1973

Eigenvalues by numerical methods

Allyn Grahame Morris

Wollongong University College

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EIGENVALUES BY NUMERICAL METHODS

by

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B.Sc. (N'cle)

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Doctor of Philosophy

in the

School of Mathematics

at

Wollongong University College,

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ABSTRACT

The theme of the thesis is the determination of the eigen-values of fourth order differential boundary value problems, with specific reference to the Orr-Sommerfeld equations of hydrodynamics.

The method adopted is to convert the continuous differential equation to a discrete algebraic problem by use of Chebyshev series, and then to solve the algebraic matrix equation,

$$|A - \lambda I| = 0$$

by an extension of the widely published QR algorithm.

Because of the nature of the physical problems considered, complex numbers appear in the differential equation, and the algebraic eigen-value problem is modified specifically for this situation.

The accuracy of the method is firstly displayed by reference to two simple problems, which have known analytic solutions. Then problems considered by several authors are solved by the present methods, and the results are compared with those previously published.
# CONTENTS

## CHAPTER 1  INTRODUCTION
1.1 Introduction  
1.2 Motivation  
1.3 The Discrete, Algebraic Problem  
1.4 Outline of Thesis  

## CHAPTER 2  THE QR ALGORITHM
2.1 The QR Algorithm for Real Matrices with Double Shift  
2.2 Hessenberg Matrices  
2.3 The QR Algorithm for a Complex Matrix  
   2.3.1 The Determination of $P_1$  
   2.3.2 The Evaluation of $A_3$  

## CHAPTER 3  RATIONAL FINITE DIFFERENCE
3.1 Introduction  
3.2 Application of Boundary Conditions  
3.3 Modifications of the Boundary Conditions  
3.4 Even and Odd Solutions
CHAPTER 4 CHEBYSHEV SERIES

4.1 Chebyshev Polynomials

4.1.1 Some General Properties

4.1.2 Even and Odd Functions

4.2 Solutions to Differential Equations Using Chebyshev Series

4.3 Boundary Conditions - Even Solution

4.4 Boundary Conditions - Odd Solution

4.5 Boundary Conditions - General Solution

4.6 The General Problem

5 RESULTS

5.1 Introduction

5.2 The Problem \[ \phi^{IV} - \lambda \phi = 0 \]

5.2.1 Even Solution: \[ \phi(1) = \phi'(1) = 0 \]

5.2.2 Odd Solution: \[ \phi(1) = \phi'(1) = 0 \]

5.3 The Problem \[ \phi^{IV} + \lambda \phi'' = 0 \]

5.3.1 Even Solution: \[ \phi(1) = \phi'(1) = 0 \]

5.3.2 Odd Solution: \[ \phi(1) = \phi'(1) = 0 \]

5.4 The Orr-Sommerfeld Equation Plane Poiseuille Flow

5.5 The Orr-Sommerfeld Equation Plane Couette Flow

5.6 The Orr-Sommerfeld Equation Plane Jet Flow
CHAPTER 5  RESULTS : Con't.

5.7 Conclusion  127
  5.7.1 Matrix Eigen-values  127
  5.7.2 Continuous to Discrete  127
  5.7.3 Eigen-values  128
  5.7.4 Eigen-functions  129
  5.7.5 General  130

6  APPENDIX  131

6.1 Analytic Solution to Test Problems  132
  6.1.1 The Problem $\phi^{iv} - \lambda \phi = 0$  132
  6.1.2 The Problem $\phi^{iv} + \lambda \phi'' = 0$  137

6.2 Direct Application of Finite Differences  142
  6.2.1 Solution to $\phi^{iv} + \lambda \phi'' = 0$  142
  6.2.2 Solution to $\phi^{iv} - \lambda \phi = 0$  148

6.3 Outline of Other Methods  153

REFERENCES  159
Section 1.1 - Introduction

The main theme of this thesis is the examination of numerical methods for finding the eigen-values \( \{\lambda\} \) of the problem

\[ L (\phi, \lambda) = 0 \]

with homogeneous boundary conditions, where \( L \) is a complex, linear differential operator, and the complex parameter \( \lambda \) occurs linearly.

The numerical problem consists of two parts. Firstly, the continuous differential equation problem is approximated by a discrete algebraic problem, and secondly, the eigen-values of the algebraic problem are found by some suitable method. Because two approximations are being used to solve one problem, large errors can often be introduced. The ideal situation is, of course, to have the error as small as possible. We therefore endeavour to choose two methods which together will achieve this aim.

Up to Chapter 4 in this thesis, the statement "solution of the problem" refers to the determination of the eigen-values of the problem.
Section 1.2 - Motivation

The work was motivated by the study of the Orr-Sommerfeld equations arising in hydro-dynamics, so the general problem has been formulated to include the Orr-Sommerfeld problem as a special case.

The Orr-Sommerfeld equations are fourth order, complex differential equations, relating perturbation of one-dimensional parallel laminar flow, with or without boundaries, to such physical constants as the Reynold's number, complex velocity, and the wave-length of the disturbance. The complex velocity of the fluid flow is the eigen-value of the problem, and this is determined for the critical Reynold's number which occurs when the imaginary part of the complex velocity is equal to zero.

The equation itself is derived from the Navier-Stokes equation of hydro-dynamics, and such a derivation may be found in Hunt "Incompressible Fluid Dynamics" (1964). Further discussion is given by Clenshaw and Elliott (1960), Lin (1946) and Lin "The Theory of Hydro-dynamic Stability" (1955). Solutions and results are also discussed by Osborne (1967), Thomas (1953), and Gallagher and Mercer (1962). In Chapter 3 and the Appendix, we outline some of these methods, up to and including the formation of the discrete algebraic eigen-value problem.
Types of methods used are finite differences (Osborne and Thomas), variational methods (Gallagher and Mercer, and Lee and Reynolds) and Chebyshev series (Clenshaw and Elliott).

The method derived in Chapter 4 uses Chebyshev series, and is similar to work set out in Fox and Parker "Chebyshev Polynomials in Numerical Analysis", Chapter 5. The method is tested with eigen-value problems having known, analytic solutions, and is then applied to the Orr-Sommerfeld equations, and our results are compared with those of the previously mentioned authors, with the accuracy of the test case answers kept in mind. Finite difference approximations are found to be crude in comparison with results derived from the Chebyshev method.

The problems considered for test cases are

$$\phi^{(iv)} - \lambda \phi = 0$$

and

$$\phi^{(iv)} + \lambda \phi'' = 0$$

where

$$\phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0$$
Section 1.3 - The Discrete, Algebraic Problem

The second step of the problem, namely the solution of the algebraic equation

$$|A - \lambda I| = 0$$

may be solved in many different ways.

We may find the eigen-values, one at a time, in descending order of magnitude, but unmodified methods of this type usually have restrictions regarding relative sizes of eigen-values, and also convergence rates are usually slow (Faddeeva "Computational Methods of Linear Algebra", and Wilkinson "The Algebraic Eigen-value Problem"). Also such methods are not applicable directly to physical problems, as these usually require the smallest eigen-value for the relevant situation.

Finally, we have methods which produce all eigen-values at once. (See Osborne (1964)). There are special methods available when the matrix is in special forms eg. Band symmetric matrices - Martin, Reinsch & Wilkinson (1970), Symmetric matrices - Rutishauser (1970), Hermitian matrices - Stewart (1969). For use with these methods, a given matrix can often be reduced to one of the special
forms: for example, Tridiagonalisation of a symmetric matrix - Martin, Reinsch and Wilkinson (1968), Tridiagonalisation of a symmetric band matrix - Schwarz (1968), and Balancing the matrix - Parlett and Reinsch (1969). These types of methods are usually quite accurate and from the vast number of methods available, we consider the LR algorithm of Rutishauser (1958) and more particularly, the QR algorithm of Francis (1961).

Since the establishment of the LR method in 1958, much has been written in the literature on improvements and shortcuts for special types of matrices. E.g. Triangularisations by LR and QR methods for real and complex matrices - Peters and Wilkinson (1970), Symmetric matrices - Martin and Wilkinson (1968), and Bowdler, Martin, Reinsch and Wilkinson (1968), Symmetric, tridiagonal matrices - Dubrulle (1970), and Reinsch and Bauer (1968). Specialisations of the general QR algorithm may be found in articles by Ruhe (1966) and Lebaud (1970), who discuss the double QR algorithm with shifts.

Wilkinson outlines in detail the basic LR and QR algorithms in "The Algebraic Eigen-value Problem" and discusses convergence in an article in 1965. Beresford Parlett in his article "Development and Use of Methods of the LR Type" (1964) gives a clear and concise approach to the QR algorithm for real matrices, with a double shift incorporated in his method.
This double shift saves the introduction of a complex matrix or the doubling in size of the original matrix, when it is real.

The results of Parlett are used as a basis for Chapter 2 which gives an extension of his method for complex matrices. Use of the double QR algorithm on a complex matrix slightly increases the rate at which two complex eigen-values may be found as compared to the single algorithm.
Section 1.4 - Outline of Thesis

Briefly, the order in which the previously mentioned ideas are presented is as follows:

Chapter 2 gives details of the elaboration of Parlett's method for the double QR algorithm to complex matrices;

Chapter 3 gives an outline of the approach to finding the eigen-values of

$$\phi^{iv} + P(y,\lambda)\phi'' + Q(y,\lambda)\phi = 0$$

using rational finite differences. P and Q are complex functions of a variable y and the eigen-value \(\lambda\) of the equation;

Chapter 4 develops a Chebyshev solution to the fourth order differential equation

$$c_0\phi^{iv} + (c_1 + c_2y + c_3y^2)\phi''' + (c_4 + c_5y + c_6y^2)\phi'' + (c_7 + c_8y + c_9y^2)\phi' + (c_{10} + c_{11}y + c_{12}y^2)\phi = 0$$

where \(c_i\) are linear, complex functions of the eigen-value;

Chapter 5 compares results of methods of Chapters 3 and 4 and the Appendix, after using the method of Chapter 2 for final determination of the eigen-values of the discrete, algebraic problem.
It is shown, even after extrapolation of the finite difference results, that the Chebyshev results are usually more accurate, and hence this is a superior method of solution. Some eigenfunctions are also displayed. The Chapter ends with a discussion of the results;

Chapter 6 is an appendix containing analytic solutions to the two test problems

\[ \phi^{iv} - \lambda \phi = 0 \]

and

\[ \phi^{iv} + \lambda \phi'' = 0 \]

which are used as a basis for a comparison of methods of Chapters 3 and 4. Also the Appendix gives a brief account of the methods of (a) Thomas (1953), who describes Plane Poiseuille Flow using the equation

\[
\begin{align*}
\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi + i\alpha R (1 - \lambda - y^2)(\phi'' - \alpha^2\phi) \\
+ 2\phi
\end{align*}
\]

\[ = 0, \]

(b) Gallagher and Mercer (1962), who describe Plane Couette Flow using the equation

\[
\left[ \frac{d^2}{dy^2} - \alpha^2 - i\alpha R (y - \lambda) \right] \left[ \frac{d^2}{dy^2} - \alpha^2 \right] \phi = 0,
\]
(c) Clenshaw and Elliott (1960), who describe the plane jet using

\[(w - \lambda)(\phi'' - \alpha^2 \phi) - \frac{d^2 w}{dy^2} = \frac{-i}{a_R} (\phi^{iv} - 2\alpha^2 \phi'' + \alpha^4 \phi)\]

where \( w = \text{sech}^2 y \), and

(d) Lee and Reynolds (1967), who, themselves, use the equations of (a), (b) and (c).

In all cases, \( \lambda \) is the eigen-value of the problem, and the boundary values are (adjusted if necessary)

\[\phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0 .\]
The QR algorithm was first presented by Francis in 1961. Given a real matrix $A \equiv A_1$, the algorithm constructs a sequence of matrices $\{A_k\}$, converging to upper triangular form if the eigen-values, $\lambda_i$, $i = 1, \ldots, n$ are real, or to block upper triangular form if complex (conjugate) eigen-values occur.

In the first step, the matrix $A_1$ is factorised as:

$$A_1 = Q_1 R_1 \quad (2.1)$$

where $Q_1$ is an orthogonal matrix and $R_1$ is an upper triangular matrix. This factorisation is always possible, and if $A_1$ is non-singular, it is unique when the sign of each diagonal element of $R_1$ is prescribed, e.g. if each element is chosen to be non-negative.

From (2.1),

$$Q_1^{-1} A_1 Q_1 = R_1 Q_1.$$

We then write

$$A_2 = R_1 Q_1 \quad (2.2)$$
and proceed in general for \( k = 1, 2, \ldots \) to factorize

\[
A_k = Q_k R_k
\]  

(2.3)

and to form

\[
A_{k+1} = R_k Q_k
\]  

(2.4)

All the matrices in the sequence \( \{A_k\} \) are similar to \( A_1 \), since

\[
A_{k+1} = Q_k^{-1} A_k Q_k
\]

\[
= Q_k^{-1} Q_{k-1}^{-1} A_{k-1} Q_{k-1} Q_k
\]

\[
= \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 
\]

\[
= Q_k^{-1} \ldots Q_1^{-1} A_1 Q_1 \ldots Q_k
\]

\[
= E_k^{-1} A_1 E_k
\]  

(2.5)

defining \( E_k \) which is orthogonal since the product of orthogonal matrices is orthogonal.

Writing

\[
H_k = R_k \ldots R_1
\]

where \( H_k \) is upper triangular, we may rewrite (2.5) as

\[
E_{k-1} A_k = A_1 E_{k-1}
\]
Then
\[ E_k H_k = E_{k-1} \Lambda_k^H H_{k-1} \]
\[ = \Lambda_1^2 E_{k-1}^H H_{k-1} \]
\[ = \Lambda_1^2 E_{k-2}^H H_{k-2} \]
\[ = \ldots \]
\[ = \Lambda_1^k \] (2.6)
since
\[ E_1 H_1 = Q_1 R_1 \]
\[ = A_1. \]

Thus \( E_k H_k \) is a QR decomposition of \( A_1^k \). Consideration of this decomposition leads to proofs of convergence of the QR algorithm which may be found in articles by Francis (1961) and Parlett (1965 and 1967). It is shown that the QR algorithm always converges. In the case of distinct eigen-values of equal modulus, convergence may be slow.

When \( A_1 \) is a singular matrix, then \( A_2 \) is not uniquely determined. However, if the rank of \( A_1 \) is \( r(<n) \), and the first \( r \) columns of \( A_1 \) are linearly independent, then the last \( n - r \) columns of \( A_2 \) will be null, and the leading \( r \times r \) principal sub-matrix will be uniquely defined, and the algorithm can then proceed on the non-singular submatrix.

Methods of accelerating convergence are always available, the main one being to shift the origin. In the convergence proof, it is shown that
A. + O(r^k)

\[ a_{ii} = \lambda_i + O(r_i^k) \]
\[ a_{i+1,i} = O(r_i^k) \]

where

\[ r_i = \max \left\{ \frac{\lambda_i}{\lambda_i - 1}, \frac{\lambda_{i+1}}{\lambda_{i+1}} \right\}, \]

are the convergence ratios, and

\[ \lambda_0 = \infty \]
\[ \lambda_{n+1} = 0. \]

These ratios give unsatisfactory linear convergence, unless the \( r_i \) are very small. If we apply the algorithm to the matrix

\[ A_k - sI, \]

where \( s \) is the shift, then the convergence ratio \( r_n \) for this matrix is given by

\[ r_n = \frac{|\lambda_n - s|}{|\lambda_{n-1} - s|} \]

and its \( (n,n-1) \) element would converge linearly to zero. There are many strategies available for the choice of \( s \), one being a direct estimate \( s \) of \( \lambda_n \), which would tend to send the element \( (n,n) \) of the matrix \( A_k - sI \) more quickly to zero, than the corresponding element of \( A_k \) to zero.

The QR algorithm may now be extended to include shifts of origin at each stage. Let these shifts be given by the sequence \( \{s_k\} \). We factorise
and write

\[ A_{k+1} = R_k Q_k + s_k I \]

for \( k = 1, 2, 3, \ldots \).

It may easily be verified that

\[ A_{k+1} = E_k^{-1} A_k E_k \]

and

\[ E_k H_k = \prod_{i=1}^{k} (A_i - s_i I) \]

where \( E_k \) and \( H_k \) are defined as before. If we define

\[ \varphi_k(\lambda) = \prod_{i=1}^{k} (\lambda - s_i) \]

then it may again be shown that the algorithm converges, except that the factor \( \lambda^k \) is replaced by \( \varphi_k(\lambda) \).

Now, for a real matrix \( A_1 \), it is desirable for all our work to remain real, while we still use the property \( s_k \rightarrow \lambda_n \). However, if an eigen-value \( \lambda_n \) is complex \((= \bar{\lambda}_{n-1})\), then we appear to be in trouble.

This problem can be overcome in the following manner. If we set out the algorithm in the usual way, we have
Now

\[ A_1 - s_1 I = Q_1 R_1 \]
\[ A_2 = R_1 Q_1 + s_1 I \]  \hspace{1cm} (2.10)
\[ A_2 - s_2 I = Q_2 R_2 \]
\[ A_3 = R_2 Q_2 + s_2 I \].

Now

\[ A_3 = Q_2^{-1} Q_1^{-1} A_1 Q_1 Q_2 \]
\[ = E_2^{-1} A_1 E_2 \]. \hspace{1cm} (2.11)

Furthermore

\[ E_2 H_2 = Q_1 Q_2 R_2 R_1 \]
\[ = Q_1 (A_2 - s_2 I) R_1 \]
\[ = Q_1 R_1 Q_1 R_1 + s_1 Q_1 R_1 - s_2 Q_1 R_1 \]
\[ \text{from (2.10)} \]
\[ = A_1 Q_1 R_1 - s_2 Q_1 R_1 \]
\[ = (A_1 - s_2 I)(A_1 - s_1 I) \]
\[ E_2 H_2 = \phi_2(A_1) \]. \hspace{1cm} (2.12)
If \( \phi_2(A_1) \) is real, then so will be \( E_2 \) and \( H_2 \). Hence, (2.11) shows \( \phi_3 \) is real. Then to make \( \phi_2(A_1) \) real, we can choose \( s_2 = \bar{s}_1 \), when \( s_1 \) is complex. Although \( A_2 \) is complex (in (2.10)), we avoid calculating it, and find \( \phi_2(A_1) \) instead, thus keeping all calculations real. We proceed to find the QR factors of \( \phi_2(A_1) \), which, by uniqueness, will in fact be \( E_2 \) and \( H_2 \), which then yield directly \( \phi_3 \), and we have bypassed \( A_2 \).

This process - calculating \( \{A_k: k \text{ odd}\} \), using the set of complex shifts \( \{s_k\} \) in the extended QR algorithm, is called "the double QR algorithm". We must at every second step calculate \( \phi_2(A_{2k-1}) \) in lieu of \( A_{2k-1} \).

It should be pointed out again that the double QR algorithm saves the introduction of a complex matrix \( A_2 \) of the form \( R + iS \), or the introduction of a \( 2n \times 2n \) real matrix

\[
\begin{bmatrix}
R & -S \\
S & R
\end{bmatrix}
\]

which gives the real and imaginary parts of the eigen-values of \( A_2 \) as its real eigen-values.
Section 2.2 - Hessenberg Matrices

A matrix $A_1$ is said to be in (upper) Hessenberg form if all the elements below the first sub-diagonal are zero

i.e. $a_{ij} = 0$, for $i > j + 1$

In the QR factorisation of $A_1$, the $j^{\text{th}}$ column $q_j$ of $Q_1$, is a linear combination of the first $j$ columns of $A_1$, and is of the same form as the $j^{\text{th}}$ column of $A_1$ when $A_1$ is in Hessenberg form, (since $a_{j+1,j} = 0$, for $j \geq 2$). Thus $Q_1$ must also be Hessenberg.

