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A tale of three varieties

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A TALE OF THREE VARIETIES

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This thesis is my own work submitted to the University of Wollongong, and has not been submitted for a degree to any other University or Institution.

Carolyn E. McPhail

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A long time ago, in a galaxy far, far away, there were three distinguished and well-studied classes of topological groups. However, little was known about the three varieties of topological groups they generate. This thesis tells the tale of these varieties. It becomes apparent that the three varieties are all distinct and that one is in fact the variety of all abelian topological groups, that is, their entire universe. The other two are comparatively small, however, no less important and interesting. A framework is established within which the three varieties are compared with each other and also come up against the varieties generated by other important classes of topological groups. Discoveries are made that will stun even the most brave....
List of Symbols

\( \mathbb{N} \)  Set of all natural numbers

\( \mathbb{Z} \)  Topological group consisting of the additive group of integers equipped with the discrete topology (Example 1.2.2(a), page 8)

\( \mathbb{R} \)  Topological group consisting of the additive group of real numbers equipped with the Euclidean topology (Example 1.2.2(b), page 8)

\( \mathbb{T} \)  Compact circle group identified with the quotient group \( \mathbb{R}/\mathbb{Z} \) (Example 1.2.2(c), page 8)

\( \mathbb{Q} \)  Topological group consisting of the additive group of rational numbers with the Euclidean topology induced by \( \mathbb{R} \) (Example 1.3.12(e), page 14)

\([0,1]\)  Topological space consisting of the unit interval \([0,1]\) with the topology induced from \( \mathbb{R} \) (Remark 1.4.11, page 19)

\( \Sigma \Omega \)  Subgroups of topological groups in \( \Omega \) (Definition 1.3.1, pages 9–10)

\( \bar{\Sigma} \Omega \)  Closed subgroup of topological groups in \( \Omega \) (Definition 1.3.2, page 10)

\( \Omega \Omega \)  Quotient groups of topological groups in \( \Omega \) (Definition 1.3.1, pages 9–10)

\( \bar{\Omega} \Omega \)  Hausdorff quotient groups of topological groups in \( \Omega \) (Definition 1.3.2, p10)

\( C \Omega \)  Cartesian products of topological groups in \( \Omega \) (Definition 1.3.1, pages 9–10)

\( P \Omega \)  Finite products of topological groups in \( \Omega \) (Definition 1.3.1, pages 9–10)

\( T(m)\)-group  Definition 1.3.4, page 11

\( \mathfrak{X}_m \)  Variety of all \( T(m)\)-groups, \( m \) a cardinal number (Definition 1.3.4, page 11)

\( \text{card}(G) \)  Cardinality of the set (space, group) \( G \) (Remark 1.3.5(b), page 11)

NSS-group  Topological group with no small subgroups (Definition 1.3.6, page 12)

\( \mathfrak{W}(\Omega) \)  Variety of topological groups generated by a class of topological groups \( \Omega \) (Definition 1.4.3, page 16)

\( c \)  Cardinality of the continuum, that is, \( \text{card}(\mathbb{R}) \) (Remark 1.4.9, page 18)

\( \mathbb{N}_0 \)  Cardinality of \( \mathbb{N} \) (Proposition 1.4.10, page 19)

\( L_m \)  Class of all locally-\( m \) groups, \( m \) an infinite cardinal number (Proposition 1.4.12, page 20)
\[ |G| \quad \text{Group underlying the the topological group } G \text{ (page 20)} \]

\[ |\Omega| \quad \text{Class of all groups underlying the topological groups in } \Omega \text{ (page 20)} \]

\[ |\Omega|_2 \quad \text{Class of all groups underlying the Hausdorff topological groups in } \Omega \text{ (page 23)} \]

\[ \mathfrak{A}_m \quad \text{Variety of all abelian } T(m)\text{-groups, } m \text{ a cardinal number (Theorem 1.6.6, page 25)} \]

\[ F(X) \quad \text{Free topological group on the space } X \text{ (Definition 2.2.1, pages 29–30)} \]

\[ FA(X) \quad \text{Free abelian topological group on the space } X \text{ (Definition 2.2.2, page 30)} \]

\[ FA[0,1] \quad \text{Free abelian topological group on } [0,1] \text{ (page 50)} \]

\[ k_\omega\text{-group} \quad \text{Definition 2.2.9, page 36} \]

\[ \mathcal{B} \quad \text{Class of all topological groups underlying Banach spaces (page 48)} \]

\[ \mathcal{A} \quad \text{Variety of all abelian topological groups (Theorem 2.4.3, page 49)} \]

\[ FLCV(X) \quad \text{Free locally convex topological vector space on the space } X \text{ (Definition 2.4.4, page 50)} \]

\[ \mathcal{M} \quad \text{Class of all abelian metrizable topological groups (Theorem 2.5.11, page 57)} \]

\[ \mathcal{N} \quad \text{Class of all abelian pseudometrizable topological groups (Theorem 2.5.16, page 60)} \]

\[ \mathcal{L}_A \quad \text{Class of all locally compact Hausdorff abelian topological groups (page 62)} \]

\[ \mathcal{L} \quad \text{Class of all locally compact Hausdorff topological groups (page 62)} \]

\[ G_o \quad \text{Connected component of the identity in the topological group } G \text{ (Theorem 3.1.4, page 62)} \]

\[ \text{LCV-space} \quad \text{A real Hausdorff locally convex topological vector space (page 67)} \]

\[ \mathcal{K}_\omega \quad \text{Class of all abelian } k_\omega\text{-groups (page 72)} \]

\[ \mathcal{C}_\sigma \quad \text{Class of all abelian } \sigma\text{-compact groups (page 77)} \]

\[ \mathcal{D} \quad \text{Class of all discrete abelian groups (Theorem 4.4.11, page 105; page 124)} \]

\[ \mathcal{P} \quad \text{Class of all abelian } \mathcal{P}\text{ groups for some property } \mathcal{P} \text{ (page 123)} \]

\[ \mathcal{L}_\mathcal{P} \quad \text{Class of all abelian locally } \mathcal{P}\text{ groups for some property } \mathcal{P} \text{ (page 123)} \]

\[ \mathcal{L}_\sigma \quad \text{Class of all abelian locally } \sigma\text{-compact topological groups (page 125)} \]
$B_S$  Class of all topological groups underlying separable Banach spaces
   (Proposition 5.3.4, page 129; page 131)

$S$  Class of all abelian separable topological groups (page 131)

$c^+$  Smallest cardinal number strictly greater than $c$ (Proposition 5.3.11, page 132)

$L_S$  Class of all abelian locally separable topological groups (page 133)

$L_m$  Class of all abelian locally-$m$ topological groups, $m$ an infinite cardinal number (page 136)

$C$  Class of all varieties respectively generated by all locally-$m$ groups for each
    infinite cardinal number $m$ (Theorem 5.4.1, page 137)

$C_m$  Class of all abelian topological groups of cardinality less than or equal to $m$,
     $m$ a cardinal number (Theorem 5.4.6, page 138)

$C_c$  Class of all abelian topological groups of cardinality less than or equal to $c$
     (page 138)

$\mathfrak{W}(\Omega)$  Wide variety of topological groups generated by the class of topological
     groups $\Omega$ (Definition 6.1.5, page 142)
Introduction

A variety of topological groups is a class of topological groups closed under the operations of forming subgroups, quotient groups and arbitrary products. The variety generated by a class of topological groups is the smallest variety containing the class. In fact, this variety consists of all topological groups topologically isomorphic to a quotient group of a subgroup of a product of groups in the given class.

This thesis is the tale of the three varieties of abelian topological groups respectively generated by three distinguished classes of topological groups: the class of all topological groups underlying Banach spaces, the class of all locally compact Hausdorff abelian groups and the “class” consisting solely of the free abelian topological group on $[0, 1]$.

The aim of our analysis is to characterize each of the varieties, determining which topological groups are contained in each, and the relationships amongst the three. We study these three varieties as the classes they are generated by are all well-studied and well documented in the literature. However, the varieties they generate are not.

Our first main result characterizes the variety generated by the class of all topological groups underlying Banach spaces. It turns out to be the variety of all abelian topological groups. In other words, every abelian topological group is a quotient group of a subgroup of a product of Banach spaces.

After examining the variety generated by all locally compact Hausdorff abelian topological groups, we have a full understanding of connected and also of locally connected topological groups in this variety. In fact, all connected groups in this variety are in the variety generated by $\mathbb{R}$, the additive group of real numbers with the Euclidean topology. In particular, then, the normed vector spaces which appear are those which are finite dimensional. Therefore, the variety generated by all locally compact Hausdorff abelian topological groups is a very long way from the variety of all abelian topological groups.
As we compare the variety generated by all locally compact Hausdorff abelian topological groups with the variety generated by the free abelian topological group on $[0, 1]$, we find that neither variety contains the other. We see that the variety generated by the free abelian topological group on $[0, 1]$ fails to contain any uncountable discrete abelian groups and therefore does not contain all locally compact Hausdorff abelian groups. However, the variety generated by the free abelian topological group on $[0, 1]$ together with the class of all discrete abelian groups does contain all locally compact Hausdorff abelian groups.

Our third variety of topological groups generated by the free abelian topological group on $[0, 1]$, produces the most surprising of all our results. This one abelian $k$-group generates every abelian $k$-group, which demonstrates how rich the free abelian topological group on $[0, 1]$ is. Further to this, however, we discover that this third variety is generated by many different free abelian topological groups. More specifically, we characterize those compact Hausdorff spaces whose free abelian topological group generates the variety.

To show how distant the varieties of topological groups respectively generated by locally compact Hausdorff abelian groups and the free abelian topological groups on $[0, 1]$ are from the variety of all abelian topological groups (generated by the class of all Banach spaces), we build a chain of varieties starting from the former two. We discover an infinite chain of varieties each of which contains the class of all locally compact Hausdorff abelian groups and the free abelian topological group on $[0, 1]$. Further, the varieties in this chain are totally ordered and their union is the variety of all abelian topological groups.

As a postscript to our main varietal analysis, we present a short chapter on the wide varieties generated by all the classes of topological groups we meet along the way. As we have quite a number of classes to consider, it is not unreasonable to expect a number of wide varieties as well. However this is not the case, as from all the classes of topological groups we consider in this thesis, only a few distinct wide varieties are generated.
Chapter 1

Basic Definitions and Results

In this chapter, the main concepts used in the thesis will be introduced, in particular, variety of groups and variety of topological groups. A few new ideas useful for our analysis shall be introduced and a brief overview of varieties generated by classes of topological groups will be given. The condition of Hausdorffness which usually appears in the definition of topological groups is noticeably absent and this will be justified when considering the class of groups underlying Hausdorff groups in a variety of topological groups.

§1.1 Varieties of Groups

It is well known that every set admits at least one group structure. Every finite set admits a finite cyclic group structure and every infinite set admits a free group structure. Hence there is a proper class (rather than a set) of non-isomorphic groups. A natural way to approach the study of groups is to consider classes of groups having similar properties (for example, abelian or nilpotent). The most important of these classes, varieties of groups, were introduced by Birkhoff in [5] and Neumann in [55]. Varieties of groups can be defined as equational classes or equivalently in terms of closure under certain operators. A standard reference covering the material that existed until 1967 concerning varieties of groups is [56]. We shall define a variety of groups in terms of operators and for this we introduce the following.

Definition 1.1.1. Let \( \mathcal{X} \) be a class of groups. The operators \( S, Q, C \) and \( P \) are defined on \( \mathcal{X} \) to give classes of groups as follows. Let \( G \) be a group. Then

- \( G \in S\mathcal{X} \) if \( G \) is isomorphic to a subgroup of a group in \( \mathcal{X} \),
- \( G \in Q\mathcal{X} \) if \( G \) is isomorphic to a quotient group of a group in \( \mathcal{X} \),
- \( G \in C\mathcal{X} \) if \( G \) is isomorphic to an infinite or a finite direct product of groups in \( \mathcal{X} \), and
- \( G \in P\mathcal{X} \) if \( G \) is isomorphic to a finite direct product of groups in \( \mathcal{X} \).
In other words, \( S\mathcal{X} \) is the class of all subgroups of groups isomorphic to members of \( \mathcal{X} \). Similarly, \( Q\mathcal{X} \) is the class of all quotient groups, and \( C\mathcal{X} \) the class of all infinite and finite products of groups isomorphic to members of \( \mathcal{X} \).

We also have “composition” of these operators. For example, \( G \in QS\mathcal{X} \) if it is isomorphic to a quotient group of a subgroup of a member of \( \mathcal{X} \).

We wish to consider classes of groups that are closed under the operations \( S \), \( Q \) and \( C \), that is, where the operators return groups that are already contained in the class. We define this as follows.

**Definition 1.1.2.** ([56], Definition 15.21) A class of groups \( \mathcal{X} \) is said to be *closed* under the operator \( S \) (respectively \( Q \), \( C \), \( P \)) if \( S\mathcal{X} \subseteq \mathcal{X} \) (respectively \( Q\mathcal{X} \subseteq \mathcal{X} \), \( C\mathcal{X} \subseteq \mathcal{X} \), \( P\mathcal{X} \subseteq \mathcal{X} \)).

We are now in a position to define formally varieties of groups.

**Definition 1.1.3.** (cf. [56], Proposition 15.23) A class of groups is said to be a *variety of groups* if it is closed under \( S \), \( Q \) and \( C \).

**Notation.** It should be noted that 1 or \( e \) will usually be used to denote the identity elements of all groups. When the group containing a given identity is not clear from the context, we use \( 1_G \) or \( e_G \) to denote the identity element.

**Examples 1.1.4.**

(a) Let \( \mathcal{X} \) be the class of all abelian groups. Clearly all subgroups, quotient groups and products of abelian groups are also abelian. Therefore, \( \mathcal{X} \) is a variety of groups.

Note that the class of all non-abelian groups is *not* a variety of groups as every group has an abelian subgroup, namely the subgroup generated by any one element of the group. Thus, this class is not closed under \( S \).
(b) Let \( n \in \mathbb{N} \) and let \( \mathcal{X}_n \) be the class of all groups that satisfy \( x^n = 1 \) for all \( x \); that is,
\[
G \in \mathcal{X}_n \text{ if and only if for all } g \in G, \ g^n = 1 \text{ where } 1 \text{ is the identity in } G.
\]
Then \( \mathcal{X}_n \) is closed under \( S, Q \) and \( C \) and is thus a variety of groups.

As we mentioned earlier, a variety of groups can be defined as an equational class. More specifically, a variety of groups is defined as the class of all groups satisfying a certain family of “laws” or “equations” [56]. We shall give a brief overview of this approach, but first, we need to introduce the appropriate notation.

**Notation.** (cf. [56]) Let \( X \) be a set of symbols \( a_1, a_2, \ldots \). We shall denote these symbols also by \( a_1, a_2, \ldots \) and we construct another set, \( X^{-1} = \{a_1^{-1}, a_2^{-1}, \ldots\} \), in one-to-one correspondence with the first set. Further, for \( n \in \mathbb{N} \), we shall use \( a_i^n \) to represent \( n \) repetitions of the symbol \( a_i \in X \) and \( a_i^{-n} \) to represent \( n \) repetitions of the symbol \( a_i^{-1} \in X^{-1} \). An expression
\[
w = w(a_{i_1}, a_{i_2}, \ldots, a_{i_m}) = x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n},
\]
where \( x_i \in \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\} \subseteq X \) and \( \mu_i \in \mathbb{Z}, \ \mu_i \neq 0 \), that is, an ordered system of a finite number of symbols of the form \( a_i^m \), \( m \in \mathbb{Z}, m \neq 0 \) is called a word. For example,
\[
w(a_1, a_2) = a_1 a_2^{-1} a_1^{-2} a_2 \text{ is a word and } w(a_1, a_2, \ldots, a_n) = a_1 a_2 \ldots a_n a_1^{-1} a_2^{-1} \ldots a_n^{-1} \text{ is a word in } n \text{ variables.}
\]
Finally, given \( w_1 = x_1^{\mu_1} x_2^{\mu_2} \ldots x_i^{\mu_i} \), and \( w_2 = x_i^{\mu_i} x_j^{\mu_j} \ldots x_m^{\mu_m} \), we shall define a third word \( w_1 w_2 \) by
\[
w_1 w_2 = x_1^{\mu_1} x_2^{\mu_2} \ldots x_i^{\mu_i} x_j^{\mu_j} \ldots x_m^{\mu_m}.
\]

Now, given a mapping \( \alpha \) of the symbols \( a_1, a_2, \ldots \) into a group \( G \) with \( \alpha(a_1) = g_1 \), \( \alpha(a_2) = g_2, \ldots \), we say that the word \( w(a_1, a_2, \ldots, a_n) \) defines the element in \( G \) given by \( w(g_1, g_2, \ldots, g_n) \). This element is called the value of the word in the group \( G \). For simplicity, we will use the convenient language mentioned earlier, where the \( a_i \) are described as variables, although the domain of these ‘variables’ is not defined: it is sometimes one group, sometimes another.
Definition 1.1.5. (cf. [56], Chapter 1, Section 2) Let $G$ be a group with identity $1$. A word $w(a_1, a_2, \ldots, a_n)$ is said to be a law in the group $G$ if $w(g_1, g_2, \ldots, g_n) = 1$ for all combinations of $g_1, g_2, \ldots, g_n \in G$. We say the group $G$ satisfies the law $w(a_1, a_2, \ldots, a_n) = 1$.

Remark 1.1.6. Abelian groups satisfy a simple law. A group $G$ is abelian if and only if it satisfies the law $w(a_1, a_2) = (a_1)^{-1}(a_2)^{-1}a_1a_2 = 1$. Thus, the variety of all abelian groups (Example 1.1.4(a)) is the class of all groups that satisfy the law $(a_1)^{-1}(a_2)^{-1}a_1a_2 = 1$.

In [56], Definition 14.1, a variety of groups is “the class of all groups satisfying each one of a given set of laws”. For example, the class of all abelian groups of exponent $n$ is the class of all groups that satisfy the laws $(a_1)^{-1}(a_2)^{-1}a_1a_2 = 1$ and $(a_3)^n = 1$. Hence, this class is a variety of groups. From the following results, we see that our definition is consistent with this definition of a variety of groups.

Proposition 1.1.7. (cf. [56], Definition 14.1) Let $\mathcal{X}$ be the class of all groups $G$ satisfying the law $w(a_1, a_2, \ldots, a_n) = 1$. Then $\mathcal{X}$ is a variety of groups.

Remarks 1.1.8. A group $G$ satisfies each one of a finite family of laws

$$\mathcal{L} = \{w_i = 1 : i = 1, 2, \ldots, n\}$$

if and only if it satisfies the law

$$w = w_1w_2\ldots w_n = 1.$$ 

Thus, from this result and Proposition 1.1.7, the class of all groups that satisfy each one of a finite set of laws is a variety. However, we have not dealt with an infinite set of laws. To do this, we use Lemma 1.1.9.

Lemma 1.1.9. Let $I$ be some index set and let $\mathcal{X}_i$ be a variety of groups for each $i \in I$. Then the intersection $\bigcap_{i \in I} \mathcal{X}_i$ is a variety of groups.
Proof. Let $G \in \bigcap_{i \in I} \mathcal{X}_i$ and let $H$ be a subgroup of $G$. Now, for each $i \in I$, $G \in \mathcal{X}_i$ and hence $H \in \mathcal{X}_i$ as each $\mathcal{X}_i$ is closed under $S$, giving, $H \in \bigcap_{i \in I} \mathcal{X}_i$. Similarly, if $K$ is a quotient group of $G$ then $K \in \bigcap_{i \in I} \mathcal{X}_i$. Finally, let $J$ be an index set, $G_j \in \bigcap_{i \in I} \mathcal{X}_i$ for each $j \in J$ and let $F$ be the product, $F = \prod_{j \in J} G_j$. Let $i \in I$, then for every $j \in J$, $G_j \in \mathcal{X}_i$. Thus, $F \in C\mathcal{X}_i \subseteq \mathcal{X}_i$. Since we took an arbitrary $i \in I$, for each $i \in I$, $F \in \mathcal{X}_i$, giving, $F \in \bigcap_{i \in I} \mathcal{X}_i$. Therefore, $\bigcap_{i \in I} \mathcal{X}_i$ is closed under $S$, $Q$ and $C$. 

Corollary 1.1.10. (cf. [56], Definition 14.1) Let $\mathcal{X}$ be the class of all groups that satisfy each one of a given set (finite or infinite) of laws. Then $\mathcal{X}$ is a variety of groups.

Proof. Let $\mathcal{L}$ be the given set of laws, indexed by $I$, that define $\mathcal{X}$; that is, $w_i = 1$ is a law in $\mathcal{L}$ for each $i \in I$. For each $i \in I$, let $\mathcal{X}_i$ be the class of groups defined by the law $w_i = 1$. By Proposition 1.1.7, each $\mathcal{X}_i$ is a variety of groups. Clearly, $\mathcal{X} = \bigcap_{i \in I} \mathcal{X}_i$ and hence by Lemma 1.1.9, $\mathcal{X}$ is a variety of groups. 

§1.2 Topological Groups

Before we turn to varieties of topological groups, we recall the definition of a topological group and the concept of topologically isomorphic. Our standard reference for this section on topological groups is [46], which contains a more detailed introduction to topological groups.

Definition 1.2.1. ([46], Chapter 1) Let $G$ be a set that is a group and a topological space. Then $G$ is said to be a topological group if

(i) the mapping $(x, y) \mapsto xy$ of $G \times G$ onto $G$ is a continuous mapping of the direct product $G \times G$ (with the product topology) onto $G$; and

(ii) the mapping $x \mapsto x^{-1}$ of $G$ onto $G$ is continuous.
Examples 1.2.2.

(a) Any group with the discrete topology is a topological group, in particular, the additive group of integers with the discrete topology, which will be denoted by $\mathbb{Z}$.

Further, any group with the indiscrete topology is also a topological group.

(b) The additive group of real numbers with the Euclidean topology forms a topological group, and it will be denoted by $\mathbb{R}$. Recall that a topological space is said to be \textit{locally compact} if every point has a compact neighbourhood. The topological group $\mathbb{R}$ is locally compact Hausdorff abelian since for each $x \in \mathbb{R}$, $[x-1, x+1]$ is a compact neighbourhood of $x$.

(c) The “circle group” consisting of the complex numbers of modulus one (that is, the set of numbers $e^{2\pi i x}, 0 < x < 1$) with multiplication (of complex numbers) and the topology induced from that of the complex plane forms a topological group. In fact this topological group, which will be denoted by $\mathbb{T}$, is a compact Hausdorff abelian group.

(d) Let $X$ be a normed vector space equipped with the norm $||\ |$ (refer [24]). Then, $X$ forms a group under the additive operation and the norm gives rise to a metric $d$ on $X$ ([24], Part I, Section 2, page 21), defined by $d(x,y) = ||x-y||$. Hence, the normed vector space $X$ has an underlying group and an underlying topological space. It is easily proved that the ‘addition mapping’ $(x,y) \mapsto x+y$ is a continuous mapping of the direct product $X \times X$ (with the product topology) onto $X$ ([24], Result 16.1) and the ‘inverse’ mapping $x \mapsto -x$ is continuous onto $X$. Therefore, the underlying group and topological space of a normed vector space together form a topological group. \hfill \blacksquare

Remark 1.2.3. Much serious work on topological groups deals only with Hausdorff topological groups. Therefore, it is important to point out that in general, our attention is \textit{not} restricted to Hausdorff topological groups. The reasons for this will be clear later.

Recall that the structure preserving maps for groups are isomorphisms (one-to-one and
onto homomorphisms) and for topologies are homeomorphisms (open, continuous, one-to-one and onto maps). We now introduce the structure preserving maps for topological groups.

**Definitions 1.2.4.** ([46], Chapter 1) Let $G_1$ and $G_2$ be topological groups. A map $f : G_1 \to G_2$ is said to be a topological isomorphism (or topological group isomorphism) if it is both an isomorphism of groups and a homeomorphism. Further, $G_1$ and $G_2$ are said to be topologically isomorphic.

It is clear from the definition that when topological groups $G_1$ and $G_2$ are topologically isomorphic, the groups $G_1$ and $G_2$ are isomorphic and the topologies $G_1$ and $G_2$ are homeomorphic. Thus, if one of these two conditions does not hold we can immediately conclude that $G_1$ and $G_2$ are not topologically isomorphic. On the other hand, the converse of this is not true. Given two topological groups $G_1$ and $G_2$ where the groups $G_1$ and $G_2$ are algebraically isomorphic and the topologies $G_1$ and $G_2$ are homeomorphic, the topological groups $G_1$ and $G_2$ need not be topologically isomorphic. For example, every infinite dimensional separable Banach space (complete normed vector space) is homeomorphic to $\ell_2$ (see [4]) and the groups underlying all such spaces are algebraically isomorphic, but they are not topologically isomorphic.

§1.3 Varieties of Topological Groups

The concept of a variety introduced in Section 1.1 will be defined in a similar fashion for topological groups. As in the earlier section, we shall use the operators $S$, $Q$, $C$ and $P$, which can now be defined on classes of topological groups.

**Definition 1.3.1.** Let $\Omega$ be a class of topological groups. The operators $S$, $Q$, $C$ and $P$ are defined on $\Omega$ to give classes of topological groups as follows. Let $G$ be a topological
group. Then

- \( G \in S(\Omega) \) if \( G \) is topologically isomorphic to a subgroup (with its induced topology) of a topological group in \( \Omega \),
- \( G \in Q(\Omega) \) if \( G \) is topologically isomorphic to a quotient group (with the quotient topology) of a topological group in \( \Omega \),
- \( G \in C(\Omega) \) if \( G \) is topologically isomorphic to an infinite or a finite direct product (with the corresponding Tychonoff product topology) of topological groups in \( \Omega \), and
- \( G \in P(\Omega) \) if \( G \) is topologically isomorphic to a finite direct product (with the corresponding Tychonoff product topology) of topological groups in \( \Omega \).

Note that the operators \( S, Q, C \) and \( P \) have different meanings when operating on groups and on topological groups, but the meaning will be clear from the context.

We mentioned in Remark 1.2.3 that our attention is not restricted to Hausdorff spaces. For example, the operator \( Q \) is defined to return all quotient groups, not just Hausdorff quotient groups. It is therefore appropriate to define the following operators.

**Definition 1.3.2.** ([7], Section 2) Let \( \Omega \) be a class of topological groups. The operators \( \overline{Q} \) and \( \overline{S} \) are defined as follows. Let \( G \) be a topological group. Then

- \( G \in \overline{Q}(\Omega) \) if \( G \) is topologically isomorphic to a Hausdorff quotient group (with the quotient topology) of a topological group in \( \Omega \), and
- \( G \in \overline{S}(\Omega) \) if \( G \) is topologically isomorphic to a closed subgroup (with its induced topology) of a topological group in \( \Omega \).

Note that \( Q(\Omega) \) is the class of all quotient groups of members of \( \Omega \), whilst \( \overline{Q}(\Omega) \) is the class of all Hausdorff quotient groups of members of \( \Omega \), that is, those where the kernel of the quotient map is closed.

Let us now introduce varieties of topological groups.
Definition 1.3.3. (cf. [38], Definition 2.1) A class of topological groups is said to be a variety of topological groups if it is closed under $S$, $Q$ and $C$.

It is clear from Example 1.1.4(a) that the class of all abelian topological groups is a variety of topological groups.

For the remainder of this section, we shall consider an example of a variety of topological groups. The example involves $T(m)$-groups introduced by Morris in [40].

First, however, recall that the index ([17], Section 1.5) of a subgroup $H$ of a group $G$ is the cardinal of the collection of cosets of $H$ in $G$. Note that the cardinals for right and left cosets of any subgroup are the same, so our definition is consistent.

Definitions 1.3.4. (cf. [40], §4) Let $m$ be some cardinal number. A topological group is said to be a $T(m)$-group if each neighbourhood of 1 contains a normal subgroup of index strictly less than $m$. Further, the class of all $T(m)$-groups will be denoted by $\mathcal{T}_m$.

Remarks 1.3.5.

(a) The inclusion of "normal" in Definition 1.3.4 did not appear in [40]. However, our definition of a $T(m)$-group has in recent years been considered to be a more useful concept [30].

(b) Let $G$ be a topological group. Then $G$ is a $T(m)$-group for every $m$ strictly greater than the cardinality of $G$ (denoted card($G$)). In particular, let $G$ be a discrete group. Then $G$ is a $T(m)$-group if and only if card($G$) is strictly less than $m$.

Further examples of $T(m)$-groups can be found by studying certain topological groups called NSS-groups. These topological groups play an important role in the solution to Hilbert's fifth problem. Hilbert's fifth problem is often regarded as follows: Is every locally euclidean group a Lie group? ([70]) Recall that a Hausdorff topological space is said to be locally euclidean ([21], Chapter 9, Part 2) if there exists a neighbourhood of the identity that is homeomorphic to a neighbourhood in $\mathbb{R}^n$ for some natural number $n$. Now, all Lie
groups are NSS-groups, so the problem was broken into two parts:

(a) Is every locally compact NSS-group a Lie group?

(b) Is every locally euclidean group a locally compact NSS-group?

Hilbert's fifth problem was answered in the affirmative and was solved completely by Montgomery and Zippin, and Gleason in 1952 (see [37] and [15]). Kaplansky gave an accessible proof of part (a) of Hilbert's fifth problem in [27].

**Definition 1.3.6.** [46] A topological group $G$ is said to have no small subgroups (or be an NSS-group) if there exists a neighbourhood of 1 which contains no subgroup other than the trivial subgroup $\{1\}$.

**Examples 1.3.7.**

(a) The topological group $\mathbb{R}$ is an NSS-group as the interval $[-1,1]$ is a neighbourhood of the identity 0 which contains no subgroup other than the trivial subgroup $\{0\}$.

(b) Similarly, a normed vector space (over the field of real numbers) is an NSS-group as the unit ball about 0 is a neighbourhood of 0 containing no subgroup other than the trivial subgroup $\{0\}$.

(c) Not every topological group is an NSS-group. Indeed an infinite product of non-trivial topological groups is not an NSS-group. Consider $G = \prod G_i$ where $G_i \neq \{1\}$ for all $i \in I$, an infinite index set. Let $N$ be a neighbourhood of the identity in $G$. By definition, $N$ contains an open neighbourhood of the identity $O = \prod O_i$ where $O_i$ is open in $G_i$ and $O_i = G_i$ for all but a finite number of members of $I$. Then there is a subgroup contained in $O$ given by $V = \prod V_i$ where $V_i = G_i$ if $O_i = G_i$ and $V_i = \{1\}$ if $O_i \neq G_i$. Clearly, $V$ is non-trivial. Thus, $\prod G_i$ is not an NSS-group.

**Proposition 1.3.8.** (cf. [47], Section 3) Let $G$ be an NSS-group. Then $G$ is a $T(m)$-group if and only if $\text{card}(G)$ is strictly less than $m$.

**Proof.** Let $G$ be a $T(m)$-group. Then every neighbourhood of the identity must contain
a normal subgroup of index strictly less than \( m \). Now, there exists a neighbourhood \( N \) of the identity that contains no (normal) subgroup other than the trivial subgroup \( \{1\} \), therefore, the index of \( \{1\} \) must be strictly less than \( m \). The index of \( \{1\} \) in the group \( G \) is equal to the cardinality of \( G \), thus, \( \text{card}(G) \) is strictly less than \( m \).

Conversely, let \( \text{card}(G) \) be strictly less than \( m \). The normal subgroup \( \{1\} \) is contained in every neighbourhood of the identity and it has index equal to the cardinality of \( G \), which is strictly less than \( m \) (see Remark 1.3.5(b)). Therefore, \( G \) is a \( T(m) \)-group. ■

**Corollary 1.3.9.** ([47], Section 3) Let \( G \) be a normed vector space. Then \( G \) is a \( T(m) \)-group if and only if \( \text{card}(G) \) is strictly less than \( m \).

*Proof.* Let \( G \) be a normed vector space. From Example 1.3.7(b), \( G \) is an NSS-group and hence, by Proposition 1.3.8, \( G \) is a \( T(m) \)-group if and only if \( \text{card}(G) < m \). ■

It is clear that for any given infinite cardinal \( m \), \( \mathfrak{T}_m \) is large, indeed, it contains a proper class of topological groups which are non-isomorphic as topological groups.

**Remark 1.3.10.** Note that it is easily shown that a subgroup of a \( T(m) \)-group is a \( T(m) \)-group; a quotient of a \( T(m) \)-group is a \( T(m) \)-group; and the product of a collection of \( T(m) \)-groups is a \( T(m) \)-group. Therefore, we have the following proposition which will be presented without proof.

**Proposition 1.3.11.** (cf. [40], Theorem 4.2) Let \( m \) be any infinite cardinal number. The class \( \mathfrak{T}_m \) of all \( T(m) \)-groups is a variety of topological groups. ■

As we have thus far only considered classes of topological groups which form varieties of topological groups, it is possible to fall under the false impression that all “nice” classes of topological groups form varieties. This is definitely not the case and we shall present a number of examples to support this point.
Examples 1.3.12. The following are examples of nice classes of topological groups which are \textit{not} varieties.

(a) The class of all finite topological groups is not closed under $C$ as an infinite product of finite topological groups is clearly not finite.

(b) The class of all countable topological groups is not closed under $C$ as an uncountable product of non-trivial topological groups is clearly not countable.

(c) The class of all discrete topological groups is not closed under $C$ as the product topology of an infinite number of non-trivial discrete topological groups is not discrete.

(d) The class of all compact groups is not closed under $S$ as a subgroup of a compact group is not necessarily compact. For example, consider the compact topological group $\mathbb{T}$. The subgroup $H$ of $\mathbb{T}$ generated by $e^{2\pi ip}$ where $p$ is irrational and $0 < p < 1$ is clearly countably infinite. Now, $H$ is not closed in $\mathbb{T}$ as every proper closed subgroup of $\mathbb{T}$ is finite ([46], Corollary 3 to Proposition 20). Therefore, $H$ is not compact as all compact subsets of $\mathbb{T}$ are closed.

(e) The class of all locally compact groups is not closed under $S$ as a subgroup of a locally compact group is not necessarily locally compact. For example, consider the locally compact topological group $\mathbb{R}$. The subgroup $\mathbb{Q}$ of rational numbers is not locally compact as there exists no compact neighbourhood of the identity 0.

(f) Before we touch on the class of all topological groups underlying normed vector spaces, we note that an uncountable product of topological groups does not necessarily satisfy the first axiom of countability. Indeed, an uncountable product of topological groups that do satisfy the first axiom of countability need not satisfy the first axiom of countability. To be precise, an uncountable product of topological groups $\prod_{i \in I} G_i$ satisfies the first axiom of countability if and only if all but a countable number of the $G_i$ are indiscrete ([29], Theorem 5, Chapter 3).

Now, note that every metrizable topological group satisfies the first axiom of count-
ability, and so, therefore, does every topological group underlying a normed vector spaces (see Example 1.2.2(d)).

We now turn to the class of all topological groups underlying normed vector spaces. Let $H$ be an uncountable product of copies of $\mathbb{R}$ which underlies the one-dimensional normed vector space. As $\mathbb{R}$ is not indiscrete, $H$ does not satisfy the first axiom of countability and hence cannot underlie a normed vector space. Therefore, the class of all topological groups underlying normed vector spaces is not closed under $C$ and thus not a variety of topological groups.

Remark 1.3.13. Towards the end of Section 1.1, we saw that varieties of groups could be defined by laws. The natural question that arises at this point is whether varieties of topological groups can also be described by "laws". An affirmative answer was given by Taylor in [64], and more information can be found in [47] and, more recently, [30].

§1.4 Generating Varieties of Topological Groups

In Example 1.3.12 we saw a number of nice classes of topological groups which are not varieties. However, we still would like to study these classes in the context of varieties of topological groups. For this, we shall introduce the concept of a variety of topological groups generated by a class of topological groups. Earlier in this chapter, we dealt with the intersection of varieties of groups (see Lemma 1.1.9), showing that it is also a variety of groups. This result can be extended to varieties of topological groups and used to define a variety generated by a class of topological groups, however, we shall use a slightly different approach to introduce a variety generated by a class of topological groups.

The following result shows that given any class $\Omega$ of topological groups there is a smallest variety, $\mathfrak{V}(\Omega)$, of topological groups containing $\Omega$. Further, every member of $\mathfrak{V}(\Omega)$ can be obtained from $\Omega$ applying each of the operators $Q$, $S$ and $C$ at most once.
Theorem 1.4.1. (cf. [7], Theorem 1) For any non-empty class \( \Omega \) of topological groups, \( QSC(\Omega) \) is closed under \( Q, S \) and \( C \) and hence is a variety of topological groups.

Remark 1.4.2. [7] It is important to note that in the proof of Theorem 1.4.1 the following results are established for \( \Omega \), a non-empty class of topological groups.

(a) \( CS(\Omega) \subseteq SC(\Omega) \),

(b) \( CQ(\Omega) \subseteq QC(\Omega) \), and

(c) \( SQ(\Omega) \subseteq QS(\Omega) \).

Further to these, due to the fact that every subgroup of an abelian topological group is normal, if \( \Omega \) is a class of abelian topological groups, then

(d) \( QS(\Omega) \subseteq SQ(\Omega) \) and hence \( QS(\Omega) = SQ(\Omega) \).

As mentioned earlier, for a non-empty class of topological groups \( \Omega \), the class \( QSC(\Omega) \) is the smallest variety of topological groups that contains the class \( \Omega \). Therefore, it is appropriate to define the variety generated by \( \Omega \) as follows.

Definition 1.4.3. (cf. [38], [7]) Let \( \Omega \) be a class of topological groups and let \( \mathfrak{V}(\Omega) \) be the smallest variety of topological groups containing \( \Omega \). Then \( \mathfrak{V}(\Omega) \) is said to be the variety generated by \( \Omega \). Further, we have \( \mathfrak{V}(\Omega) = QSC(\Omega) \).

Note that each element of \( \mathfrak{V}(\Omega) \) is obtained from successive applications of \( S, Q \) and \( C \) to elements in \( \Omega \), which is the reason for saying the variety is generated by \( \Omega \).

Recall we have not placed any Hausdorff restriction on the definition of topological groups. However, it is often of interest to study the Hausdorff topological groups contained in a variety. The following theorem turns out to be a useful tool for identifying Hausdorff topological groups within varieties of topological groups.
Theorem 1.4.4. ([7], Theorem 2) If $\Omega$ is any class of topological groups and $G$ is a Hausdorff topological group in $\mathfrak{B}(\Omega)$ then $G \in SCQSP(\Omega)$. $lacksquare$

Theorem 1.4.4 can be reduced even further when in an abelian world, noting that $Q$ and $S$ commute (see [7], Remark after Theorem 2). Thus, for $\Omega$ a class of abelian topological groups, we use Remark 1.4.2(d) and Theorem 1.4.4 to obtain the following corollary.

Corollary 1.4.5. ([7], Remark after Theorem 2) If $\Omega$ is any class of abelian topological groups and $G$ is a Hausdorff topological group in $\mathfrak{B}(\Omega)$ then $G \in SCQSP(\Omega)$. $lacksquare$

We note that Corollary 1.4.5 may not work for the non-abelian case.

In Proposition 1.3.11, we saw that the class of all $T(m)$-groups is a variety of topological groups. Thus, we have $\mathfrak{B}(\mathfrak{T}_m) = \mathfrak{T}_m$. We shall introduce what appears to be a new concept of locally-$m$ which has a similar flavour to the $T(m)$-group property, but where the class of all locally-$m$ groups is not a variety of topological groups. However, we will see they are very useful.

Definition 1.4.6. Let $m$ be any infinite cardinal number. A topological group is said to be a locally-$m$ group if it has a neighbourhood of the identity of cardinality less than or equal to $m$. $lacksquare$

Remark 1.4.7. We note that if $G$ is a locally-$m$ group, then every neighbourhood of the identity in $G$ contains a neighbourhood of the identity of cardinality less than or equal to $m$.

As mentioned earlier, for any given cardinal number $m$, the class of all locally-$m$ groups does not form a variety of topological groups. However, the class of all locally-$m$ topological groups is closed under all but one of the operators, as can be seen in the next proposition, which is introduced in this thesis.
Proposition 1.4.8. Let $m$ be any infinite cardinal number. The class of all locally-$m$ groups is closed under $S$, $Q$ and $P$ but not $C$.

Proof. Let $G$ be a locally-$m$ group. It is clear that any subgroup $H$ of $G$ is also a locally-$m$ group. If $K$ is a quotient group of $G$ with quotient homomorphism $f : K \to G$, and $O$ is a neighbourhood of the identity in $G$ with cardinality $\text{card}(O) \leq m$, then $f(O)$ is clearly a neighbourhood of the identity in $K$ and $\text{card}(f(O)) \leq m$. Thus $K$ is a locally-$m$ group. Further, if $G_1, G_2, \ldots, G_n$ are all locally-$m$ groups with $O_1, O_2, \ldots, O_n$ the open neighbourhoods of the respective identities, each with cardinality less than or equal to $m$, then $O = \prod_{i=1}^{n} O_i$ is a neighbourhood of the identity in $\prod_{i=1}^{n} G_i$ and $\text{card}(O) \leq m$. Thus, $\prod_{i=1}^{n} G_i$ is a locally-$m$ group. Therefore, the class of all locally-$m$ groups is closed under $S$, $Q$ and $P$.

Finally, let $G$ be a discrete group such that $\text{card}(G) > m$. Then $G$ is a locally-$m$ group. Choose an infinite index set $I$. The product $\prod_{i \in I} G_i$, where $G_i = G$ for each $i \in I$, is not a locally-$m$ group as the smallest open set clearly has cardinality greater than or equal to $\text{card}(G)$.

Remarks 1.4.9.

(a) Any discrete group, in particular $\mathbb{Z}$, is a locally-$m$ group for every infinite cardinal $m$.

(b) The topological group $\mathbb{R}$ is clearly a locally-$c$ group, as is $T$, where $c$ is the cardinality of the continuum.

(c) A topological group of cardinality $n$ is a locally-$m$ group for all $m$ greater than or equal to $n$.

There is a large class of topological groups that are locally-$c$ groups. Every locally euclidean topological group is a locally-$c$ group; that is, every Lie group is a locally-$c$ group (cf. Hilbert's fifth problem mentioned earlier: [70], [37], [15]).

We saw in Corollary 1.3.9 that a normed vector space $N$ is a $T(m)$-group if and only if the cardinality of $N$ is strictly less than $m$. A similar result is true for locally-$m$ groups.
**Proposition 1.4.10.** Let \( m \) be any infinite cardinal number. A normed vector space \( N \) is a locally-\( m \) group if and only if the cardinality of \( N \) is less than or equal to \( m \).

**Proof.** Without loss of generality, we take \( N \) to be non-trivial. Let \( U \) be a neighbourhood of the identity of cardinality less than or equal to \( m \). Then \( U \) contains an open sphere, \( B_r \) of radius \( r \), about the identity. Now \( N = \bigcup_{n=1}^{\infty} nB_r \) and so if we denote the cardinality of \( N \) by \( \aleph_0 \), \( \text{card}(B_r) \leq \text{card}(N) \leq \aleph_0 \cdot \text{card}(B_r) = \text{card}(B_r) \). Thus, \( \text{card}(B_r) = \text{card}(N) \) and as \( \text{card}(B_r) \leq \text{card}(U) \leq m \), \( \text{card}(N) \leq m \).

The previous proposition can now be used to show that each class of locally-\( m \) groups properly contains the class of locally-\( n \) groups when \( n \) is an infinite cardinal less than or equal to \( m \). Indeed, in Chapter 5, we shall show that an analogous containment still holds when considering the varieties generated by each class.

**Remark 1.4.11.** Let \( m \) be an infinite cardinal number greater than or equal to \( c \). We remark that there exists a normed vector space \( N \) of cardinality \( m \). Firstly, if \( m = c \), choose \( N = \mathbb{R} \). For \( m > c \), let \( I \) be an index set of cardinality \( m \), let \( X = \prod_{i \in I} X_i \) where each \( X_i = [0,1] \) (the unit interval \([0,1]\) with the topology induced from \( \mathbb{R} \)) and let \( C(X) \) be the Banach space of all continuous functions \( X \rightarrow \mathbb{R} \) (see [60], Chapter 9, Section 46, Example 7). Consider the projection map \( p_j : X \rightarrow X_j \) for each \( j \in I \). Clearly, \( p_i \neq p_j \) whenever \( i \neq j \) and each \( p_j \) is in \( C(X) \). Therefore, \( \text{card}(C(X)) \geq \text{card}I = m \). We further note that the dimension of a Banach space \( B \) is equal to \( \text{card}(B) \) if \( \text{card}(B) > c \). (If \( S \) is a basis set, \( B = \text{gp}(\mathbb{R}.S) \) where \( \mathbb{R}.S = \bigcup_{s \in S} \{ rs : r \in \mathbb{R} \} \) giving \( \text{card}(B) = c.\text{card}(S) = c.\text{dim}B > c \) which implies \( \text{dim}B > c \) and so \( \text{card}(B) = \text{dim}B \).) Therefore, \( C(X) \) has dimension greater than or equal to \( m \). Choose any \( m \) linearly independent vectors in \( C(X) \) and let \( N \) be the normed vector space spanned by these vectors. Then \( N \) has cardinality \( m \).

**Notation.** In this thesis we shall use \( \subset \) to represent proper containment.
Proposition 1.4.12. Let $m$ and $n$ be cardinals greater than or equal to $c$ and let $L_m$ and $L_n$ denote the classes of all locally-$m$ and locally-$n$ groups, respectively. Then $L_m \subset L_n$ if and only if $m < n$.

