Vortex sheet technique for the computation of unsteady flows

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University of Wollongong

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VORTEX SHEET TECHNIQUE FOR THE COMPUTATION OF UNSTEADY FLOWS

A thesis submitted in fulfilment of the requirements for the award of the degree

Ph.D.

from

THE UNIVERSITY OF WOLLONGONG

by

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The author wishes to dedicate this thesis to his parents and eldest brother, Ching-Hsin Yu, for they have exerted important influences on his life.
This thesis is concerned with the study of vortex sheet technique for the computation of unsteady flows. Development has been made on the discretization of the vortex sheet by replacing the constant distribution of vorticity in each element with a linear distribution. This refinement increases accuracy and reduces the strength of the singularities at the boundaries of the discretized sheet elements, which is brought about by the process of discretization.

This numerical method is used in the study of the following unsteady flow phenomena: (i) the roll-up of a vortex sheet and the analyses of a spiral core; (ii) the shedding of vorticity from a semi-infinite plate; and (iii) the transient heat transfer around a heated plate with flow separation.

The following are the findings of this study.

(1) The circumferential average of spiral core structure is stable as seen from the result in which the geometrical centre of the core coincides with the centroid of the core.

(2) During the time interval following the initial shedding of vorticity, the growth rate of the separation bubble near the leading edge of a semi-infinite plate, is a constant, and is independent of the Strouhal number.

(3) Unlike many well known vortex sheets which have a strong tendency to roll up, the vortex sheet generated from the leading edge of a semi-infinite plate shows the structure of the small-scale spirals, which are suspected of being caused by Kelvin-Helmholtz's instability. These small-scale spirals become very prominent in the present results when the flow is perturbed under resonance frequency. These results
are in good agreement with the results from a well known experiment done by Pierce (1961).

(4) The rolling up of the vortex sheet has strong influences on the heat transfer coefficient and the temperature distribution near the point of flow separation.

(5) The maximum value of instantaneous Nusselt numbers decreases with time while the location of the maximum value of instantaneous Nusselt numbers moves downstream. Contrary to the expectation of many works, the points of flow reattachment and the maximum Nusselt numbers do not coincide.

It is evident that the applications of the vortex sheet technique are very suitable for the computation of unsteady flow. The free shear layer is represented by a vortex sheet and the evolution of the vortex sheet is traced and used to study the properties, such as velocity, stream function and velocity potential, in the flow field. These also allow further investigation into the other characteristics of the flow field, such as forces, temperature, and energy.
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1. INTRODUCTION

The rapid development of computer technology during the past few decades has brought a strong impact to the study of fluid mechanics. The use of computer technology, for its high speed capability, has opened the way for the study of unsteady flow phenomena. In experimental research, there are many transient properties which are still difficult to measure. Computer simulation provides an efficient way of gaining knowledge about unsteady flow. There are various numerical techniques in this area of application and one of them is the vortex sheet method. The present work is concerned with the development and applications of the vortex sheet method.

The mechanics of unsteady two-dimensional vortex sheets are of fundamental importance in obtaining an understanding of the large-scale behaviour of two-dimensional and quasi-three-dimensional thin shear layers in a variety of fluid mechanical applications. Examples include

(1) the phenomenon of a typhoon eye;

(2) the process of combustion blast;

(3) the roll-up of the approximately two-dimensional shear layer shed by a elliptically loaded wing;

(4) the roll-up of the vortex sheet about a Delta wing;

(5) the unsteady separated flow past a sharp-edged body such as a building, tower, bridge, automobile, fins of a heat exchanger, solar collectors, hills, chimneys, and blades of a turbine;
(6) the unsteady separated flow past a circular cylinder such as cables in the air or under water, tubes in the heater exchanger, and submarine in the sea;

(7) vortex development behind stenoses and cardiac valves.

In these and other examples, the vortex sheet is usually regarded as a thin shear layer in which a two-dimensional line distribution of circulation $\Gamma$ is concentrated. This gives rise to an initial-value problem describing the vortex-sheet motion. Since circulation is often expressed by a linear integration of vorticity along the sheet, it is more convenient to calculate the strength of vorticity than to calculate the circulation. Actually, vorticity has provided a powerful qualitative description for many of the important phenomena of fluid mechanics. The formation and separation of boundary layers have been so described in terms of the production, convection and diffusion of vorticity. The dissipation of energy at a rate practically independent of the viscosity in turbulent flows is explained by the amplification of vorticity and by the stretching of vortex lines. The lift on a wing is explained by the bound vorticity and trailing vortex structure. Most recently the concept of the coherent structure in turbulent shear flows has led to the visualization of such flows as a superposition of organized, 'deterministic' vortices whose evolution and interaction is the turbulence.