Now $A_2 = R_1 Q_1$

Then by virtue of the fact that $R_1$ is upper triangular in shape, the $j^{\text{th}}$ row of $A_2$ is a linear combination of rows $j$ to $n$ of $Q_1$.

Thus $A_2$ will also be Hessenberg in form, and the QR algorithm preserves Hessenberg form.

Moreover, any matrix may be reduced to Hessenberg form by similarity transformations, and several satisfactory methods are given by Wilkinson (1959) and (1969).

In future, we will assume $A_1$ is a Hessenberg matrix.

An important property of the QR algorithm for Hessenberg matrices is that we do not need to calculate all of the matrix $\phi_2(A_{2k-1})$. 

Parlett (1967) explains that the first column of \( \phi_2(A_1) \) (\( q_1 \), say) has at most three non-zero elements, which occur in the leading positions, so that we can write

\[
q_1 = \begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
o \\
\vdots \\
o
\end{bmatrix}
\]

and, with the matrix \( A_1 \) in Hessenberg form, we do not need to calculate any further columns of \( \phi_2(A_1) \).

Francis calculates \( A_3 \) in the sequence \( \{ A_1, A_3, A_5, \ldots \} \) by the following method. We firstly find an orthogonal matrix \( P_1 \) whose first column is \( q_1 \). If \( P_1^T \) is the transpose of \( P_1 \), we write

\[
B_1 = P_1^T A_1 P_1.
\]

It is further possible to find an orthogonal matrix \( P_2 \) whose first column is \( e_1 \), whereby the first column of

\[
B_2 = P_2^T B_1 P_2
\]

is in Hessenberg form. Columns 2 and 3 are still to be restored to Hessenberg form. Choose an orthogonal matrix \( P_3 \), where column 1 is again \( e_1 \), whence
and columns 1 and 2 are now in Hessenberg form. Continuing this process, we produce a final Hessenberg matrix $B_{n-1}$ such that

$$B_{n-1} = P_{n-1}^T \cdots P_1^T A_1 P_1 \cdots P_{n-1}$$

Note that column 1 of $P_1 \cdots P_{n-1}$ is $q_1$, and by a result in Parlett,

$$B_{n-1} = A_3.$$ 

Moreover, $P_1 \cdots P_{n-1}$ is the unique C factor of $\phi_2(A_1)$.

The choice for $P_i$ is

$$P_i = I - 2 \omega_i \omega_i^T$$

where $\omega_i$ is a unit vector. $P_i$ is then both orthogonal and symmetric. This is further discussed in section 2.3.
Section 2.3 - The QR Algorithm for a Complex Matrix

Section 2.3.1 - The Determination of $P_1$.

By virtue of the fact that by using a double shift for a real matrix, only $A_1$, and the first column of $\phi_2(A_1)$, and $A_3$ need to be calculated in the first two steps (first double step) of the QR algorithm for $A_1$ in Hessenberg form, it is anticipated that if $A_1$ is complex, then by using the same procedure, we again eliminate the calculation of $A_2$. For real matrices, the double shift was introduced to save computations using complex matrices, (or to save increasing size), but even though $A_1$ is complex, we may use this procedure to save the calculation of every second step (virtually), providing $A_1$ is put into Hessenberg form.

We now proceed to outline the modifications of the QR algorithm with double shift if $A_1$ is complex.

There is always a variety of shifts available to accelerate convergence. However, we use a simple shift in all cases, as this is not the problem involved. By experiment, a simple shift suitable for most practical purposes is found to be the complex number $s$, where $s$ is one of the eigen-values of the last $2 \times 2$ principal sub-matrix, and is given by

$$s = \frac{1}{2} (a_{n-1,n-1} + a_{n,n}) + \sqrt{d}$$

where

$$d = \frac{1}{2} (a_{n-1,n-1} - a_{n,n})^2 + a_{n-1,n} a_{n,n-1}.$$
and $a_{ij}$ are the entries in $A_i$.

When this has been determined, all that remains is to construct the matrices $\{P_i\}$ in section 2.2. It should be pointed out that in section 2.2, all matrices dealt with were real. Now that we have non-real matrices, we will require the matrices $\{P_i\}$ to be Hermitian and unitary, instead of symmetric and orthogonal as in section 2.2. Thus for any matrix $P_i$, we require

\[
\begin{align*}
\begin{cases}
P_i^* &= P_i \\
\text{and} \quad P_i^* P_i &= I
\end{cases}
\end{align*}
\]

(2.12)

where

\[
P_i^* = \overline{P_i}^T
\]

Let

\[
P_i = I - \alpha v v^*
\]

(2.13)

Then if $P_i$ is Hermitian, we have, by inspection, that

\[
\alpha^* = \alpha
\]

so that $\alpha$ is real.

Because $P_i$ is to be unitary,

\[
(I - \alpha v v^*) (I - \alpha v v^*) = I
\]

giving

\[
\alpha (\alpha v^* v - 2) = 0
\]
Excluding the trivial case \( a = 0 \), for which \( P_1 = I \), we see that \( P_1 \) will be both Hermitian and unitary if

\[
\alpha = \frac{2}{v^* v} \quad (2.14)
\]

Let us consider \( P_1 \).

Write

\[
\psi = \begin{bmatrix}
1 \\
\psi_1 \\
\psi_2 \\
0 \\
\vdots \\
0
\end{bmatrix}
\quad (2.15)
\]

then from (2.14)

\[
1 + \psi_1^* \psi_1 + \psi_2^* \psi_2 = \frac{2}{\alpha} \quad (2.16)
\]

Now from (2.13) and (2.15), the first column of \( P_1 \) is given by

\[
g_1 = P_1 \psi = \begin{bmatrix}
1 - \alpha \\
- \alpha \psi_1 \\
- \alpha \psi_2 \\
0 \\
\vdots \\
0
\end{bmatrix} \quad (2.17)
\]
Let us write

\[
\begin{bmatrix}
1 - \alpha \\
- \alpha \psi_1 \\
- \alpha \psi_2 \\
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
\gamma_1/K \\
\gamma_2/K \\
\gamma_3/K \\
0 \\
\vdots \\
0
\end{bmatrix}
\] (2.18)

where the right-hand side is the first column of \( \phi_2(A_1) \) multiplied by \( 1/K \), and is proportional to the first column of \( P_1 \).

As the matrix \( P_1 \) is unitary,

\[
\left( \begin{array}{ccc}
\gamma_1/K \\
\gamma_2/K \\
\gamma_3/K \\
\end{array} \right)
\cdot
\left( \begin{array}{ccc}
\bar{\gamma}_1/K \\
\bar{\gamma}_2/K \\
\bar{\gamma}_3/K \\
\end{array} \right)^T
= 1
\]

\[
\therefore \gamma_1 \bar{\gamma}_1 + \gamma_2 \bar{\gamma}_2 + \gamma_3 \bar{\gamma}_3 = K \bar{K}
\] (2.19)

and from (2.18) we obtain

\[
\begin{align*}
- \alpha &= \pm \frac{\gamma_1}{K} - 1 \\
- \alpha \psi_1 &= \pm \frac{\gamma_2}{K} \\
- \alpha \psi_2 &= \pm \frac{\gamma_3}{K}
\end{align*}
\] (2.20)

Taking complex conjugates in (2.20), and multiplying each equation by its conjugate, then adding, we get
\[ \alpha^2 (1 + \psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2) \]

\[ = \frac{\gamma_1 \gamma_1}{KK} \pm \frac{\gamma_1}{K} \pm \frac{\gamma_1}{K} + 1 + \frac{\gamma_2 \gamma_2}{KK} + \frac{\gamma_3 \gamma_3}{KK} \]  

(2.21)

Using (2.16) and (2.19), (2.21) yields

\[ \alpha = 1 \pm \frac{1}{2} \left( \frac{\gamma_1}{K} + \frac{\bar{\gamma}_1}{K} \right) \]  

(2.22)

From the first equation in (2.20), since \( \alpha \) and 1 are real, \( \frac{\gamma_1}{K} \) must be real, and we write

\[ \frac{\gamma_1}{K} = \frac{1}{\beta} \]  

(2.23)

where \( \beta \) is real, and using (2.19) and (2.23) is given by

\[ \beta = \sqrt{1 + \frac{\gamma_2 \bar{\gamma}_2 + \gamma_3 \bar{\gamma}_3}{\gamma_1 \bar{\gamma}_1}} \]  

(2.24)

Hence, (2.22) becomes

\[ \alpha = 1 + \frac{1}{\beta} \]

where the positive sign is chosen to avoid cancellation of errors.

Again using (2.20) we may obtain expressions for \( \psi_1 \) and \( \psi_2 \).

Thus finally, to calculate the real constants \( \alpha \) and \( \beta \), and the complex constants \( K, \psi_1 \) and \( \psi_2 \), we write
$\gamma_k = \gamma_{rk} + i \gamma_{ik}$, (k = 1, 2, 3)

$K = K_r + i K_i$

$\psi_k = \psi_{rk} + i \psi_{ik}$, (k = 1, 2)

Case (i), when $\gamma_1 \neq 0$

$$\beta = \left\{ \frac{1 + \gamma_{r2}^2 + \gamma_{i2}^2 + \gamma_{r3}^2 + \gamma_{i3}^2}{\gamma_{r1}^2 + \gamma_{i1}^2} \right\}^{1/2}$$

$$K_r = \beta \gamma_{r1}$$

$$K_i = \beta \gamma_{i1}$$

$$\psi_1 = \frac{\gamma_2}{\gamma_1} \cdot \frac{1}{1 + \beta}$$

$$\psi_2 = \frac{\gamma_3}{\gamma_1} \cdot \frac{1}{1 + \beta}$$

and $\alpha = \frac{1}{\beta} + 1$

Case (ii), when $\gamma_1 = 0$

Using equation (2.20)

$$\alpha = 1$$

$$\psi_1 = \gamma_2/K$$

$$\psi_3 = \gamma_3/K$$
K no longer satisfies (2.23), but satisfies only (2.19). There is no need for $\beta$ in this case, and $K$ may be chosen to be real, whence

$$K = K_r = \left( \gamma_{r_2}^2 + \gamma_{i_2}^2 + \gamma_{r_3}^2 + \gamma_{i_3}^2 \right)^{1/2},$$

$$K_1 = 0.$$

**Case (iii), when $K = 0$**

In the special case of $K = 0$, we may set

$$\alpha = 2$$

$$\psi_1 = 0$$

$$\psi_2 = 0$$

However, if the method is used only on matrices with non-zero sub-diagonal elements, then this case will not arise.
Section 2.3.2 - The Evaluation of $A_3$.

We have at this stage evaluated the first column of $P_1$. According to section 2.2, we must find the rest of $P_1$ so as to form $B_1$ from the equation

$$B_1 = P_1^* A_1 P_1$$

so that we may proceed to calculate the sequences $\{P_k\}$ and $\{B_k\}$ to form $A_3$. However, all that is needed at each step is the $i$th column of $P_i$, and we may alter the $B_i$ in each successive step, by overwriting. We proceed to show how this can be done.

The method follows the ideas used for real matrices, but in our present case, where $A_1$ is complex, we modify the results of Parlett to ensure $P_1$ is Hermitian and unitary.

Firstly, consider $P_2$, whose function is to make the first column of

$$B_2 = P_2^* B_1 P_2$$

(2.26)

into Hessenberg form.

The only requirement on $P_2$ is that the first column is $e_1$, so that $P_1 P_2 e_1$ will still be $q_1$. We need the first column of $B_2$ to be in Hessenberg form, so that
from (2.26), which defines \( \mathbf{b}_1 \).

Now \( \mathbf{b}_1 \), the first column of \( \mathbf{B}_1 \), has four non-zero elements, so we may write

\[
\begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

We already have \( \mathbf{p}_1 \) such that

\[
\mathbf{p}_1 \mathbf{e}_1 = \mathbf{q}_1
\]

from (2.17), and since \( \mathbf{p}_1 \) is Hermitian and unitary,

\[
\mathbf{e}_1 = \mathbf{p}_1 \mathbf{q}_1
\]  

(2.28)
Note that $P_2$ will also be Hermitian. Hence $P_2$ must satisfy

$$B_2 e_1 = P_2 b_1$$

from (2.27).

If we ignore the first row on both sides of (2.29), we can see that in its remaining $n-1$ rows, the column vector $B_2 e_1$ is a multiple of $e_1$ (in $n-1$ dimensions), so that (2.29) has the same form as (2.28). We may therefore use the same procedure for $P_2$ as for $P_1$, by simply shifting everything down one row, and by now having $\gamma_1, \gamma_2, \text{ and } \gamma_3$ defined as in $b_1$.

We may write

$$P_2 = I - \alpha vv^*$$

and set

$$v = \begin{bmatrix} 0 \\ 1 \\ \psi_1 \\ \psi_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
for the procedure. In general, all the $P^i$ may be derived by the above algorithm. For $P^i$, $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the last three non-identically zero elements of the $(i-1)^{th}$ column of $B_{i-1}$, except for $i = 1$ and $n-1$.

The last step

$$A_3 = B_{n-1} = P^*_{n-1} B_{n-2} P_{n-1}$$

has the possibility $\gamma_1 = \gamma_2 = 0$, and $\gamma_3 \equiv 0$ always. We must invoke case (iii) of section 2.3.1. That is, $K = 0$, so put $\psi_1 = \psi_2 = 0$, and $\alpha = 2$. Remember $B_0 = A_1$.

Whence the general step of the algorithm

$$B_j = P^*_j B_{j-1} P_j$$

for $j = 1, \ldots, n-1$, using only $K, \psi_1, \psi_2$ and $\alpha$ at each stage, may be displayed by writing

$$v_j = e_j + \psi_1 e_{j+1} + \psi_2 e_{j+2}$$

and computing

$$C_{j-1} = [I - v_j \alpha v^*_j] B_{j-1}$$

and then

$$B_j = C_{j-1} [I - v_j \alpha v^*_j]$$
by using the simple steps

\[ \eta^T = \alpha v_j^* b_{j-1} \]

\[ c_{j-1} = b_{j-1} - v_j \eta^T \]

\[ \xi = \alpha c_{j-1} v_j \]

\[ b_j = c_{j-1} - \xi v_j^* \]

This procedure takes us from \( A_1 \) to \( A_3 \) by avoiding \( A_2 \), and calculates only the first column of \( \phi_2(A_1) \) on the way, in place of the whole of \( A_2 \). The steps may then be repeated to take us from \( A_3 \) to \( A_5 \), and in general from \( A_{2k-1} \) to \( A_{2k+1} \) for \( k = 1, 2, 3, \ldots \), until the algorithm has converged.
Section 3.1 - Introduction

Let us consider the general fourth order differential equation

\[ \phi'''' + P(y)\phi'' + Q(y)\phi = 0 \quad (3.1) \]

with boundary conditions

\[ \phi(1) = \phi(-1) = \phi'(1) = \phi'(-1) = 0 \quad (3.2) \]

Rewriting (3.1), using operator D, i.e.

\[ \frac{d}{dy} \equiv D \]

we obtain

\[ \{D^4 + P(y)D^2 + Q(y)\}\phi = 0 \quad (3.3) \]

Introducing the central difference operator (cf. Hildebrand - "Introduction to Numerical Analysis"), we have

\[ (hD)^4 = \delta^4 - \frac{1}{6}\delta^6 + \frac{7}{240}\delta^8 - \frac{41}{7560}\delta^{10} + \ldots. \quad (3.4) \]

and

\[ (hD)^2 = \delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 + \ldots. \quad (3.5) \]

Osborne (1967) rewrites these two equations, using rational finite difference operators

\[ (hD)^4 = \frac{\delta^4}{1 + \frac{1}{6}\delta^2 - \frac{1}{720}\delta^4} - \frac{1}{3024}\delta^{10} + \ldots. \quad (3.6) \]
and
\[(hD)^2 = \frac{\delta^2 + \frac{1}{12}\delta^4}{1 + \frac{1}{6}\delta^2 - \frac{1}{720}\delta^6} + 0(\delta^6) \quad (3.7)\]

We treat the denominator as a post-multiplier, and proceed to make a change of variable

\[\phi = (1 + \frac{1}{6}\delta^2 - \frac{1}{720}\delta^4)\psi \quad (3.8)\]

(3.6), (3.7) and (3.8) now give

\[h^4D^4\phi = \delta^4\psi\]
\[h^2D^2\phi = (\delta^2 + \frac{1}{12}\delta^4)\psi\]
\[\phi = (1 + \frac{1}{6}\delta^2 - \frac{1}{720}\delta^4)\psi\]

(after neglecting terms of order higher than \(\delta^4\)), and these three equations may be written in matrix form

\[
\begin{bmatrix}
h^4D^4\phi \\
h^2D^2\phi \\
\phi
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{12} & 1 & 0 \\
-\frac{1}{720} & \frac{1}{6} & 1
\end{bmatrix}
\begin{bmatrix}
\delta^4\psi \\
\delta^2\psi \\
\psi
\end{bmatrix} \quad (3.9)
\]

Multiplying (3.3) by \(h^4\), we get

\[(h^4D^h\phi) + h^2P(h^2D^2\phi) + h^4Q\phi = 0\]

or

\[
\begin{bmatrix}
1 & h^2P & h^4Q
\end{bmatrix}
\begin{bmatrix}
h^4D^\phi \\
h^2D^2\phi \\
\phi
\end{bmatrix} = 0 \quad (3.10)
\]
Again referring to Hildebrand, we have

\[
\begin{align*}
\delta^4 \psi_r &= \psi_{r-2} - 4\psi_{r-1} + 6\psi_r - 4\psi_{r+1} + \psi_{r+2} \\
\delta^2 \psi_r &= \psi_{r-1} - 2\psi_r + \psi_{r+1} \\
\psi_r &= \psi_r
\end{align*}
\]

which, when rewritten are

\[
\begin{bmatrix}
\delta^4 \psi_r \\
\delta^2 \psi_r \\
\psi_r
\end{bmatrix} = 
\begin{bmatrix}
1 & -4 & 6 & -4 & 1 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_{r-2} \\
\psi_{r-1} \\
\psi_r \\
\psi_{r+1} \\
\psi_{r+2}
\end{bmatrix}
\]

(3.11)

Combining (3.9), (3.10) and (3.11), we get

\[
\begin{bmatrix}
1 & h^2 P & h^4 Q
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{12} & 1 & 0 \\
-\frac{1}{720} & \frac{1}{6} & 1
\end{bmatrix}
\begin{bmatrix}
1 & -4 & 6 & -4 & 1 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_{r-2} \\
\psi_{r-1} \\
\psi_r \\
\psi_{r+1} \\
\psi_{r+2}
\end{bmatrix} = 0
\]

(3.12)
Section 3.2 - Application of Boundary Conditions

Let the interval (-1, 1) be divided into \( n+1 \) equal strips, where

\[
\begin{align*}
\phi_0 &= \phi(-1) \\
\phi_{n+1} &= \phi(1) \\
(n+1)h &= 2
\end{align*}
\]  

(3.13)  
(3.14)

If we use (3.8), and neglect \( \delta^h \psi \), we obtain

\[
\phi_0 = (1 + \frac{1}{6}\delta^2)\psi_0
\]  

(3.15)

and since

\[
\phi(-1) = \phi_0 = 0
\]

then

\[
0 = \psi_0 + \frac{1}{6}(\psi_{-1} - 2\psi_0 + \psi_1)
\]

which gives

\[
\psi_{-1} + 4\psi_0 + \psi_1 = 0
\]  

(3.16)

Similarly, since

\[
\phi(1) = 0,
\]

using (3.14), we get

\[
\psi_n + 4\psi_{n+1} + \psi_{n+2} = 0
\]  

(3.17)

To use the other two boundary conditions involving the first derivative, introducing the averaging operator \( \mu \), we have

\[
hD\phi = \frac{\mu\delta}{1 + \frac{1}{6}\delta^2} \phi + O(\delta^5)
\]
which gives

\[ hD\phi_0 = \frac{\mu \delta}{1 + \frac{1}{6} \delta^2} \phi_0 \]

and using (3.15), we obtain

\[ hD\phi_0 = \mu \delta \psi_0 \]

Now

\[ \phi'(-1) = 0 \]

implies

\[ D\phi_0 = 0 \]

hence

\[ \mu \delta \psi_0 = 0 \]

\[ \therefore \psi_{-1} - \psi_1 = 0 \]

i.e.

\[ \psi_{-1} = \psi_1 \quad (3.18) \]

Similarly, since

\[ \phi'(1) = D\phi_{n+1} = 0 \]

then

\[ \psi_n = \psi_{n+2} \quad (3.19) \]

Substituting (3.18) into (3.16), we get

\[ \psi_0 = -\frac{1}{2} \psi_1 \quad (3.20) \]

and similarly

\[ \psi_{n+1} = -\frac{1}{2} \psi_n \quad (3.21) \]
We then write (3.12) in the form

\[
(1 + \frac{h^2p}{12} - \frac{h^4Q}{720})\psi_{r-2} + (-4 + \frac{2h^2p}{3} + \frac{31h^4Q}{180})\psi_{r-1} + (6 - \frac{3h^2p}{2} + \frac{79h^4Q}{120})\psi_r
+ (-4 + \frac{2h^2p}{3} + \frac{31h^4Q}{180})\psi_{r+1} + (1 + \frac{h^2p}{12} - \frac{h^4Q}{720})\psi_{r+2}
\]

\[
= 0 \quad (3.22)
\]

where \( r = 1, 2, \ldots, n \).

