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2014

# Exact and approximate solutions for options with time-dependent stochastic volatility

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## Publication Details

Goard, J. (2014). Exact and approximate solutions for options with time-dependent stochastic volatility. *Applied Mathematical Modelling*, 38 (11-12), 2771-2780.

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## Abstract

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## Keywords

solutions, approximate, options, time, volatility, exact, dependent, stochastic

## Disciplines

Engineering | Science and Technology Studies

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# Exact and Approximate Solutions for Options with Time-dependent Stochastic Volatility

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## Abstract

In this paper it is shown how symmetry methods can be used to find exact solutions for European option pricing under a time-dependent 3/2-stochastic volatility model  $dv = kv(A(t) - v)dt + bv^{\frac{3}{2}}dZ$ . This model with  $A(t)$  constant has been proven by many authors to outperform the Heston model in its ability to capture the behaviour of volatility and fit option prices. Further, singular perturbation techniques are used to derive a simple analytic approximation suitable for pricing options with short tenor, a common feature of most options traded in the market.

*Keywords and phrases:* stochastic volatility, volatility model, option pricing

*Mathematics subject classification:* 91G20, 35A22, 22E05

# 1 Introduction

Stochastic volatility models have become particularly popular for derivative pricing and hedging over the last 20 years. Empirical evidence on underlying asset prices and on their derivatives strongly suggest that asset volatility is stochastic and that the Black-Scholes (BS) model [1] is inadequate to describe features of asset returns such as skewness, leptokurtosis and pronounced conditional heteroskedasticity. The literature supports the use of stochastic volatility in an effort to reproduce the implied volatility smile observed in markets and to avoid many of the shortcomings of the constant variance diffusion assumed by BS.

Work on stochastic volatility models include that of Hull and White [2], Scott [3] and Wiggins [4], whose pricing solutions depended on extensive use of numerical techniques to solve two-dimensional partial differential equations (PDEs). Assuming that volatility is uncorrelated with the spot asset, Stein and Stein [5] manage to analytically value options. However, volatility shocks are known to be negatively correlated with asset price shocks so that when volatility increases, stock prices decrease and vice-versa. This is commonly referred to as leverage and at least partially accounts for a skewed distribution for the asset price. Thus without correlation, the prices cannot hope to capture such important skewness effects arising from such correlation.

One of the most popular stochastic volatility models to date is the mean-reverting model of Heston [6]. Using Fourier inversion methods, Heston provides a closed form solution for the price of European options when the spot asset is correlated with volatility, and variance  $v$ , follows the ‘square-root’ process  $dv = k(\theta - v)dt + \sigma\sqrt{v}dZ$ , which is commonly referred to as the ‘Heston’ model. [Note that here and elsewhere in the paper,  $Z$  and also  $Z_1$  and  $Z_2$  will refer to Wiener processes under a real measure  $P$  while  $\tilde{Z}$ ,  $\tilde{Z}_1$  and  $\tilde{Z}_2$  will refer to Wiener processes under a corresponding equivalent risk-neutral measure  $Q$ .] However, numerous empirical studies show that the Heston model is misspecified. Chacko and Viceira

[7] performed a comprehensive empirical analysis on variance models of the form

$$dv = (a + bv)dt + cv^\gamma dZ \quad (1.1)$$

where the instantaneous standard deviation of variance was allowed to be proportional to any power of variance. Using the estimation technique of spectral GMM (generalised method of moments) they found that the best value of  $\gamma$  was between 1 and 2, with the standard errors indicating that the differences between the values found for  $\gamma$  and one half (as in the Heston model) were statistically significant. This means that compared to the Heston model, the volatility of the variance process is more highly sensitive to the level of variance. Other empirical studies have yielded similar results. For example, Jones [8] analyses the more general CEV (constant elasticity of variance) model using a bivariate series of returns and an at-the-money short maturity option. He finds that periods of high volatility coincide with periods of volatile volatility which is in disagreement with the Heston model, in which the volatility of instantaneous volatility is not level-dependent. In particular a number of his specification tests favour the nonaffine CEV model over the Heston model with best estimates of gamma as in (1.1), that are greater than one.

More recently, using various samples of S&P500 index return data, Chrisoffersen *et al* [9] compare the performance of the Heston model with 5 simple alternatives, all of which can be described by:

$$dv = kv^a(\theta - v)dt + \sigma v^b dZ \quad a = \{0, 1\}, \quad b = \{1/2, 1, 3/2\}. \quad (1.2)$$

They found that ‘the 3/2N’ model ( $a = 1, b = 3/2$ ) and ‘the ONE’ model ( $a = 0, b = 1$ ) performed best as far as maximising model fit for the samples. The Heston model ranked 4th or 5th. Using out-of-the-money S&P500 index option data for 1996-2004, in minimising option implied volatility mean square error, the ONE and 3/2N model outperformed the

Heston model for both in -sample and out-of-sample experiments. Thus the ‘3/2N’ model namely

$$dS = \mu S dt + \sqrt{v} S dZ_1 \quad (1.3a)$$

$$dv = (\omega v - \theta v^2) dt + \xi v^{\frac{3}{2}} dZ_2, \quad (1.3b)$$

has proven empirically to outperform the Heston model. Similarly, Goard and Mazur [10] show that the 3/2-model outperforms the Heston (and all other models considered in their test) in its ability to fit the VIX time series. The novel features of (1.3b) are 1) the specification for the diffusion having a high power law of 1.5 which can reduce the heteroskedasticity of volatility<sup>1</sup> and 2) a nonlinear drift so that it exhibits substantial nonlinear mean-reverting behaviour when the volatility is above its long-run mean. Hence after a large volatility spike, the volatility can potentially quickly decrease while after a low volatility period it can be slow to increase. It has also been shown that with (1.3b), with  $\theta > 0$ ,  $v$  will always remain positive. Further aspects of the model have been studied by Lewis [12], who using a risk-adjustment via utility theory provides an analytic solution for option prices under this model, namely for a call option with strike price  $K$ , time to maturity  $\tau$  and with  $\gamma \leq 1$  and  $\gamma(1 - \gamma)\xi^2 \leq (\theta + \xi^2/2)^2$

$$C(S, v, \tau) = S - \frac{K e^{-r\tau}}{2\pi} \int_{ik_i - \infty}^{ik_i + \infty} e^{-iuy} \frac{H(u, v, \tau)}{u^2 - iu} du \quad (1.4)$$

$$\text{where } 0 < k_i < 1, \quad y = \log(S/K) + r\tau,$$

$$H(u, v, \tau) = \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta)} \left[ X\left(\frac{\bar{\omega}}{v}, \omega\tau\right) \right]^\alpha M \left[ \alpha, \beta, -X\left(\frac{\bar{\omega}}{v}, \omega\tau\right) \right],$$

$$X(x, t) = \frac{x}{e^t - 1}, \quad \mu = \frac{1}{2}(1 + \hat{\theta}), \quad \delta = [\mu^2 + \bar{c}]^{\frac{1}{2}},$$

$$\alpha = -\mu + \delta, \quad \beta = 1 + 2\delta, \quad \bar{\omega} = \frac{2\omega}{\xi^2}, \quad \bar{c} = \frac{(u^2 - iu)}{\xi^2},$$

$$\hat{\theta}(u) = -1 + \frac{2}{\xi^2} [\sqrt{(\theta + \xi^2/2)^2 - \gamma(1 - \gamma)\xi^2} + (1 - \gamma + iu)\rho\xi].$$

where  $M(a, b, x)$  is the Kummer-M confluent hypergeometric function (see e.g. Abramowitz

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<sup>1</sup>See Campbell *et al* [11] for a discussion on this for the short interest rate model of the same form.

and Stegun [13]) and  $r$  is the risk-free interest rate. Heston [14] also independently developed the solution but with 2 integral functions corresponding to 2 cumulative distribution functions.

