An Aleksandrov type theorem for k-convex functions

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Abstract
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AN ALEXSANDROV TYPE THEOREM FOR $k$-CONVEX FUNCTIONS

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In this note we show that $k$-convex functions on $\mathbb{R}^n$ are twice differentiable almost everywhere for every positive integer $k > n/2$. This generalises Alexsandrov's classical theorem for convex functions.

1. INTRODUCTION

A classical result of Alexsandrov [1] asserts that convex functions in $\mathbb{R}^n$ are twice differentiable almost everywhere, (see also [3, 8] for more modern treatments). It is well known that Sobolev functions $u \in W^{2,p}$, for $p > n/2$ are twice differentiable almost everywhere. The following weaker notion of convexity known as $k$-convexity was introduced by Trudinger and Wang [12, 13]. Let $\Omega \subset \mathbb{R}^n$ be an open set and $C^2(\Omega)$ be the class of continuously twice differentiable functions on $\Omega$. For $k = 1, 2, \ldots, n$ and a function $u \in C^2(\Omega)$, the $k$-Hessian operator, $F_k$, is defined by

$$F_k[u] := S_k(\lambda(\nabla^2 u)),$$

where $\nabla^2 u = (\partial_{ij} u)$ denotes the Hessian matrix of the second derivatives of $u$, $\lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the vector of eigenvalues of an $n \times n$ matrix $A \in \mathbb{R}^{n \times n}$ and $S_k(\lambda)$ is the $k$-th elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Alternatively we may write

$$F_k[u] = [\nabla^2 u]_k,$$

where $[A]_k$ denotes the sum of the $k \times k$ principal minors of an $n \times n$ matrix $A$, which may also be called the $k$-trace of $A$. The study of $k$-Hessian operators was initiated by Caffarelli, Nirenberg and Spruck [2] and Ivochkina [6] and further developed by Trudinger and Wang [10, 12, 13, 14, 15].

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A function \( u \in C^2(\Omega) \) is called \( k \)-convex in \( \Omega \) if \( F_j[u] \geq 0 \) in \( \Omega \) for \( j = 1, 2, \ldots, k \); that is, the eigenvalues \( \lambda(\nabla^2 u) \) of the Hessian \( \nabla^2 u \) of \( u \) lie in the closed convex cone given by

\[
\Gamma_k := \{ \lambda \in \mathbb{R}^n : S_j(\lambda) \geq 0, \ j = 1, 2, \ldots, k \}. \tag{1.4}
\]

(see [2] and [13] for the basic properties of \( \Gamma_k \).) We notice that \( F_1[u] = \Delta u \), is the Laplacian operator and 1-convex functions are subharmonic. When \( k = n \), \( F_n[u] = \det(\nabla^2 u) \), the Monge-Ampère operator and \( n \)-convex functions are convex. To extend the definition of \( k \)-convexity for non-smooth functions we adopt a viscosity definition as in [13]. An upper semi-continuous function \( u : \Omega \to [-\infty, \infty) \) (\( u \neq -\infty \) on any connected component of \( \Omega \)) is called \( k \)-convex if \( F_j[q] \geq 0 \), in \( \Omega \) for \( j = 1, 2, \ldots, k \), for every quadratic polynomial \( q \) for which the difference \( u - q \) has a finite local maximum in \( \Omega \). Henceforth, we shall denote the class of \( k \)-convex functions in \( \Omega \) by \( \Phi^k(\Omega) \). When \( k = 1 \) the above definition is equivalent to the usual definition of subharmonic function, see for example ([5, Section 3.2]) or ([7, Section 2.4]). Thus \( \Phi^1(\Omega) \) is the class of subharmonic functions in \( \Omega \). We notice that \( \Phi^k(\Omega) \subset \Phi^1(\Omega) \subset L^1_{loc}(\Omega) \) for \( k = 1, 2, \ldots, n \), and a function \( u \in \Phi^n(\Omega) \) if and only if it is convex on each component of \( \Omega \). Among other results Trudinger and Wang [13] (Lemma 2.2) proved that \( u \in \Phi^k(\Omega) \) if and only if

\[
\int_\Omega u(x) \left( \sum_{i,j=1}^n a_{ij} \partial_i \partial_j \phi(x) \right) \, dx \geq 0 \tag{1.5}
\]

for all smooth compactly supported functions \( \phi \geq 0 \), and for all constant \( n \times n \) symmetric matrices \( A = (a_{ij}) \) with eigenvalues \( \lambda(A) \in \Gamma_k^* \), where \( \Gamma_k^* \) is the dual cone defined by

\[
\Gamma_k^* := \{ \lambda \in \mathbb{R}^n : \langle \lambda, \mu \rangle \geq 0 \text{ for all } \mu \in \Gamma_k \}. \tag{1.6}
\]

In this note we prove the following Aleksandrov type theorem for \( k \)-convex functions.