Equations (3.18), (3.19), (3.20) and (3.21) give us the means to eliminate \( \psi_{-1}, \psi_0, \psi_{n+1} \) and \( \psi_{n+2} \) in the special cases of (3.22) where \( r = 1, 2, n-1 \) and \( n \). From this, we may set up the matrix equation

\[
(A - \lambda B)\psi = 0
\]

where

\[
\psi^T = [\psi_1, \ldots, \psi_n]
\]

and solve directly for the eigen-values, by pre-multiplying by \( B^{-1} \) and then using the QR algorithm as in Chapter 2 on the matrix \( B^{-1}A \). In general, in physical situations, \( B \) will be non-singular.

In applications of (3.22), we must keep in mind that \( P \) and \( Q \) are functions of \( y \), and hence are functions of \( r \), so they must be calculated separately for each value of \( r \).
Section 3.3 - Modifications to the Boundary Conditions

In practical situations, it is found that by including the terms of order $\delta^4$ in (3.8) when applying the boundary conditions, the accuracy of results obtained are usually better in the 4th significant figure. So the following results are derived and used as a means of comparison of finite difference methods against Chebyshev methods.

We get from the boundary conditions that

$$\left\{ 1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4 \right\} \psi_0 = 0$$

$$\mu \delta \psi_0 = 0$$

$$\left\{ 1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4 \right\} \psi_{n+1} = 0$$

$$\mu \delta \psi_{n+1} = 0$$

which imply

$$-\psi_{-2} + 124 \psi_{-1} + 474 \psi_0 + 124 \psi_1 - \psi_2 = 0$$

$$\psi_{-1} = \psi_1$$

$$-\psi_{n-1} + 124 \psi_n + 474 \psi_{n+1} + 124 \psi_{n+2} - \psi_{n+3} = 0$$

$$\psi_{n+2} = \psi_n$$

The condition that the derivatives are zero is extended to

$$\psi_{-2} = \psi_2$$

and

$$\psi_{n+3} = \psi_{n-1}$$
which only imposes a particular form of the function outside the range of interest. Thus

\[
\begin{align*}
\psi_{-1} &= \psi_1 \\
\psi_0 &= -\frac{124}{237}\psi_1 + \frac{1}{237}\psi_2 \\
\psi_{n+2} &= \psi_n \\
\psi_{n+1} &= -\frac{124}{237}\psi_n + \frac{1}{237}\psi_{n-1}
\end{align*}
\]

(3.23)

and these boundary conditions are then applied to (3.22), in the cases \( r = 1, 2, n-1, \) and \( n. \)
Section 3.4 - Even and Odd Solutions

In certain problems considered later, it is found that the differential equations possess solutions separable into even and odd parts, which are themselves solutions. So to further increase accuracy of solutions for these cases, the following modifications are introduced.

It should be noted, that for the even (or odd) solution, we need consider only the interval (0,1), and when we divide this interval into strips, then for a given size matrix, we have effectively doubled the number of strips in the usual interval (-1,1).

In practice, it is convenient to divide the interval (0,1) into \( n \) equal strips, rather than \( n+1 \) strips, in order to avoid certain notational problems associated with the matrix representation. Thus we put

\[
\begin{align*}
\phi_0 &= \phi(0) \\
\phi_n &= \phi(1) \\
\text{nh} &= 1
\end{align*}
\]

If \( \phi \) is an even function

\[
\phi_{-r} = \phi_r \quad , \quad r = 1, 2, \ldots
\]

and if \( \phi \) is an odd function, then

\[
\phi_{-r} = -\phi_r \quad , \quad r = 1, 2, \ldots
\]
The boundary conditions reduce to

\[ \phi(1) = \phi'(1) = 0 \]

i.e.

\[ \phi_n = \phi'_n = 0 \]

Using the ideas described in the preceding sections, including
the change of variable (3.8), together with modified results from (3.23)
we find that the matrix representation of the problem can be found using
(3.22), with \( r = 0, 1, \ldots, n-1 \), and end conditions

\[
\begin{align*}
\psi_{-2} &= \pm \psi_2 \\
\psi_{-1} &= \pm \psi_1 \\
\psi_{n+1} &= \psi_{n-1} \\
\psi_n &= -\frac{124}{237}\psi_{n-1} + \frac{1}{237}\psi_{n-2}
\end{align*}
\]

The matrix representation is then

\[(A - \lambda B)\psi = 0\]

where

\[\psi^T = [\psi_0, \psi_1, \ldots, \psi_{n-1}].\]

Results using this method are given in Chapter 5, and
are compared with those obtained with the method of Chapter 4.
Section 4.1 - Chebyshev Polynomials

Section 4.1.1 - Some General Properties

The Chebyshev Polynomials are defined by

\[ T_k(x) = \cos(k \cos^{-1} x), \ k = 0, 1, 2, \ldots. \quad (4.1) \]

This implies the first few polynomials are

\[ T_0(x) = 1 \]
\[ T_1(x) = x \]
\[ T_2(x) = 2x^2 - 1, \text{ etc.} \]

We note that \( T_k(x) \) is an even function for \( k \) even, and an odd function for \( k \) odd.

It may be shown that

\[ u(x) = T_k(x) \]

is a solution of the differential equation

\[ (1-x^2)u'' - xu' + k^2u = 0. \]

Using (4.1),

\[ T_{-k}(x) = \cos(-k \cos^{-1} x) = \cos(k \cos^{-1} x) \]

\[ T_{-k}(x) = T_k(x). \]
Also,
\[ T_{k+m}(x) + T_{|k-m|}(x) = 2T_m(x)T_k(x) \]
giving as a special case
\[ T_{k+1}(x) + T_{|k-1|}(x) = 2xT_k(x). \]

Further,
\[ 4x^2T_k(x) = 2xT_{k+1}(x) + 2xT_{|k-1|}(x) \]
\[ = T_{|k-2|}(x) + 2T_k(x) + T_{k+2}(x) \]
and in general
\[ (2x)^p T_k(x) = \sum_{j=0}^{p} \binom{p}{j} T_{|k-p+2j|}(x) \] (4.2)

Also, using (4.1),
\[
\begin{align*}
T_k(1) &= 1 \\
T_k(-1) &= (-1)^k \\
T_{2k+1}(0) &= 0
\end{align*}
\] (4.3)

and
\[
\int T_k(x)dx = \frac{1}{2} \left[ T_{k+1}(x) - \frac{T_{k-1}(x)}{k+1} - (1-\delta_{k1}) \frac{T_{k-1}(x)}{k-1} \right] + \text{constant} \] (4.4)

We can approximate a function \( f(x) \) under certain conditions by using a series
\[
f(x) \approx f_n(x) = \sum_{k=0}^{n} (1-\frac{1}{2}\delta_{k0})a_k T_k(x)
\]
where the $a_k$ are determined by some specific conditions related to the function itself. In particular, if $f(x)$ is the solution to a differential equation with boundary conditions, then this information may determine the $a_k$.

If the sequence $\{f(x)\}$ is convergent, then we may write

$$f(x) = \sum_{k=0}^{\infty} (1 - \frac{1}{2} \delta_{k0}) a_k T_k(x) \quad (4.5)$$

and, assuming $f(x)$ is differentiable, then we write

$$f'(x) = \sum_{k=0}^{\infty} (1 - \frac{1}{2} \delta_{k0}) a'_k T_k(x) \quad (4.6)$$

By integrating (4.6) and using (4.4), we obtain the basic relationship

$$a'_k = a'_{k+2} + 2(k+1)a_{k+1} \quad , \quad k \geq 0$$

which may be extended in general for a function which is continuously differentiable, to give

$$a^{(i)}_k = a^{(i)}_{k+2} + 2(k+1)a^{(i-1)}_{k+1} \quad , \quad k \geq 0 \quad (4.7)$$

where $a^{(i)}_k$ denotes the $k^{th}$ co-efficient of the series for the $i^{th}$ derivative of $f(x)$. (See Clenshaw (1957)).

It may be shown quite simply that we can write

$$a_{-k} = a_k$$

and in general

$$a^{(i)}_{-k} = a^{(i)}_k$$

(4.8)

In the following we will often write $T_k \equiv T_k(x)$.
Section 4.1.2 - Even and Odd Functions

When a function and its derivative are written as sums of Chebyshev polynomials as in (4.5) and (4.6), we note that, using (4.3)

\[
\begin{align*}
  f(l) &= \frac{1}{2}a_0 + a_1 + a_2 + a_3 + a_4 + \ldots \quad (4.9) \\
  f(-l) &= \frac{1}{2}a_0 - a_1 + a_2 - a_3 + a_4 - \ldots \\
  f'(l) &= \frac{1}{2}a'_1 + a'_2 + a'_3 + a'_4 + \ldots \\
  f'(-l) &= \frac{1}{2}a'_0 - a'_1 + a'_2 - a'_3 + a'_4 - \ldots \\
  f(0) &= \frac{1}{2}a_0 - a_2 + a_4 - a_6 + \ldots \quad (4.10)
\end{align*}
\]

If \( f(x) \) is an even function then, because \( T_k(x) \) is an even polynomial when \( k \) is even, and is an odd polynomial when \( k \) is odd, we can write

\[
\begin{align*}
  f(x) &= \frac{1}{2}a_0 T_0 + a_2 T_2 + a_4 T_4 + \ldots \\
  f'(x) &= a'_1 T_1 + a'_3 T_3 + a'_5 T_5 + \ldots \quad (4.11)
\end{align*}
\]

noting that

\[
\begin{align*}
  f(1) &= f(-1) = \frac{1}{2}a_0 + a_2 + a_4 + \ldots \quad (4.11a) \\
  f'(1) &= -f'(-1) = a'_1 + a'_3 + a'_5 + \ldots
\end{align*}
\]
Similarly, if \( f(x) \) is an odd function, we can write

\[
\begin{align*}
  f(x) &= a_1 T_1 + a_3 T_3 + a_5 T_5 + \ldots. \\
  f'(x) &= \frac{1}{2} a_0 T_0 + a_2 T_2 + a_4 T_4 + \ldots.
\end{align*}
\]  

(4.12)

giving

\[
\begin{align*}
  f(1) &= -f(-1) = a_1 + a_3 + a_5 + \ldots. \\
  f'(1) &= f'(-1) = \frac{1}{2} a_0' + a_2' + a_4' + \ldots.
\end{align*}
\]  

(4.12a)
Section 4.2 - Solution to Differential Equations Using Chebyshev Series

We now proceed to develop a method using Chebyshev Series for finding the eigen-values of general differential equations with polynomial co-efficients in the independent variable. The method shall be used on the fourth order differential equation

\[ c_0 \phi^{iv} + (c_1 + c_2 y + c_3 y^2)\phi''' + (c_4 + c_5 y + c_6 y^2)\phi'' \]
\[ + (c_7 + c_8 y + c_9 y^2)\phi' + (c_{10} + c_{11} y + c_{12} y^2)\phi \]
\[ = 0 \quad (4.13) \]

with boundary conditions

\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0 \]

where the \( c_i \) are complex, linear functions of the eigen-value of the problem and \( \phi(y) \) is a complex valued function of the real variable \( y \).

For simplicity sake, we will describe the method as it is used to resolve the eigen-values for the special case of (4.13) in the form:

\[ \epsilon \phi^{iv} + (\alpha + \beta y^2)\phi'' + (\gamma + \delta y^2)\phi = 0 \quad (4.14) \]

with boundary conditions

\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0. \]

The general result for (4.13) is obtained similarly and is given in Section 4.6. It must be noted that \( \alpha, \beta, \gamma, \delta \) and \( \epsilon \) are complex, linear functions of the eigen-value of the equation.
We assume the solution to (4.14) can be expressed as a Chebyshev series

\[ \phi(y) = \sum_{k=0}^{\infty} a_k T_k(y) + a_{1,1} T_1(y) + a_{2,2} T_2(y) + \ldots + a_{k,k} T_k(y) + \ldots \quad (4.15) \]

Further, we write

\[ \phi''(y) = \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a''_{k,k} T_k(y) \]

and

\[ \phi^{iv}(y) = \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a^{iv}_{k,k} T_k(y) \]

Substitute into (4.14),

\[ \varepsilon \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a^{iv}_{k,k} T_k(y) + (\alpha + \beta y^2) \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a''_{k,k} T_k(y) \]

\[ + (\gamma + \delta y^2) \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a_{k,k} T_k(y) = 0. \]

Using (4.2), this becomes

\[ \varepsilon \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a^{iv}_{k,k} T_k(y) + \alpha \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a''_{k,k} T_k(y) \]

\[ + \beta \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a''_{k,k} T_k(y) + \gamma \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a_{k,k} T_k(y) \]

\[ + \delta \sum_{k=0}^{\infty} (1-\frac{1}{2}\delta k \, 0) a_{k,k} T_k(y) = 0. \]
Considering co-efficients of $T_k$, we obtain

\[ \varepsilon a_k^{iv} + \alpha a_k'' + \frac{1}{4}\beta(a_k'' + 2a_k'' + a_k'+) + \gamma a_k 
+ \frac{1}{4}\delta(a_k'' + 2a_k' + a_k'+) = 0 \quad (4.16) \]

for $k > 0$, remembering that negative subscripts can be dealt with by (4.8). We now attempt to rewrite (4.16) in terms of $a_k^{(1)}$ only, and while doing this, we keep the co-efficient of $\varepsilon$ strictly to $a_k^{(1)}$, so that all other co-efficients will be symmetric about $k$. In order to achieve this, we must up-date and down-date the subscripts of (4.16), and we get respectively

\[ \varepsilon a_k^{iv} + \alpha a_k'' + \frac{1}{4}\beta(a_k'' + 2a_k'' + a_k'+) + \gamma a_k + 1 
+ \frac{1}{4}\delta(a_{k-1} + 2a_k + a_{k+1}) = 0 \quad \text{, for } k \geq -1 \quad (4.17) \]

and

\[ \varepsilon a_k^{iv} + \alpha a_k'' + \frac{1}{4}\beta(a_k'' + 2a_k'' + a_k'+) + \gamma a_k + 1 
+ \frac{1}{4}\delta(a_k' - 3 + 2a_k' + a_k'+) = 0 \quad \text{, for } k \geq 1 \quad (4.18) \]
Now, subtracting (4.17) from (4.18),
\[ \varepsilon (a_{k-1}^{iv} - a_{k+1}^{iv}) + \alpha (a_{k-1}'' - a_{k+1}'') + \frac{\beta}{\kappa} \left[ a_{k-3}'' - a_{k-1}'' \right] + 2(a_{k-1}'' - a_{k+1}'') + (a_{k+1}'' - a_{k+3}'') + \gamma (a_{k-1} - a_{k+1}) + \frac{\delta}{\kappa} [a_{k-3} + a_{k-1} - a_{k+1} - a_{k+3}] \]
\[ = 0 \] (4.19)

Using (4.7), (4.19) becomes
\[ 2\varepsilon a_k'' + 2\alpha a_k' + \frac{\beta}{\kappa} \left[ 2(k-2)a_{k-2}' + 2.2ka_k' + 2(k+2)a_{k+2}' \right] + \gamma (a_{k-1} - a_{k+1}) + \frac{\delta}{\kappa} [a_{k-3} + a_{k-1} - a_{k+1} - a_{k+3}] \]
\[ = 0 \] , for \( k \geq 1 \) (4.20)

Now,
\[ 2(k-2)a_{k-2}' + 2.2ka_k' + 2(k+2)a_{k+2}' \]
\[ = 2k(a_{k-2}' + 2a_k' + a_{k+2}') - 4(a_{k-2}' - a_{k+2}') \]
\[ = 2k(a_{k-2}' + 2a_k' + a_{k+2}') - 4[(a_{k-2}' - a_k') + (a_k' - a_{k+2}')] \]
\[ = 2k(a_{k-2}' + 2a_k' + a_{k+2}') - 4[2(k-1)a_{k-1}' + 2(k+1)a_{k+1}'] \]

after using (4.7).