The ability of vortex sheet technique also leads to the realization that almost all irrotational motion, such as surface waves on uniform irrotational fluid or air bubbles of moderate size in water, may be treated as problems of vortex motion. Thus the free surface of water waves or the boundary of a two-phase flow can be considered as a vortex sheet whose position satisfies an integro-differential equation.
However, it can hardly be said that the behaviour of unsteady flow is thoroughly understood, although numerical methods have been developed and some digital computer solutions of the equations for unsteady inviscid flow have become available. In general, unsteady flow phenomena cannot be predicted from a knowledge of steady flow phenomena; and in particular, the unsteady flows which contain flow separation are extremely complex. In other words, it is necessary to study further in this field in order to gain more understanding on the nature of unsteady flows.

The aim of the present report is to provide a more generalized representation of the vorticity distribution as a tool to investigate various unsteady flows. Before developing a new vortex sheet technique, the earlier vortex methods for the flow simulation are reviewed in Chapter 2. The details of the present numerical model, which make use of the linear distribution of vorticity in each sheet element, are described in Chapter 3. The roll-up of the vortex sheets shed from an elliptical loaded wing have been further studied by the present numerical method. Furthermore, the study of the flows inside vortex core gives results for comparisons with that of earlier studies. These are discussed in Chapter 4. In Chapter 5, the problems of unsteady flow around a semi-infinite plate have been investigated using the vortex sheet technique together with the Schwarz-Christoffel transformation. Different situations of unsteady flow and the decay of vorticities are also considered in the present numerical experiments. In addition, not only the evolution of the vortex sheet has been traced with the time, the other instantaneous characteristics influenced by the roll-up vortex sheet, i.e., temperature distribution and heat transfer coefficient around a heated plate, have also been investigated in Chapter 6. Finally the concluding remarks and recommendations for the further works are presented in Chapter 7.
2. A REVIEW OF VORTEX METHODS FOR FLOW SIMULATION

Many of the results which established our knowledge about vortex methods can be found in the review by Fink and Soh (1974), Clements and Maull (1975), Saffman and Baker (1979), Leonard (1980), Saffman (1981), Aref (1983) and Sarpkaya (1989). The methods of representing a vortex sheet can be classified into three categories. The first is a system of point vortices which has been used extensively. The second is an array of vortex sheet elements which emphasize the continuous characteristics in the flow. The third is a continuous distribution of vorticity over a finite-area which is also called vortex patches. None of these systems is entirely satisfactory. Aref (1983) pointed out that vortex patches provide a convenient mathematical model to study the effects of finite vortex cores. Point vortices is the simplest system to represent a vortex sheet on that one degree of freedom is required for each vortex. It must be noted that all these three representations of vortex sheet are singular solutions of the two-dimensional Euler equation. In order to understand the outline of vortex methods for flow simulation, research in this area is reviewed in five sections. These are (i) the point vortex method, (ii) the vortex sheet element method, (iii) the cloud-in-cell method, (iv) the rediscretization technique, and (v) the other vortex methods, which include the redistribution technique, the higher-order panel method, the higher-order discretization scheme and the others which involve special smoothing techniques.

2.1 The point vortex method

The first attempt to simulate a flow by a vortex method was carried out by Rosenhead (1931), who approximated the motion of a two-dimensional vortex sheet by following the movement of a system of point vortices. Thus, the
vorticity originally concentrated along a line (vortex sheet) was concentrated even further into a finite number of point vortices. In this point vortex method the scalar vorticity field $\omega$ has the representation

$$\omega(x,t) = \sum_{i=1}^{N} \Gamma_i \delta [x-x_i(t)]$$

(2.1.1)

where $\delta$ is the two-dimensional Dirac delta function, $x_i = (x_i,y_i)$ are the locations of the $N$ vortices, and the $\Gamma_i$ are their respective circulations. In general, the circulations $\Gamma$ of the considered flow field is defined by

$$\Gamma = \int \omega \, dx$$

(2.1.2)

To satisfy the inviscid vorticity transport equation,

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = 0$$

(2.1.3)

or

$$\frac{D \omega}{Dt} = 0$$

(2.1.4)

the velocity of each vortex must be given by the value of the velocity field at its present location,

$$\frac{d \mathbf{x}_i}{dt} = \mathbf{u}(\mathbf{x}_i,t)$$

(2.1.5)

and the boundary condition at a solid surface with unit normal $\mathbf{n}$ is
\[
\mathbf{u} \cdot \mathbf{n} \bigg|_{\text{surface}} = 0
\]  
(2.1.6)

The velocity field is computed as the solution to the Poisson equation

\[
\nabla^2 \mathbf{u} = - \nabla \times (\omega \hat{\mathbf{e}}_z)
\]  
(2.1.7)

where \(\hat{\mathbf{e}}_z\) is the unit vector in the \(z\)-direction. If the two-dimensional flow field has no interior boundaries and the fluid is at rest at infinity, the solution to the Poisson equation may be written as the Biot-Savart integral,

\[
\mathbf{u}(\mathbf{x}, t) = -\frac{1}{2\pi} \int \frac{(\mathbf{x} - \mathbf{x}') \times \hat{\mathbf{e}}_z \omega(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d\mathbf{x}'
\]  
(2.1.8)