**Theorem 1.1.** Let \( k > n/2 \), \( n \geq 2 \) and \( u : \mathbb{R}^n \to [-\infty, \infty) \) (\( u \neq -\infty \) on any connected subsets of \( \mathbb{R}^n \)), be a \( k \)-convex function. Then \( u \) is twice differentiable almost everywhere. More precisely, we have the Taylor's series expansion for \( \mathcal{L}^n_\times x \) almost everywhere,

\[
|u(y) - u(x) - \langle \nabla u(x) y - x \rangle - \frac{1}{2} \langle \nabla^2 u(x)(y-x),(y-x) \rangle| = o(|y-x|^2), \tag{1.7}
\]

as \( y \to x \).

In Section 3 (see Theorem 3.2.), we also prove that the absolutely continuous part of the \( k \)-Hessian measure (see [12, 13]) \( \mu_k[u] \), associated to a \( k \)-convex function for \( k > n/2 \) is represented by \( F_k[u] \). For the Monge-Ampère measure \( \mu[u] \) associated to a convex function \( u \), a similar result is obtained in [16].

To conclude this introduction we note that it is equivalent to assume only \( F_k[q] \geq 0 \), in the definition of \( k \)-convexity [13]. Moreover \( \Gamma_k \) may also be characterised as the closure of the positivity set of \( S_k \) containing the positive cone \( \Gamma_n \), [2].
2. Notations and Preliminary Results

Throughout the text we use the following standard notations. \(| \cdot |\) and \((\cdot, \cdot)\) will stand for the Euclidean norm and inner product in \(\mathbb{R}^n\), and \(B(x, r)\) will denote the open ball in \(\mathbb{R}^n\) of radius \(r\) centred at \(x\). For measurable \(E \subset \mathbb{R}^n\), \(\mathcal{L}^n(E)\) will denote its Lebesgue measure. For a smooth function \(u\), the gradient and Hessian of \(u\) are denoted by \(\nabla u = (\partial_1 u, \ldots, \partial_n u)\) and \(\nabla^2 u = (\partial_{ij} u)_{1 \leq i,j \leq n}\) respectively. For a locally integrable function \(f\), the distributional gradient and Hessian are denoted by \(Df = (D_1 f, \ldots, D_n f)\) and \(D^2 u = (D_{ij} u)_{1 \leq i,j \leq n}\) respectively.

For the convenience of the readers, we cite the following Hölder and gradient estimates for \(k\)-convex functions, and the weak continuity result for \(k\)-Hessian measures, [12, 13].

**Theorem 2.1.** ([13, Theorem 2.7].) For \(k > n/2\), \(\Phi^k(\Omega) \subset C^{0,\alpha}(\Omega)\) with \(\alpha := 2 - n/k\) and for any subdomain \(\Omega' \subset \subset \Omega\), \(u \in \Phi^k(\Omega)\), there exists \(C > 0\), depending only on \(n\) and \(k\) such that

\[
\sup_{x \neq y} \frac{d_{x,y}^{n+\alpha} |u(x) - u(y)|}{|x - y|^\alpha} \leq C \int_{\Omega'} |u|, \tag{2.1}
\]

where \(d_x := \text{dist}(x, \partial \Omega')\) and \(d_{x,y} := \min\{d_x, d_y\}\).

**Theorem 2.2.** ([13, Theorem 4.1].) For \(k = 1, \ldots, n\), and \(0 < q < nk/(n - k)\), the space of \(k\)-convex functions \(\Phi^k(\Omega)\) lies in the local Sobolev space \(W^{1,q}_{\text{loc}}(\Omega)\). Moreover, for any \(\Omega' \subset \subset \Omega'' \subset \subset \Omega\) and \(u \in \Phi^k(\Omega)\) there exists \(C > 0\), depending on \(n\), \(k\), \(q\), \(\Omega'\) and \(\Omega''\), such that

\[
\left( \int_{\Omega''} |Du|^q \right)^{1/q} \leq C \int_{\Omega''} |u|. \tag{2.2}
\]