However, as Jones [8] and many other authors have found from empirical studies, although the CEV processes greatly improve over the square-root specification, they still fail to match observed prices of short-dated options and there still remain questions regarding long-term memory in volatility.

In Section 2 of this paper we outline the Lie symmetries method to solve PDEs and in Section 3 of this paper show how symmetries can be used to find solutions to European options on an underlying with price  $S$ , under the time-dependent risk-neutral 3/2-model

$$dS = rSdt + \sqrt{v}Sd\tilde{Z}_1 \quad (1.5a)$$

$$dv = kv(A(t) - v)dt + bv^{\frac{3}{2}}d\tilde{Z}_2, \quad (1.5b)$$

with non-zero correlation,  $\rho$ , between price and volatility shocks, with  $|\rho| < 1$ . This is a promising candidate for a realistic model for volatility from a financial viewpoint. It not only includes a diffusion term in agreement with empirical findings but also includes a realistic time-dependent drift component in which we are free to choose the moving target. Thus model (1.5a,b) with its infinitely more degrees of freedom would be able to describe a wide range of volatility data. For example, using a truncated Fourier series for  $A(t)$  would be able to pick up the main trends of cyclic behaviour from historical records. Further the free function of time lends itself to calibration so that theoretical and current market prices can be matched for all maturities (see e.g Wilmott [15] for discussions on calibration). [We note that with  $F_t$  denoting the futures price with maturity  $T$  at time  $t$ ,  $X_t = \ln(\frac{F_t}{F_0})$ ,  $\langle X_t \rangle = \int_0^t v_s ds$ , Carr and Sun [16] provide the joint conditional Fourier Laplace transform of  $X_T$  and  $\langle X_T \rangle - \langle X_t \rangle$ . This relies in part in ‘guessing’ certain forms for transformations.] The form of the exact solution however still involves the valuation of an integral with complex arguments. This means that care needs to be taken in numerical calculations and calibration of market and theoretical

prices may not be easy. In Section 4 of this paper we obtain a simple, analytic approximation for prices of put options with short tenor. This approximation provides quick and accurate values for options with short expiries, such as one or two months. In fact it is these types of options that dominate the options markets. Then in Section 5, we compare the exact and approximate solutions using different maturities and parameters. In Section 6 we present a short conclusion.

## 2 Method

A symmetry of a differential equation is a transformation mapping an arbitrary solution to another solution of the differential equation. The classical Lie groups of point invariance transformations depend on continuous parameters and act on the system's graph space that is co-ordinatised by the independent and dependent variables. As these symmetries can be determined by an explicit computational algorithm (known as Lie's algorithm or Lie's classical method), many automated computer algebra packages (see e.g Sherring [17]) have been developed to find them. Thus they are the most extensively used of all symmetries.

If a PDE is an invariant under a point symmetry, one can often find **similarity solutions** or **invariant solutions** which are invariant under some subgroup of the full group admitted by the PDE. These solutions result from solving a reduced equation in fewer variables.

In essence, the **classical method** for finding symmetry reductions of a second-order PDE in one dependent variable  $P$  and 3 independent variables  $(S, v, \tau)$

$$\Delta(S, v, \tau, P_S, P_v, P_\tau, P_{SS}, P_{Sv}, P_{S\tau}, P_{vv}, P_{v\tau}, P_{\tau\tau}) = 0, \quad (2.1)$$



is to find a one-parameter Lie group of transformations in infinitesimal form

$$S^* = S + \varepsilon\Theta(S, v, \tau, P) + O(\varepsilon^2) \quad (2.2a)$$

$$v^* = v + \varepsilon V(S, v, \tau, P) + O(\varepsilon^2) \quad (2.2b)$$

$$\tau^* = \tau + \varepsilon\bar{T}(S, v, \tau, P) + O(\varepsilon^2) \quad (2.2c)$$

$$P^* = P + \varepsilon\eta(S, v, \tau, P) + O(\varepsilon^2) \quad (2.2d)$$

which leaves (2.1) invariant. The coefficients  $\Theta, V, \bar{T}$  and  $\eta$  of the infinitesimal symmetry are often referred to as the ‘infinitesimals’. This invariance requirement is determined by

$$G^{(2)}\Delta|_{\Delta=0} = 0, \quad (2.3)$$

where

$$G = \Theta(S, v, \tau, P)\frac{\partial}{\partial S} + V(S, v, \tau, P)\frac{\partial}{\partial v} + \bar{T}(S, v, \tau, P)\frac{\partial}{\partial \tau} + \eta(S, v, \tau, P)\frac{\partial}{\partial P} \quad (2.4)$$

are vector fields which span the associated Lie algebra, and are called the **infinitesimal generators** of the transformation (2.2a-d) and  $G^{(2)}$  is the second extension (or second prolongation) of  $G$ , extended to the second jet space, co-ordinatised by  $S, v, \tau, P, P_S, P_v, P_\tau, P_{SS} \dots$  (see Chapter 2 in the book of Bluman and Kumei [18]).

Equation (2.3) is a polynomial equation in a set of independent functions of the derivatives of  $P$ . As the equation must be true for arbitrary values of these independent functions, their coefficients must vanish, leading to an over-determined linear system of equations, known as the determining equations for the coefficients  $\Theta(S, v, \tau, P), V(S, v, \tau, P), \bar{T}(S, v, \tau, P)$  and  $\eta(S, v, \tau, P)$ . Then for known functions  $\Theta, V, \bar{T}, \eta$ , invariant solutions  $P$  corresponding to

(2.2a-d) satisfy the invariant surface condition (ISC)

$$\Omega = \Theta(S, v, \tau, P) \frac{\partial P}{\partial S} + V(S, v, \tau, P) \frac{\partial P}{\partial v} + \bar{T}(S, v, \tau, P) \frac{\partial P}{\partial \tau} - \eta(S, v, \tau, P) = 0, \quad (2.5)$$

which when solved as a first-order PDE by the method of characteristics, yields the functional form of the similarity solution in terms of an arbitrary function, i.e

$$P = q(S, v, \tau, \phi(z_1, z_2)),$$

where

$$z_1 = z_1(S, v, \tau), \quad z_2 = z_2(S, v, \tau),$$

and where  $\phi$  is an arbitrary function of invariants  $z_1, z_2$  for the symmetry. Substituting this functional form into (2.1) produces a quotient PDE in two independent variables which one solves for the function  $\phi(z_1, z_2)$ .