By substituting (2.1.1) into (2.1.8), it is found that \(x_i\) are the solution to the following system of \(2N\) nonlinear ODE's:

\[
\frac{d\mathbf{x}_i}{dt} = -\frac{1}{2\pi} \sum_{j \neq i} \frac{\mathbf{x}_i - \mathbf{x}_j \times \hat{\mathbf{e}}_z \Gamma_j}{|\mathbf{x}_i - \mathbf{x}_j|^2}
\]  
(2.1.9)

All results of Rosenhead (1931) were calculated by hand. Rosenhead gave an estimate of the errors introduced at each step of the calculation. He pointed out that the method could not be used for a large number of steps since "the accumulation of the errors introduced at each step may become large enough to destroy the value of the approximation". Westwater (1935) made the attempt to compute the unsteady motion of two-dimensional vortex sheets. He replaced the vortex sheet by a finite number of point vortices whose positions were calculated
numerically. As his calculations lead to very realistic rolling-up patterns, it was believed for a long time that the behaviour of an actual vortex sheet can be followed by means of its simplest substitute - an array of vortices in an inviscid fluid. However, Birkhoff & Fisher (1959) criticized this idea in the case studied by Rosenhead; their criticism is based upon the energy conservation law and the ergodic nature of motion of a vortex system in an inviscid fluid. Birkhoff & Fisher (1959) and Hama & Burke (1960) revealed that, in regions of concentration of vorticity, the position of the vortices became so random that the vortex sheet would cross over itself - a physical impossibility. Birkhoff & Fisher (1959) concluded that the absence of viscosity from this model was responsible for the random movement. However, Hama and Burke (1960) performed a perturbation analysis of this model and found that the fundamental modes of a sinusoidally perturbed sheet had a non-uniform vorticity distribution which had not been represented by the row of evenly spaced or equal strength vortices previously used to model the sheet. On repeating the calculation using unevenly spaced vortices, they found a smoother development of the flow. At later times the random motion of the vortices still became apparent.

However, point vortex models have been used in numerical calculations by several authors. For example, Abernathy & Kronauer (1962), who studied the formation of the von Karman vortex street; and Chorin & Bernard (1973), who used point vortices for studies of slight viscous flows. With the advent of computers, many research workers used more point vortices with more accurate time integration schemes in order to achieve "better" results. In many cases the vortices became a chaotic state of motion. Takami (1964) and Moore (1971) applied the discrete-vortex approximation to re-examine the Westwater's problem and found that strong randomization of the vortices takes place near the tip of the sheet in contrast to Westwater's previous computation. Birkhoff (1962) used N=20, fourth order Runge-Kutta time integration and various initial
conditions to examine the Rosenhead's point-vortex approximation and obtained an irregular point motion. Chorin & Bernard (1973) reported that a point vortex approximation to a vortex sheet cannot be taken too literally, since a point vortex induces a velocity field which becomes unbounded, and cannot approximate a bounded field in any reasonable norm. However, they pointed out that as soon as the velocity field of the point vortices is smoothed out and made bounded, the approximation becomes reasonable.

By introducing a tip vortex to represent the tightly rolled portion of the vortex sheet, Moore (1974) eliminated the chaotic motion which was a feature of some earlier studies and calculated the details of the outer portion of the spiral. He mentioned that in his work and the work of Takami, there are two possibilities for failing to obtain a smooth spiral structure in their earlier studies. The first is that the exact solution of the discrete system converges to the solution of the equations governing the continuous sheet as the number of point vortices tends to infinity. The chaotic motion would then be due to a failure of the numerical method to integrate the discrete system correctly. This failure could spring from either the use of too crude an integration method - for example, the time step might be too large - or because the discrete equations themselves are numerically unstable in the sense that small perturbations, always present in a numerical calculation, are rapidly amplified. The second possibility is that the exact solution of the discrete system does not converge to the solution of the equation governing the continuous sheet as the number of point vortices tends to infinity.

After a careful analysis, Fink & Soh (1974) pointed out that a point-vortex representation implicitly involves the neglect of logarithmic terms which represent the local contribution in the Cauchy principal-value integrals for the self-induced velocity field of a vortex sheet (details will be discussed in § 2.2 The vortex sheet element method). They argued that the logarithmic terms,
although initially vanishingly small, amplify through small sheet distortions, leading eventually to the chaotic motion observed by most authors. The proposed remedy is to rediscretize the sheet at every time step by replacing the current set of vortices with another equi-distantly spaced set which also represents discretely the same vorticity distribution. However, the contention that Helmholtz instability is the fundamental difficulty in vortex sheet calculations has been contested by Fink and Soh (1978), who claim that integration error causes the chaotic motion. In support of this, Fink and Soh introduced a rediscretization method aimed at improving the accuracy of the evaluation of the integro-differential equation and found that chaotic motion disappeared. Moreover, a deliberately introduced disturbance did not amplify or cause chaotic motion.