Hence (4.20) becomes
\[ 2\varepsilon k a_k^m + 2\alpha k a_k'^1 + \frac{1}{2}\beta [2k(a'_k - 2 + 2a'_k + a'_k) - 4(2(k+1)a_{k+1} - 2(k+1)a_{k+2}) + 2(2k+1)\alpha a_{k+1}] + \gamma (a_{k-1} - a_{k+1}) + \frac{1}{2}\delta [a_{k-3} + a_{k-1} - a_{k+1} - a_{k+1} - a_{k+2}] = 0 \quad \text{for } k \geq 1. \]

On simplifying, we obtain after dividing throughout by 2k
\[ \varepsilon a_k^m + \alpha a_k'^1 + \frac{1}{2}\beta [a'_k - 2 + 2a'_k + a'_k] - \beta \left[ \frac{k-2}{k} a_{k-2} + \frac{k}{k+1} a_k + \frac{k+2}{k+1} a_{k+2} \right] + \gamma \left[ \frac{a_k - a_{k+1}}{2k} \right] + \delta \left[ \frac{a_{k-3} - a_{k-1} + a_{k+1} - a_{k+3}}{4k} \right] + 0 \quad (4.21) \]

Up-dating and down-dating the subscripts of (4.21) by 1, subtracting the former from the latter, and using (4.7), we obtain
\[ 2\varepsilon k a_k'' + 2\alpha k a_k' + \frac{1}{2}\beta [2(2k-2)a_{k-2} + 2.2ka_k - 2(2k+2)a_{k+2}] \]
\[ - \beta \left[ \frac{k-2}{k-1} a_{k-2} + \frac{k}{k-1} a_k - \frac{k}{k+1} a_{k+2} \right] + \gamma \left[ \frac{a_k - a_{k-2}}{2k-2} \right] + \frac{\delta}{8} \left[ \frac{a_{k-4}}{k-1} - \frac{a_{k-2} - a_{k+2}}{k+1} - a_k - \frac{1}{k-1} - \frac{1}{k+1} \right] + \frac{a_{k+2} \left( \frac{1}{k-1} + \frac{1}{k+1} \right) + \frac{a_{k+4}}{k+1}}{k} \]
\[ = 0 \]

On simplifying, and dividing throughout by 2k, we obtain
\begin{align*}
\varepsilon a''_k + \alpha a'_k + \beta \left[ \frac{a_{k-2}}{k-1} \frac{(k-3)(k-2)}{k} + a_k \frac{2(k^2-3)}{(k-1)(k+1)} + a_{k+2} \frac{(k+3)(k+2)}{k(k+1)} \right] \\
+ \gamma \left[ \frac{a_{k-2}}{k-1} - a_k \frac{2}{(k-1)(k+1)} + a_{k+2} \right] \\
+ \delta \left[ \frac{a_{k-4}}{k-1} + a_{k-2} \frac{2}{(k-1)(k+1)} - a_k \frac{2}{(k-1)(k+1)} \right] \\
+ \frac{a_{k+2}}{(k-1)(k+1)} + \frac{a_{k+4}}{k(k+1)} \\
= 0 \tag{4.22}
\end{align*}

Again, up-dating and down-dating the subscripts of (4.22) by 1, subtracting the former from the latter, using (4.7), simplifying and dividing by \(2k\), we obtain

\begin{align*}
\varepsilon a'_k + \frac{\alpha}{2} \left[ \frac{a_{k-1}}{k} - \frac{a_{k+1}}{k} \right] + \beta \left[ \frac{a_{k-3}}{k} \frac{(k-4)(k-3)}{(k-2)(k-1)k} + \frac{(k^2+3k-16)}{(k-2)k(k+1)} a_{k-1} \right] \\
+ \frac{a_{k+1}}{k} \frac{(-k^2+3k+16)}{(k-1)k(k+2)} - a_{k+3} \frac{(k+3)(k+4)}{k(k+1)(k+2)} \right] + \gamma \left[ \frac{a_{k-3}}{(k-2)(k-1)k} \right] \\
+ \frac{3a_{k-1}}{(k-2)k(k+1)} + \frac{3a_{k+1}}{(k-1)k(k+2)} - \frac{a_{k+3}}{k(k+1)(k+2)} \right] \\
+ \frac{\delta}{32} \left[ \frac{a_{k-5}}{(k-2)(k-1)k} - \frac{a_{k-3}(k-5)}{(k-2)(k-1)k(k+1)} - \frac{2a_{k-1}(k+4)}{(k-2)k(k+1)(k+2)} \right] \\
+ \frac{2a_{k+1}(k-4)}{(k-2)(k-1)k(k+2)} + \frac{a_{k+3}(k+5)}{(k-1)k(k+1)(k+2)} - \frac{a_{k+5}}{k(k+1)(k+2)} \right] \\
= 0 \text{ , for } k \geq 3 \tag{4.23}
\end{align*}
Repeating the same procedure on (4.23) as used on (4.22) we obtain

\[
\begin{align*}
&\alpha_{k-6} \left\{ \frac{\delta}{64(k-3)(k-2)(k-1)k} \right\} \\
&+ \alpha_{k-4} \left\{ \frac{\beta(k-5)(k-4)}{16(k-3)(k-2)(k-1)k} + \frac{\gamma}{16(k-3)(k-2)(k-1)k} \right\} \\
&- \frac{\delta(k-5)}{32(k-3)(k-2)(k-1)k(k+1)} \\
&+ \alpha_{k-2} \left\{ \frac{\alpha}{4(k-1)k} + \frac{\beta(5k-19)}{8(k-3)(k-1)k(k+1)} - \frac{\gamma}{4(k-3)(k-1)k(k+1)} \right\} \\
&- \frac{\delta(k+17)}{64(k-3)(k-1)k(k+1)(k+2)} \\
&+ \alpha_{k} \left\{ \frac{\alpha}{2(k-1)(k+1)} - \frac{\beta(k-22)}{8(k-2)(k-1)(k+1)(k+2)} \right\} \\
&+ \frac{3\gamma}{8(k-2)(k-1)(k+1)(k+2)} + \frac{\delta(k^{2}-19)}{32(k-3)(k-2)(k-1)(k+1)(k+2)(k+3)} \\
&+ \alpha_{k+2} \left\{ \frac{\alpha}{4k(k+1)} - \frac{\beta(5k+19)}{8(k-1)k(k+1)(k+3)} - \frac{\gamma}{4(k-1)k(k+1)(k+3)} \right\} \\
&- \frac{\delta(k-17)}{64(k-2)(k-1)k(k+1)(k+3)} \\
&+ \alpha_{k+4} \left\{ \frac{\beta(k+4)(k+5)}{16k(k+1)(k+2)(k+3)} + \frac{\gamma}{16k(k+1)(k+2)(k+3)} \right\} \\
&- \frac{\delta(k+5)}{32(k-1)k(k+1)(k+2)(k+3)} \\
&+ \alpha_{k+6} \left\{ \frac{\delta}{64k(k+1)(k+2)(k+3)} \right\} \\
&= 0 \quad , \text{for } k \geq 4 \quad (4.24)
\end{align*}
\]
A matrix $W$ is now set up to give the factors relating to the co-efficients of the series solution and the co-efficients of the differential equation. This is derived from (4.24) with a multiplication throughout by $(k-2)(k+2)$, to balance the final matrix about 1.

At this stage, we may observe the "skew-symmetry" of $k$ in $W$.

$W$ is exhibited on the next page:
<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{k-6}$</td>
<td></td>
<td></td>
<td>(k-5)(k-4)(k+2)</td>
<td>$\frac{k+2}{16(k-3)(k-1)} k$</td>
<td>$-\frac{(k-5)(k+2)}{64(k-3)(k-1)k}$</td>
</tr>
<tr>
<td>$a_{k-4}$</td>
<td></td>
<td></td>
<td>$\frac{(5k-19)(k-2)(k+2)}{8(k-3)(k-1)k(k+1)}$</td>
<td>$\frac{k+2}{16(k-3)(k-1)} k$</td>
<td>$-\frac{(k-2)(k+2)}{32(k-3)(k-1)k(k+1)}$</td>
</tr>
<tr>
<td>$a_{k-2}$</td>
<td></td>
<td></td>
<td>$\frac{-(k-2)(k+2)}{2(k-1)(k+1)}$</td>
<td>$\frac{-(k^2-22)}{8(k-1)(k+1)}$</td>
<td>$\frac{3}{8(k-1)(k+1)}$</td>
</tr>
<tr>
<td>$a_k$</td>
<td>(k-2)(k+2)</td>
<td></td>
<td>$\frac{-(k-2)(k+2)}{2(k-1)(k+1)}$</td>
<td>$\frac{-(k^2-22)}{8(k-1)(k+1)}$</td>
<td>$\frac{3}{8(k-1)(k+1)}$</td>
</tr>
<tr>
<td>$a_{k+2}$</td>
<td>(k-2)(k+2)</td>
<td></td>
<td>$\frac{-(5k+19)(k-2)(k+2)}{8(k-1)k(k+1)(k+3)}$</td>
<td>$\frac{-(k-2)(k+2)}{4(k-1)k(k+1)(k+3)}$</td>
<td>$\frac{3}{8(k-1)(k+1)}$</td>
</tr>
<tr>
<td>$a_{k+4}$</td>
<td>(k-2)(k+4)(k+5)</td>
<td></td>
<td>$\frac{k-2}{16k(k+1)(k+3)}$</td>
<td>$\frac{k-2}{16k(k+1)(k+3)}$</td>
<td>$\frac{-(k-2)(k+5)}{32(k-1)k(k+1)(k+3)}$</td>
</tr>
<tr>
<td>$a_{k+6}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\frac{k-2}{64k(k+1)(k+3)}$</td>
</tr>
</tbody>
</table>
Section 4.3 - Boundary Conditions - Even Solution

Examination of equation (4.14) shows that an even solution $\phi(y)$ exists. Firstly, we seek this solution. We can write

$$\phi(y) = \frac{1}{2}a_0T_0 + a_2T_2 + a_4T_4 + \ldots.$$

with boundary conditions

$$\phi(1) = \phi'(1) = 0.$$

If we set $k = 4, 6, 8, \ldots$ in (4.24), the first co-efficient is $a_{-2}$. Using (4.8) we have $a_{-2} = a_2$ to replace $a_{-2}$, and we can use the boundary conditions to eliminate $a_0$ and $a_2$ as follows.

Consider

$$\phi(1) = 0.$$

Then by (4.11a)

$$\frac{1}{2}a_0 = -\sum_{k=1}^{\infty} a_{2k}$$

This then eliminates $a_0$ in (4.24) for $k = 4$ and $k = 6$.

Using the second condition

$$\phi'(1) = 0$$

we use (4.12)

$$a_1' + a_3' + a_5' + \ldots = 0.$$
\[ (a_1' - a_3') + 2(a_3' - a_5') + 3(a_5' - a_7') + 4(a_7' - a_9') + \ldots. \]
\[ = 0 \]

and applying (4.7), gives

\[ 2.2a_2 + 2.4a_4 + 3.6a_6 + 4.8a_8 + \ldots. \]
\[ = 0. \]

Hence,

\[ a_2 = - \sum_{k=2}^{\infty} k^2 a_{2k} \quad (4.26) \]

and this eliminates \( a_2 \) from (4.24) for \( k = 4, 6, 8, \ldots. \)

The main assumption in the Chebyshev method is that \( \phi(y) \) is approximated well by a small number of terms in the Chebyshev polynomial expansion, with the co-efficients \( \{a_i\} \) rapidly decreasing in magnitude. This is the idea introduced by Clenshaw (1957) and described by Fox and Parker (1968).

Thus we truncate the series for \( \phi(y) \) depending on the size of the matrix being constructed, and assume that all co-efficients are zero after this point.
We can now write down the discrete algebraic matrix representation of the differential equation, and solve the matrix eigen-value problem corresponding to the continuous boundary value problem.

A very important factor to note when setting up the matrix representation for the even solution, is that for an $n \times n$ matrix, we are infact dealing with $2n + 2$ terms of the series solution, remembering all the odd terms are zero. This is a major facet in the use of this method, in as much as we double the length of the series, yet use the same size matrix.

Now remembering that $\alpha, \beta, \gamma, \delta$ and $\epsilon$ are complex, linear functions of the eigen-value $\lambda$, the problem to date is of the form

$$(A - \lambda B)\hat{a} = \hat{0}$$

where

$$\hat{a}^T = [a_4, a_6, \ldots]$$

and the matrices $A$ and $B$ are found using (4.8), (4.24), (4.25) and (4.26), with $k = 4, 6, \ldots$.

$A$ is complex and we can write

$$A = A_r + iA_i$$

and for the problems considered, $B$ is either real, or pure imaginary, and the physical conditions guarantee that $B$ is non-singular.
Thus we have an algebraic eigen-value problem in standard form

\[(D - \lambda I)\vec{a} = 0\]

where

\[D = B^{-1}(A_r + iA_i)\]  \hspace{1cm} (4.27)

and where the real matrices \(B^{-1}A_r\) and \(B^{-1}A_i\) are found in a single Gaussian elimination calculation, viz.

\[
\begin{pmatrix}
A_r & A_i & B \\
\end{pmatrix} \rightarrow \begin{pmatrix}
B^{-1}A_r & B^{-1}A_i & I \\
\end{pmatrix}
\]

We may now reduce \(D\) to Hessenberg form, and then apply the QR algorithm as in Chapter 2.
Section 4.4 - Boundary Conditions - Odd Solution

We now note that (4.14) admits an odd solution \( \phi(y) \). We can write

\[
\phi(y) = a_1 T_1 + a_3 T_3 + a_5 T_5 + \ldots.
\]

\[
\phi'(y) = \frac{1}{2} a_0' T_0 + a_2' T_2 + a_4' T_4 + \ldots.
\]

with boundary conditions

\[
\phi(1) = \phi'(1) = 0.
\]

The matrix representation of the problem is obtained using (4.24) with \( k = 3, 5, 7, \ldots \).

Use of (4.8) eliminates negative subscripts and the boundary conditions are used to eliminate \( a_1 \) and \( a_3 \) as follows.

Since \( \phi'(1) = 0 \), use of (4.12a) gives

\[
\frac{1}{2} a_0' + a_2' + a_4' + a_6' + \ldots = 0
\]

\[
\therefore \frac{1}{2}.1.(a_0' - a_2') + \frac{1}{2}.3.(a_2' - a_4') + \frac{1}{2}.5.(a_4' - a_6') + \ldots = 0
\]

and applying (4.7)

\[
1^2 a_1 + 2^2 a_3 + 3^2 a_5 + \ldots = 0
\]
Since \( \phi(1) = 0 \), then, also from (4.12a),

\[
a_1 + a_3 + a_5 + \ldots = 0
\]

and from these last two equations

\[
a_1 = \frac{1}{2} \sum_{k=2}^{\infty} (k+2)(k-1)a_{2k+1} \quad (4.28)
\]

\[
a_3 = -\frac{1}{2} \sum_{k=2}^{\infty} k(k+1)a_{2k+1} \quad (4.29)
\]
Section 4.5 - Boundary Conditions - General Solution

The general solution of (4.14) is obtained by writing

\[ \phi(y) = \frac{1}{2}a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots \]

with

\[ \phi'(y) = \frac{1}{2}a_0'^2 + a_1'^2 + a_2'^2 + a_3'^2 + \ldots \]

with boundary conditions

\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0 \]

Using (4.4), these boundary conditions lead to

\[ \frac{1}{2}a_0 + a_1 + a_2 + a_3 + a_4 + \ldots = 0 \]
\[ \frac{1}{2}a_0 - a_1 + a_2 - a_3 + a_4 - \ldots = 0 \]
\[ \frac{1}{2}a_0' + a_1' + a_2' + a_3' + a_4' + \ldots = 0 \]
\[ \frac{1}{2}a_0' - a_1' + a_2' - a_3' + a_4' - \ldots = 0 \]

from which we obtain the equivalent results

\[ \frac{1}{2}a_0 + a_2 + a_4 + \ldots = 0 \]
\[ a_1 + a_3 + a_5 + \ldots = 0 \]
\[ \frac{1}{2}a_0' + a_2' + a_4' + \ldots = 0 \]
\[ a_1' + a_3' + a_5' + \ldots = 0 \]
Now these have arisen in the discussion of the boundary conditions in sections 4.3 and 4.4, leading to (4.25), (4.26), (4.28) and (4.29). Thus these four results giving expressions for $\frac{1}{2}a_0$, $a_1$, $a_2$ and $a_3$ which express the boundary conditions, together with (4.8), can be used to obtain the appropriate matrix representation of the general problem with $k = 3, 4, 5, 6, \ldots$. 
Section 4.6 - The General Problem

In the preceding sections, the solution of

\[ \varepsilon \phi^{iv} + (\alpha + 3y^2)\phi'' + (\gamma + \delta y^2)\phi = 0 \]  \hspace{1cm} (4.14)

\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0 \]

has been discussed, in preference to the tedious description required for the solution of the more general problem

\[ c_0 \phi^{iv} + (c_1 + c_2y + c_3y^2)\phi'' + (c_4 + c_5y + c_6y^2)\phi'' + (c_7 + c_8y + c_9y^2)\phi' + (c_{10} + c_{11}y + c_{12}y^2)\phi = 0 \]  \hspace{1cm} (4.13)

\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0 \]

However, this more general problem has been investigated and the more complicated matrix \( W \) which corresponds to this case is given at the end of the thesis.

It should be noted, that if no \( c_i = 0 \) \( (i = 0, 1, \ldots, 12) \) then we cannot find an even (odd) function which is a solution of the problem. However, it will be possible to find such solutions when

\[ c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = 0, \]

and the case for which we also have

\[ c_2 = c_8 = 0 \]

reduces to (4.14).
Section 5.1 - Introduction

This Chapter contains the numerical results obtained using the methods discussed in Chapters 2, 3 and 4. Basically, this involves the conversion of the boundary value problem

\[ L(\phi, \lambda) = 0 \]

to a discrete matrix eigen-value problem by either Chebyshev series methods (Chapter 4) or rational finite differences (Chapter 3), and the solution to the matrix problem by the QR algorithm (Chapter 2).

The results are discussed with reference to the order, \( n \), of the matrix used to obtain the eigen-values. Thus the statement "\( n = 12 \)" means that a matrix of order 12 was used. In terms of the discussions of Chapters 3 and 4, the implications are as follows:

When finite difference methods (Chapter 3) are used

(a) to find the general solution of

\[ L(\phi, \lambda) = 0 \]
\[ \phi(\pm 1) = \phi'(\pm 1) = 0 \]

then \((-1, 1)\) is divided into \( n + 1 \) equal intervals;
(b) to find the even (or odd) solution of

\[ L(\phi, \lambda) = 0 \]

\[ \phi(1) = \phi'(1) = 0 \]

then \((0, 1)\) is divided into \(n\) equal intervals.

When the solution is expressed as the sum of Chebyshev polynomials (Chapter 4)

(a) to find the general solution of

\[ L(\phi, \lambda) = 0 \]

\[ \phi(\pm 1) = \phi'(\pm 1) = 0 \]

then the expansion involves the terms with co-efficients

\(a_0, a_1, a_2, \ldots, a_{n+2}, a_{n+3}\)

(b) to find the even solution of

\[ L(\phi, \lambda) = 0 \]

\[ \phi(1) = \phi'(1) = 0 \]

then the expansion involves the terms with co-efficients

\(a_0, a_2, \ldots, a_{2n}, a_{2n+2}\)
(c) to find the odd solution of

\[ L(\phi, \lambda) = 0 \]
\[ \phi(1) = \phi'(1) = 0 \]

then the expansion involves the terms with co-efficients

\[ a_1, a_3, \ldots, a_{2n+1}, a_{2n+3} \]

When finite difference methods are used, the direct results for the eigen-values are always inferior to those obtained using Chebyshev methods. In sections 5.2 and 5.3 we underline the figures of agreement by these two methods with the exact results, to display the exactness of the Chebyshev method. Even when the finite difference results are extrapolated using \( h^4 \) - extrapolation, the results are still not as good as the original Chebyshev answers.
Section 5.2 - The Problem $\phi^{iv} - \lambda \phi = 0$

A discussion of the exact solution of this problem is given in Section 6.1.

Section 5.2.1 - Even Solution: $\phi(1) = \phi'(1) = 0$

We begin by examining the first four (4) eigen-values in the case $n = 8$ for the Chebyshev series method.

<table>
<thead>
<tr>
<th>EXACT</th>
<th>CHEBYSHEV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.1285243 \times 10$</td>
<td>$3.1285244 \times 10$</td>
</tr>
<tr>
<td>$9.1360188 \times 10^2$</td>
<td>$9.1360187 \times 10^2$</td>
</tr>
<tr>
<td>$5.5709629 \times 10^3$</td>
<td>$5.5710953 \times 10^3$</td>
</tr>
<tr>
<td>$1.9263028 \times 10^4$</td>
<td>$1.9213938 \times 10^4$</td>
</tr>
</tbody>
</table>

Even with a matrix of order 8, agreement in the first two (2) eigen-values is very good (7 significant figures).

For the $n = 12$ case, we compare the exact result with the rational difference result, and with the Chebyshev result for the first eight (8) eigen-values.
We can see that for the first four (4) eigen-values, 7 significant figures of accuracy are held in the Chebyshev case, while only four (4) figures are held in the very first eigen-value for the Rational method.