Further, for an initial-value problem with the initial condition  $P(S, v, 0) = j(S, v)$ , then we need a linear combination of generators such that condition (2.5) is satisfied at  $\tau = 0$ ,  $P = j(S, v)$  i.e

$$\begin{aligned} &\Theta(S, v, 0, j(S, v))P_S(S, v, 0) + \bar{T}(S, v, 0, j(S, v))P_\tau(S, v, 0) \\ &+ V(S, v, 0, j(S, v))P_v(S, v, 0) = \eta(S, v, 0, j(S, v)). \end{aligned} \quad (2.6)$$

$P_S(S, v, 0)$  and  $P_v(S, v, 0)$  can be found from the final condition. As well, for evolution equations  $P_\tau(S, v, 0)$  can be found from the governing PDE (see Goard [19] for details).

### 3 Exact Solution for European Put Option Price under the Time-Dependent 3/2-Model

We assume that the risk-neutral process for the asset price is given by (1.5a,b) where  $\text{corr}(d\tilde{Z}_1, d\tilde{Z}_2) = \rho dt$ ,  $r$  is the constant risk-free rate and  $v$  the variance. In (1.5b),  $A(t)$  is the long-run target of the variance process,  $k$  is the reversion rate to the long-run target and  $b$  is the volatility of volatility.

**Theorem 3.1:** The value of a European put option with strike  $K$  and expiry  $T$ , when the underlying asset follows the risk-neutral process (1.5a,b), is given by

$$P(S, v, \tau) = e^{-r\tau} [K + \psi(Se^{r\tau}, v\bar{T}(\tau))] \quad (3.1)$$

where

$$\bar{T}(\tau) = \begin{cases} \frac{1}{r} \left[ e^{k \int_0^\tau a(u) du} - \frac{a_0}{a(\tau)} - a_0 e^{k \int_0^\tau a(u) du} \int_0^\tau \left\{ \frac{a'(u)}{a(u)^2} e^{-k \int_0^u a(u_1) du_1} \right\} du \right], & \text{if } a = a(\tau) \\ \frac{1}{r} (e^{ak\tau} - 1), & \text{if } a \text{ is constant,} \end{cases}$$

$$\begin{aligned} a(\tau) &= A(T - \tau), \quad a_0 = a(0), \\ \psi(x, y) &= \frac{-K}{2\pi} \int_{iw_i - \infty}^{iw_i + \infty} \left[ x^{-iw} y^{-m} \frac{K^{iw}}{w^2 - iw} \frac{\Gamma(m + 1 - \frac{c_1}{b})}{\Gamma(2m + 1 - \frac{c_1}{b})} \left( \frac{2ka_0}{b^2 r} \right)^m \right. \\ &\quad \left. \times M \left( m, 2m + 1 - \frac{c_1}{b}, \frac{-2ka_0}{b^2 r y} \right) \right] dw \end{aligned}$$

(see Note 1 below),

$M$  is the confluent hypergeometric function, (see e.g Abramowitz and Stegun [13]),

$$\text{and } \tau = T - t, \quad 0 < w_i < 1, \quad c_1 = -b - 2\rho iw - \frac{2k}{b}, \quad i^2 = -1, \quad m = \frac{c_1 + \sqrt{c_1^2 - 4wi + 4w^2}}{2b}.$$

**Note 1:** In practice, it can be convenient to perform the integration in the real plane and so we compute  $\psi(x, y)$  with  $w \rightarrow [i/2 + w_r]$  and use

$$\psi(x, y) = \frac{-K}{\pi} \int_0^{\bar{w}} \mathbb{R} \left\{ x^{-i[i/2+w_r]} y^{-m} \frac{K^{i[i/2+w_r]}}{[i/2 + w_r]^2 - i[i/2 + w_r]} \right. \\ \left. \frac{\Gamma(m + 1 - \frac{c_1}{b})}{\Gamma(2m + 1 - \frac{c_1}{b})} \left( \frac{2ka_0}{b^2r} \right)^m M \left( m, 2m + 1 - \frac{c_1}{b}, \frac{-2ka_0}{b^2ry} \right) \right\} dw_r$$

where  $\bar{w}$  is sufficiently large.

**Proof:** Given that the stock price follows (1.5a,b), the price of a European put option  $P(S, v, \tau)$  with expiry  $T$  and strike  $K$  satisfies the following PDE

$$\frac{vS^2}{2} \frac{\partial^2 P}{\partial S^2} + \rho bv^2 S \frac{\partial^2 P}{\partial S \partial v} + \frac{b^2 v^3}{2} \frac{\partial^2 P}{\partial v^2} + rS \frac{\partial P}{\partial S} + kv(a(\tau) - v) \frac{\partial P}{\partial v} - rP - \frac{\partial P}{\partial \tau} = 0 \quad (3.2a)$$

where  $a(\tau) = A(T - \tau)$ , and subject to

$$P(S, v, 0) = \max(K - S, 0) \quad (3.2b)$$

$$\lim_{S \rightarrow 0} P(S, v, \tau) = Ke^{-r\tau} \quad (3.2c)$$

$$\lim_{S \rightarrow \infty} P(S, v, \tau) = 0 \quad (3.2d)$$

$$P(S, 0, \tau) = \max(Ke^{-r\tau} - S, 0) \quad (3.2e)$$

$$\lim_{v \rightarrow \infty} P(S, v, \tau) = Ke^{-r\tau} \quad (3.2f)$$

where  $\tau = T - t$ .

The boundary conditions in the  $S$  direction are the standard conditions for European put options (see e.g Wilmott [15]). The boundary condition at  $v = 0$  is based on the riskless asset growing argument and the boundary condition as  $v \rightarrow \infty$  is explained by Zhu and Chen [20] and is based on using the BS value for the put in the limit as  $\sigma = \sqrt{v} \rightarrow \infty$  i.e  $Ke^{-r\tau}$ .

Using Lie's classical symmetries method, we find PDE (3.2a) has a finite-dimensional

symmetry with generator

$$G = F(v, \tau)P \frac{\partial}{\partial P} + \bar{T}(\tau) \frac{\partial}{\partial \tau} - v\bar{T}'(\tau) \frac{\partial}{\partial v} - Sg(\tau) \left(1 + \frac{2\rho}{b}\right) \frac{\partial}{\partial S}$$

where  $F(v, \tau)$ ,  $\bar{T}(\tau)$  and  $g(\tau)$  satisfy the determining equations listed in Appendix A. From these determining equations and with consideration of the given initial and boundary conditions we use the following admitted symmetry generator:

$$G = -f(\tau)P \frac{\partial}{\partial P} + \bar{T}(\tau) \frac{\partial}{\partial \tau} - v\bar{T}'(\tau) \frac{\partial}{\partial v} - Sf(\tau) \frac{\partial}{\partial S}$$

where  $\bar{T}(\tau) = \frac{1}{r}f(\tau)$  and

$$f(\tau) = e^{k \int_0^\tau a(u) du} - \frac{a(0)}{a(\tau)} - a(0)e^{k \int_0^\tau a(u) du} \int_0^\tau \left( \frac{a'(u)}{a(u)^2} e^{-k \int_0^u a(u_1) du_1} \right) du. \quad (3.3)$$

The corresponding invariant solution is

$$P = \phi(x, y)e^{-r\tau}, \quad x = Se^{r\tau}, \quad y = v\bar{T}(\tau). \quad (3.4)$$

Substitution of this into (3.2a) we find that  $\phi(x, y)$  needs to satisfy the reduced equation

$$x^2\phi_{xx} + 2\rho bxy\phi_{xy} + b^2y^2\phi_{yy} - 2k(qa_0 + y)\phi_y = 0 \quad (3.5)$$

where  $a_0 = a(0)$ ,  $q = \frac{1}{r}$  and as  $\bar{T}(0) = 0$ , from (3.2b-f) the initial and boundary conditions for  $\phi$  are given as  $\phi(x, 0) = \max(K - x, 0)$ ,  $\phi(0, y) = K$ ,  $\lim_{x \rightarrow \infty} \phi(x, y) = 0$ ,  $\lim_{y \rightarrow \infty} \phi(x, y) = K$ .