Moore (1981) provided the detailed analysis of the point-vortex method. He used a uniform circular vortex sheet to examine the point vortex method and found that chaotic motion, which often arises when the point vortex representation is used, is due to the amplification of numerically introduced disturbances. At the same time he pointed out that the linear smoothing method of Longuet-Higgins and Cokelet (1976) and the rediscretization method of Fink and Soh (1978) are able to reduce the instability. Additionally, by substituting van der Vooren's (1980) formula into the integro-differential equation governing the evolution of a vortex sheet, Moore (1981) also showed the error term in the point vortex approximation, which was introduced by Rosenhead (1931). He also cited that the error in the point vortex approximation is less than $e_N$, where

$$e_N = \frac{\Gamma_e}{4\pi N} \max I \frac{\partial^2 z}{\partial \Gamma^2} |_g \left\| I \frac{\partial z}{\partial \Gamma} |_g \right\| ^2$$

(2.1.10)
Thus the point vortex approximation is a consistent approximation because the truncation error $e_N$ goes to zero as the number of integration points goes to infinity, although it is a crude approximation for any practical value of $N$.

In order to examine Rosenhead's point-vortex approximation, Krasny (1986) used discrete Fourier analysis, which have been used by Moore (1981), to seek the source of difficulty experienced in the previous numerical studies. It is shown that perturbations introduced spuriously by computer roundoff error are responsible for the irregular point-vortex motion that occurs at a smaller time as the number of points increased. Krasny stated that the source of computational error can be controlled through either higher precision arithmetic or a new filtering technique (see § 2.5.5 The filtering technique).

The equations of motion for the point vortices have the disadvantage if compared with that for cloud-in-cell or vortex-in-cell method (see § 2.3 The cloud-in-cell method or vortex-in-cell method) that the number of operations per time step necessary to compute the vortex velocities from their positions scales as $N^2$, where $N$ is the number of vortices. The comprehensive calculation by Ashurst (1979) employed an increasing number of point vortices, from 1 to 800 as the calculation progressed, and required 250 hours on a CDC 6600 computer. Furthermore, the steeply increasing velocity that a point vortex produces in its immediate vicinity makes the equations of motion a rather stiff system of differential equations. And although integration schemes now exist that can handle such systems satisfactorily, too much time is spent computing details of close vortex encounters that are usually of little relevance to smoothed-out and average vorticity which is the objective of the calculation. As a check on the accuracy of the discrete-vortex method, van de Vooren (1980) suggested that the Hamiltonian and vorticity centroid could be used as invariants. Since the set of
discrete vortices is changing at each time step, Bromilow & Clements (1984) suggested that the Hamiltonian, in its usual form, is of no significance.

2.1.1 Some applications of the point-vortex method

Gerrard (1967) was the first to apply the discrete vortex model to the flow about the circular cylinder. He mentioned that

"A particularly important part of the vortex model of the flow and the most difficult aspect of the design of the model, is the positioning of the points of appearance of the elementary vortices representing the vortex sheets and the determination of their strength".

Most investigators utilized the Kutta condition at the edges of the plate in order to relate the strength of the nascent vortices with their locations. Clements (1973) used a discrete vortex model to approximate the free shear layers in two-dimensional inviscid flow. The motion of the shear layers is computed from the velocities of the discrete vortices, which in turn are derived through a Schwarz-Christoffel transformation of the boundary. Kuwahara (1973) used a discrete-vortex approximation to simulate two-dimensional vortex shedding behind an inclined flat plate. The strengths of the point vortices are determined from the Kutta condition. Each point vortex moves in the velocity field induced from other vortices and the original irrotational flow. Successive point vortices form a configuration which simulates a pair of vortex sheets separating from both edges of the plate. The randomization of vortices inherent to the discrete-vortex approximation will lead to a turbulent wake. Kuwahara (1978) used the same method to simulate two-dimensional vortex shedding behind a circular cylinder. The boundary layer is divided into partitions and each of them is replaced by a point vortex with the same circulation as the corresponding partition. The
position of the generated vortex is supposed to be the centre of the partition along the flow and to be a small distance $\varepsilon$ apart from the surface of the cylinder. He mentioned that the $\varepsilon$, the distance between the generation point of the nascent vortices and the surface of the cylinder must be about half the thickness of the boundary layer. He also suggested that if the body has sharp edges like a flat plate, then the generation of vortices can be determined by the Kutta condition. However, if the body has no such edges but is smooth like a circular cylinder, the flow can be simulated by replacing the boundary by a point vortex array.

One of the most successful point-vortex models was presented by Sarpkaya (1975) for the calculation of flow separating from an inclined plate. He determined the strength of the nascent vortices $\delta \Gamma$ in terms of the average (say $U_m$) of the velocities of the first four vortices in each shear layer using the relation $\delta \Gamma = (1/2) U_m^2 \delta t$, where $\delta t$ is the time interval for the introduction of the nascent vortices into the wake. Kiya & Arie (1980) simulated a unsteady separated flow from a plate with the discrete-vortex model and introduced a "velocity point" in the vicinity of each edge of the plate to determine the strength of the nascent vortices by the velocity $U_s$ at this point with the relation $\delta \Gamma = (1/2) U_s^2 \delta t$.