**Theorem 2.3.** ([13, Theorem 1.1].) For any \(u \in \Phi^k(\Omega)\), there exists a Borel measure \(\mu_k[u] \) in \(\Omega\) such that

(i) \(\mu_k[u](V) = \int_V F_k[u](x) \, dx\) for any Borel set \(V \subset \Omega\), if \(u \in C^2(\Omega)\) and

(ii) if \((u_m)_{m \geq 1}\) is a sequence in \(\Phi^k(\Omega)\) converging in \(L^1_{\text{loc}}(\Omega)\) to a function \(u \in \Phi^k(\Omega)\), the sequence of Borel measures \((\mu_k[u_m])_{m \geq 1}\) converges weakly to \(\mu_k[u]\).

Let us recall the definition of the dual cones, [11]

\[
\Gamma_k^* := \{ \lambda \in \mathbb{R}^n : \langle \lambda, \mu \rangle \geq 0 \text{ for all } \mu \in \Gamma_k \},
\]

which are also closed convex cones in \(\mathbb{R}^n\). We notice that \(\Gamma_j^* \subset \Gamma_k^*\) for \(j \leq k\) with \(\Gamma_n^* = \Gamma_n = \{ \lambda \in \mathbb{R}^n : \lambda_i \geq 0, \; j = 1, 2, \ldots, n \}\), \(\Gamma_1^*\) is the ray given by

\[
\Gamma_1^* = \{ t(1, \ldots, 1) : t \geq 0 \},
\]
and $\Gamma_2^*$ has the following interesting characterisation,

\[
\Gamma_2^* = \left\{ \lambda \in \Gamma_n : |\lambda|^2 \leq \frac{1}{n-1} \left( \sum_{i=1}^{n} \lambda_i \right)^2 \right\}.
\]

We use this explicit representation of $\Gamma_2^*$ to establish that the distributional derivatives $D_{ij}u$ of the $k$-convex function $u$ are signed Borel measures for $k \geq 2$, (see also [13]).

**Theorem 2.4.** Let $2 \leq k \leq n$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a $k$-convex function. Then there exist signed Borel measures $\mu^{ij} = \mu^{ij}$ such that

\[
\int_{\mathbb{R}^n} u(x) \partial_{ij} \phi(x) \, dx = \int_{\mathbb{R}^n} \phi(x) \, d\mu^{ij}(x), \quad \text{for } i, j = 1, 2, \ldots, n,
\]

for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

**Proof:** Let $k \geq 2$ and $u \in \Phi^k(\mathbb{R}^n)$. Since $\Phi^k(\mathbb{R}^n) \subset \Phi^2(\mathbb{R}^n)$ for $k \geq 2$, it is enough to prove the theorem for $k = 2$. Let $u$ be a 2-convex function in $\mathbb{R}^n$. For $A \in \mathcal{S}^{n \times n}$, the space of $n \times n$ symmetric matrices, define the distribution $T_A : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$, by

\[
T_A(\phi) := \int_{\mathbb{R}^n} u(x) \sum_{i,j} a_{ij} \partial_{ij} \phi(x) \, dx.
\]

By (1.5), $T_A(\phi) \geq 0$ for $A \in \mathcal{S}^{n \times n}$ with eigenvalues $\lambda(A) \in \Gamma_2^*$, and $\phi \geq 0$. Therefore, by Riesz representation (see for example [9, Theorem 2.14] or [3, Theorem 1, Section 1.8]), there exist a Borel measure $\mu^A$ in $\mathbb{R}^n$, such that

\[
T_A(\phi) = \int_{\mathbb{R}^n} \phi \sum_{i,j} a_{ij} D_{ij}u \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^A,
\]

for all $\phi \in C_c^\infty(\mathbb{R}^n)$ and all $n \times n$ symmetric matrices $A$ with $\lambda(A) \in \Gamma_2^*$. In order to prove that the second order distributional derivatives $D_{ij}u$ of $u$ to be signed Borel measures, we need to make special choices for the matrix $A$. By taking $A = I_n$, the identity matrix, $\lambda(A) \in \Gamma_1^* \subset \Gamma_2^*$, we obtain a Borel measure $\mu^I$ such that

\[
\int_{\mathbb{R}^n} \phi \sum_{i=1}^{n} D_{ii}u \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^I,
\]