Proof. Note that for each cardinal $k$ greater than or equal to $c$, there exists a normed vector space of cardinality $k$. Now, if $m < n$ then, clearly, $L_m \subseteq L_n$. Further, there exists a normed vector space $N$ of cardinality $n$, which is contained in $L_n$ but not $L_m$, by Proposition 1.4.10. Therefore, $L_m \subset L_n$. Conversely, let $L_m \subset L_n$ and suppose $m \geq n$. There exists a normed vector space $N'$ of cardinality $m$ and by Proposition 1.4.10, $N'$ is a locally-$m$ group, but not a locally-$n$ group. This clearly leads to a contradiction and so $m < n$. 

The significance of studying varieties generated by locally-$m$ groups (for an infinite cardinal $m$) shall become apparent in Chapter 5.

§1.5 Groups in Varieties of Topological Groups

So far in this chapter, we have considered varieties of groups and varieties of topological groups. As they have similar definitions (closed under subgroups, quotients and products), it is of interest to establish relationships between them so various results about varieties of groups can be transferred easily to varieties of topological groups. First, we shall introduce a few concepts that will also be useful later.

Notation. Let $\Omega$ be a variety of topological groups and let $G \in \Omega$. Then $|G|$ will denote the group obtained from $G$ by "forgetting" the topology. We call $|G|$ the group underlying $G$. Further, $|\Omega|$ will denote the class of all groups underlying the topological groups contained in the variety $\Omega$.
Definition 1.5.1. ([46], Chapter 1) Let $I$ be an index set and let $G_i$ be a topological group with identity $1_{G_i}$ for each $i \in I$. Then the restricted direct product (or weak direct product), denoted $\prod_{i=1}^{\infty} G_i$, is the subgroup of $\prod_{i=1}^{\infty} G_i$ consisting of elements $\prod_{i=1}^{\infty} g_i$, with $g_i = 1_{G_i}$ for all but a finite number of members of $I$, with the topology induced as a subspace of $\prod_{i=1}^{\infty} G_i$.

Proposition 1.5.4 turns out to be an extremely useful result in our analysis of different varieties of topological groups—as we shall soon see. We need the following two lemmas to assist us in proving Proposition 1.5.4.

Lemma 1.5.2. ([46], Proposition 15) Let $I$ be an index set and let $G_i$ be a topological group for each $i \in I$. Then $\prod_{i=1}^{\infty} G_i$ is a normal dense subgroup of $\prod_{i\in I} G_i$.

Lemma 1.5.3. Let $G$ be a topological group and let $H$ be a normal dense subgroup of $G$. Then the quotient topological group $G/H$ is indiscrete.

Proof. Let $f : G \to G/H$ be the quotient mapping from $G$ onto $G/H$. Suppose there is a proper non-empty open set in $G/H$. Then, by translation, there is an open set $U$ in $G/H$ such that $H \not\subseteq U$. Now, $f^{-1}(U)$ is open in $G$. Therefore, $f^{-1}(U) \cap H \neq \emptyset$; that is, there exists an element $x$ such that $x \in f^{-1}(U)$ and $x \in H$. Therefore, $f(x) = H$ and hence $H \in U$. This is a contradiction and so there are no proper open sets in $G/H$. Hence, $G/H$ is indiscrete.

Proposition 1.5.4. ([38], Lemma 2.7) Let $\Omega$ be a variety of topological groups and let $G \in \Omega$. Then $|G|$ with the indiscrete topology is contained in $\Omega$.

Proof. Consider the countable product $\prod_{i=1}^{\infty} G_i$ where $G_i = G$ for each $i \in \mathbb{N}$. Then from Lemma 1.5.2, the restricted direct product $\prod_{i=1}^{\infty} G_i$ is a normal dense subgroup of $\prod_{i\in I} G_i$. Therefore, by Lemma 1.5.3, the quotient topological group $K = \prod_{i=1}^{\infty} G_i / \prod_{i=1}^{\infty} G_i$ is indiscrete.
Now, let $\rho : \prod_{i=1}^{\infty} G_i \to K$ be the quotient mapping from $\prod_{i=1}^{\infty} G_i$ to $K$ and consider the homomorphism $f : G \to K$ given by $f(g) = \rho((g,g,g,g,\ldots))$ for all $g \in G$. For each $g \in G$, $g \neq e$, $g$ is not contained in the kernel of $f$; that is, the kernel of $f$ is the set $\{e\}$. Therefore, $f$ is a one-to-one homomorphism from $|G|$ to $|K|$ and hence $|G|$ can be embedded in $|K|$. Since $\Omega$ is a variety of topological groups and $K \in QC(\Omega)$, we have $K \in \Omega$ and thus $|G|$ with the induced topology from $K$ is also contained in $\Omega$; that is, $|G|$ with the indiscrete topology is contained in $\Omega$.

Remark 1.5.5. It is worth remarking that varieties of groups can be regarded as special cases of varieties of topological groups. If we let $\mathcal{X}$ be a variety of groups and we let $\Omega$ be the class of topological groups consisting of elements of $\mathcal{X}$ each with the corresponding indiscrete topology, then $\Omega$ is a variety of topological groups. In other words, given a variety of groups, it can be considered as a variety of topological groups if each group is given the indiscrete topology. (cf. [7].)

We now turn to the converse problem. Given a variety of topological groups, is the underlying class of groups a variety of groups? The following proposition answers this in the affirmative. The result follows from Definitions 1.1.3 and 1.3.3, thus we shall omit the proof.

**Proposition 1.5.6.** (cf. [38], Remark 2.3) Let $\Omega$ be a variety of topological groups. Then the underlying class of groups $|\Omega|$ is a variety of groups.

Note that although we have referred to a direct proof of Proposition 1.5.6, the result also follows from Proposition 1.5.4.

Remark 1.5.7. From the results in Proposition 1.5.4, Remark 1.5.5 and Proposition 1.5.6, it can be readily seen that most varieties of topological groups contain a smaller variety of topological groups consisting of the underlying groups each with the indiscrete topology.
§1.6 Hausdorffness in Varieties of Topological Groups

Much work on topological groups is restricted to Hausdorff topological groups. On several occasions, including Remark 1.2.3, we stressed that our attention is not focused on Hausdorff topological groups. We conclude the chapter with some new results, Theorems 1.6.6, 1.6.7 and 1.6.8, concerning classes of groups underlying Hausdorff groups in different varieties. In this way, we justify our decision to not include Hausdorffness in our definition of topological groups.

In Proposition 1.5.6, we saw that for a variety of topological groups $\Omega$, $|\Omega|$ is a variety of groups. We will let $|\Omega|_2$ denote the class of all groups that, with a Hausdorff topology, appear in $\Omega$. It is of interest to know if the class of groups $|\Omega|_2$ is always a variety of groups. To answer this problem, we will turn our attention back to $T(m)$-groups. Firstly, we present a more general version of Proposition 1.3.11.

**Proposition 1.6.1.** (cf. [40], Theorem 4.2) Let $V$ be a variety of groups and for any infinite cardinal number $m$, let $\mathfrak{V}_m$ be the class of all $T(m)$-groups $G$ such that $|G| \in V$. Then $\mathfrak{V}_m$ is a variety of topological groups.

**Proof.** Let $G \in \mathfrak{V}_m$. Then a subgroup $H$ and quotient $K$ of $G$ are both $T(m)$-groups (see Remark 1.3.10). Further, $|H|$ and $|K|$ are both contained in $V$, therefore $H, K \in \mathfrak{V}_m$. Similarly, if $G_i \in \mathfrak{V}_m$ for each $i \in I$, then $\prod_{i \in I} G_i \in \mathfrak{V}_m$. Therefore, $\mathfrak{V}_m$ is closed under $S$, $Q$ and $C$, giving a variety of topological groups.

We shall see below that the variety of all $T(\aleph_0)$-groups, $\mathfrak{X}_{\aleph_0}$, has the property that $|\mathfrak{X}_{\aleph_0}|_2$ is not a variety of groups. To do so, we require the following definition.

**Definition 1.6.2.** [35] A group is said to be *residually finite* if the intersection of all of its normal subgroups with finite index is the identity.

Another way of interpreting this definition is to say that a group is said to be residually
finite if it is isomorphic to a subgroup of a product of finite groups. It turns out that the (algebraic) property of residually finite characterizes those groups that underlie Hausdorff topological groups contained in $\mathbb{T}_{\aleph_0}$, as we shall soon see. However, firstly, we shall say a group $G$ admits a Hausdorff topology if $G$ with the topology form a topological group.

**Lemma 1.6.3.** Let $G$ be a group. Then $G$ admits a Hausdorff group topology which makes it a $T(\aleph_0)$-group if and only if $G$ is residually finite.

**Proof.** Let $G$ be a group that admits a Hausdorff group topology which makes it a $T(\aleph_0)$-group. Let $\{U_i : i \in I\}$ be the family of all neighbourhoods of the identity $1$ in $G$. Then for each $U_i$ there exists a normal subgroup $N_i$ of $G$ such that $1 \in N_i \subseteq U_i$ and $\text{card}(G/N_i)$ is finite. Further, let $\phi_i : G \to G/N_i$ be the canonical homomorphism from $G$ onto $G/N_i$ and let $\Phi : G \to \prod_{i \in I} G/N_i$ be given by $\Phi(g) = \prod_{i \in I} \phi_i(g)$. Since the topology on $G$ is Hausdorff, $\bigcap_{i \in I} U_i = \{1\}$ and so $\Phi$ is a one-to-one homomorphism from $G$ into a product of finite groups. Therefore, $G$ is residually finite.

Conversely, let $G$ be a residually finite group. Then there exists an index set $I$ and finite groups $F_i$ for each $i \in I$ such that $G$ is isomorphic to a subgroup of $\prod_{i \in I} F_i$. Now, for each $i \in I$, $F_i$ with the discrete topology is a Hausdorff $T(\aleph_0)$-group. Since products of Hausdorff $T(\aleph_0)$-groups and subgroups of Hausdorff $T(\aleph_0)$-groups are Hausdorff $T(\aleph_0)$-groups, then $G$ with the induced topology from $\prod_{i \in I} F_i$ is a Hausdorff $T(\aleph_0)$-group.

To prove Theorem 1.6.6 concerning the variety of all $T(\aleph_0)$-groups, we also need the following result in group theory.

**Lemma 1.6.4.** Let $G$ be a non-trivial divisible group. Then $G$ is not residually finite.

**Proof.** Suppose $G$ is residually finite. Then $G$ is isomorphic to a subgroup of $\prod_{i \in I} F_i$, where $I$ is an index set and each $F_i$ is a finite group. Let $p_i$ be the projection of $G$ into $F_i$. As $G$ is divisible and $F_i$ is finite, $p_i(G)$ is a finite divisible group and hence is the trivial group. Thus, $G$ is trivial which is a contradiction. Therefore, $G$ is not residually finite.
Remark 1.6.5. From Lemma 1.6.4, we know for example that $\mathbb{R}$, the additive group of real numbers, is not a residually finite group since it is divisible. However, every free (abelian) group is residually finite (see [35], Problems for Section 2.4, #24).

We now turn to the variety of all $T(\aleph_0)$-groups.

Theorem 1.6.6. Let $\mathcal{T}_{\aleph_0}$ be the variety of all $T(\aleph_0)$-groups and let $\mathcal{A}_{\aleph_0}$ be the variety of all abelian $T(\aleph_0)$-groups. Then $|\mathcal{T}_{\aleph_0}|_2$ and $|\mathcal{A}_{\aleph_0}|_2$ are not varieties of groups.

Proof. By Remark 1.6.5 and Lemma 1.6.3, $X_{\aleph_0}$ contains each free group with some Hausdorff group topology. So each free group is in $|\mathcal{T}_{\aleph_0}|_2$. Suppose that $|\mathcal{T}_{\aleph_0}|_2$ is a variety of groups. Then $|\mathcal{T}_{\aleph_0}|_2$ must be the variety of all groups, as every group is a quotient of a free group. However, as remarked above, $\mathbb{R}$ is not residually finite and hence by Lemma 1.6.3 is not contained in $|\mathcal{T}_{\aleph_0}|_2$. Thus, $|\mathcal{T}_{\aleph_0}|_2$ is not a variety of groups. The abelian case is proved similarly.  

We shall now examine $T(m)$-groups for $m > \aleph_0$ and see that there is a proper class of varieties of topological groups $\mathcal{W}$ such that $|\mathcal{W}|_2$ is not a variety of groups. However, we shall find that there is also a proper class of varieties of topological groups $\mathcal{W}$ such that $|\mathcal{W}|_2$ is a variety of groups.

Theorem 1.6.7. Let $m$ be an infinite cardinal and let $\mathcal{T}_m$ be the variety of all $T(m)$-groups. Then the class of groups $|\mathcal{T}_m|_2$ is not a variety of groups.

Proof. Every free group is a $T(\aleph_0)$-group and therefore is a $T(m)$-group. Suppose $|\mathcal{T}_m|_2$ is a variety of groups. Then $|\mathcal{T}_m|_2$ contains all groups. However, by [20] for example, for each cardinal $m$ there exists a simple group $G$ of cardinality strictly greater than $m$. Thus $G$ is not contained in $|\mathcal{T}_m|_2$ and so $|\mathcal{T}_m|_2$ is not a variety of groups.  

Up to this point we have only presented varieties of topological groups $\mathcal{W}$ where the class of groups underlying Hausdorff groups contained in $\mathcal{W}$ do not form varieties of groups.
1. Basic Definitions and Results

However, this is not always the case, as we see in Theorem 1.6.8.

We shall denote by \( \mathbb{Z}(p^{\infty}) \) the set of all complex numbers \( \exp \left( \frac{2\pi ik}{p^n} \right) \), where \( p \) is a fixed prime, \( k \) runs through all integers, and \( n \) through all nonnegative integers. Note that \( \mathbb{Z}(p^{\infty}) \) is a (countable) abelian group under multiplication.

**Theorem 1.6.8** Let \( m \) be any cardinal number such that \( m > \aleph_0 \) and let \( \mathfrak{A}_m \) be the variety of all abelian \( T(m) \)-groups. Then the class of groups \( [\mathfrak{A}_m]_2 \) is a variety of groups; indeed \( [\mathfrak{A}_m]_2 \) is the variety of all abelian groups.

**Proof.** We know that the group of rational numbers \( \mathbb{Q} \) under addition is an abelian group of cardinality \( \aleph_0 < m \). Therefore, \( \mathbb{Q} \) with the discrete topology forms a (Hausdorff) \( T(m) \)-group. Hence, \( \mathbb{Q} \in [\mathfrak{A}_m]_2 \). Similarly, \( \mathbb{Z}(p^{\infty}) \in [\mathfrak{A}_m]_2 \).

Let \( G \) be an abelian group. Then \( G \) can be embedded in a divisible abelian group \( H \) (see, for example, [19], Theorem A.15). Further, \( H \) is isomorphic to a subgroup of a product of copies of \( \mathbb{Q} \) and \( \mathbb{Z}(p^{\infty}) \) where \( p \) ranges over all primes (see, for example, [19], Theorem A.14). Now, \( G \) with the topology induced by the product of copies of \( \mathbb{Q} \) and \( \mathbb{Z}(p^{\infty}) \) each with the discrete topology forms a Hausdorff \( T(m) \)-group. Therefore, \( G \in [\mathfrak{A}_m]_2 \). Thus, \( [\mathfrak{A}_m]_2 \) is the variety of all abelian groups.

In Proposition 1.5.6 we mentioned that the class of groups underlying a variety of topological groups is itself a variety of groups. Therefore, results pertaining to varieties of groups are readily applied to varieties of topological groups. Theorems 1.6.6 and 1.6.7 show that when only considering Hausdorff topological groups, the analogous result is false. If we were to restrict our attention to Hausdorff topological groups results pertaining to varieties of groups would not come "free". However, we see in Theorem 1.6.8 that abelian \( T(m) \)-groups behave a little "better" when \( m > \aleph_0 \). In this thesis most of the work will concern abelian topological groups.
The Variety Generated by the Class of All Banach Spaces

In this chapter, the variety of topological groups generated by the class of topological groups underlying Banach Spaces comes under the microscope. Some interesting—even surprising—results will be presented. In the final section, we answer a few naturally arising questions about classes of topological groups that are quotients of subgroups of Banach spaces.

§2.1 Pseudometrics

Definition 2.1.1. ([24], Part I, Section 2) A pseudometric on a set $X$ is a mapping $\rho$ from the cartesian product $X \times X$ to the set of non-negative real numbers such that for all points $x, y$ and $z$ in $X$,

(a) $\rho(x, y) = \rho(y, x)$

(b) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality)

(c) $\rho(x, x) = 0$.

Note that a metric is a pseudometric which has the added condition that if $\rho(x, y) = 0$ then $x = y$.

A topology on $X$ can be derived from a pseudometric $\rho$ in the same manner in which a topology is derived from a metric. More precisely, let $\alpha > 0$ be a real number and $x \in X$. The open sphere of radius $\alpha$ about $x$ is defined to be $B_{\alpha}(x) = \{y \in X : \rho(x, y) < \alpha\}$ and the family of all such open spheres is a basis for the topology induced by $\rho$ on $X$.

Examples 2.1.2. ([24], Part I, Section 2)

(a) Let $X$ be a set and let $\rho(x, y) = 0$ for all $x, y \in X$. Then $\rho$ is a pseudometric and it induces the indiscrete topology on $X$. 
2. Banach Spaces

(b) If $f$ is any mapping from a set $X$ to the set of all real numbers, then a pseudometric $ho$ on $X$ can be defined by $\rho(x, y) = |f(x) - f(y)|$.

Remark 2.1.3. Let $I$ be some index set. Assume that $\{\rho_i : i \in I\}$ is a family of pseudometrics on a set $X$. Then a subbasis for a topology on $X$ is given by the sets of the form $\{y : \rho_i(x, y) < \alpha \text{ for each } i \in F\}$ where $x \in X$, $\alpha > 0$ is a real number and $F$ is a finite subset of $I$. This topology is called the topology induced by the $\rho_i$.

Note that all we have seen so far on pseudometrics can be applied to groups; that is, we can define a pseudometric on a group $G$ that defines a topology on $G$. However, not every pseudometric on a group defines a topological group.

Definitions 2.1.4. ([19], Chapter II, §8) A (pseudo)metric $\rho$ on a group $G$ is said to be left invariant if $\rho(az, ay) = \rho(z, y)$ for all $a, z, y \in G$. If $\rho(xa, ya) = \rho(x, y)$ for all $a, x, y \in G$, then $\rho$ is said to be right invariant. If $\rho$ is both left and right invariant, it is said to be two-sided invariant.

It is well-known that every two-sided invariant pseudometric on a group gives rise to a topological group (the proof follows easily from [19], §4, Chapter II).

Definition 2.1.5. ([46], Chapter 8, Exercise Set Sixteen, 5(i)) Let $X$ be a topological space and $d$ a (pseudo)metric on $X$. Then $d$ is said to be a continuous (pseudo)metric on $X$ if the map $d : X \times X \to \mathbb{R}$ is continuous, where $X \times X$ denotes the product of two copies of the topological space $X$. We say that $X$ admits a continuous (pseudo)metric.

Theorem 8.2 of [19] tells us that the topology of a topological group can be described in terms of continuous left invariant pseudometrics (see [19], Comment 8.14). Note that in an abelian topological group, a left invariant pseudometric is also right invariant. Therefore, we have the following result.
Proposition 2.1.6. ([19], Theorem 8.2; cf. [29], Chapter 6, Problem Q and Corollary 17)
The topology of any abelian topological group is defined by the family of all continuous
two-sided invariant pseudometrics on the group. ■

Remark 2.1.7. It is well-known that the following statements about a topological space $X$ are equivalent.

(i) $X$ is completely regular;

(ii) the topology on $X$ is defined by the family of all continuous pseudometrics on $X$; and

(iii) the topology on $X$ is defined by a family of continuous pseudometrics on $X$.

To see this, we note that a space is completely regular if and only if it is homeomorphic
to a subspace of a product of pseudometrizable spaces (see [29], Chapter 4, Problem L).
Further, as every topological group has its topology defined by a family of invariant
continuous pseudometrics (Proposition 2.1.6), it is clear that every topological group is
completely regular.
Finally, it will become evident that every completely regular space is a subspace of a
topological group, namely the free abelian topological group on $X$.

§2.2 Free Abelian Topological Groups

In this section we shall briefly consider the free abelian topological group on a completely
regular space as discussed by Graev in [16]. However, we must first define the free and
free abelian topological groups on a completely regular topological space.

Definition 2.2.1. [16] Let $X$ be a completely regular space with a distinguished point $e$ in $X$. We shall call the topological group $F(X)$ a Graev free topological group on the space $X$ if it has the following properties:

(1) $X$ is a subspace of $F(X)$;
2. Banach Spaces

(2) $X$ generates $F(X)$ algebraically;

(3) for any continuous mapping $\phi$ of $X$ into any topological group $G$ which maps the point $e$ onto the identity element of $G$, there exists a continuous homomorphism $\Phi$ of $F(X)$ into $G$ such that $\Phi(x) = \phi(x)$ on $X$. ■

The definition of the free abelian topological group on a completely regular space, $X$, is analogous.

Definition 2.2.2. [16] Let $X$ be a completely regular space with a distinguished point $e$ in $X$. We shall call the topological group $FA(X)$ a Graev free abelian topological group on the space $X$ if it has the following properties:

(1) $X$ is a subspace of $FA(X)$;

(2) $X$ generates $FA(X)$ algebraically;

(3) for any continuous mapping $\phi$ of $X$ into any abelian topological group $G$ which maps the point $e$ onto the identity element of $G$, there exists a continuous homomorphism $\Phi$ of $FA(X)$ into $G$ such that $\Phi(x) = \phi(x)$ on $X$. ■

Most of the analysis in this thesis will deal with the (Graev) free abelian topological group on $X$, $FA(X)$.

Remarks 2.2.3.

(a) $|FA(X)|$, the group underlying $FA(X)$, is the free abelian group on $X\setminus\{e\}$ ([48], Proposition [47], page 376).

Proof. Let $\phi$ be any mapping of $X\setminus\{e\}$ into any group $H$. Equip $H$ with the indiscrete topology so $H$ is now a topological group. Further, define $\phi(e)$ to be the identity in $H$. We now have a continuous mapping of $X$ into a topological group $H$ as any mapping into an indiscrete space is continuous. Therefore, there exists a (continuous) homomorphism $\Phi$ of $FA(X)$ into $H$ which extends $\phi$ on $X\setminus\{e\}$. Therefore $FA(X)$ is, algebraically, the free abelian group on $X\setminus\{e\}$.
(b) The topological group $FA(X)$ is the finest topological group topology on the free abelian group on $X \setminus \{e\}$ that satisfies property (1) in Definition 2.2.2. In [16], Graev showed the existence of the Graev free abelian topological group, $FA(X)$, on each completely regular Hausdorff space $X$. To do so, observe that it suffices to find one topological group topology on $|FA(X)|$, the free abelian group on the set $X \setminus \{e\}$, which satisfies conditions (1) and (2), since the sum of all such topologies will satisfy condition (3). As a first step, note that every completely regular space is determined by a family of continuous pseudometrics (see Remark 2.1.7). Each continuous pseudometric on $X$ can be extended to a two-sided invariant pseudometric on $|FA(X)|$. We shall refer to this extension as the Graev extension of the pseudometric. The Graev extension determines a topological group topology on $|FA(X)|$. The sum of all such topologies on $|FA(X)|$ gives a topological group topology on $|FA(X)|$, and this topological group will be called the Graev abelian topological group on $X$ and it clearly satisfies conditions (1) and (2).

c) Any two free (free abelian) topological groups on $X$ are topologically isomorphic; that is, $F(X)$ is unique up to topological isomorphism. In particular, $F(X)$ does not depend on the choice of the point $e$ in the space $X$. (See [16].)

Proof. Let $X$ be a completely regular space, let $F$ be the free (free abelian) topological group on $X$ with distinguished point $e$ and $F'$ the free (free abelian) topological group on $X$ with distinguished point $f$. Let $\phi$ be the map from $X$ into $F'$ given by $\phi(x) = xe^{-1}$ for $x \in X$. Clearly, $\phi$ is continuous and $\phi(e) = ee^{-1} = f$. Thus, there exists a continuous homomorphism $\Phi : F \to F'$ such that $\Phi|_X$ is $\phi$. Similarly, let $\psi$ be the map from $X$ into $F$ given by $\psi(x) = xf^{-1}$ for $x \in X$. Then $\psi$ is continuous, $\psi(f) = e$, and there exists a continuous homomorphism $\Psi : F' \to F$ such that $\Psi|_X$ is $\psi$. Now, let $x \in X \subseteq F$, consider $\Psi \circ \Phi : F \to F$. We have

$$\Psi(\Phi(x)) = \Psi(xe^{-1}) = \psi(x)\psi(e)^{-1} = x f^{-1}(e f^{-1})^{-1} = xe^{-1} = x$$

as $e \in F$ is the identity element. As $X$ generates $F$, $\Psi \circ \Phi$ is the identity map from $F$ onto $F$. Similarly, $\Phi \circ \Psi$ is the identity map from $F'$ onto $F'$ and so both $\Psi$ and
\( \Phi \) are continuous isomorphisms, indeed, \( \Phi = \Psi^{-1} \). Finally, for an open set \( O \) in \( F \), \( \Phi(O) = \Psi^{-1}(O) \) which is open in \( F' \). Hence \( \Phi \) is a topological isomorphism and so \( F \) and \( F' \) are topologically isomorphic.

**Remark 2.2.4.** [16]

We shall summarize, in a sense, the Graev approach of extending a pseudometric from a space \( X \) with distinguished point \( e \) to \( F \), the free abelian group on the set \( X \setminus \{e\} \). Let \( \rho \) denote the pseudometric on \( X \) and let \( T \) be the topology on \( X \) induced by \( \rho \). We wish to extend the pseudometric \( \rho \) to a pseudometric \( \rho' \) on \( F \) in such a way that \( \rho' \) defines the topology \( T' \) on \( F \) and \( T' \) induces the topology \( T \) on \( X \) as a subspace of \((F,T')\).

Define the pseudometric \( \rho' \) on the set \( X \cup X^{-1} \) as follows. Let \( x, y \in X \) and \( x^{-1}, y^{-1} \in X^{-1} \) where \( X^{-1} = \{x^{-1} : x \in X\} \). Then

\[
\rho'(x, y) = \rho(x, y);
\rho'(x^{-1}, y^{-1}) = \rho(x, y);
\rho'(x^{-1}, y) = \rho'(x, y^{-1}) = \rho(x, e) + \rho(y, e).
\]

We now extend the pseudometric \( \rho' \) from \( X \cup X^{-1} \) to the full group \( F \). Let \( a, b \in F(X) \), \( a \neq b \) such that the reduced representation in the form of words are given by

\[
a = x_1x_2\ldots x_m \quad \text{and} \quad b = y_1y_2\ldots y_n
\]

where \( x_i, y_j \in X \cup X^{-1}, i = 1,\ldots,m, j = 1,\ldots,n \) and each \( x_i \) and \( y_j \) is different from \( e \).

We shall consider all possible (not necessarily reduced) representations of \( a \) and \( b \) in the form of words of equal length with elements of \( X \cup X^{-1} \):

\[
a = a_1a_2\ldots a_s
\]

\[
b = b_1b_2\ldots b_s
\]

where \( a_i, b_i \in X \cup X^{-1} \) for \( i = 1,\ldots,s \). We define \( \rho'(a, b) \) to be the infimum over all such representations of \( a \) and \( b \) of \( \sum_{i=1}^{s} \rho(a_i, b_i) \).

Graev goes on to show that there are representations for \( a \) and \( b \) that give the value

\( \rho'(a, b) \); that is, there is a minimum value for \( \sum_{i=1}^{s} \rho(a_i, b_i) \). Further, these representations have length at most \( m + n \) and only use the letters \( x_1, x_2,\ldots, x_n, y_1, y_2,\ldots, y_m \).
Note that Graev dealt with both the free topological group and the free abelian topological group on a completely regular Hausdorff space $X$. Further, we note that in general, Graev’s construction of taking the sum of all topologies on $|F(X)|$ resulting from extending pseudometrics on $X$ is not the Graev free topological group on $X$ (see [50], Theorem 2). However, as we see in the following proposition, for the abelian case Graev’s construction actually gives the free topology.

**Proposition 2.2.5.** ([48], pp 378–379; [50], Proposition 1) The Graev abelian topological group on a completely regular Hausdorff space $X$ is the Graev free abelian topological group on $X$.

**Proof.** The Graev free abelian topological group on $X$, given by $FA(X)$ is defined by the family of all two-sided invariant continuous pseudometrics (Proposition 2.1.6). Let the family of such pseudometrics be $\{\sigma_j : j \in J\}$ where $J$ is some index set, and for each $j \in J$ let $\rho_j$ be the restriction of $\sigma_j$ to $X$ with the Graev extension of $\rho_j$ to $|FA(X)|$ denoted by $\rho'_j$. Now, for $x, y \in X$, we have

$$\rho'_j(x, y) = \sigma_j(x, y)$$

$$= \sigma_j(x^{-1}, y^{-1})$$

$$= \rho'_j(x^{-1}, y^{-1}).$$

Further,

$$\rho'_j(x^{-1}, y) = \rho'_j(x, e) + \rho'_j(y, e)$$

$$= \sigma_j(x, e) + \sigma_j(y, e)$$

$$= \sigma_j(x^{-1}, e) + \sigma_j(y, e) \geq \sigma_j(x^{-1}, y)$$

Therefore, for each $a, b \in X \cup X^{-1}$, $\sigma_j(a, b) \leq \rho'_j(a, b)$. Let $u$ and $v$ be words in $FA(X)$ and let $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_n$ be the representations for $u$ and $v$ respectively where $a_i, b_i \in X \cup X^{-1}$ for each $i = 1, \ldots, n$ and $\rho'_j(u, v) = \sum_{i=1}^{n} \rho'_j(a_i, b_i)$. Then, we have

$$\rho'_j(u, v) \geq \sum_{i=1}^{n} \sigma_j(a_i, b_i)$$

$$= \sum_{i=1}^{n} \sigma_j(a_i b_i^{-1}, e)$$

$$\geq \sigma_j(a_1 a_2 \ldots a_n, b_1 b_2 \ldots b_n) = \sigma_j(u, v).$$
Therefore, for each \(w_1, w_2 \in FA(X)\), \(\sigma_j(w_1, w_2) \leq \rho'_j(w_1, w_2)\).

Now, let \((|FA(X)|, T_1)\) be the Graev free abelian topological group on \(X\) defined by \(\{\sigma_j : j \in J\}\), and let \((|FA(X)|, T_2)\) be the Graev abelian topological group on \(X\). Clearly, \(T_2 \subseteq T_1\). Take \(U = \{w \in FA(X) : \sigma_j(w, e) < \varepsilon\}\), an open sphere about the identity in \(T_1\), and consider \(O = \{w \in FA(X) : \rho'_j(w, e) < \varepsilon\}\). If \(w \in O\), then \(\sigma_j(w, e) \leq \rho'_j(w, e) < \varepsilon\) and so \(w \in U\). Therefore, \(e \in O \subseteq U\) and so \(U\) is open in \(T_2\). Thus, \(T_1 \subseteq T_2\) and the result follows.

From this point on, we shall refer to the Graev free abelian topological group as simply the free abelian topological group. Our reasons for choosing the Graev free abelian topological group (as opposed to Markov [36]) will be seen in Chapter 3, when we first consider \(FA[0, 1]\), the free abelian topological group on a connected space. To complete our brief overview of free abelian topological groups, we present the following results which will be used in later analysis.

The first result is similar to the known result that every abelian topological group \(G\), with \(\mathcal{P}\) the family of all pseudometrics on \(G\), can be embedded in the product \(\prod_{\rho \in \mathcal{P}} (|G|, \rho)\), where \(|G|\) is the group underlying \(G\). In the case of the free abelian topological group, \(FA(X)\), on a completely regular Hausdorff space \(X\), we see that \(FA(X)\) can be embedded in the product over the family of pseudometrics \(\mathcal{R}\), where \(\mathcal{R}\) consists only of the pseudometrics on \(FA(X)\) that are Graev extensions of continuous pseudometrics on \(X\). Indeed, this is another way of presenting Remark 2.2.3(b).

**Proposition 2.2.6.** Let \(X\) be a completely regular topological space whose topology is defined by the family \(\{\rho_i : i \in I\}\) of pseudometrics. Then the free abelian topological group on \(X\), \(FA(X)\), can be embedded as a topological subgroup of the product \(H = \prod_{i \in I} (|FA(X)|, \rho'_i)\), where \(|FA(X)|\) is the free group on \(X\setminus\{e\}\) and \(\rho'_i\) is the Graev extension of \(\rho_i\) for each \(i \in I\).

**Proof.** Let \(f : FA(X) \to H\) be given by \(f(w) = \prod_{i \in I} w_i\) where \(w \in FA(X)\) and \(w_i = w\).
for each \( i \in I \). The mapping \( f \) is clearly a one-to-one homomorphism. A subbasis at \( e \) for the topology of \( FA(X) \) is given by the open spheres about \( e \) in \( \rho_i, i \in I \). Consider the open sphere \( B_j(e) \) in \( \rho_j \) where \( B_j(e) = \{ w \in FA(X) : \rho_j(w, e) < \varepsilon \} \), for \( j \in I \) and some \( \varepsilon > 0 \). Then \( f(\bigcap_{i \neq j} B_j(e)) = \bigcap_{i \in I} \rho_i \cap f(FA(X)) \) where \( U_j = B_j(e) \) and \( U_i = FA(X) \) for each \( i \neq j \). Clearly this is open in \( f(FA(X)) \) and so \( f \) is an open mapping. Finally, let \( O = \bigcap_{i \in I} O_i \) be an open set in \( H \), where \( O_i \) is open in \( (|FA(X)|, \rho_i') \) and \( O_i = |FA(X)| \) for all \( i \in I \setminus J, J \subseteq I \) a finite set. Then \( f^{-1}(O) = \bigcap_{i \in I} O_i = \bigcap_{j \in J} O_j \) is open in \( FA(X) \). Therefore, \( f \) is continuous and the result follows.

**Lemma 2.2.7.** ([38], Lemma 2.11) Let \( X \) be a completely regular space and \( F \) a topological group algebraically generated by \( X \) such that \( X \) is a subspace of \( F \). If \( f \) is a homomorphism from \( F \) into a topological group \( G \) such that \( f|_X \) is open then \( f \) is open.

**Proof.** Let \( a \in F \) and let \( O \) be open in the subspace \( aX \). Then, for some \( U \) open in \( F \),

\[
O = aX \cap U = a(X \cap a^{-1}U) = aO_1
\]

where \( O_1 \) is open in \( X \). Now, since \( f \) is a homomorphism, \( f(O) = f(a)f(O_1) \). Further, \( f(O_1) \) is open in \( G \) and hence, \( f(O) = f(a)f(O_1) \) is open in \( G \).

Clearly \( F = \bigcup_{a \in F} aX \). Let \( A \) be open in \( F \) then

\[
f(A) = f\left( A \cap \left( \bigcup_{a \in F} aX \right) \right) = \bigcup_{a \in F} f(A \cap aX).
\]

Now, for each \( a \in F \), \( A \cap aX \) is open in \( aX \) and hence \( f(A \cap aX) \) is open in \( G \). Therefore, \( f(A) \) is also open in \( G \).

We note that all topological groups are completely regular (see Remark 2.1.7) and so the free abelian topological group on a Hausdorff topological group makes sense. Further, we have the following relationship between a topological group \( G \) and the free abelian topological group on \( G \).
Proposition 2.2.8. ([38], Theorem 2.12) An abelian topological group $G$ is a quotient group of $FA(G)$, the free abelian topological group on the underlying topological space of $G$.

Proof. Let $\phi : G \to G$ be the identity map. As $\phi$ is continuous and $FA(G)$ is the free abelian topological group, there exists a continuous homomorphism $\Phi$ of $FA(G)$ into $G$ such that $\Phi|_G = \phi$. Clearly $\Phi$ is surjective as $\phi$ is surjective. As $\phi$ is an open mapping, Lemma 2.2.7 implies $\Phi$ is open and the result follows. ■

We shall briefly define the concepts of $k_\omega$-space and $k_\omega$-group here, but they will be studied in more detail in Chapter 3. What we present here will be the results necessary for the analysis in this chapter.

Definitions 2.2.9. [63] A topological space $X$ is said to be a $k_\omega$-space with $k_\omega$-decomposition $X = \bigcup_{n=1}^{\infty} X_n$ if it is a Hausdorff space with compact subsets $X_n$ for $n = 1, 2, \ldots$, such that

(i) $X = \bigcup_{n=1}^{\infty} X_n$;

(ii) $X_n \subseteq X_{n+1}$ for all $n$;

(iii) a subset $A$ of $X$ is closed if and only if $A \cap X_n$ is compact (or closed) for all $n$.

Further, a topological group that is a $k_\omega$-space is said to be a $k_\omega$-group. ■

The following remark presents an equivalent definition of a $k_\omega$-space. We saw in condition (iii) of Definition 2.2.9 that a subset $A$ of a $k_\omega$-space $X = \bigcup_{n=1}^{\infty} X_n$ is closed if and only if $A \cap X_n$ is compact. Here we see an equivalent definition where "closed" and "compact" are replaced with "open".

Remarks 2.2.10.

(a) Let $X$ be a Hausdorff topological space with compact sets $X_n$ such that $X = \bigcup_{n=1}^{\infty} X_n$ and $X_n \subseteq X_{n+1}$ for all $n$. Then the following two statements are equivalent.

(i) $X$ is a $k_\omega$-space.

(ii) A subset $A$ of $X$ is open if and only if $A \cap X_n$ is open in $X_n$ for all $n$. 

(b) We note from Section 2 of [34] that if $K$ is any compact subspace of a $k_\omega$-space $X$ with $k_\omega$-decomposition $X = \bigcup_{n=1}^{\infty} X_n$, $X_n$ compact for each $n \in \mathbb{N}$, then $K \subseteq X_n$ for some $n \in \mathbb{N}$.

**Examples 2.2.11.**

(a) Every compact Hausdorff space $Y$ is a $k_\omega$-space with $k_\omega$-decomposition $Y = \bigcup_{n=1}^{\infty} X_n$ where each $X_n = Y$.

(b) The topological space $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n,n]$ is an example of a $k_\omega$-space which is not a compact space.

(c) A discrete $k_\omega$-space, $D$, is countable since if $D = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is compact in $D$, then each $X_n$ must be finite.

(d) Every connected locally compact Hausdorff group $G$ is a $k_\omega$-group ([34], Section 2) with $k_\omega$-decomposition $G = \bigcup_{n=1}^{\infty} K^n$ where $K$ is any compact symmetric neighbourhood of the identity in $G$ ([46], Corollaries 1 and 2 to Proposition 8).

**Proposition 2.2.12.** ([16], Theorem 4, Part II, §7) Let $X$ be a compact Hausdorff topological space. Then $F(X)$, the free (free abelian) topological group on $X$, is a $k_\omega$-group with $k_\omega$-decomposition $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$, where $F_n(X)$ is the set of all words of length less than or equal to $n$ with respect to $X$.

§2.3 *The Enflo Property*

In [11], Enflo characterized those metric spaces which can be embedded isometrically in a Banach space. The metrics on these topological groups have a property which we shall call the *Enflo property*. The Enflo property turns out to be the key in our analysis. We shall extend it to pseudometrics.
Definition 2.3.1: The Enflo Property. (cf. [11], Theorem 2.4.1) Let $(G, \rho)$ be a pseudometric abelian topological group. Then the pseudometric $\rho$ is said to have the Enflo property if it is invariant and $\rho(x^2, e) = 2\rho(x, e)$ for all $x \in G$.

The Enflo property on a pseudometric says more than is at first apparent. The following lemma gives the result that for an invariant pseudometric $\rho$ with the Enflo property on a group $G$, $\rho(x^n, e) = |n|\rho(x, e)$ for any $n \in \mathbb{Z}$.

Lemma 2.3.2. ([11], Section 2.4) Let $G$ be an abelian topological group that admits an invariant pseudometric $\rho$ such that $\rho(x^2, e) = 2\rho(x, e)$ for all $x \in G$. Then for each $n \in \mathbb{Z}$, $\rho(x^n, e) = |n|\rho(x, e)$.

Proof. Firstly, for $m \in \mathbb{N}$, if $\rho(x^m, e) \leq m\rho(x, e)$, then

$$\rho(x^{m+1}, e) = \rho(x^m, x^{-1})$$

$$\leq \rho(x^m, e) + \rho(e, x^{-1})$$

$$\leq m\rho(x, e) + \rho(x, e)$$

$$= (m + 1)\rho(x, e).$$

Therefore, using the fact that $\rho(x^2, e) = 2\rho(x, e)$ and Mathematical Induction, we have $\rho(x^n, e) \leq n\rho(x, e)$ for all $n \in \mathbb{N}$.

We now use Mathematical Induction over $m$ to prove $\rho(x^k, e) = k\rho(x, e)$ for all natural numbers $k \leq 2^m$. Clearly, the result is true for $m = 1$. Now, suppose $\rho(x^k, e) = k\rho(x, e)$ for all $k \leq 2^m$, $k \in \mathbb{N}$. Note that

$$\rho(x^{2^m+1}, e) = \rho((x^{2^m})^2, e) = 2\rho(x^{2^m}, e) = 2^{m+1}\rho(x, e).$$

Let $2^m < k < 2^{m+1}$, $k \in \mathbb{N}$, and consider

$$\rho(x^{2^{m+1}}, e) = k\rho(x, e) + (2^{m+1} - k)\rho(x, e).$$

As $0 < 2^{m+1} - k < 2^m$, we have

$$\rho(x^{2^{m+1}}, e) - \rho(x^{2^{m+1} - k}, e) = k\rho(x, e).$$

(1)
Further,
\[
\rho(x^{2m+1}, e) = \rho(x^{2m+1-k}, x^{-k}) 
\leq \rho(x^{2m+1-k}, e) + \rho(x^k, e).
\]
Therefore, we have
\[
\rho(x^{2m+1}, e) - \rho(x^{2m+1-k}, e) \leq \rho(x^k, e).
\]
Thus, by (1), \( k \rho(x, e) \leq \rho(x^k, e) \) and by our earlier work, \( k \rho(x, e) \geq \rho(x^k, e) \). Hence, \( \rho(x^k, e) = k \rho(x, e) \) for all \( k \leq 2m+1 \) and by Mathematical Induction, the result follows.

Therefore, we have \( \rho(x^n, e) = n \rho(x, e) \) for all \( n \in \mathbb{N} \). Noting that \( \rho(x^0, e) = \rho(e, e) = 0 \) and for \( m \in \mathbb{N} \),
\[
\rho(x^{-m}, e) = \rho(x^m, e) = m \rho(x, e)
\]
it follows that \( \rho(x^n, e) = |n| \rho(x, e) \) for all \( n \in \mathbb{Z} \).

We wish to extend Lemma 2.3.2 to rational powers of elements of \( G \). However, we must first establish that for \( g \in G \), the expression \( g^r \), where \( r \in \mathbb{Q} \), is defined.

**Lemma 2.3.3.** Let \( G \) be an abelian topological group that admits an invariant pseudometric \( \rho \) with the Enflo property. If \( g, h \in G \) such that there exists \( n \in \mathbb{N} \) with \( g^n = h^n \), then \( \rho(g, h) = 0 \). Further, if \( \rho \) is a metric, then \( g = h \).

**Proof.** As \( g^n = h^n, (gh^{-1})^n = e \) and so \( \rho((gh^{-1})^n, e) = 0 \). Thus,
\[
0 = \rho((gh^{-1})^n, e) = n \rho(gh^{-1}, e) = n \rho(g, h)
\]
and so \( \rho(g, h) = 0 \). If \( \rho \) is a metric, then \( g = h \).

A consequence of the previous lemma is that if \( \rho \) is a metric, and \( g^{1/n} \) exists for some \( g \in G \), then it is unique. However, if \( \rho \) is not a metric, then for any two elements \( x, y \in G \) such that \( x^n = y^n = g, \rho(x, y) = 0 \).

The following lemma gives the result that for a pseudometric \( \rho \) with the Enflo property on a group \( G \), \( \rho(x^r, e) = |r| \rho(x, e) \) for any \( r \in \mathbb{Q} \) where \( x^r \) is defined in \( G \).
Lemma 2.3.4. ([11], Section 2.4) Let $G$ be an abelian topological group that admits an invariant pseudometric $\rho$ with the Enflo property. Then if $r = \frac{m}{n} \in \mathbb{Q}$ and for $x \in G$ there exists $y \in G$ such that $y^n = x^m$, then $\rho(y, e) = |r|\rho(x, e)$. Further, if $\rho$ is a metric, then there exists at most one $y$ such that $x^r = y$ and $\rho(x^r, e) = |r|\rho(x, e)$.

Proof. From Lemma 2.3.2, we have $\rho(y^n, e) = |n|\rho(y, e) = \rho(x^m, e) = |m|\rho(x, e)$ and so $\rho(y, e) = |\frac{m}{n}|\rho(x, e)$. If $\rho$ is metric, then for any $z \in G$ such that $z^n = x^m$, $z = y$ and so $x^r = y$ uniquely, thus $\rho(x^r, e) = |r|\rho(x, e)$.