Nagano et al. (1982) used a discrete-vortex model to analyse a two-dimensional flow past a rectangular prism and mentioned that three main causes of the loss in vorticity were (i) the destruction of vorticity near the wall due to the interaction with the boundary layer along the body, (ii) the cancellation of opposite-signed vortices due to mixing and (iii) the dissipation of vorticity due to three-dimensional turbulence. Kiya, Sasaki & Arie (1982) used a discrete-vortex model to simulate the separation bubble and suggested to reduce the circulation of elemental vortices as a function of their ages in order to represent the three-dimensional deformation of vortex filaments. Their calculation, when compared
with wind tunnel measurements (Ota & Kon (1974); Ota & Itasaka (1976); Ota & Narita (1978); Hiller & Cherry (1981); and Kiya, Sasaki & Arie (1982)), showed reasonable predictions for the time-mean and r.m.s. values of the velocity, the surface-pressure fluctuations, and the correlations between these fluctuating components over most of the separation bubble.

Bromilow & Clements (1984) used a discrete-vortex model, incorporated an amalgamation and rediscretization technique, to study the interaction of two and three growing line vortices of different strengths and to assess the suitability of the method of such simulation.

By using a discrete-vortex model, Cooper et al. (1986) predicted the velocity field around the semi-infinite plate with a square leading edge. The fluid was assumed to be inviscid, incompressible and irrotational everywhere except at the position of the line vortices. The Kutta condition determined the initial vortex positions. The vortices were convected according to the local velocity resulting from the irrotational flow and the remaining vortices. Additionally, they studied the heat transfer of a bluff body in separated flow. Thompson et al. (1986) combined finite-difference and discrete-vortex methods to study acoustically perturbed two-dimensional separated flow around a heated plate. The complex velocity potential is given by (a) the steady irrotational flow, (b) the periodic fluctuating flow (simulating an acoustic field), and (c) the flow induced by the vortices and their images. The results indicate that by decreasing the reattachment length, a substantial increase (up to 60%) in the heat transfer can be achieved. Distortion of the isotherms near the large-scale vortex structures is due to the local induced rotational motion that moves fluid up and down as it passes over such structure. Hence, the isotherms are further apart behind a vortex and they are forced to be closer together immediately in front of a vortex.
Thompson et al. (1987) used a discrete vortex model combined with the finite element solution of the acoustic field and a theory by Howe (1975) to predict the sources of sound in the flow over a rectangular cylinder in a duct. The flow is assumed to be two-dimensional, incompressible, inviscid, isentropic and irrotational except at points where elemental line vortices are embedded in the flow. The boundary layer around the bluff body is replaced by discrete surface vortices which provide a zero velocity at the surface. At each time step, the newly created surface vortices are free to be convected into the flow and new surface vortices are created. Their results showed that the dominant acoustic source region is found near the trailing edge of the cylinder.

Downie, Bearman & Graham (1988) applied a discrete vortex method as a local solution to model vortex shedding from the bilges of a barge hull of rectangular cross-section. They then used an analytic method for predicting the barge's coupled motion in three degrees of freedom, which includes the effects of the main component of viscous damping. Evidently the discrete vortex method has provided a time domain solution for separated flow about bluff bodies. At separation, the Kutta-Joukowski condition is satisfied and the rigid boundaries are modelled by conformal mapping techniques or by the use of distributions of surface singularities (the boundary integral equation method).
2.2 The vortex sheet element method

Although the discrete vortex technique has been successfully applied in many physical or engineering simulations, there are still some difficulties in convergence; i.e., growing randomness in the locations of the equivalent vortices is such that the line joining them, in the same sequence as in the initial state, crossed itself and could no longer represent a vortex sheet (see Birkhoff (1962); Takami (1964); and Moore (1971)). Fink & Soh (1978) recommended a more careful process of discretization, which is free of artificial smoothing procedures. It includes consideration of Cauchy principal value integrals and higher-order terms which are neglected in discrete vortex method. The conjugate complex velocity \( u(z) - iv(z) \) induced by a vortex sheet of strength \( \gamma(s) \) situated on the contour \( C \) is given by

\[
\frac{1}{2\pi i} \int_C \frac{\gamma(s)}{z - z(s)} \, ds
\]  

(2.2.1)

In the process of discretization, a vortex sheet is described by \( n \) "segments" and its motion is determined by the induced velocities of these segments. Consider a pivotal point \( z^*_k \) which lies within the \( k \)th segment extending from \( z_{k-1/2} \) to \( z_{k+1/2} \). The velocity induced by the \( k \)th segment to point \( z_j \) is given by

\[
\Delta q_{jk} = \frac{1}{2\pi i} \int_{\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{\gamma_k(\sigma)}{z_j - z(\sigma)} \, d\sigma
\]  