for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Therefore, the trace of the distributional Hessian $D^2u$, is a Borel measure. For each $i = 1, \ldots, n$, let $A_i$ be the diagonal matrix with all entries 1 but the $i$-th diagonal entry being 0. Then by the characterisation of $\Gamma_2^*$ in (2.3), it follows that $\lambda(A_i) \in \Gamma_2^*$. Hence there exist a Borel measure $\mu^i$ in $\mathbb{R}^n$ such that

\[
\int_{\mathbb{R}^n} \phi \sum_{j \neq i} D_{jj}u \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^i,
\]
for all \( \phi \in C^2_c(\mathbb{R}^n) \). From (2.6) and (2.7) it follows that the diagonal entries \( D_u = \mu^1 - \mu^i = \mu^i \) are signed Borel measure and

\[
\int_{\mathbb{R}^n} u \partial_{ij} \phi \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^i,
\]

for all \( \phi \in C^2_c(\mathbb{R}^n) \). Let \( \{e_1, \ldots, e_n\} \) be the standard orthonormal basis in \( \mathbb{R}^n \) and for \( a, b \in \mathbb{R}^n \), \( a \otimes b := (a^t b^t) \), denotes the \( n \times n \) rank-one matrix. For \( 0 < t < 1 \) and \( i \neq j \), let us define \( A_{ij} := I_n + t[e_i \otimes e_j + e_j \otimes e_i] \). By a straightforward calculation, it is easy to see that the vector of eigenvalues is \( \lambda(A_{ij}) = (1 - t, 1 + t, 1, \ldots, 1) \in \Gamma_2^* \), for \( 0 < t < (n/2(n - 1))^{1/2} \). Note that for this choice of \( A_{ij} \)

\[
\sum_{k,l=1}^n a^{kl} \partial_{kl} \phi = \sum_{k=1}^n \partial_{kk} \phi + 2t \partial_{ij} \phi.
\]

Thus for \( i \neq j \), (2.5) and (2.6) yields

\[
\int_{\mathbb{R}^n} u \partial_{ij} \phi \, dx = \frac{1}{2t} \left[ \int_{\mathbb{R}^n} u \sum_{k,l=1}^n a^{kl} \partial_{kl} \phi \, dx - \int_{\mathbb{R}^n} u \sum_{k=1}^n \partial_{kk} \phi \, dx \right]
= \frac{1}{2t} \left[ \int_{\mathbb{R}^n} \phi \, d\mu^{A_{ij}} - \int_{\mathbb{R}^n} \phi \, d\mu^I \right]
\]

(2.9)

where

\[
\mu^{ij} := \frac{1}{2t} (\mu^{A_{ij}} - \mu^I) = \frac{1}{2t} \left( \mu^{A_{ij}} - \sum_{k=1}^n \mu^{kk} \right).
\]

Therefore \( D_{ij} u = \mu^{ij} \) are signed Borel measures and satisfy the identity (2.4). \( \square \)

A function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) is said to have locally bounded variation in \( \mathbb{R}^n \) if for each bounded open subset \( \Omega' \) of \( \mathbb{R}^n \),

\[
\sup \left\{ \int_{\Omega'} f \, \text{div} \phi \, dx : \phi \in C^1_c(\Omega'; \mathbb{R}^n), \ |\phi(x)| \leq 1 \text{ for all } x \in \Omega' \right\} < \infty.
\]

We use the notation \( BV_{\text{loc}}(\mathbb{R}^n) \) to denote the space of such functions. For the theory of functions of bounded variation readers are referred to \([4, 17, 3]\).

**Theorem 2.5.** Let \( n \geq 2, k > n/2 \) and \( u : \mathbb{R}^n \to (-\infty, \infty) \), be a \( k \)-convex function. Then \( u \) is differentiable almost everywhere \( \mathcal{L}^n \) and \( \partial_i u \in BV_{\text{loc}}(\mathbb{R}^n) \), for all \( i = 1, \ldots, n \).