For the $n = 30$ case, we compare the first sixteen (16) eigen-values calculated by the Chebyshev and Rational methods with the Exact results.
<table>
<thead>
<tr>
<th>EXACT</th>
<th>CHEBYSHEV</th>
<th>RATIONAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.1285243 \times 10$</td>
<td>$3.1285241 \times 10$</td>
<td>$3.1285339 \times 10$</td>
</tr>
<tr>
<td>$9.1360188 \times 10^2$</td>
<td>$9.1360199 \times 10^2$</td>
<td>$9.1361742 \times 10^2$</td>
</tr>
<tr>
<td>$5.5709629 \times 10^3$</td>
<td>$5.5709629 \times 10^3$</td>
<td>$5.5711941 \times 10^3$</td>
</tr>
<tr>
<td>$1.9263028 \times 10^4$</td>
<td>$1.9263028 \times 10^4$</td>
<td>$1.9264466 \times 10^4$</td>
</tr>
<tr>
<td>$4.9587695 \times 10^4$</td>
<td>$4.9587696 \times 10^4$</td>
<td>$4.9593227 \times 10^4$</td>
</tr>
<tr>
<td>$1.0648069 \times 10^5$</td>
<td>$1.0648069 \times 10^5$</td>
<td>$1.0649578 \times 10^5$</td>
</tr>
<tr>
<td>$2.0221556 \times 10^5$</td>
<td>$2.0221557 \times 10^5$</td>
<td>$2.0224504 \times 10^5$</td>
</tr>
<tr>
<td>$3.5140367 \times 10^5$</td>
<td>$3.5140368 \times 10^5$</td>
<td>$3.5143572 \times 10^5$</td>
</tr>
<tr>
<td>$5.7099420 \times 10^5$</td>
<td>$5.7099421 \times 10^5$</td>
<td>$5.7095209 \times 10^5$</td>
</tr>
<tr>
<td>$8.8027416 \times 10^5$</td>
<td>$8.8027416 \times 10^5$</td>
<td>$8.7989774 \times 10^5$</td>
</tr>
<tr>
<td>$1.3008683 \times 10^6$</td>
<td>$1.3008684 \times 10^6$</td>
<td>$1.2994561 \times 10^6$</td>
</tr>
<tr>
<td>$1.8567394 \times 10^6$</td>
<td>$1.8567394 \times 10^6$</td>
<td>$1.8526323 \times 10^6$</td>
</tr>
<tr>
<td>$2.5741878 \times 10^6$</td>
<td>$2.5741879 \times 10^6$</td>
<td>$2.5637990 \times 10^6$</td>
</tr>
<tr>
<td>$3.4818518 \times 10^6$</td>
<td>$3.4818484 \times 10^6$</td>
<td>$3.4579334 \times 10^6$</td>
</tr>
<tr>
<td>$4.6107075 \times 10^6$</td>
<td>$4.6108025 \times 10^6$</td>
<td>$4.5593863 \times 10^6$</td>
</tr>
<tr>
<td>$5.9940687 \times 10^6$</td>
<td>$5.9923663 \times 10^6$</td>
<td>$5.8899696 \times 10^6$</td>
</tr>
</tbody>
</table>
Here, we see that down as far as the thirteenth eigen-value for the Chebyshev case, we have 7 significant figures of accuracy, while for the rational problem we have only 5 figures of accuracy in the first eigen-value.

Finally, by considering only the smallest eigen-value, we compare convergence of the two methods for various cases from $n = 8$ to $n = 27$.

Actual eigen-value = 31.285243

<table>
<thead>
<tr>
<th>n</th>
<th>CHEBYSHEV</th>
<th>RATIONAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>31.285244</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>31.285244</td>
<td>31.286328</td>
</tr>
<tr>
<td>18</td>
<td>31.285244</td>
<td>31.285565</td>
</tr>
<tr>
<td>27</td>
<td>31.285242</td>
<td>31.285339</td>
</tr>
</tbody>
</table>

Examination of the given results in the rational case, suggests that $h^k$ - extrapolation is appropriate. (The sub-divisions in the three (3) cases give $h_1 = \frac{1}{12}$, $h_2 = \frac{1}{18}$ and $h_3 = \frac{1}{27}$, so that $h_2 = \frac{2}{3}h_1$, $h_3 = \frac{2}{3}h_2$).

The $h^k$ - extrapolated result using 31.286328 and 31.285565 is 31.28538.

The $h^k$ - extrapolated result using 31.285565 and 31.285339 is 31.28529.
For the first two (2) eigen-values, the eigen-functions are calculated using Gaussian elimination, and the results are tabulated below.

Co-efficients in the expansion

\[ \phi(y) = \frac{1}{2}a_0 T_0(y) + \sum_{k=1}^{\infty} a_{2k} T_{2k}(y) \]

<table>
<thead>
<tr>
<th></th>
<th>( \lambda = 31.235244 )</th>
<th>( \lambda = 913.50188 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}a_0 )</td>
<td>0.36671373</td>
<td>-0.25588471</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-0.49560691</td>
<td>-0.09858039</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>0.13311714</td>
<td>0.69003370</td>
</tr>
<tr>
<td>( a_6 )</td>
<td>-0.00439119</td>
<td>-0.39451818</td>
</tr>
<tr>
<td>( a_8 )</td>
<td>0.00016909</td>
<td>0.06542670</td>
</tr>
<tr>
<td>( a_{10} )</td>
<td>-0.000000190</td>
<td>-0.00688165</td>
</tr>
<tr>
<td>( a_{12} )</td>
<td>0.000000003</td>
<td>0.00042367</td>
</tr>
<tr>
<td>( a_{14} )</td>
<td></td>
<td>-0.00001976</td>
</tr>
<tr>
<td>( a_{16} )</td>
<td></td>
<td>-0.00000364</td>
</tr>
<tr>
<td>( a_{18} )</td>
<td></td>
<td>-0.00000002</td>
</tr>
</tbody>
</table>
Values of $\phi(y)$ for $0 \leq y \leq 1$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\lambda = 31.285244$</th>
<th>$\lambda = 913.60188$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.978723</td>
<td>0.850878</td>
</tr>
<tr>
<td>0.2</td>
<td>0.916446</td>
<td>0.446915</td>
</tr>
<tr>
<td>0.3</td>
<td>0.817762</td>
<td>-0.094639</td>
</tr>
<tr>
<td>0.4</td>
<td>0.690113</td>
<td>-0.617805</td>
</tr>
<tr>
<td>0.5</td>
<td>0.543484</td>
<td>-0.974969</td>
</tr>
<tr>
<td>0.6</td>
<td>0.390010</td>
<td>-1.072424</td>
</tr>
<tr>
<td>0.7</td>
<td>0.243521</td>
<td>-0.901572</td>
</tr>
<tr>
<td>0.8</td>
<td>0.119072</td>
<td>-0.547691</td>
</tr>
<tr>
<td>0.9</td>
<td>0.032492</td>
<td>-0.175633</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
The analytic solution to this problem is

\[ \phi(y) = \frac{\sinh \alpha \cos \alpha y + \sin \alpha \cosh \alpha y}{\sinh \alpha + \sin \alpha} \]

where \( \alpha = \sqrt{\lambda} \)

and \( \alpha \) is the solution of \( \tan \alpha = -\tanh \alpha \). The eigen-function has been normalised so that \( \phi(0) = 1 \). (See Section 6.1).

The values of \( \phi(y) \) given in the above table agree exactly with the analytic solution, to the given number of figures.
Section 5.2.2 - Odd Solution $\phi(1) = \phi'(1) = 0$.

This problem was examined using the Chebyshev series method with various sized matrices, and the agreement of calculated eigen-values with the analytic solutions was of the same order as noted in Section 5.2.1 for the even solution. In particular, the first two (2) eigen-values associated with the odd solution were found exactly as $237.72106$ and $2496.4874$, using a matrix of order 10. The eigen-functions associated with the eigen-values are given below.

Co-efficients in the expansion

$$\phi(y) = \sum_{k=1}^{\infty} a_{2k-1} T_{2k-1}(y)$$

<table>
<thead>
<tr>
<th>$\lambda = 237.72106$</th>
<th>$\lambda = 2496.4874$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$1.00000000$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$-1.57409682$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$0.62837651$</td>
</tr>
<tr>
<td>$a_7$</td>
<td>$-0.05809595$</td>
</tr>
<tr>
<td>$a_9$</td>
<td>$0.00394457$</td>
</tr>
</tbody>
</table>
Co-efficients in the expansion

\[ \phi(y) = \sum_{k=1}^{\infty} a_{2k-1} T_{2k-1}(y) : \text{Con't.} \]

<table>
<thead>
<tr>
<th></th>
<th>( \lambda = 237.72106 )</th>
<th>( \lambda = 2496.4874 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{11} )</td>
<td>-0.00013204</td>
<td>0.04966584</td>
</tr>
<tr>
<td>( a_{13} )</td>
<td>0.00000379</td>
<td>-0.00461815</td>
</tr>
<tr>
<td>( a_{15} )</td>
<td>-0.00000007</td>
<td>0.00031414</td>
</tr>
<tr>
<td>( a_{17} )</td>
<td></td>
<td>-0.00001566</td>
</tr>
<tr>
<td>( a_{19} )</td>
<td></td>
<td>0.00000062</td>
</tr>
</tbody>
</table>

Values of \( \phi(y) \) for \( 0 \leq y \leq 1 \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( \lambda = 237.72106 )</th>
<th>( \lambda = 2496.4874 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.908348</td>
<td>-1.757198</td>
</tr>
<tr>
<td>0.2</td>
<td>1.686420</td>
<td>-2.669865</td>
</tr>
<tr>
<td>0.3</td>
<td>2.225020</td>
<td>-2.296855</td>
</tr>
</tbody>
</table>
Values of $\phi(y)$ for $0 \leq y \leq 1$: Con't.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\lambda = 237.72106$</th>
<th>$\lambda = 2496.4874$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>2.454087</td>
<td>-0.809817</td>
</tr>
<tr>
<td>0.5</td>
<td>2.355229</td>
<td>1.092751</td>
</tr>
<tr>
<td>0.6</td>
<td>1.967089</td>
<td>2.527543</td>
</tr>
<tr>
<td>0.7</td>
<td>1.383152</td>
<td>2.864498</td>
</tr>
<tr>
<td>0.8</td>
<td>0.742889</td>
<td>2.058646</td>
</tr>
<tr>
<td>0.9</td>
<td>0.218423</td>
<td>0.732331</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

The analytic solution of this problem is

$$\phi(y) = \sinh \alpha \sin \alpha y - \sin \alpha \sinh \alpha y$$

where $\alpha = \frac{1}{\lambda}$

and $\alpha$ is the solution of $\tan \alpha = \tanh \alpha$. The eigen-function is normalised so that the co-efficient $a_1$ in the Chebyshev series is equal to 1.
Section 5.3 - The Problem \( \phi^{iv} + \lambda \phi'' = 0 \)

The analytic solution of this problem is discussed in Section 6.1.

Section 5.3.1 - Even Solution \( \phi(1) = \phi'(1) = 0 \)

We give a comparison for the first six (6) eigen-values for the case \( n = 12 \) and then for the first sixteen (16) eigen-values for the case \( n = 27 \) for both the Chebyshev and rational difference methods, and compare these results with the exact results.

<table>
<thead>
<tr>
<th>EXACT</th>
<th>CHEBYSHEV</th>
<th>RATIONAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.8696044</td>
<td>9.8696044</td>
<td>9.8694107</td>
</tr>
<tr>
<td>3.9478417 x 10</td>
<td>3.9478418 x 10</td>
<td>3.9465922 x 10</td>
</tr>
<tr>
<td>8.8826439 x 10</td>
<td>8.8826440 x 10</td>
<td>8.8682322 x 10</td>
</tr>
<tr>
<td>1.5791367 x 10^2</td>
<td>1.5791362 x 10^2</td>
<td>1.5709091 x 10^2</td>
</tr>
<tr>
<td>2.4674011 x 10^2</td>
<td>2.4675089 x 10^2</td>
<td>2.4354531 x 10^2</td>
</tr>
<tr>
<td>3.5530575 x 10^2</td>
<td>3.5475095 x 10^2</td>
<td>3.4560000 x 10^2</td>
</tr>
</tbody>
</table>

We note that 7 figures of accuracy are held in the first
four (4) eigen-values for the Chebyshev case, but only four (4) figures of agreement can be shown using the rational method for even the first eigen-value.

<table>
<thead>
<tr>
<th>EXACT</th>
<th>CHEBYSHEV</th>
<th>RATIONAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.8696044</td>
<td>9.8696044</td>
<td>9.8695969</td>
</tr>
<tr>
<td>3.9478417 x 10</td>
<td>3.9478418 x 10</td>
<td>3.9477934 x 10</td>
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<td>8.8826439 x 10</td>
<td>8.8826440 x 10</td>
<td>8.8820918 x 10</td>
</tr>
<tr>
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<td>1.5791367 x 10^2</td>
<td>1.5788254 x 10^2</td>
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<td>2.4674011 x 10^2</td>
<td>2.4662079 x 10^2</td>
</tr>
<tr>
<td>3.5530575 x 10^2</td>
<td>3.5530576 x 10^2</td>
<td>3.5494753 x 10^2</td>
</tr>
<tr>
<td>4.8361061 x 10^2</td>
<td>4.8361062 x 10^2</td>
<td>4.8270164 x 10^2</td>
</tr>
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<td>6.3165468 x 10^2</td>
<td>6.3165468 x 10^2</td>
<td>6.2961529 x 10^2</td>
</tr>
<tr>
<td>7.9943795 x 10^2</td>
<td>7.9943796 x 10^2</td>
<td>7.9527273 x 10^2</td>
</tr>
<tr>
<td>9.8696044 x 10^2</td>
<td>9.8696044 x 10^2</td>
<td>9.7906155 x 10^2</td>
</tr>
<tr>
<td>1.1942221 x 10^3</td>
<td>1.1942221 x 10^3</td>
<td>1.1801168 x 10^3</td>
</tr>
<tr>
<td>1.4212230 x 10^3</td>
<td>1.4212237 x 10^3</td>
<td>1.3972589 x 10^3</td>
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<td>1.6679631 x 10^3</td>
<td>1.6679429 x 10^3</td>
<td>1.6289271 x 10^3</td>
</tr>
<tr>
<td>1.9344424 x 10^3</td>
<td>1.9348065 x 10^3</td>
<td>1.8731126 x 10^3</td>
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<td>2.2206609 x 10^3</td>
<td>2.2167191 x 10^3</td>
<td>2.1272950 x 10^3</td>
</tr>
<tr>
<td>2.5266187 x 10^3</td>
<td>2.5612091 x 10^3</td>
<td>2.3883906 x 10^3</td>
</tr>
</tbody>
</table>
Again, we note that seven (7) figures are held for the first twelve (12) eigen-values in the Chebyshev case, but only five (5) figures are held even in the first eigen-value by the rational method. (The full eight (8) hold for the first eigen-value for the Chebyshev case).

Finally, we compare the first eigen-value for cases $n = 12, 18$ and $27$:

$$\begin{align*}
\text{Exact eigen-value} & = 9.8696044 \\
\text{CHEBYSHEV} & = 9.8696044, 9.8694107 \\
\text{RATIONAL} & = 9.8695662, 9.8695969
\end{align*}$$


Again, Gaussian elimination was used to calculate the eigen-functions, and the results are tabulated below.
The Even Eigen-functions of $\phi^{iv} + \lambda \phi'' = 0$, $\phi(1) = \phi'(1) = 0$.

Co-efficients in the expansion

$$\phi(y) = \frac{1}{2}a_0 T_0(y) + \sum_{k=1}^{\infty} a_{2k} T_{2k}(y)$$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>9.8696044</th>
<th>39.47842</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}a_0$</td>
<td>0.34787891</td>
<td>0.38986155</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.48543393</td>
<td>-0.28788036</td>
</tr>
<tr>
<td>$a_4$</td>
<td>0.15142457</td>
<td>-0.31568048</td>
</tr>
<tr>
<td>$a_6$</td>
<td>-0.01454597</td>
<td>0.27768840</td>
</tr>
<tr>
<td>$a_8$</td>
<td>0.00069612</td>
<td>-0.07329532</td>
</tr>
<tr>
<td>$a_{10}$</td>
<td>-0.00002010</td>
<td>0.01013846</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>0.00000039</td>
<td>-0.00088337</td>
</tr>
<tr>
<td>$a_{14}$</td>
<td>0.000005342</td>
<td></td>
</tr>
<tr>
<td>$a_{16}$</td>
<td>0.00000238</td>
<td></td>
</tr>
<tr>
<td>$a_{18}$</td>
<td>0.00000008</td>
<td></td>
</tr>
</tbody>
</table>
The analytic solution to this problem is

$$\phi(y) = \cos \alpha y - \cos \alpha$$

where $\alpha = \frac{n\pi}{\lambda^2}.$

The eigen-function has been normalised so that $\phi(0) = 1$ (See Section 6.1) for the first eigen-value, and $\phi(.5) = 1$ for the second eigen-value (since $\phi(0) = 0$ in this case). Notice the symmetry of the results about $y = .5$ for this case (as expected).
Section 5.3.2 - Odd Solution $\phi(1) = \phi'(1) = 0$.

This problem was also examined using the Chebyshev series method with various sized matrices and the agreement of calculated eigen-values with the analytic solution was of the same order as noted in Section 5.3.1 for the even solution.

The first two (2) eigen-values were found exactly as 20.190729 and 59.679516 using a matrix of order 10. The associated eigen-functions are given below.

Co-efficients in the expansion

$$\phi(y) = \sum_{k=1}^{\infty} a_{2k-1} T_{2k-1}(y)$$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$a_1$</th>
<th>$a_3$</th>
<th>$a_5$</th>
<th>$a_7$</th>
<th>$a_9$</th>
<th>$a_{11}$</th>
<th>$a_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20.190729$</td>
<td>1.00000000</td>
<td>-1.64282456</td>
<td>0.74907228</td>
<td>-0.11505988</td>
<td>0.00926122</td>
<td>-0.00046468</td>
<td>0.00001601</td>
</tr>
<tr>
<td>$59.67952$</td>
<td>1.00000000</td>
<td>-0.90727902</td>
<td>-0.78563334</td>
<td>0.97653959</td>
<td>-0.33922204</td>
<td>0.06233792</td>
<td>-0.00730968</td>
</tr>
</tbody>
</table>
Co-efficients in the expansion

\[
\phi(y) = \sum_{k=1}^{\infty} a_{2k-1} T_{2k-1}(y): \text{Con't.}
\]

| \(a_{15}\) | \(-0.00000040\) | 0.00060179 |
| \(a_{17}\) | \(-0.00003692\) |
| \(a_{19}\) | 0.00000176 |
| \(a_{21}\) | 0.00000007 |

Values of \(\phi(y)\) for \(0 \leq y \leq 1\).

<table>
<thead>
<tr>
<th>(y)</th>
<th>(\lambda = 20.190729)</th>
<th>(\lambda = 59.67952)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>1.027871</td>
<td>-0.967988</td>
</tr>
<tr>
<td>0.2</td>
<td>1.889119</td>
<td>-1.295434</td>
</tr>
<tr>
<td>0.3</td>
<td>2.450202</td>
<td>-0.705432</td>
</tr>
<tr>
<td>0.4</td>
<td>2.637169</td>
<td>0.558091</td>
</tr>
<tr>
<td>0.5</td>
<td>2.450346</td>
<td>1.868849</td>
</tr>
</tbody>
</table>
Values of $\phi(y)$ for $0 \leq y \leq 1$: Con't

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\lambda = 20.190729$</th>
<th>$\lambda = 59.67952$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>1.964265</td>
<td>2.573743</td>
</tr>
<tr>
<td>0.7</td>
<td>1.312874</td>
<td>2.363621</td>
</tr>
<tr>
<td>0.8</td>
<td>0.662938</td>
<td>1.448784</td>
</tr>
<tr>
<td>0.9</td>
<td>0.180934</td>
<td>0.439601</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

The analytic solution to this problem is

$$\phi(y) = \sin \alpha y - y \sin \alpha$$

where $\alpha = \frac{\lambda}{\pi}$

is the solution to $\alpha = \tan \alpha$.

$\phi(y)$ is normalised so that the co-efficient $a_1$ in the Chebyshev series expansion of the function is such that $a_1 = 1$.

Note again the function agrees with the analytic solution to all published figures. (See Section 6.1).
Section 5.4 - The Orr-Sommerfeld Equation

Plane Poiseuille Flow

The problem of plane Poiseuille flow has been studied by Thomas (1953) using a modified finite differences method which is described in Section 6.3. He presents numerical results for the solution of the Orr-Sommerfeld boundary value problem

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi - i\alpha R\{(1 - \lambda - y^2)(\phi'' - \alpha^2\phi) + 2\phi\} = 0$$

$$\phi(1) = \phi'(1) = 0$$

where the even solution is sought.