(2.2.2)

where \( \sigma = s - s_k \) and \( s_k \) measured along the sheet from a convenient origin to the pivotal point \( z_k \). The limits of the integral involve the arc length \( \Delta s_k = l s_k + 1/2 - s_k \) \( l \). Therefore, equation (2.2.1) becomes
The vorticity density within each segment is expressed as a convergent series

\[ \gamma_k(\sigma) = \sum_{p=0}^{\infty} \frac{\gamma_k^{(p)}(\sigma)}{p!} \]  

(2.2.4)

where \( \gamma_k^{(p)} \) are the pth derivatives of \( \gamma_k \) with respects to \( \sigma \) at \( z_k \). Since the vortex sheet is represented by \( n \) "segments" and the velocity of each segment is determined by the Cauchy principal values of the integral, the technique described by Fink & Soh (1978) is termed to be the "vortex sheet element method" in order to distinguish from the point vortex method. In the vortex sheet element method of Fink & Soh (1978), the sheet element is straight and has a constant distribution of vorticity.

Using the concept of irreversibility in the generation of vorticity and a vortex sheet starting at a distance very close to the cylinder, Soh (1983) simulated the flow separation from the surface of a circular cylinder. As the generation of vorticity is an irreversible process, vorticities in the boundary layer cannot be destroyed. The reduction of net vorticity is brought about by the generation of vorticity of opposite sign. The sheet segment is considered to be straight and has a constant distribution of vorticity. Soh reported that the separation rate of the vorticity is dependent on the spacing between the initial vortex sheets and the cylinder. Round off errors have the same effect of viscous diffusion. The symmetry of the flow will not persist for very long in time. The wake will eventually be broken down to form oscillating wake structures as shown in the works of Faltinsen & Pettersen (1982) and Sarpkaya & Shoaff (1979).
Using the vortex sheet element model, Soh, Hourigan & Thompson (1988) represented a laminar boundary layer by a vortex sheet and determined the shedding rate and location of shed vorticity. In their results, they showed the process of capturing of small vortex sheet spirals by a larger spiral. Additionally, making the comparison between symmetrical and asymmetrical flows, they indicated that the vortex sheet technique of flow simulation is capable of representing both roll-up and random structures in the flow.

Vortex sheet element method has also been applied to the calculation of free surface waves. Longuet-Higgins & Cokelet (1976) developed a numerical technique for the computation of surface waves in water. The only independent variables in their problem are the coordinates and velocity potential of marked particles at the free surface. At each time-step the following integral equation is solved for the new normal component of velocity \( \frac{\partial \phi}{\partial n} \),

\[
\int_C \frac{\partial \phi}{\partial n} \ln R \, ds = \mathbf{P} \int_C \phi \, d\alpha - \pi \phi_0
\]  

(2.2.5)

where \( \frac{\partial \phi}{\partial n} \) is the normal component of velocity, \( \mathbf{P} \) indicates the principal value of the integral. For a typical point, \( P \), on the boundary, its polar coordinates with respect to an arbitrary point \( Q(r_0,\theta_0) \) in the interior of contour \( C \), are \((R,\alpha)\). The velocity potential of \( Q \) is denoted by \( \phi_0 \).

For solving the normal component of velocity \( \frac{\partial \phi}{\partial n} \), Longuet-Higgins & Cokelet (1976) approximated the integrals in equation (2.2.5) by linear sums and distinguished the singular terms from the sums. The non-singular integrand terms are calculated by a 4-point quadrature formula. The \( \frac{\partial \phi}{\partial n} \) in the singular terms are approximated by differentiating a four-order Lagrangian interpolation
polynominal. The arclengths are approximated by cubic splines. (Details see Longuet-Higgins & Cokelet (1976))

The method of Longuet-Higgins & Cokelet (1976) has the advantage that the marked particles become concentrated near regions of sharp curvature. The method was tested on a free, steady wave of finite amplitude and applied to unsteady waves, produced by initially applying an asymmetric distribution of pressure to a symmetric, progressive wave. The freely running wave then steepens and overturns. Longuet-Higgins & Cokelet demonstrated that the surface remains rounded till well after the overturning takes place. They also pointed out that all numerical computations using a rectangular grid have an important practical limitation. To obtain acceptable accuracy, the grid-spacing must be reasonably small, at least compared to the local scale of the flow or the radius of curvature of the free surface.

Soh (1985) computed for two dimensional nonlinear large amplitude water wave by representing the free surface with vortex sheet elements. The flow field enclosed by the free surface is irrotational and incompressible and the velocity of the flow is calculated from the induced velocity of the vortex sheet. In this study, Soh demonstrated three practical examples of vortex sheet element method: (i) The collision of a solitary wave with a vertical wall; (ii) The breaking of a dam; and (iii) Collapse of a column of water. From the comparisons with experimental data, the vortex sheet element method can be applied to simulate the large amplitude waves although it has a slight underestimation of the peak amplitude of the waves.
2.3 The cloud-in-cell method or vortex-in-cell method

The idea of the "cloud-in-cell" method comes from the "particle-in-cell" method, which has been used in plasma simulation (Birdsall & Fuss (1969); Buneman (1973)). Christiansen (1973) used the cloud-in-cell method to simulate the motion of the flow due to a distribution of "vorticity". He replaced the particles in the particle-in-cell method with vortices in his algorithm. After that, the cloud-in-cell method acquired the new name of "vortex-in-cell" method. In fact, the essential of these numerical techniques, such as the area-weighting rule, are similar.