Remark 2.3.5. Note that a metric abelian topological group, $(G, d)$, where $d$ has the Enflo property is torsion-free; that is, every element other than the identity $e$ has infinite order. Let $g \in G$ such that $g \neq e$. Suppose $g^n = e = e^n$ for some $n \in \mathbb{N}$. By Lemma 2.3.3, $g = e$ which is a contradiction, so $g$ has infinite order. Note that this implies $G$ is trivial or infinite.

We now present the result by Enflo that has attracted our attention.

Theorem 2.3.6. ([11], Theorem 2.4.1) Let $(G, d)$ be a metric abelian topological group. If the metric $d$ is invariant and has the property $d(x^2, e) = 2d(x, e)$ for all $x \in G$, then $G$ is topologically isomorphic to a subgroup of a Banach space.

Proof. We first recall that by Lemma 2.3.4, if $x^r \in G$ for $r \in \mathbb{Q}$ and $x \in G$, then $d(x^r, e) = |r|d(x, e)$. Further, by Remark 2.3.5, $G$ is torsion-free.

Form the set of formal powers $F = \{x^r : r \in \mathbb{Q}, x \in G\}$ and define the relation $\sim$ on $F$ by $x^{r_1} \sim y^{r_2}$ if there exists $p \in \mathbb{N}$ such that $x^{pr_1} = y^{pr_2} \in G$. It is easily shown that $\sim$ is in fact an equivalence relation and so we form $N$, the set of equivalence classes of $F$ under $\sim$. Define addition in $N$ as follows.

$$x^{p_1/q_1} + y^{p_2/q_2} = (x^{p_1q_2}y^{p_2q_1})^{1/q_1q_2}.$$ 

Take $x^{p_1/q_1} \sim (x')^{p_1/q_1}$ where $x^{np_1/q_1} = (x')^{np_1/q_1}$ and also take $y^{p_2/q_2} \sim (y')^{p_2/q_2}$ where
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\[ y^{mp_2/q_2} = (y')^{m_{s_2}/t_2}. \] Then

\[
(x^{p_1/q_1} y^{p_2/q_1})^{nm/q_1 q_2} = x^{m(p_1/q_1)} y^{n(m p_2/q_2)}
\]

\[
= (x')^{m_{s_1}/t_1} (y')^{n(m_{s_2}/t_2)}
\]

\[
= ((x')^{s_1 t_2} (y')^{s_2 t_1})^{nm/t_1 t_2}
\]

and hence \( x^{p_1/q_1} + y^{p_2/q_2} \sim (x')^{s_1/t_1} + (y')^{s_2/t_2} \). Therefore, the definition of addition is independent of the choice of member of the equivalence class. Define scalar multiplication of \( N \) over \( \mathbb{Q} \) by \( \frac{r}{s}(x^{p/q}) = x^{rp/sq} \). This is consistent with addition and again, is independent of the choice of member of the equivalence class. It is routine to show that \( N \) with addition and scalar multiplication forms a vector space over \( \mathbb{Q} \). We define a norm on \( N \) by putting \( \|x^r\| = |r|d(x, e) \). If \( x^r \sim y^s \) with \( x^{pr} = y^{ps} \in G \), then by applying Lemma 2.3.4, we have

\[
|p|\|x^r\| = |pr|d(x, e)
\]

\[
= d(x^{pr}, e)
\]

\[
= d(y^{ps}, e)
\]

\[
= |ps|d(y, e)
\]

\[
= |p|\|y^s\|
\]

giving \( \|x^r\| = \|y^s\| \). Thus, the definition is independent of the choice of member of the equivalence class. Clearly, \( \| \| \) satisfies the conditions of a norm and so \( N \) is a normed vector space over \( \mathbb{Q} \).

Let \( f : G \to N \) be given by \( f(g) = g \in N \) for all \( g \in G \). Now, \( f \) is clearly a homomorphism. To show that \( f \) is one-to-one, we take \( f(g) = f(h) \). Then \( g \sim h \), that is, there exists \( p \in \mathbb{N} \) such that \( g^p = h^p \in G \). This gives \( g^p h^{-p} = e \) and hence \( (gh^{-1})^p = e \). As \( G \) is torsion-free, we have \( gh^{-1} = e \) giving \( g = h \) and so \( f \) is one-to-one. Now, the topology on \( N \) is defined by the metric

\[
d^*(x^r, e) = \|x^r\| = |r|d(x, e)
\]

and when \( x^r \in G \), \( d^*(x^r, e) = d(x^r, e) \). Thus, for \( B_\varepsilon(e) = \{ g \in G : d(g, e) < \varepsilon \} \), the open sphere about \( e \) in \( G \), \( f(B_\varepsilon(e)) = f(G) \cap O \) where \( O = \{ x^r \in N : d^*(x^r, e) < \varepsilon \} \), and so \( f(B_\varepsilon(e)) \) is open in \( f(G) \subseteq N \). Finally, as \( f \) is one-to-one, \( f^{-1}(f(G) \cap O) = B_\varepsilon(e) \) and so \( f \) is continuous on \( G \). Therefore, \( f \) is an embedding and so \( G \) is topologically isomorphic to a subgroup of \( N \).
To complete the proof, we let $B$ be the completion of $N$. Then $B$ is a complete normed vector space over $\mathbb{R}$, that is, a Banach space, and we have $G$ embedded as a topological subgroup in $B$.

It is possible to extend Theorem 2.3.6 to show that for $(G, \rho)$, a pseudometric abelian topological group where $\rho$ has the Enflo property, $G$ is topologically isomorphic to a subgroup of a seminormed topological vector space. The proof is almost identical to that for Theorem 2.3.6. However we present a more general result dealing with Hausdorff abelian topological groups and avoid the detailed approach of Theorem 2.3.6. First, we recall the following result by Graev.

**Proposition 2.3.7.** (cf. [19], 8.17) Let $G$ be a Hausdorff abelian topological group. Then $G$ is topologically isomorphic to a subgroup of a product of topological groups each of which has its topology determined by an invariant metric.

**Remark 2.3.8.** ([19], 8.17) Note that in the proof of Proposition 2.3.7 in [19], $G$ is a subgroup of a product of quotient groups of $G$, $|G/A|$, where $|A|$ is a normal subgroup of $|G|$ and for some pseudometric $\rho$ on $G$, $A = \{x \in G : \rho(x, e) = 0\}$. The metric $d$ on $|G/A|$ is then given by $d(xA, yA) = \rho(x, y)$ and it is noted that the topology $d$ induces on $|G/A|$ may be different from the quotient topology of $G/A$.

As we are working in an abelian world, it is convenient to combine Proposition 2.3.7 with the following lemma to see that a Hausdorff abelian topological group $G$ can be embedded as a topological subgroup of a product of abelian metrizable groups.

**Lemma 2.3.9.** Let $G$ be an abelian topological group such that $G$ is a subgroup of a product $\prod_{i \in I} H_i$, for some index set $I$, where each $H_i$ is a Hausdorff topological group which is not necessarily abelian. Then $G$ is a subgroup of a product $\prod_{i \in I} A_i$ where for each $i \in I$, $A_i$ is a closed abelian subgroup of $H_i$. 
Proof. For each $i \in I$, consider the projection homomorphism $p_i : G \to H_i$. Then $p_i(G)$ is an abelian subgroup of $H_i$ and hence $A_i = \overline{p_i(G)}$ is a closed abelian subgroup of $H_i$ ([46], Proposition 5, page 8). Therefore, $G$ is a subgroup of $\prod_{i \in I} A_i$. \hfill \blacksquare

We now have the following interesting result.

**Theorem 2.3.10.** A Hausdorff abelian topological group can be embedded in a Hausdorff locally convex topological vector space if and only if there is a family of pseudometrics, each with the Enflo property, that define the topology on $G$.

Proof. Let $G$ be a Hausdorff abelian topological group with the topology defined by the family $\{p_i : i \in I\}$ of pseudometrics, each with the Enflo property. We first note that by Proposition 2.3.7 and Lemma 2.3.9, $G$ can be embedded in a product of metrizable abelian topological groups, $\prod_{j \in J} H_j$. Now, we also note that each $H_j$ is algebraically a quotient group of $G$, say $|G/A_j|$ where $A_j$ is a normal subgroup of $G$, and the metric on $H_j$ is defined by $d_j(aA_j, bA_j) = p_j(a, b)$ for some $p_j \in \{p_i : i \in I\}$. (See Remark 2.3.8.) Now, $d_j(aA_j, aA_j) = p_j(a^2, e) = 2 \rho_j(a, e)$ and $d_j(aA_j, A_j) = \rho_j(a, e)$. This gives $d_j(aA_j, A_j) = 2d_j(aA_j, A_j)$, and hence $d_j$ has the Enflo property. Therefore, each $H_j$, $j \in J$, is topologically isomorphic to a subgroup of a Banach space, by Theorem 2.3.6. Thus, $G$ is a subgroup of a product of Banach spaces, and hence $G$ is a subgroup of a Hausdorff locally convex topological vector space.

Conversely, let $G$ be a subgroup of a Hausdorff locally convex topological vector space, $L$. Then $L$ is defined by a family of seminorms, $\{q_i : i \in I\}$ (see [58], Chapter 1, §4) and hence the topology on $L$ is defined by the family of invariant pseudometrics $\{\rho_i : \rho_i(x, y) = q_i(x - y), x, y \in G\}$. Note that $\rho_i(2x, e) = q_i(2x) = 2q_i(x) = 2\rho_i(x, e)$ and so each pseudometric has the Enflo property. Thus the topology on $G$ is defined by a family of pseudometrics, each with the Enflo property. \hfill \blacksquare

The most useful example of a pseudometrizable topological group where the pseudometric has the Enflo property is found in the world of free abelian topological groups. The
following proposition proves that the Graev extension of a pseudometric (or metric) has the Enflo property.

**Proposition 2.3.11.** (cf. [50], Proposition 5) Let $X$ be a completely regular space which has a continuous pseudometric $p$ and let $p'$ be the Graev extension of $p$ on $F$, the free abelian group on the set $X \setminus \{e\}$. Then for each $a \in F$, $p'(a^2, e) = 2p'(a, e)$.

**Proof.** If $a = e$, the result is trivial, so assume $a \neq e$. Note that

$$\rho'(a^2, e) \leq \rho'(a^2, a) + \rho'(a, e)$$

$$= \rho'(a, e) + \rho'(a, e)$$

$$= 2\rho'(a, e).$$

Consider $a \in F$ with the reduced representation of $a^2$ given by

$$a^2 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}, \quad x_i \in X \setminus \{e\}, \quad \varepsilon_i \in \mathbb{Z} \setminus \{0\},$$

where $x_i \neq x_j$ for $i \neq j$. First, we shall prove that each $\varepsilon_i$ is an even integer.

Let $j \in \{1, \ldots, n\}$ such that $x_j^{\varepsilon_j}$ appears in the reduced form of $a^2$. Then there is a mapping $\psi$ from the space $X$ to the cyclic group of order 2, $\mathbb{Z}_2 = \{1, -1\}$, given by $\psi(x_j) = -1$ and $\forall x \in X \setminus \{x_j\}$, $\psi(x) = 1$. Now, $\psi$ extends to a homomorphism $\Psi$ from $F$ to $\mathbb{Z}_2$ such that $\Psi(a^2) = [\Psi(a)]^2 = 1$, regardless of the value of $\Psi(a)$. Now,

$$\Psi(a^2) = \Psi(x_1^{\varepsilon_1} \ldots x_j^{\varepsilon_j} \ldots x_n^{\varepsilon_n})$$

$$= [\psi(x_1)]^{\varepsilon_1} \ldots [\psi(x_j)]^{\varepsilon_j} \ldots [\psi(x_n)]^{\varepsilon_n}$$

$$= (-1)^{\varepsilon_j} = 1.$$ 

Therefore, $\varepsilon_j$ must be an even integer, for each $j = 1, \ldots, n$.

Now, let the value of $\rho'(a^2, e)$ be given by the representations (of equal length) $W_1$ and $W_2$ for $a^2$ and $e$ respectively. Write $W_2$ under $W_1$ and specify the cancellation order for $W_1$; that is, for each $i = 1, \ldots, n$, choose $|\varepsilon_i|$ occurrences of $x_i^{\alpha_i}$, $\alpha_i = \frac{\varepsilon_i}{|\varepsilon_i|}$, in $W_1$ that will appear in the reduced form of $W_1$. Now, using Graev's algorithm ([16], pp 313–314), consider the element $x_i^{\alpha_i}$ that appears in the reduced representation of $W_1$. The element of $W_2$ that stands directly below $x_i^{\alpha_i}$ shall be called $u_1$. The element of $W_1$ which stands
directly above $u_1^{-1}$ shall be called $u_2$. Continue to write down a column consisting of one element from $W_1$ and one element from $W_2$. We stop when we encounter either $e$ or one of the $x_j^{\alpha_j}$ that appears in the reduced representation of $W_1$. We call the resulting array a line arrangement. If we start with $x_i^{\alpha_i}$, there are three possibilities.

1. \[
\begin{pmatrix}
x_i^{\alpha_i} & u_2 & u_2^{-1} & \ldots & u_2^{-1} & x_j^{\alpha_j} \\
u_1 & u_1^{-1} & u_3 & \ldots & u_2n+1 & u_2n+1^{-1}
\end{pmatrix}
\]

where $\alpha_j = \frac{\varepsilon_j}{|\varepsilon_j|}$ and $x_j^{\alpha_j}$ is in the reduced representation of $a^2$

2. \[
\begin{pmatrix}
x_i^{\alpha_i} & u_2 & u_3 & \ldots & u_2n \\
u_1 & u_1^{-1} & u_2 & \ldots & e
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
x_i^{\alpha_i} & u_2 & u_3 & \ldots & e \\
u_1 & u_1^{-1} & u_2 & \ldots & u_2n+1
\end{pmatrix}
\]

Note that $u_i \in X \cup X^{-1}$ for every $i$.

For case 1,

$$\rho(x_i^{\alpha_i}, u_1) + \rho(u_1^{-1}, u_2) + \rho(u_2^{-1}, u_3) + \ldots + \rho(u_2n+1, x_j^{\alpha_j}) \geq \rho(x_i^{\alpha_i}, x_j^{-\alpha_j})$$

for which we consider the following two possibilities.

Case A. $\alpha_i = \alpha_j$. Then $\rho(x_i^{\alpha_i}, x_j^{\alpha_j}) = \rho(x_i^{\alpha_i}, e) + \rho(x_j^{\alpha_j}, e)$. Therefore, if in $W_1$ and $W_2$ we replace (using commutativity) the line arrangement of case 1 by

$$1'. \begin{pmatrix} x_i^{\alpha_i} & x_j^{\alpha_j} \\ e & e \end{pmatrix}$$

the sum of distances for $\rho'(a^2, e)$ remains unchanged and we still have representations for $a^2$ and $e$. For $x_i^{\alpha_i}$ and $x_j^{\alpha_j}$ we now have two separate line arrangements, namely

4. \[
\begin{pmatrix}
x_i^{\alpha_i} \\
e
\end{pmatrix}
\]

5. \[
\begin{pmatrix}
x_j^{\alpha_j} \\
e
\end{pmatrix}
\]

Case B. $\alpha_i = -\alpha_j$. Then $\rho(x_i^{\alpha_i}, x_j^{-\alpha_j}) = \rho(x_i^{\alpha_i}, x_i^{\alpha_i}) + \rho(x_j^{\alpha_j}, x_i^{-\alpha_i})$. Note that in this case $x_i \neq x_j$. If in $W_1$ and $W_2$ we replace the line arrangement of case 1 by

$$6. \begin{pmatrix} x_i^{\alpha_i} & x_j^{\alpha_j} \\ x_i^{\alpha_i} & x_i^{-\alpha_i} \end{pmatrix}$$
the sum of distances for $\rho'(a^2, e)$ remains unchanged and we still have representations for $a^2$ and $e$.

Similarly, for cases 2 and 3 it can be shown that we still have representations for $a^2$ and $e$, and the sum of distances for $\rho'(a^2, e)$ remains unchanged if we replace the line arrangement by

$$7. \left\{ \begin{array}{l} x_i^\alpha_i \\ e \end{array} \right.$$

We denote the new representations of $a^2$ and $e$ by $W'_1$ and $W'_2$ respectively. We can now use the above method to obtain all disjoint line arrangements from the representations $W'_1$ and $W'_2$. Each arrangement is of the form $\left\{ \begin{array}{l} x_i^\alpha_i \\ e \end{array} \right.$ (we call this a “single”) or $\left\{ \begin{array}{l} x_i^\alpha_i \\ x_j^\alpha_j \\ x_i^{-\alpha_i} \\ x_j^{-\alpha_j} \end{array} \right.$ (we call this a “double”) where $x_i^\alpha_i$ and $x_j^\alpha_j$ appear in the reduced representation of $a^2$.

We now separate the line arrangements into two disjoint sets, $A$ and $B$, so each set will give us representations for $a$ and $e$. First, consider all pairs of line arrangements that are identical and place one line arrangement from each pair in $A$ and the other in $B$. Before we deal with the remaining line arrangements, we note the following.

1. For each $x_i^\alpha_i$, there remains at most one single line arrangement of the form $\left\{ \begin{array}{l} x_i^\alpha_i \\ e \end{array} \right.$.
2. There remain an even number of line arrangements containing $x_i^\alpha_i$ in the top row for each $i = 1, \ldots, n$.
3. If $x_i^\alpha_i$ appears with $x_j^\alpha_j$ in the top line of a double line arrangement, and if $x_j^\alpha_j$ appears with $x_k^\alpha_k$ in the top line of another double line arrangement, then $x_i^\alpha_i$ does not appear with $x_k^\alpha_k$ in the top line of any double line arrangement.

From the remaining line arrangements, we choose a double line arrangement with $x_i^\alpha_i$ and $x_j^\alpha_j$ on the top row. Place this line arrangement in the set $A$. Now, $x_j^\alpha_j$ appears in at least one other remaining line arrangement, which we place in the set $B$. If this line arrangement is a double with $x_j^\alpha_j$ and $x_k^\alpha_k$ on the top row, choose a line arrangement containing $x_k^\alpha_k$ and place it in the set $A$. If this line arrangement is a double with $x_k^\alpha_k$ and $x_h^\alpha_h$ on the top row, choose a line arrangement containing $x_h^\alpha_h$ and place it in the set $B$. Continue in this way until the next line arrangement we choose is either a double line arrangement to be placed in set $B$ with $x_i^\alpha_i$ in the top row or a single line arrangement.

In the former case, we stop. In the latter case, we must choose another line arrangement.
containing \( x_i^{\alpha_i} \) to be placed in set \( B \). If this line arrangement is a single line arrangement, we stop. Otherwise, we have a double line arrangement with \( x_i^{\alpha_i} \) and \( x_j^{\alpha_j} \) on the top row, so we choose a further line arrangement containing \( x_i^{\alpha_i} \) to be placed in \( A \). Continue as described before until we choose a single line arrangement, at which point we stop. Repeat the process until all line arrangements have been allocated to either \( A \) or \( B \).

If we write the line arrangements from \( A \) next to each other, we have a representation for \( a \) written over a representation for \( e \). The same occurs for the line arrangements contained in \( B \). Clearly, one of these sets of representations for \( a \) and \( e \) will give \( \rho'(a, e) \leq \frac{1}{2} \rho'(a^2, e) \).

Therefore, \( \rho'(a^2, e) = 2 \rho'(a, e) \). ■

We now use Proposition 2.3.11 to form a link between free abelian topological groups and Banach spaces, that foreshadows the method of proving our main result in Section 2.4.

Corollary 2.3.12. Let \((X, d)\) be a metric space and let \( F \) be the free abelian group on the set \( X \setminus \{e\} \), for \( e \in X \). Further, let \( d' \) be the Graev extension of \( d \) to \( F \). Then \((F, d')\) is a topological group which is topologically isomorphic to a subgroup of a Banach space.

Proof. The result follows immediately from Proposition 2.3.11 and Theorem 2.3.6. ■

§2.4 The Variety Generated by Banach Spaces

We now turn our attention to the variety of topological groups generated by the class of all topological groups underlying Banach spaces. The key to unlocking this variety is to consider the free abelian topological group generated by a completely regular Hausdorff space. Recall that a completely regular space \( X \) has its topology determined by a family of continuous pseudometrics (see Remark 2.1.7), and by Proposition 2.3.11 the Graev extension of each of these pseudometrics has the Enflo property. Thus, we have the following result.
Theorem 2.4.1. Let $X$ be a completely regular Hausdorff topological space. Then $\text{FA}(X)$, the free abelian topological group on $X$, can be embedded in a Hausdorff locally convex topological vector space.

Proof. By Remark 2.1.7, the topology on $X$ is defined by the family $\{\rho_i : i \in I\}$ of continuous pseudometrics. Proposition 2.2.5 tells us that the topology on $\text{FA}(X)$, the free abelian topological group on $X$, is defined by the family $\{\rho'_i : i \in I\}$ of invariant pseudometrics where each $\rho'_i$ is the Graev extension of $\rho_i$. Further, by Proposition 2.3.11, each $\rho'_i$ has the Enflo property, thus we can apply Theorem 2.3.10 to get the result. 

We need one final tool before we present the main theorem of this section concerning the variety generated by the class of all Banach spaces.

Proposition 2.4.2. Let $G$ be a topological group and let $\overline{\{e\}}$ be the closure of $\{e\}$ in $G$. Then $G$ is topologically isomorphic to a subgroup of $G/\overline{\{e\}} \times |G|_I$.

Proof. We first note that $\overline{\{e\}}$ is a closed normal subgroup of $G$ ([46], Corollary to Proposition 5). Let $f : G \to G/\overline{\{e\}}$ be the quotient homomorphism onto $G/\overline{\{e\}}$ and let $h : G \to G/\overline{\{e\}} \times |G|_I$ be the mapping defined by $h(g) = (f(g), g)$ for each $g \in G$. We must show that $h$ is a continuous one-to-one homomorphism, which is open from $G$ to $h(G)$. Clearly, $h$ is a homomorphism. Further, if $h(g_1) = h(g_2)$, then $(f(g_1), g_1) = (f(g_2), g_2)$ and hence $g_1 = g_2$, showing that $h$ is one-to-one. Let $O$ be an open set in $G/\overline{\{e\}} \times |G|_I$. Then $O = O_1 \times O_2$ where $O_1$ is open in $G/\overline{\{e\}}$ and $O_2$ is open in $|G|_I$. Now, if $O_2 = \emptyset$ then $h^{-1}(U) = \emptyset$, which is open in $G$. On the other hand, if $O_2 = |G|$ then $h^{-1}(U) = f^{-1}(O_1)$, which is also open in $G$ and hence $h$ is continuous. Finally, let $O$ be an open set in $G$. Then $h(O) = (f(O) \times O) \cap h(G) = (f(O) \times |G|) \cap h(G)$, which is open in $h(G)$. Therefore, $h$ is an embedding and the result follows.

Notation. We shall denote by $\mathcal{B}$ the class of all topological groups which underlie Banach Spaces.
Banach spaces are a very rich and special class of topological groups. It is interesting and somewhat surprising that the variety of topological groups generated by them is the class of all abelian topological groups. We are now able to establish this result.

**Theorem 2.4.3.** The variety of topological groups generated by the class of all topological groups that underlie Banach spaces is exactly the variety of all abelian topological groups.

**Proof.** Let \( \mathcal{A} \) be the variety of all abelian topological groups. Clearly, \( \mathcal{V}(\mathcal{B}) \subseteq \mathcal{A} \).

Let \( G \) be a Hausdorff abelian topological group. Then \( G \) is completely regular Hausdorff, and by Theorem 2.4.1, \( FA(G) \), the free abelian topological group on \( G \) (as a topological space), is topologically isomorphic to a subgroup of a Hausdorff locally convex topological vector space, \( L \). Now, \( L \) is topologically isomorphic to a subgroup of a product of Banach spaces (see [58], Corollary to Proposition 19, Chapter V) and hence \( L \in \mathcal{V}(\mathcal{B}) \). Therefore, \( FA(G) \in \mathcal{V}(\mathcal{B}) \). Now, by Proposition 2.2.8, \( G \) is a quotient group of \( FA(G) \), giving \( G \in \mathcal{V}(\mathcal{B}) \). Thus, \( \mathcal{V}(\mathcal{B}) \) contains all Hausdorff abelian topological groups.

We note that \( \mathcal{V}(\mathcal{B}) \) contains the topological groups \( \mathbb{R} \) and \( \mathbb{Z} \) and so the circle group \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) is in \( \mathcal{V}(\mathcal{B}) \). But it is well-known that every abelian group is algebraically isomorphic to a subgroup of a product of copies of the divisible group \( \mathbb{T} \) ([46], Corollary to Proposition 17 and proof to Theorem 1). Hence, every abelian group appears in \( \mathcal{V}(\mathcal{B}) \) with some topology. Therefore, by Proposition 1.5.4, \( \mathcal{V}(\mathcal{B}) \) contains all abelian groups with the indiscrete topology.

Finally, by Proposition 2.4.2, for any abelian topological group \( H \), \( H \) is a subgroup of the product of \( H/\{e\} \times |H| \) where \( |H| \) is the group \( |H| \) with the indiscrete topology.

Clearly, \( H/\{e\} \) is a Hausdorff abelian topological group and is thus contained in \( \mathcal{V}(\mathcal{B}) \). Also, \( |H| \in \mathcal{V}(\mathcal{B}) \). Therefore, \( H \in \mathcal{V}(\mathcal{B}) \).

Therefore, \( \mathcal{V}(\mathcal{B}) \) contains all abelian topological groups, hence \( \mathcal{V}(\mathcal{B}) = \mathcal{A} \).

It should be made clear that the variety of topological groups generated by \( \mathcal{B} \) is **not** the same as the variety of topological vector spaces generated by \( \mathcal{B} \). The latter is the smaller of the two as \( Q \) and \( S \) are more restrictive in the world of topological vector spaces.
The beauty of Theorem 2.4.3 lies in the fact that we now have a complete picture of $\mathfrak{D}(B)$, the variety generated by the class of all Banach spaces. Indeed, we see that the variety generated by the class of all locally compact abelian topological groups is contained in $\mathfrak{D}(B)$, as is $\mathfrak{D}(FA[0,1])$, the variety generated by $FA[0,1]$, the free abelian topological group on $[0,1]$. For the moment we shall leave the comparison of our three varieties as the analysis of the latter two varieties will be performed in Chapters 3 and 4.

Theorem 2.4.3 was first published in [49] by the author of this thesis and her supervisor, however, the proof in that paper is different to the one presented in this thesis. The proof in the paper involves free locally convex topological vector spaces.

**Definition 2.4.4. ([13],[33])** Let $X$ be a completely regular space with distinguished point $e$. The real locally convex topological vector space $FLCV(X)$ is said to be a free locally convex topological vector space on the space $X$ if it has the following properties:

1. $X$ is a subspace of $FLCV(X)$;
2. $X$ is a (vector space) basis for $FLCV(X)$;
3. for any continuous mapping $\phi$ of $X$ into any locally convex topological vector space $V$ such that $\phi(e) = 0$, there exists a continuous linear transformation $\Phi$ of $FLCV(X)$ into $V$ such that $\Phi(x) = \phi(x)$ on $X$.

We note that the free locally convex topological vector space on the space $X$, $FLCV(X)$, always exists and is unique. Further, if $X$ is Hausdorff, then $FLCV(X)$, where $X$ is completely regular, is Hausdorff. (See [33] and [13]).

The key to the alternative proof of Theorem 2.4.3 as presented in [49], referred to the following well known, but not obvious result. Tkachenko, in [66], first introduced the result, however, Uspenskiĭ [68] found Tkachenko's proof was incomplete and corrected it. The proof was quite technical, considering seminorms and pseudometrics. We present the theorem with an easy proof using the result by Enflo, Theorem 2.3.6. In this way we see that in fact, the proof to Theorem 2.4.3 presented in [49] is just a modification of the one presented in this thesis.
Theorem 2.4.5. ([66], Theorem 3; [68]; [33], Theorem 2.3) Let $X$ be a completely regular space and let $FLCV(X)$ be the free locally convex topological vector space on $X$. Then the subgroup of $FLCV(X)$ that is algebraically generated by $X$ is (with the induced topology) topologically isomorphic to the free abelian topological group on $X$.

Proof. Firstly, by Theorem 2.4.1, $FA(X)$, the free abelian topological group on $X$, is a subgroup of a Hausdorff locally convex topological vector space, $L$. Now, consider the identity map $\phi: X \to X$, from $X$ to $X$ as a subspace of $FA(X) \subseteq L$. There exists a continuous linear transformation $\Phi$ of $FLCV(X)$ into $L$ such that $\Phi(x) = x$ for all $x \in X$. Let $F = gp(X \setminus \{e\}) = gp(X)$ be the abelian subgroup of $FLCV(X)$ algebraically generated by $X \setminus \{e\}$. Now, obviously $\Phi|_F$ is an algebraic isomorphism of $F$ onto $|FA(X)|$.

Further, as $\Phi$ is continuous, $\Phi|_F$ is a continuous homomorphism; that is, $|FA(X)|$ is a continuous one-to-one homomorphic image of $F$. Therefore, $F$ is a topological group algebraically isomorphic to $FA(X)$ that induces the given topology on $X$ and has a finer topology than $FA(X)$. However, $FA(X)$ has the finest group topology that induces the given topology on $X$, and so $F$ is exactly $FA(X)$ and the result follows.

Theorem 2.4.5, as it appears in [66], further states that for a completely regular Hausdorff space $X$, $FA(X)$ appears as a closed subgroup of $FLCV(X)$. We shall use a compactification of $X$ (see [29], Chapter 5) to prove this and that $X$ also appears as a closed subspace of $FLCV(X)$. The latter result is similar to a result by Corson (see [19], 8.21) which states that if $X$ is a completely regular space, then there is a complex topological linear space $L$ such that $X$ is a closed subset of $L$. Our result improves this by saying that $X$ sits in real topological vector space as a closed subset.

Corollary 2.4.6. ([66], Theorem 3; cf. [19], 8.21) Let $X$ be a completely regular Hausdorff topological space. Then $X$ is a closed subspace and $FA(X)$, the free abelian topological group on $X$, is a closed subgroup of $FLCV(X)$.

Proof. Firstly, we note that there exists a compact Hausdorff topological space $\beta X$ such that $X$ is a dense subspace of $\beta X$ ([29], Chapter 5, in particular Theorem 24). Then,
we have $X$ a subspace of $FLCV(\beta X)$ and $F = \text{gp}(X)$, the group generated by $X$, a subset of the same. Consider $\overline{F}$ in $FLCV(\beta X)$. As $X$ is dense in $\beta X$, $\beta X \subseteq \overline{F}$ and so $\overline{F} = \text{gp}(\beta X)$. Now, by Theorem 2.4.5, $\text{gp}(\beta X) = FA(\beta X)$ in $FLCV(\beta X)$ giving $\overline{F} = FA(\beta X)$. Proposition 2.2.12 tells us that $FA(\beta X)$ is a $k_\omega$-group. A $k_\omega$-group is Weil complete ([22], Theorem 2) and is therefore closed in any Hausdorff topological group of which it is a subgroup (cf. [6], Chapter II, §3, No. 4, Proposition 8). Thus, $\overline{F} = FA(\beta X)$. Now, let $V(X) \subseteq FLCV(\beta X)$ be the vector space generated by $X$. Then $\overline{F} \cap V(X) = FA(\beta X) \cap V(X) = F$. Therefore, $F$ is closed in $V(X)$, the vector space generated by $X$, as a subspace of $FLCV(\beta X)$.

Let $\phi : X \to \beta X$ be the identity map from $X$ to $\beta X$. As $FLCV(X)$ is a free locally convex topological vector space, there exists a continuous linear transformation $\Phi$ from $FLCV(X)$ into $FLCV(\beta X)$ which extends $\phi$. Clearly, $\Phi^{-1}(\beta X) = X$ and as $\beta X$ is closed in $FLCV(\beta X)$, $X$ is closed in $FLCV(X)$. Note that $\Phi$ is a one-to-one mapping, and also that $\Phi^{-1}(V(X)) = FLCV(X)$. Therefore, $\Phi$ is a continuous mapping onto $V(X)$. Now, $F$ is closed in $V(X)$ and so $\Phi^{-1}(F)$ is closed in $FLCV(X)$. Further, $\Phi^{-1}(F) = FA(X)$ and the result follows.

**Remark 2.4.7.** Markov [36] introduced the topic of free topological groups to study the problem of whether there exist non-normal Hausdorff topological groups. Markov solved the problem in the affirmative, but we can see it also follows easily from Corollary 2.4.6. Recall that a topological space $X$ is said to be normal if for any two disjoint closed sets $C_1$ and $C_2$, there exist open sets $U_1$ and $U_2$ such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$. Further, we note that normality is a condition preserved under closed subspaces. Now, by Corollary 2.4.6, if $X$ is a completely regular Hausdorff topological space, it is a closed subspace of $FA(X)$, the free abelian topological group on $X$. Therefore, if we let $X$ be a completely regular Hausdorff non-normal topological space (for example, the Deleted Tychonoff Plank, [62] Example 87) then $FA(X)$ is a non-normal Hausdorff topological group, for if it were normal, $X$ would be normal. In a similar fashion, we see that $FLCV(X)$ is a non-normal Hausdorff locally convex topological vector space.
§2.5 Further Results

We recall the concept of a topological group which is uniformly free from small subgroups, introduced by Enflo [11], and present a number of useful results concerning these topological groups.

**Definition 2.5.1.** ([11], Section 2.1) A Hausdorff topological group $G$ is said to be uniformly free from small subgroups if it has a neighbourhood of the identity, $U$, such that for every neighbourhood of the identity, $V$, there exists a positive integer $n_V$ with the property that $x \notin V \implies x^n \notin U$ for some $n \leq n_V$.

**Example 2.5.2.** We note that all discrete groups are uniformly free from small subgroups as $\{e\}$ is a neighbourhood that satisfies the property in Definition 2.5.1. Further, Morris and Pestov observed that Banach spaces and Lie groups are also uniformly free from small subgroups ([52], Theorem 2.7).

Morris and Pestov in [52] reformulated the concept of a group uniformly free from small subgroups in the following, more convenient form.

**Proposition 2.5.3.** ([52], Proposition 2.3) A Hausdorff topological group $G$ is uniformly free from small subgroups if and only if for some neighbourhood of the identity, $U$, the sets

$$(1/n)U = \{x \in G : \forall k = 1, 2, \ldots, n, x^k \in U\}$$

form a neighbourhood basis at the identity.

The following corollaries follow routinely from Proposition 2.5.3.

**Corollary 2.5.4.** A subgroup of a topological group uniformly free from small subgroups is also uniformly free from small subgroups.
Corollary 2.5.5. [52] A topological group uniformly free from small subgroups is an NSS-group.

Proof. Let $G$ be a topological group that is uniformly free from small subgroups and let $U$ be a neighbourhood of the identity such that $(1/n)U$ form a neighbourhood basis at the identity. Let $H$ be a subgroup contained in $U$ and $h \in H$, $h \neq e$. Then $h^n \in U$ for all $n \in \mathbb{N}$. Therefore, $H \subseteq (1/n)U$ for all $n \in \mathbb{N}$. Thus, each open neighbourhood $V$ of the identity contains $H$. However, $G$ is Hausdorff. Therefore, $H = \{e\}$ and so $G$ is an NSS-group. 

We now present a clever result that further characterizes the free abelian topological group on a completely regular space that admits a metric. The following proposition is, in effect, a corollary to Proposition 2.3.11 and Corollary 2.3.12.

Remark 2.5.6. First, recall that a topological space $X$ admits a continuous metric if there exists a continuous metric that defines a topology on $X$ which is coarser than the given topology. Further, if a completely regular Hausdorff space $Y$ whose topology is determined by a family of pseudometrics $\{\rho_i : i \in I\}$ also admits a continuous metric $d$, then the topology of $Y$ is determined by the family of continuous metrics

$$\{d_i : d_i = \rho_i + d, i \in I\}. $$

Proposition 2.5.7. Let $X$ be a completely regular space that admits a continuous metric. Then the Graev free abelian topological group on $X$ has no small subgroups.

Proof. Let $X$ be defined by the family of all continuous metrics $\{d_i : i \in I\}$. If one of the metrics $d_i$ is extended to a metric $d'_i$ on the free abelian group $F$ on $X\setminus\{e\}$ using Graev’s method, then, by Corollary 2.3.12, the topological group $(F, d'_i)$ is topologically isomorphic to a subgroup of a Banach space. Therefore, $(F, d'_i)$ is a subgroup of a topological group uniformly free from small subgroups and so, by Corollary 2.5.4, $(F, d'_i)$ is also uniformly free from small subgroups. Hence, by Corollary 2.5.5, $(F, d'_i)$ has no small subgroups. Since
the free abelian topological group on $X$, $FA(X)$ is the group $F$ with a finer topology than that induced by $d_i$; $FA(X)$ has no small subgroups. ■

Before presenting our next interesting result, we state the following lemma, again from the work by Morris and Pestov.

**Proposition 2.5.8.** ([52], Corollary 3.5) If $\Omega$ is any class of topological groups and $G$ a topological group uniformly free from small subgroups in $SC(\Omega)$, then $G \in SP(\Omega)$. ■

Note that from Proposition 2.5.8, we can deduce that if a topological group $G$ is uniformly free from small subgroups and contained in $\mathcal{Y}(\Omega)$ for some class $\Omega$ of topological groups, then $G \in QSP(\Omega)$.

We know from Theorem 2.4.3 that every abelian topological group is a quotient of a subgroup of a product of Banach spaces. It is of interest to refine this further and see that any abelian topological group that is uniformly free from small subgroups is, in fact, a quotient of a subgroup of a Banach space.

**Corollary 2.5.9.** If an abelian topological group $G$ is uniformly free from small subgroups then it is topologically isomorphic to a Hausdorff quotient group of a subgroup of a Banach space.

**Proof.** Firstly, Theorem 2.4.3 implies that $G \in QSC(B)$. By Proposition 2.5.8, $G$ is topologically isomorphic to a Hausdorff quotient group of a subgroup of a finite product of Banach spaces. Since a finite product of Banach spaces is a Banach space, the result follows. ■

In particular, as all Lie groups are uniformly free from small subgroups, they are contained in $QS(B)$. As in the abelian case the operators $Q$ and $S$ commute (Remark 1.4.2(d)), all Lie groups are contained in $SQ(B)$. However, Lie groups are complete and so they are all
We note that from Proposition 2.5.3, any topological group, \( G \), that is uniformly free from small subgroups has a countable basis at the identity. Therefore, by Theorem 8.3 of [19], \( G \) is metrizable. This leads us to ask the natural question of whether the class \( \bar{Q}S(B) \) is the class of all metrizable topological groups. We shall answer this shortly, but we need the following result to do so.

**Lemma 2.5.10.** Let \( G \) be a metrizable abelian topological group with invariant metric \( d \). Then \( G \) is a quotient group of \( (F,d') \) where \( F \) is free group on \( G \setminus \{e\} \) and \( d' \) is the Graev extension of the metric \( d \) onto \( F \).

**Proof.** To avoid confusion, we will use \( xy \) to denote the multiplication of \( x, y \in G \) and \( u \cdot v \) to denote the multiplication of \( u, v \in F \). Also, we will denote by \( x^{-1} \) the inverse of \( x \) in \( G \) and by \( \bar{w} \) the inverse of \( w \) in \( F \).

Let \( f : (F,d') \to G \) be the mapping defined as follows. For each \( x \in G \), \( f(x) = x \) and \( f(x) = x^{-1} \). Then, for a word in \( F \), \( w = u_1 \cdot u_2 \cdot \ldots \cdot u_n \), \( f(w) = f(u_1)f(u_2)\ldots f(u_n) \). Clearly, \( f \) is an onto homomorphism and \( f|_G \) is the identity mapping, which is open. Therefore, by Lemma 2.2.7, \( f \) is an open homomorphism from \( (F,d') \) onto \( G \). To complete the proof, we must show that \( f \) is continuous and for this, we need prove only that \( f \) satisfies the property \( d'(w_1,w_2) \geq d(f(w_1), f(w_2)) \) for \( w_1, w_2 \in F \). Before we proceed, we note that as \( d \) is invariant, for \( z_1, \ldots, z_n \in G \),

\[ d(z_1,e) + d(z_2,e) + \ldots + d(z_n,e) \geq d(z_1z_2 \ldots z_n,e). \]

Now, we have for all \( x, y \in G \),

\[ d'(x,y) = d(x,y) = d(f(x),f(y)); \]
\[ d'((x,y) = d(x,y) = d(x^{-1},y^{-1}) = d(f(x),f(y)); \]
\[ d'(\bar{x},\bar{y}) = d(x,e) + d(y,e) \geq d(x^{-1},y) = d(f(x),f(y)). \]

Therefore, for all \( a, b \in G \cup \bar{G} \), \( d'(a,b) \geq d(f(a), f(b)) \). Now, let \( w \in F \) be a word with \( w = a_1a_2 \ldots a_n \) and \( e = b_1b_2 \ldots b_n \), \( a_i, b_i \in G \cup \bar{G} \) for \( i = 1, \ldots, n \), the representations of
2. Banach Spaces

w and e that achieve \( d'(w, e) \). Thus, we have

\[
d'(w, e) = \sum_{i=1}^{n} d'(a_i, b_i) \\
\geq \sum_{i=1}^{n} d(f(a_i), f(b_i)) \\
= \sum_{i=1}^{n} d(f(a_i b_i), e) \\
\geq d(f(a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n), e) \\
= d(f(w), e).
\]

Finally, for two words \( w_1, w_2 \in FA(G) \),

\[
d'(w_1, w_2) = d'(w_1 \overline{w_2}, e) \\
\geq d(f(w_1) f(\overline{w_2}), e) \\
= d(f(w_1) [f(w_2)]^{-1}, e) \\
= d(f(w_1), f(w_2)).
\]

Therefore, \( f \) is a quotient mapping from \((FA(G), d')\) onto \( G \).

As hinted earlier, the next theorem tells us the interesting fact that \( \overline{QS}(B) \) is indeed the class of all metrizable abelian topological groups.

**Theorem 2.5.11.** The class \( \overline{QS}(B) \) is the class of all metrizable abelian topological groups.

**Proof.** Firstly, let \( \mathcal{M} \) denote the class of all metrizable abelian topological groups. We note that every subgroup of a Banach space is a metrizable topological group. Further, a Hausdorff quotient group of a metrizable topological group is also metrizable ([24], Part II, Section 25). Therefore, \( \overline{QS}(B) \subseteq \mathcal{M} \).

Let \( G \in \mathcal{M} \), with \( d \) the associated metric on \( X \). Let \( F \) be the free abelian group generated by \( G \setminus \{e\} \) and let \( d' \) be Graev's extension of \( d \) to \( F \) as described earlier. By Corollary 2.3.12, \((F, d')\) is isomorphic to a subgroup of a Banach space. Further, using Lemma 2.5.10, \( G \) is a quotient group of \((F, d')\). As \( G \) is Hausdorff, \( G \in \overline{QS}(B) \) and the result follows.
Noting that when dealing with abelian topological groups, $Q$ and $S$ commute, we have the following consequence of Theorem 2.5.11.

**Corollary 2.5.12.** The class $\overline{Q \ S}(B)$ is the class of all complete metrizable abelian topological groups. □

As the final result in this section, we determine the class $QS(B)$. We first need the next few results, which do not require the "abelian" restriction used throughout the chapter, and so we prove the general result.

**Lemma 2.5.13.** Every indiscrete topological group is a quotient of a metrizable topological group.

*Proof.* Let $G_I$ be a group $G$ with the indiscrete topology, and for $i \in \mathbb{N}$, let $G_i$ be $G$ with the discrete space. We know that each $G_i$ is metrizable and we also know that $H = \prod_{i=1}^{\infty} G_i$ is a metrizable topological group. Now, let $K = \prod_{i=1}^{\infty} G_i$ be the restricted direct product, then by Lemmas 1.5.2 and 1.5.3, $H/K$ is an indiscrete group. Further, as we saw in the proof to Proposition 1.5.4, $G_I$ can be embedded as a topological group in $H/K$ and so $G_I \in SQ(M)$ where $M$ is the class of all metrizable topological groups. Now, by Remark 1.4.2(c), $G_I \in QS(M)$. However, a subgroup of a metrizable group is also metrizable. Therefore, $G_I \in Q(M)$. □

In Proposition 2.4.2, we saw that a topological group $G$ is topologically isomorphic to a subgroup of $G / \{e\} \times |G|_I$ where $\{e\}$ is the closure of $\{e\}$ in $G$ and $|G|_I$ is the group $G$ with the indiscrete topology. In the case that $G$ is a pseudometrizable topological group, we shall see shortly that $G / \{e\}$ is a metrizable topological group. In Chapter 4, we shall generalize this by introducing a Metrification Mechanism for pseudometrizable topological spaces, showing such a space is a subspace of a product of a metrizable space and an indiscrete space.
Lemma 2.5.14. Let $G$ be a group topologised by an invariant pseudometric. Then $G / \{e\}$ is metrizable by a two-sided invariant metric.