Lee and Reynolds (1967) include a discussion of this problem in their article, which reviews various methods for solving the Orr-Sommerfeld equation (see Section 6.3.)

We examine the Thomas problem for varying values of $\alpha$, $R$ and $n$, and do this using both the Chebyshev and the rational finite difference method, comparing with Thomas' results and (in some cases) the results of Lee and Reynolds.
\( a = 1.0, \ R = 100 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.47751 - 0.16271i</td>
<td>0.47848 - 0.163021</td>
</tr>
<tr>
<td>18</td>
<td>0.47751 - 0.16271i</td>
<td>0.47849 - 0.162961</td>
</tr>
<tr>
<td>27</td>
<td>0.47751 - 0.16271i</td>
<td>0.47849 - 0.162951</td>
</tr>
</tbody>
</table>

Thomas: 0.47849 - 0.16297i

Lee and Reynolds: 0.47849 - 0.16294i

Extrapolated finite diff.: 0.47849 - 0.162951

(1st Pair)

0.47849 - 0.162951

(2nd Pair)

It should be noted that Thomas did not publish results for this set of data, but Lee and Reynolds calculated both results, the first by Thomas' scheme.
(ii) \( \alpha = 1.0, \ R = 1600 \)

<table>
<thead>
<tr>
<th>n</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.32336 - 0.02557i</td>
<td>0.32260 - 0.02931i</td>
</tr>
<tr>
<td>18</td>
<td>0.32320 - 0.02541i</td>
<td>0.32298 - 0.02693i</td>
</tr>
<tr>
<td>27</td>
<td>0.32320 - 0.02541i</td>
<td>0.32366 - 0.02636i</td>
</tr>
</tbody>
</table>

Thomas: 0.3231 - 0.02621i

Extrapolated finite diffs:

\[ \begin{align*}
\text{(1st Pair)} & : 0.32296 - 0.02634i \\
\text{(2nd Pair)} & : 0.32310 - 0.02622i
\end{align*} \]
(iii) $\alpha = 1.0, \ R = 2,500$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.30107 - 0.01365i</td>
<td>0.30134 - 0.01942i</td>
</tr>
<tr>
<td>18</td>
<td>0.30131 - 0.01332i</td>
<td>0.30097 - 0.01546i</td>
</tr>
<tr>
<td>27</td>
<td>0.30131 - 0.01332i</td>
<td>0.30108 - 0.01446i</td>
</tr>
</tbody>
</table>

Thomas: 0.3011 - 0.01421
Lee and Reynolds: 0.30101 - 0.01425i
Extrapolated finite diffs:
*(1st Pair)*
0.30087 - 0.014491

*(2nd Pair)*
0.30111 - 0.014213i
(iv) \( a = 1.0, \ R = 6,400 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( 0.25708 + 0.00283i )</td>
<td>( 0.25771 - 0.00895i )</td>
</tr>
<tr>
<td>18</td>
<td>( 0.25708 + 0.00189i )</td>
<td>( 0.25776 - 0.00247i )</td>
</tr>
<tr>
<td>27</td>
<td>( 0.25710 + 0.00190i )</td>
<td>( 0.25695 + 0.0009i )</td>
</tr>
</tbody>
</table>

Thomas: \( 0.2569 + 0.0009i \)

Extrapolated finite diffs:

(1st Pair) \( 0.25777 - 0.00087i \)

(2nd Pair) \( 0.25675 + 0.00072i \)
\( \alpha = 1.05, \quad R = 8,000 \)

<table>
<thead>
<tr>
<th>n</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( .25314 + .00362i )</td>
<td>( .25505 - .00667i )</td>
</tr>
<tr>
<td>18</td>
<td>( .25254 + .00279i )</td>
<td>( .25432 - .00256i )</td>
</tr>
<tr>
<td>27</td>
<td>( .25259 + .00278i )</td>
<td>( .25262 + .00052i )</td>
</tr>
</tbody>
</table>

Thomas: \( .2524 + .0017i \)

Extrapolated finite diffs: \( .25414 - .00155i \)
(1st Pair)

\( .25520 + .00130i \)
(2nd Pair)
\( \alpha = 1.0, \ R = 10,000 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( .23866 + .00501i )</td>
<td>( .24263 - .00287i )</td>
</tr>
<tr>
<td>18</td>
<td>( .23765 + .00480i )</td>
<td>( .23994 - .00102i )</td>
</tr>
<tr>
<td>27</td>
<td>( .23768 + .00474i )</td>
<td>( .23780 + .00239i )</td>
</tr>
</tbody>
</table>

Thomas: \( .23753 + .00374i \)

Lee and Reynolds: \( .23764 + .0036i \)
(vii) \( \alpha = 1.0, \ R = 35,000 \)

<table>
<thead>
<tr>
<th>n</th>
<th>Chebyshev</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( .18665 + .00106i )</td>
<td>( .22661 + .00186i )</td>
</tr>
<tr>
<td>18</td>
<td>( .18821 - .00015i )</td>
<td>( .19262 - .00618i )</td>
</tr>
<tr>
<td>27</td>
<td>( .18857 - .00022i )</td>
<td>( .19202 - .00404i )</td>
</tr>
</tbody>
</table>

Thomas: \( .1886 - .0009i \)

The eigen-function corresponding to the first eigen-value for each of the cases \( \alpha = 1, \ R = 100 \) and \( \alpha = 1, \ R = 10,000 \) is tabulated below.
(a) \( \alpha = 1, \quad R = 100, \quad \lambda = 0.4775108 - 0.1627056i \)

Co-\( \text{efficients in the expansion} \)

\[
\phi(y) = \frac{1}{2}a_0 T_0(y) + \sum_{k=1}^\infty a_{2k} T_{2k}(y)
\]

<table>
<thead>
<tr>
<th></th>
<th>REAL</th>
<th>IMAGINARY</th>
<th>MODULUS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_0)</td>
<td>0.380117</td>
<td>-0.041363</td>
<td>0.382361</td>
</tr>
<tr>
<td>(a_2)</td>
<td>-0.510294</td>
<td>0.028459</td>
<td>0.511037</td>
</tr>
<tr>
<td>(a_4)</td>
<td>0.126966</td>
<td>0.037089</td>
<td>0.132272</td>
</tr>
<tr>
<td>(a_6)</td>
<td>0.008710</td>
<td>-0.029518</td>
<td>0.030776</td>
</tr>
<tr>
<td>(a_8)</td>
<td>-0.007013</td>
<td>0.004629</td>
<td>0.008403</td>
</tr>
<tr>
<td>(a_{10})</td>
<td>0.001629</td>
<td>0.001013</td>
<td>0.001918</td>
</tr>
<tr>
<td>(a_{12})</td>
<td>-0.000079</td>
<td>-0.000355</td>
<td>0.000363</td>
</tr>
<tr>
<td>(a_{14})</td>
<td>-0.000044</td>
<td>0.000047</td>
<td>0.000064</td>
</tr>
<tr>
<td>(a_{16})</td>
<td>0.000009</td>
<td>0.000000</td>
<td>0.000009</td>
</tr>
</tbody>
</table>
Values of $\phi(y)$ for $0 \leq y \leq 1$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\phi$ REAL</th>
<th>$\phi$ FAG</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.984153</td>
<td>-0.008218</td>
</tr>
<tr>
<td>0.2</td>
<td>0.935445</td>
<td>-0.03822</td>
</tr>
<tr>
<td>0.3</td>
<td>0.851571</td>
<td>-0.061614</td>
</tr>
<tr>
<td>0.4</td>
<td>0.732262</td>
<td>-0.090721</td>
</tr>
<tr>
<td>0.5</td>
<td>0.583121</td>
<td>-0.106855</td>
</tr>
<tr>
<td>0.6</td>
<td>0.417868</td>
<td>-0.101860</td>
</tr>
<tr>
<td>0.7</td>
<td>0.256910</td>
<td>-0.075890</td>
</tr>
<tr>
<td>0.8</td>
<td>0.122300</td>
<td>-0.039600</td>
</tr>
<tr>
<td>0.9</td>
<td>0.032330</td>
<td>-0.010049</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>-0.000000</td>
</tr>
</tbody>
</table>

The eigen-functions were normalised so that

$$\phi(0) = 1.0.$$
\( a = 1, \quad R = 10^{3}, \quad \lambda = 0.2376787 - 0.0047366i \)

Co-efficients in the expansion of \( \phi(y) \)

<table>
<thead>
<tr>
<th>( b_a )</th>
<th>REAL</th>
<th>IMAGINARY</th>
<th>MODULUS</th>
</tr>
</thead>
<tbody>
<tr>
<td>3a_0</td>
<td>0.507738</td>
<td>0.005727</td>
<td>0.507770</td>
</tr>
<tr>
<td>a_2</td>
<td>-0.520729</td>
<td>0.005785</td>
<td>0.520761</td>
</tr>
<tr>
<td>a_4</td>
<td>-0.023052</td>
<td>-0.001695</td>
<td>0.023068</td>
</tr>
<tr>
<td>a_6</td>
<td>0.013241</td>
<td>-0.004418</td>
<td>0.013959</td>
</tr>
<tr>
<td>a_8</td>
<td>0.012790</td>
<td>-0.004956</td>
<td>0.013716</td>
</tr>
<tr>
<td>a_10</td>
<td>0.006968</td>
<td>-0.003283</td>
<td>0.007702</td>
</tr>
<tr>
<td>a_12</td>
<td>0.002284</td>
<td>-0.000709</td>
<td>0.002392</td>
</tr>
<tr>
<td>a_14</td>
<td>0.000325</td>
<td>0.001119</td>
<td>0.001165</td>
</tr>
<tr>
<td>a_16</td>
<td>0.000140</td>
<td>0.001479</td>
<td>0.001485</td>
</tr>
<tr>
<td>a_18</td>
<td>0.000309</td>
<td>0.000857</td>
<td>0.000911</td>
</tr>
<tr>
<td>a_20</td>
<td>0.000135</td>
<td>0.000170</td>
<td>0.000259</td>
</tr>
<tr>
<td>a_22</td>
<td>-0.000051</td>
<td>-0.000093</td>
<td>0.000106</td>
</tr>
<tr>
<td>a_24</td>
<td>-0.000146</td>
<td>-0.000038</td>
<td>0.000154</td>
</tr>
<tr>
<td>a_26</td>
<td>-0.000077</td>
<td>0.0000942</td>
<td>0.000088</td>
</tr>
<tr>
<td>a_28</td>
<td>0.000007</td>
<td>0.000034</td>
<td>0.000034</td>
</tr>
<tr>
<td>a_30</td>
<td>0.000023</td>
<td>-0.000006</td>
<td>0.000024</td>
</tr>
<tr>
<td>a_32</td>
<td>0.000033</td>
<td>-0.000016</td>
<td>0.000016</td>
</tr>
<tr>
<td>a_34</td>
<td>-0.000008</td>
<td>-0.000003</td>
<td>0.000009</td>
</tr>
<tr>
<td>a_36</td>
<td>-0.000004</td>
<td>0.000004</td>
<td>0.000006</td>
</tr>
<tr>
<td>y</td>
<td>( \phi ) REAL</td>
<td>( \phi ) IMAG</td>
<td>( \phi ) REAL</td>
</tr>
<tr>
<td>-----</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>0.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.991867</td>
<td>0.000000</td>
<td>0.991868</td>
</tr>
<tr>
<td>0.2</td>
<td>0.967251</td>
<td>0.000329</td>
<td>0.967255</td>
</tr>
<tr>
<td>0.3</td>
<td>0.925489</td>
<td>0.000747</td>
<td>0.925497</td>
</tr>
<tr>
<td>0.4</td>
<td>0.865403</td>
<td>0.001338</td>
<td>0.865417</td>
</tr>
<tr>
<td>0.5</td>
<td>0.785165</td>
<td>0.002125</td>
<td>0.785190</td>
</tr>
<tr>
<td>0.6</td>
<td>0.682009</td>
<td>0.003118</td>
<td>0.682046</td>
</tr>
<tr>
<td>0.7</td>
<td>0.551533</td>
<td>0.004308</td>
<td>0.551578</td>
</tr>
<tr>
<td>0.8</td>
<td>0.384143</td>
<td>0.005372</td>
<td>0.384155</td>
</tr>
<tr>
<td>0.9</td>
<td>0.167347</td>
<td>0.020108</td>
<td>0.166567</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

Results were normalised so that \( \phi(0) = 1 \).
Section 5.5 - The Orr Sommerfeld Equation

Plane Couette Flow

The problem of plane Couette flow has been solved numerically by Gallagher and Mercer (1962) using a method of Galerkin type to find the eigen-values associated with the equation

\[
\begin{bmatrix}
\frac{d^2}{dy^2} - \alpha^2 - i\alpha R(y - \xi) \\
\frac{d^2}{dy^2} - \alpha^2
\end{bmatrix}
\begin{bmatrix}
\phi \\
\phi'
\end{bmatrix} = 0
\]

\[
\phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0.
\]

Following a change of variable in order to apply their method of solution, Gallagher and Mercer tabulated \( \lambda \) given by

\[
\lambda = C - \frac{1}{2} \pi b i (1 + \xi)
\]

\[
C = -4\alpha^2 / \pi^2
\]

\[
b = -8\alpha R / \pi^3
\]

(See Section 6.3).

We have found \( \xi \) directly from the above differential equation and have then calculated and tabulated \( \lambda \) following Gallagher and Mercer.

When the eigen-function \( \phi(y) \) is tabulated, it is a solution of the above differential equation for \(-1 \leq y \leq 1\), and it was calculated directly, by applying Gaussian elimination to the matrix constructed by the Chebyshev recurrence relation.
In the ensuing results tabulated for $\lambda$, "*" indicates the results derived by Lee and Reynolds, who also reviewed the problem by using their variational scheme.
\( \alpha = 0.5 \)

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(iii) \( \alpha = 2.0 \) Con't.

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<th>Chebyshev</th>
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20 22.7923 + 72.2381i
30 22.8011 + 72.23551
It should be noted that on each set of calculations, the eigen-values, as expected, were either real, or complex conjugate pairs.

Remember "*" indicates the result obtained by Lee and Reynolds.

We now tabulate the values of the eigen-function \( \phi(y) \) for \(-1 \leq y \leq 1\) for three cases of \( \alpha \) and \( \alpha R \). In each case, the values were normalized so that \( \phi(0) = 1 \).
(i) \[ \alpha = 1, \quad \alpha_R = 1, \quad \lambda = 3.3698667 \]

Co-efficients in the expansion

\[ \phi(y) = \frac{1}{2} a_0 T_0(y) + \sum_{k=1}^{\infty} a_k T_k(y) \]

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Values of $\phi(y)$ for $-1 \leq y \leq 1$

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Values of $\phi(y)$ for $-1 \leq y \leq 1$: Con't.

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\( g = 4, \quad \alpha_R = 20, \quad \lambda = 3.0725401 \)

Co-efficients in the expansion

\[
\phi(y) = \frac{1}{2} a_0 T_0(y) + \sum_{k=1}^{\infty} a_k T_k(y)
\]

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Values of $\phi(y)$ for $-1 \leq y \leq 1$

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Values of $\phi(y)$ for $-1 \leq y \leq 1$: Con't.

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\( g = 8, \quad g_R = 300, \quad \lambda = 22.801104 + 72.235532i \)

Co-efficients in the expansion

\[ \phi(y) = \frac{1}{2}a_0 T_0(y) + \sum_{k=1}^{\infty} a_k T_k(y) \]

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(iii) $a = 8, \ a_R = 300, \ \lambda = 22.801104 + 72.235532i$: Con't.

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\( \alpha = 8, \quad \alpha R = 300, \quad \lambda = 22.801104 + 72.235532i: \) Cont.

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Values of $\phi(y)$ for $-1 \leq y \leq 1$

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Values of $\phi(y)$ for $-1 \leq y \leq 1$: Con't.

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Section 5.6 - Orr-Sommerfeld Equation

Plane Jet Flow

Clenshaw and Elliott (1960) considered the problem of the plane jet, by solving the Orr-Sommerfeld equation

\[(w - \lambda)(\phi'' - \alpha^2 \phi) - \frac{d^2 w}{dy^2} \phi = \frac{-i}{\alpha R} (\phi iv - 2\alpha^2 \phi'' + \alpha^4 \phi)\]

where \(w = \text{sech}^2 y\)

and \(\phi(\pm\infty) = \phi'(\pm\infty) = 0.\)

Following a change of variable \(t = \tanh y\) we obtain \(w = 1 - t^2\), and the problem is reduced to the solution of a differential equation with polynomial co-efficients on the interval \(-1 \leq t \leq 1\), and the solution is written as a series in terms of Chebyshev polynomials (see Section 6.3).

The calculations for this problem did not use directly the method of Chapter 4 in setting up the matrix, but instead, we used the results derived by Clenshaw and Elliott for the co-efficients in the series expansion, and applied the QR algorithm directly.
(a) \( \alpha = 1.0, \quad R = 12.58 \)

Using the QR algorithm for various \( n \), we obtained

\[
\begin{array}{c|c}
 n & \lambda \\
\hline
7 & 0.39130 - 0.00638i \\
12 & 0.39428 - 0.00136i \\
27 & 0.39348 - 0.000002i \\
\end{array}
\]

Clenshaw and Elliott: \( 0.3933 - 0i \)
Lee and Reynolds (Chebyshev): \( 0.39415 - 0.00072i \)
Lee and Reynolds (Variational): \( 0.39373 - 0.00046i \)

(b) \( \alpha = 1.5, \quad R = 36.21 \)

\[
\begin{array}{c|c}
 n & \lambda \\
\hline
12 & 0.54287 - 0.00005i \\
27 & 0.54286 - 0.000001i \\
\end{array}
\]

Clenshaw and Elliott: \( 0.5429 + 0i \)

In both cases, we see the imaginary part is close to zero, which is the actual value set by Clenshaw and Elliott.
Also, since Clenshaw and Elliott fixed $\alpha$ and the imaginary part of $\lambda(=0)$, the fact that we are using only the 4 significant figure published value for $R$, which was probably rounded from a more accurate result, did not allow us to obtain the imaginary part of $\lambda$ as exactly zero. However, the result we obtained is in far better agreement with Clenshaw and Elliott, than either result obtained by Lee and Reynolds.

The eigen-function in each case is calculated, and results are tabulated below.
(a) $\alpha = 1, \ R = 12.58, \ \lambda = -0.1065226 - 0.0000021$

Co-efficients in the expansion

$$\phi(y) = \frac{1}{4} a_0 T_0(y) + \sum_{k=1}^{\infty} a_{2k} T_{2k}(y)$$

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\( a = 1, \quad R = 12.58, \quad \lambda = -0.1065226 - 0.0000021; \quad \text{Con't.} \)

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Values of $\phi(y)$ for $0 \leq y \leq 1$

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$\phi(y)$ was normalised so that $\phi(0) = 1$. 
Co-efficients in the expansion

\[ \phi(y) = \frac{1}{2}a_0 T_0(y) + \sum_{k=1}^{\infty} a_{2k} T_{2k}(y) \]

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<th></th>
<th>REAL</th>
<th>IMAGINARY</th>
<th>MODULUS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2a_0</td>
<td>0.634838</td>
<td>0.045860</td>
<td>0.636493</td>
</tr>
<tr>
<td>a_2</td>
<td>-0.558705</td>
<td>-0.026034</td>
<td>0.559311</td>
</tr>
<tr>
<td>a_4</td>
<td>-0.151321</td>
<td>-0.039361</td>
<td>0.156356</td>
</tr>
<tr>
<td>a_6</td>
<td>0.051184</td>
<td>0.025268</td>
<td>0.057081</td>
</tr>
<tr>
<td>a_8</td>
<td>0.013443</td>
<td>-0.007706</td>
<td>0.015495</td>
</tr>
<tr>
<td>a_{10}</td>
<td>0.006084</td>
<td>0.000170</td>
<td>0.006087</td>
</tr>
<tr>
<td>a_{12}</td>
<td>0.002152</td>
<td>0.000989</td>
<td>0.002369</td>
</tr>
<tr>
<td>a_{14}</td>
<td>0.000786</td>
<td>0.000535</td>
<td>0.000951</td>
</tr>
<tr>
<td>a_{16}</td>
<td>0.000378</td>
<td>0.000212</td>
<td>0.000433</td>
</tr>
<tr>
<td>a_{18}</td>
<td>0.000245</td>
<td>0.000070</td>
<td>0.000255</td>
</tr>
<tr>
<td>a_{20}</td>
<td>0.000187</td>
<td>0.000018</td>
<td>0.000188</td>
</tr>
<tr>
<td>a_{22}</td>
<td>0.000150</td>
<td>0.000002</td>
<td>0.000150</td>
</tr>
<tr>
<td>a_{24}</td>
<td>0.000121</td>
<td>-0.000002</td>
<td>0.000121</td>
</tr>
</tbody>
</table>
(b) \( \alpha = 1.5, \quad R = 36.21, \quad \lambda = 0.04286129 - 0.00000121 \): Con't.

