to Ackermann [1]:

Definition 2.1. $\phi(m, n, 0) = m + n, \ \phi(m, 0, 1) = 0, \ \phi(m, 0, 2) = 1, \ \phi(m, 0, r) = m, \ \text{for } r > 2 \ \phi(m, n, r) = \phi(m, \phi(m, n-1, r), r-1), \ \text{for } n > 0, r > 0$.

This gives $\phi(m, n, 1) = mn, \ \phi(m, n, 2) = m^n, \ \phi(m, n, 3) = m^{m^{m^{m^{\ldots}}}}(n \text{ ms})$. Ackermann considered such functions to clarify Hilbert’s proposed proof of the continuum hypothesis. It is also one of the earliest and simplest examples of a total function that is...
Definition 3.1. The Ackermann function, usually called the Ackermann function.  

\( \text{ack} \) or \( \phi \) see for example Giesler [5]. Knuth [7] and Conway and Guy [4] have other notations for the \( \phi \) or \( \text{ack} \) function.

Note that Ackermann’s \( \phi(m, n, r) \) is related to, but not the same as, what is these days usually called the Ackermann function.

3. A general Horadam-style recurrence

A general Horadam recurrence, motivated by the discussion in Section 1, is given by:

**Definition 3.1.** \( s_{i,0}(a, b; p, q) = a, \quad s_{i,1}(a, b; p, q) = b, \)

\[ s_{i,n+1}(a, b; p, q) = \phi(\phi(s_{i,n}(a, b; p, q), p, i + 1)), \phi(s_{i,n-1}(a, b; p, q), q, i + 1), i). \]

We will usually write \( s_{i,n}(a, b; p, q) \) as \( s_{i,n} \).

Clearly \( s_{1,n} = w_n(a, b; p, q) \), \( s_{2,n} = z_n \) and \( s_{3,n+1} = (s_{3,n}^{s_{3,n}}, s_{3,n})^{-1} \), where there are \( p \) \( s_{3,n} \)'s and \( q \) \( s_{3,n-1} \)'s.

Unless the meaning of repeated exponentiation can somehow be generalised, this, of course, requires \( p \) and \( q \) to be positive integers.

(Note that our notation would have been neater, giving \( s_{1,n} = w_n(a, b; p, q) \), if we had \( q \) for \( -q \) on the right hand side of the recurrence in Definition 1.1, as this gives \( s_{1,n} = w_n! \))

4. \( s_{m,n} \) in simple terms or in terms of \( s_{j,n} \) where \( j < m \)?

\( s_{1,n} \) can be expressed as:

If \( n \geq 0, p^2 \neq -4q \), \( C = (p + \sqrt{(p^2 + 4q)})/2 \) and \( D = (p + \sqrt{(p^2 + 4q)})/2 \),

\[ s_{1,n} = \left( \frac{b - aC}{C - D} \right) C^n + \left( \frac{b - aD}{D - C} \right) D^n. \]

If \( n \geq 0, s_{1,n}(a, b, p, -p^2/4) = nb(p/2)^{n-1} - (n - 1)a(p/2)^n. \)

(See Horadam [6] pp 161,175 and Bunder [3]).

In Section 2, \( s_{2,n}(= z_n) \) was given in terms of \( w_n(0, 1; p, -q) \) and \( w_n(1, 0; p, -q) \), we also have:

\[ s_{1,n} = w_n(a, b; p, q) = aw_n(0, 1; p, -q) + bw_n(0, 1; p, -q), \]

so we might expect \( s_{3,n} = \left( b^{-a} \right) \), where there are \( w_n(0, 1; p, -q) \) \( b \)'s and \( w_n(1, 0; p, -q) \) \( a \)'s. However the examples below show that this is not generally the case. Even in simple cases such as \( i = 3 \) and \( p, q < 5 \), there seems to be no simple expressions for \( s_{i,n} \), nor one in terms of \( s_{j,n} \) where \( j < i \).

**Example 4.1.** Let \( p = -q = 1 \) then

\[ < s_{1,n} >= < a, b, a + b, a + 2b, 2a + 3b, ... > \]

\[ < z_n >= < s_{2,n} >= < a, b, ab, ab^2, a^2b^2, ... > \]

\[ s_{3,n+1} = s_{3,n}^{s_{3,n-1}} \text{ and } < s_{3,n} >= < a, b, b^a, b^{ab}, b^{ab^2}, b^{ab^{2+a}}, b^{ab^{3+a}}, ... > \].
Example 4.2. Let $p = 3, q = -2$ then

\[
<s_{1,n} > = < a, b, 2a + 3b, 6a + 11b, 22a + 39b, \ldots > \\
<s_{2,n} > = < a, b, a^2b^3, a^6b^{11}, a^{22}b^{39}, \ldots > \\
<s_{3,n} > = < a, b, b^{a}a^{b}, \ldots >
\]

\[
s_{3,n+1} = \left( \frac{s_{3,n}}{s_{3,n-1}} \right)^{s_{3,n}}
\]

and $< s_{3,n} > = < a, b, b^{a}a^{b}, \ldots >$.

5. Summary

A sequence of functions $< s_{1,n}, s_{2,n}, \ldots >$ has been defined, (with $s_{1,n}$ the Horadam function $w_{n}(a, b; p, -q)$ and $s_{2,n} = z_{n}$), each element of which is generated by a Horadam like recurrence relation, with higher order operations than the previous one. The first two of these can be represented in terms of elementary arithmetical functions, $z_{n}$ can also be written in terms of $w_{n}$. Later functions in the sequence, it seems, cannot be represented in terms of such elementary functions except for specific values of $n$. Perhaps later work, maybe with new notation, can change this situation.

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