Proof. Let $\rho$ be the invariant pseudometric on $G$. We note that $\{e\} = \{x : \rho(x, e) = 0\}$, as this is the smallest closed set containing $e$. Further, $x\{e\} = \{y : \rho(x, y) = 0\}$ for all $x \in G$. Now, let $d$ be a metric on $G / \{e\}$ defined by

$$d(x\{e\}, y\{e\}) = \inf \{\rho(a, b) : a \in x\{e\}, b \in y\{e\}\},$$

where $x, y \in G$. Note that for $a \in x\{e\}$ and $b \in y\{e\}$,

$$\rho(a, b) \leq \rho(a, x) + \rho(x, y) + \rho(y, b) = \rho(x, y)$$

and

$$\rho(x, y) \leq \rho(x, a) + \rho(a, b) + \rho(b, y) = \rho(a, b)$$

giving $\rho(a, b) = \rho(x, y)$. Therefore, $d(x\{e\}, y\{e\}) = \rho(x, y)$. Now, let $f : G \to G / \{e\}$ be the quotient homomorphism. Take $O$ an open neighbourhood of the identity in $G / \{e\}$. Then $f^{-1}(O)$ is open in $G$ and there exists an open sphere $B_\varepsilon(e) = \{g \in G : \rho(g, e) < \varepsilon\}$, for some $\varepsilon > 0$, such that $e \in B_\varepsilon(e) \subseteq f^{-1}(O)$. We thus have

$$\{e\} \in f(B_\varepsilon(e)) = \{g\{e\} : d(g\{e\}, \{e\}) < \varepsilon\} \subseteq O.$$

Hence, every open set in $G / \{e\}$ is open in the topology induced by $d$ on $G / \{e\}$. Further, let $U = \{g\{e\} : d(g\{e\}, \{e\}) < \delta\}$, for some $\delta > 0$, be an open sphere about the identity in $G / \{e\}$ under $d$. Then $f^{-1}(U) = \{g \in G : \rho(g, e) < \delta\}$ which is open in $G$ and so $U$ is open in $G / \{e\}$. Therefore, the quotient topology on $G / \{e\}$ is indeed defined by the metric $d$.

Finally, for $x\{e\}, y\{e\}, a\{e\} \in G / \{e\}$, we have

$$d(a\{e\}x\{e\}, a\{e\}y\{e\}) = d(ax\{e\}, ay\{e\})$$

$$= \rho(ax, ay)$$

$$= \rho(x, y)$$

$$= d(x\{e\}, y\{e\}).$$
Therefore, $d$ is left invariant. Similarly, $d$ is right invariant, and so $d$ is a two-sided invariant metric.

These two lemmas allow us to prove the following result that seems obvious, but is not trivial to prove.

**Proposition 2.5.15.** Every pseudometrizable topological group, $(G, \rho)$, is a quotient of a metrizable topological group.

**Proof.** From Proposition 2.4.2, $G$ is a subgroup of the product $G / \{e\} \times |G|_I$ where $|G|_I$ is the group $G$ with the indiscrete topology. In Lemma 2.5.13, we saw that $|G|_I \in Q(\mathcal{M})$ where $\mathcal{M}$ is the class of all metrizable topological groups. Further, by Lemma 2.5.14, $G / \{e\} \in \mathcal{M}$. Therefore, $G \in SPQ(\mathcal{M})$. We use Remark 1.4.2 to simplify this expression to $G \in QSP(\mathcal{M})$. Noting that finite products and subgroups of metrizable spaces yield only metrizable spaces, the result follows.

We are now in a position to establish our previous conjecture that $QS(B)$ is the class of all abelian pseudometrizable topological groups.

**Theorem 2.5.16.** The class $QS(B)$ is the class of all pseudometrizable abelian topological groups.

**Proof.** Let $\mathcal{N}$ denote the class of all pseudometrizable abelian topological groups. As mentioned in the proof to Theorem 2.5.11, a subgroup of a Banach space is a metrizable topological group. Further, any quotient group of a metrizable topological group is a pseudometrizable. Therefore, $QS(B) \subseteq \mathcal{N}$.

Let $X \in \mathcal{N}$. Then by Proposition 2.5.15, $X$ is a quotient of a metrizable topological group $Y$. Further, by Theorem 2.5.11, $Y$ is a quotient of a subgroup of a Banach space. Therefore, we have $X \subseteq Q[QS(B)] = QS(B)$. Hence, $\mathcal{N} \subseteq QS(B)$ and the result follows.
Chapter 3

The Variety Generated by All Locally Compact Abelian Groups

The variety generated by the class of all locally compact Hausdorff abelian groups, \( \mathfrak{V}(\mathbb{L}_A) \) is next in line for analysis. We describe the connected groups, locally connected groups and locally convex topological vector spaces which are contained in the variety—and we find they are quite restricted. As we compare \( \mathfrak{V}(\mathbb{L}_A) \) to \( \mathfrak{V}(\mathbb{F}_A[0,1]) \), the variety generated by the free abelian topological group on \([0,1]\), we extend our analysis to varieties generated by classes of \( k_\omega \)-groups and varieties generated by classes of \( \sigma \)-compact groups.

§3.1 Connected Groups in the Variety

As we are studying locally compact abelian groups, it is fitting to recall the Principal Structure Theorem for these groups.

**Theorem 3.1.1: Principal Structure Theorem.** ([46], Theorem 25) Every locally compact Hausdorff abelian group has an open subgroup topologically isomorphic to \( \mathbb{R}^n \times K \), for some compact Hausdorff group \( K \) and non-negative integer \( n \).

Given we will be focusing on connected groups in particular, we present the following, which is an immediate consequence of the Principal Structure Theorem.

**Proposition 3.1.2.** ([46], Theorem 26) Every connected locally compact Hausdorff abelian group is topologically isomorphic to \( \mathbb{R}^n \times K \), where \( K \) is a compact connected group and \( n \) is a nonnegative integer.

Given Proposition 3.1.2, it is interesting that a similar result applies to connected topo-
logical groups which are closed subgroups of products of locally compact abelian groups, as shown by Morris in [44].

**Proposition 3.1.3.** ([44], Theorem, p123) Let $G$ be a closed subgroup of a product of locally compact Hausdorff abelian groups. If $G$ is connected, then it is topologically isomorphic to $\mathbb{R}^N \times K$, for some cardinal number $N$ and some compact connected Hausdorff abelian group $K$.

We now extend this result even further to show that the connected component of any complete Hausdorff topological group contained in the variety generated by the class of all locally compact abelian groups is also of the form $\mathbb{R}^N \times K$.

**Notation.** For the remainder of this thesis, we shall use $\mathcal{L}_A$ to denote the class of all locally compact Hausdorff abelian topological groups. Further, $\mathcal{L}$ shall be used to denote the class of all locally compact (not necessarily abelian) Hausdorff topological groups.

**Theorem 3.1.4.** Let $G$ be a complete Hausdorff topological group contained in the variety generated by all locally compact Hausdorff abelian groups, $\mathfrak{V}(\mathcal{L}_A)$. Then the connected component of the identity in $G$ is topologically isomorphic to $\mathbb{R}^N \times K$, where $N$ is some cardinal number and $K$ is a connected compact Hausdorff abelian topological group.

**Proof.** Let $G_0$ denote the connected component of the identity of $G$. Recall that $G_0$ is a closed subgroup of $G$. By Corollary 1.4.5, $G \in SC\overline{Q}P(\mathcal{L}_A)$. Since $G$ is complete, $G \in \overline{SCQ}P(\mathcal{L}_A)$ and hence $G_0 \in \overline{SCQ}P(\mathcal{L}_A)$. However, $\mathcal{L}_A$ is closed under $\overline{Q}$ and $P$. Therefore, $G_0 \in \overline{SC}(\mathcal{L}_A)$ and by Proposition 3.1.3, the result follows.

As all results so far have dealt with closed subgroups or complete groups, we shall find the following result useful in our analysis.
Proposition 3.1.5. (cf. [7], Corollary 1 to Theorem 2) Let \( \Omega \) be a family of locally compact Hausdorff groups. Then every Hausdorff topological group \( G \) in \( \mathfrak{V}(\Omega) \) has a completion \( \hat{G} \) in \( \mathfrak{V}(\Omega) \).

**Proof.** By Theorem 1.4.4, \( G \in SCQ \bar{S}P(\Omega) \). Now, the class of all locally compact Hausdorff groups is closed under \( \bar{Q}, \bar{S} \) and \( P \) and so \( G \) is topologically isomorphic to a subgroup of a product, \( H \), of locally compact Hausdorff abelian groups contained in \( \bar{Q} \bar{S}P(\Omega) \). As every product of locally compact Hausdorff groups is complete and every closed subgroup of a complete group is complete, \( \hat{G} = \bar{G} \) as a subgroup of \( H \). Thus, \( \hat{G} \in \mathfrak{V}(\Omega) \).

We shall say a variety is *closed under completions* if every Hausdorff topological group contained in the variety has a completion in the variety.

We note that Proposition 3.1.5 in fact says that a variety generated by *any* class of locally compact Hausdorff groups (not necessarily all) is closed under completions. Therefore, it applies immediately to a variety generated by locally compact Hausdorff abelian groups.

Now, although Proposition 3.1.3 reveals a substantial amount about some connected groups in \( \mathfrak{V}(L_A) \), we can use it, combined with Proposition 3.1.5, to obtain the following theorem.

**Theorem 3.1.6.** Let \( G \) be a connected Hausdorff abelian topological group in \( \mathfrak{V}(L_A) \). Then \( \hat{G} \), the completion of \( G \), is topologically isomorphic to \( \mathbb{R}^\kappa \times K \) where \( K \) is a compact connected Hausdorff abelian group and \( \kappa \) is some cardinal number. Further, if \( G \) is also a metrizable group, \( \kappa \leq \aleph_0 \).

**Proof.** By Proposition 3.1.5, \( \hat{G} \in \mathfrak{V}(L_A) \), where \( \hat{G} = \bar{G} \) as a closed subgroup of a Hausdorff topological group contained in \( \mathfrak{V}(L_A) \). Thus, \( \hat{G} \) is a complete connected Hausdorff group and so by Theorem 3.1.4, \( \hat{G} \) is topologically isomorphic to \( \mathbb{R}^\kappa \times K \) where \( K \) is compact connected abelian and \( \kappa \) is some cardinal number.

If \( G \) is also a metrizable group, then \( \hat{G} \) would also be metrizable and topologically isomorphic to \( \mathbb{R}^\kappa \times K \). Now, as a product of topological groups can be metrizable if and
only if the product is (essentially) a countable product (as it must have a countable basis at the identity), we have $\hat{G}$ topologically isomorphic to $\mathbb{R}^N \times K$ where $N \leq \aleph_0$ and $K$ is a compact connected metrizable abelian group.

The first corollary to Theorem 3.1.6 concerns a further characterization of all connected abelian topological groups contained in $\mathfrak{B}(L_A)$.

**Corollary 3.1.7.** Let $G$ be a connected abelian topological group contained in $\mathfrak{B}(L_A)$. Then $G$ is topologically isomorphic to a subgroup of $\mathbb{R}^N \times K$ where $N$ is some cardinal and $K$ is a compact connected abelian group.

**Proof.** By Proposition 2.4.2, $G$ is topologically isomorphic to a subgroup of the product $G/\{e\} \times |G|_I$ where $|G|_I$ is the group underlying $G$ with the indiscrete topology. By Theorem 3.1.6, $G/\{e\}$ is topologically isomorphic to a subgroup of $\mathbb{R}^N \times K'$ where $K'$ is a compact connected Hausdorff abelian group and $N$ is some cardinal number. Therefore, $G$ is topologically isomorphic to a subgroup of $\mathbb{R}^N \times K' \times |G|_I$. Noting that $K = K' \times |G|_I$ is a compact connected abelian group, we have the result.

The next corollary shows, in a sense, how small $\mathfrak{B}(L_A)$ is.

**Corollary 3.1.8.** Let $G$ be a connected topological group contained in $\mathfrak{B}(L_A)$. Then $G \in \mathfrak{B}(\mathbb{R})$.

**Proof.** By Corollary 3.1.7, $G$ is topologically isomorphic to a subgroup of $\mathbb{R}^N \times K$ where $N$ is some cardinal and $K$ is a connected compact abelian topological group. Now, by Proposition 2.4.2, $K$ is topologically isomorphic to $K/\{e\} \times |K|_I$, where $e$ is the identity in $K$ and $|K|_I$ is the group underlying $K$ with the indiscrete topology. As $K/\{e\}$ is compact Hausdorff, $K/\{e\}$ is a subgroup of a product of copies of $\mathbb{T}$ ([46], Corollary 1 to Theorem 14) and is thus contained in $\mathfrak{B}(\mathbb{R})$. It is well-known that every abelian group is algebraically isomorphic to a subgroup of a product of copies of the group $\mathbb{T}$ ([46], Corollary to Proposition 17 and proof to Theorem 1) and so appears in $\mathfrak{B}(\mathbb{R})$ with some
topology. Therefore, by Proposition 1.5.4, every indiscrete abelian topological group is contained in \( \mathcal{U}(\mathbb{R}) \). Thus, \(|K|_f \in \mathcal{U}(\mathbb{R})\) and so \( K \in \mathcal{U}(\mathbb{R}) \), indeed, \( G \in \mathcal{U}(\mathbb{R}) \). □

**Remark 3.1.9.** We note that by Theorem 3.1.4, if \( G \) is a complete Hausdorff topological group contained in \( \mathcal{U}(\mathcal{L}_A) \) then \( G_0 \), the connected component of the identity in \( G \), is topologically isomorphic to \( \mathbb{R}^\aleph \times K \) where \( \aleph \) is some cardinal and \( K \) is a connected compact Hausdorff abelian group. We also notice that both \( \mathbb{R}^\aleph \) and \( K \) are divisible groups, and hence their product is divisible. Now, if \( D \) is a divisible subgroup of \( A \), an abelian group, then \( A \) is algebraically isomorphic to \( D \times A/D \). For topological groups, however, this is not always the case. In any case, if \( D \) is an open subgroup, the result holds ([46], Proposition 18) and \( A/D \) is a discrete group. Thus, we have the following result concerning locally connected locally compact abelian groups.

**Proposition 3.1.10.** ([46], Theorem 33) Every locally connected locally compact Hausdorff abelian group \( G \) is topologically isomorphic to \( \mathbb{R}^a \times K \times D \) where \( \aleph \) is some cardinal, \( K \) is a connected Hausdorff locally connected compact abelian group, \( D \) is a discrete abelian group and \( a \geq 0 \). □

We can now extend Proposition 3.1.10 to complete Hausdorff locally connected groups in \( \mathcal{U}(\mathcal{L}_A) \).

**Theorem 3.1.11.** Let \( G \) be a complete Hausdorff locally connected abelian topological group in \( \mathcal{U}(\mathcal{L}_A) \). Then \( G \) is topologically isomorphic to \( \mathbb{R}^\aleph \times K \times D \) where \( \aleph \) is some cardinal, \( K \) is a connected Hausdorff locally connected compact abelian group and \( D \) is a discrete abelian group.

**Proof.** By Theorem 3.1.4, \( G_o \) is topologically isomorphic to \( \mathbb{R}^\aleph \times K \) where \( \aleph \) is some cardinal and \( K \) is a compact connected Hausdorff topological group. Now, \( \mathbb{R}^\aleph \times K \) is divisible ([46], Corollary 1 to Theorem 31) and \( G_o \) is open as \( G \) is locally connected. Therefore, \( G \) is topologically isomorphic to \( G_o \times G/G_o \). As \( G_o \) is open, \( G/G_o \) is a discrete
abelian group and hence $G$ is topologically isomorphic to $\mathbb{R}^N \times K \times D$ where $D$ is a discrete abelian group.

Up to this point we have dealt with $\mathfrak{V}(\mathcal{L}_A)$, the variety generated by the class of all locally compact Hausdorff abelian groups. We shall consider abelian groups in $\mathfrak{V}(\mathcal{L})$, the variety generated by all locally compact Hausdorff (not necessarily abelian) groups. For this we use Lemma 2.3.9.

The first result concerning $\mathfrak{V}(\mathcal{L})$ is an extension of both Theorem 3.1.4 and Corollary 3.1.8 from $\mathfrak{V}(\mathcal{L}_A)$ to $\mathfrak{V}(\mathcal{L})$. Recall that the commutator subgroup $G'$ of a group $G$ is the subgroup of $G$ generated by $\{x^{-1}y^{-1}xy : x, y \in G\}$ (see [17], Chapter 9, Section 9.2). Further, the quotient group $G/G'$ is abelian.

**Theorem 3.1.12.** Let $G$ be any topological group in $\mathfrak{V}(\mathcal{L})$, the variety generated by the class $\mathcal{L}$ of all locally compact Hausdorff (not necessarily abelian) topological groups. Let $G'$ be the commutator subgroup of $G$. Then $(G/G')_o$, the connected component of $G/G'$, is contained in $\mathfrak{V}(\mathbb{R})$.

**Proof.** Firstly, we note that $H = G/G'$ is a Hausdorff abelian topological group. Therefore, by Theorem 1.4.4, $H \in SCQ \mathfrak{S}P(\mathcal{L})$. However, $\mathcal{L}$ is closed under $\mathfrak{Q}, \mathfrak{S}$ and $P$ and so $H$ is a subgroup of a product $\prod_{i \in I} L_i$ where each $L_i$ is a locally compact Hausdorff topological group. Therefore, by Lemma 2.3.9, $H$ is a subgroup of the product $\prod_{i \in I} A_i$, where each $A_i$ is a locally compact Hausdorff abelian topological group. Hence $H \in \mathfrak{V}(\mathcal{L}_A)$, where $\mathcal{L}_A$ is the class of all locally compact Hausdorff abelian topological groups. Let $H_o$ denote the connected component of $H$. As $H_o$ is a subgroup of $H$, $H_o$ is in $\mathfrak{V}(\mathcal{L}_A)$. Finally, as $H_o$ is connected, by Corollary 3.1.8, $H_o$ is in $\mathfrak{V}(\mathbb{R})$.

We now present a couple of results that follow from Theorem 3.1.12 concerning connected groups contained in $\mathfrak{V}(\mathcal{L})$.

**Notation.** The next corollary will use the same notation as used in Theorem 3.1.12.
Corollary 3.1.13.

(i) Let \( G \) be a connected topological group contained in \( \mathfrak{B}(\mathcal{L}) \). Then \( G/\overline{G'} \) is contained in \( \mathfrak{B}(\mathbb{R}) \).

(ii) Let \( H \) be an abelian connected topological group contained in \( \mathfrak{B}(\mathcal{L}) \). Then \( H \) is contained in \( \mathfrak{B}(\mathbb{R}) \).

**Proof.** As \( G/\overline{G'} \) is connected Hausdorff abelian topological group, (i) follows from Theorem 3.1.12.

From part (i), \( H/\{e\} \in \mathfrak{B}(\mathbb{R}) \). By Proposition 2.4.2, \( H \) is topologically isomorphic to a subgroup of \( H/\{e\} \times |H|_I \). We saw earlier that \(|H|_I \in \mathfrak{B}(\mathbb{R})\) (see proof to Corollary 3.1.8) and so \( H \in \mathfrak{B}(\mathbb{R}) \), giving (ii). □

Part (ii) of Corollary 3.1.13 implies that there are no abelian connected topological groups in \( \mathfrak{B}(\mathcal{L}) \) outside those generated by \( \mathbb{R} \). Indeed, every abelian connected topological group contained in \( \mathfrak{B}(\mathcal{L}) \) is contained in \( \mathfrak{B}(\mathcal{L}_A) \). In other words, to consider connected abelian topological groups in \( \mathfrak{B}(\mathcal{L}) \) we do not need to go outside those generated by locally compact abelian groups.

§3.2 Normed Vector Spaces in the Variety

We first start with locally convex topological vector spaces. For the purposes of this thesis, a real Hausdorff locally convex topological vector space shall be referred to as an LCV-space. Note that all LCV-spaces are path-connected, and hence connected. Further, a locally compact LCV-space is finite dimensional (see [58], Chapter III, §3, Theorem 2). Therefore, we shall use the results from Section 3.1 and consider LCV-spaces contained in \( \mathfrak{B}(\mathcal{L}_A) \). Indeed, we shall present a slightly modified version of the result found in [42] that such LCV-spaces have the weak topology. First, we present a number of useful results from work by Morris.
Lemma 3.2.1. ([41], Lemma 2) Let $G$ be a discrete group contained in $\mathfrak{B}(\mathbb{R})$. Then $G$ is finitely generated.

\[ \]

Proposition 3.2.2. ([42], Theorem) Let $E$ be an LCV-space. Then $E$ has its weak topology if and only if every discrete subgroup of $E$ is finitely generated.

\[ \]

The following result is an extension of work by Kaplan, who dealt with necessary and sufficient conditions for an LCV-space to be topologically isomorphic to a product of copies of $\mathbb{R}$ ([26], Theorem 2).

Proposition 3.2.3. ([23], Theorem A) Let $G$ be topologically isomorphic to a closed subgroup of a product $\prod_{i \in I} R_i$ of copies of $\mathbb{R}$. If $G$ is also connected, then $G$ is topologically isomorphic to a product of copies of $\mathbb{R}$.

\[ \]

Given Corollary 3.1.13(ii) and the fact that LCV-spaces are connected abelian, for the moment we shall present our results in terms of $\mathfrak{M}(\mathcal{L})$, the variety generated by the class of all locally compact (not necessarily abelian) topological groups. The next result shows that LCV-spaces contained in $\mathfrak{M}(\mathcal{L})$ are very restricted.

Theorem 3.2.4. (cf. [42], Corollary) Let $G$ be an LCV-space contained in $\mathfrak{M}(\mathcal{L})$. Then $G \in \mathfrak{M}(\mathbb{R})$ and thus, $G$ has the weak topology. Further, $\hat{G}$ is topologically isomorphic to $\mathbb{R}^N$ for some cardinal $\aleph$.

\[ \]

Proof. As $G$ is connected abelian, by Corollary 3.1.13(ii), $G \in \mathfrak{M}(\mathbb{R})$. Now, every discrete subgroup $D$ of $E$ is contained in $\mathfrak{M}(\mathbb{R})$ and so by Lemma 3.2.1, $D$ is finitely generated. Therefore, by Proposition 3.2.2, $G$ has the weak topology.

As $G$ has the weak topology, it is topologically isomorphic to a subgroup of the product $\prod_{i \in I} R_i$ where each $R_i$ is a copy of $\mathbb{R}$. Now, $\hat{G} = \overline{G}$ as a closed subgroup of $\prod_{i \in I} R_i$ and it is also connected. Thus, the result follows from Proposition 3.2.3.
Corollary 3.2.5. Let $G$ be a metrizable LCV-space contained in $\mathfrak{U}(L)$. Then $\hat{G}$ is a Fréchet space (that is, a complete metrizable LCV-space) topologically isomorphic to $\mathbb{R}^n$ where $n \leq \aleph_0$.

**Proof.** The result follows from Theorem 3.2.4 and noting that a metrizable product of topological groups must be a countable product.

We now turn our attention to normed vector spaces contained in $\mathfrak{U}(L)$, which we originally set out to consider. Indeed, the following result is an extension of the work of Remus and Trigos-Arrieta [57].

**Theorem 3.2.6.** Let $G$ be a normed vector space contained in $\mathfrak{U}(L)$ (or $\mathfrak{U}(L_A)$). Then $G$ is finite dimensional and so topologically isomorphic to $\mathbb{R}^n$ for some nonnegative integer $n$.

**Proof.** By Proposition 3.1.5, $\hat{G}$ (a Banach space) is contained in $\mathfrak{U}(L)$. Now, by Theorem 3.2.4, $\hat{G}$ is topologically isomorphic to a product of copies of $\mathbb{R}$; that is, $\hat{G} \in C(\mathbb{R})$. However, $\hat{G}$ is uniformly free from small subgroups, and so by Proposition 2.5.8, $\hat{G} \in P(\mathbb{R})$; that is, $\hat{G}$ is finite dimensional. Therefore, $G$ is finite dimensional and hence topologically isomorphic to $\mathbb{R}^n$ for some nonnegative integer $n$ (see [24], Part II, Section 20, Corollary 20.4).

**Remark 3.2.7.** To summarize, $\mathfrak{U}(L)$ contains only those LCV-spaces with the weak topology, only those complete LCV-spaces of the form $\mathbb{R}^N$ for some cardinal $N$, only those Fréchet spaces of the form $\mathbb{R}^{\aleph_0}$ or $\mathbb{R}^n$, $n \in \mathbb{N}$, and only those Banach spaces of the form $\mathbb{R}^n$, $n \in \mathbb{N}$, with no other normed vector spaces appearing.

We now see that $\mathfrak{U}(L)$ is a long way from being the variety of all topological groups—it contains only finite dimensional normed vector spaces. Similarly, $\mathfrak{U}(L_A)$ is a long way from being the variety of all abelian topological groups. This is further quantified and exposed in later chapters.
The final result in this section extends Theorem 3.2.6 to a wider class of groups using results of Morris and Pestov. Recall that a topological group $G = (G, T)$ is said to be \textit{locally minimal} if there exists a neighbourhood of the identity, $V$, with the property that whenever $T'$ is a Hausdorff group topology on $G$ coarser than $T$ and the interior of $V$ in $T'$ is non-empty, then $T' = T$. Note also that a group which is uniformly free from small subgroups is both locally minimal and an NSS-group.

**Proposition 3.2.8.** ([52], Theorem 3.10) If $G$ is a locally minimal topological group and has no small normal subgroups and $G \in \mathfrak{W}(\Omega)$ where $\Omega$ is any class of topological groups, then $G \in \overline{SPQ}(\Omega)$, indeed, $G \in \overline{SQP}(\Omega)$. If $\Omega$ is a class of abelian topological groups, then $G \in \overline{SQP}(\Omega)$.

Clearly, we have the following.

**Proposition 3.2.9.** Let $\Omega$ be any class of locally compact groups and let $G$ be a complete locally minimal topological group that has no small subgroups in $\mathfrak{W}(\Omega)$. Then $G$ is a locally compact group. Further, if $G$ is a Banach space, then $G$ is finite dimensional.

\textbf{Proof.} By Proposition 3.2.8, $G \in \overline{SQSP}(\mathcal{L})$. Now, the class of all locally compact groups is closed under $Q$, $\overline{S}$ and $P$ and therefore, $G$ is a locally compact group. To complete the proof, we note that a locally compact normed vector space is finite dimensional (see [58], Chapter III, §3, Theorem 2).

§3.3 $k_\omega$-Groups in the Variety

We now compare $\mathfrak{W}(L_A)$ with $\mathfrak{W}(FA[0,1])$, the variety generated by the free abelian topological group on $[0,1]$. We will first establish the fact that $\mathfrak{W}(FA[0,1]) \not\subseteq \mathfrak{W}(L_A)$. We shall deal with the reverse containment later.

Before we step into our main analysis in this section, we recall the following result from the work of Katz, Morris and Nickolas. This result is not at all obvious.
Proposition 3.3.1. ([28], Theorem 1) The free abelian topological group on (0,1) is topologically isomorphic to a subgroup of $FA[0,1]$, the free abelian topological group on [0,1].

We will presently show that $FA[0,1]$ is not contained in $\mathcal{U}(\mathcal{L}_A)$. However, we need a number of results, the first of which is about free abelian topological groups from the work by Mack, Morris and Ordman.

Lemma 3.3.2. ([34], Theorem 3) Let $X$ be a $k_\omega$-space and let $Y$ be a closed (and hence a $k_\omega$-space) subspace of $X$. Then $gp(Y)$ as a subgroup of $F(X)$, the free (free abelian) topological group on $X$, is the free (respectively free abelian) topological group on $Y$, $F(Y)$.

The following is a corollary to Proposition 3.3.1.

Corollary 3.3.3. The free abelian topological group on $\mathbb{Z}$ is topologically isomorphic to a subgroup of the free abelian topological group on [0,1].

Proof. By Proposition 3.3.1, $FA[0,1]$ contains $FA(0,1)$ as a subgroup. Now, $\mathbb{Z}$ is homeomorphic to a closed subspace of (0,1) and thus, by Lemma 3.3.2, $FA(\mathbb{Z})$ is a subgroup of $FA(0,1)$. Therefore, $FA(\mathbb{Z})$ is topologically isomorphic to a subgroup of $FA[0,1]$.

Recall that we denote by $FA(X)$ the Graev free abelian topological group generated by $X$, and so the identity element in $FA(X)$ is contained in $X$, $X^{-1}$, $X^n$ and $X^{-n}$ for each $n \in \mathbb{N}$. Thus, if $X$ is a connected space, as in the case of [0,1], we have the following result proved by Graev in [16].

Lemma 3.3.4. ([16], Part I, §6, Result A) The Graev free abelian topological group on a connected space $X$ is connected.
We are now in a position to prove that $FA[0,1] \not\in \mathfrak{V}(\mathcal{L}_A)$, establishing that $\mathfrak{V}(FA[0,1])$ is not contained in $\mathfrak{V}(\mathcal{L}_A)$.

**Proposition 3.3.5.** The topological group $FA[0,1]$ is not contained in $\mathfrak{V}(\mathcal{L}_A)$.

*Proof.* Suppose $FA[0,1] \in \mathfrak{V}(\mathcal{L}_A)$. Then as $[0,1]$ is connected, by Lemma 3.3.4, $FA[0,1]$ is also connected and so from Corollary 3.1.8, $FA[0,1] \in \mathfrak{V}(\mathbb{R})$. Now, any discrete subgroup $D$ of $FA[0,1]$ is also contained in $\mathfrak{V}(\mathbb{R})$ and hence by Lemma 3.2.1, $D$ is finitely generated. However, by Corollary 3.3.3, $FA(\mathbb{Z})$ is a discrete subgroup of $FA[0,1]$ which is not finitely generated. Thus, we have a contradiction and so $FA[0,1]$ is not contained in $\mathfrak{V}(\mathcal{L}_A)$. 

Recall that $FA[0,1]$ is a $k_w$-group (see Definition 2.2.9 and Proposition 2.2.12). We shall turn our attention to varieties generated by classes of $k_w$-groups rather than limit ourselves to $FA[0,1]$. In Chapter 4, the full reason for this decision will become apparent. We now present a number of results concerning $k_w$-groups and $\sigma$-compact groups. The concept of $\sigma$-compact is the necessary and sufficient condition for a locally compact Hausdorff abelian group to be a $k_w$-group.

It is routine to show that the property of being a $k_w$-group is preserved under closed subgroups, quotient groups and finite products.

**Proposition 3.3.6.** ([14], Results 4, 11, 14) The class of all $k_w$-groups is closed under $\overline{S}$, $\overline{Q}$ and $P$. 

Note that the class of all $k_w$-groups is not closed under $S$ or $C$ and so does not form a variety of topological groups.

**Notation.** We shall use $\mathcal{K}_w$ to denote the class of all abelian $k_w$-groups.

To compare varieties generated by abelian $k_w$-groups to the class of all locally compact
abelian groups, we first consider discrete groups in the varieties. Note that \( \mathcal{L}_A \) contains all discrete abelian groups, including those with arbitrarily large cardinality. As discrete groups are uniformly free from small subgroups (see Definition 2.5.1 and Example 2.5.2), we shall use Proposition 3.2.8 in our analysis.

**Proposition 3.3.7.** Let \( \Omega \) be a class of \( k_\omega \)-groups. Then each discrete group \( D \) in \( \mathbb{V}(\Omega) \) is countable.

**Proof.** As \( D \) is a locally minimal NSS-group, by Proposition 3.2.8, \( D \in S\overline{Q} \overline{S}P(\Omega) \).

Now, \( D \) is a locally compact subgroup of a Hausdorff space (products of Hausdorff are Hausdorff) and hence \( D \) is a closed subgroup; that is, \( D \in \overline{S} \overline{Q} \overline{S}P(\Omega) \) ([46], Proposition 7). Now, by Proposition 3.3.6, the class of all \( k_\omega \)-groups is closed under \( Q, S \) and \( P \), and hence, \( D \) is a \( k_\omega \)-group. Thus, \( D \) is countable (Examples 2.2.11(c)).

**Corollary 3.3.8.** The variety of topological groups generated by \( FA[0,1] \) does not contain the class \( \mathcal{L}_A \).

**Proof.** As \( FA[0,1] \) is a \( k_\omega \)-group, Proposition 3.3.7 implies that \( \mathbb{V}(FA[0,1]) \) contains no uncountable discrete groups. Therefore, \( \mathcal{L}_A \not\subseteq \mathbb{V}(FA[0,1]) \), as \( \mathcal{L}_A \) contains discrete groups of all cardinalities.

Corollary 3.3.8 clearly implies that \( \mathbb{V}(\mathcal{L}_A) \not\subseteq \mathbb{V}(FA[0,1]) \).

A further consequence of Proposition 3.3.7 is that any discrete abelian group in \( \mathbb{V}(\mathcal{K}_\omega) \) is countable.

**Corollary 3.3.9.** Let \( \mathcal{K}_\omega \) be the class of all abelian \( k_\omega \)-groups. Then a discrete abelian group \( D \) is contained in \( \mathbb{V}(\mathcal{K}_\omega) \) if and only if \( D \) is countable.

**Proof.** A countable discrete abelian group is clearly an abelian \( k_\omega \)-group and thus is in \( \mathbb{V}(\mathcal{K}_\omega) \). Conversely, by Proposition 3.3.7, a discrete abelian group contained in \( \mathbb{V}(\mathcal{K}_\omega) \) is countable.
We introduce the concept of $\sigma$-compact which will be of interest in this section, and even more so in later chapters.

**Definition 3.3.10.** A topological space is said to be $\sigma$-compact if it is a countable union of compact sets.

**Examples 3.3.11.**

(a) $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ and each $[-n, n]$ is compact. Thus, $\mathbb{R}$ is $\sigma$-compact.

(b) Let $X$ be a $k_\omega$-space such that $X = \bigcup_{n=1}^{\infty} X_n$, where each $X_n$ is compact. Clearly $X$ is $\sigma$-compact.

(c) A discrete $\sigma$-compact space, $D$, is countable since if $D = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is compact in $D$, then each $X_n$ must be finite.

**Remark 3.3.12.** It is clear that $\sigma$-compactness is preserved under closed subspaces, quotient spaces and finite products. Therefore, the class of all abelian $\sigma$-compact topological groups, $C_\sigma$, is closed under $\mathfrak{S}$, $Q$ and $P$.

We now present a result which characterizes locally compact Hausdorff abelian groups contained in $\mathfrak{B}(K_\omega)$.

**Theorem 3.3.13.** Let $G$ be a locally compact Hausdorff abelian topological group. Then the following are equivalent.

(i) $G$ is $\sigma$-compact;

(ii) $G$ is a $k_\omega$-group;

(iii) $G \in \mathfrak{B}(K_\omega)$;

(iv) Every open subgroup of $G$ has countable index.

**Proof.** (i) $\Rightarrow$ (ii) follows from Result 10 in [14]. (See also [46], Exercise Set Twelve, 4(ii)).
(ii) $\iff$ (iii) is trivial.

(iii) $\implies$ (iv): Let $A$ be an open subgroup of $G$. Clearly, $G/A$ is a discrete abelian group in $\mathfrak{B}(K)$. By Corollary 3.3.9, $G/A$ is countable and hence $A$ has countable index.

(iv) $\implies$ (i): By the Principal Structure Theorem, $G$ has an open subgroup $H$ topologically isomorphic to $\mathbb{R}^n \times K$ for some nonnegative integer $n$ and compact abelian group $K$. Then the index of $H$ in $G$ is countable. Now, $\mathbb{R}^n \times K$ is $\sigma$-compact as each $\mathbb{R}$ and $K$ are $\sigma$-compact and $\sigma$-compactness is preserved under finite products. Therefore, $H$ is a countable union of compact sets. Thus, $G$ is the countable union of countable unions of compact sets, and hence $G$ is $\sigma$-compact.

Remark 3.3.14. Note that Theorem 3.3.13 implies that any locally compact Hausdorff abelian group, $G$, contained in the variety generated by $\Omega$, where $\Omega$ is any class of abelian $k_\omega$-groups, is $\sigma$-compact, for if $G \in \mathfrak{B}(\Omega)$, then $G \in \mathfrak{B}(K)$.

Turning our attention to $\mathfrak{B}(L)$, we can use results in Section 3.1 to obtain a description of the connected topological groups in $\mathfrak{B}(K) \cap \mathfrak{B}(L)$.

Proposition 3.3.15. The class of all connected groups in $\mathfrak{B}(L)$ is contained in $\mathfrak{B}(K)$. 

Proof. Let $G$ be a connected group in $\mathfrak{B}(L)$. Then by Corollary 3.1.8, $G \in \mathfrak{B}(\mathbb{R})$. Now, $\mathbb{R}$ is an abelian $k_\omega$-group and therefore, $\mathbb{R} \in \mathfrak{B}(K)$ and hence, $G \in \mathfrak{B}(\mathbb{R}) \subseteq \mathfrak{B}(K)$.

We note that $\mathfrak{B}(\mathbb{R})$ is properly contained in $\mathfrak{B}(K) \cap \mathfrak{B}(L)$ as the latter contains the discrete group $FA(\mathbb{Z})$ which, being countably infinitely generated, is not in $\mathfrak{B}(\mathbb{R})$ (see Lemma 3.2.1).

We shall now focus more closely on normed vector spaces (including Banach spaces) contained in varieties generated by classes of $k_\omega$-groups, and later normed vector spaces contained in the variety generated by the class of all abelian $\sigma$-compact groups. Firstly, we recall the following result about metrizable $k_\omega$-spaces.
Remark 3.3.16. ([14], Result 21) A metrizable \( k_\omega \)-space is locally compact.

Following from Remark 3.3.16, we present the following result concerning normed vector spaces which are also \( k_\omega \)-groups.

Proposition 3.3.17. Let \( N \) be a normed vector space that is also a \( k_\omega \)-group. Then \( N \) is finite dimensional, that is, \( N \cong \mathbb{R}^n \), for some \( n \in \mathbb{N} \).

Proof. As \( N \) is a normed vector space, it is metrizable. By Remark 3.3.16, \( N \) is locally compact. The only normed vector spaces which are locally compact are those of finite dimension (see [58], Chapter III, §3, Theorem 2). Therefore, \( N \cong \mathbb{R}^n \) (see [24], Part II, Section 20, Corollary 20.4).

Before we turn to normed vector spaces in the variety generated by all abelian \( k_\omega \)-groups, we must consider completions in the variety.

Earlier we saw that \( \mathfrak{M}(\Omega) \) is closed under completions for \( \Omega \) a class of locally compact Hausdorff topological groups. The same can be said if \( \Omega \) is a class of \( k_\omega \)-groups.

Proposition 3.3.18. Let \( \Omega \) be a family of \( k_\omega \)-groups. Then every Hausdorff topological group \( G \) in \( \mathfrak{M}(\Omega) \) has a completion \( \widehat{G} \) in \( \mathfrak{M}(\Omega) \).

Proof. By Theorem 1.4.4, \( G \in SC\overline{Q} SP(\Omega) \). Now, the class of all \( k_\omega \)-groups is closed under \( Q, \overline{S} \) and \( P \) (see Proposition 3.3.6) and so \( G \) is topologically isomorphic to a subgroup of a product of \( k_\omega \)-groups contained in \( \overline{Q} \overline{SP}(\Omega) \), \( K = \prod_{i \in I} K_i \). Now \( K \) is complete as \( k_\omega \)-groups are complete ([22], Theorem 2) and so \( \widehat{G} = \overline{G} \in \mathfrak{M}(\Omega) \) as a subgroup of \( K \).

As an immediate consequence of Proposition 3.3.18, we have the following.

Corollary 3.3.19. The variety of topological groups generated by \( \mathcal{K}_\omega \), the class of all abelian \( k_\omega \)-groups, is closed under completions.
We have already seen that a normed vector space that is also a $k_\omega$-group is finite dimensional. A more general result holds, namely that normed vector spaces contained in varieties generated by $k_\omega$-groups are also finite dimensional. Firstly, we present the following result about locally minimal topological groups with no small subgroups.

**Proposition 3.3.20.** Let $\Omega$ be a class of $k_\omega$-groups and let $G$ be a complete locally minimal topological group that has no small subgroups (e.g., Banach space, Lie group or discrete group) such that $G \in \mathfrak{V}(\Omega)$. Then $G$ is a $k_\omega$-group.

**Proof.** By Proposition 3.2.8, $G \in \overline{S} \ Q \ SP(\Omega)$. Now, by Proposition 3.3.6, the class of all $k_\omega$-groups is closed under $\overline{Q}$, $\overline{S}$ and $P$. Therefore, $G$ is a $k_\omega$-group. 

We can now prove the finite dimensional restriction on normed vector spaces contained in varieties generated by $k_\omega$-groups, indeed, $\mathfrak{V}(K_\omega)$.

**Theorem 3.3.21.** Every normed vector space contained in $\mathfrak{V}(K_\omega)$ is finite dimensional.

**Proof.** Let $N$ be a normed vector space contained in $\mathfrak{V}(K_\omega)$. The completion $\tilde{N}$ of $N$ is a Banach space contained in $\mathfrak{V}(K_\omega)$ as $\mathfrak{V}(K_\omega)$ is closed under completions (Corollary 3.3.19). Now, by Proposition 3.3.20, $\tilde{N}$ is a $k_\omega$-group and so by Proposition 3.3.17, $\tilde{N}$ is finite dimensional. Hence $N$ is finite dimensional; that is, $N = \tilde{N} = \mathbb{R}^m$ for some nonnegative integer $m$.

Turning our attention to the variety generated by the class of all abelian $\sigma$-compact groups, we consider Banach spaces in this variety.

**Notation.** We shall let $C_\sigma$ denote the class of all abelian $\sigma$-compact topological groups. Clearly $C_\sigma$ has $K_\omega$ as a proper subclass.

**Proposition 3.3.22.** Every Banach space contained in the variety generated by a class of $\sigma$-compact groups is a $\sigma$-compact group.
Proof. Let \( \Omega \) be a class of \( \sigma \)-compact groups and let \( B \) be a Banach space such that \( B \in \mathcal{V}(\Omega) \). By Proposition 3.2.8, \( B \in SQ SP(\Omega) \). We know that \( \sigma \)-compactness is preserved under \( \overline{Q}, \overline{S} \) and \( P \) (see Remark 3.3.12) and so \( B \in S(\Omega') \) where \( \Omega' \) is a class of \( \sigma \)-compact groups. As \( B \) is complete, \( B \in \overline{S}(\Omega') \) and hence \( B \) is \( \sigma \)-compact.

The question we now address is whether an infinite dimensional Banach space can be \( \sigma \)-compact. We will see shortly that this is not possible, but first we recall the following result about \( \sigma \)-compact products.

Lemma 3.3.23. ([19], Theorem 3.9) Let \( I \) be an index set and for each \( i \in I \) let \( X_i \) be a topological space. Then \( \prod_{i \in I} X_i \) is \( \sigma \)-compact if and only if all of the \( X_i \) are \( \sigma \)-compact and all but a finite number of them are compact.

Theorem 3.3.24. The variety of topological groups generated by the class of all abelian \( \sigma \)-compact groups contains no infinite dimensional Banach spaces.

Proof. Suppose \( B \) is an infinite dimensional Banach space contained in \( \mathcal{V}(C_{\sigma}) \). By Proposition 3.3.22, \( B \) is \( \sigma \)-compact. As \( B \) is also metrizable, \( B \) is separable (see [24], Part I, Section 10, Result 10.3). By Theorem 5.2 of Chapter VI in [4], which says that any separable Fréchet space is homeomorphic to \( \mathbb{R}^\kappa \), \( B \) is homeomorphic to an infinite product of \( \sigma \)-compact spaces, none of which are compact. Thus, by Lemma 3.3.23, \( B \) cannot be \( \sigma \)-compact. Therefore, \( B \notin \mathcal{V}(C_{\sigma}) \).

Given Theorem 3.3.24, it may come as a surprise, that \( \mathcal{V}(C_{\sigma}) \) does contain a number of infinite dimensional normed vector spaces. Note firstly that there exist countable dimensional normed vector spaces. Let \( B \) be an infinite dimensional Banach space and let \( S \) be any countable set of linearly independent vectors in \( B \). Then the vector space \( N \) generated by \( S \), is a subspace of \( B \) and so is normed. We now prove the following.

Proposition 3.3.25. The variety generated by the class of all abelian \( \sigma \)-compact groups
contains every countable dimensional normed vector space.

Proof. Let $N$ be a countable dimensional normed vector space and let $S$ be a countable basis for $N$. Then $N = \text{gp}(\mathbb{R}.S)$ where $\mathbb{R}.S = \bigcup_{s \in S} \{rs : r \in \mathbb{R}\}$. Further,

$$\{rs : r \in \mathbb{R}\} = \bigcup_{i=1}^{\infty} \{rs : r \in [-i, i]\}.$$ 

Clearly, $\{rs : r \in [-i, i]\}$ is compact for each $s \in S$ and so $N$ is a countable union (of countable unions of countable unions) of compact sets and hence is $\sigma$-compact; that is, $N \in \mathcal{C}_\sigma$. Thus, $N \in \mathfrak{A}(\mathcal{C}_\sigma)$.

We are now able show that, in contrast to varieties generated by locally compact abelian groups and varieties generated by $k_\omega$-groups, the variety generated by the class of all abelian $\sigma$-compact groups is not closed under completions.

**Proposition 3.3.26.** The variety generated by the class of all $\sigma$-compact abelian groups is not closed under completions.

Proof. By Proposition 3.3.25, $\mathfrak{A}(\mathcal{C}_\sigma)$ contains a countable dimensional normed vector space $N$. However, $\hat{N}$ is an infinite dimensional Banach space, and by Theorem 3.3.24, cannot be contained in $\mathfrak{A}(\mathcal{C}_\sigma)$. Therefore, $N$ has no completion in $\mathfrak{A}(\mathcal{C}_\sigma)$ and the result follows.

It is now clear that $\mathfrak{A}(\mathcal{C}_\sigma)$ and $\mathfrak{A}(\mathcal{K}_\omega)$ are not the same variety. However, as every $k_\omega$-group is $\sigma$-compact, we have the following relationship between the two varieties.