<table>
<thead>
<tr>
<th>REAL</th>
<th>IMAGINARY</th>
<th>MODULUS</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_{26}</td>
<td>0.000099</td>
<td>-0.000002</td>
</tr>
<tr>
<td>a_{28}</td>
<td>0.000079</td>
<td>-0.000002</td>
</tr>
<tr>
<td>a_{30}</td>
<td>0.000064</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{32}</td>
<td>0.000051</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{34}</td>
<td>0.000041</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{36}</td>
<td>0.000032</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{38}</td>
<td>0.000025</td>
<td>-0.000002</td>
</tr>
<tr>
<td>a_{40}</td>
<td>0.000019</td>
<td>-0.000002</td>
</tr>
<tr>
<td>a_{42}</td>
<td>0.000015</td>
<td>-0.000002</td>
</tr>
<tr>
<td>a_{44}</td>
<td>0.000011</td>
<td>-0.000002</td>
</tr>
<tr>
<td>a_{46}</td>
<td>0.000008</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{48}</td>
<td>0.000005</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{50}</td>
<td>0.000003</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{52}</td>
<td>0.000002</td>
<td>-0.000001</td>
</tr>
<tr>
<td>a_{54}</td>
<td>0.000001</td>
<td>-0.000000</td>
</tr>
</tbody>
</table>
Values of $\phi(y)$ for $0 \leq y \leq 1$

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\phi$ REAL</th>
<th>$\phi$ IMAGINARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>1.007546</td>
<td>0.009107</td>
</tr>
<tr>
<td>0.2</td>
<td>1.026818</td>
<td>0.033567</td>
</tr>
<tr>
<td>0.3</td>
<td>1.048093</td>
<td>0.065482</td>
</tr>
<tr>
<td>0.4</td>
<td>1.056359</td>
<td>0.093929</td>
</tr>
<tr>
<td>0.5</td>
<td>1.032987</td>
<td>0.108267</td>
</tr>
<tr>
<td>0.6</td>
<td>0.957776</td>
<td>0.102387</td>
</tr>
<tr>
<td>0.7</td>
<td>0.811283</td>
<td>0.078799</td>
</tr>
<tr>
<td>0.8</td>
<td>0.578718</td>
<td>0.049232</td>
</tr>
<tr>
<td>0.9</td>
<td>0.265922</td>
<td>0.024016</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

$\phi(y)$ was normalised so that $\phi(0) = 1$. 
Section 5.7 - Conclusion

Section 5.7.1 - Matrix Eigen-values

Because of its stability, the QR algorithm was chosen as a method for determining the eigen-values of the matrices formed from the physical problem at hand. In order to deal with complex matrices, the algorithm was modified from that described by Parlett (1967). (See Chapter 2).

Section 5.7.2 - Continuous to Discrete

To actually obtain the matrix from the continuous differential equation, a method using Chebyshev series was set out in Chapter 4, and the results are set out in the earlier sections of this Chapter.
Section 5.7.3 - Eigen-values

For the two test cases considered, 7 significant figures of accuracy were obtained for the first eigen-value, and for a matrix of size \( n \), this accuracy was usually held for at least the first \( \frac{1}{3}n \) eigen-values. The modified rational finite difference approach in most cases only gave 4 significant figures of accuracy for even the first 2 eigen-values.

Bearing this order of accuracy in mind, we observe at least 3 figures of agreement in most of the Orr-Sommerfeld problems as published by other authors. Probably the worst agreement of the Chebyshev method, and the best agreement of the rational method with the published results, was in the Thomas problem for \( \alpha = 1, \ R = 100 \). This is probably due to consistency of finite difference type methods (bearing in mind, that Thomas used finite differences).

Otherwise the Chebyshev method gave very consistent accuracy with the published results, particularly with those results surveyed by Lee and Reynolds.
Section 5.7.4 - Eigen-functions

When the eigen-function is calculated, it should be noted that the Chebyshev method leads to a solution over the whole interval \(-1 \leq y \leq 1\), whereas the finite difference method leads to solutions only at a finite set of points.

The eigen-functions for the first two eigen-values for each test problem were accurate to all stated figures.

The eigen-function for the Thomas problem \(a = 1, R = 10,000\) agreed very well with the corresponding results of Thomas.

For the two eigen-function determinations for the Gallagher and Mercer problems where \(\lambda\) was real, we note the form \(\phi(y) = a + ib\) for \(0 \leq y \leq 1\) and \(\phi(y) = a - ib\) for \(0 \geq y \geq -1\), as we would normally expect from consideration of the given differential equation.
The justification for the use of the Chebyshev series method is the fact that the series converges in all cases considered, and the convergence was such that sufficient accuracy could be obtained using only a small number of terms.

An idea of the rapidity of convergence of the Chebyshev method may be obtained by considering the number of terms of the series obtained when evaluating the eigen-functions. In no cases, did the problem ever show signs of instability, and even if the series was slow to converge (e.g. Clenshaw and Elliott, $\alpha = 1.5$, $R = 36.21$) the convergence was smooth.

The combination of the Chebyshev series method of converting the continuous analytic problem to the discrete algebraic problem and the QR algorithm for finding the eigenvalues of the matrix, has proven to be a very stable, and quickly convergent method.
CHAPTER 6

Appendix

In Section 1 of this Chapter we discuss the analytic solution to the two problems referred to as "Test Problems" in earlier sections.

In Section 2, the application of finite difference methods to the two test problems is discussed, and it is shown how some of the results can be found directly.

Section 3 contains brief outlines of solutions to the Orr-Sommerfeld problems, as presented by Thomas, Gallagher and Mercer, Clenshaw and Elliott, and Lee and Reynolds.
Section 6.1 - Analytic Solutions to Test Problems

Section 6.1.1 - The Problem \( \phi'''' - \lambda \phi = 0 \)

The general solution to the differential equation is 
\[ \phi(y) = A \cosh \alpha y + B \cos \alpha y + C \sinh \alpha y + D \sin \alpha y \]
where
\[ \alpha = \lambda^{\frac{1}{4}}. \]

Applying the boundary conditions in the order
\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0 \]
we obtain
\[
\begin{align*}
0 &= A \cosh \alpha + B \cos \alpha + C \sinh \alpha + D \sin \alpha \\
0 &= A \sinh \alpha - B \sin \alpha + C \cosh \alpha + D \cos \alpha \\
0 &= A \cosh \alpha + B \cos \alpha + C \sinh \alpha - D \sin \alpha \\
0 &= -A \sinh \alpha + B \sin \alpha + C \cosh \alpha - D \cos \alpha
\end{align*}
\]

(6.1)

For non trivial solutions in \( A, B, C \) and \( D \), then

\[
\begin{vmatrix}
\cosh \alpha & \cos \alpha & \sinh \alpha & \sin \alpha \\
\sinh \alpha & -\sin \alpha & \cosh \alpha & \cos \alpha \\
\cosh \alpha & \cos \alpha & -\sinh \alpha & -\sin \alpha \\
-\sinh \alpha & \sin \alpha & \cosh \alpha & \cos \alpha \\
\end{vmatrix}
= 0.
\]
Solving this we get
\[ \tan \alpha = \pm \tanh \alpha, \]
which give the fourth roots of the eigen-values.

If we want only an even solution, then put
\[ \alpha = \beta = \gamma \]
in the general solution, and then from the first two equations in (6.1)
\[ \tan \alpha = -\tanh \alpha \quad (6.2) \]

Similarly for the odd solution,
\[ \tan \alpha = \tanh \alpha \quad (6.3) \]

The roots of (6.2) and (6.3) give the fourth roots of the eigen-values for the even and odd solution of the differential equations.

Numerical solutions to (6.2) and (6.3) respectively giving the first 16 even and odd eigen-values correct to 8 significant figures are set out below:
<table>
<thead>
<tr>
<th>EVEN</th>
<th>ODD</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1285243 x 10</td>
<td>2.3772106 x 10^2</td>
</tr>
<tr>
<td>5.1366188 x 10^2</td>
<td>2.4364874 x 10^3</td>
</tr>
<tr>
<td>5.5709629 x 10^3</td>
<td>1.0067582 x 10^4</td>
</tr>
<tr>
<td>1.9253028 x 10^4</td>
<td>3.1730966 x 10^4</td>
</tr>
<tr>
<td>4.9887695 x 10^4</td>
<td>7.4003949 x 10^4</td>
</tr>
<tr>
<td>1.0648969 x 10^5</td>
<td>1.4863447 x 10^5</td>
</tr>
<tr>
<td>2.0221556 x 10^5</td>
<td>2.6912343 x 10^5</td>
</tr>
<tr>
<td>3.5140367 x 10^5</td>
<td>4.5124799 x 10^5</td>
</tr>
<tr>
<td>5.7099420 x 10^5</td>
<td>7.1312524 x 10^5</td>
</tr>
<tr>
<td>8.8327416 x 10^5</td>
<td>1.0752141 x 10^6</td>
</tr>
<tr>
<td>1.3383683 x 10^6</td>
<td>1.5603052 x 10^6</td>
</tr>
<tr>
<td>1.8557394 x 10^6</td>
<td>2.1335313 x 10^6</td>
</tr>
<tr>
<td>2.5741578 x 10^6</td>
<td>3.9023616 x 10^6</td>
</tr>
<tr>
<td>3.4213518 x 10^6</td>
<td>4.0166033 x 10^6</td>
</tr>
<tr>
<td>4.6137075 x 10^6</td>
<td>5.2684015 x 10^6</td>
</tr>
<tr>
<td>5.9940637 x 10^6</td>
<td>6.7822388 x 10^6</td>
</tr>
</tbody>
</table>
The exact eigen-functions corresponding to the first two even eigen-values are displayed below (for \(0 \leq y \leq 1\)):

<table>
<thead>
<tr>
<th>(y)</th>
<th>(\lambda = 31.285243)</th>
<th>(\lambda = 913.60168)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.978723</td>
<td>0.850678</td>
</tr>
<tr>
<td>0.2</td>
<td>0.916446</td>
<td>0.446915</td>
</tr>
<tr>
<td>0.3</td>
<td>0.817762</td>
<td>-0.094639</td>
</tr>
<tr>
<td>0.4</td>
<td>0.690113</td>
<td>-0.617805</td>
</tr>
<tr>
<td>0.5</td>
<td>0.543484</td>
<td>-0.574569</td>
</tr>
<tr>
<td>0.6</td>
<td>0.390010</td>
<td>-1.072424</td>
</tr>
<tr>
<td>0.7</td>
<td>0.243521</td>
<td>-0.901572</td>
</tr>
<tr>
<td>0.8</td>
<td>0.119072</td>
<td>-0.547691</td>
</tr>
<tr>
<td>0.9</td>
<td>0.032492</td>
<td>-0.175633</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
The exact eigen-functions corresponding to the first two odd eigen-values are displayed below (for $0 \leq y \leq 1$);

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\lambda = 237.72106$</th>
<th>$\lambda = 2496.4874$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>9.987632</td>
<td>380.839804</td>
</tr>
<tr>
<td>0.2</td>
<td>18.542855</td>
<td>578.643262</td>
</tr>
<tr>
<td>0.3</td>
<td>24.464967</td>
<td>457.300303</td>
</tr>
<tr>
<td>0.4</td>
<td>26.983652</td>
<td>175.512685</td>
</tr>
<tr>
<td>0.5</td>
<td>25.396666</td>
<td>-236.333293</td>
</tr>
<tr>
<td>0.6</td>
<td>21.628917</td>
<td>-547.797391</td>
</tr>
<tr>
<td>0.7</td>
<td>15.208305</td>
<td>-620.326288</td>
</tr>
<tr>
<td>0.8</td>
<td>8.163354</td>
<td>-446.172913</td>
</tr>
<tr>
<td>0.9</td>
<td>2.401350</td>
<td>-158.719063</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

The solutions given here and those given in Section 5.2 are in constant ratio $10.99539$ for $\lambda = 237.72106$ and $216.7313$ for $\lambda = 2496.4874$. 
Section 6.1.2 - The Problem \( \frac{\lambda v}{\lambda} + \lambda \phi'' = 0 \).

The general solution to the differential equation is

\[
\phi = A \cos \alpha y + B + C \sin \alpha y + D y
\]

where

\[
\alpha = \sqrt{\lambda}
\]

Applying the boundary conditions in the order

\[
\phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0
\]

we obtain

\[
\begin{align*}
0 &= A \cos \alpha + B + C \sin \alpha + D \\
0 &= -A \alpha \sin \alpha + C \alpha \cos \alpha + D \\
0 &= A \cos \alpha + B - C \sin \alpha - D \\
0 &= A \alpha \sin \alpha + C \alpha \cos \alpha + D
\end{align*}
\]

(6.4)

For non-trivial solutions in \( A, B, C \) and \( D \),

\[
\begin{vmatrix}
\cos \alpha & 1 & \sin \alpha & 1 \\
-\alpha \sin \alpha & 0 & \alpha \cos \alpha & 1 \\
\cos \alpha & 1 & -\sin \alpha & -1 \\
\alpha \sin \alpha & 0 & \alpha \cos \alpha & 1
\end{vmatrix} = 0.
\]

Solving this, we obtain

\[
4 \alpha \sin \alpha (\sin \alpha - \alpha \cos \alpha) = 0.
\]
For non-trivial solutions

$$\sin \alpha (\sin \alpha - \alpha \cos \alpha) = 0$$

\[ \therefore \quad \alpha = n\pi, \quad n = 1, 2, \ldots. \]

or \( \alpha = \tan \alpha \)

For an even solution, put

\[ C = D = 0 \]

in the general solution, and from the first two equations of (6.4),

\[ \alpha = n\pi \]

\[ \therefore \quad \lambda = n^2\pi^2, \quad n = 1, 2, \ldots. \] (6.5)

Similarly, for the odd solutions,

\[ \alpha = \tan \alpha \] (6.6)
The first 16 solutions of these equations are set out below:

<table>
<thead>
<tr>
<th>EVEN</th>
<th>ODD</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.8696044</td>
<td>2.0190728 × 10</td>
</tr>
<tr>
<td>3.9478417 × 10</td>
<td>5.9679516 × 10</td>
</tr>
<tr>
<td>8.8826433 × 10</td>
<td>1.1889387 × 10²</td>
</tr>
<tr>
<td>1.5791367 × 10²</td>
<td>1.9785782 × 10²</td>
</tr>
<tr>
<td>2.4674011 × 10²</td>
<td>2.9655441 × 10²</td>
</tr>
<tr>
<td>3.5530575 × 10²</td>
<td>4.1498998 × 10²</td>
</tr>
<tr>
<td>4.8361061 × 10²</td>
<td>5.5316465 × 10²</td>
</tr>
<tr>
<td>6.3165468 × 10²</td>
<td>7.1107845 × 10²</td>
</tr>
<tr>
<td>7.9943795 × 10²</td>
<td>8.883142 × 10²</td>
</tr>
<tr>
<td>9.8696044 × 10²</td>
<td>1.0861236 × 10³</td>
</tr>
<tr>
<td>1.1942221 × 10³</td>
<td>1.3032549 × 10³</td>
</tr>
<tr>
<td>1.4212230 × 10³</td>
<td>1.5401254 × 10³</td>
</tr>
<tr>
<td>1.6679631 × 10³</td>
<td>1.7967352 × 10³</td>
</tr>
<tr>
<td>1.9344424 × 10³</td>
<td>2.0730642 × 10³</td>
</tr>
<tr>
<td>2.2206609 × 10³</td>
<td>2.3591723 × 10³</td>
</tr>
<tr>
<td>2.5266187 × 10³</td>
<td>2.6849397 × 10³</td>
</tr>
</tbody>
</table>
The exact eigen-functions for the first two even eigen-values are displayed below (for $0 \leq y \leq 1$):

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\lambda = 2.826044$</th>
<th>$\lambda = 39.78417$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.975528</td>
<td>0.095492</td>
</tr>
<tr>
<td>0.2</td>
<td>0.904508</td>
<td>0.345492</td>
</tr>
<tr>
<td>0.3</td>
<td>0.793893</td>
<td>0.654509</td>
</tr>
<tr>
<td>0.4</td>
<td>0.654508</td>
<td>0.904509</td>
</tr>
<tr>
<td>0.5</td>
<td>0.500000</td>
<td>1.000000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.345492</td>
<td>0.904509</td>
</tr>
<tr>
<td>0.7</td>
<td>0.206107</td>
<td>0.654509</td>
</tr>
<tr>
<td>0.8</td>
<td>0.095492</td>
<td>0.345492</td>
</tr>
<tr>
<td>0.9</td>
<td>0.024472</td>
<td>0.095492</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
The exact eigen-functions for the first two odd eigen-values are given below (for $0 \leq y \leq 1$):

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\lambda = 20.190729$</th>
<th>$\lambda = 59.67952$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.531934</td>
<td>0.598773</td>
</tr>
<tr>
<td>0.2</td>
<td>0.977731</td>
<td>0.301323</td>
</tr>
<tr>
<td>0.3</td>
<td>1.268124</td>
<td>0.436363</td>
</tr>
<tr>
<td>0.4</td>
<td>1.364891</td>
<td>-0.345221</td>
</tr>
<tr>
<td>0.5</td>
<td>1.268193</td>
<td>-1.156024</td>
</tr>
<tr>
<td>0.6</td>
<td>1.016623</td>
<td>-1.592054</td>
</tr>
<tr>
<td>0.7</td>
<td>0.679490</td>
<td>-1.462078</td>
</tr>
<tr>
<td>0.8</td>
<td>0.343109</td>
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The solutions given here and those given in Section 5.3 are in constant ratio $0.517559$ for $\lambda = 20.190729$ and $0.613575$ for $\lambda = 59.67952$. 
Section 6.2 - Direct Application of Finite Differences

Section 6.2.1 - Solution to \( \phi^{iv} + \lambda \phi''' = 0 \).

Using

\[
\begin{align*}
\delta^4_D &= \frac{\delta^4}{1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4} \\
\delta^2_D &= \frac{\delta^2 + \frac{1}{12} \delta^4}{1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4}
\end{align*}
\]

(see Chapter 3), the equation

\[
\phi^{iv} + \lambda \phi''' = 0
\]

becomes

\[
\left\{ \begin{array}{c}
\delta^4 \\
1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4
\end{array} + \lambda h^2 \quad \begin{array}{c}
\delta^2 \\
1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4
\end{array} \right\} \phi_r = 0.
\]

Writing

\[
\phi_r = (1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4) \psi_r
\]

we obtain

\[
\delta^4 \psi_r + \lambda h^2 (1 + \frac{1}{12} \delta^2) \delta^2 \psi_r = 0
\]

whose general solution is

\[
\psi_r = A + B \cos \theta + C r + D \sin \theta
\]
where

\[ 2 \cos \theta = \frac{2 - \frac{5}{6} \lambda h^2}{1 + \frac{1}{12} \lambda h^2} \]  

(6.8)

If we consider only the even solution, i.e.

\[ \psi_r = A + B \cos \theta \]

the boundary conditions

\[ \phi(1) = \phi'(1) = 0 \]

give

\[ A + B \cos(n-1)\theta = A + B \cos(n+1)\theta \]  

(6.9)

and

\[ A + B \cos(n-2)\theta = 124A + 124B \cos(n-1)\theta \]
\[ + 237A + 237B \cos n\theta \]  

(6.10)

using results in Chapter 3.