Letting  $x = \exp(X)$ ,  $y = \exp(bY)$ , (3.5) becomes

$$\phi_{XX} + \phi_{YY} + 2\rho\phi_{XY} - \phi_X - b\phi_Y - \frac{2k}{b}\phi_Y - \frac{2kqa_0e^{-bY}}{b}\phi_Y = 0 \quad (3.6)$$

to be solved on the infinite domain subject to

$$\lim_{Y \rightarrow -\infty} \phi(X, Y) = \max(K - e^X, 0) \quad (3.7)$$

and that  $\phi \rightarrow 0$  as  $X \rightarrow \infty$ . Taking the generalised Fourier Transform<sup>2</sup> of (3.6) and (3.7) with respect to  $X$  where  $\mathfrak{F}\{\phi(X, Y)\} = F(w, Y)$  we get that for  $Im(w) < 0$ ,

$$\frac{\partial^2 F}{\partial Y^2} - \frac{\partial F}{\partial Y} \left[ b + 2\rho iw + \frac{2k}{b} + \frac{2kqa_0 e^{-bY}}{b} \right] + (iw - w^2)F = 0 \quad (3.8a)$$

$$\lim_{Y \rightarrow -\infty} F(w, Y) = \frac{-K^{1+iw}}{w^2 - iw}. \quad (3.8b)$$

The solution to the above (see e.g Polyanin and Zaitsev [21]) is

$$F(w, Y) = -e^{-mbY} \frac{\Gamma(m+1 - \frac{c_1}{b})}{\Gamma(2m+1 - \frac{c_1}{b})} \left( \frac{2kqa_0}{b^2} \right)^m \frac{K^{1+iw}}{w^2 - iw} M \left( m, 2m+1 - \frac{c_1}{b}, \frac{-2kqa_0 e^{-bY}}{b^2} \right) \quad (3.9a)$$

where

$$c_1 = -b - 2\rho iw - \frac{2k}{b} \text{ and } m = \frac{c_1 + \sqrt{c_1^2 - 4wi + 4w^2}}{2b}. \quad (3.9b)$$

Taking the Fourier inverse of (3.9a) then gives

$$\phi(X, Y) = \frac{-1}{2\pi} \int_{iw_i - \infty}^{iw_i + \infty} \left[ e^{-iwX} e^{-mbY} \frac{K^{1+iw}}{w^2 - iw} \frac{\Gamma(m+1 - \frac{c_1}{b})}{\Gamma(2m+1 - \frac{c_1}{b})} \left( \frac{2kqa_0}{b^2} \right)^m M \left( m, 2m+1 - \frac{c_1}{b}, \frac{-2ka_0 q e^{-bY}}{b^2} \right) \right] dw$$

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<sup>2</sup>Given that  $k_1, k_2 \in \mathbb{R}$ ,  $k_1 < k_2$  and  $\int_{-\infty}^{\infty} e^{-k_1 x} |g(x)| dx < \infty$ ,  $\int_{-\infty}^{\infty} e^{-k_2 x} |g(x)| dx < \infty$ , then the generalised Fourier transform  $\mathfrak{F}g(z) = \int_{-\infty}^{\infty} e^{izx} g(x) dx$ ,  $z \in \mathbb{C}$  exists and is analytic for all  $z$  in the strip  $\{z \in \mathbb{C} : k_1 < Im(z) < k_2\}$ . The inversion formula is given by  $g(x) = \int_{iu - \infty}^{iu + \infty} e^{-izx} \mathfrak{F}g(z) dz$  with  $k_1 < u < k_2$ .

and hence with  $X = \ln x$ ,  $Y = \frac{1}{b} \ln y$ ,  $q = \frac{1}{r}$  we get

$$\begin{aligned} \phi(x, y) = \frac{-1}{2\pi} \int_{iw_i - \infty}^{iw_i + \infty} \left[ x^{-iw} y^{-m} \frac{K^{1+iw}}{w^2 - iw} \frac{\Gamma(m + 1 - \frac{c_1}{b})}{\Gamma(2m + 1 - \frac{c_1}{b})} \left( \frac{2ka_0}{rb^2} \right)^m \right. \\ \left. \times M \left( m, 2m + 1 - \frac{c_1}{b}, \frac{-2ka_0}{b^2 r y} \right) \right] dw. \quad (3.10) \end{aligned}$$

The solution for the European put option price in its natural domain of definition, from (3.4) is then given by  $P(S, v, \tau) = \phi(Se^{r\tau}, v\bar{T}(\tau))e^{-r\tau}$  where  $\bar{T}(\tau) = \frac{1}{r}f(\tau)$  and  $f(\tau)$  is given in (3.3) and  $\phi(x, y)$  is given in (3.10).

This solution agrees with the European put solution  $P_1$  in Lewis [12] when  $a(t)$  is constant and  $\gamma = 1$  so that (1.3a,b) (with  $\mu$  replaced by  $r$ ), corresponds to a risk-neutral process.

However, as stated and shown by Lewis [12], in practice we do the  $w$ -integration in  $0 < Im(w) < 1$  as within this strip the integrand is well-behaved with  $Re(w^2 - iw) \geq 0$ . To do this, we use the put-call parity,  $P = Ke^{-r\tau} - [S - C(S, v, \tau)]$ , where the expression in square brackets represents the covered call option with payoff  $\min(S_T, K)$ . The payoff of the covered call has a transform which is simply the negative of the payoff transform of the put option, but with the restriction  $0 < Im(w) < 1$ . This then leads to the solution (3.1). We note that this solution satisfies (3.2b-f). ##

A comparison of put option prices when  $v = 0.2$  using the BS formula and (3.1) with  $a(\tau) = 0.3$  and  $a(\tau) = 0.2[\cos(4\pi\tau) + \sin(4\pi\tau)] + 0.4$  is given in Figure 1. Other parameter values used were  $K = 20$ ,  $b = 2$ ,  $T = 1$ ,  $k = 10$ ,  $\rho = -0.75$ ,  $\bar{w} = 1000$ . Calculations were performed using the mathematics computation package MAPLE [22] with the integration performed numerically with a relative error tolerance of  $0.5 \times 10^{-9}$ . From the figure, we see that in this example with the negative correlation, out-of-the-money put option prices with the stochastic volatility models are higher than BS prices.