The cloud-in-cell method (CIC) and vortex-in-cell (VIC) method retain the Lagrangian treatment of the vorticity field and at the same time they solve the Poisson equation for the velocity field in a fixed Eulerian mesh. If one integrates analytically the Poisson's equation relating the velocity field to the vorticity field in two or three dimensions, the result is a direct Biot-Savart integration or summation over all the vortices making the velocity field. With N vortices, a time step in a direct summation scheme will involve evaluating N-1 terms for the velocity which displaces a single vortex, and this totalled N(N-1) terms per time step. For large N, such a code will be very time consuming. This places a rather modest upper limit on the number of elements which can be used in a fixed CPU time environment.

Consider this model in two dimensions. Suppose a given vortex, say at \((x_n, y_n)\) with circulation \(\Gamma_n\), resides within a certain mesh cell, and contributes incremental vorticity, \(\delta \omega(l)\), to each of the four mesh points at the corners \((l=1,2,3,4)\) according to the area weighting scheme,
\[ \delta \omega (l) = \frac{A_1 \Gamma_n}{h_x h_y} \]  

(2.3.1)

where \( h_x, h_y \) are the mesh spacing in the x, y direction respectively and the \( A_1 \) represent four areas of interaction between the mesh and the square-shaped vortex, and CIC method is often referred to as the area-weighting technique (see scheme diagram in figure 2.3.1). This method assumes each "point" vortex to possess uniform vorticity within a unit square such that the corresponding unit amount of vorticity can be credited to the surrounding mesh points through a bilinear interpolation. A smoothing of the vorticity distribution will result from this interpolation.

![Area weighting scheme for the cloud-in-cell method](image)

**Figure 2.3.1** Area weighting scheme for the cloud-in-cell method

After all the vorticity has been distributed among the mesh points, a finite difference scheme for the Poisson equation in stream function, \( \psi \), is solved. For example, one might solve

\[
\left( \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} \right) \psi_{i,j} = -\omega_{i,j} \]  

(2.3.2)
where $\delta^2/\delta x^2$ and $\delta^2/\delta y^2$ are the three-point central difference operators. From the stream function, velocities at the mesh points can be computed; for example, by central differences,

$$u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2h_y}$$  \hspace{1cm} (2.3.3)

and

$$v_{i,j} = \frac{-\psi_{i+1,j} - \psi_{i-1,j}}{2h_x}$$  \hspace{1cm} (2.3.4)

and then bilinear interpolation (area weighting) can be used to determine the velocity of vortex $n$,

$$u_n = \sum_{l=1}^{4} u(l) \frac{A_l}{h_x h_y}$$  \hspace{1cm} (2.3.5)

where $h_x$, $h_y$ are the grid spacing assumed uniform in the $x$, $y$ directions respectively. The vortex markers $(x_n, y_n)$ are moved forward in time by the first order Euler time integration,

$$x_n (t+\Delta t) = x_n(t) + u_n \Delta t$$  \hspace{1cm} (2.3.6)

$$y_n (t+\Delta t) = y_n(t) + v_n \Delta t$$  \hspace{1cm} (2.3.7)

where $\Delta t$ is the time step. In this way, the vorticity distribution at $t+\Delta t$ can be computed and the procedure repeats to give the motion of the vorticity.
In the vortex-in-cell (VIC) method, one discretizes the interface into discrete vortex elements with zero core radius and calculates the velocity field from the vorticity stream function on an Eulerian grid. Since the stream function is independent of the vortex core radius, the mathematical singularity inherent in the Green's function method is avoided. The vorticity stream function is found by the fast Fourier transform (FFT) method, which also uses the reality of the physical variables. The number of operations in the FFT is proportional to $M_x M_y \ln(M_x M_y)$, where $M_x$ and $M_y$ are the numbers of mesh points in the $x$ and $y$ directions, which are independent of the total number of discrete vortices $N$. Therefore the VIC method can treat a problem with a sharp interface using a very large number of vortices without increasing the computational effort, which is not possible by the Green's function method.

A disadvantage of the CIC or VIC method is having to account for grid effect. The method and its fine-scale behaviour are sensitive to the size of the mesh, the surface boundary conditions, the number of vortices and time step. The flow features of scale smaller than the grid cannot be accurately resolved. Thus the CIC or VIC simulation may not reproduce the actual flow as correctly as those methods such as the point vortex method and the vortex sheet element method which calculate the flow situation of all discrete sections directly.

