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Directed Graphs and $k$-graphs: Topology of the Path Space and How It Manifests in the Associated $C^*$-algebra

A thesis submitted in fulfilment of the requirements for the award of the degree

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I, Samuel B.G. Webster, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Samuel B.G. Webster
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Abstract

Directed graphs and their higher-rank analogues provide an intuitive framework for the analysis of a broad class of $C^*$-algebras which we call graph algebras. Kumjian, Pask, Raeburn and Renault built a groupoid $\mathcal{G}_E$ from the infinite-path space of a locally finite directed graph $E$, and used the theory of groupoid $C^*$-algebras to define the graph $C^*$-algebra. Local finiteness was required so that $\mathcal{G}_E$ was locally compact and $r$-discrete, with unit space homeomorphic to the infinite path space of $E$. Similarly, the higher-rank graphs of Kumjian and Pask were initially studied with similar restrictive hypotheses in order to use groupoid based analysis of their higher-rank $C^*$-algebras. In particular, the topology on the path space of a directed graph or higher-rank graph is crucial in the analysis of graph $C^*$-algebras.

Drinen and Tomforde defined a process called desingularisation which can be used to extend many results about the $C^*$-algebras of locally finite directed graphs to those of arbitrary directed graphs. Drinen and Tomforde construct from an arbitrary directed graph $E$ a row-finite directed graph $\hat{E}$ with no sources such that $C^*(E)$ embeds in $C^*(\hat{E})$ as a full corner. Subsequently, Farthing developed a partial desingularisation for higher-rank graphs, which constructs from a row-finite higher-rank graph $\Lambda$ with sources a row-finite higher-rank graph $\tilde{\Lambda}$ with no sources such that $C^*(\Lambda)$ embeds in $C^*(\tilde{\Lambda})$ as a full corner.

In Chapter 2, we construct a topology on the path space of an arbitrary directed graph $E$ and prove that it is locally compact and Hausdorff. We show that there is a homeomorphism $\phi_\infty$ from a subspace of the infinite-path space of the Drinen-Tomforde desingularisation $\hat{E}$ onto the boundary-path space $\partial E$ of $E$. We then show that there is a commutative $C^*$-subalgebra $D_E$ of $C^*(E)$ which is homeomorphic to the continuous functions on $\partial E$. Concluding our results on directed graphs, we show that the embedding of $C^*(E)$ in $C^*(\hat{E})$ restricts to an embedding of $D_E$ in $D_{\hat{E}}$ which implements $\phi_\infty$. In Chapter 3, we develop a modification of Farthing’s desingularisation procedure for row-finite higher-rank graphs which contains cleaner proofs of her results. We use this modification to prove analogues for higher-rank graphs of the results from Chapter 2.