**Proof:** Observe that for \( k > n/2 \), we can take \( n < q < nk/(n - k) \) and by the gradient estimate (2.2), we conclude that \( k \)-convex functions are differentiable \( \mathcal{L}^n \) almost
everywhere $x$. Let $\Omega' \subset \mathbb{R}^n$, $\phi = (\phi^1, \ldots, \phi^n) \in C^1_c(\Omega'; \mathbb{R}^n)$ such that $|\phi(x)| \leq 1$ for $x \in \Omega'$. Then by integration by parts and the identity (2.4), we have for $i = 1, \ldots, n$,

$$\int_{\Omega'} \frac{\partial u}{\partial x_i} \text{div} \phi \, dx = -\sum_{j=1}^n \int_{\Omega'} u \frac{\partial^2 \phi^j}{\partial x_i \partial x_j} \, dx = -\sum_{j=1}^n \int_{\Omega'} \phi^j \, d\mu^{ij} \leq \sum_{j=1}^n |\mu^{ij}|(\Omega') < \infty,$$

where $|\mu^{ij}|$ is the total variation of the Radon measure $\mu^{ij}$. This proves the theorem.

### 3. Twice Differentiability

Let $u$ be a $k$-convex function, $k \geq 2$. Then by the Theorem 2.4, we have $D^2 u = (\mu^{ij})_{i,j}$, where $\mu^{ij}$ are Radon measures. By Lebesgue's Decomposition Theorem, we may write

$$\mu^{ij} = \mu^{ij}_{ac} + \mu^{ij}_s \text{ for } i, j = 1, \ldots, n,$$

where $\mu^{ij}_{ac}$ is absolutely continuous with respect to $\mathcal{L}^n$ and $\mu^{ij}_s$ is supported on a set with Lebesgue measure zero. Let $u_{ij}$ be the density of the absolutely continuous part, that is, $d\mu^{ij}_{ac} = u_{ij} \, dx$, $u_{ij} \in L^1_{\text{loc}}(\mathbb{R}^n)$. Set $u_{ij} := \partial_{ij} u$, $\nabla^2 u := \partial_{ij} u = (u_{ij})_{i,j} \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $[D^2 u]_s := (\mu^{ij}_{ac})_{i,j}$. Thus the vector valued Radon measure $D^2 u$ can be decomposed as $D^2 u = [D^2 u]_{ac} + [D^2 u]_s$, where $d[D^2 u]_{ac} = \nabla^2 u \, dx$. Now we are in a position prove theorem 1.1. To carry out the proof, we use a similar approach to Evans and Gariepy, see [3, Section 6.4].

**Proof of Theorem 1.1:** Let $n \geq 2$ and $u$ be a $k$-convex function on $\mathbb{R}^n$, $k > n/2$. Then by Theorem 2.4, and Theorem 2.5, we have for $\mathcal{L}^n$ almost everywhere $x$

$$\lim_{r \to 0} \int_{B(x,r)} |\nabla u(y) - \nabla u(x)| \, dy = 0, \tag{3.1}$$
$$\lim_{r \to 0} \int_{B(x,r)} |\nabla^2 u(y) - \nabla^2 u(x)| \, dy = 0 \tag{3.2}$$

and

$$\lim_{r \to 0} \frac{[D^2 u]_s(B(x,r))}{r^n} = 0. \tag{3.3}$$

where $\int_E f \, dx$ we denote the mean value $(\mathcal{L}^n(E))^{-1} \int_E f \, dx$. Fix a point $x$ for which (3.2)–(3.3) holds. Without loss generality we may assume $x = 0$. Then following similar calculations as in the proof of [3, Theorem 1, Section 6.4], we obtain,

$$\int_{B(r)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0) y, y \rangle \right| \, dy = o(r^2), \tag{3.4}$$
as $r \to 0$. In order to establish

$$\text{(3.5)} \quad \sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0)y, y \rangle \right| = o(r^2) \quad \text{as } r \to 0,$$

we need the following lemma.

**Lemma 3.1.** Let $h(y) := u(y) - u(0) - \langle \nabla u(0), y \rangle - \langle \nabla^2 u(0)y, y \rangle/2$. Then there exists a constant $C > 0$ depending only on $n$, $k$ and $|\nabla^2 u(0)|$, such that for any $0 < r < 1$

$$\text{(3.6)} \quad \sup_{y,z \in B(r)} \frac{|h(y) - h(z)|}{|y - z|^\alpha} \leq \frac{C}{r^\alpha} \int_{B(2r)} |h(y)| \, dy + C r^{2-\alpha},$$

where $\alpha := (2 - n/k)$.

