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Cancellation laws for BCI-algebra, atoms and p-semisimple BCI-algebras

Abstract
We derive cancellation laws for BCI-algebras and for p-semisimple BCI-algebras, show that the set of all atoms of a BCI-algebra is a p semisimple BCI-algebra and that in a p-semisimple BCI-algebra and = are the same.

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CANCELLATION LAWS FOR BCI-ALGEBRA, ATOMS AND P-SEMISIMPLE BCI-ALGEBRAS

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Abstract. We derive cancellation laws for BCI-algebras and for p-semisimple BCI-algebras, show that the set of all atoms of a BCI-algebra is a p-semisimple BCI-algebra and that in a p-semisimple BCI-algebra ≤ and = are the same.

1. Introduction. BCI-algebras, first introduced by Iséki in [1], can be defined as follows:

**Definition 1** An algebra \((X; \ast, 0)\) of type \((2, 0)\) is a BCI-algebra if for all \(x, y, z \in X\).

BCI-1 \((x \ast y) \ast (x \ast z) \leq z \ast y\)

BCI-2 \(x \ast (x \ast y) \leq y\)

BCI-3 \(x \leq x\)

BCI-4 \(x \leq y\) and \(y \leq x\) imply \(x = y\)

BCI-5 \(x \leq y\) iff \(x \ast y = 0\)

The following well known properties of BCI-algebras are used below.

(1) \((x \ast y) \ast z = (x \ast z) \ast y\)

(2) \(0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)\)

(3) \(x \ast 0 = x\)

(4) \(x \ast (x \ast (x \ast y)) = x \ast y\)

(5) \(x \ast x = 0\)

(6) \(x \leq 0 \Rightarrow x = 0\).


**Theorem 1** If \((X; \ast, 0)\) is a BCI-algebra and \(x, y, z \in X\) then:

(i) \(x \ast y \leq x \ast z \Rightarrow 0 \ast y = 0 \ast z\);

(ii) \(y \ast x \leq z \ast x \Rightarrow 0 \ast y = 0 \ast z\).

**Proof** (i) If \(x \ast y \leq x \ast z\), by BCI-5,

\[(x \ast y) \ast (x \ast z) = 0\]

and so by BCI-1 and BCI-5,

\[0 \ast (z \ast y) = 0\]

and by (2),

\[(0 \ast z) \ast (0 \ast y) = 0\].
Hence by $BCI$-5

$$0 \ast z \leq 0 \ast y.$$

We now apply the same cancellation procedure to this as we did to $x \ast y \leq x \ast z$, this time “cancelling” the 0 to give:

$$0 \ast y \leq 0 \ast z$$

$$\therefore \ 0 \ast y = 0 \ast z.$$

(ii) If $y \ast x \leq z \ast x$, by $BCI$-5,

$$(y \ast x) \ast (z \ast x) = 0.$$ 

$BCI$-1 and (1) give

$$((y \ast x) \ast (z \ast x)) \ast (y \ast z) = 0$$

so

$$0 \ast (y \ast z) = 0$$

giving, as above,

$$0 \ast y \leq 0 \ast z.$$ 

As in (i) this gives $0 \ast y = 0 \ast z$.

**Corollary** If $\langle X; \ast, 0 \rangle$ is a $BCI$-algebra and $x, y, z \in X$ then

(i) $x \ast y = x \ast z \Rightarrow 0 \ast y = 0 \ast z$

(ii) $y \ast x = z \ast x \Rightarrow 0 \ast y = 0 \ast z$.

We have two further properties resulting from the above cancellation laws:

**Theorem 2** If $\langle X; \ast, 0 \rangle$ is a $BCI$-algebra and $x, y, z \in X$ then:

(i) $x \leq x \ast z \Rightarrow 0 \leq z$

(ii) $x \ast y \leq x \Rightarrow 0 \leq y$.

**Proof** (i) If $x \leq x \ast z$, by (3) $x \ast 0 \leq x \ast z$ and so by Theorem 1 (i) $0 \ast z = 0 \ast 0$. This gives $0 \ast z = 0$ i.e. $0 \leq z$.

(ii) If $x \ast y \leq x$, by (3), $x \ast y \leq x \ast 0$ and so by Theorem 1 (ii) $0 \ast y = 0 \ast 0 = 0$, so $0 \leq y$.

3. **P-Semisimple Algebras.** These were introduced by Lei and Xi in [2] as follows:

**Definition 2** A $BCI$-algebra $\langle X; \ast, 0 \rangle$ is p-semisimple if

$$(\forall x \in X)(0 \ast x = 0 \Rightarrow x = 0).$$

In these algebras we find that $\leq$ becomes the same as $=$.

**Theorem 3** If $\langle X; \ast, 0 \rangle$ is a p-semisimple $BCI$-algebra and $x, y \in X$ then if $x \leq y$ also $x = y$.

**Proof** If $x \leq y$, $x \ast y = 0$ by $BCI$-5. Also by (5), $x \ast y = x \ast x$, so by the corollary to Theorem 1, $0 \ast y = 0 \ast x$.

