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The strong relevance logics

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Abstract
The tautology p - q - p is not a theorem of the various relevance logics (see Anderson and Belnap [1]) because q is not considered to be relevant in the derivation of final p. We can take this lack of relevance to mean simply that p-q-p could have been proved without q and its -, i.e., p-p. By the same criterion we could say that in ((p- p) -q) -q p-p is not relevant. In general we will say that any theorem A of an implicational logic is strongly relevant if there is no subpart B ! which can be removed from A, leaving the rest still a theorem of the same logic. Such a subpart B - is said to be superfluous.

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THE STRONG RELEVANCE LOGICS

Introduction

The tautology

\[ p \to q \to p \]

is not a theorem of the various relevance logics (see Anderson and Belnap [1]) because \( q \) is not considered to be relevant in the derivation of final \( p \). We can take this lack of relevance to mean simply that \( p \to q \to p \) could have been proved without \( q \) and its \( \to \), i.e., \( p \to p \).

By the same criterion we could say that in

\[ (p \to p) \to q \to q \]

\( p \to p \) is not relevant.

In general we will say that any theorem \( A \) of an implicational logic is strongly relevant if there is no subpart \( B \to \) which can be removed from \( A \), leaving the rest still a theorem of the same logic. Such a subpart \( B \to \) is said to be superfluous.

The strongly relevant form of a logic

If \( L \) is an implicational logic, the theorems of the strongly relevant form \( SR(L) \) of \( L \) are obtained from the theorems of \( L \) by reducing them to strongly relevant theorems by means of the algorithm given below.

The algorithm requires the notion of depth. A \( wfr A \) is said to have depth 0 in \( A \).

If \( B = B_1 \to \ldots \to B_m \to p \) has depth \( d \) in \( A \) any \( B_i \) has depth \( d + 1 \) in \( A \).
The relevance algorithm

To change a theorem $A$ of a logic $L$ to its strongly relevant form, $SR(A)$, in the logic $SR(L)$, proceed in the following way for $d = 1, 2, \ldots$

Remove all superfluous $B \rightarrow s$ of depth $d$ from $A$ from the left. Then remove any superfluous $B \rightarrow s$ of levels less than $d + 1$ from the reduced $A$, starting from depth 1.

Here are some examples from Classical Logic.

1. In $(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow p \rightarrow r$ there are no superfluous subparts of depth 1 and the only one of depth 2 is $q \rightarrow r$. The removal leaves

$$(p \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow p \rightarrow r.$$  

Now $(p \rightarrow q) \rightarrow r$ of depth 1 is superfluous. Removing this yields:

$$(p \rightarrow r) \rightarrow p \rightarrow r.$$  

Now the first $p \rightarrow r$ (depth 2) is superfluous and when removed gives

$$r \rightarrow p \rightarrow r,$$

which then is reduced to

$$r \rightarrow r.$$  

2. In $((p \rightarrow q) \rightarrow p) \rightarrow p$ the $(p \rightarrow q) \rightarrow r$ of depth 2 is all that can be removed yielding

$$p \rightarrow p.$$  

Strongly relevant forms of logics

We will name logics by the combinators associated with their axioms:

<table>
<thead>
<tr>
<th>Combinator</th>
<th>Axiom</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\vdash p \rightarrow p$</td>
</tr>
<tr>
<td>B</td>
<td>$\vdash (p \rightarrow q) \rightarrow (r \rightarrow p) \rightarrow r \rightarrow q$</td>
</tr>
<tr>
<td>B'</td>
<td>$\vdash (p \rightarrow q) \rightarrow (q \rightarrow r) \rightarrow p \rightarrow r$</td>
</tr>
<tr>
<td>C</td>
<td>$\vdash (p \rightarrow q \rightarrow r) \rightarrow q \rightarrow p \rightarrow r$</td>
</tr>
<tr>
<td>S</td>
<td>$\vdash (p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow p \rightarrow r$</td>
</tr>
</tbody>
</table>
\[ W \vdash (p \rightarrow p \rightarrow q) \rightarrow p \rightarrow q \]
\[ K \vdash p \rightarrow q \rightarrow p \]

In general, relevance logics are those without \( K \). First we need two lemmas.