**Proposition 3.3.27.** The variety generated by the class of all abelian $k_\omega$-groups is properly contained in the variety generated by the class of all abelian $\sigma$-compact groups.

Proof. Let $\mathcal{K}_\omega$ be the class of all abelian $k_\omega$-groups and let $\mathcal{C}_\sigma$ be the class of all abelian $\sigma$-compact groups. As each $k_\omega$-group is $\sigma$-compact, $\mathfrak{A}(\mathcal{K}_\omega) \subseteq \mathfrak{A}(\mathcal{C}_\sigma)$. Now, $\mathfrak{A}(\mathcal{K}_\omega)$ is closed under completions (Corollary 3.3.19) but $\mathfrak{A}(\mathcal{C}_\sigma)$ is not closed under completions (Proposition 3.3.26). Thus, $\mathfrak{A}(\mathcal{K}_\omega) \neq \mathfrak{A}(\mathcal{C}_\sigma)$ and the result follows.
We summarize some results from this section in the following theorem.

**Theorem 3.3.28.** Let $E$ be an LCV-space, $N$ a normed vector space and $B$ a Banach space.

(i) $E \in \mathcal{V}(\mathbb{R})$ if and only if it has the weak topology. $N \in \mathcal{V}(\mathbb{R})$ if and only if it is finite dimensional.

(ii) $E \in \mathcal{V}(\mathcal{L}_\mathcal{A})$ if and only if it has the weak topology. $N \in \mathcal{V}(\mathcal{L}_\mathcal{A})$ if and only if it is finite dimensional.

(iii) $N \in \mathcal{V}(\mathcal{K}_\omega)$ if and only if it is finite dimensional.

(iv) $B \in \mathcal{V}(\mathcal{C}_\sigma)$ if and only if it is finite dimensional. If $N$ is countable dimensional, then $N \in \mathcal{V}(\mathcal{C}_\sigma)$. 

\[\blacksquare\]
Chapter 4

The Variety Generated by $FA[0, 1]$

In this chapter we analyze the variety generated by $FA[0, 1]$, the free abelian topological group on $[0, 1]$. In the process, we not only compare $QJ(FA[0, 1])$ to the variety generated by the class of all locally compact Hausdorff abelian groups but also to the variety generated by the class of all abelian $k_\omega$-groups with some surprising results. Finally, we consider the variety generated by the free abelian topological group on a compact Hausdorff space and present an interesting characterization of this variety.

§4.1 Results on Free Abelian Topological Groups

In our analysis, we require the following tools concerning free abelian topological groups. The first result is a slight extension of the well-known (though not necessarily explicitly stated in the literature) Corollaries 4.1.2 and 4.1.3.

Proposition 4.1.1. Let $X$ and $(Y, T)$ be completely regular topological spaces such that there exists a quotient mapping $\phi : X \to (Y, T)$. Let $G = \text{gp}(Y)$, an abelian topological group algebraically generated by $Y$ such that the topology $T_G$ on $G$ is the finest topological group topology that induces $T$ on $Y$. Then there exists a quotient homomorphism $\Phi$ from $FA(X)$, the free abelian topological group on $X$, to $(G, T_G)$, such that $\Phi|_X = \phi$.

Proof. Let $\Phi : FA(X) \to (G, T_G)$ be the continuous homomorphism from $FA(X)$ onto $G$ that extends naturally from $\phi$. Let $T_G'$ be the quotient topology on $G$ induced by $\Phi$. Clearly, $T_G \subseteq T_G'$ and $T \subseteq T'$, where $T'$ is the topology induced by $T_G$ on $Y$. Now, $\Phi|_X$ is a continuous map from $X$ onto $(Y, T')$ where $\Phi|_X(x) = \phi(x)$ for each $x \in X$. Then $T' \subseteq T$, as $T$ is the quotient topology induced by $\phi$. Therefore, $T = T'$ and hence $T_G \subseteq T_G'$ as $T_G$ is the finest topology that induces $T$ on $Y$. Thus, $T_G = T_G'$ and the result follows.
The following two corollaries are special cases of Proposition 4.1.1.

**Corollary 4.1.2.** Let $X$ and $Y$ be completely regular topological spaces such that there exists a quotient mapping $\phi : X \to Y$. Then there exists a quotient homomorphism $\Phi : FA(X) \to FA(Y)$ where $FA(X)$ and $FA(Y)$ are the free abelian topological groups on $X$ and $Y$ respectively.

**Corollary 4.1.3.** Let $X$ be a completely regular space and $G$ a topological group algebraically generated by $X$ and having the finest topological group topology that induces the given topology on $X$. Then $G$ is a quotient group of $FA(X)$, the free abelian topological group on $X$.

In Lemma 3.3.2 we saw that for a $k_\omega$-space $X$ with closed subspace $Y$, $FA(Y)$ is a subgroup of $FA(X)$. We can extend this to obtain a similar result for $X$ any completely regular space with compact subspace $Y$. We first present a general result showing that a subset $S$ of $FA(X)$ is closed if for each $n \in \mathbb{N}$, the set of words in $S$ with reduced representation of length less than or equal to $n$ is compact. The proof of this is a standard application of the Stone-Čech compactification (cf. [18]).

**Notation.** Let $X$ be a completely regular space and let $n \in \mathbb{N}$. We shall denote by $FA_n(X)$ the set of all words in $FA(X)$ whose reduced representation has length less than or equal to $n$ with respect to $X$. Similarly, $F_n(X)$ represents the set of all words in $F(X)$ whose reduced representation has length at most $n$ with respect to $X$.

**Proposition 4.1.4.** Let $X$ be a completely regular Hausdorff space and let $S$ be a subset of $FA(X)$ such that $S \cap FA_n(X)$ is compact for all $n \in \mathbb{N}$. Then $S$ is closed in $FA(X)$.

**Proof.** Let $\beta X$ be the Stone-Čech compactification of $X$ and let $FA(\beta X)$ be the free abelian topological group on $\beta X$. Then the natural map $\phi : X \to \beta X \subseteq FA(\beta X)$ can be extended to a continuous, one-to-one homomorphism $\Phi : FA(X) \to FA(\beta X)$. Now,
clearly $FA(\beta X) = \bigcup_{n=1}^{\infty} FA_n(\beta X)$, indeed this is the $k_\omega$-decomposition of $FA(\beta X)$.

Now, consider $\Phi(S) \subseteq FA(\beta X)$. We have the following.

$$\Phi(S) \cap FA_n(\beta X) = \Phi(S \cap FA_n(X))$$

As $S \cap FA_n(X)$ is compact, $\Phi(S) \cap FA_n(\beta X)$ is compact. Therefore, $\Phi(S)$ is closed in $FA(\beta X)$, showing $\Phi^{-1}(\Phi(S))$ is closed in $FA(X)$. The proof is completed noting that $\Phi^{-1}(\Phi(S)) = S$ as $\Phi$ is one-to-one.

As a consequence of Proposition 4.1.4, we have the following result alluded to earlier.

**Notation.** Let $X$ be a subset of a group $G$. Then we shall denote the subset $\bigcup_{i=1}^{n} (X \cup X^{-1})^i$ of $G$ by $gp_n(X)$.

**Corollary 4.1.5.** Let $X$ be a completely regular Hausdorff space and let $K$ be a compact subspace of $X$. Let $G$ be the subgroup of $FA(X)$, the free abelian topological group on $X$, algebraically generated by $K$. Then $G$ is topologically isomorphic to $FA(K)$, the free abelian topological group on $K$.

**Proof.** Clearly, $G = \bigcup_{n=1}^{\infty} gp_n(K)$. Note that $gp_n(K)$ is compact, and hence closed in $FA(X)$, for each $n$ as $K$ is compact. Let $A \subseteq G$ be such that $A \cap gp_n(K)$ is compact for each $n \in \mathbb{N}$. Clearly $A \cap gp_n(K) = A \cap gp_n(X)$ and so by Proposition 4.1.4, $A$ is closed in $FA(X)$. Thus, $A$ is closed in $G$ and so, by Proposition 2.2.12, $G$ is topologically isomorphic to $FA(K)$.

**Remark 4.1.6.** Proposition 4.1.4 and Corollary 4.1.5 are also true with $F(X)$ replacing $FA(X)$ and $F_n(X)$ replacing $FA_n(X)$.

We next present the following interesting theorem which characterizes the free abelian topological groups which are topologically isomorphic to a closed subgroup of $FA[0,1]$.

We note that for $X$ a completely regular Hausdorff topological space, if $FA(X)$ is a
closed subgroup of $FA[0,1]$, then it, and hence also $X$, must be a $k_\omega$-space. Further, if $X = \bigcup_{n=1}^{\infty} X_n$ is a $k_\omega$-subspace of the $k_\omega$-group $FA[0,1]$, then each compact space $X_n$ is in $FA_m[0,1]$ for some $m$ (see Remark 2.2.10(b)). As $FA[0,1]$ is finite dimensional and metric, so too is each $X_n$. So the following theorem gives the necessary and sufficient condition for free abelian topological groups which are topologically isomorphic to a closed subgroup of $FA[0,1]$.

**Theorem 4.1.7.** ([33], Theorem 4.1) Let $X$ be a completely regular Hausdorff topological space. If $X = \bigcup_{n=1}^{\infty} X_n$ is $k_\omega$-space such that each $X_n$ is metrizable and finite dimensional, then $FA(X)$, the free abelian topological group on $X$, is topologically isomorphic to a subgroup of $FA[0,1]$.

Further to Lemma 3.3.2, Corollary 4.1.5 and Theorem 4.1.7, we have the following result by Mack, Morris and Ordman.

**Lemma 4.1.8.** ([34], Theorem 3) Let $X = \bigcup_{n=1}^{\infty}$ be a $k_\omega$-space. Let $Y \subseteq FA(X)$ be a closed subspace containing $e$ such that $Y \setminus \{e\}$ freely generates $gp(Y)$, the subgroup of $FA(X)$ generated by $Y$. Let $Y = \bigcup_{n=1}^{\infty} Y_n$ be a $k_\omega$-decomposition of $Y$. If for each natural number $n$ there is a natural number $m$ such that $gp(Y) \cap gp_n(X_n) \subseteq gp_m(Y_m)$, then $gp(Y)$ is the free abelian topological group on $Y$, $FA(Y)$, and both $FA(Y)$ and $Y$ are closed subsets of $FA(X)$.

The following new lemma will be used a number of times in this thesis.

**Notation.** Let $X$ and $Y$ be disjoint topological spaces. We shall denote by $X \cup Y$ the free union of $X$ and $Y$; that is, $X \cup Y$ is the set $X \cup Y$ with a topology $T$ such that $X$, and all its open subspaces, and $Y$, and all its open subspaces, are open in $T$. Further, for each $n = 1, 2, \ldots$, let $Y_n$ be a topological space disjoint from each of $Y_1, Y_2, \ldots, Y_{n-1}$. We shall denote by $\bigcup_{n=1}^{\infty} Y_n$ the free union over $Y_n$. 
Lemma 4.1.9. Let $X = \bigcup_{n=1}^{\infty} X_n$ be a $k_\omega$-space. For each $n$, let $Y_n$ be a space homeomorphic to $X_n$ and disjoint from each of $Y_1, Y_2, \ldots, Y_{n-1}$. Put $Y = \bigcup_{n=1}^{\infty} Y_n$, the free union of the $Y_n$. Then $FA(X)$ is a quotient group of $FA(Y)$.

Proof. For each $n \in \mathbb{N}$, let $f_n : Y_n \rightarrow X_n$ be the homeomorphism from $Y_n$ onto $X_n$. Define the mapping $\phi : Y \rightarrow X$ as follows. For each $y \in Y$, there is a unique $n \in \mathbb{N}$ such that $y \in Y_n$, so let $\phi(y) = f_n(y)$. Clearly $\phi$ is an onto mapping. We shall show it is also a quotient mapping. Let $O$ be open in $X$. Now, $\phi^{-1}(O) = \bigcup_{n=1}^{\infty} (\phi^{-1}(O) \cap Y_n)$. Further, for each $n \in \mathbb{N}$, $\phi^{-1}(O) \cap Y_n = f_n^{-1}(O)$, which is open in $Y_n$ and hence open in $Y$. Therefore, $\phi^{-1}(O)$ is the union of open sets in $Y$ and hence it is open in $Y$. So $\phi$ is continuous. Now let $U$ be a subset of $X$ such that $\phi^{-1}(U)$ is open in $Y$. Then $\phi^{-1}(U) \cap Y_n$ is open in $Y_n$ for each $n \in \mathbb{N}$. But, $f_n \left( \phi^{-1}(U) \cap Y_n \right) = U \cap X_n$ is open in $X_n$ for each $n \in \mathbb{N}$. As $X = \bigcup_{n=1}^{\infty} X_n$ is a $k_\omega$-space, $U$ is open in $X$. Thus, $\phi$ is a quotient mapping from $Y$ onto $X$ and hence, by Corollary 4.1.2, $FA(X)$ is a quotient group of $FA(Y)$. 

§4.2 Inside the Variety Generated by $FA[0,1]$ 

We begin our analysis of $\mathfrak{V}(FA[0,1])$ with some straight-forward results following from Proposition 4.2.1, which was first presented in Chapter 3.

Proposition 4.2.1. ([28], Theorem 1) The free abelian topological group on $(0,1)$ is topologically isomorphic to a subgroup of $FA[0,1]$, the free abelian topological group on $[0,1]$. 

Proposition 4.2.2. The topological groups $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{T}$ are all contained in $\mathfrak{V}(FA[0,1])$.

Proof. Noting that $\mathbb{R}$ and $(0,1)$ are homeomorphic as topological spaces, $FA(\mathbb{R})$ and $FA(0,1)$ are topologically isomorphic as topological groups. Therefore, from Proposition 4.2.1, $FA(\mathbb{R}) \in \mathfrak{V}(FA[0,1])$ and so $\mathbb{R} \in \mathfrak{V}(FA[0,1])$ as $\mathbb{R}$ is a quotient group of $FA(\mathbb{R})$. 


(Proposition 2.2.8). It follows then, that $\mathbb{Z}$ and $\mathbb{T}$ are also contained in $\mathfrak{V}(FA[0,1])$ as they are, respectively, a subgroup and a quotient group of $\mathbb{R}$. 

Using Proposition 4.2.2, we see that every abelian group appears in $\mathfrak{V}(FA[0,1])$ with some topology and so we have the following useful result.

**Corollary 4.2.3.** Every abelian group appears in $\mathfrak{V}(FA[0,1])$ with the indiscrete topology.

**Proof.** It is well-known that every abelian group $G$ is algebraically isomorphic to a subgroup of a product of copies of the group $\mathbb{T}$ ([46], Corollary to Proposition 17 and proof to Theorem 1). Thus by Proposition 4.2.2, every abelian group $G$ is contained in $\mathfrak{V}(FA[0,1])$ with some topological group topology. By Proposition 1.5.4, $G_I \in \mathfrak{V}(FA[0,1])$ where $G_I$ is the group $G$ equipped with the indiscrete topology.

Recall that every compact Hausdorff abelian topological group is a subgroup of a product of copies of $\mathbb{T}$ ([46], Corollary 1 to Theorem 14). Thus, we have that every compact Hausdorff abelian group is contained in $\mathfrak{V}(FA[0,1])$.

**Corollary 4.2.4.** Every compact abelian topological group is in $\mathfrak{V}(FA[0,1])$.

**Proof.** Noting that every compact Hausdorff abelian group $G$ is topologically isomorphic to a subgroup of a product of copies of $\mathbb{T}$ ([46], Corollary 1 to Theorem 14), it follows from Proposition 4.2.2 that $G \in \mathfrak{V}(FA[0,1])$. Now, let $H$ be a compact abelian group which is not necessarily Hausdorff. Then by Proposition 2.4.2, $H$ is topologically isomorphic to a subgroup of $H \setminus \{e\} \times |H|_I$ where $|H|_I$ is the group underlying $H$ with the indiscrete topology. Clearly $H \setminus \{e\}$ is compact Hausdorff abelian and therefore contained in $\mathfrak{V}(FA[0,1])$. Further, by Corollary 4.2.3, $|H|_I \in \mathfrak{V}(FA[0,1])$, giving $H \in \mathfrak{V}(FA[0,1])$. 

As every finite group is compact, the next corollary follows immediately from Corollary 4.2.4.
Corollary 4.2.5. Every finite abelian group is in $\mathfrak{V}(FA[0, 1])$ with some topology.

In Chapter 3 we saw that $FA[0, 1] \not\subseteq \mathfrak{V}(\mathcal{L}_A)$ and so $\mathfrak{V}(FA[0, 1]) \not\subseteq \mathfrak{V}(\mathcal{L}_A)$. However, given Proposition 4.2.2, we have $\mathfrak{V}(\mathbb{R}) \subseteq \mathfrak{V}(\mathcal{L}_A) \cap \mathfrak{V}(FA[0, 1])$ and this leads to the following result concerning the connected groups in $\mathfrak{V}(\mathcal{L}_A)$.

Proposition 4.2.6. Every connected abelian topological group $G$ in $\mathfrak{V}(\mathcal{L}_A)$ is in $\mathfrak{V}(FA[0, 1])$.

Proof. By Corollary 3.1.8, $G$ contained in $\mathfrak{V}(\mathcal{L}_A)$ is in $\mathfrak{V}(\mathbb{R})$. Therefore, by Proposition 4.2.2, $G$ is also contained in $\mathfrak{V}(FA[0, 1])$.

Recall that $FA[0, 1]$ is a $k_\omega$-group and so a number of results from Section 3.3 apply here. For example, Theorem 3.3.21 implies that any normed vector space contained in $\mathfrak{V}(FA[0, 1])$ is finite dimensional. Also, from Proposition 3.3.7, any discrete abelian group contained in $\mathfrak{V}(FA[0, 1])$ is countable. However, we will not spend any more time on this type of application as all results concerning $\mathfrak{V}(\mathcal{K}_\omega)$ will apply to $\mathfrak{V}(FA[0, 1])$ in the most unexpected way.

For the moment, we shall turn to the question of which locally compact abelian topological groups, apart from countable discrete abelian groups and connected groups, are contained in $\mathfrak{V}(FA[0, 1])$. An application of Remark 3.3.14 to $\mathfrak{V}(FA[0, 1])$ immediately tells us that any locally compact Hausdorff abelian topological group $G$ contained in $\mathfrak{V}(FA[0, 1])$ must be $\sigma$-compact; that is, $G$ must be a $k_\omega$-group as every locally compact Hausdorff $\sigma$-compact group is a $k_\omega$-group (see Theorem 3.3.13). We in fact will show every metrizable locally compact $\sigma$-compact abelian group is contained in $\mathfrak{V}(FA[0, 1])$. However, we first make a number of comments that shall be useful in proving this proposition.

Remark 4.2.7. Let $X$ be a topological space contained in a topological group $G$ such that $G$ has the finest group topology that induces the given topology on $X$. Let $Y$ be
a topological space such that $X \subseteq Y$ and $Y \subseteq G$. Then $G$ clearly has the finest group topology that induces the given topology on $Y$, for if there were a finer topology on $G$ that induces the given topology on $Y$, it would also induce the given topology on $X$ and we would have a contradiction.

**Remark 4.2.8.** Let $D$ be a countable discrete topological space, $K$ a compact Hausdorff space and $n$ a nonnegative integer. Then from the following argument, we have that the free union $(\mathbb{R}^n \times K) \cup D$ is a closed subspace of $\mathbb{R}^{n+1} \times K$.

\[
\mathbb{R}^n \times K \cup D \subseteq (\mathbb{R}^n \times K) \cup (D \times K) \\
= (\mathbb{R}^n \cup D) \times K \\
\subseteq (\mathbb{R} \times \mathbb{R}^{n-1} \cup D \times \mathbb{R}^{n-1}) \times K \\
= (\mathbb{R} \cup D) \times \mathbb{R}^{n-1} \times K \\
\subseteq \mathbb{R}^2 \times \mathbb{R}^{n-1} \times K
\]
since $D$ is countably infinite discrete, and we can identify $x \in \mathbb{R}$ with $(x, 0)$ and $d_i \in D$ with $(0, i)$, $i$ a positive integer and each containment is a closed subspace.

**Proposition 4.2.9.** Every metrizable locally compact abelian $\sigma$-compact group $G$ is in $\overline{Q}(\mathbb{F}A[0, 1])$ and therefore also in $\mathfrak{B}(\mathbb{F}A[0, 1])$.

**Proof.** Let $G$ be a metrizable locally compact abelian Hausdorff $\sigma$-compact group. By the Principle Structure Theorem, $G$ contains an open subgroup $H$ topologically isomorphic to $\mathbb{R}^n \times K$ where $K$ is compact and $n$ is a nonnegative integer. Further, $K$ is metric and $G/H$ is discrete $\sigma$-compact, and hence countable. Note that $G$ is the finest topology which induces the given topology on $H$. Choose one element out of each coset of $H$ different from $H$ and form the set $D$ which is clearly discrete countable. Clearly $H$ and $D$ are disjoint, so $H \cup D = H \cup D$ is a subspace of $G$, indeed, by Remark 4.2.7, $G$ has the finest group topology which induces the given topology on $H \cup D$. Further, $G = \text{gp}(H \cup D)$ and so by Corollary 4.1.3, $G$ is a quotient group of $\mathbb{F}A(H \cup D)$. By Remark 4.2.8, $H \cup D$ is a closed subspace of $\mathbb{R}^{n+1} \times K$ and applying Lemma 3.3.2 we see that $\mathbb{F}A(H \cup D)$ is topologically isomorphic to a subgroup of $\mathbb{F}A(\mathbb{R}^{n+1} \times K)$. As $K$ is compact metric, it is a continuous
image of $C$, the Cantor space $\{0,1\}^{\mathbb{N}_0}$ ([29], Problem O(e), Chapter 5). Indeed, as $C$ is compact Hausdorff, $K$ is a quotient space of $C$ (cf. [29], Chapter 3, Theorem 8) and so $\mathbb{R}^{n+1} \times K$ is a quotient space of $\mathbb{R}^{n+1} \times C$. Therefore, by Corollary 4.1.2, $FA(\mathbb{R}^{n+1} \times K)$ is a quotient group of $FA(\mathbb{R}^{n+1} \times C)$. Note that $C$ is a closed subspace of $\mathbb{R}$, thus, $\mathbb{R}^{n+1} \times C$ is a closed subspace of $\mathbb{R}^{n+2}$, giving $FA(\mathbb{R}^{n+1} \times C)$ a subgroup of $FA(\mathbb{R}^{n+2})$ by Lemma 3.3.2. Finally, by Theorem 4.1.7, $FA(\mathbb{R}^{n+2})$ is topologically isomorphic to a subgroup of $FA[0,1]$. Thus, $G \in \mathcal{Q}S(FA[0,1]) = \mathcal{Q}S(FA[0,1])$ and so $G \in \mathcal{Q}S(FA[0,1])$.

Recall that every locally compact abelian $\sigma$-compact group is a $k_\omega$-group. Therefore, Proposition 4.2.9 essentially says that every locally compact abelian metrizable group that is also a $k_\omega$-group is contained in $\mathcal{Q}(FA[0,1])$. We can in fact generalize this to show that every submetrizable $k_\omega$-group is contained in $\mathcal{Q}(FA[0,1])$—to which Proposition 4.2.9 is a direct consequence. However, the proof to Proposition 4.2.9 relied heavily on structure theory for locally compact abelian groups, while the proof of the generalized result does not.

**Proposition 4.2.10.** Let $X$ be a submetrizable $k_\omega$-space. Then $FA(X) \in \mathcal{Q}S(FA[0,1])$.

**Proof.** Let $X = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is compact metric. Further, let $Y$ be the free union $\bigcup_{n=1}^{\infty} Y_n$ where each $Y_n$ is homeomorphic to $X_n$, and disjoint from each of $Y_1, Y_2, \ldots, Y_{n-1}$. Then by Lemma 4.1.9, $FA(X)$ is a quotient group of $FA(Y)$. Now, for each $n \in \mathbb{N}$, $Y_n$ is a quotient space of $C = \{0,1\}^{\mathbb{N}_0}$, the Cantor space. Thus, let $B = \bigcup_{n=1}^{\infty} B_n$ where each $B_n$ is a homeomorphic copy of $C$ contained in $[n + \frac{1}{4}, n + \frac{3}{4}]$. In a method similar to that in Lemma 4.1.9 it can be shown that $Y$ is a quotient space of $B$, giving $FA(Y)$ a quotient group of $FA(B)$. Therefore, $FA(X)$ is a quotient group of $FA(B)$. Now, for each $n \in \mathbb{N}$, $B_n$ is a closed subspace of $[n + \frac{1}{4}, n + \frac{3}{4}]$ and so $B$ is homeomorphic to a closed subspace of $\mathbb{R}$. Thus, by Lemma 3.3.2, $FA(B)$ is a subgroup of $FA(\mathbb{R})$, which is also a subgroup of $FA[0,1]$ (see Proposition 4.2.1). Finally, note that $FA(B)$ and $FA(\mathbb{R})$ are both $k_\omega$-groups (see Proposition 2.2.12 and [34], Corollary 1) and hence complete ([22], Theorem 2). Therefore, $FA(B)$ is a closed subgroup of $FA(\mathbb{R})$ and $FA(\mathbb{R})$ is a closed subgroup of $FA[0,1]$. So, we have $FA(X) \in \mathcal{Q}S(FA[0,1])$. ■
We can now deduce the following characterization of $\overline{Q} \overline{S}(FA[0,1])$.

**Theorem 4.2.11.** The class of all abelian submetrizable $k_\omega$-groups, $\mathcal{K}$, is precisely $\overline{Q} \overline{S}(FA[0,1])$. In particular, $FA(X) \in \overline{Q} \overline{S}(FA[0,1])$ if and only if $X$ is a submetrizable $k_\omega$-space. Further, each submetrizable abelian $k_\omega$-group is in $\mathcal{W}(FA[0,1])$.

**Proof.** Firstly, let $G \in \mathcal{K}$. By Proposition 4.2.10, $FA(G) \in \overline{Q} \overline{S}(FA[0,1])$. Further, by Proposition 2.2.8, $G$ is a quotient group of $FA(G)$ and so $G \in \overline{Q} \overline{S}(FA[0,1])$ giving $\mathcal{K} \subseteq \overline{Q} \overline{S}(FA[0,1])$. Conversely, let $G \in \overline{Q} \overline{S}(FA[0,1])$. Then there exists a submetrizable $k_\omega$-group $H = \bigcup_{n=1}^{\infty} H_n$ such that $\phi : H \to G$ is a quotient homomorphism, and the kernel of $\phi$ is a closed subgroup of $H$. We note that the $k_\omega$-decomposition of $G$ is given by $G = \bigcup_{n=1}^{\infty} \phi(H_n)$. Further, $\phi : H_n \to \phi(H_n)$ is a quotient mapping and as $\phi(H_n)$ is Hausdorff, $\phi(H_n)$ is compact metric. Thus, $G$ is a submetrizable $k_\omega$-group and we have $\mathcal{K} = \overline{Q} \overline{S}(FA[0,1]) \subseteq \mathcal{W}(FA[0,1])$.

Next, by Proposition 4.2.10, for $X$ a submetrizable $k_\omega$-space, $FA(X) \in \overline{Q} \overline{S}(FA[0,1])$. Conversely, let $FA(X) \in \overline{Q} \overline{S}(FA[0,1])$ for some completely regular Hausdorff space $X$. By Corollary 2.4.6, $X$ is a closed subspace of $FA(X)$. Finally, noting that closed subspaces and Hausdorff quotient spaces of submetrizable $k_\omega$-spaces are also submetrizable $k_\omega$-spaces, we have that $X$ is a submetrizable $k_\omega$-space and the result is proven. 

Theorem 4.2.11 should be contrasted with the Leiderman, Pestov and Morris result that $FA(X) \in \overline{S}(FA[0,1])$ if and only if $X = \bigcup_{n=1}^{\infty} X_n$ is a submetrizable $k_\omega$-space with each $X_n$ finite dimensional (see Theorem 4.1.7).

As finite products of submetrizable $k_\omega$-groups are also submetrizable $k_\omega$-groups, it is also true that $\overline{Q} \overline{S}(FA[0,1]) = \overline{Q} \overline{SP}(FA[0,1])$. A natural question to ask at this point is whether the variety generated by $FA[0,1]$ in fact contains the class of all $k_\omega$-groups. We will address this problem shortly.
§4.3 The Metrification Mechanism

The Metrification Mechanism proved next allows us to reduce many problems to the metric case, though why this is the case is not immediately evident.

Theorem 4.3.1: Metrification Mechanism. Let \((X, \rho)\) be a pseudometrizable topological space. Then \((X, \rho)\) is a subspace of the product of a metrizable space \((Y, d)\) and the set \(X\) with the indiscrete topology.

Proof. Define the equivalence relation \(\sim\) on \(X\) by \(x \sim y\) if and only if \(\rho(x, y) = 0\), for \(x, y \in X\) and let \([x] = \{y \in X : \rho(x, y) = 0\}\) denote the equivalence class of \(x\) under \(\sim\). Let \(Y\) be the set of all equivalence classes and define \(f : X \to Y\) by \(f(x) = [x]\) for all \(x \in X\). We note that \(f\) is surjective and we put the quotient topology on \(Y\) so that \(U\) is open in \(Y\) if and only if \(f^{-1}(U)\) is open in \(X\). Consider \(O\) open in \(X\). Then 

\[
\{y \in f^{-1}(f(O)) : f(x) = [x] \text{ for some } x \in O, \text{ that is, } \rho(x, y) = 0. \text{ Now, } x \in O \text{ and so there exists } \alpha > 0 \text{ such that } x \in B_\alpha(x, \rho) \subseteq O, \text{ where } B_\alpha(x, \rho) = \{z \in X : \rho(x, z) < \alpha\}. \text{ Clearly, } y \in B_\alpha(x, \rho) \text{ also, and hence } y \in O. \text{ Therefore, } f^{-1}(f(O)) = O, \text{ which is open. Therefore, } f(O) \text{ is open in } Y \text{ and so } f \text{ is an open mapping.}
\]

We define \(d\) on \(Y\) by 

\[
d([x], [y]) = \inf \{\rho(a, b) : a \in [x], b \in [y]\}
\]

for all \([x], [y] \in Y\). However, we note that for \(a \in [x]\) and \(b \in [y]\),

\[
\rho(a, b) \leq \rho(a, x) + \rho(x, y) + \rho(y, b) = \rho(x, y)
\]

and

\[
\rho(x, y) \leq \rho(x, a) + \rho(a, b) + \rho(b, y) = \rho(a, b).
\]

giving \(\rho(a, b) = \rho(x, y)\). Therefore, our definition for \(d\) reduces to \(d([x], [y]) = \rho(x, y)\) for each \([x], [y] \in Y\). From the definition, \(d\) is clearly a pseudometric. To see that \(d\) is a metric, we take \([x], [y] \in Y\) with \(d([x], [y]) = \rho(x, y) = 0\). Then \(x \in [y]\) and so \([x] = [y]\).

Next we need to show that \(d\) defines the topology on \(Y\). Let \(x \in X\) with \([x] \in Y\). Consider \(B_\alpha(x, \rho) = \{z \in X : \rho(x, z) < \alpha\}\) the open sphere of radius \(\alpha\) about \(x \in X\) under \(\rho\). Let
$B'_\alpha([x],d) = \{ [y] \in Y : d([x],[y]) < \alpha \}$ denote the open sphere of radius $\alpha$ about $[x] \in Y$ under $d$. To prove the result, it is enough to show that $B'_\alpha([x],d) = f(B_\alpha(x,\rho))$. Now, $f(B_\alpha(x,\rho)) = \{ [z] : \rho(x,z) < \alpha \}$ and $B'_\alpha([x],d) = \{ [y] \in Y : \rho(x,y) < \alpha \}$ and so the two sets are equal.

Finally, we shall show that $(X,\rho)$ is indeed homeomorphic to a subspace of the product $H = (Y,d) \times X_I$ where $X_I$ is the set $X$ with the indiscrete topology. Consider the mapping $g : (X,\rho) \to H$ given by $g(x) = (f(x),x)$ for each $x \in X$. If $g(x_1) = g(x_2)$, then $\langle f(x_1),x_1 \rangle = \langle f(x_2),x_2 \rangle$ and so $x_1 = x_2$ showing that $g$ is one-to-one. Let $U$ be an open set in $H$. Then $U = O_1 \times O_2$ where $O_1$ is open in $(Y,d)$ and $O_2$ is either $\emptyset$ or $X$. Now, if $O_2 = \emptyset$, then $g^{-1}(U) = \emptyset$, which is open in $(X,\rho)$. On the other hand, if $O_2 = X$, then $g^{-1}(U) = f^{-1}(O_1)$, which is open in $(X,\rho)$. Therefore, $g$ is continuous. Finally, if $O$ is an open set in $(X,\rho)$, then $g(O) = (f(O) \times O) \cap g(X) = (f(O) \times X) \cap g(X)$, which is open in $g(X)$. Therefore, $(X,\rho)$ is homeomorphic to $g(X)$, a subspace of $H$.

**Notation.** If $(X,\rho)$ and $(Y,d)$ are as in the Metrification Mechanism, we refer to $(Y,d)$ as the metrification of $(X,\rho)$.

**Remarks 4.3.2.**

(a) We note that the metrification $(Y,d)$ of a pseudometrizable space $(X,\rho)$ is indeed a quotient space of $(X,\rho)$.

(b) Given the Metrification Mechanism for $(X,\rho)$, a pseudometrizable topological space, there exists a metrizable topological space, $Y$, such that $FA(Y)$ is a quotient of $FA(X)$. This is simply an application of the Metrification Mechanism and Corollary 4.1.2.

(c) In Proposition 2.4.2 we saw that a topological group $G$ is a subgroup of the product of the Hausdorff topological group, $G \setminus \{e\}$ and $|G|_I$, the group $|G|$ with the indiscrete topology. Further, in Lemma 2.5.14, we showed that for a pseudometrizable topological group $(G,\rho)$, the quotient group $G \setminus \{e\}$ is a metrizable topological group.
Indeed, noting that \( \overline{\{e\}} = \{x : \rho(x, e) = 0\} \), it is clear that \( G / \overline{\{e\}} \) with the metric defined in Lemma 2.5.14 is the metrification of \((G, \rho)\). Therefore, the metrification of a pseudometrizable topological group is a topological group.

For \( X \) a completely regular space, we wish to study the metrification of \((|FA(X)|, \rho')\) where \(|FA(X)|\) is the group underlying \(FA(X)\) and \(\rho'\) is the Graev extension of \(\rho\), a pseudometric on \( X \). To do this we need the following lemma.

**Lemma 4.3.3.** Let \((X, \rho)\) be a pseudometric space and let \(\rho'\) be the Graev extension of \(\rho\) to \(FA(X)\), the free abelian group on \(X\). Let \(x = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, \varepsilon_i = \pm 1\) for \(i = 1, \ldots, n\), be the reduced representation of a word in \(FA(X)\) such that \(\rho'(x, e) = 0\) and for each \(i = 1, \ldots, n, \rho(x_i, e) \neq 0\). Then \(x\) can be written the form \( x = u_1u_2 \cdots u_m \) where each \(u_i = z_1^{s_1}z_2^{-1}, z_1 \in \{x_i : \varepsilon_i = 1\} \subseteq X\) and \(z_{i_2} \in \{x_j : \varepsilon_j = -1\} \subseteq X\) and \(\rho(z_{i_1}, z_{i_2}) = 0\).

**Proof.** We first recall that for the Graev extension of \(\rho\),

\[
\rho'(x, e) = \inf \left\{ \sum_{i=1}^m \rho(a_i^{\eta_i}, b_i^{\gamma_i}) : x = a_1^{\eta_1} \cdots a_m^{\eta_m}, e = b_1^{\gamma_1} \cdots b_m^{\gamma_m} \right\}.
\]

Further, this infimum is achieved and the representations for \(x\) and \(e\) that give this infimum have length at most \(n + 1\) and use only the letters \(x_1, \ldots, x_n, e\) (see Remark 2.2.4). We also note that if \(a_1^{\eta_1} \cdots a_{n+1}^{\eta_{n+1}}\) and \(b_1^{\gamma_1} \cdots b_{n+1}^{\gamma_{n+1}}\) are representations for \(x\) and \(e\) respectively which give \(\rho'(x, e) = 0\), then for each \(i = 1, \ldots n + 1, \rho(a_i^{\eta_i}, b_i^{\gamma_i}) = 0\). Using these constraints we will construct our new representations for \(x\) and \(e\) that give \(\rho'(x, e) = 0\).

Take any \(x_i^{\varepsilon_i}\) that is found in the reduced representation for \(x\) and place it in the new representations for both \(x\) and \(e\). We now need to place an element in the new representation for \(e\) that will cancel \(x_i^{-\varepsilon_i}\), namely \(x_i^{-\varepsilon_i}\). Further, the corresponding element we place in the new representation for \(x\) must be zero distance from \(x_i^{-\varepsilon_i}\). Choose \(x_j^{\varepsilon_j}\) from the reduced representation of \(x\) such that \(\varepsilon_j = -\varepsilon_i\) and \(\rho(x_i, x_j) = 0\). We note that given the constraints mentioned earlier, either \(x_j^{\varepsilon_j}\) will exist or \(\rho(x_i, e) = 0\). However, \(\rho(x_i, e) \neq 0\), therefore \(x_j^{\varepsilon_j}\) must exist, or we would obtain a non-zero value for \(\rho'(x, e)\). Place \(x_i^{-\varepsilon_i}\) in the new representation for \(e\) and place \(x_j^{\varepsilon_j}\) in the new representation for \(x\) over \(x_i^{-\varepsilon_i}\), so we have
We continue in this way until all the letters in the reduced representation for $x$ have been placed in a pair in the new representation for $x$. We note that each pairing has one of two forms:

$$
\begin{align*}
(1) & \quad z_1 \ z_2^{-1} \quad \text{or} \quad (2) \quad z_1^{-1} \ z_2 \\
(2') & \quad z_2 \ z_1^{-1}
\end{align*}
$$

where $z_1, z_2 \in \{x_1, \ldots, x_n\}$, $\rho(z_1, z_2) = 0$ and $z_1, z_2^{-1}$ appear in the reduced representation of $x$. In the case of (2), use the commutativity of $FA(X)$ to write as

$$
(2') \quad z_2 \ z_1^{-1}.
$$

We now have representations for $x$ and $e$ of the form

$$
\begin{align*}
x : & \quad z_1 \ z_2^{-1} \ z_3 \ z_4^{-1} \ \cdots \ \ z_{2m-1} \ z_{2m}^{-1} \\
e : & \quad z_1 \ z_1^{-1} \ z_3 \ z_3^{-1} \ \cdots \ \ z_{2m-1} \ z_{2m-1}^{-1}
\end{align*}
$$

where for each $i = 1, \ldots, m$, $\rho(z_{2i-1}, z_{2i}) = 0$, with $z_{2i-1} \in \{x_i : \epsilon_i = 1\} \subseteq X$ and $z_{2i} \in \{x_j : \epsilon_j = -1\} \subseteq X$. If for each $i = 1, 2, \ldots, m$, we let $u_i = z_{2i-1}z_{2i}^{-1}$, then $x = u_1u_2 \ldots u_m$ and the result follows.

The following proposition applies the Metrification Mechanism to $(X, \rho)$, a pseudometrizable topological space, and $(|FA(X)|, \rho')$, the free abelian group on $X\setminus\{e\}$ with the Graev extension of $\rho$. We see that if $(Y, d)$ is the metrification of $(X, \rho)$ and we take the free abelian algebraic group on $Y$ with the Graev extension of $d$, the result is exactly the metrification of $(|FA(X)|, \rho')$.

**Proposition 4.3.4.** Let $X$ be a pseudometrizable topological space, with pseudometric $\rho$. Let $(Y, d)$ be the metrification of $(X, \rho)$. Let $\rho'$ and $d'$ be the Graev extensions of $\rho$ and $d$ to $F_X$, the group underlying $FA(X, \rho)$, and $F_Y$, the group underlying $FA(Y, d)$, respectively. Then the metrification of $(F_X, \rho')$ is a topological group and is topologically isomorphic to $(F_Y, d')$; indeed $(F_Y, d')$ is isometrically isomorphic to the metrification of $(F_X, \rho')$. 
Proof. The notation for \((X, \rho)\) and \((Y, d)\) will be as in the Metrification Mechanism, Theorem 4.3.1. Note that \([e] \in Y\) is the identity element in \(F_y\). We shall denote by \(\left(\frac{F_y}{\{e\}}, h\right)\) the metrification of \((F_x, \rho')\) as per Remark 4.3.2(c).

Let \(w \in F_Y\) have reduced representation \(w = [x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n}\), where \(e_i = \pm 1\) for each \(i = 1, \ldots, n\). We define the map \(f : F_Y \to F_X / \{e\}\) by

\[
f([x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n}) = \{x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}\}.
\]

We need to show that for each \(w \in F_Y\), \(f(w)\) is independent of the choice of coset representative of \([x_1], \ldots, [x_n]\); that is, \(f\) is well-defined. Firstly, let \(w = [x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n}\) in \(F_Y\) with \(e_i = \pm 1\) for each \(i = 1, \ldots, n\) be the reduced representation of a word in \(F_Y\).

Further, let \([y_1]^{e_1} [y_2]^{e_2} \ldots [y_n]^{e_n}\) be another reduced representation for \(w\); that is, for each \(i = 1, \ldots, n\), \([x_i] = [y_i]\) and for some (even all) \(i, x_i \neq y_i\). Then, for each \(i = 1, \ldots, n\),

\[
\rho'(x_i, y_i) = 0 = \rho'(x_i^{e_i}, y_i^{e_i}) \quad \text{and so} \quad x_i^{e_i} [e] = y_i^{e_i} [e].
\]

Thus, we have

\[
f([x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n}) = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} [e]
\]

and so \(f\) is well-defined. Clearly \(f\) is a group homomorphism. We shall show that \(f\) is a topological group isomorphism from \((F_Y, d')\) onto \(\left(\frac{F_X}{\{e\}}, h\right)\).

First, take \(x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} [e] \in F_X / \{e\}\), \(e_i = \pm 1\) for \(i = 1, \ldots, n\). Then we have \([x_1], [x_2], \ldots, [x_n] \in Y\) and so \([x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n} \in F_Y\) with

\[
f([x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n}) = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} [e].
\]

Therefore, \(f\) is surjective.

To show \(f\) is one-to-one, let

\[
w_1 = [x_1]^{e_1} [x_2]^{e_2} \ldots [x_n]^{e_n}, \quad e_i = \pm 1 \text{ for each } i = 1, \ldots, n
\]

and

\[
w_2 = [y_1]^{\eta_1} [y_2]^{\eta_2} \ldots [y_m]^{\eta_m}, \quad \eta_i = \pm 1 \text{ for each } i = 1, \ldots, m,
\]

be words in \(F_Y\) in their reduced representation. We note that if \([x_i] = [x_j], \ i \neq j\), then \(e_i = e_j\) and similarly, \([y_i] = [y_j]\) implies \(\eta_i = \eta_j\). Also, \(\rho(x_i, e) \neq 0\) for each
\[ i = 1, \ldots, n \text{ and } \rho(y_j, e) \neq 0 \text{ for each } j = 1, \ldots, m. \] Now, let \( f(w_1) = f(w_2). \) Then

\[ x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n} \{e\} = y_1^{\eta_1}y_2^{\eta_2} \cdots y_m^{\eta_m} \{e\} \text{ and so } \rho'(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}, y_1^{\eta_1}y_2^{\eta_2} \cdots y_m^{\eta_m}) = 0. \] By the invariance of \( \rho', \rho'(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_m^{-\eta_m} \cdots y_2^{-\eta_2}y_1^{-\eta_1}, e) = 0. \] By Lemma 4.3.3, the word

\[ x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_m^{-\eta_m} \cdots y_2^{-\eta_2}y_1^{-\eta_1} \] can be written in the form \( u_1u_2 \cdots u_q \) where \( u_i = z_{2i-1}z_{2i}^{-1} \) such that \( \rho(z_{2i-1}, z_{2i}) = 0 \) with \( z_{2i-1} \in \{ \epsilon_i : \epsilon_i = 1 \} \cup \{ \eta_j : \eta_j = -1 \} \subseteq X \) and \( z_{2i} \in \{ \epsilon_i : \epsilon_i = -1 \} \cup \{ \eta_j : \eta_j = 1 \} \subseteq X. \) Suppose for some \( u_i, z_{2i-1} = \epsilon_i \) and \( z_{2i} = \eta_j, i \neq j. \) Clearly \( \epsilon_i \neq \eta_j. \) However, as \( \rho(z_{2i-1}, z_{2i}) = 0, [z_{2i-1}] = [z_{2i}], \) that is \( [\epsilon_i] = [\eta_j], \) implying \( \epsilon_i = \eta_j. \) This is contradiction, so for each \( u_i, \) both \( z_{2i-1} \) and \( z_{2i} \) cannot be letters from \{\( x_1, \ldots, x_n\}\}. Similarly, for each \( u_i, \) both \( z_{2i-1} \) and \( z_{2i} \) cannot be letters from \{\( y_1, \ldots, y_m\}\}. Therefore, for each \( u_i \) one letter is from \( x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n} \) and the other from \( y_m^{-\eta_m} \cdots y_2^{-\eta_2}y_1^{-\eta_1}. \) This implies that each \( x_i^{e_i} \) is paired with a \( y_j^{-\eta_j} \) such that \( \rho(x_i, y_j) = 0, \) giving \([x_i] = [y_j].\) Further, \( m = n \) and using commutativity of \( F_y, \) we can form representations of \( w_1 \) and \( w_2 \) by taking one letter from each pair \([x_i] = [y_i].\) This gives \( w_1 = w_2 \) and so \( f \) is one-to-one.