It is possible to solve directly for the eigen-values from (6.9).

\[ \cos (n-1) \theta - \cos (n+1) \theta = 0 \]

\[ \therefore 2 \sin \theta \sin n\theta = 0 \]

and for non-trivial solutions, \( n \theta = k\pi \).

i.e. \( \theta = \frac{k\pi}{n} = k\pi h \), \( k = 1, 2, \ldots, n \).
Using this result and (5.8), the eigen-values are determined by

\[ \lambda^* = \frac{12(1 - \cos k\pi h)}{h^2(5 + \cos k\pi h)} \]

\[ = \frac{6}{6 - 2 \sin^2(lk\pi h)} \cdot \frac{\sin^2(lk\pi h)}{(lk\pi h)^2} \cdot k^2 \pi^2 \]

\[ k = 1, 2, \ldots, n \quad (6.11) \]

To determine the corresponding eigen-functions for this problem, consider (6.10) which gives

\[ A = \frac{(-1)^{k-1}}{360} \left\{ 237 + 124 \cos \frac{k\pi}{n} - \cos \frac{2k\pi}{n} \right\} B \]

Then we can write

\[ \psi_r = \frac{(-1)^k}{360} \left\{ 237 + 124 \cos \frac{k\pi}{n} - \cos \frac{2k\pi}{n} \right\} + \cos r \frac{k\pi}{n} \]

Substitution of this into (6.7) yields

\[ \phi_r = \frac{(-1)^{k-1}}{360} \left\{ 237 + 124 \cos \frac{k\pi}{n} - \cos \frac{2k\pi}{n} \right\} + \]
\[ \cos r \frac{k\pi}{n} \left\{ 1 + \frac{1}{3} \left( \cos \frac{k\pi}{n} - 1 \right) + \frac{1}{180} \left( \cos \frac{k\pi}{n} - 1 \right)^2 \right\} \quad (6.12) \]

where

\[ \begin{cases} k = 1, 2, \ldots, n \\
   r = 0, 1, \ldots, n. \end{cases} \]
The more elementary approach using standard finite differences instead of rational finite differences gives

\[ \delta^4 \phi + \lambda h^2 \delta^2 \phi = 0 \]

\[ \therefore \phi_{r-2} - (4 - \lambda h^2) \phi_{r-1} + (6 - 2\lambda h^2) \phi_r - (4 - \lambda h^2) \phi_{r+1} \]

\[ + \phi_{r+2} = 0 \]

whose general solution is

\[ \phi_r = A + B \cos r \theta + C r + D \sin r \theta \]

where in this case,

\[ 2 \cos \theta = 2 - \lambda h^2 \]

Again, considering only the even solution,

\[ \phi_r = A + B \cos r \theta \]

the boundary conditions

\[ \phi(1) = \phi'(1) = 0 \]

give

\[ \phi_n = 0 = A + B \cos n \theta \]

(6.14)

and

\[ A + B \cos(n-1)\theta = A + B \cos(n+1)\theta \]

(6.15)
Directly from (6.15), we obtain

\[ n\theta = k\pi, \text{ i.e. } = \frac{k\pi}{n} = k\pi h, \quad k = 1, 2, \ldots, n \]

Hence from (6.13)

\[ \lambda^{**} = \frac{2}{h^2} (1 - \cos k\pi h) \]

\[ = \frac{\sin^2\left(\frac{1}{2}k\pi h\right)}{(\frac{1}{2}k\pi h)^2}, k = 1, 2, \ldots, n \]

(6.16)

From (6.11) and (6.16) it is obvious that

\[ \lim_{h \to 0} \lambda^{*} = \lim_{h \to 0} \lambda^{**} = k^2\pi^2 \]

and that

\[ \lambda^{**} \leq \lambda^{*} < k^2\pi^2 \]

Thus both the simple finite difference method and the rational finite difference method give approximations which tend to the exact eigen-value from below, with the rational difference method giving the better approximation.

For \( n = 12, 16, 27 \) the approximations to the first two eigen-values in the even case, as calculated from (6.11) are:
These are the same as calculated using the complete theory of Chapter 3 followed by the calculation of the eigenvalues of the resulting matrix. This agreement also occurs for the later entries. See the tables in Section 5.3 for the complete set of results.

The $h^k$-extrapolation of these results is discussed in Chapter 5.
Section 6.2.2 - Solution to $\phi^{iv} - \lambda \phi = 0$

This time we first consider the simple finite difference approximation

$$h^4 D^4 \equiv \delta^4$$

This problem leads to

$$\phi_{r-2} - 4\phi_{r-1} + (6 - \lambda h^4)\phi_r - 4\phi_{r+1} + \phi_{r+2} = 0.$$  

If an even solution is sought, try

$$\phi_r = \cos r \theta.$$  

Then,

$$2 \cos r \theta \cos 2\theta - 8 \cos r \theta \cos \theta + (6 - \lambda h^4) \cos r \theta = 0$$

which gives

$$2 \cos 2\theta - 8 \cos \theta + (6 - \lambda h^4) = 0$$

(assuming $\cos r \theta \neq 0$)

so that

$$\cos \theta = 1 \pm \sqrt[2]{\lambda} h^2.$$  

Put

$$\cos \theta_1 = 1 - \sqrt[2]{\lambda} h^2$$

(6.18)
and

$$\cos \theta_2 = 1 + \frac{1}{2\sqrt{\lambda}} h^2.$$  

Clearly $\theta_2$ is complex. We may write

$$\theta_2 = i\xi$$

whence

$$\cosh \xi = 1 + \frac{1}{2\sqrt{\lambda}} h^2 \quad (6.19)$$

Let us now write

$$\theta = \theta_1.$$  

Thus

$$\phi_r = A \cos r \theta + B \cosh r \xi$$

where $\theta$ and $\xi$ are given by (6.18) and (6.19).

Applying the usual boundary conditions for the even solution, viz.

$$\phi(1) = \phi'(1) = 0$$

we get

$$A \cos n \theta + B \cosh n \xi = 0$$

and
A \cos(n-1)\theta + B \cosh(n-1)\xi = A \cos(n+1)\theta + B \cosh(n+1)\xi, 

the second reducing to 

A \sin n \theta \sin \theta - B \sinh n \xi \sinh \xi = 0.

Eliminating \( A \) and \( B \), for non-trivial solutions we get 

\[-\cos n \theta \sinh n \xi \sinh \xi = \cosh n \xi \sin n \theta \sin \theta \quad (6.20)\]

and from (6.17) we get 

\[\cos \theta + \cosh \xi = 2 \quad (6.21)\]

We can eliminate \( \xi \) between (6.21) and (6.20), and then apply Newton's method, or the method of regula falsi to determine \( \theta \), and then obtain \( \lambda \) from (6.18).

Now applying rational differences as in Section 6.2.1, we get, using the substitution (6.7)

\[\delta^4 \psi_r - \lambda h^4 \left( 1 + \frac{1}{6} \delta^2 - \frac{1}{720} \delta^4 \right) \psi_r = 0\]

giving

\[(1 + \frac{\lambda h^4}{720}) \psi_{r-2} - (4 + \frac{31\lambda h^4}{180}) \psi_{r-1} + (6 - \frac{79\lambda h^4}{120}) \psi_r - (4 + \frac{31\lambda h^4}{180}) \psi_{r+1} + (1 + \frac{\lambda h^4}{720}) \psi_{r+2} = 0.\]
For the even solution, try

\[ \psi_r = \cos r \theta \]

giving, when \( \cos r \theta \neq 0 \),

\[ 2\left(1 + \frac{\lambda h^4}{720}\right) \cos 2\theta - 2\left(4 + \frac{31 \lambda h^4}{180}\right) \cos \theta + (6 - \frac{79 \lambda h^4}{120}) = 0 \]

\[ \therefore \left(1 + \frac{\lambda h^4}{720}\right) \cos^2 \theta - 2\left(1 + \frac{31 \lambda h^4}{720}\right) \cos \theta + (1 - \frac{119 \lambda h^4}{720}) = 0 \]

(6.22)

Applying the usual boundary conditions, we get

\[ A \sin n \theta \sin \theta - B \sinh n \xi \sinh \xi = 0 \]

and

\[ A \cos n \theta (237 + 124 \cos \theta - \cos 2\theta) + \]

\[ B \cosh n \xi (237 + 124 \cosh \xi - \cosh 2\xi) = 0 \]

where \( \theta \) and \( \xi \) are real.

Eliminating \( A \) and \( B \), non-trivial solutions will exist if

\[ \begin{align*}
-\cos n \theta (237 + 124 \cos \theta - \cos 2\theta) \sinh n \xi \sinh \xi \\
= \cosh n \xi (237 + 124 \cosh \xi - \cosh 2\xi) \sin n \theta \sin \theta
\end{align*} \]

Also from (6.22) (5.23)
\[
\begin{align*}
\cos \theta + \cosh \xi &= \frac{2(1 + \frac{31 \lambda h^4}{720})}{(1 + \frac{\lambda h^4}{720})} \\
\cos \theta \cosh \xi &= \frac{(1 - \frac{119 \lambda h^4}{720})}{(1 + \frac{\lambda h^4}{720})}
\end{align*}
\]

so that the method of regula falsi may be used on (6.23) to determine \( \lambda \).

For \( n = 12, 18 \) and 27, the approximations to the first two eigen-values in the even case, as calculated from (6.23), are

\[
\begin{array}{ccc}
n = 12 & n = 18 & n = 27 \\
31.286328 & 31.285565 & 31.285339 \\
913.77301 & 913.65392 & 913.61742
\end{array}
\]

These are the same as calculated by the method of Chapter 3 followed by the calculation of the eigen-values of the resulting matrix. See Section 5.2 for the complete set of results.
Section 6.3 - Outline of Other Methods

In this section, we briefly outline methods used by other authors in their approaches to solving various forms of the Orr-Sommerfeld equations. We have applied the method of Chapters 2, 3 and 4 to these problems, and used our results as a method of comparison with other authors.
Section 6.3.1 - The Method of Thomas (1953)

Thomas considers the Orr-Sommerfeld equation in the form

\[ i \psi'' - 2a^2 \psi'' + a^4 \psi + i a R ((1 - \lambda - y^2)(\psi'' - a^2 \psi) + 2\psi) = 0 \]  

(6.24)

where

\[ \psi(l) = \psi'(l) = \psi(-l) = \psi'(-l) = 0 \]

where \( R \) is the Reynolds number of the fluid flow, \( \frac{2\pi}{a} \) is the wave-length of the disturbance, whose profile is given by \( \psi \), and \( \lambda/U_0 \) is the complex velocity of the disturbance, where \( U_0 \) is the speed of the undisturbed flow in the \( x \) direction. This equation describes Plane Poiseuille flow, and is solved by Thomas using a simple finite difference method, incorporating the Gauss-Jackson-Numerov method.
Section 6.3.2 - The Method of Gallagher and Mercer (1962)

This method deals with Plane Couette flow between two plates, which is given by the equation

$$\left[ \frac{d^2}{dy^2} - \alpha^2 - i\alpha R(y - \lambda) \right] \left[ \frac{d^2}{dy^2} - \alpha^2 \right] \phi = 0 \quad (6.25)$$

with

$$\phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0.$$ 

This equation was set up by Lin (1955). After a change of variable

$$y_1 = \frac{1}{2\pi} (y+1)$$

Gallagher and Mercer use the Galerkin method to solve the equation, with a set of orthogonal functions defined by

$$\frac{d^4 Y}{dy^4} = \mu^4 Y$$

(see Chandrasekhar and Reid (1957)).

The eigen-values of (6.25) are always either real, or pairs of complex conjugates.
Section 6.3.3 - The Method of Clenshaw and Elliott (1960)

In studying the flow of a laminar jet, the equation to be considered is

\[
(w - \lambda) \left( \frac{d^2 \phi}{dy^2} - \alpha^2 \phi \right) - \frac{d^2 w}{dy^2} = \frac{-i}{\alpha R} \left( \frac{d^4 \phi}{dy^4} - 2 \alpha^2 \frac{d^2 \phi}{dy^2} + \alpha^4 \phi \right) \tag{6.26}
\]

Clenshaw and Elliott determine the eigen-values by a Chebyshev series solution after changing the variable from \( y \) to \( t \) using

\[
t = \tanh y
\]

which alters the range from \((-\infty, \infty)\) to \((-1, 1)\), this being necessary for Chebyshev polynomials by the definition (4.1).

The system of equations derived is

\[
A(k) a_{k-4} + B(k) a_{k-2} + C(k) a_k + B(-k) a_{k+2} + A(-k) a_{k+4} = 0 \tag{6.27}
\]

where

\[
A(k) = -k(k-1)(k-6) + i(k-1)(k-2)(k-3)(k-4)
\]

\[
B(k) = 2K(k^2-4k) + 4K\alpha^2 - \beta(k-1)(k-2) - 4i(k-1)(k-2)(k^2-2k+2+2\alpha^2)
\]

\[
C(k) = 2K(2-k^2) + 2\beta(k^2+2\alpha^2) + 2i(3k^4+5k^2+8k^2\alpha^2+8\alpha^4)
\]
and

\[ K = \alpha R, \quad \beta = 2K(2\lambda - 1). \]

(6.27) was solved by a method outlined in Chapter 5 of Fox and Parker (Chebyshev Polynomials in Numerical Analysis) and we compared the results by using the QR algorithm on the matrix set up after applying the usual boundary conditions to (6.27).
Section 6.3.4 - The Method of Lee and Reynolds (1967)

The method used by Lee and Reynolds is a variational scheme, and is applied to each of the equations of Sections 6.3.1, 6.3.2 and 6.3.3 in turn. The method uses adjoint operators, and for the Orr-Sommerfeld equations, such an operator is developed in an article by Stuart (1960).

Approximating functions used for the various equations were:

Thomas problem: \( f_n = (1-y^2)^2 y^{2(n-1)} \)

Gallagher and Mercer: \( f_n = (1-y^2)^2 y^{n-1} \)

Clenshaw and Elliott: \( f_n = \text{sech}^n ay \)
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Note: The table is truncated for brevity. The full table would continue with similar entries for each row, representing various mathematical expressions and calculations.
SUPPLEMENT TO Ph.D. THESIS

"EIGENVALUES BY NUMERICAL METHODS"

by

ALLYN GRAHAME MORRIS

B.Sc. (N'cle)

JUNE, 1974
The main theme of the thesis was the study of numerical methods for the calculation of eigenvalues of boundary value problems. The process of calculation involved two steps:

(i) conversion of the continuous differential equation which is linear in the eigenvalue to a discrete matrix problem; and
(ii) the calculation of the eigenvalues of the resultant matrix.

By restricting consideration to problems where the coefficients of the derivatives in the differential equation were polynomials in the independent variable, and where the domain of the independent variable was the interval \(-1 \leq y \leq 1\), it was possible to derive a Chebyshev polynomial approach to (i). It should be noted that the boundary value problem was allowed to be complex valued, and the boundary conditions imposed were

\[ \phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0. \]

Then a modified version of the QR algorithm for complex matrices was used to determine the eigenvalues of the problem in step (ii).

Step (i) was also performed using rational finite differences, and results by both methods compared.

Once the not unconsiderable amount of manual work for constructing the recurrence relations for the boundary value problem had been performed, the computer calculations for step (i) were minimal, and the bulk of the calculations were in step (ii). The ratio of time taken to
perform steps (i) and (ii) was approximately 1:20. Early calculations were performed in FORTRAN II on the IBM 1620 computer at Wollongong University College, in 12 digit precision. However, this computer restricted the matrix size to $10 \times 10$ (ie. 10 non zero terms of the Chebyshev series expansion), and the calculation time was also very slow.

Later calculations were computed in FORTRAN IV on the IBM 360-50 computer at the Australian Atomic Energy Commission at Lucas Heights. 16 digit precision was used here, and for a typical $27 \times 27$ matrix, total time for calculation of all the eigenvalues was approximately 50 secs.

As there was no noticeable time difference in step (i) for either the Chebyshev or the finite difference method, comparison of methods lies in the accuracy of the results.

Two "test" problems, which have known analytic solutions, were first tested using various sized matrices, viz.

$$\phi^{iv} + \lambda \phi'' = 0$$

$$\phi^{iv} - \lambda \phi = 0$$

and odd, even and general solutions were individually computed. In each case, the Chebyshev method proved superior, the process having converged to at least 8 significant figures in the first (smallest) eigenvalue for a matrix of order 8. (ie. for the use of approximately 8 non-zero terms in the Chebyshev expansion.)
As the formulation of the problem allowed for complex valued functions, it was decided to further test the technique on some well-known, complex valued physical problems. The fourth order differential equations of fluid dynamics referred to as the Orr-Sommerfeld problems lent themselves to this situation very appropriately, as they had polynomial coefficients and were linear in the eigenvalue.

The types of flow considered as examples have been

(i) Plane Couette flow;
(ii) Plane Poiseuille flow; and
(iii) Plane Jet flow.

Different techniques have been used by relevant authors in their solutions to their respective problems. One method, that used by Clenshaw and Elliott, also used Chebyshev polynomials, but calculations were done by an iterative method. For the solution of a physical problem, this method is perhaps quite suitable, as often only the first eigenvalue is required, but the present method is more general and all the eigenvalues are known immediately, if necessary. There are many other techniques available for the calculation of just the first few eigenvalues of a matrix. Even when determining all the eigenvalues of a matrix, the QR algorithm is slower than, say, the LR algorithm. However, in a situation which requires all the eigenvalues, the QR algorithm has the advantage of stability. As well, the convergence rate of this method can be increased by a suitable choice of a shift of the origin.
Although in the thesis, calculations were restricted to fourth order differential equations, whose coefficients were polynomials in the independent variable, the technique may be extended to deal with equations of any order, and coefficients may be any function of the independent variable.

As the theme of the thesis was the calculation of eigenvalues by numerical methods, the attention paid to the Orr-Sommerfeld problems was by way of illustration of these numerical methods. The concern with physical stability under certain conditions, eg infinitesimal disturbances, [Davey], heat and non-linear stability [Ellingsen] is another self contained topic, distinct from the theme of the present thesis.

Further work on eigenvalue calculation under certain conditions is briefly outlined below.

Work by Jordinson on the Orr-Sommerfeld equation for mean Blasius flow, discusses stability in a space amplified case. He studied the equation

$$\frac{i}{R} \left( \frac{d^2}{dz^2} - \alpha^2 \right)^2 \phi + (au - \beta) \left( \frac{d^2}{dz^2} - \alpha^2 \right) \phi - \alpha \phi \frac{d^2}{dz^2} u = 0$$

with boundary condition

$$\phi(0) = \phi'(0) = 0$$

and $\phi \sim \exp(-\alpha z)$ for $z$ large, with the upper boundary approximated at $z = 6$. This is non-linear in the eigenvalue $\alpha$, which is the complex
wavenumber, and was solved by a Numerov Technique developed by Osborne (using rational finite differences). The problem can be made linear in the eigenvalue by writing $\alpha u = \nu$, and then $c = \alpha^2$, but a further change of the independent variable would be necessary to give the boundary conditions at ±1. Eagles studied Jeffery-Hamel flow, and showed that under certain conditions the Orr-Sommerfeld equation of Thomas for Plane Poiseuille flow could be obtained.

In conclusion, it is observed that while there are a multitude of avenues of research available in connection with the Orr-Sommerfeld problem, (mostly the physical situation), none are directly related to the main theme of the thesis, which is solely a numerical treatment of eigenvalue problems. The Orr-Sommerfeld problem is used only as an appropriate example for calculations.
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