**Proof:** Let $\Lambda := |\nabla^2 u(0)|$ and define $g(y) := h(y) + \Lambda |y|^2/2$. Since $\Lambda |y|^2/2 - u(0) - \langle \nabla u(0), y \rangle - \langle \nabla^2 u(0)y, y \rangle/2$ is convex and the sum of two $k$-convex functions are $k$-convex (follows from (1.4)), we conclude that $g$ is $k$-convex. Applying the Hölder estimate in (2.1) for $g$ with $\Omega' = B(2r)$, there exists $C := C(n, k) > 0$, such that

$$\text{(3.7)} \quad \sup_{y,z \in B(r)} \frac{|g(y) - g(z)|}{|y - z|^\alpha} \leq C \int_{B(r)} |g(y)| \, dy + C r^{n+2},$$

where $d_{y,z} := \min \left\{ \text{dist}(y, \partial B(2r)), \text{dist}(z, \partial B(2r)) \right\}$. Therefore the estimate (3.6) for $h$ follows from the estimate (3.7) and the definition of $g$. \qed

**Proof of Theorem 1.1.** To prove (3.5), take $0 < \varepsilon, \delta < 1$, such that $\delta^{1/n} \leq 1/2$. Then there exists $r_0$ depending on $\varepsilon$ and $\delta$, sufficiently small, such that, for $0 < r < r_0$

$$\mathcal{L}^n \left\{ z \in B(r) : |h(z)| \geq \varepsilon r^2 \right\} \leq \frac{1}{\varepsilon r^2} \int_{B(r)} |h(z)| \, dz = o(r^n) \quad \text{by (3.4)}$$

$$\text{(3.8)} \quad < \delta \mathcal{L}^n (B(r))$$

Set $\sigma := \delta^{1/n} r$. Then for each $y \in B(r/2)$ there exists $z \in B(r)$ such that

$$|h(z)| \leq \varepsilon r^2 \quad \text{and} \quad |y - z| \leq \sigma.$$
Hence for each \( y \in B(r/2) \), we obtain by (3.4) and (3.6),
\[
|h(y)| \leq |h(z)| + |h(y) - h(z)| \\
\leq \varepsilon r^2 + C|y - z|^{\sigma} \left( \frac{1}{r^n} \int_{B(2r)} |h(y)| \, dy + r^{2-\sigma} \right) \\
\leq \varepsilon r^2 + C\delta^{\alpha/n} r^{\alpha} \left( \frac{1}{r^n} \int_{B(2r)} |h(y)| \, dy + r^{2-\sigma} \right) \\
\leq \varepsilon r^2 + C\delta^{\alpha/n} \left( \int_{B(2r)} |h(y)| \, dy + r^2 \right) \\
= r^2 (\varepsilon + C\delta^{\alpha/n}) + o(r^2) \quad \text{as} \quad r \to 0
\]

By choosing \( \delta \) such that, \( C\delta^{\alpha/n} = \varepsilon \), we have for sufficiently small \( \varepsilon > 0 \) and \( 0 < r < r_0 \),
\[
\sup_{B(r/2)} |h(y)| \leq 2\varepsilon r^2 + o(r^2) .
\]

Hence
\[
\sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0) y, y \rangle \right| \, dy = o(r^2) \quad \text{as} \quad r \to 0 .
\]

This proves (1.7) for \( x = 0 \) and hence \( u \) is twice differentiable at \( x = 0 \). Therefore \( u \) is twice differentiable at almost every \( x \) and satisfies (1.7), for which (3.2)–(3.3) holds. This proves the theorem.

Let \( u \) be a \( k \)-convex function and \( \mu_k[u] \) be the associated \( k \)-Hessian measure. Then \( \mu_k[u] \) can be decomposed as the sum of a regular part \( \mu_k^{ac}[u] \) and a singular part \( \mu_k^s[u] \).

As an application of the Theorem 1.1, we prove the following theorem.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( u \in \mathcal{F}_k(\Omega), k > n/2 \). Then the absolutely continuous part of \( \mu_k[u] \) is represented by the \( k \)-Hessian operator \( F_k[u] \). That is
\[
\mu_k^{ac}[u] = F_k[u] \, dx.
\]

**Proof:** Let \( u \) be a \( k \)-convex function, \( k > n/2 \) and \( u_\varepsilon \) be the mollification of \( u \).
Then by (1.5) and the properties of mollification (see for example [3, Theorem 1, Section 4.2]) it follows that \( u_\varepsilon \in \mathcal{F}_k(\Omega) \cap C^\infty(\Omega) \). Since \( u \) is twice differentiable almost everywhere (by Theorem 1.1) and \( u \in W^{2,1}_{loc}(\Omega) \) (by Theorem 2.5), we conclude that \( \nabla^2 u_\varepsilon \to \nabla^2 u \) in \( L^1_{loc} \).