**Lemma 1.** If \( K, B, B' \) and \( I \) hold in \( L \) and

(i) \( Q \) is in a positive position in \( A(Q) \) then
\[ \vdash_L A(Q) \rightarrow A(P \rightarrow Q); \]

(ii) \( Q \) is in a negative position in \( A(Q) \) then
\[ \vdash_L A(P \rightarrow Q) \rightarrow A(Q). \]

**Proof.** We prove (i) and (ii), where only one instance of \( Q \) or \( P \rightarrow Q \) is being replaced, by induction on the depth \( d \) of \( Q \) in \( A(Q) \).

If \( d = 0 \) \( A(Q) = Q \) and \( \vdash_L Q \rightarrow (P \rightarrow Q) \)

If \( d > 0 \) and \( A(Q) = A_1 \rightarrow \ldots \rightarrow A_i(Q) \rightarrow B \), there are 2 cases:

If \( d \) is odd, \( Q \) is in a negative position in \( A(Q) \) and in a positive position in \( A_i(Q) \), so by the induction hypothesis
\[ \vdash A_i(Q) \rightarrow A_i(P \rightarrow Q). \]

By \( B' \)
\[ \vdash (A_i(P \rightarrow Q) \rightarrow B) \rightarrow A_i(Q) \rightarrow B \]

and by \( B \) applied \( i - 1 \) times we get
\[ \vdash A(P \rightarrow Q) \rightarrow A(Q). \]

If \( d \) is odd, \( Q \) is in a positive position in \( A(Q) \) and in a negative position in \( A_i(Q) \), so by the induction hypothesis
\[ \vdash A_i(P \rightarrow Q) \rightarrow A_i(Q) \]

and by \( B \) and \( B' \) we obtain as above:
\[ \vdash A(Q) \rightarrow A(P \rightarrow Q). \]

Multiple copies of \( Q \) and \( P \rightarrow Q \) can be replaced in \( A(Q) \) and \( A(P \rightarrow Q) \) by repeating this procedure.

**Lemma 2.** If \( K, B, B' \) and \( I \) hold in \( L \), then the Relevance Algorithm will reduce any theorem \( A \) of \( L \) that is not \( p \rightarrow p \).

**Proof.** If \( A \) has a negative part of the form \( P \rightarrow Q \), write \( A = B(P \rightarrow Q) \), then by Lemma 1 (ii) \( P \rightarrow \) is superfluous in \( A \).

As \( A \) has a superfluous part, the Relevance Algorithm will reduce it (though not necessarily that part first, or even at all).
If $A$ has no negative part of the form $P \rightarrow Q$, it must be of the form:

$$A = p_1 \rightarrow p_2 \ldots \rightarrow p_n$$

where at least one $p_i = p_n$.

Unless $n = 1$, the Relevance Algorithm reduces this $A$ to $p_n \rightarrow p_n$

**Theorem 1.** If $K$, $B$, $B'$ and $I$ hold in $L$ then

$$SR(L) = \{ p \rightarrow p | \text{p is a variable} \}.$$

**Proof.** By Lemma 2, the Relevance Algorithm will reduce the length of any theorem that is not $p \rightarrow p$. Thus the algorithm will reduce any theorem to $p \rightarrow p$.

It can probably easily be shown that if $K$, $B$, $B'$ (but not $I$) hold in $L$, then

$$SR(L) = \{ p \rightarrow q \rightarrow p | \text{p, q are variables} \}.$$

The same holds if $L$ has $K$ and $B$ or $K$ and $B'$, but not $I$ nor even $K$.

**Theorem 2.** $SR(KI) = \{ p \rightarrow p | \text{p is a variable} \}$.

**Proof.** It is easy to show that every theorem $T$ of $KI$-logic is of the form

$$T_1 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow A$$

or

$$T_2 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow B \rightarrow A.$$

We can also assume that $A$ in $T_1$, and $A$ and $B \rightarrow A$ in $T_2$ are not theorems of $KI$ logic, since in that case we would have:

$B \rightarrow A$ or $A = C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow A_1$

or $A$ or $B \rightarrow A = C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow B_1 \rightarrow A$, so that $T_1 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow A_1$,

$T_2 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow B \rightarrow C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow A_1$,

$T_1 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow B_1 \rightarrow A_1$

$T_2 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow B \rightarrow C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow B_1 \rightarrow A_1$

$T_2 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow A_1$

or $T_2 = B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A \rightarrow C_1 \rightarrow \ldots \rightarrow C_k \rightarrow A_1 \rightarrow B_1 \rightarrow A_1$

which are in the above forms but with $A_1$ smaller than $A$.