To complete the proof, we show that \( d' \) and \( h \) are isometric. We must show that for two words \( w_1 \) and \( w_2 \) in \( F_y, \) the equality \( d'(w_1, w_2) = h(f(w_1), f(w_2)) \) holds. Note that for \( x(e) \) and \( y(e) \) in \( F_X \setminus \{e\}, h(x(e), y(e)) = \rho'(x, y). \)

Firstly, we must prove that \( \rho'(x^\epsilon, y^\eta) = d'([x]^\epsilon, [y]^\eta), \) where \( x^\epsilon, y^\eta \in X \cup X^{-1} \) and \( [x]^\epsilon, [y]^\eta \in Y \cup Y^{-1}. \) In the case that \( \epsilon = \eta, \)

\[ \rho'(x^\epsilon, y^\eta) = \rho(x, y) \]

\[ = d([x], [y]) \]

\[ = d'([x]^\epsilon, [y]^\eta). \]

If \( \epsilon \neq \eta, \)

\[ \rho'(x^\epsilon, y^\eta) = \rho(x, e) + \rho(y, e) \]

\[ = d([x], [e]) + d([y], [e]) \]

\[ = d'([x]^\epsilon, [y]^\eta). \]

Next, we will show that for a word \( w \in F_y, \) \( d'(w, [e]) = h(f(w), f([e])). \) Let the reduced representation of \( w \) be given by \( w = [x_1]^{e_1}[x_2]^{e_2} \cdots [x_n]^{e_n}, \epsilon_i = \pm 1 \text{ for } i = 1, \ldots, n \) and let

\[ w = [a_1]^{\eta_1}[a_2]^{\eta_2} \cdots [a_{n+1}]^{\eta_{n+1}}, \eta_i = \pm 1 \text{ for } i = 1, \ldots, n + 1 \] and

\[ [e] = [b_1]^\xi_1[b_2]^\xi_2 \cdots [b_{n+1}]^\xi_{n+1}, \xi_i = \pm 1 \text{ for } i = 1, \ldots, n + 1 \]
be the representations for $w$ and $[e]$ respectively, such that
\[ d'(w, [e]) = \sum_{i=1}^{n+1} d'([a_i]^{\gamma_i}, [b_i]^{\epsilon_i}). \]

As $\rho'(a_i^{\eta_i}, b_i^{\xi_i}) = d'([a_i]^{\eta_i}, [b_i]^{\xi_i})$, we have
\[ d'(w, [e]) = \sum_{i=1}^{n+1} \rho'(a_i^{\eta_i}, b_i^{\xi_i}) \geq \rho'(a_1^{\eta_1} a_2^{\eta_2} \ldots a_{n+1}^{\eta_{n+1}}, b_1^{\xi_1} b_2^{\xi_2} \ldots b_{n+1}^{\xi_{n+1}}). \]

Now,
\[
\begin{align*}
 f(w) &= x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n} \{e\} \\
 &= a_1^{\eta_1} a_2^{\eta_2} \ldots a_{n+1}^{\eta_{n+1}} \{e\},
\end{align*}
\]
and
\[
\begin{align*}
 f([e]) &= \{e\} \\
 &= b_1^{\xi_1} b_2^{\xi_2} \ldots b_{n+1}^{\xi_{n+1}} \{e\}.
\end{align*}
\]
Further,
\[
h(f(w), f([e])) = h(a_1^{\eta_1} a_2^{\eta_2} \ldots a_{n+1}^{\eta_{n+1}} \{e\}, b_1^{\xi_1} b_2^{\xi_2} \ldots b_{n+1}^{\xi_{n+1}} \{e\})
\]
\[
= \rho'(a_1^{\eta_1} a_2^{\eta_2} \ldots a_{n+1}^{\eta_{n+1}}, b_1^{\xi_1} b_2^{\xi_2} \ldots b_{n+1}^{\xi_{n+1}})
\]
and hence $d'(w, [e]) \geq h(f(w), f([e]))$.

Conversely, let
\[
\begin{align*}
 c_1^{\gamma_1} c_2^{\gamma_2} \ldots c_{n+1}^{\gamma_{n+1}}, \quad \gamma_i = \pm 1 \text{ for } i = 1, \ldots, n + 1 \quad \text{and} \\
 d_1^{\xi_1} d_2^{\xi_2} \ldots d_{n+1}^{\xi_{n+1}}, \quad \epsilon_i = \pm 1 \text{ for } i = 1, \ldots, n + 1
\end{align*}
\]
be the representations for $x = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$ and $e$ (in $F_X$) respectively, such that
\[
\rho'(x, e) = \sum_{i=1}^{n+1} \rho'(c_i^{\gamma_i}, d_i^{\xi_i})
\]
\[
= \sum_{i=1}^{n+1} d'([c_i]^{\gamma_i}, [d_i]^{\xi_i}) \geq d'([c_1]^{\gamma_1} [c_2]^{\gamma_2} \ldots [c_{n+1}]^{\gamma_{n+1}}, [d_1]^{\xi_1} [d_2]^{\xi_2} \ldots [d_{n+1}]^{\xi_{n+1}}).
\]
Now as $c_1^{\gamma_1} c_2^{\gamma_2} \ldots c_{n+1}^{\gamma_{n+1}} = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$, we have
\[
\begin{align*}
 f([c_1]^{\gamma_1} [c_2]^{\gamma_2} \ldots [c_{n+1}]^{\gamma_{n+1}}) &= c_1^{\gamma_1} c_2^{\gamma_2} \ldots c_{n+1}^{\gamma_{n+1}} \{e\} \\
 &= x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n} \{e\} \\
 &= f([x_1]^{\epsilon_1} [x_2]^{\epsilon_2} \ldots [x_n]^{\epsilon_n}) \\
 &= f(w).
\end{align*}
\]
Therefore, \( w = [c_1]^{\gamma_1} [c_2]^{\gamma_2} \ldots [c_{n+1}]^{\gamma_{n+1}} \). Similarly, \([e] = [d_1]^{\epsilon_1} [d_2]^{\epsilon_2} \ldots [d_{n+1}]^{\epsilon_{n+1}}\) and we have

\[
\begin{align*}
& h(f(w), f([e])) = h(x(e), e) \\
& = \rho'(x, e) \\
& \geq d'(w, [e]),
\end{align*}
\]

giving equality.

Finally, let \( w_1, w_2 \in FA(Y) \). Then

\[
d'(w_1, w_2) = d'(w_1w_2^{-1}, [e]) \\
= h(f(w_1w_2^{-1}), f([e])) \\
= h(f(w_1)f(w_2)^{-1}, [e]) \\
= h(f(w_1), f(w_2)).
\]

Proposition 4.3.4 deals only with the metrification of \((|FA(X)|, \rho')\), that is, it deals only with the metrification of \(|FA(X)|\) with one of the defining pseudometrics. We now establish the connection between \(FA(X)\) and a family of free abelian topological groups on metric spaces.

**Proposition 4.3.5.** Let \( X \) be a topological space whose topology is induced by the family of pseudometrics, \( \{\rho_i : i \in I\} \), where \( I \) is some index set. Then \( FA(X) \) is a subgroup of a product of free abelian topological groups on metric spaces and an indiscrete abelian group.

**Proof.** For each \( i \in I \), let \((Y_i, d_i)\) be the metrification of \((X, \rho_i)\). Let \((|FA(X)|, \rho'_i)\) be the free abelian group on \( X \setminus \{e\} \) with \( \rho'_i \) the Graev extension of the pseudometric \( \rho_i \). Further, let \((|FA(Y_i)|, d'_i)\) be the free abelian group on \( Y_i \setminus \{[e]_i\} \) with \( d'_i \) the Graev extension of the metric \( d_i \). By Proposition 4.3.4, \((|FA(Y_i)|, d'_i)\) is the metrification of \((|FA(X)|, \rho'_i)\) and there exists a topological group embedding \( g_i : (|FA(X)|, \rho'_i) \to (|FA(Y_i)|, d'_i) \times K_i \) where \( K_i \) is \(|FA(X)|\) with the indiscrete topology, such that \( g_i(X) \) is a subspace of \( Y_i \times K_i \) (that is, \( X \) is topologically isomorphic to a subspace of \( Y_i \times K_i \)).

Let \( H = \prod_{i \in I} (|FA(X)|, \rho'_i) \). By Proposition 2.2.6, \( FA(X) \) is topologically isomorphic to
a subgroup of $H$ where the embedding $f : FA(X) \to H$ is given by $f(w) = \prod_{i \in I} w_i$, $w_i = w \in FA(X)$ for each $i \in I$.

Consider $\Phi : FA(X) \to \prod_{i \in I} [([FA(Y_i), d'_i] \times K_i]$ given by $\Phi(w) = \prod_{i \in I} g_i(w)$. Using $f$ and projection mappings, it is clear that $\Phi$ is continuous. Also, as each $g_i$ is one-to-one and open on the image of $(|FA(X)|, \rho'_i)$, $\Phi$ is also one-to-one and open on $\Phi(FA(X))$. Thus, $\Phi$ is a topological group embedding and $\Phi(X)$ is a subspace of $\prod_{i \in I} [(Y_i, d_i) \times K_i]$; that is, $X$ can be embedded as a topological space in $\prod_{i \in I} [(Y_i, d_i) \times K_i]$.

For each $i \in I$, let $FA(Y_i, d_i)$ be the free abelian topological group on the metric space $(Y_i, d_i)$. We note that both $(|FA(Y_i)|, \rho'_i)$ and $FA(Y_i, d_i)$ induce the topology $(Y_i, d_i)$ on $Y_i$, with $FA(Y_i, d_i)$ having the finer topology. Now, let $\Psi : FA(X) \to \prod_{i \in I} (FA(Y_i, d_i) \times K_i)$ be given by $\Psi(w) = \Phi(w)$ for each $w \in FA(X)$. Let $T'$ be the topology on $|FA(X)|$ for which $\Psi$ is a (topological) embedding. If we denote the topology on $FA(X)$, the free abelian topological group on $X$, by $T$, then clearly $T \subseteq T'$. Considering $\Psi(X) = \Phi(X)$, we note $\Psi(X)$ is a subspace of $\prod_{i \in I} ((Y_i, d_i) \times K_i)$. Therefore, $(|FA(X)|, T')$ induces the same topology on $X$ as does $FA(X)$ (with $T$). However, $FA(X)$ has the finest group topology that induces the original topology on $X$, giving $T' = T$. Thus, $FA(X)$ is topologically isomorphic to a subgroup of $\prod_{i \in I} (FA(Y_i, d_i) \times K_i)$. Further, $\prod_{i \in I} (FA(Y_i, d_i) \times K_i)$ is topologically isomorphic to $\prod_{i \in I} FA(Y_i, d_i) \times K$, where $K = \prod_{i \in I} |FA(X)|I$, an indiscrete abelian group, and the result follows.

Indeed, if in Proposition 4.3.5 the space $X$ is also Hausdorff, Proposition 4.3.5 can be refined further, "dropping" the indiscrete abelian group as follows.

**Theorem 4.3.6.** Let $X$ be a completely regular Hausdorff space. Then $FA(X)$ is topologically isomorphic to a subgroup of a product of free abelian topological groups on metric spaces.

**Proof.** Let the topology on $X$ be defined by the family of pseudometrics $\{\rho_i : i \in I\}$ and for each $i \in I$, let $(Y_i, d_i)$ be the metrification of $(X, \rho_i)$. By Proposition 4.3.5, $FA(X)$, the free abelian topological group on $X$, is topologically isomorphic to a subgroup of
\[ \prod_{i \in I} FA(Y_i, d_i) \times K \] where \( K \) is an indiscrete abelian group. Let \( \Phi \) be the topological group isomorphism of \( FA(X) \) onto its image in \[ \prod_{i \in I} FA(Y_i, d_i) \times K, \] as in Proposition 4.3.5. Let \( p \) be the projection of \[ \prod_{i \in I} FA(Y_i, d_i) \times K \] onto \[ \prod_{i \in I} FA(Y_i, d_i) \] and consider \( \gamma = p \circ \Phi. \) Clearly, \( \gamma \) is a continuous homomorphism of \( FA(X) \) into \[ \prod_{i \in I} FA(Y_i, d_i). \] As \( K \) is indiscrete, \( \gamma \) is an open mapping of \( FA(X) \) onto its image \( \gamma(FA(X)) \) in \[ \prod_{i \in I} FA(Y_i, d_i). \] To complete the proof, we show that \( \gamma \) is one-to-one. Let \( w \in FA(X) \) such that \( w \neq e. \) As \( FA(X) \) is Hausdorff, there exists \( k \in I \) such that \( \rho'_k(w, e) \neq 0. \) Let \( ([FA(Y_k)], d'_k) \) be the group underlying \( FA(Y_k) \) with \( d'_k \) the Graev extension of the metric \( d_k. \) As \( ([FA(Y_k)], d'_k) \) is the metrification of \( ([FA(X)], \rho'_k) \) (see Proposition 4.3.4), \( d'_k([w]_k, [e]_k) \neq 0, \) where \([w]_k\) and \([e]_k\) are the respective images of \( w \) and \( e \) in \( FA(Y_k). \) This implies that \([w]_k \neq [e]_k\) for some \( k \in I \) and so \( \gamma(w) \) is not the identity in \( \prod_{i \in I} FA(Y_i, d_i). \) Therefore, \( \gamma \) is one-to-one and the result follows.

The final result in this section extends Theorem 4.3.6 to the special case where \( X \) is a compact Hausdorff space.

**Theorem 4.3.7.** Let \( X \) be any compact Hausdorff space. Then \( FA(X) \) is topologically isomorphic to a subgroup of a product of free abelian topological groups on compact metric spaces \( Y_i, \) where each \( Y_i \) is a quotient space of \( X. \) Further, each \( FA(Y_i) \) is a quotient group of \( FA(X). \)

**Proof.** Let the topology on \( X \) be defined by the family of pseudometrics \( \{\rho_i : i \in I\} \) and for each \( i \in I, \) let \( (Y_i, d_i) \) be the metrification of \( (X, \rho_i). \) Note that in the proof to Theorem 4.3.6, \( FA(X) \) is topologically isomorphic to a subgroup of the product \[ \prod_{i \in I} FA(Y_i, d_i) \] where \( FA(Y_i, d_i) \) is the free abelian topological group on \( (Y_i, d_i). \) Observe, also that \( (Y_i, d_i) \) is a continuous image of \( (X, \rho_i) \), indeed of \( X, \) and so \( (Y_i, d_i) \) is compact. Thus, \( (Y_i, d_i) \) is a quotient space of \( X \) and so \( FA(Y_i, d_i) \) is a quotient of \( FA(X) \) by Corollary 4.1.2.
§4.4 $\mathfrak{V}(FA[0,1])$: The Whole Story

At the end of Section 4.2, we alluded to the possibility that $\mathfrak{V}(FA[0,1])$ and $\mathfrak{V}(K_\omega)$ were in fact the same variety. In this section we shall see that the free abelian topological group on any compact Hausdorff space is contained in $\mathfrak{V}(FA[0,1])$. This result will then allow us to prove the surprising fact that the equality $\mathfrak{V}(FA[0,1]) = \mathfrak{V}(K_\omega)$ indeed holds.

A useful tool in our analysis is the Hahn-Mazurkiewicz Theorem. We will state it in the following way.

**Theorem 4.4.1: Hahn-Mazurkiewicz Theorem.** ([3], Part II, Chapter 1, §1, page 100) The Hausdorff continuous images of $[0,1]$ are precisely those which are compact connected locally connected metrizable.

As a first step towards establishing that $FA(X)$ is contained in $\mathfrak{V}(FA[0,1])$ for every compact Hausdorff space $X$, we consider the case when $X$ is a compact metric space.

**Lemma 4.4.2.** (cf. [45], Corollary to Theorem 1) Let $Y$ be a compact metric topological space. Then $FA(Y)$ is contained in $\mathfrak{V}(FA[0,1])$.

**Proof.** As $Y$ satisfies the second axiom of countability and is regular, it can be embedded in $[0,1]^{\aleph_0}$ ([10], Chapter IX, Section 9, Corollary 9.2). As $Y$ is compact, by Corollary 4.1.5, $FA(Y)$ is a closed subgroup of $FA[0,1]^{\aleph_0}$. Now, noting that $[0,1]^{\aleph_0}$ is a compact connected locally connected metrizable topological space, by the Hahn-Mazurkiewicz Theorem, it is a continuous image of $[0,1]$, and hence a quotient space of $[0,1]$. Thus, by Corollary 4.1.2, $FA[0,1]^{\aleph_0}$ is a quotient group of $FA[0,1]$ and the result follows.

We now use the Metrification Mechanism to extend Lemma 4.4.2 to every compact Hausdorff topological space.
Theorem 4.4.3. Let $X$ be any compact Hausdorff topological space. Then $FA(X)$ is contained in $\mathfrak{B}(FA[0,1])$.

Proof. By Theorem 4.3.7, $FA(X)$ is topologically isomorphic to a subgroup of a product of free abelian topological groups on compact metric spaces. From Lemma 4.4.2, each of these free abelian topological groups is in $\mathfrak{B}(FA[0,1])$ and the result follows.

We note that an immediate consequence of Theorem 4.4.3 is that for every compact topological group $G$, $FA(G)$ is contained in $\mathfrak{B}(FA[0,1])$. Further, as $G$ is a quotient of $FA(G)$, $G$ is also contained in $\mathfrak{B}(FA[0,1])$ (cf. Corollary 4.2.4).

The next corollary is a special case of Theorem 4.4.3.

Corollary 4.4.4. The free abelian topological group on $[0,1]^N$, for an arbitrary cardinal number $N$, is contained in $\mathfrak{B}(FA[0,1])$, the variety generated by $FA[0,1]$.

We now turn our attention back to the problem of establishing that $\mathfrak{B}(FA[0,1]) = \mathfrak{B}(\mathcal{K}_\omega)$, where $\mathcal{K}_\omega$ is the class of all abelian $k_\omega$-groups. Recall that by Lemma 4.1.9, the free abelian topological group on a $k_\omega$-space $X$ is a quotient of the free abelian topological group on the space $Y$ where $Y$ is the free union over the $k_\omega$-decomposition of $X$. Therefore, we use the next few results to show that the free abelian topological group on a space $Y$ which is the free union of compact Hausdorff spaces is contained in $\mathfrak{B}(FA[0,1])$.

Proposition 4.4.5. Let $X$ be any compact Hausdorff topological space and let $Y$ be a subspace of $X$ such that $Y = \bigcup_{n=1}^{\infty} Y_n$ where each $Y_n$ is compact Hausdorff (and disjoint from $Y_1, Y_2, \ldots, Y_{n-1}$). Then $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA(X)$.

Proof. Without loss of generality, let $e \in Y_1 \subseteq X$. Let $Z$ be the subspace of $FA(X)$ defined by $Z = \bigcup_{n=1}^{\infty} Z_n$ where $Z_n = \{y^n : y \in Y_n\}$. Now, $Z\{e\}$ freely generates gp($Z$). Further, $Z \cap FA_n(X) = \bigcup_{i=1}^{n} Z_i$ which is compact. Therefore, $Z$ is closed in $FA(X)$ by Proposition 4.1.4. A similar argument shows that for each $n \in \mathbb{N}$, $Z \setminus Z_n$ is closed in $FA(X)$ and so $Z_n$
is open in $Z$. Thus, $Z$ is a free union of the spaces $Z_n$, $n = 1, 2, \ldots$. So $Z$ is homeomorphic to $Y$. Next, let $Z'_n = Z_1 \cup Z_2 \cup \ldots \cup Z_n$. Then $Z$ is a $k_\omega$-space with $k_\omega$-decomposition $Z = \bigcup_{n=1}^{\infty} Z'_n$. Now, from the definition of $Z_n$ we see that $\text{gp}(Z) \cap FA_n(X) \subseteq \text{gp}_n(Z'_n)$ and so by Lemma 4.1.8, $\text{gp}(Z)$ is closed in $FA(X)$ and is the free abelian topological group on $Z$, $FA(Z)$. Further, as $Z$ is homeomorphic to $Y$, $FA(Y)$ is topologically isomorphic to $FA(Z)$. Thus, $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA(X)$. ■

**Corollary 4.4.6.** Let $Y = \bigcup_{n=1}^{\infty} Y_n$ be a free union of a countably infinite family of compact Hausdorff spaces. Then there exists a compact Hausdorff space $X$ such that $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA(X)$.

*Proof.* As $Y$ is completely regular Hausdorff, it can be embedded in $X = [0,1]^N$ for some cardinal $N$ ([29], Chapter 4, Theorem 7). As $[0,1]$ is compact Hausdorff, so too is $X$ and the result follows from Proposition 4.4.5.

**Corollary 4.4.7.** Let $Y = \bigcup_{n=1}^{\infty} Y_n$ be a free union of compact Hausdorff spaces $Y_n$. Then $FA(Y) \in \mathfrak{B}(FA[0,1])$.

*Proof.* By Corollary 4.4.6, there exists a compact Hausdorff space $X$ such that $FA(Y)$ is topologically isomorphic to a subgroup of $FA(X)$. By Theorem 4.4.3, $FA(X)$ is in $\mathfrak{B}(FA[0,1])$ and the result follows.

We are now in a position to show that the free abelian topological group on any $k_\omega$-space in contained in $\mathfrak{B}(FA[0,1])$, which in turn will allow us to prove that every $k_\omega$-group is contained in $\mathfrak{B}(FA[0,1])$.

**Theorem 4.4.8.** Let $X$ be a $k_\omega$-space. Then $FA(X) \in \mathfrak{B}(FA[0,1])$.

*Proof.* Let $X = \bigcup_{n=1}^{\infty} X_n$ be the $k_\omega$-decomposition of $X$ and let $Y = \bigcup_{n=1}^{\infty} Y_n$ where each $Y_n$ is homeomorphic to $X_n$. By Lemma 4.1.9, $FA(X)$ is a quotient of $FA(Y)$ and by Corollary 4.4.7, $FA(Y) \in \mathfrak{B}(FA[0,1])$. Therefore, $FA(X) \in \mathfrak{B}(FA[0,1])$. ■
We now present the main theorem of this section.

**Theorem 4.4.9.** The variety of topological groups generated by $FA[0,1]$ is precisely the variety generated by the class of all abelian $k_\omega$-groups; that is, $\mathfrak{V}(FA[0,1]) = \mathfrak{V}(\mathcal{K}_\omega)$.

**Proof.** Let $G$ be an abelian $k_\omega$-group. By Theorem 4.4.8, the free abelian topological group $FA(G)$ is in $\mathfrak{V}(FA[0,1])$. Further, by Proposition 2.2.8, $G$ is a quotient of $FA(G)$ and so $G \in \mathfrak{V}(FA[0,1])$. Therefore, $\mathcal{K}_\omega \subseteq \mathfrak{V}(FA[0,1])$ and so $\mathfrak{V}(\mathcal{K}_\omega) \subseteq \mathfrak{V}(FA[0,1])$.

Finally, note that $FA[0,1]$ is a $k_\omega$-group, giving $\mathfrak{V}(FA[0,1]) \subseteq \mathfrak{V}(\mathcal{K}_\omega)$ and the result follows. ■

Theorem 4.4.9 tells us that $FA[0,1]$ is a rich enough $k_\omega$-group to generate every $k_\omega$-group. Therefore, every result concerning $\mathfrak{V}(\mathcal{K}_\omega)$ in Section 3.3 is a result about $\mathfrak{V}(FA[0,1])$—no "application" necessary. We can conclude that $\mathfrak{V}(FA[0,1])$ is an interesting structure to study as it is more easily analyzed than $\mathfrak{V}(\mathcal{K}_\omega)$, being a singly-generated variety, that is, generated by a single topological group. The question still remains: which other $k_\omega$-groups singly-generate $\mathfrak{V}(\mathcal{K}_\omega)$? Section 4.5 begins to answer this and again, we see some fascinating results concerning $\mathfrak{V}(FA[0,1])$.

Before we move onto Section 4.5, however, we shall return briefly to the comparison between $\mathfrak{V}(\mathcal{L}_A)$ and $\mathfrak{V}(FA[0,1])$. We noted earlier that $\mathfrak{V}(\mathcal{L}_A) \nsubseteq \mathfrak{V}(FA[0,1])$ as the variety $\mathfrak{V}(FA[0,1])$ does not contain discrete abelian groups of large cardinality. Indeed, $\mathfrak{V}(FA[0,1])$ contains only those locally compact Hausdorff abelian groups that are $k_\omega$-groups (see Theorem 3.3.13). We are now in a position to show that in fact, discrete groups are all we need to add to $\mathfrak{V}(FA[0,1])$ to generate all locally compact Hausdorff abelian groups. We use the following result by Morris and Ward.

**Proposition 4.4.10.** ([43], Theorem 6) Let $X$ and $Y$ be completely regular spaces. Then $FA(X \cup Y)$ is topologically isomorphic to $FA(X) \times FA(Y)$. ■
Theorem 4.4.11. Let $\mathcal{D}$ be the class of all discrete abelian topological groups. Then $\mathfrak{L}(\mathcal{L}_A)$ is properly contained in $\mathfrak{L}(FA[0, 1] \cup \mathcal{D})$.

Proof. Let $G$ be a locally compact Hausdorff abelian topological group. By the Principal Structure Theorem (Theorem 3.1.1), $G$ has an open subgroup $H$ topologically isomorphic to $\mathbb{R}^n \times K$ where $K$ is compact Hausdorff abelian group and $n$ a nonnegative integer. Note that as $H$ is an open subgroup of $G$, $G$ has the finest group topology which induces the given topology on $H$. Choose one element from each coset of $H$ different from $H$ and form the set $D$ which, as $H$ is open, is clearly discrete and disjoint from $H$. Thus, $H \cup D = H \cup \mathcal{D}$ is a subspace of $G$, and by Remark 4.2.7, $G$ has the finest topology which induces the given topology on $H \cup D$. Further, $G = \text{gp}(H \cup D)$ and so by Corollary 4.1.3, $G$ is a quotient group of $FA(H \cup D)$. By Proposition 4.4.10, $FA(H \cup D)$ is topologically isomorphic to $FA(H) \times FA(D)$. Note that $FA(H)$ is a $k_\omega$-group ([34], Corollary 1) and $FA(D)$ is discrete, so $FA(H) \times FA(D) \in \mathfrak{L}(FA[0, 1] \cup \mathcal{D})$. Therefore, $G \in \mathfrak{L}(FA[0, 1] \cup \mathcal{D})$ and so $\mathfrak{L}(\mathcal{L}_A) \subseteq \mathfrak{L}(FA[0, 1] \cup \mathcal{D})$. We see that $\mathfrak{L}(\mathcal{L}_A)$ is a proper subvariety of $\mathfrak{L}(FA[0, 1] \cup \mathcal{D})$ as $FA[0, 1] \not\in \mathfrak{L}(\mathcal{L}_A)$ (Proposition 3.3.5).

Remark 4.4.12. Let $\mathcal{F}$ be the class of all free abelian topological groups on discrete spaces. Then Theorem 4.4.11 can be extended by noting that

$$\mathfrak{L}(FA[0, 1] \cup \mathcal{D}) = \mathfrak{L}(FA[0, 1] \cup \mathcal{F}).$$

§4.5 The Variety Generated by the Free Abelian Topological Group on a Compact Space

The free abelian topological group on $[0, 1]$ is a well-known group and it seems sensible to study the variety it generates. We have seen in Section 4.4 that the variety it generates contains all $k_\omega$-groups. Now in this section we shall see the same variety is singly-generated by many other free abelian topological groups.
An interesting consequence of Theorem 4.4.3 is that for every compact Hausdorff topological space, $X$, $\mathcal{V}(FA(X)) \subseteq \mathcal{V}(FA[0,1])$. At this point, we must question whether the converse is also true, giving equality. However, we shall soon see that if $X$ is a countable compact space, then $FA[0,1] \not\subseteq \mathcal{V}(FA(X))$. At first glance, therefore, we have a negative answer. Thus, we address the question of which compact Hausdorff topological spaces, $X$, give $\mathcal{V}(FA(X)) = \mathcal{V}(FA[0,1])$, and find that this is true for a very wide class of compact Hausdorff spaces.

**Proposition 4.5.1.** Let $X$ be any countable completely regular space. Then $\mathbb{R}$ is not in $\mathcal{V}(FA(X))$ and so $\mathcal{V}(FA[0,1]) \not\subseteq \mathcal{V}(FA(X))$.

**Proof.** As $X$ is a countable set, the free abelian topological group $FA(X)$ on $X$ is also countable. Therefore, $FA(X)$ is a $T(c)$-group. Recall that any topological group contained in a variety generated by $T(c)$-groups is also a $T(c)$-group (see Remark 1.3.10). Now, from Example 1.3.7(a) and Proposition 1.3.8, $\mathbb{R}$ is a $T(m)$-group if and only if $m$ is strictly greater than $c$ and thus cannot be contained in $\mathcal{V}(FA(X))$. The result follows from the fact that $\mathbb{R} \in \mathcal{V}(FA[0,1])$ (see Proposition 4.2.2).

From Theorem 4.4.3 and Proposition 4.5.1, we can deduce that for $X$ a countable compact Hausdorff space, $\mathcal{V}(FA(X))$ is a proper subvariety of $\mathcal{V}(FA[0,1])$.

As immediate consequences of Proposition 4.5.1, we have the following two results. The interest in the first will be made clear shortly.

**Remark 4.5.2.** Let $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ as a subspace of $[0,1]$. Then $\mathbb{R}$ is not in $\mathcal{V}(FA(S))$.

**Corollary 4.5.3.** The variety generated by the free abelian topological group on $\mathbb{Q}$ does not contain the variety generated by $FA[0,1]$, the free abelian topological group on $[0,1]$.

**Proof.** By Proposition 4.5.1, $\mathbb{R} \not\in \mathcal{V}(FA(\mathbb{Q}))$ and so $\mathcal{V}(FA[0,1])$ cannot be contained in $\mathcal{V}(FA(\mathbb{Q}))$. 
Given Corollary 4.5.3, it is also worth noting that \( \mathcal{V}(FA(\mathbb{Q})) \) does not even contain \( \mathcal{V}(\mathbb{R}) \), which is a "small" variety of topological groups.

Speaking of "small" varieties of topological groups, the following proposition is somewhat interesting in that it leads us to show that there is a "smallest" variety generated by \( FA(X) \) where \( X \) is an infinite compact Hausdorff space. Indeed, this "smallest" variety is contained in every other variety generated by \( FA(X) \) where \( X \) is an infinite compact Hausdorff space.

**Proposition 4.5.4.** Let \( S = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \) as a subspace of \([0,1]\), and let \( X \) be any infinite compact Hausdorff space. Then \( \mathcal{V}(FA(S)) \subseteq \mathcal{V}(FA(X)) \).

**Proof.** Firstly, consider \( X \) any infinite (including countably infinite) compact metric space. As \( X \) is not discrete, let \( x \in X \) be a non-isolated point and let \( d \) be the metric on \( X \). Construct a sequence of elements in \( X \) as follows. Let \( y_1 \in X, y_1 \neq x \) and let \( \varepsilon_1 = d(y_1, x) \). Next, choose \( y_2 \in \{ y : d(y, x) < \varepsilon_1 \}, y_2 \neq x \) (which is possible as \( x \) is not isolated), and let \( \varepsilon_2 = d(y_2, x) \). In general, choose \( y_i \in \{ y : d(y, x) < \varepsilon_i \}, y_i \neq x \), and let \( \varepsilon_i = d(y_i, x) \). Finally, define \( f : S \to X \) by \( f(0) = x \) and \( f\left(\frac{1}{n}\right) = y_n \) for each \( n \in \mathbb{N} \). Clearly, \( f \) is one-to-one. Now, let \( f(S) = Y = \{x\} \cup \{y_n : n \in \mathbb{N}\} \subseteq X \). For \( y_i \in Y \), we know \( \varepsilon_{i+1} < d(y_i, x) < \varepsilon_{i-1} \) and so take \( \delta = \min\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{i-1} - \varepsilon_i\} \) which means the set \( \{ z : d(z, y_i) < \delta \} \) contains \( y_i \) but not \( x \) nor \( y_j \) for every \( j \neq i \). Therefore, each \( y_n \) is an isolated point in \( Y \) and we note that \( f^{-1}(\{y_n\}) = \{\frac{1}{n}\} \), which is open in \( S \). Let \( B_\varepsilon(x) = \{ y \in X : d(x, y) < \varepsilon \} \) be an open sphere about \( x \) in \( X \), for some \( \varepsilon > 0 \). Then \( f^{-1}(B_\varepsilon(x) \cap Y) = \{0\} \cup \left\{ \frac{1}{n} : \varepsilon_n < \varepsilon \right\} \), which is clearly open in \( S \). Therefore, \( f \) is continuous. Given the compactness of \( S \) and \( Y \), to see \( f \) is an open mapping onto \( Y \), it is enough to note in this case that \( f \) is one-to-one. Therefore, \( S \) is homeomorphic to \( Y \); that is, \( S \) can be embedded in \( X \). By Corollary 4.1.5, \( FA(S) \) is a subgroup of \( FA(X) \) and so \( FA(S) \in \mathcal{V}(FA(X)) \).

Now, let \( X \) be any infinite Hausdorff compact space, and let \( \rho \) be one of the pseudometrics on \( X \). Then the identity map \( f : X \to (X, \rho) \) is continuous. Let \((Y,d)\) be the metrification of \((X,\rho)\). Then there is a quotient map \( g : (X, \rho) \to (Y, d) \). Clearly, the composition
$h = g \circ f$ is a continuous map from $X$ onto $(Y, d)$ and as $X$ (and hence $(Y, d)$) is compact, $h$ is a quotient map. Therefore, by Corollary 4.1.2, $FA(Y, d)$ is a quotient of $FA(X)$ and hence $FA(Y, d) \in \mathfrak{V}(FA(X))$. As we saw earlier, $FA(S)$ is a subgroup of $FA(Y, d)$ and so $FA(S) \in \mathfrak{V}(FA(X))$, giving the result.

To establish that $\mathfrak{V}(FA(S))$ is the smallest variety generated by the free abelian topological group on an infinite compact Hausdorff space, we present Corollary 4.5.5, which is an immediate consequence of Proposition 4.5.4.

**Corollary 4.5.5.** Let $C$ be the class of all infinite compact Hausdorff topological spaces and let $S$ be as in Proposition 4.5.4. Then $\bigcap_{X \in C} \mathfrak{V}(FA(X)) = \mathfrak{V}(FA(S))$.

At this stage, we are no closer to determining for which compact Hausdorff topological spaces $X$ the equality $\mathfrak{V}(FA(X)) = \mathfrak{V}(FA[0,1])$ holds. As a first step, we shall consider the Cantor space, $\{0,1\}^{\aleph_0}$, which is a compact metric space.

**Remark 4.5.6.** ([12], Exercises 6.2.A(c)) The Cantor space can be characterized as follows. Every non-empty compact totally disconnected perfect (that is, has no isolated points) metrizable space is homeomorphic to the Cantor space.

**Proposition 4.5.7.** Let $G = \{0,1\}^{\aleph_0}$ be the Cantor space. Then

$$\mathfrak{V}(FA(G)) = \mathfrak{V}(FA[0,1]).$$

**Proof.** As $G$ is a compact metric space, by Lemma 4.4.2, $FA(G) \in \mathfrak{V}(FA[0,1])$ and hence $\mathfrak{V}(FA(G)) \subseteq \mathfrak{V}(FA[0,1])$.

Now, there exists a continuous mapping $\phi$ of $G$ onto $[0,1]$ (see [12], Exercise 3.2.B, page 193) and as both $G$ and $[0,1]$ are compact, $\phi$ is a quotient mapping. Therefore, by Corollary 4.1.2, there exists a quotient homomorphism from $FA(G)$ onto $FA[0,1]$ and the result follows.
As we are dealing with varieties of topological groups, classes which are closed under subgroups, and we have Corollary 4.1.5 as a tool, the compact Hausdorff space $X$ need only have the Cantor space as a subspace to obtain the result that $\mathfrak{B}(FA[0,1]) = \mathfrak{B}(FA(X))$.

**Proposition 4.5.8.** Let $K$ be any compact Hausdorff space that contains the Cantor space. Then $\mathfrak{B}(FA(K)) = \mathfrak{B}(FA[0,1])$.

*Proof.* By Theorem 4.4.3, $FA(K) \in \mathfrak{B}(FA[0,1])$ and hence $\mathfrak{B}(FA(K)) \subseteq \mathfrak{B}(FA[0,1])$. Now, let $G$ be the Cantor space contained in $K$. Then $G$ is a compact subspace of $K$ and hence, by Corollary 4.1.5, $FA(G)$ is a subgroup of $FA(K)$. Thus, $FA(G) \in \mathfrak{B}(FA(K))$ giving $\mathfrak{B}(FA(G)) \subseteq \mathfrak{B}(FA(K))$. However, $\mathfrak{B}(FA(G)) = \mathfrak{B}(FA[0,1])$ by Proposition 4.5.7, and the result follows.

The technique used in Proposition 4.5.7 to show that $FA[0,1] \in \mathfrak{B}(FA(G))$, where $G$ is the Cantor space, suggests also that if we can find a quotient mapping from a space $X$ onto $[0,1]$, then $FA[0,1] \in \mathfrak{B}(FA(X))$ by Corollary 4.1.2. Hence we have the following theorem.

**Theorem 4.5.9.** Let $X$ be a compact connected Hausdorff space with at least two points. Then $\mathfrak{B}(FA(X)) = \mathfrak{B}(FA[0,1])$.

*Proof.* Using Theorem 4.4.3 we immediately have $\mathfrak{B}(FA(X)) \subseteq \mathfrak{B}(FA[0,1])$.

As $X$ is completely regular, there exists a continuous mapping $\phi : X \to [0,1]$ such that $\phi(a) = 0$ and $\phi(b) = 1$ for some $a, b \in X$, $a \neq b$. Now, as $X$ is connected, $\phi(X)$ is also connected. As $\phi(X)$ is connected and contains 0 and 1, $\phi(X) = [0,1]$. Thus, $\phi$ is a continuous map of $X$ onto $[0,1]$. As $X$ is compact, $[0,1]$ is a quotient space of $X$. We now apply Corollary 4.1.2, to obtain $FA[0,1]$ as a quotient group of $FA(X)$. Therefore, $FA[0,1] \in \mathfrak{B}(FA(X))$ and the result follows.

A similar argument yields a slightly stronger result.
Proposition 4.5.10. Let $X$ be a compact Hausdorff space which is not totally disconnected. Then $\mathfrak{B}(FA(X)) = \mathfrak{B}(FA[0,1])$.

It is appropriate at this point to take a brief look at the problem at hand in the context of compact topological groups.

The following proposition from [31] will give us the desired result for all compact topological groups. Recall that the weight of a topological space is the cardinality of the smallest basis for the topology ([21], Appendix 4, Definition A4.7).

Proposition 4.5.11. ([31], Chapter 24, Theorem 1.45; [8], Theorem 3.2) Let $G$ be an infinite compact Hausdorff group. Then there is a continuous map from $G$ onto $[0,1]^{w(G)}$, where $w(G)$ is the weight of $G$.

Theorem 4.5.12. Let $G$ be an infinite compact Hausdorff topological group. Then $\mathfrak{B}(FA(G)) = \mathfrak{B}(FA[0,1])$.

Proof. Firstly, by Theorem 4.4.3 we have $\mathfrak{B}(FA(G)) \subseteq \mathfrak{B}(FA[0,1])$. By Proposition 4.5.11, there exists a continuous map from $G$ onto $[0,1]^{w(G)}$, where $w(G)$ is the weight of $G$. Using projection maps, there is clearly a continuous map from $G$ onto $[0,1]$. Therefore, there is a quotient map from $G$ onto $[0,1]$ and by Corollary 4.1.2, $FA[0,1]$ is a quotient group of $FA(G)$. Therefore, $FA[0,1] \in \mathfrak{B}(FA(G))$ and so $\mathfrak{B}(FA(G)) = \mathfrak{B}(FA[0,1])$.

Returning to topological spaces, for all compact Hausdorff spaces, $X$, which are not totally disconnected we have $\mathfrak{B}(FA(X)) = \mathfrak{B}(FA[0,1])$. We shall now consider various topological spaces which are totally disconnected, starting with compact metric spaces.

Proposition 4.5.13. Let $X$ be a compact metric space that has a subspace, $Y$, with no isolated points. Then $\mathfrak{B}(FA[0,1]) = \mathfrak{B}(FA(\overline{Y})) = \mathfrak{B}(FA(X))$.

Proof. Once again, $\mathfrak{B}(FA(X)) \subseteq \mathfrak{B}(FA[0,1])$.

Clearly, $\overline{Y}$ is a compact metric space. If $\overline{Y}$ is not totally disconnected, then by Proposition
4.5.10, \( \mathfrak{V}(FA(\overline{Y})) = \mathfrak{V}(FA[0,1]) \). As \( \overline{Y} \) is a compact subspace of \( X \), then \( FA(\overline{Y}) \) is a subgroup of \( FA(X) \) giving \( \mathfrak{V}(FA(\overline{Y})) \subseteq \mathfrak{V}(FA(X)) \) and the result follows.

For \( \overline{Y} \) totally disconnected, we shall show that \( \overline{Y} \) is homeomorphic to the Cantor space. Suppose \( \overline{Y} \) has an isolated point \( y \). Then, there is an open set, \( O \), in \( X \) such that \( O \cap \overline{Y} = \{y\} \). Therefore, \( y \) is not a limit point for \( Y \) and hence \( y \in Y \); that is, \( y \) is an isolated point in \( Y \), which is a contradiction. Therefore, \( \overline{Y} \) has no isolated points. Thus, by Remark 4.5.6, \( \overline{Y} \) is homeomorphic to the Cantor space and the result follows from Proposition 4.5.7 and Proposition 4.5.8.

We should note at this point, that Proposition 4.5.13 can also be deduced directly from the following result, which gives a stronger case for \( X \). The following theorem uses a result proved by Hausdorff in 1914.

**Theorem 4.5.14.** Let \( X \) be any completely regular Hausdorff space with a Hausdorff quotient space \( Y \) which has a subspace \( Z \) which

(i) is complete metrizable; and

(ii) has no isolated points.

Then \( \mathfrak{V}(FA[0,1]) \subseteq \mathfrak{V}(FA(X)) \).

**Proof.** We know that \( Z \) is a complete metric space with no isolated points. Then \( Z \) contains a subspace homeomorphic to the Cantor space \( G \) ([12], Exercise 4.5.5, page 361). Thus, \( Y \) has \( G \) as a compact Hausdorff subspace. Now, by Corollary 4.1.5, \( FA(G) \) is a subspace of \( FA(Y) \). Therefore, \( \mathfrak{V}(FA(G)) \subseteq \mathfrak{V}(FA(Y)) \). Now, clearly \( FA(Y) \) is a quotient of \( FA(X) \). Therefore, we have

\[
\mathfrak{V}(FA(G)) \subseteq \mathfrak{V}(FA(Y)) \subseteq \mathfrak{V}(FA(X)).
\]

Noting that from Proposition 4.5.7, \( \mathfrak{V}(FA(G)) = \mathfrak{V}(FA[0,1]) \), the result follows.

So far, we have shown that \( \mathfrak{V}(FA[0,1]) = \mathfrak{V}(FA(X)) \) for \( X \) compact Hausdorff not totally disconnected and \( X \) compact metric with a subspace that has no isolated points.
We are left to consider $X$ compact Hausdorff totally disconnected and $X$ compact metric such that every subspace has isolated points (which implies totally disconnected). This leads us to present the following definition, which will, in due course, assist us in the non-metrizable case, as well. We thank Vladimir Uspenskii for drawing our attention to scattered spaces.

**Definition 4.5.15.** ([32], Chapter I, §9, Part VI) A topological space $X$ is said to be **scattered** if every non-empty subspace of $X$ has an isolated point.

**Proposition 4.5.16.** [69] The product of two scattered topological spaces is also a scattered topological space.

*Proof.* Let $X$ and $Y$ be scattered spaces and $A$ a non-empty subspace of $X \times Y$. Let $p_X : X \times Y \to X$ be the projection mapping onto $X$ and consider $p_X(A)$, a subspace of $X$. Since $X$ is scattered, $p_X(A)$ has an isolated point, $a$. Let $A_a = \{ y \in Y : (a, y) \in A \} \subseteq Y$. Clearly, $A_a$ is non-empty and, as it is a subspace of the scattered space $Y$, it has an isolated point, $b$. Now, let $O \subseteq X$ be a neighbourhood of $a$ such that $O \cap p_X(A) = \{a\}$ and let $U \subseteq Y$ be a neighbourhood of $b$ such that $U \cap A_a = \{b\}$. Then $(a, b) \subseteq O \times U$ and $(O \times U) \cap A = \{(a, b)\}$. Thus, $A$ has an isolated point and so $X \times Y$ is scattered. \[\blacksquare\]

To prove the necessary theorems that complete our analysis of $\mathcal{M}(FA(X))$ for $X$ compact Hausdorff, we present the Tietze-Urysohn Theorem. (See Remark 2.4.7 for the definition of a normal space and note that a compact Hausdorff topological space is normal.)