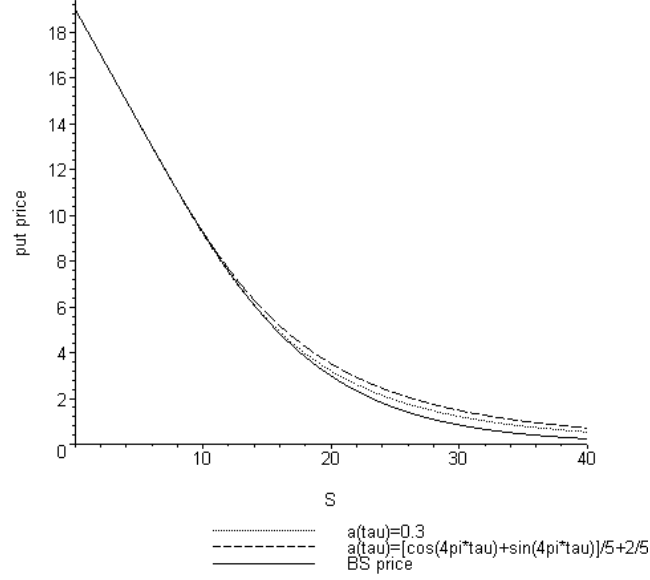


Figure 1: Put option values with  $T = 1$ ,  $K = 20$  using BS formula (bold line) and 3/2 stochastic volatility formula (3.1) with  $a(\tau) = 0.3$  (dotted line) and  $a(\tau) = 0.2[\cos(4\pi\tau) + \sin(4\pi\tau)] + 0.4$  (dashed line).

## 4 Analytic Approximation

To find an approximation to (3.2a-f) for small  $\tau$  we follow the method outlined by Howison [23]. We let  $\tau = \epsilon t'$  where  $0 < \epsilon \ll 1$  and assume the solution can be expanded as a series

$$P(S, v, \tau) = \sum_{i=0}^{\infty} \epsilon^i P_i(S, v, t'). \quad (4.1)$$

Substituting (4.1) into (3.2a) we get

$$-\frac{\partial P}{\partial \tau} + \epsilon r S \frac{\partial P}{\partial S} + \frac{\epsilon v S^2}{2} \frac{\partial^2 P}{\partial S^2} + \epsilon \rho b v^2 S \frac{\partial^2 P}{\partial S \partial v} + \frac{\epsilon b^2 v^3}{2} \frac{\partial^2 P}{\partial v^2} + \epsilon k v (a(\tau) - v) \frac{\partial P}{\partial v} - \epsilon r P = 0 \quad (4.2)$$



Upon equating coefficients of  $\epsilon^0$  and  $\epsilon^1$ , and with consideration of corresponding boundary and initial conditions, we get that

$$P_0(S, v, t') + \epsilon P_1(S, v, t') = \begin{cases} K - S - rKt'\epsilon & S - K \ll \epsilon K \\ 0 & S - K \gg \epsilon K \end{cases} \quad (4.3)$$

However, the above solution is not differentiable at  $S = K$  and as we expect large Gamma i.e.  $\frac{\partial^2 P}{\partial S^2}$  near the strike, this ‘outer’ solution is not valid near  $S = K$ . For the ‘inner’ solution where  $S$  is near  $K$ , the second-order derivative with respect to  $S$  needs to be included in the differential system. We introduce the inner variable  $x = \frac{S-K}{\epsilon^{\frac{1}{2}}K}$  and rescale  $P$  to  $P = \epsilon^{\frac{1}{2}}KQ$ .

This leads to the equation

$$\begin{aligned} \frac{\partial Q}{\partial t'} = \epsilon^{\frac{1}{2}}r(x\epsilon^{\frac{1}{2}} + 1)\frac{\partial Q}{\partial x} + \frac{v}{2}(x\epsilon^{\frac{1}{2}} + 1)^2\frac{\partial^2 Q}{\partial x^2} + \epsilon^{\frac{1}{2}}\rho bv^2(x\epsilon^{\frac{1}{2}} + 1)\frac{\partial^2 Q}{\partial x\partial v} + \epsilon\frac{b^2v^3}{2}\frac{\partial^2 Q}{\partial v^2} \\ + \epsilon kv(a(\epsilon t') - v)\frac{\partial Q}{\partial v} - \epsilon rQ \end{aligned} \quad (4.4)$$

subject to  $Q(x, v, 0) = \max(-x, 0)$ ,  $\lim_{x \rightarrow \infty} Q(x, v, \tau) = 0$ ,  $\lim_{x \rightarrow -\infty} Q(x, v, \tau) \rightarrow -x - rt'\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}})$ .

We now expand

$$Q(x, v, \tau) = \sum_{i=0}^{\infty} \epsilon^{\frac{i}{2}} Q_i(x, v, t') \quad (4.5)$$

and substitute this form into (4.4). Equating terms of  $O(1)$  we get

$$\frac{\partial Q_0}{\partial t'} = \frac{v}{2} \frac{\partial^2 Q_0}{\partial x^2} \quad (4.6)$$

subject to  $Q_0(x, v, 0) = \max(-x, 0)$ ,  $\lim_{x \rightarrow \infty} Q_0(x, v, t') = 0$ ,  $\lim_{x \rightarrow -\infty} Q_0(x, v, t') \rightarrow -x$ .

PDE (4.6) admits the finite-dimensional Lie group of transformations with infinitesimal generators given by  $G_1 = 2t'vx\frac{\partial}{\partial x} + 2(t')^2v\frac{\partial}{\partial t'} + (-t'v - x^2)Q_0\frac{\partial}{\partial Q_0}$ ,  $G_2 = x\frac{\partial}{\partial x} + 2t'\frac{\partial}{\partial t'}$ ,  $G_3 = vt'\frac{\partial}{\partial x} - xQ_0\frac{\partial}{\partial Q_0}$ ,  $G_4 = \frac{\partial}{\partial x}$ ,  $G_5 = x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}$ ,  $G_6 = \frac{\partial}{\partial t'}$ ,  $G_7 = Q_0\frac{\partial}{\partial Q_0}$ , and where each generator  $G_1 - G_7$  can be multiplied by an arbitrary function of  $v$ . We note that while the Lie symmetry

algebra is six-dimensional for the heat equation, as  $Q_0$  is a function of  $v$  as well as  $x$  and  $t'$ , we get a seven-dimensional algebra for (4.6), even though there are no derivatives of  $v$ .

With consideration of the initial and boundary conditions we use the symmetry with generator  $\Gamma = \Gamma_2 + \Gamma_7 = x \frac{\partial}{\partial x} + 2t' \frac{\partial}{\partial t'} + Q_0 \frac{\partial}{\partial Q_0}$ . This leads to an invariant solution of the form  $Q_0 = (t')^{\frac{1}{2}} \phi(z)$  where  $z = \frac{x}{\sqrt{t'}}$ . Substitution of this invariant form into (4.6) yields the reduced equation

$$v\phi'' + z\phi' - \phi = 0$$

which needs to be solved subject to  $\lim_{z \rightarrow \infty} \phi = 0$ ,  $\lim_{z \rightarrow -\infty} \phi \rightarrow -z$ .