Now

$$SR(T_2) = SR(A \rightarrow B \rightarrow A) = SR(A \rightarrow A)$$

$$SR(T_1) = SR(A \rightarrow A).$$

Let

$$A = A_1 \rightarrow A_2,$$
then
\[ SR(A \to A) = SR((A_1 \to A_2) \to A_1 \to A_2) \]
\[ = SR((A_1 \to A_2) \to A_2) \]
\[ = SR(A_2 \to A_2) \]
or
\[ SR(A \to A) = A_2 \to (A_1 \to A_2) \]
\[ = SR(A_2 \to A_2). \]

We can continue this reduction till we get \( p \to p \) for some variable \( p \).

The same result probably holds for \( \text{KBI} \) and \( \text{KB'} \).

For logics without \( \text{K} \) the situation is much more complex as is shown below:

**Lemma 4.**

(i) \( SR(\text{BB'}IW) \not\subseteq SR(\text{BCI}) \cup SR(\text{BCIW}) \cup SR(\text{BB'}I); \)

(ii) \( SR(\text{BB'}I) \cap SR(\text{BCI}) \not\subseteq SR(\text{BCIW}) \cup SR(\text{BB'}I); \)

(iii) \( SR(\text{BCIW}) \cap SR(\text{BCI}) \not\subseteq SR(\text{BB'}IW) \cup SR(\text{BB'}I); \)

(iv) \( SR(\text{BCIW}) \not\subseteq SR(\text{BCI}); \)

(v) \( SR(\text{BB'}I) \not\subseteq SR(\text{BCI}). \)

**Proof.**

\[
((p \to q) \to p) \to (p \to q) \to (p \to q) \to q
\]
is a theorem of \( SR(\text{BB'}I) \) and \( SR(\text{BCI}) \) but not of \( SR(\text{BCIW}) \) nor \( SR(\text{BB'}IW) \) wherein it is reduced to

\[
((p \to q) \to p) \to (p \to q) \to q.
\]

Hence (ii) holds.

The last formula above is a theorem of \( SR(\text{BB'}IW) \) but not of \( SR(\text{BCIW}) \) where it is reduced to

\[
p \to (p \to q) \to q.
\]

Neither is it a theorem of \( SR(\text{BCI}) \) or \( SR(\text{BB'}I) \). Hence (i) holds.

This last formula above is a theorem of \( SR(\text{BCIW}) \) and \( SR(\text{BCI}) \), but not of \( SR(\text{BB'}IW) \) or \( SR(\text{BB'}I) \), so (iii) holds.

\[
(p \to (p \to (p \to q))) \to p \to q
\]
is a theorem of \( SR(\text{BCIW}) \) but not of \( SR(\text{BCI}) \), so (iv) holds.
\[(p \to r \to q) \to ((p \to q) \to r) \to p \to (p \to q) \to q.\]

is a theorem of \( SR(\text{BB'T}) \) but not of \( SR(\text{BCIW}) \).

**Theorem 3.** The systems \( SR(\text{BCIW}) \), \( SR(\text{BB'IW}) \), \( SR(\text{BCI}) \), and \( SR(\text{BB'I}) \) are mutually independent.

**Proof.** By Lemma 4.

We should note that the relevance requirements here, although similar, are stronger than those in [2] where effectively only superfluous subparts of depth 1 have been removed.

The work can be extended to logics with the connectives \( \land \) and \( \lor \) where parts \( \land B, B\land, B\lor \) and \( \lor B \) can be superfluous.

Again all theorems of positive classical, intuitionistic and \( \text{BCK} \) logics reduce to the form \( p \to p \). For relevance logics, as before, the situation is more complex.

**References**


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