**Theorem 4.5.17.** (Tietze-Urysohn Theorem, [12], Theorem 2.1.8, page 97) Every continuous function from a closed subspace $M$ of a normal space $X$ to $[0,1]$ or $\mathbb{R}$ is continuously extendable over $X$. \[\blacksquare\]

**Remark 4.5.18.** In the proof of the following proposition, we need to find two open non-empty sets in a topological space that have disjoint closures. For a topological space
X that is regular and Hausdorff, we point out that you can indeed find two open non-empty sets that have disjoint closures. To see this let a, b ∈ X. Then there exist open sets U_1 and U_2 such that a ∈ U_1, b ∈ U_2 and U_1 ∩ U_2 = ∅. Note also that U_1 ⊆ X \ U_2 and so U_1 ⊆ X \ U_2 giving b ∉ U_1. Given the regularity of X, there exist open sets O_1, O_2 such that U_1 ⊆ O_1 and b ∈ O_2 with O_1 ∩ O_2 = ∅. Now, we have O_2 ⊆ X \ O_1 and so O_2 ⊆ X \ O_1. Therefore, we have two open sets U_1 and O_2 such that U_1 ∩ O_2 = ∅.

As each subset A of X is also regular and Hausdorff, the procedure above also applies to A.

Recall we mentioned earlier that if we can find a quotient map from a compact Hausdorff space X onto [0,1], then FA[0,1] ∈ Q(FA(X)). Therefore, the key to answering our question seems to be found in the following proposition.

**Proposition 4.5.19.** [69] A compact Hausdorff space admits a continuous mapping onto [0,1] if and only if it is not scattered.

**Proof.** Let X be a compact Hausdorff space that admits a continuous (closed) mapping f : X → [0,1]. Let Y be a subspace of X such that f|_Y : Y → [0,1] is one-to-one and onto. Then Y is a closed subspace of X such that f(Y) = [0,1] and no proper closed subset of Y is mapped onto [0,1]. Suppose p ∈ Y is an isolated point. Then Y \ {p} is a proper closed (in X) subset of Y and hence f(Y \ {p}) ≠ [0,1]. Therefore, f(Y \ {p}) = [0,1] \ {f(p)} is a closed subspace of [0,1], and so [0,1] has an isolated point, which is impossible. Thus, Y has no isolated points and so X is not scattered.

Conversely, let X be a compact Hausdorff space that is not scattered. Let Y be a subspace of X that has no isolated points. Further, Y is compact Hausdorff and has no isolated points. Thus, if we can show there exists an onto mapping f : Y → [0,1], then by Theorem 4.5.17, f can be extended over X. Therefore, without loss of generality, we shall assume that X has no isolated points.

Let V_0 and V_1 be any two open non-empty subsets of X with disjoint closures (see Remark 4.5.18). Find V_{00} and V_{01}, two open non-empty subsets of V_0 with disjoint closures in V_0,
as well as \( V_{10} \) and \( V_{11} \), two open non-empty subsets of \( V_1 \) with disjoint closures in \( V_1 \).

Continuing in this manner, we construct open non-empty sets \( V_s \) for each finite sequence \( s \) of \( \{0,1\} \). Note that the construction is possible as \( X \) has no isolated points and so no open set is a singleton set.

Construct the set \( F \subseteq X \) consisting of all points \( x \in X \) such that \( x \in \bigcap_{n=1}^{\infty} V_{g_1g_2...g_n} \) for some infinite sequence \( g_1g_2...g_n... \) where \( g_n \in \{0,1\} \) for each \( n \in \mathbb{N} \). We will show that \( F \) is closed. Let \( f \in \overline{F} \). Then, there exists an infinite sequence \( f_1, f_2, ..., f_n, ... \), that converges to \( f \) where \( f_\alpha \in F \) for each \( \alpha = 1, 2, ..., \) Now, \( \overline{F} \subseteq \overline{V_0} \cup \overline{V_1} \) and so \( f \) belongs to exactly one of \( \overline{V_0} \) and \( \overline{V_1} \), which we shall denote by \( \overline{V_{g_1}} \). Note that for each \( n \in \mathbb{N} \), there exists \( \alpha \in \mathbb{N} \), \( \alpha > n \) such that \( f_\alpha \in \overline{V_{g_1}} \). Thus, there exists an infinite sequence contained in \( \overline{V_{g_1}} \cap F \) that converges to \( f \), giving \( \overline{f} \in \overline{V_{g_1}} \cap F \subseteq \overline{V_{g_10}} \cup \overline{V_{g_11}} \). As before, we have \( f \) belonging to exactly one of \( \overline{V_{g_10}} \) and \( \overline{V_{g_11}} \), denoted by \( \overline{V_{g_1g_2}} \). Continuing this way, it is clear that \( f \in \bigcap_{n=1}^{\infty} V_{g_1g_2...g_n} \) for some infinite sequence \( g_1g_2...g_n... \) where \( g_n \in \{0,1\} \) for each \( n \in \mathbb{N} \). Therefore, \( f \in F \) and so \( F \) is closed.

We define a mapping \( \phi \) from \( F \) to \( \{0,1\}^{\infty} \), the Cantor space as follows. Let \( x \in F \), \( x \in \bigcap_{n=1}^{\infty} V_{g_1g_2...g_n} \), then \( \phi(x) = (g_1, g_2, ..., g_n, ...) \). Clearly, \( \phi \) is onto. To show that \( \phi \) is continuous, we first note that for a finite sequence \( s \) of \( \{0,1\} \), \( (\overline{V_{s0}} \cup \overline{V_{s1}}) \cap F = V_s \cap F \), and this is open in \( F \). Now, let a basic open set in the Cantor space be given by \( O = \bigcap_{i=1}^{\infty} O_i \) where \( O_i = \{0,1\} \) for all but a finite number of values for \( i \), and let \( m \) be the largest value for which \( O_m \neq \{0,1\} \). Let \( K = \prod_{i=1}^{m} O_i \) and let \( k = (k_1, ..., k_m) \in K \). We claim that \( \phi^{-1}(O) = \left( \bigcup_{k \in K} V_{k_1k_2...k_m} \right) \cap F \), which is open in \( F \). Let \( x \in \phi^{-1}(O) \). Then \( x \in F \) and \( x \in \bigcap_{n=1}^{\infty} V_{g_1g_2...g_n} \) for some infinite sequence \( (g_1, g_2, ..., g_n, ...) \in O \). Clearly, \( (g_1, g_2, ..., g_m) \in K \) and \( x \in \overline{V_{g_1g_2...g_m}} \) or \( x \in \overline{V_{g_1g_2...g_m}} \), giving \( x \in V_{g_1g_2...g_m} \). Conversely, let \( x \in V_{k_1k_2...k_m} \cap F \) where \( k = (k_1, k_2, ..., k_m) \in K \). Now, \( x \in \bigcap_{n=1}^{\infty} V_{h_1h_2...h_n} \) for some infinite sequence \( h = (h_1, h_2, ..., h_n, ...) \) in the Cantor space. Suppose \( h_1 \neq k_1 \), then \( x \in \overline{V_0} \) and \( x \in \overline{V_1} \), which is not possible. Thus, \( h_1 = k_1 \). Similarly, for each \( i = 1, ..., m \), \( h_i = k_i \) and so \( h \in O \) giving \( \phi(x) = h \in O \), that is, \( x \in \phi^{-1}(O) \). Therefore, \( \phi \) is continuous.

Therefore we have a continuous map from a closed subspace \( F \) of \( X \) onto the Cantor...
space, which in turn admits a map onto \([0,1]\); that is, \(F\) admits a continuous map onto \([0,1]\). Thus, by the Tietze-Urysohn Theorem, \(X\) admits a continuous map onto \([0,1]\). 

It seems at this stage, that for \(X\) compact Hausdorff, \(\mathfrak{U}(FA[0,1]) = \mathfrak{U}(FA(X))\) if and only if \(X\) is not scattered. The proof of this is not as straightforward as we might first expect. We now introduce the concept of a Peano curve and present four lemmas that will assist us in proving our conjecture, the main theorem for this section.

**Definition 4.5.20.** A non-singleton continuous Hausdorff image of \([0,1]\) is called a Peano curve.

**Remark 4.5.21.** We note that a Peano curve \(P\) is not just non-singleton, but is, in fact, uncountable. As \(P\) is a continuous image of a compact connected space, \(P\) is also compact connected. Further, as \(P\) is Hausdorff, this implies that \(P\) is a completely regular Hausdorff connected topological space. Thus, there exists a continuous mapping \(\Phi : P \to [0,1]\) such that \(\phi(P)\) is connected and hence contains an interval. Therefore, \(P\) must be uncountable.

Our first lemma concerning Peano curves essentially shows that if a product of topological spaces contains a Peano curve, at least one of the factors also contains a Peano curve—though not necessarily the same curve.

**Lemma 4.5.22.** If \(\prod_{i \in I} R_i\), where each \(R_i\) is a Hausdorff topological space, contains a Peano curve (a non-singleton continuous Hausdorff image of \([0,1]\)), then for some \(i \in I\), \(R_i\) contains a Peano curve.

**Proof.** Firstly, we note that for each \(j \in I\), the projection mapping \(p_j : \prod_{i \in I} R_i \to R_j\) given by \(p_j \left( \prod_{i \in I} r_i \right) = r_j\), is continuous onto \(R_j\). Now, let \(f : [0,1] \to \prod_{i \in I} R_i\) be a continuous mapping such that \(f[0,1] \subseteq \prod_{i \in I} R_i\) is a Peano curve. Clearly, for each \(j \in I\), \(p_j \circ f = h_j : [0,1] \to R_j\) is a continuous mapping into \(R_j\). Further, if \(h_j[0,1] \subseteq R_j\) were a
singleton set for each \( j \in J \), then \( f[0,1] \subseteq \bigcap_{j \in J} h_j[0,1] \) would be a singleton set. However, \( f[0,1] \) is a Peano curve, and therefore, cannot be a singleton set. Thus, for some \( j \in J \), \( h_j[0,1] \subseteq R_j \) is not a singleton set. Finally, we note that as \( R_j \) is Hausdorff, \( h_j[0,1] \) is also Hausdorff. Therefore, \( R_j \) contains a Peano curve. 

**Lemma 4.5.23.** Every non-empty open subset \( O \) of a Peano curve \( P \), contains a Peano curve.

**Proof.** By the Hahn-Mazurkiewicz Theorem, \( P \) is compact connected locally connected metrizable and so \( O \) contains a connected neighbourhood of a point \( a \in O \). If this neighbourhood were a singleton, then \( \{a\} \) would be a closed and open subset of the connected space \( P \), which is a contradiction. Therefore, \( O \) is uncountable as it contains a non-singleton connected completely regular (as \( P \) is completely regular) Hausdorff space. Let \( f : [0,1] \rightarrow P \) be a continuous map onto \( P \). Consider \( f^{-1}(O) \). As \( O \) is open, \( f^{-1}(O) \) is an open set in \([0,1]\) and hence is a countable union of open intervals. Suppose the image of each interval were a singleton. Then \( O \) would be countable. However, we know \( O \) is uncountable and therefore, the image of one of the intervals in \( f^{-1}(O) \) is not singleton. Let \( [a, b] \subseteq f^{-1}(O) \) be such an interval (if the only one is open, take a smaller closed interval). Then \( f([a, b]) \subseteq O \) and hence \( O \) contains a Peano curve. 

The next lemma presents a result along similar lines to Lemma 4.5.22 in that it shows that if a union of two topological spaces (such that both are closed in the union) contains a Peano curve, then one of the two topological spaces contains a Peano curve.

**Lemma 4.5.24.** Let \( P \) be a Peano curve contained in \( A \cup B \) where \( A \) and \( B \) are closed in the Hausdorff space \( A \cup B \). Then \( A \) or \( B \) contains a Peano curve.

**Proof.** We know that \( (A \cup B) \setminus B = A \setminus B \) is open in the space \( A \cup B \). Either \( (A \setminus B) \cap P \) is non-empty or \( P \subseteq B \) (in which case we are done). We note that \( (A \setminus B) \cap P \) is an open subset of \( P \) and so by Lemma 4.5.23, \( (A \setminus B) \cap P \) contains a Peano curve. So either \( A \) contains a Peano curve or \( B \) contains a Peano curve.
Our final lemma makes the curious point that if a quotient of a compact space $X$ contains a Peano curve, then $[0,1]$ is a quotient of $X$.

**Lemma 4.5.25.** Let $\phi : X \to Y$ be a quotient mapping from $X$ onto $Y$ where both $X$ and $Y$ are compact Hausdorff spaces. Further, let $f : [0,1] \to Y$ be a non-trivial continuous mapping into $Y$. Then $[0,1]$ is a quotient space of $X$.

**Proof.** Clearly, $f([0,1])$ is a compact connected Hausdorff space contained in $Y$. Therefore, there exists a continuous mapping $g$ of $f([0,1])$ onto $[0,1]$. Thus, by the Tietze-Urysohn Theorem, $g$ can be extended to a continuous mapping of $Y$ onto $[0,1]$. Finally, we see that $g \circ \phi : X \to [0,1]$ is a continuous surjective map and hence is a quotient map. $lacksquare$

We now are at the main theorem for this section, answering our previous question of which compact Hausdorff topological groups $X$ give $\mathfrak{Q}(FA(X)) = \mathfrak{Q}(FA[0,1])$.

**Theorem 4.5.26.** Let $X$ be a compact Hausdorff space. Then the following conditions are equivalent.

(i) $X$ is not a scattered space;

(ii) $\mathfrak{Q}(FA(X)) = \mathfrak{Q}(FA[0,1])$;

(iii) $\mathfrak{Q}(FA(X))$ contains $FA[0,1]$;

(iv) $\mathfrak{Q}(FA(X))$ contains a Hausdorff group which is not totally path disconnected.

(v) $\mathfrak{Q}(FA(X))$ contains $\mathbb{R}$;

(vi) $\mathfrak{Q}(FA(X))$ contains $\mathbb{T}$.

**Proof.** $(i) \implies (ii)$: As $X$ is not scattered, by Proposition 4.5.19, there exists a continuous mapping $f$ from $X$ onto $[0,1]$. Now, $f$ is a quotient mapping and by Corollary 4.1.2, $FA[0,1]$ is a quotient group of $FA(X)$. Thus, $\mathfrak{Q}(FA[0,1]) \subseteq \mathfrak{Q}(FA(X))$ and we have equality from Theorem 4.4.3.

$(ii) \implies (iii)$ is trivial.

$(iii) \implies (iv)$ is trivial.

We now complete the proof that the first four statements are equivalent by showing that
(iv) implies (i).

(iv) \implies (i): Let \( G \) be a non-totally path-disconnected Hausdorff topological group contained in \( \mathcal{Q}(FA(X)) \). Let \( f : [0, 1] \to G \) be a non-singleton continuous mapping and let \( P = f([0, 1]) \), then \( P \) is a Peano curve. Now, by Corollary 1.4.5, \( G \in SC\mathcal{Q}P(FA(X)) \). Thus, \( G \) is a subgroup of \( H = \prod_{i \in I} H_i \) where each \( H_i \in \mathcal{Q}P(FA(X)) \), and so \( H \) contains a Peano curve. By Lemma 4.5.22, there exists \( i \in I \) such that \( H_i \in \overline{\mathcal{Q}P(FA(X))} \) contains a Peano curve. Note that \( FA(X) \times FA(X) \) is topologically isomorphic to \( FA(X_1 \cup X_2) \) where \( X_1 \) and \( X_2 \) are copies of \( X \) (Proposition 4.4.10). Therefore, for \( K \in P(FA(X)) \), \( K \) is topologically isomorphic to \( FA(X_1 \cup X_2 \cup \ldots \cup X_n) \) where each \( X_i \) is a copy of \( X \). Thus, we have \( H_i \in \overline{\mathcal{Q}(FA(X_1 \cup X_2 \cup \ldots \cup X_n))} \) and \( H_i \) contains a Peano curve. Noting that \( Y = X_1 \cup X_2 \cup \ldots \cup X_n \) is compact, \( FA(Y) \) is a \( k_\omega \)-group and hence \( H_i \) is a \( k_\omega \)-group. Let \( \theta : FA(Y) \to H_i \) be the quotient homomorphism onto \( H_i \). Then a \( k_\omega \)-decomposition of \( H_i \) is given by \( H_i = \bigcup_{j=1}^{\infty} \mathrm{gp}_j(\theta(Y)) \), as \( \mathrm{gp}_j(\theta(Y)) = \theta(FA_j(Y)) \). Further, every compact subspace of \( H_i \) lies in \( \mathrm{gp}_m(\theta(Y)) \) for some \( m \in \mathbb{N} \) (see Remark 2.2.10(b)). So, we choose \( m \) to be the smallest value such that \( \mathrm{gp}_m(\theta(Y)) \) contains the Peano curve \( P_i \) in \( H_i \). Let \( g : [0, 1] \to H_i \) be the mapping such that \( P_i = g([0, 1]) \). Then \( g : [0, 1] \to \mathrm{gp}_m(\theta(Y)) \) is continuous and non-trivial. We now have

\[
U = P_i \cap (\mathrm{gp}_m(\theta(Y)) \setminus \mathrm{gp}_{m-1}(\theta(Y)))
\]

is an open non-empty subset of the Peano curve \( P_i \) and so, by Lemma 4.5.23, \( U \) contains a Peano curve. Consider

\[
\mathrm{gp}_m(\theta(Y)) \setminus \mathrm{gp}_{m-1}(\theta(Y)) = \theta(FA_m(Y)) \setminus \theta(FA_{m-1}(Y)) \subseteq \theta(FA_m(Y)) \setminus FA_{m-1}(Y) = A_1 \cup A_2 \cup \ldots \cup A_2^m = A
\]

where each \( A_k = \theta(Y^{\varepsilon_1}Y^{\varepsilon_2} \cdots Y^{\varepsilon_m}) = \theta(Y)^{\varepsilon_1} \cdots \theta(Y)^{\varepsilon_m} \), \( \varepsilon_i = \pm 1 \) for each \( i = 1, \ldots, m \). Now, each \( A_k \) is compact and \( A \) contains a Peano curve. Thus, by Lemma 4.5.24, some \( A_k \) contains a Peano curve, \( P_k \). Now,

\[
A_k = \theta(X_1 \cup \ldots \cup X_n)^{\varepsilon_1} \cdots \theta(X_1 \cup \ldots \cup X_n)^{\varepsilon_m} = [\theta(X_1)^{\varepsilon_1} \cup \ldots \cup \theta(X_n)^{\varepsilon_1}] \cdots [\theta(X_1)^{\varepsilon_m} \cup \ldots \cup \theta(X_n)^{\varepsilon_m}]]
\]
and so we can see that $A_k$ is the union of closed sets of the type $\theta(X_{t_1})^{e_1} \cdots \theta(X_{t_m})^{e_m}$ where each $X_{t_i}$ is a copy of $X$. So, by Lemma 4.5.24, for some collection $l_1, \ldots, l_m$, $\theta(X_{t_1})^{e_1} \cdots \theta(X_{t_m})^{e_m}$, which is homeomorphic to $\theta(X^n)$, contains a Peano curve. As $X^n$ is compact, $\theta(X^n)$ is a compact quotient space of $X^n$. Applying Lemma 4.5.25, $[0,1]$ is a quotient space of $X^n$ and so $X^n$ is not scattered. Finally, if $X$ were scattered, by Proposition 4.5.16, then $X^n$ would also be scattered. Therefore, $X$ is not scattered.

To complete the proof, we use statements (iii) and (iv) to link (v) and (vi) with the rest

$$(iii) \implies (v)$$ follows from Propositions 4.2.1 and 2.2.8, as for the proof to Proposition 4.2.2.

$$(v) \implies (vi)$$ and $$(vi) \implies (iv)$$ are trivial, completing the proof.

So we have for $X$ a compact Hausdorff space, $\mathfrak{V}(FA[0,1]) = \mathfrak{V}(FA(X))$ if and only if $X$ is not scattered. Without taking the analysis too much further, we point out the following interesting point that comes from this section.

**Corollary 4.5.27.** Let $X$ be a compact Hausdorff space. Then $\mathfrak{V}(FA(X))$ is the variety generated by all $k_\omega$-groups if and only if $X$ is not scattered.
Chapter 5

Building a Varietal Structure

The variety generated by the class of all Banach spaces is the variety of all abelian topological groups. On the other hand, neither $\mathcal{FA}[0,1]$ nor the class of all locally compact abelian groups were enough to generate all abelian topological groups. In this chapter, we consider varieties of topological groups generated by classes of topological groups that contain both $\mathcal{FA}[0,1]$ and $\mathcal{L}_A$. We discover a whole class of such varieties and find a hierarchy of proper containment leading from $\mathfrak{V}(\mathcal{FA}[0,1])$ and $\mathfrak{V}(\mathcal{L}_A)$ up to the variety of all abelian topological groups.

§5.1 Locally $\mathcal{P}$ Groups

In our search for varieties of topological groups that contain $\mathfrak{V}(\mathcal{L}_A)$ and $\mathfrak{V}(\mathcal{FA}[0,1])$, we examine varieties generated by classes of locally $\mathcal{P}$ groups, where $\mathcal{P}$ is a property of topological spaces, such as $\sigma$-compact or separable.

Definition 5.1.1. Let $\mathcal{P}$ be a property of topological spaces. A topological space $G$ is said to be locally $\mathcal{P}$ if each neighbourhood of each point contains a neighbourhood of that point with property $\mathcal{P}$.

Clearly, a topological group $G$ is locally $\mathcal{P}$ if each neighbourhood of the identity $e$ contains a neighbourhood of the identity with property $\mathcal{P}$.

The local properties we shall be considering share a number of useful characteristics. Therefore, we introduce the concept of a “marrang”\(^1\) property, which has these characteristics.

\(^1\) Marrang is the Wiradjuri word for “good” or “friend”. Wiradjuri is one of the largest aboriginal language groupings in New South Wales, being spoken over much of the central southern region of the state. [65]
Definition 5.1.2. Let $\mathcal{P}$ be a topological property. Then $\mathcal{P}$ is said to be a marrang property if

(i) $\mathcal{P}$ is preserved under finite product spaces;
(ii) $\mathcal{P}$ is preserved under quotient spaces;
(iii) every singleton space $\{x\}$ has property $\mathcal{P}$;
(iv) a topological space that has property $\mathcal{P}$ is also locally $\mathcal{P}$; and
(v) any topological group $G$ algebraically generated by a subspace $X$ with property $\mathcal{P}$, also has property $\mathcal{P}$.

Lemma 5.1.3. Let $\mathcal{P}$ be a marrang property. Then a topological group $G$ is locally $\mathcal{P}$ if and only if $G$ has an open subgroup with the property $\mathcal{P}$.

Proof. Let $G$ have an open subgroup $H$ with the property $\mathcal{P}$. By part (iv) of Definition 5.1.2, $H$ is locally $\mathcal{P}$. Let $U$ be a neighbourhood of the identity $e$ of $G$. Then $U \cap H$ is clearly a neighbourhood of $e$ in $H$ and so contains a neighbourhood of $e$ with the property $\mathcal{P}$. Therefore, $U$ contains a neighbourhood of $e$ with property $\mathcal{P}$, implying $G$ is locally $\mathcal{P}$. Conversely, if $G$ is a locally $\mathcal{P}$ group, then there exists a neighbourhood $U$ of the identity with property $\mathcal{P}$. From part (v) of Definition 5.1.2, the open subgroup $\text{gp}(U)$ has property $\mathcal{P}$ and the result follows.

The previous lemma gives an alternative definition for a locally $\mathcal{P}$ group when $\mathcal{P}$ is a marrang property. We shall use it in all cases without reference.

Remark 5.1.4.

(a) Let $\mathcal{P}$ be a marrang property. Then a connected locally $\mathcal{P}$ group has the property $\mathcal{P}$.

(b) Let $\mathcal{P}$ be a marrang property and $X$ a completely regular space with property $\mathcal{P}$. Then $FA(X)$, the free abelian topological group on $X$ has property $\mathcal{P}$ from part (v) of Definition 5.1.2.
(c) If $\mathcal{P}$ is a marrang property, then the class of all abelian locally $\mathcal{P}$ groups is closed under $Q$ and $P$. Let $G$ be a locally $\mathcal{P}$ group with open subgroup $H$ that has property $\mathcal{P}$. Clearly, if $F$ is a quotient topological group of $G$ with quotient homomorphism $f : G \to F$, then $f(H)$ is an open subgroup of $F$ with property $\mathcal{P}$ and so $F$ is locally $\mathcal{P}$. Further, let $G_i$ be a locally $\mathcal{P}$ group for each $i = 1, 2, \ldots, n$, and let $H_i$ be an open subgroup of $G_i$ with property $\mathcal{P}$. Clearly $\prod_{i=1}^{n} H_i$ is an open subgroup of $\prod_{i=1}^{n} G_i$ that has property $\mathcal{P}$ and so $\prod_{i=1}^{n} G_i$ is locally $\mathcal{P}$.

(d) It follows from part (iii) of Definition 5.1.2 that if $\mathcal{P}$ is a marrang property, every discrete space is locally $\mathcal{P}$.

We shall see in the next few sections that the properties of $\sigma$-compactness and separability are marrang properties. For the moment, we present one simple example, followed by a number of general results concerning varieties generated by locally $\mathcal{P}$ groups, where $\mathcal{P}$ is a marrang property.

Recall that for an infinite cardinal $m$, a locally-$m$ group is a topological group with a neighbourhood of the identity of cardinality less than or equal to $m$ (see Definition 1.4.6). From Remark 1.4.7, we see that a group $G$ is locally-$m$ if and only if it is locally $\mathcal{P}_1$, where $\mathcal{P}_1$ is the property of having cardinality less than or equal to $m$. We note now that if $m$ is an infinite cardinal number, $\mathcal{P}_1$ is a marrang property.

**Proposition 5.1.5.** Let $m$ be any infinite cardinal number. The property $\mathcal{P}_1$ of having cardinality less than or equal to $m$ is a marrang property.

**Proof.** Clearly, $\mathcal{P}_1$ is preserved by finite product spaces and quotient spaces and $\{x\}$ has the property $\mathcal{P}_1$. Further if a space $Y$ has cardinality less than or equal to $m$, every subset has cardinality less than or equal to $m$ and so $Y$ is locally-$m$. For $X$ an infinite set, $\text{card}(\text{gp}(X)) = \aleph_0 \cdot \text{card}(X) = \text{card}(X)$. Hence, if $X$ has property $\mathcal{P}_1$, any topological group generated by $X$ also has property $\mathcal{P}_1$. Thus, $\mathcal{P}_1$ is a marrang property. 

\[ \blacksquare \]
In Remark 5.1.4(a) we noted that a connected locally $\mathcal{P}$ group has the property $\mathcal{P}$. We extend this to a similar result concerning connected Hausdorff topological groups in varieties generated by locally $\mathcal{P}$ groups.

**Notation.** Let $\mathcal{P}$ be a marrang property. We shall denote by $\Psi$ the class of all abelian topological groups with property $\mathcal{P}$ and by $\mathcal{L}_\mathcal{P}$ the class of all abelian locally $\mathcal{P}$ topological groups.

**Proposition 5.1.6.** Let $\mathcal{P}$ be a marrang property. Let $G$ be a connected Hausdorff topological group. Then $G \in \Psi(\mathcal{L}_\mathcal{P})$, the variety generated by the class of all abelian locally $\mathcal{P}$ groups, if and only if $G \in \Psi(\Psi)$, the variety generated by the class of all abelian topological groups with property $\mathcal{P}$.

**Proof.** Clearly, if $G \in \Psi(\Psi)$, then $G \in \Psi(\mathcal{L}_\mathcal{P})$ (see Definition 5.1.2(iv)). Let $G \in \Psi(\mathcal{L}_\mathcal{P})$. Then by Corollary 1.4.5, $G \in SCQ\mathcal{P}(\mathcal{L}_\mathcal{P})$. By Remark 5.1.4(c), $\mathcal{L}_\mathcal{P}$ is closed under $Q$ and $P$ and so $G \in SC(\mathcal{L}_\mathcal{P})$. Therefore, $G$ is topologically isomorphic to a subgroup of $\prod_{i \in I} L_i$ for some index set $I$, where each $L_i$ has an open (and closed) subgroup, $H_i$, with property $\mathcal{P}$. For $j \in I$, let $p_j$ be the projection map from $\prod_{i \in I} L_i$ onto $L_j$. As $p_j(G)$ is connected, $p_j(G)$ is a subgroup of $H_j$. Therefore, $G$ is topologically isomorphic to a subgroup of $\prod_{i \in I} H_i$, where each $H_i$ has property $\mathcal{P}$. Thus, $G \in \Psi(\Psi)$, giving the result. $lacksquare$

As an immediate consequence of Proposition 5.1.6, we have the following result concerning normed vector spaces.

**Corollary 5.1.7.** Let $\mathcal{P}$ be a marrang property and let $N$ be a normed vector space. Then $N \in \Psi(\mathcal{L}_\mathcal{P})$ if and only if $N \in \Psi(\Psi)$. $lacksquare$

The question we now address is whether the variety generated by $\mathcal{L}_\mathcal{P}$ for some marrang property $\mathcal{P}$ is generated by $\Psi$ and a class of discrete groups. The answer is affirmative.
Notation. We shall denote by $\mathcal{D}$ the class of all discrete abelian groups.

**Theorem 5.1.8.** Let $\mathcal{P}$ be a marrang property. The variety of topological groups generated by $\mathcal{L}_\mathcal{P}$, the class of all abelian locally $\mathcal{P}$ groups, is equal to $\mathfrak{V}(\mathfrak{P} \cup \mathcal{D})$, the variety generated by the class of all abelian topological groups with property $\mathcal{P}$ and the class of all discrete abelian groups.

**Proof.** As all abelian topological groups with property $\mathcal{P}$ and all discrete abelian groups are locally $\mathcal{P}$ (see Remark 5.1.4(d)), $\mathfrak{V}(\mathfrak{P} \cup \mathcal{D}) \subseteq \mathfrak{V}(\mathcal{L}_\mathcal{P})$.

Let $G \in \mathcal{L}_\mathcal{P}$. Then $G$ has an open subgroup $H$ which is has property $\mathcal{P}$. Note that $G$ has the finest group topology which induces the given topology on $H$. Choose one element out of each coset of $H$ different from $H$ and form the set $D$ which is clearly discrete and disjoint from $H$. Thus, $H \cup D = H \cup D$ is a subspace of $G$, and by Remark 4.2.7, $G$ has the finest topology which induces the given topology on $H \cup D$. Further, $G = \text{gp}(H \cup D)$ and so by Corollary 4.1.3, $G$ is a quotient group of $FA(H \cup D)$. By Proposition 4.4.11, $FA(H \cup D)$ is topologically isomorphic to $FA(H) \times FA(D)$. By Remark 5.1.4(b), $FA(H)$ has property $\mathcal{P}$ and we know $FA(D)$ is discrete, so $FA(H) \times FA(D) \in \mathfrak{V}(\mathfrak{P} \cup \mathcal{D})$. Therefore, $G \in \mathfrak{V}(\mathfrak{P} \cup \mathcal{D})$ and the result follows.

§5.2 Locally $\sigma$-Compact Groups

Recall from Proposition 3.3.27 that the variety generated by $\mathcal{K}_\omega$, the class of all abelian $k_\omega$-groups, is properly contained in the variety generated by $\mathcal{C}_\sigma$, the class of all abelian $\sigma$-compact groups. We now examine a class of topological groups which includes all $k_\omega$-groups, indeed all $\sigma$-compact groups, and all locally compact Hausdorff abelian groups: locally $\sigma$-compact groups. We shall define locally $\sigma$-compact groups in the manner of Definition 5.1.1.

**Definition 5.2.1.** A topological space $G$ is said to be *locally $\sigma$-compact* if each neighbourhood of each point contains a $\sigma$-compact neighbourhood of that point.
It is useful to know that $\sigma$-compactness is a marrang property. The following proposition proves this result.

**Proposition 5.2.2.** The property of $\sigma$-compactness is a marrang property.

**Proof.** By Remark 3.3.12, $\sigma$-compactness is preserved under quotient spaces and finite products. It is also clear that every singleton $\{x\}$ is a $\sigma$-compact space.

Let $Y$ be a $\sigma$-compact space and $N$ an open neighbourhood of a point $y \in Y$. Then $N$ contains a closed neighbourhood $C$ of $y$ and by Remark 3.3.12, $C$ is $\sigma$-compact. Therefore, $Y$ is locally $\sigma$-compact.

Finally, let $X$ be a $\sigma$-compact space and let $G = \text{gp}(X)$. Then $G = \bigcup_{n=1}^{\infty} (X \cup X^{-1})^n$. Clearly, $X \cup X^{-1}$ is $\sigma$-compact, as is $(X \cup X^{-1})^n$ for each $n \in \mathbb{N}$. Therefore, $G$ is a countable union of countable unions of compact sets and so is $\sigma$-compact.

Therefore, $\sigma$-compactness is a marrang property. 

In most cases it will be easier to use the equivalent definition of a locally $\sigma$-compact group from Lemma 5.1.3, and we will do so without reference.

From Remark 5.1.4, we see that a connected locally $\sigma$-compact group $G$ is $\sigma$-compact. Further, the class of all abelian locally $\sigma$-compact groups, $\mathcal{L}_\sigma$, is closed under $Q$ and $P$.

**Example 5.2.3.** Every locally compact Hausdorff abelian group $G$ is locally $\sigma$-compact as by the Principal Structure Theorem (Theorem 3.1.1), $G$ has an open subgroup topologically isomorphic to the $\sigma$-compact group $\mathbb{R}^n \times K$, $K$ a compact abelian group and $n$ a nonnegative integer.

**Notation.** We shall denote by $\mathcal{L}_\sigma$ the class of all abelian locally $\sigma$-compact topological groups.

**Remark 5.2.4.** We know already that $\mathcal{L}_\sigma$ is closed under the operators $Q$ and $P$. As we shall be considering the variety generated by $\mathcal{L}_\sigma$, it is of interest to know also that $\mathcal{S}$
preserves the property of locally $\sigma$-compact. This can be seen by taking an abelian locally $\sigma$-compact group $G$ with open $\sigma$-compact subgroup $H$ and $K$ a closed subgroup of $G$. Then $K \cap H$ is a closed subgroup of $H$ and is therefore $\sigma$-compact. So $K$ has $K \cap H$ as an open $\sigma$-compact subgroup and so is locally $\sigma$-compact.

We showed in Theorems 3.2.6 and 3.3.24 that neither $\mathfrak{W}(L_\lambda)$ nor $\mathfrak{W}(C_\sigma)$ contain infinite dimensional Banach spaces. We shall show that the variety generated by $L_\sigma$ also contains no infinite dimensional Banach spaces. First, as $\sigma$-compactness is a marrang property, we can apply Proposition 5.1.6 and Corollary 5.1.7.

**Proposition 5.2.5.** Let $G$ be a connected Hausdorff topological group. Then $G \in \mathfrak{W}(L_\sigma)$ if and only if $G \in \mathfrak{W}(C_\sigma)$.

**Proposition 5.2.6.** The variety generated by the class of all abelian locally $\sigma$-compact groups, $\mathfrak{W}(L_\sigma)$, contains no infinite dimensional Banach spaces. Indeed, any Banach space $B$ in $\mathfrak{W}(L_\sigma)$ is $\sigma$-compact.

**Proof.** Let $B$ be a Banach space in $\mathfrak{W}(L_\sigma)$. By Corollary 5.1.7, $B \in \mathfrak{W}(C_\sigma)$. The result follows from Proposition 3.3.22 and Theorem 3.3.24.

In Proposition 3.3.26 we saw that the variety generated by the class of all abelian $\sigma$-compact groups is not closed under completions. The argument relied on the fact that $\mathfrak{W}(C_\sigma)$ contains a separable normed vector space $N$ of countable dimension, whose completion, $\hat{N}$, is a separable Banach space. As $\mathfrak{W}(C_\sigma)$ contains no infinite dimensional Banach spaces, $\hat{N}$ is not contained in $\mathfrak{W}(C_\sigma)$. Using exactly the same argument, we see that $\mathfrak{W}(L_\sigma)$ is not closed under completions and so we have the following result.

**Proposition 5.2.7.** The variety generated by the class of all abelian locally $\sigma$-compact groups is not closed under completions.
We now have a variety of topological groups that properly contains both $\mathfrak{V}(\mathcal{L}_A)$ and $\mathfrak{V}(C_\sigma)$. In fact, this new variety is generated by the union of $\mathcal{L}_A$ and $C_\sigma$.

**Theorem 5.2.8.** The variety of topological groups generated by $\mathcal{L}_\sigma$, the class of all abelian locally $\sigma$-compact groups, is equal to $\mathfrak{V}(C_\sigma \cup \mathcal{D})$, the variety generated by $C_\sigma \cup \mathcal{D}$.

Indeed, $\mathfrak{V}(\mathcal{L}_\sigma) = \mathfrak{V}(C_\sigma \cup \mathcal{L}_A)$.

**Proof.** By Proposition 5.2.2, $\sigma$-compactness is a marrang property. Therefore, by Theorem 5.1.8, $\mathfrak{V}(\mathcal{L}_\sigma) = \mathfrak{V}(C_\sigma \cup \mathcal{D})$. As all abelian $\sigma$-compact groups and all locally compact Hausdorff abelian groups are locally $\sigma$-compact (see Example 5.2.3), we have that $\mathfrak{V}(C_\sigma \cup \mathcal{D}) \subseteq \mathfrak{V}(C_\sigma \cup \mathcal{L}_A) \subseteq \mathfrak{V}(\mathcal{L}_\sigma)$ and the result follows. 

Even though $\mathfrak{V}(\mathcal{L}_\sigma)$ contains both $\mathfrak{V}(\mathcal{L}_A)$ and $\mathfrak{V}(C_\sigma)$, $\mathfrak{V}(\mathcal{L}_\sigma)$ still does not produce any infinite dimensional Banach spaces. To complete this section, we present a summarizing theorem which gives a diagramatic representation of the relationships amongst all varieties studied so far. Recall we denote by $\mathcal{A}$ the variety of all abelian topological groups.

**Notation.** In the following and later theorems, a variety of topological groups $\mathcal{V}$ appears linked by $\mid$, $\setminus$ or $\setminus$ below a variety $\mathcal{V}'$ if and only if $\mathcal{V}$ is a **proper** subvariety of $\mathcal{V}'$.

**Theorem 5.2.9.**

$$
\mathcal{A} = \mathfrak{V}(\mathcal{B})
\downarrow
\mathfrak{V}(C_\sigma \cup \mathcal{L}_A) = \mathfrak{V}(C_\sigma \cup \mathcal{D}) = \mathfrak{V}(\mathcal{L}_\sigma)
\downarrow
\mathfrak{V}(\mathcal{L}_A) \quad \mathfrak{V}(C_\sigma)
\downarrow
\mathfrak{V}(\mathcal{D}) \quad \mathfrak{V}(\mathcal{K}_\omega) = \mathfrak{V}(FA[0,1])
$$

**Proof.** By Proposition 5.2.6, $\mathfrak{V}(\mathcal{L}_\sigma)$ contains no infinite dimensional Banach spaces and so is a proper subvariety of $\mathcal{A}$. 

We saw in Proposition 3.1.5 that \( \mathfrak{U}(\mathcal{L}_A) \) is closed under completions, whereas \( \mathfrak{U}(\mathcal{L}_\sigma) \) is not closed under completions (Proposition 5.2.7). Thus, \( \mathfrak{U}(\mathcal{L}_A) \neq \mathfrak{U}(\mathcal{L}_\sigma) \) and so \( \mathfrak{U}(\mathcal{L}_A) \) is a proper subvariety of \( \mathfrak{U}(\mathcal{L}_\sigma) \).

Noting that \( \mathfrak{U}(\mathcal{L}_\sigma) \) contains all discrete abelian groups (indeed, all groups in \( \mathcal{L}_A \)), while \( \mathfrak{U}(\mathcal{L}_\sigma) \) contains no uncountable discrete abelian groups (discrete groups are contained in \( \mathfrak{Q} \mathcal{SP}(\mathcal{L}_\sigma) \) and so are \( \sigma \)-compact, hence countable), \( \mathfrak{U}(\mathcal{L}_\sigma) \neq \mathfrak{U}(\mathcal{L}_\sigma) \). Thus, \( \mathfrak{U}(\mathcal{L}_\sigma) \) is properly contained in \( \mathfrak{U}(\mathcal{L}_\sigma) \).

We saw in Proposition 3.3.27 that \( \mathfrak{U}(\mathcal{K}_\omega) \) is properly contained in \( \mathfrak{U}(\mathcal{L}_A) \).

Finally, suppose \( \mathcal{R} \) were contained in \( \mathfrak{U}(\mathcal{D}) \). By Corollary 1.4.5, \( \mathcal{R} \in \mathfrak{SP} \mathcal{Q}(\mathcal{D}) \). As quotient groups and finite products of discrete groups are discrete, \( \mathcal{R} \in \mathfrak{SP} \mathcal{Q}(\mathcal{D}) \); that is, \( \mathcal{R} \) is topologically isomorphic to a subgroup of \( \prod_{i \in I} D_i \) where for each \( i \in I, \) \( D_i \) is discrete and \( I \) some index set. For \( j \in I \), let \( p_j \) be the projection mapping from \( \prod_{i \in I} D_i \) onto \( D_j \). Clearly, \( p_j(\mathcal{R}) \) is a connected subgroup of \( D_j \) and therefore, must be a singleton set. Further, \( \mathcal{R} \) is topologically isomorphic to a subgroup of \( \prod_{i \in I} p_i(\mathcal{R}) \), which implies that \( \mathcal{R} \) is a singleton set. This is clearly a contradiction, and so \( \mathcal{R} \notin \mathfrak{U}(\mathcal{D}) \). Therefore, \( \mathfrak{U}(\mathcal{D}) \) is a proper subvariety of \( \mathfrak{U}(\mathcal{L}_A) \).

### 5.3 Separable and Locally Separable Groups

In our search for varieties that fall in between \( \mathfrak{U}(\mathcal{L}_\sigma) \) and \( \mathcal{A} \), we turn our attention to varieties generated by separable groups and locally separable groups. We therefore will make use of the following well-known result concerning topological groups generated by separable spaces, the proof of which is straightforward.

**Lemma 5.3.1.** Let \( G \) be an abelian topological group algebraically generated by a separable subspace \( X \). Then \( G \) is separable.

Much of our work thus far has revolved around the free abelian topological group on
a completely regular space $X$. This section is no different, so we present the following separability result concerning $FA(X)$, the free abelian group on $X\setminus\{e\}$, with the Graev extension of one of the continuous pseudometrics on $X$. The result follows immediately from Lemma 5.3.1.

**Corollary 5.3.2.** Let $(X, \rho)$ be a separable (completely regular) pseudometrizable topological space. Let $F$ be the free abelian group on $X\setminus\{e\}$ for some $e \in X$, and let $\rho'$ be the Graev extension of $\rho$ onto $F$. Then $(F, \rho')$ is separable.

**Remark 5.3.3.** As we are about to consider separable topological groups, we note that separability is, in general, not preserved by subgroups. However, open subgroups of separable spaces are separable (see [24], Part I, Section 3, Result 3.11), and all subgroups of separable metrizable spaces are separable.

Recall the fact that every separable Banach space is a quotient group of the separable Banach space $\ell_1$ (see [24], Part I, Section 2). Thus, we have the following result.

**Proposition 5.3.4.** The variety of topological groups generated by the topological group underlying the separable Banach space $\ell_1$ is precisely the variety of topological groups generated by the topological groups underlying the class of all separable Banach spaces.

**Proof.** Let $\mathcal{B}_S$ denote the class of all topological groups underlying separable Banach spaces. As $\ell_1 \in \mathcal{B}_S$, $\mathfrak{W}(\ell_1) \subseteq \mathfrak{W}(\mathcal{B}_S)$. Further, as every separable Banach space is a quotient group of $\ell_1$ ([24], Part II, Section 22, Result 22.6), $\mathcal{B}_S$ is contained in $\mathfrak{W}(\ell_1)$. Thus, $\mathfrak{W}(\mathcal{B}_S) \subseteq \mathfrak{W}(\ell_1)$ and the result follows.

We can extend this, using the Metrification Mechanism, to show that $\mathfrak{W}(\ell_1)$ contains all separable abelian topological groups. We first make a note of the fact that every indiscrete abelian group is contained in $\mathfrak{W}(\ell_1)$. 

Lemma 5.3.5. Every indiscrete abelian topological group is contained in $\mathfrak{V}(\ell_1)$.

Proof. Noting that $\mathbb{R}$ is a separable Banach space, we have $\mathbb{R}, \mathbb{T} \in \mathfrak{V}(\ell_1)$. It is well-known that every abelian group is algebraically isomorphic to a subgroup of a product of copies of the divisible group $\mathbb{T}$ and so is contained in $\mathfrak{V}(\ell_1)$ with some topological group topology. It follows, then, from Proposition 1.5.4, that every indiscrete abelian group appears in $\mathfrak{V}(\ell_1)$.

Next, we show that $\mathfrak{V}(\ell_1)$ contains all free abelian topological groups on separable spaces.

Proposition 5.3.6. Let $X$ be a separable completely regular topological space. Then $FA(X)$, the free abelian topological group on $X$, is contained in $\mathfrak{V}(\ell_1)$.