Hence we get that  $\phi(z) = \frac{\sqrt{v}}{\sqrt{2\pi}} \exp(-\frac{z^2}{2v}) - \frac{z}{2} \operatorname{erfc}(\frac{z}{\sqrt{2v}})$  and so

$$Q_0(x, v, t') = \frac{\sqrt{vt'}}{\sqrt{2\pi}} \exp(-\frac{x^2}{2vt'}) - \frac{x}{2} \operatorname{erfc}(\frac{x}{\sqrt{2vt'}}).$$

Now collecting terms of  $O(\epsilon^{1/2})$  we get that  $Q_1(x, v, t')$  satisfies

$$\frac{\partial Q_1}{\partial t'} = \frac{v}{2} \frac{\partial^2 Q_1}{\partial x^2} + xv \frac{\partial^2 Q_0}{\partial x^2} + r \frac{\partial Q_0}{\partial x} + \rho b v^2 \frac{\partial^2 Q_0}{\partial x \partial v} \quad (4.7)$$

subject to  $Q_1(x, v, 0) = 0$ ,  $\lim_{x \rightarrow \infty} Q_1(x, v, t') = 0$ ,  $\lim_{x \rightarrow -\infty} Q_1(x, v, t') = -rt'$ . The solution to this problem is <sup>3</sup>

$$Q_1(x, v, t') = -\frac{rt'}{2} \operatorname{erfc}(\frac{x}{\sqrt{2t'v}}) - \frac{\sqrt{2}\rho b}{8\sqrt{\pi}} xv^{\frac{1}{2}}(t')^{\frac{1}{2}} \exp(-\frac{x^2}{2t'v}) + \frac{v^{\frac{1}{2}}(t')^{\frac{1}{2}}x}{2\sqrt{2\pi}} \exp(-\frac{x^2}{2t'v}).$$

The two-term inner expansion can then be found by  $Q_0(x, v, t') + \sqrt{\epsilon}Q_1(x, v, t')$ .

We then match the inner and outer solutions to get a solution that is uniformly valid by calculating ‘outer + inner - common’ where ‘common’ is that part of the solution that is common to both. In this case as  $\epsilon \rightarrow 0$  the inner solution is the same as the outer solution and so the outer expansion is in fact the common expansion. This means that the inner expansion

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<sup>3</sup>For the terms  $xv(Q_0)_{xx}$  and  $r(Q_0)_x$  we use the results [23] that (i) if  $u_t = \frac{1}{2}u_{xx}$  and  $v_t = \frac{1}{2}v_{xx} + u$  then a particular solution is  $v = tu$  and (ii) if  $u_t = \frac{1}{2}u_{xx}$  and  $v_t = \frac{1}{2}v_{xx} + xu$  then a particular solution is  $v = xtu + \frac{1}{2}t^2u_x$ .

is uniformly valid. In terms of the original variables our approximate solution to (3.2a-f) is then

$$P(S, v, \tau) = \frac{\sqrt{v\tau}}{\sqrt{\pi}} e^{-\frac{(S-K)^2}{2v\tau K^2}} \left[ \frac{K}{\sqrt{2}} - \frac{\sqrt{2}}{8} \rho b (S-K) + \frac{(S-K)}{2\sqrt{2}} \right] - \frac{1}{2} \operatorname{erfc} \left( \frac{S-K}{K\sqrt{2v\tau}} \right) (S-K+r\tau K). \quad (4.8)$$

We note that it is not difficult to find the next term in the expansion (4.5). Collecting terms of  $O(\epsilon)$  in (4.4) we get that  $Q_2(x, v, t')$  needs to satisfy

$$\frac{\partial Q_2}{\partial t'} = \frac{v}{2} \frac{\partial^2 Q_2}{\partial x^2} + q(x, v, t') \quad (4.9)$$

where

$$q(x, v, t') = \begin{aligned} & xv \frac{\partial^2 Q_1}{\partial x^2} + r \frac{\partial Q_1}{\partial x} + \rho b v^2 \frac{\partial^2 Q_1}{\partial x \partial v} + \frac{v}{2} x^2 \frac{\partial^2 Q_0}{\partial x^2} + \rho b v^2 x \frac{\partial^2 Q_0}{\partial x \partial v} + \frac{b^2 v^3}{2} \frac{\partial^2 Q_0}{\partial v^2} + \\ & rx \frac{\partial Q_0}{\partial x} + kv(a_0 - v) \frac{\partial Q_0}{\partial v} - rQ_0 \end{aligned} \quad (4.10)$$

subject to  $Q_2(x, v, 0) = 0$ ,  $\lim_{x \rightarrow \pm\infty} Q_2(x, v, t') = 0$ , and where  $a_0 = a(0)$ .

The solution for  $Q_2$  is

$$\begin{aligned} Q_2(x, v, t') &= \frac{1}{\sqrt{2\pi v}} \int_0^{t'} \frac{1}{\sqrt{t'-z}} \int_{-\infty}^{\infty} q(\xi, v, z) e^{\frac{-(x-\xi)^2}{2v(t'-z)}} d\xi dz \\ &= \frac{e^{-\frac{x^2}{2t'v}}}{96\sqrt{2\pi}} \left[ \frac{3x^4}{\sqrt{t'v}} (\rho b - 2)^2 + 2\sqrt{t'} x^2 \left\{ \sqrt{v} (-\rho^2 b^2 - 4 + 2b^2) + \frac{12r}{\sqrt{v}} (\rho b - 2) \right\} \right. \\ &\quad \left. + (t')^{\frac{3}{2}} \left\{ v^{\frac{3}{2}} (12\rho b - 24k - 4 - 4b^2 - 7\rho^2 b^2) + 24\sqrt{v} (-r\rho b + ka_0 - 2r) \right. \right. \\ &\quad \left. \left. + \frac{48r^2}{\sqrt{v}} \right\} \right]. \end{aligned}$$

Using  $Q_0, Q_1$  and  $Q_2$  then leads to the following approximate solution in terms of the original variables:

$$\begin{aligned}
P(S, v, \tau) = & \frac{\sqrt{v\tau}}{\sqrt{\pi}} e^{-\frac{(S-K)^2}{2v\tau K^2}} \left[ \frac{K}{\sqrt{2}} - \frac{\sqrt{2}}{8} \rho b (S-K) + \frac{(S-K)}{2\sqrt{2}} \right] - \frac{1}{2} \operatorname{erfc} \left( \frac{S-K}{K\sqrt{2v\tau}} \right) (S-K+r\tau K) \\
& + \frac{e^{-\frac{(S-K)^2}{2v\tau K^2}}}{96\sqrt{2\pi}} \left[ \frac{3(S-K)^4}{K^3\sqrt{v\tau}} (\rho b - 2)^2 + \frac{2\sqrt{\tau}(S-K)^2}{K} \left\{ \sqrt{v}(-\rho^2 b^2 - 4 + 2b^2) + \frac{12r}{\sqrt{v}}(\rho b - 2) \right\} \right. \\
& + K(\tau)^{\frac{3}{2}} \left\{ v^{\frac{3}{2}}(12\rho b - 24k - 4 - 4b^2 - 7\rho^2 b^2) + 24\sqrt{v}(-r\rho b + ka_0 - 2r) \right. \\
& \left. \left. + \frac{48r^2}{\sqrt{v}} \right\} \right]. \tag{4.11}
\end{aligned}$$

## 5 Comparison of Exact and Approximate Solutions

We now test the accuracy of the new approximations (4.8) and (4.11) with the exact solution (3.1). Firstly, using parameter values  $k = 32.88, b = 7.9, a_0 = 0.1147, \rho = -0.7321$  as given in [9]<sup>4</sup>, as well as  $K = 20, v = 0.1, r = 0.05$  a plot showing the comparison of solutions (3.1) and (4.8) at times to expiry  $\tau = \frac{1}{12}$  years and  $\tau = \frac{1}{6}$  years is given in Figure 2.