As $(0 \ast x) \ast (0 \ast x) = 0$, we have $(0 \ast y) \ast (0 \ast x) = 0$ and by (2), $0 \ast (y \ast x) = 0$.

As $BCI$-algebras are closed under $\ast$, $y \ast x \in X$, so if the algebra is p-semisimple, $y \ast x = 0$.

By $BCI$-4, $x = y$.

Our cancellation laws can now be strengthened.

**Theorem 4** If $\langle X; \ast, 0 \rangle$ is a p-semisimple $BCI$-algebra and $x, y, z \in X$ then:

(i) $x \ast y \leq x \ast z \Rightarrow y = z$;

(ii) $y \ast x \leq z \ast x \Rightarrow y = z$.

**Proof** (i) If $x \ast y \leq x \ast z$, by Theorem 1(i) we get $0 \ast z = 0 \ast y$ and so $(0 \ast z) \ast (0 \ast y) = 0$.

By (2) this gives $0 \ast (z \ast y) = 0$, so if the algebra is p-semisimple we have $z \ast y = 0$ i.e. $z \leq y$.

The result then follows from Theorem 3.

(ii) Similar.
Corollary If $\langle X; *, 0 \rangle$ is a p-semisimple $BCI$-algebra and $x, y, z \in X$ then
(i) $x * y = x * z \implies y = z$;
(ii) $y * x = z * x \implies y = z$.


Definition 3 An element of a $BCI$-algebra $\langle X; *, 0 \rangle$ is an atom if

$$(\forall x \in X)(x * a = 0 \implies x = a)$$

Definition 4 $L(X) = \{x \in X \mid x$ is an atom of $X\}$

Meng and Xin prove in [3]:

Theorem 5 If $\langle X; *, 0 \rangle$ is a $BCI$-algebra then
(i) $a$ is an atom iff $a = 0 * (0 * a)$;
(ii) $(\forall x \in X) \ 0 * x \in L(X)$.

(Theorem 5(i) also follows from (4) and (i).)

The following simple representation of $L(X)$ results:

Theorem 6 $L(X) = \{0 * x \mid x \in X\}$.

Meng and Xin prove that $L(X)$ is a $BCI$-algebra. The following result of Lei and Xi [2]:

Theorem 7 If $\langle X; *, 0 \rangle$ is a $BCI$-algebra then $X$ is p-semisimple iff

$(\forall x \in X) \ 0 * (0 * x) = x$.

and Theorem 5(i) give us:

Theorem 8 If $\langle X; *, 0 \rangle$ is a $BCI$-algebra $\langle L(X); *, 0 \rangle$ is a p-semisimple $BCI$-algebra.

A final result on $L(X)$ is the following:

Theorem 9 If $\langle X; *, 0 \rangle$ is a $BCI$-algebra then $L(L(X)) = L(X)$.

Proof By Theorem 6,

$$L(L(X)) = \{0 * x \mid x \in L(X)\} = \{0 * (0 * y) \mid y \in X\}$$

Similarly

$$L(L(L(X))) = \{0 * (0 * (0 * z)) \mid z \in X\},$$

so by (4)

$$L(L(L(X))) = L(X).$$

Hence as $L(L(L(X))) \subseteq L(L(X)) \subseteq L(X)$ we have $L(L(X)) = L(X)$.

5. Powers. In [2] Lei and Xi define a new operation $+$ by:

Definition 5 $x + y = x * (0 * y)$

and show that if $\langle X; *, 0 \rangle$ is a p-semisimple $BCI$-algebra then $\langle X, + \rangle$ is an abelian group.

In [3] Meng and Wei use the same operation to define powers of elements by:

$x^1 = x$

$x^{n+1} = x * (0 * x^n)$,

(though $m x$ instead of $x^m$ might have been in better keeping with $+$).

The following are new properties of this form of exponentiation:

Theorem 10 If $x$ is an element of a $BCI$-algebra $\langle X; *, 0 \rangle$ then:

(i) $(0 * x)^n = 0 * x^n$;
(ii) $(0 * x)^n = (\ldots((0 * x) * x)\ldots) * x$

(where there are $n$ $x$s on the right hand side).

Proof (i) By induction on $n$. 

\( n = 1 \) - obvious.

Assuming (i) for \( n \),

\[
(0 \ast x)^{n+1} = (0 \ast x) \ast (0 \ast (0 \ast x)^n)
\]
\[
= (0 \ast x) \ast (0 \ast (0 \ast x^n))
\]
\[
= 0 \ast (x \ast (0 \ast x^n))
\]
\[
= 0 \ast x^{n+1}
\]

(ii) By induction on \( n \).

\( n = 1 \) - obvious.

Assuming (ii) for \( n \), by (c) above, (1) and (4):

\[
(0 \ast x)^{n+1} = (0 \ast (0 \ast (0 \ast x^n))) \ast x
\]
\[
= (0 \ast x^n) \ast x
\]
\[
= (0 \ast x)^n \ast x
\]
\[
= (\ldots ((0 \ast x) \ast x) \ldots) \ast x.
\]

as required.

REFERENCES