Proof. Let $\{\rho_i : i \in I\}$ be the family of all continuous pseudometrics on $X$. Clearly each $(X, \rho_i)$ is a separable space. Let $(Y_i, d_i)$ be the metrification of $(X, \rho_i)$ for each $i \in I$. Then $(Y_i, d_i)$ is separable metrizable. Further, let $F_i$ be the free abelian group on $Y_i \setminus \{e_i\}$ and let $d'_i$ be the Graev extension of $d_i$ onto $F_i$. From Corollary 5.3.2, $(F_i, d'_i)$ is separable and by Corollary 2.3.12, $(F_i, d'_i)$ is a topological group which is topologically isomorphic to a subgroup of a Banach space, indeed, a separable Banach space. Therefore, $(F_i, d'_i) \in \mathfrak{V}(\mathcal{B}_S) = \mathfrak{V}(\ell_1)$. By Proposition 2.2.6, $FA(X)$ is topologically isomorphic to a subgroup of the product $P = \prod_{i \in I} (|FA(X)|, \rho'_i)$ where $|FA(X)|$ is the group underlying $FA(X)$ and $\rho'_i$ is the Graev extension of $\rho_i$ for each $i \in I$. Now, by Proposition 4.3.4, $(F_i, d'_i)$ is the metrification of $(|FA(X)|, \rho'_i)$ and so $P$ can be embedded in the product $\prod_{i \in I} (F_i, d'_i) \times H$ where $H$ is an indiscrete abelian group. By Lemma 5.3.5, $H$ is contained in $\mathfrak{V}(\ell_1)$, thus, $FA(X)$ can be embedded as a topological group in a product of topological groups contained in $\mathfrak{V}(\ell_1)$ giving $FA(X) \in \mathfrak{V}(\ell_1)$.

We are now in a position to show that $\mathfrak{V}(\ell_1)$ contains every separable abelian topological group. We therefore, present the following theorem.
Notation. We shall denote by $S$ the class of all abelian separable topological groups and by $B_S$ the class of all topological groups underlying separable Banach spaces.

**Theorem 5.3.7.** The following varieties are equal.

(i) $\mathcal{V}(\ell_1)$, the variety generated by $\ell_1$;

(ii) $\mathcal{V}(B_S)$, the variety generated by the class of all topological groups underlying separable Banach spaces; and

(iii) $\mathcal{V}(S)$, the variety generated by the class of all abelian separable topological groups.

**Proof.** We have already established that $\mathcal{V}(\ell_1) = \mathcal{V}(B_S)$. Clearly, $\mathcal{V}(\ell_1) \subseteq \mathcal{V}(S)$. Now, let $G$ be an abelian separable topological group. By Proposition 5.3.6, $FA(G) \in \mathcal{V}(\ell_1)$. By Proposition 2.2.8, $G$ is a quotient group of $FA(G)$ and so $G \in \mathcal{V}(\ell_1)$. Therefore, $S \subseteq \mathcal{V}(\ell_1)$ and thus $\mathcal{V}(S) = \mathcal{V}(\ell_1)$. ■

Although $\mathcal{V}(\ell_1)$ contains all separable topological groups, it does not contain any non-separable normed vector spaces.

**Lemma 5.3.8.** Any normed vector space contained in $\mathcal{V}(S)$, the variety generated by the class of all abelian separable topological groups, is separable.

**Proof.** Firstly, by Theorem 5.3.7, $\mathcal{V}(S) = \mathcal{V}(\ell_1)$. Let $N$ be a normed vector space in $\mathcal{V}(\ell_1)$. As $N$ is uniformly free from small subgroups, by Proposition 3.2.8, $N \in SQP(\ell_1)$. We note that $\ell_1$ is a separable metric topological group. Further, finite products and Hausdorff quotients of separable metrizable groups are separable metrizable. Hence, $N$ is a subgroup of a separable metrizable topological group, and by Remark 5.3.3 is therefore separable. ■

We will now show that $\mathcal{V}(S)$ contains the variety generated by the class of all abelian $\sigma$-compact groups.
Proposition 5.3.9. Let $X$ be a $\sigma$-compact completely regular topological space. Then $FA(X)$, the free abelian topological group on $X$, is contained in $\mathfrak{B}(S)$, the variety generated by the class of all abelian separable metrizable topological groups.

Proof. Let $\{\rho_i : i \in I\}$ be the family of all continuous pseudometrics on $X$. Clearly each $(X, \rho_i)$ is a $\sigma$-compact pseudometrizable space and is therefore separable ([24], Part I, Section 10, Result 10.3). By Proposition 2.2.6, $FA(X)$ is topologically isomorphic to a subgroup of the product $P = \prod_{i \in I} (|FA(X)|, \rho'_i)$ where $|FA(X)|$ is the group underlying $FA(X)$ and $\rho'_i$ is the Graev extension of $\rho_i$ for each $i \in I$. By Corollary 5.3.2, $(|FA(X)|, \rho'_i)$ is separable for each $i \in I$, and therefore, contained in $\mathfrak{B}(S)$. Hence, $FA(X) \in \mathfrak{B}(S)$. □

Theorem 5.3.10. The variety generated by the class of all abelian separable groups properly contains the variety generated by the class of all abelian $\sigma$-compact groups.

Proof. Let $G$ be an abelian $\sigma$-compact group. By Proposition 5.3.9, $FA(G)$ is contained in $\mathfrak{B}(S)$. Further, by Proposition 2.2.8, $G$ is a quotient group of $FA(G)$ and so contained in $\mathfrak{B}(S)$. Therefore, $\mathfrak{B}(C_\sigma) \subseteq \mathfrak{B}(S)$, where $C_\sigma$ is the class of all abelian $\sigma$-compact groups. By Theorem 3.3.24, $\mathfrak{B}(C_\sigma)$ only contains finite dimensional Banach spaces. However, $\mathfrak{B}(S)$ contains, for example, $\ell_1$, and so $\mathfrak{B}(C_\sigma)$ is a proper subvariety of $\mathfrak{B}(S)$. □

We will now compare $\mathfrak{B}(L_\sigma)$ and $\mathfrak{B}(S)$.

Proposition 5.3.11. The variety generated by $S$ is not contained in nor does it contain the variety generated by $L_\sigma$.

Proof. By Proposition 5.2.6, $\mathfrak{B}(L_\sigma)$ contains no infinite dimensional Banach spaces and so $\mathfrak{B}(S)$ is not contained in $\mathfrak{B}(L_\sigma)$. On the other hand, $\ell_1$ is a separable metrizable topological group and so has cardinality $c$ ($\ell_1$ is a subspace of $[0,1]^{\aleph_0}$; see [29], Chapter 4, Theorem 17). Thus, $\ell_1$ is a $T(c^+)$-group, where $c^+$ is the smallest cardinal strictly greater than $c$, and so every topological group contained in $\mathfrak{B}(S) = \mathfrak{B}(\ell_1)$ is a $T(c^+)$-group (see Remark 1.3.10); that is, every discrete group contained in $\mathfrak{B}(S)$ has cardinality strictly less than $c^+$ (see Remark 1.3.5(b)). However, $\mathfrak{B}(L_\sigma)$ contains every discrete group and so is not contained in $\mathfrak{B}(S)$. □
As an aside, we note the following interesting result which is a corollary to the proof of Proposition 5.3.11.

**Corollary 5.3.12.** Every abelian $\sigma$-compact group, indeed every abelian $k_\omega$-group, is a $T(c^+)$-group.  

We see now that although $\mathcal{B}(S)$ contains $\mathcal{B}(C_\sigma)$, it does not encompass $\mathcal{B}(L_A)$. Thus, we introduce the concept of locally separable topological groups, to find a variety that contains all locally $\sigma$-compact groups and all separable abelian groups.

**Definition 5.3.13.** A topological space $G$ is said to be locally separable if each neighbourhood of each point contains a separable neighbourhood of that point.

As with $\sigma$-compactness, we prove that separability is a marrang property.

**Proposition 5.3.14.** The property of separability is a marrang property.

**Proof.** Clearly, separability is preserved under quotient spaces and finite products, and each singleton space $\{x\}$ is separable.

Let $Y$ be a separable space and $N$ an open neighbourhood of a point $y \in Y$. As open subspaces of separable spaces are separable, ([24], Part I, Section 3, Result 3.11), $N$ is a separable neighbourhood of $y$. Therefore, $Y$ is locally separable.

Finally, by Lemma 5.3.1, any topological group algebraically generated by a separable space is separable.

Therefore, $\sigma$-compactness is a marrang property.

**Notation.** We use $L_S$ to denote the class of all abelian locally separable topological groups.

We will use the equivalent definition of a locally separable group from Lemma 5.1.3 without
reference (as with locally $\sigma$-compact).

From Definition 5.1.2(iv), we note that a separable group is locally separable which implies that $S$, the class of all abelian separable groups, is contained in $\mathfrak{W}(L_S)$. Also, from Remark 5.1.4, a connected locally separable group $G$ is separable. Finally, the class of all abelian locally separable groups, $L_S$, is closed under $Q$ and $P$.

We shall see that the class of all abelian locally separable groups generates a variety that contains $L_\sigma$ and $S$ as well as $L_A$ and $C_\sigma$. In fact, $\mathfrak{W}(L_S)$ is generated by the union of $S$ and $D$, which further implies $\mathfrak{W}(L_S)$ is generated by the union of $S$ and $L_\sigma$.

**Theorem 5.3.15.** The variety of topological groups generated by $L_S$, the class of all abelian locally separable groups, is equal to $\mathfrak{W}(S \cup D)$, the variety generated by $S \cup D$.

Indeed, $\mathfrak{W}(L_S) = \mathfrak{W}(S \cup L_\sigma)$.

**Proof.** By Proposition 5.3.14, separability is a marrang property. Therefore, by Theorem 5.1.8, $\mathfrak{W}(L_S) = \mathfrak{W}(S \cup D)$. Now, Theorem 5.3.10 clearly implies that $C_\sigma \subseteq \mathfrak{W}(S)$, and so $\mathfrak{W}(C_\sigma \cup D) \subseteq \mathfrak{W}(S \cup D)$. By Theorem 5.2.8, $\mathfrak{W}(C_\sigma \cup D) = \mathfrak{W}(L_\sigma)$ giving $L_\sigma \subseteq \mathfrak{W}(S \cup D)$. Thus, as $D \subseteq L_\sigma$, $\mathfrak{W}(S \cup D) \subseteq \mathfrak{W}(S \cup L_\sigma) \subseteq \mathfrak{W}(L_S)$ and the result follows. ■

**Corollary 5.3.16.** The variety generated by the class of all abelian locally $\sigma$-compact groups is properly contained in $\mathfrak{W}(L_S)$, the class of all abelian locally separable groups.

**Proof.** By Theorem 5.3.15, $\mathfrak{W}(L_\sigma) \subseteq \mathfrak{W}(L_S)$. Recall from Proposition 5.2.6 that $\mathfrak{W}(L_\sigma)$ contains no infinite dimensional Banach spaces, but all separable Banach spaces are locally separable. Therefore, $\mathfrak{W}(L_\sigma)$ is properly contained in $\mathfrak{W}(L_S)$. ■

**Remark 5.3.17.** In Section 5.4, we shall easily see that $\mathfrak{W}(L_S)$ is properly contained in $\mathfrak{W}(B)$, the variety generated by the class of all Banach spaces. However, for completion, we make a note of this fact here. As separability is a marrang property, we can apply Corollary 5.1.7 to a normed vector space $N$ contained in $\mathfrak{W}(L_S)$ and see that $N \in \mathfrak{W}(S)$. However, by Lemma 5.3.8, $N$ is separable. Therefore, $\mathfrak{W}(L_S)$ clearly does not contain all
normed vector spaces and so is properly contained in $\mathfrak{B}(B)$.

We again summarize the results achieved so far in the following theorem, which extends Theorem 5.2.9.

**Theorem 5.3.18.**

$$\mathcal{A} = \mathfrak{B}(B)$$

$$\mathfrak{B}(L_S) = \mathfrak{B}(S \cup D) = \mathfrak{B}(S \cup L_\sigma)$$

$$\mathfrak{B}(C_\sigma \cup L_A) = \mathfrak{B}(C_\sigma \cup D) = \mathfrak{B}(L_\sigma)$$

$$\mathfrak{B}(S) = \mathfrak{B}(B_S) = \mathfrak{B}(\ell_1)$$

$$\mathfrak{B}(D) = \mathfrak{B}(K_\omega) = \mathfrak{B}(FA[0,1])$$

$\blacksquare$

§5.4 **Locally-$m$ Groups**

In this section, we shall complete our chain of varieties. The "missing link" will be a chain of varieties generated by classes of locally-$m$ groups, $m$ an infinite cardinal. These are topological groups which were introduced in Chapter 1. In the process, we will prove the following theorem.

**Theorem 5.4.1.** There exists a proper class $\mathcal{C}$ of varieties of topological groups such that

(i) the smallest variety in the class is $\mathfrak{V}(L_c)$, the variety generated by the class of all abelian locally-$c$ groups, and

(ii) if $V_1, V_2 \in \mathcal{C}$, then $V_1$ is properly contained in $V_2$ or $V_2$ is properly contained in $V_1$. 
Notation. Let $m$ be an infinite cardinal. We shall denote by $\mathcal{L}_m$ the class of all abelian locally-$m$ groups.

From Proposition 1.4.12 we can see that for $m$ and $n$ infinite cardinals greater than or equal to $c$, $\mathcal{L}_m$ is properly contained in $\mathcal{L}_n$ if and only if $m < n$. This result extends with relative ease to the varieties generated by each class. We first prove a result analogous to Proposition 1.4.10.

**Proposition 5.4.2.** Let $m$ be an infinite cardinal. A normed vector space $N$ is contained in $\mathfrak{B}(\mathcal{L}_m)$ if and only if the cardinality of $N$ is less than or equal to $m$.

**Proof.** Clearly, a normed vector space $N$ of cardinality less than or equal to $m$ is contained in $\mathfrak{B}(\mathcal{L}_m)$. Conversely, let $N$ be a normed vector space contained in $\mathfrak{B}(\mathcal{L}_m)$. As $N$ is a subgroup of a Banach space, $N$ is uniformly free from small subgroups. Therefore, by Proposition 3.2.8, $N \in S^Q P(\mathcal{L}_m)$. We saw in Proposition 1.4.8 that $\mathcal{L}_m$ is closed under $S$, $Q$ and $P$. Therefore, $N \in \mathcal{L}_m$ and so by Proposition 1.4.10, $N$ has cardinality less than or equal to $m$. 

**Proposition 5.4.3.** Let $m$ and $n$ be infinite cardinals greater than or equal to $c$. Then $\mathfrak{B}(\mathcal{L}_m)$ is a proper subvariety of $\mathfrak{B}(\mathcal{L}_n)$ if and only if $m < n$.

**Proof.** Note that for each cardinal $k$ greater than or equal to $c$, there exists a normed vector space of cardinality $k$ (see Remark 1.4.11). From Proposition 1.4.12, if $m < n$ then $\mathcal{L}_m \subseteq \mathcal{L}_n$ and so $\mathfrak{B}(\mathcal{L}_m) \subseteq \mathfrak{B}(\mathcal{L}_n)$. Further, by Proposition 5.4.2, there exists a normed vector space $N$ of cardinality $n$ such that $N \in \mathfrak{B}(\mathcal{L}_n)$ but $N \notin \mathfrak{B}(\mathcal{L}_m)$. Therefore, $\mathfrak{B}(\mathcal{L}_m) \subset \mathfrak{B}(\mathcal{L}_n)$.

Conversely, let $\mathfrak{B}(\mathcal{L}_m) \subset \mathfrak{B}(\mathcal{L}_n)$ and suppose $m \geq n$. If $m = n$ then $\mathfrak{B}(\mathcal{L}_m) = \mathfrak{B}(\mathcal{L}_n)$. If $m > n$, there exists a normed vector space $N'$ of cardinality $m$ and by Proposition 5.4.2, $N' \in \mathfrak{B}(\mathcal{L}_m)$, but $N' \notin \mathfrak{B}(\mathcal{L}_n)$. Both cases clearly lead to a contradiction, so $m < n$.

We now have a family of varieties that satisfy all the conditions of Theorem 5.4.1: those
generated by the classes of locally-$m$ groups for cardinal numbers greater than or equal to $c$.

**Proof of Theorem 5.4.1.** Let $\mathcal{C}$ be the class of all varieties $\mathcal{V}(\mathcal{L}_m)$ where $m$ is an infinite cardinal number greater than or equal to $c$. For every cardinal $m \geq c$, $\mathcal{V}(\mathcal{L}_c) \subseteq \mathcal{V}(\mathcal{L}_m)$ and so $\mathcal{V}(\mathcal{L}_c)$ is the smallest variety in the class $\mathcal{C}$. Now, let $V_1$ and $V_2$ be two varieties contained in $\mathcal{C}$. Then there exist distinct infinite cardinal numbers greater than or equal to $c$, $m$ and $n$, such that $V_1 = \mathcal{V}(\mathcal{L}_m)$ and $V_2 = \mathcal{V}(\mathcal{L}_n)$. If $m < n$, by Proposition 5.4.3, $V_1 \subseteq V_2$; if $m > n$, $V_2 \subseteq V_1$ and the theorem is proven.

It turns out that the class $\mathcal{C}$ given in the above proof is the class of varieties needed to complete our structure. We shall show that the variety generated by the class of all abelian locally separable topological groups, $\mathcal{V}(\mathcal{L}_s)$, is a proper subvariety of $\mathcal{V}(\mathcal{L}_c)$.

**Proposition 5.4.4.** Let $G$ be an abelian topological group with a neighbourhood of the identity which is separable. Then $G \in \mathcal{V}(\mathcal{L}_c)$ where $\mathcal{V}(\mathcal{L}_c)$ is the variety generated by the class of all abelian locally-c groups.

**Proof.** Let $\{\rho'_i : i \in I\}$ be the family of all continuous pseudometrics on $G$. Let $U$ be a neighbourhood of the identity in $G$ which is separable and let $\rho$ be a continuous pseudometric on $G$ in which $U$ is a neighbourhood of the identity. We shall consider $\{\rho_i : \rho_i = \rho'_i + \rho, i \in I\}$, a family of continuous pseudometrics on $G$ that also defines the topology on $G$. Then $G$ is topologically isomorphic to a subgroup of the product $\prod_{i \in I} (G, \rho_i)$ and we note that $U$ is a separable neighbourhood of the identity in each $(G, \rho_i)$. Let $G_i = (G, \rho_i, d_i)$ be the metrification of $(G, \rho_i)$ for each $i \in I$, with $f_i : G \to G_i$ the quotient homomorphism from $G$ onto $G_i$. Theorem 4.3.1 and Remark 4.3.2(c) clearly imply that $G$ is a subgroup of the product $\prod_{i \in I} G_i \times H$ where $H$ is an abelian indiscrete group. We note that each $G_i$ has a separable neighbourhood of the identity in $G_i$, namely $f_i(U)$, the continuous image of $U$ in $G_i$. Further, $f_i(U)$ is metrizable and we know that separable metrizable topological spaces have cardinality less than or equal to $c$ (as
subspaces of $[0, 1]^{\aleph_0}$, see [29], Chapter 4, Theorem 17). Therefore, for each $i \in I$, $G_i$ is an abelian locally-$c$ group and so is in $\mathfrak{V}(L_c)$. Noting that every discrete abelian group is a locally-$c$ group and so in $\mathfrak{V}(L_c)$, by Proposition 1.5.4 every indiscrete abelian group is in $\mathfrak{V}(L_c)$. Therefore, $G$ is in $\mathfrak{V}(L_c)$, as required.

Theorem 5.4.5. The variety $\mathfrak{V}(L_c)$ properly contains $\mathfrak{V}(L_S)$, the variety generated by all abelian locally separable topological groups.

Proof. Noting that every locally separable topological group has at least one separable neighbourhood of the identity, Proposition 5.4.4 implies $\mathfrak{V}(L_S) \subseteq \mathfrak{V}(L_c)$.

Let $N$ be a normed vector space of cardinality $c$ that is not separable (for example, $\ell_{\infty}$; see [60], Chapter 9, Section 46, Example 6 and [9], Proof to Corollary 4.17). From Remark 5.3.17 we see that $\mathfrak{V}(L_S)$ contains no non-separable normed vector spaces, and so $N \notin \mathfrak{V}(L_S)$. However, $N \in \mathfrak{V}(L_c)$ and the result follows.

Before we present our final theorem to complete the chain of varieties, we note $\mathfrak{V}(L_m)$ for an infinite cardinal $m$ is generated by $C_m \cup D$ where $C_m$ is the class of all abelian topological groups of cardinality less than or equal to $m$.

Theorem 5.4.6. Let $m$ be an infinite cardinal number and let $C_m$ be the class of all abelian topological groups of cardinality less than or equal to $m$. The variety of topological groups generated by $L_m$ is equal to $\mathfrak{V}(C_m \cup D)$.

Proof. The result follows from Proposition 5.1.5 and Theorem 5.1.8.

We now, finally, can complete the chain of varieties leading from $\mathfrak{V}(L_A)$ and $\mathfrak{V}(FA[0, 1])$ up to $\mathfrak{V}(B)$.

Notation. We shall denote by $C_c$ the class of all abelian topological groups of cardinality less than or equal to $c$. 
Theorem 5.4.7.

\[ A = \mathcal{B}(B) \]

\[ \mathcal{B}(L_m) = \mathcal{B}(C_m \cup D), \quad m > c \]

\[ \mathcal{B}(L_c) = \mathcal{B}(C_c \cup D) \]

\[ \mathcal{B}(L_S) = \mathcal{B}(S \cup D) = \mathcal{B}(S \cup L_\sigma) \]

\[ \mathcal{B}(C_\sigma \cup L_A) = \mathcal{B}(C_\sigma \cup D) = \mathcal{B}(L_\sigma) \]

\[ \mathcal{B}(L_A) \quad \mathcal{B}(C_\sigma) \]

\[ \mathcal{B}(D) \quad \mathcal{B}(K_\omega) = \mathcal{B}(FA[0,1]) \]
Chapter 6

Wide Varieties

A wide variety allows continuous homomorphic images, as well as products and subgroups. In this chapter we shall perform a brief analysis on the wide varieties of topological groups generated respectively by the class of all Banach spaces, the class of all locally compact abelian groups and $FA[0,1]$. In the process, we shall consider the wide varieties generated by the particular classes of abelian topological groups presented in Chapter 5.

§6.1 Wide Varieties of Topological Groups

The concept of a wide variety of topological groups is very similar to the concept of a variety of topological groups. The difference between the two is that wide varieties include continuous homomorphic images whereas varieties only include open continuous homomorphic images (that is, quotients). Before we formally define wide varieties, we introduce the operator $H$ on a class of topological groups.

**Definition 6.1.1.** Let $\Omega$ be a class of topological groups. The operator $H$ is defined on $\Omega$ to give the class of topological groups as follows. Let $G$ be a topological group. Then $G \in H(\Omega)$ if $G$ is topologically isomorphic to a continuous homomorphic image of a topological group in $\Omega$.

Let us now define wide varieties of topological groups, as introduced by Taylor in [64].

**Definition 6.1.2.** [64] A class of topological groups is said to be a variety of topological groups if it is closed under $S$, $H$ and $C$. 

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Examples 6.1.3.

(a) It is clear that the class of all abelian topological groups is a wide variety of topological groups as any continuous homomorphic image of an abelian topological group is also abelian.

(b) Let $m$ be a cardinal number. The class of all $T(m)$-groups is a wide variety of topological groups. This is seen by considering a $T(m)$-group $G$ with continuous homomorphism $f : G \to K$ from $G$ onto $K$. Clearly, for $U$ an open neighbourhood of the identity in $K$, $f^{-1}(U)$ is an open neighbourhood of the identity in $G$ and thus contains a normal subgroup, $N$ of index strictly less than $m$. The image $f(N)$ in $K$ is also a normal subgroup of index strictly less than $m$ and $f(N) \subseteq U$. Thus, $K$ is a $T(m)$-group. ■

In Section 1.4, we presented varieties of topological groups generated by classes of topological groups. In a similar fashion, we shall define the wide variety of topological groups generated by a class of topological groups.

The following theorem is analogous to Theorem 1.4.1, and, as with varieties, suggests there is a smallest wide variety of topological groups that contains a non-empty class $\Omega$ of topological groups. It indicates also, that just as with varieties generated by classes of topological groups, any topological group contained in this smallest wide variety can be obtained by just one application of each of the operators $H$, $S$ and $C$.

**Theorem 6.1.4.** ([64], Proposition 0.2) *For any non-empty class $\Omega$ of topological groups, $HSC(\Omega)$ is closed under $H$, $S$ and $C$ and hence is a wide variety of topological groups.*

**Proof.** From Remark 1.4.2 we see that for a non-empty class of topological groups $\Omega$, $CS(\Omega) \subseteq SC(\Omega)$. Now, if $G \in SH(\Omega)$, then there exists $K \in \Omega$ and continuous homomorphism $f$ defined on $K$ such that $G$ is is topologically isomorphic to a subgroup of $f(K)$. Clearly, $f^{-1}(G)$ is a subgroup of $K$ and $G = f(f^{-1}(G))$. Therefore, $G \in HS(\Omega)$. Next, if $G \in CH(\Omega)$, then for each $i \in I$, $I$ an index set, there exists $K_i \in \Omega$ and
continuous homomorphism $f_i$ defined on $K_i$ such that $G$ is topologically isomorphic to the product $\prod_{i \in I} f_i(K_i)$. Let $g$ be the continuous homomorphism defined on $\prod_{i \in I} K_i$ as $g\left(\prod_{i \in I} k_i\right) = \prod_{i \in I} f_i(k_i)$. Clearly, $G$ is the image of $g$ and so $G \in HC(\Omega)$.

Now, for $\Omega$, a non-empty class of topological groups, we have

$$S(HSC(\Omega)) \subseteq HSSC(\Omega) = HSC(\Omega),$$

$$H(HSC(\Omega)) = HSC(\Omega),$$

and

$$C(HSC(\Omega)) \subseteq HSCC(\Omega) \subseteq HSCC(\Omega) = HSC(\Omega).$$

Therefore, $HSC(\Omega)$ is closed under $S$, $H$ and $C$ and so is a wide variety of topological groups.

We now define the wide variety generated by $\Omega$, a class of topological groups, in the same way as $\mathfrak{W}(\Omega)$ was defined.

**Definition 6.1.5.** (cf. [64]) Let $\Omega$ be a class of topological groups and let $\mathfrak{W}(\Omega)$ be the smallest wide variety containing $\Omega$. Then $\mathfrak{W}(\Omega)$ is said to be the **wide variety generated by $\Omega$**. Further, we have $\mathfrak{W}(\Omega) = HSC(\Omega)$.

Note that a quotient group of a topological group $G$ is a continuous image of $G$ and so it is clear that the variety generated by $\Omega$, a class of topological groups, is contained in the wide variety generated by $\Omega$.

**Proposition 6.1.6.** Let $\Omega$ be a class of topological groups. Then $\mathfrak{W}(\Omega)$ is contained in $\mathfrak{W}(\Omega)$, the wide variety generated by $\Omega$.

For a class of topological groups $\Omega$, there is a closer relationship between the topological groups in $\mathfrak{W}(\Omega)$ and those in $\mathfrak{W}(\Omega)$ than it seems at first.

**Proposition 6.1.7.** Let $\Omega$ be a class of topological groups and let $G \in \mathfrak{W}(\Omega)$. Then $G$ is a continuous one-to-one homomorphic image of $G'$ where $G' \in \mathfrak{W}(\Omega)$. 
Proof. By the definition of $\mathfrak{W}(\Omega)$, $G \in HSC(\Omega)$. Therefore, $G$ is a continuous homomorphic image of $K \in SC(\Omega)$. Let $f : K \to G$ be the continuous homomorphism of $K$ onto $G$ and let $\mathcal{T}$ be the given topology on $G$. Further, let $\mathcal{T}'$ be the quotient topology induced on $|G|$ by $f$. Clearly, $(|G|, \mathcal{T}') = G'$ is a quotient group of $K$ and so is contained in $\mathfrak{W}(\Omega)$. Finally, we note that $\mathcal{T}'$ is finer than $\mathcal{T}$ and so the identity mapping $i : G' \to G$ is a continuous, one-to-one homomorphism, giving the result.

A useful corollary to Proposition 6.1.7 concerns discrete groups contained in wide varieties.

**Corollary 6.1.8.** Let $\Omega$ be a class of topological groups. A discrete group $D$ is contained in $\mathfrak{W}(\Omega)$ if and only if $D \in \mathfrak{W}(\Omega)$.

Proof. Clearly, if $D \in \mathfrak{W}(\Omega)$, then $D \in \mathfrak{W}(\Omega)$. Let $D \in \mathfrak{W}(\Omega)$. By Proposition 6.1.7, $D$ is a continuous one-to-one homomorphic image of a group $G \in \mathfrak{W}(\Omega)$. As the topology on $G$ is finer than the topology on $D$, $G$ must be a discrete group. Therefore, $D$ is topologically isomorphic to $G$ and so $D \in \mathfrak{W}(\Omega)$.

Another interesting application of Proposition 6.1.7 is a characterization of the LCV-spaces contained in $\mathfrak{W}(\mathbb{R})$.

**Proposition 6.1.9.** Let $E$ be a LCV-space contained in $\mathfrak{W}(\mathbb{R})$, the wide variety generated by $\mathbb{R}$. Then $E \in \mathfrak{W}(\mathbb{R})$.

Proof. By Proposition 6.1.7, there exists a topological group $G \in \mathfrak{W}(\mathbb{R})$ such that $E$ is a continuous, one-to-one homomorphic image of $G$; that is, there exists $f : G \to E$ such that $f$ is a continuous isomorphism. Let $D$ be a discrete subgroup of $E$, then $f^{-1}(D)$ is a discrete subgroup of $G$. By Lemma 3.2.1, $f^{-1}(D)$ is finitely generated and hence, $D$ is finitely generated. Therefore, by Proposition 3.2.2, $E$ has the weak topology and so is contained in $\mathfrak{W}(\mathbb{R})$.

We have presented here a very brief overview of wide varieties, giving only those results
that are needed in this section. More information on wide varieties can be found in Taylor’s work [64] and more recently, in work by Kopperman, Mislove, Morris, Nickolas, Pestov and Svetlichny in [30] and by Morris, Nickolas and Pestov in [51].

§6.2 The Wide Varieties Generated by Banach Spaces and Locally Compact Abelian Groups

The first two wide varieties of topological groups we consider are those generated (independently) by the class of all Banach spaces and the class of all locally compact abelian groups. For both classes, we easily characterize the wide varieties they generate.

Remark 6.2.1. We know that $\mathfrak{W}(B)$, the variety generated by the class of all topological groups underlying Banach spaces, is exactly the variety of all abelian topological groups. Thus, $\mathfrak{W}(B) = \mathfrak{W}(B) = \mathcal{A}$ where $\mathcal{A}$ is the variety of all abelian topological groups.

Proposition 6.2.2. The wide variety of topological groups generated by the class of all locally compact abelian groups is precisely the variety of all abelian topological groups.

Proof. Clearly, $\mathfrak{W}(\mathcal{L}_A) \subseteq \mathcal{A}$ where $\mathcal{L}_A$ is the class of all locally compact abelian groups and $\mathcal{A}$ is the variety of all abelian topological groups. Let $G$ be an abelian topological group and let $|G|_D$ be the group underlying $G$ equipped with the discrete topology. Clearly, $|G|_D$ is contained in $\mathcal{L}_A$ and the identity homomorphism $i : |G|_D \to G$ is continuous. Therefore, $G \in \mathfrak{W}(\mathcal{L}_A)$ and so $\mathcal{A} \subseteq \mathfrak{W}(\mathcal{L}_A)$, giving the result.

Corollary 6.2.3. The variety of topological groups generated by $\mathcal{L}_A$ is properly contained in the wide variety of topological groups generated by $\mathcal{L}_A$.

Proof. By Proposition 3.2.9, $\mathfrak{W}(\mathcal{L}_A)$ contains no infinite dimensional Banach spaces, whereas by Proposition 6.2.2, $\mathfrak{W}(\mathcal{L}_A)$ contains all Banach spaces. The result then follows from Proposition 6.1.6.
Remark 6.2.4. We note that if a class of topological groups \( \Omega \), contains \( L_A \)—even \( D \), the class of all discrete abelian groups—then \( W(\Omega) \) is the variety of all abelian topological groups. Therefore, considering the diagram at the end of Chapter 5, we see that the following wide varieties of topological groups are all equal to the variety of all abelian topological groups.

(i) \( W(B) \), generated by all Banach spaces;
(ii) \( W(L_m) \), generated by all locally-\( m \) groups for \( m \geq c \);
(iii) \( W(L_S) \), generated by all locally separable groups;
(iv) \( W(L_\sigma) \), generated by all locally \( \sigma \)-compact groups;
(v) \( W(L_A) \), generated by all locally compact abelian topological groups; and
(vi) \( W(D) \), generated by all discrete abelian topological groups.

That is about all there is to say concerning \( W(B) \) and \( W(L_A) \). Therefore, we shall move on to more interesting wide varieties.

§6.3 The Wide Variety Generated by \( FA[0,1] \)

The wide variety generated by \( FA[0,1] \) turns out to be the most interesting of the three wide varieties in question. As we consider this wide variety, \( C_\sigma \), the class of all \( \sigma \)-compact groups makes an appearance, and we discover that \( W(FA[0,1]) = W(C_\sigma) \). We first present a number of wide varieties of topological groups which are obviously equal to \( W(FA[0,1]) \).

Proposition 6.3.1. The following wide varieties of topological groups are equal.

(i) \( W(FA[0,1]) \);
(ii) \( W(K_\omega) \), where \( K_\omega \) is the class of all abelian \( k_\omega \)-groups;
(iii) \( W(FA(X)) \), where \( X \) is a compact Hausdorff non-scattered space.

Proof. From Theorems 4.4.9 and 4.5.26, \( W(FA[0,1]) = W(K_\omega) = W(FA(X)) \), where \( X \) is a compact Hausdorff non-scattered space, and the result follows.

\[ \Box \]
Recall that the variety generated by $FA[0,1]$ contains every countable discrete abelian topological group (see Corollary 3.3.9) and so we have the following result.

**Proposition 6.3.2.** The wide variety of topological groups generated by $FA[0,1]$ contains all countable abelian topological groups.

**Proof.** By Theorem 4.4.9, $\mathcal{W}(FA[0,1]) = \mathcal{W}(K_\omega)$ and so by Corollary 3.3.9, $\mathcal{W}(FA[0,1])$ contains every countable discrete abelian topological group. Clearly every countable abelian topological group $G$ is a continuous homomorphic image of $|G|_D$, the group underlying $G$ with the discrete topology, and as $|G|_D \in \mathcal{W}(FA[0,1]) \subseteq \mathcal{W}(FA[0,1])$, the result follows. 

We now turn to the problem of showing that $\mathcal{W}(C_\sigma) = \mathcal{W}(FA[0,1])$. To do this, we need the following two lemmas.

**Lemma 6.3.3.** Let $X$ be a $\sigma$-compact space. Then there exists a $k_\omega$-space $Y$ such that $X$ is the continuous image of $Y$.

**Proof.** Let $X = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is compact. Let $H_n$ be a homeomorphic copy of $X_n$, disjoint from $H_1, H_2, \ldots, H_{n-1}$, with $f_n : H_n \to X_n$ the corresponding homeomorphism.

Let $Y = \bigcup_{n=1}^{\infty} H_n$ be the free union of the $H_n$. Now, let $Y_n = H_1 \cup H_2 \cup \ldots \cup H_n$, and note that each $Y_n$ is compact. Clearly, $Y = \bigcup_{n=1}^{\infty} Y_n$ and we will show that with this decomposition, $Y$ is a $k_\omega$-space. Let $A$ be a subset of $Y$ such that $A \cap Y_n$ is compact for each $n \in \mathbb{N}$. Clearly, for each $i \in \mathbb{N}$, $A \cap H_i$ is compact and so $H_i \setminus A$ is open in $H_i$, indeed, $H_i \setminus A$ is open in $Y$. Noting that $Y \setminus A = \bigcup_{i=1}^{\infty} (H_i \setminus A)$, $A$ is closed in $Y$. So $Y = \bigcup_{n=1}^{\infty} Y_n$ is a $k_\omega$-decomposition of $Y$. Finally, let $f : Y \to X$ be the mapping defined as follows. If $y \in Y$ then $y \in H_n$ for some $n \in \mathbb{N}$ and so define $f(y) = f_n(y)$. The mapping $f$ is clearly onto, and for an open set $U$ in $X$, $f^{-1}(U) = \bigcup_{n=1}^{\infty} f_n^{-1}(U)$, which is open in $Y$. Therefore, $X$ is the continuous image of $Y$, a $k_\omega$-space.

The following result is analogous to Corollary 4.1.2.
Lemma 6.3.4. Let $X$ and $Y$ be completely regular spaces such that there exists a continuous mapping $\phi : X \to Y$ from $X$ onto $Y$. Then there exists a continuous homomorphism $\Phi : FA(X) \to FA(Y)$ from $FA(X)$ onto $FA(Y)$, where $FA(X)$ and $FA(Y)$ are the free abelian topological groups on $X$ and $Y$ respectively.

Proof. Let $\Phi$ be the continuous homomorphism from $FA(X)$ into $FA(Y)$ that extends naturally from $\phi$, according to the definition of a free abelian topological group. To show that $\Phi$ is an onto homomorphism, take $w \in FA(Y)$ such that $w = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_n^{\varepsilon_n}$, $y_i \in Y$ and $\varepsilon_i = \pm 1$ for each $i = 1, \ldots, n$. For each $y_i$, there exists an $x_i \in X$ such that $\phi(x_i) = \Phi(x_i) = y_i$. Further,

$$
\Phi(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) = \Phi(x_1)^{\varepsilon_1} \Phi(x_2)^{\varepsilon_2} \cdots \Phi(x_n)^{\varepsilon_n} = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_n^{\varepsilon_n} = w.
$$

Therefore, $\Phi$ is an onto continuous homomorphism. $
$

We now have an interesting characterization of $\mathcal{M}(FA[0,1])$ showing it is the same as the wide variety generated by $\mathcal{C}_\sigma$.

Theorem 6.3.5. The wide variety of topological groups generated by $FA[0,1]$ is exactly the wide variety generated by $\mathcal{C}_\sigma$, the class of all abelian $\sigma$-compact groups.

Proof. As $FA[0,1]$ is a $k_\omega$-group, it is $\sigma$-compact and so $FA[0,1] \in \mathcal{M}(\mathcal{C}_\sigma)$ giving $\mathcal{M}(FA[0,1]) \subseteq \mathcal{M}(\mathcal{C}_\sigma)$. Now, let $G$ be an abelian $\sigma$-compact group then by Lemma 6.3.3, there exists a $k_\omega$-space $X$ such that $G$ (as a topological space) is a continuous image of $X$. Further, $FA(G)$ is a continuous homomorphic image of $FA(X)$, by Lemma 6.3.4. We know that $FA(X)$ is a $k_\omega$-group (see [34], Corollary 1) and therefore is contained in $\mathcal{M}(FA[0,1])$ and hence also in $\mathcal{M}(FA[0,1])$. Thus, $FA(G) \in \mathcal{M}(FA[0,1])$ and as $G$ is a quotient group of $FA(G)$ (Proposition 2.2.8), $G \in \mathcal{M}(FA[0,1])$ also. Hence, $\mathcal{M}(\mathcal{C}_\sigma) \subseteq \mathcal{M}(FA[0,1])$ and the result follows.
Corollary 6.3.6. The variety generated by $FA[0, 1]$ is properly contained in the wide variety generated by $FA[0, 1]$.

Proof. By Proposition 3.3.27, $\mathfrak{W}(FA[0, 1])$ is properly contained in $\mathfrak{W}(C_\sigma)$, which is contained in $\mathfrak{W}(FA[0, 1])$. Therefore, $\mathfrak{W}(FA[0, 1])$ is properly contained in $\mathfrak{W}(F)$.

In summary, the following wide varieties are all equal.

(a) $\mathfrak{W}(FA[0, 1])$, the wide variety generated by $FA[0, 1]$;
(b) $\mathfrak{W}(K_\omega)$, the wide variety generated by the class of all abelian $k_\omega$-groups;
(c) $\mathfrak{W}(C_\sigma)$, the wide variety generated by the class of all abelian $\sigma$-compact groups.

In Chapter 5, we considered the variety of topological groups generated by $S$, the class of all separable abelian groups. As we have considered the wide varieties generated by every other class of topological groups studied in Chapter 5, it is appropriate to also examine $\mathfrak{W}(S)$. First, given the fact that $\mathfrak{W}(S) = \mathfrak{W}(B_S) = \mathfrak{W}(\ell_1)$ (see Theorem 5.3.7), we have the following proposition.

Proposition 6.3.7. The following wide varieties of topological groups are equal.

(i) $\mathfrak{W}(\ell_1)$;
(ii) $\mathfrak{W}(B_S)$, where $B_S$ is the class of all separable Banach spaces;
(iii) $\mathfrak{W}(S)$, where $S$ is the class of all separable topological groups.

Note that $\mathfrak{W}(S) = \mathfrak{W}(\ell_1)$ is not the wide variety of all abelian topological groups, as $\mathfrak{W}(\ell_1)$ is a class of $T(c^+)$-groups, and there exist discrete groups of all sizes.

Proposition 6.3.8. The wide variety generated by $\ell_1$ is properly contained in the wide variety of all abelian topological groups.

Proof. Recall that $\ell_1$ has cardinality $c$ and so every topological group contained in $\mathfrak{W}(\ell_1)$ is a $T(c^+)$-group (see Corollary 1.3.9 and Remark 1.3.10). By Corollary 6.1.8, a discrete group contained in $\mathfrak{W}(\ell_1)$ must also be contained in $\mathfrak{W}(\ell_1)$; that is, every discrete group
contained in \( \mathfrak{W}(\ell_1) \) has cardinality strictly less than \( c^+ \) (see Remark 1.3.5(b)). Therefore, \( \mathfrak{W}(\ell_1) \) does not contain all discrete abelian topological groups and is therefore properly contained in the wide variety of all abelian topological groups. \[ \square \]

We now summarize our wide variety results in the following diagram, as we did for varieties in Chapter 5.

\[
\begin{array}{c}
\mathfrak{W}(\mathcal{B}) = \mathfrak{W}(\mathcal{D}) = \mathfrak{W}(\mathcal{L}_A) = \mathfrak{W}(\mathcal{L}_\sigma) = \mathfrak{W}(\mathcal{L}_m) \text{ for all } m \geq c \\
\mathfrak{W}(\mathcal{C}) = \mathfrak{W}(\mathcal{C}_\omega) = \mathfrak{W}(C_\sigma) = \mathfrak{W}(S) = \mathfrak{W}(B_\sigma) = \mathfrak{W}(\ell_1)
\end{array}
\]

The appearance of the question mark in the previous diagram is due to the fact that we do not know the relationship between \( \mathfrak{W}(C_\sigma) \) and \( \mathfrak{W}(S) \). We leave this as an open question.

**Open Question.** In the structure given in Chapter 5, we saw that \( \mathfrak{B}(C_\sigma) \) is properly contained in \( \mathfrak{B}(S) \), the variety generated by the class of all separable abelian topological groups. Thus, it is clear that \( \mathfrak{W}(C_\sigma) \subseteq \mathfrak{W}(S) \). However, the question remains whether these two wide varieties are equal.

We note that \( \mathbb{R}^{\mathbb{N}_0} \) is a separable metric space whose topology is defined by the family \( \{\rho_i : i \in I\} \) of all continuous pseudometrics. Let \( d \) be the metric on \( \mathbb{R}^{\mathbb{N}_0} \). Then the topology on \( \mathbb{R}^{\mathbb{N}_0} \) is defined by the family \( \{d_i : d_i = \rho_i + d, i \in I\} \) of continuous metrics. Therefore, the free abelian topological group on \( \mathbb{R}^{\mathbb{N}_0} \) can be defined using the Graev extension of each \( d_i \); that is, \( FA(\mathbb{R}^{\mathbb{N}_0}) \) is topologically isomorphic to a subgroup of \( \prod_{i \in I} (|FA(\mathbb{R}^{\mathbb{N}_0})|, d'_i) \) where \( |FA(\mathbb{R}^{\mathbb{N}_0})| \) is the group underlying \( FA(\mathbb{R}^{\mathbb{N}_0}) \) and \( d'_i \) is the Graev extension of the metric \( d_i \) (cf. Remark 2.2.3(b), Proposition 2.2.6 and Remark 2.5.6). From Corollary 5.3.2, for each \( i \in I \), \( (|FA(\mathbb{R}^{\mathbb{N}_0})|, d'_i) \) is a separable metric space and thus is in \( \mathfrak{B}(S) \). Therefore, \( FA(\mathbb{R}^{\mathbb{N}_0}) \in \mathfrak{B}(S) \).

We also know that \( FA(\ell_1) \) is topologically isomorphic to \( FA(\mathbb{R}^{\mathbb{N}_0}) \) as \( \ell_1 \) is homeomorphic to \( \mathbb{R}^{\mathbb{N}_0} \) (see [24], Part I, Section 2). Therefore, \( \ell_1 \in \mathfrak{B}(FA(\mathbb{R}^{\mathbb{N}_0})) \), giving \( \mathfrak{B}(S) = \mathfrak{B}(FA(\mathbb{R}^{\mathbb{N}_0})) \), indeed, \( \mathfrak{W}(S) = \mathfrak{W}(FA(\mathbb{R}^{\mathbb{N}_0})) \).
We already know that $\mathcal{W}(C_\sigma) = \mathcal{W}(FA[0, 1])$. Therefore our question of whether $\mathcal{W}(C_\sigma)$ equals $\mathcal{W}(S)$ can be reduced to whether $\mathcal{W}(FA[0, 1])$ equals $\mathcal{W}(FA(\mathbb{R}^\omega))$, that is, whether $FA(\mathbb{R}^\omega) \in \mathcal{W}(FA[0, 1])$. \hfill $\blacksquare$
References


References