From Figure 2, it can be seen that while as expected, a smaller  $\tau$  produces the better fit, the approximate solution (4.8) still provides a reasonably good fit to the solution (3.1) even for  $\tau = \frac{2}{12}$ . It can also be observed that when  $S$  is near the strike  $K$ , the approximate solution (4.8) slightly undervalues in-the-money options and slightly overvalues out-of-the-money options.

Now using the same parameter values we compute signed relative errors i.e.  $\frac{P_{true} - P_{approx}}{P_{true}} \times 100\%$  for different interest rates  $r$ , using approximate solutions (4.8) and (4.11). The results are listed in Table 1.

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<sup>4</sup>These parameter estimates were obtained by minimising option implied volatility mean square errors using option quotes between 4/1/1996 - 31/12/2004.

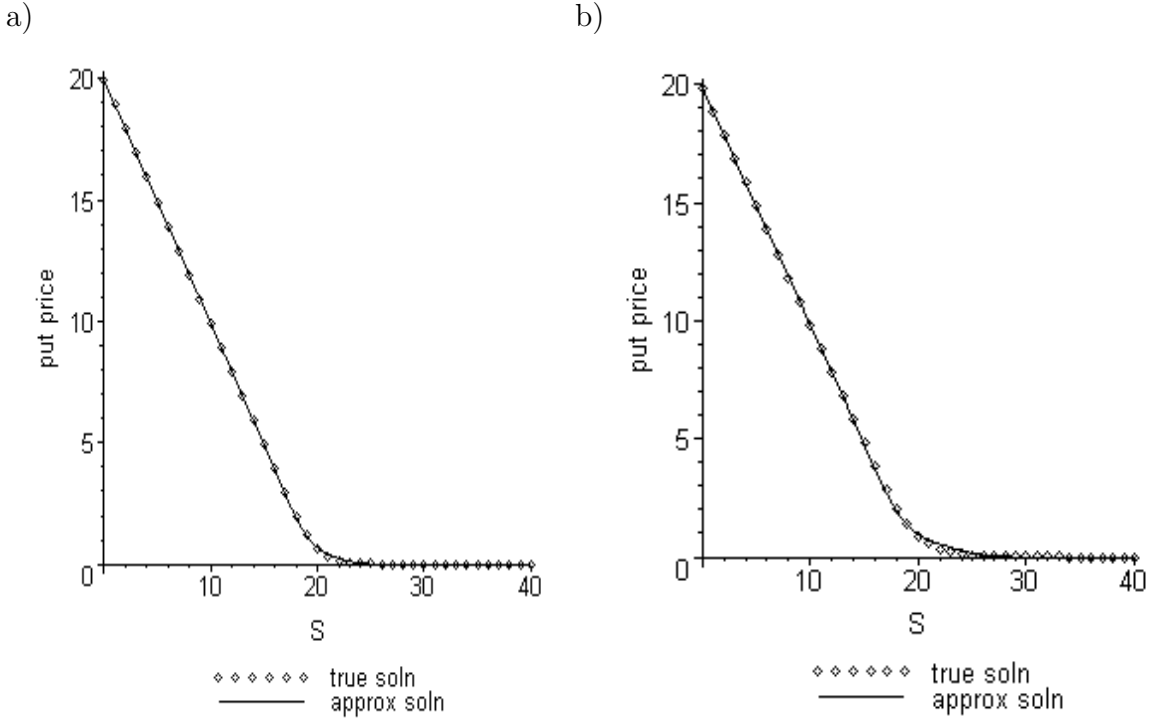


Figure 2: Exact solution (3.1) versus approximate solution (4.8) at times to expiry a)  $\tau = \frac{1}{12}$  and b)  $\tau = \frac{2}{12}$ . Other parameters used were  $k = 32.88, b = 7.9, a_0 = 0.1147, \rho = -0.7321, K = 20, v = 0.1, r = 0.05$ .

In particular, from Table 1 we see that:

As expected, the higher-order approximation (4.11) mostly yields better results than (4.8), especially near at-the-money options. The approximation produces relative errors for in- and at-the-money options less than  $\frac{1}{4}\%$  for  $\tau = 1/12$  and less than 1% for  $\tau = 2/12$ . Approximation (4.8) however still produces good results with relative errors less than 1% for  $\tau = 1/12$  and less than 1.3% for  $\tau = 2/12$  for in-the-money options.

We should point out that while the relative errors for out-of-the-money options ( $S = 22$ ) appear large, in absolute terms the errors are quite small as the actual option values themselves are very small (see Figure 2). For example, an 11.38% relative error when  $\tau = \frac{1}{12}, r = 0.1, S = 22$ , corresponds to an absolute error  $< 0.02$ .

$r$	$S$	$\tau = \frac{1}{12}$		$\tau = \frac{2}{12}$	
		Eqn(4.8)	Eqn(4.11)	Eqn(4.8)	Eqn(4.11)
0.01	15	0.13%	-0.19%	1.01%	-0.47%
	17	0.75%	0.03%	1.26%	0.93%
	20	-4.09%	-0.15%	-8.09%	0.13%
	22	-11.08%	-11.4%	-24.06%	-14.4%
0.05	15	0.13%	-0.16%	0.98%	-0.4%
	17	0.69%	0.04%	1.08%	0.9%
	20	-3.9%	-0.22%	-7.96%	-0.14%
	22	-11.89%	-11.36%	-25.19%	-14.9%
0.1	15	0.14%	-0.16%	0.97%	-0.31%
	17	0.64%	-0.24%	0.94%	0.89%
	20	-3.3%	-0.31%	-7.14%	-0.43%
	22	-12.57%	-11.38%	-25.8%	-15.67%

Table 1: Signed relative errors of approximations.

## 6 Conclusion

By far the most popular stochastic volatility model in the literature seems to be the Heston model, as it generates exact solutions to derivative prices. While the Heston model does generate non-Gaussian returns, empirically it has been found that the skewness and kurtosis it generates are too small to be consistent with equity index returns (see Jones [8]). The 3/2-stochastic model (1.5b) has shown to empirically perform better than the Heston model. It has a higher power-law in the diffusion term of 1.5 which can reduce the heteroskedasticity of volatility. As well, the drift is a quadratic rather than linear function of volatility. As in this case the mean-reversion speed is a linear function of the volatility, the speed of reversion increases with volatility. This generates a balancing effect of a stronger mean reversion with higher volatility. However many unanswered questions remain on the fit of market prices. We have shown how to use symmetries to systematically find the similarly simple solution to European options when the underlying stochastic model includes an arbitrary function of time. This solution however still involves the evaluation of integrals. We have also derived simple analytic approximations for the solution in terms of standard, transcendental functions.

These can be used to generate fast and accurate answers for options with short tenor. Such options with short tenor are extremely popular in the market and so these approximations could be very useful to practitioners.

## References

- [1] F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.* **81** (1973) 637-659.
  
- [2] J. Hull, A. White, The pricing of options on assets with stochastic volatility, *J. Financ.* **42** (1987) 281-300.
  