Let \( \mu_k[u_\varepsilon] \) and \( \mu_k[u] \) be the Hessian measures associated to the functions \( u_\varepsilon \) and \( u \) respectively. Then by weak continuity Theorem 2.3 ([13, Theorem 1.1]), \( \mu_k[u_\varepsilon] \) converges to \( \mu_k[u] \) in measure and \( \mu_k[u_\varepsilon] = F_k[u_\varepsilon] \, dx \). It follows that for any compact set \( E \subset \Omega \),
\[
\mu_k[u](E) \geq \limsup_{\varepsilon \to 0} \mu_k[u_\varepsilon](E) = \limsup_{\varepsilon \to 0} \int_E F_k[u_\varepsilon] .
\]
Since $F_k[u_x] \geq 0$ and $F_k[u_x](x) \to F_k[u](x)$ almost everywhere, by Fatou's Lemma, for every relatively compact measurable subset $E$ of $\Omega$, we have

\begin{equation}
\int_E F_k[u] \leq \liminf_{\varepsilon \to 0} \int_E F_k[u_x].
\end{equation}

Therefore by Theorem 3.1, [13], it follows that $F_k[u] \in L^1_{\text{loc}}(\Omega)$. Let $\mu_k[u] = \mu_k[u_x] + \mu_k^s[u]$, where $\mu_k^s[u] = h \, dx$, $h \in L^1_{\text{loc}}(\Omega)$ and $\mu_k^s[u]$ is the singular part supported on a set of Lebesgue measure zero. We would like to prove that $h(x) = F_k[u_x](x) \mathcal{L}^n$ almost everywhere $x$. By taking $E := B(x, r)$, from (3.10) and (3.11), we obtain

\begin{equation}
\int_{B(x, r)} F_k[u] \, dy \leq \frac{\mu_k[u_x](B(x, r))}{\mathcal{L}^n(B(x, r))} = \int_{B(x, r)} h \, dy + \frac{\mu_k^s[u](B(x, r))}{\mathcal{L}^n(B(x, r))}.
\end{equation}

Hence by letting $\varepsilon \to 0$, we obtain

\begin{equation}
F_k[u_x](x) \leq h(x) \quad \mathcal{L}^n \text{ almost everywhere } x.
\end{equation}

To prove the reverse inequality, let us recall that $h$ is the density of the absolutely continuous part of the measure $\mu_k[u]$, that is for $\mathcal{L}^n$ almost everywhere $x$

\begin{equation}
h(x) = \lim_{r \to 0} \frac{\mu_k^a[u_x](B(x, r))}{\mathcal{L}^n(B(x, r))} = \lim_{r \to 0} \frac{\mu_k[u_x](B(x, r))}{\mathcal{L}^n(B(x, r))}.
\end{equation}

Since $\mu_k^s[u]$ is supported on a set of Lebesgue measure zero,

$$
\mu_k^s[u_x](\partial B(x, r)) = 0, \quad \mathcal{L}^1 \text{ almost everywhere } r > 0.
$$

Therefore by the weak continuity of $\mu_k[u_x]$ (see for example Theorem 1, [3, Theorem 1]), we conclude that

\begin{equation}
\lim_{\varepsilon \to 0} \mu_k[u_x](B(x, r)) = \mu_k[u_x](B(x, r)), \quad \mathcal{L}^1 \text{ almost everywhere } r > 0.
\end{equation}

Let $\delta > 0$. Then for $\varepsilon < \varepsilon' = \varepsilon(\delta)$ and for $\mathcal{L}^1$ almost everywhere $r > 0$, $\mathcal{L}^n$ almost everywhere $x$

\begin{align*}
h(x) &\leq \lim_{r \to 0} \frac{(1 + \delta)\mu_k[u_x](B(x, r))}{\mathcal{L}^n(B(x, r))} \\
&= (1 + \delta) \lim_{r \to 0} \int_{B(x, r)} F_k[u_x] \, dy \\
&= (1 + \delta) F_k[u_x](x)
\end{align*}

(3.16)

By letting $\varepsilon \to 0$ and finally $\delta \to 0$, we obtain

$$
h(x) \leq F_k[u_x](x), \quad \mathcal{L}^n \text{ almost everywhere } x.
$$

This proves the theorem.
REFERENCES