- [3] L.O. Scott, Option pricing when the variance changes randomly: Theory, estimation and an application, *J. Financ. Quant. Anal.* **22** (1987) 419-438.
  
- [4] J. Wiggins, Option values under stochastic volatility: Theory and empirical estimates, *J. Financ. Econ.* **19** (1987) 351-372.
  
- [5] E.M. Stein, J.C. Stein, Stock price distributions with stochastic volatility: An analytical approach, *Rev. Financ. Stud.* **4** (1991) 727-752.
  
- [6] S.L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.* **6**, (1993) 327-343.

- [7] G. Chacko, L.M. Viceira, Spectral GMM estimation of continuous-time processes, *J. Econometrics* **116** (2003) 259-292.
- [8] C. Jones, The dynamics of stochastic volatility: evidence from underlying and options markets, *J. Econometrics* **116**, (2003) 181-224.
- [9] P. Chrisoffersen, K. Jacobs, K. Mimouni, Volatility dynamics for the S&P500: Evidence from realized volatility, daily returns and option prices, *Rev. Financ. Stud.* **23(8)** (2010) 3141-3189.
- [10] J. Goard, M. Mazur, Stochastic volatility models and the pricing of VIX options, to appear in *Math. Financ.* (2012).
- [11] J.Y. Campbell, A.W. Lo, A.C. MacKinlay, *The econometrics of financial markets*, Princeton University Press, Princeton, New Jersey, (1996).
- [12] A.L. Lewis, *Option valuation under stochastic volatility*, Finance Press, Newport Beach CA, (2000).
- [13] M. Abramowitz, M. I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, (1965).



- [14] S. Heston, A simple new formula for options with stochastic volatility, Manuscript: John M. Olin, School of Business, Washington University, (1997).
- [15] P. Wilmott, Derivatives: The theory and practice of financial engineering, John Wiley and Sons, New York, (1997).
- [16] P. Carr, J. Sun, A new approach for option pricing under stochastic volatility, Rev. Deriv. Res. **10** (2007) 87-150.
- [17] J. Sherring, DIMSYM users manual, La Trobe University, Melbourne, (1993).
- [18] G.W. Bluman, S. Kumei, Symmetries and differential equations, Springer-Verlag, New York, (1989).
- [19] J.M. Goard, Noninvariant boundary conditions, Appl. Anal. **82(5)** (2003) 473-481.
- [20] S. Zhu, W. Chen, A predictor-corrector scheme based on the ADI method for pricing American puts with stochastic volatility, Comput. Math. Appl. **62** (2011) 1-26.
- [21] A.D. Polyanin, V.F. Zaitsev, Handbook of exact solutions of ordinary differential equations, CRC Press, Boca Raton, Florida (1995).
- [22] Maplesoft, Maple 12 Users manual, Maplesoft, Waterloo, Canada (2008).
- [23] S. Howison, Matched asymptotic expansions in financial engineering, J. Eng. Math. **53** (2005) 385-406.

## 7 Appendix : Symmetries of (3.2a)

With the help of Dimsym [17] we find that equation (3.2a) has the finite-dimensional symmetry with generator  $G = F(v, \tau)P \frac{\partial}{\partial P} + \bar{T}(\tau) \frac{\partial}{\partial \tau} - v\bar{T}'(\tau) \frac{\partial}{\partial v} - Sg(\tau)(1 + \frac{2\rho}{b}) \frac{\partial}{\partial S}$  where  $F(v, \tau), \bar{T}(\tau), a(\tau)$  and  $g(\tau)$  satisfy the following determining equations:

1. 
$$g''''[2a''ak\rho - a''br - 3(a')^2k\rho + 2a'a^2k^2\rho - a'abkr] + g''''[-2a'''ak\rho + a'''br + 4a''a'k\rho - 2a''a^2k^2\rho + a''abkr - 4(a')^2ak^2\rho + (a')^2bkr] + g''[3a'''a'k\rho + 4(a'')^2k\rho - 3a''a'ak^2\rho + 2a''a'bkr - 2a''a^3k^3\rho + a''a^2bk^2r + 9(a')^3k^2\rho + (a')^2a^2k^3\rho + 2(a')^2abk^2r - 2a'a^4k^4\rho + a'a^3bk^3r] + g'[a'''a'ak^2\rho - 2a'''a'bkr + 2a'''a^3k^3\rho - a'''a^2bk^2r - 2(a'')^2ak^2\rho + 3(a'')^2bkr + a''(a')^2k^2\rho - 13a''a'a^2k^3\rho + 4a''a'abk^2r + 2a''a^4k^4\rho - a''a^3bk^3r + 14(a')^3ak^3\rho - 2(a')^3bk^2r - 4(a')^2a^3k^4\rho + 2(a')^2a^2bk^3r] = 0$$
2. 
$$g''''[a'v\{b^3 + b^2\rho + bk - 2b\rho^2 + 2k\rho\} - a'ak(b + 2\rho)] + g''[a''v\{-b^3 - b^2\rho - bk + 2b\rho^2 - 2k\rho\} + a'v\{-ab^3k - ab^2k\rho - abk^2 + 2abk\rho^2 - 2ak^2\rho\} + a''ak(b + 2\rho) + a'a^2k^2(b + 2\rho)] + g'[a''av\{b^3k + 3b^2k\rho + bk^2 + 2bk\rho^2 + 2k^2\rho\} - a''vb^2r(b + 2\rho) - a''a^2k^2(b + 2\rho) + (a'')^2v\{-2b^3k - 5b^2k\rho - 2bk^2 - 2bk\rho^2 - 4k^2\rho\} + 2(a')^2ak^2(b + 2\rho) + 2a'a^2vk^2b\rho(b + 2\rho) - a'avk\rho b^2(b + 2\rho)] + F_\tau[-a''vb^3r + 2a''avb^2k\rho - 3(a')^2vb^2k\rho + 2a'va^2b^2k^2\rho - a'vab^3kr] = 0$$
3. 
$$\bar{T}(\tau)[2a''abk^2\rho r - a''b^2kr^2 - 3(a')^2bk^2\rho r + 2a'a^2bk^3\rho r - a'ab^2k^2r^2] + g''''(-2abk\rho - 4ak\rho^2 + b^2r + 2b\rho r) + g''(3a'bkr + 6a'k\rho^2) + g'(b + 2\rho)(a'ak^2\rho - 2a'bkr + 2a^3k^3\rho - a^2bk^2r) = 0$$
4. 
$$(b + 2\rho)[g'''a' - g''a'' - g''a'ak + g'a''ak - 2g'(a')^2k] + F_v(v, \tau)v^2b^2[-2a''ak\rho + a''br + 3(a')^2k\rho - 2a'a^2k^2\rho + a'abkr] = 0$$

### Figure captions

Figure 1. Put option values with  $T = 1$ ,  $K = 20$  using BS formula (bold line) and 3/2 stochastic volatility formula (3.1) with  $a(\tau) = 0.3$  (dotted line).

Figure 2. Exact solution (3.1) versus approximate solution (4.8) at times to expiry a)  $\tau = \frac{1}{12}$  and b)  $\tau = \frac{2}{12}$ . Other parameters used were  $k = 32.88, b = 7.9, a_0 = 0.1147, \rho = -0.7321, K = 20, v = 0.1, r = 0.05$ .