To counteract unwanted grid effects, Wang (1977) did some extensive two-dimensional simulations in which he claimed to have improved on the CIC method by using cubic splines for interpolation and subsequently applying a Guassian shape factor or "filter" in wave vector space. This provides each vortex with a finite core and distributes vorticity within the core with cylindrical symmetry; he then obtained the results which is insensitive to grid effect.
Murman & Stremel (1982) mentioned that there are several advantages to the CIC method. The solution of the velocity field can be accomplished by a standard finite difference method. For the velocity field calculation, a singular vortex marker is effectively distributed over a mesh cell volume with a constant vorticity, and thus the velocity singularity of the Biot-Savart method is automatically eliminated by the finite mesh spacing feature inherent in a discrete solution, that is, the artificial viscosity is a result of the truncation error. The Lagrangian treatment of the vorticity field retains the desirable feature that the vortex wake remains well defined throughout the computational domain. A simple redistribution and interpolation scheme is used to yield the velocity field which is required for the evaluation of the vorticity field.

At the same time Murman & Stremel (1982) also pointed out that the principal disadvantage of the CIC method is that the numerical errors are created by the interpolation and distribution process. The accuracy in the determination of the velocity field is limited by the usual truncation error of finite difference solutions.

With regard to the computational effort, Leonard (1980) analysed the CPU time per time step for the CIC method and found that the operation count per time step for this portion of the computation is $O(N)$. Therefore, he cited that vortex-in-cell methods can be used to improve dramatically the computational efficiency if a large number of vortex elements are required. Zufiria (1989) stated that the introduction of the grid in the CIC method produces a considerable reduction in the computational cost at the expense of losing spatial accuracy in the results. Therefore, Zufiria focused his attention on the spatial resolution and showed that the effect of the grid in the CIC method is qualitatively equivalent to the effect of the surface tension. Zufiria also gave a notice that the loss of accuracy comes essentially from the vorticity smoothing
and velocity interpolation step. As the Poisson solver is quite accurate, he therefore suggested that new distribution and interpolation schemes should be considered to improve the behaviour of the vortex-in-cell method.

2.3.1 Some applications of the VIC or CIC method

Christiansen (1973) first used the cloud-in-cell (CIC) method to simulate the motion of a continuous hydrodynamic fluid and found a number of interesting results, in particular the behaviour of the interaction of finite sized vortex structures. He suggested that an incompressible flow field should be thought of in terms of the mutual interaction of a set vortex elements (action-at-a-distance model), rather than in terms of the velocity field itself (field model). Christiansen & Zabusky (1973) used a vortex-in-cell approach to study vortices interactions. Fusion of vortex structures together with elongation and rotation were observed in the development of the vortex sheet. However, in Christiansen's original VIC method, the vorticity of the point vortices is assigned to the corners of the mesh block which surround the vortices. The distribution of the vorticity of a vortex to the four corner of the mesh block is by an area weighting rule.

Meng & Thomson (1978) used two numerical methods to investigate some nonlinear hydrodynamic problems. One is based upon the well-known Green's function method, which is a Lagrangian method using the Biot-Savart law; the other is the vortex-in-cell (VIC) method, which is a Lagrangian-Eulerian method. Both methods treat the interface as a shear layer and represent it by a distribution of point vortices. The VIC method applies the FFT (fast Fourier transform) to solve the streamfunction - vorticity equation in an Eulerian grid, and computational efficiency is further improved by using the reality properties of the physical variables.
Baker (1979) applied the CIC method to simulate the roll-up of a vortex sheet and artificially introduced small perturbations to the grid. He found that once the process of vortex amalgamation is well under way, the emerging large scale behaviour is relatively insensitive to the precise details of the initial perturbations. The most striking features are the emergence of a small scale structure and the smoothing of the spiral core. Additionally, he mentioned that the main disadvantage in using this technique is the loss of detail of the spiral nature of the vortex sheet roll up. The advantage is that it can prevent the vortex sheet from kinking and lead to the formation of a spiral. Baker (1980) showed that the iteration scheme necessary to calculate the vortex-sheet strength at every time step converged, and produced several computational results extending well into the nonlinear regime with good precision. Coupling the iteration scheme with a cloud-in-cell calculation of vortex velocities, a code capable of following collective interactions between several interfacial structures is produced.

Aref & Siggia (1980) calculated the rolling up of the two-dimensional shear layer by the cloud-in-cell method to obtain the r.m.s. velocity and the Reynolds shear stress, which are much larger than experimental results. Couët, Buneman & Leonard (1981) stated that VIC is sometimes referred as CIC. They simulated the three-dimensional incompressible flows with a vortex-in-cell method, which include a quadratic spline interpolation scheme and a filtering technique. The velocity field is calculated by creating a mesh-record of the vorticity field, and then integrating a Poisson's equation via the fast Fourier transform (FFT) to generate a mesh-record of the velocity. They concluded that their Lagrangian treatment of the vorticity allows the motion of vortex rings over long distances without diffusive effects caused by differencing of convective derivatives. No undesirable grid effects or numerical instabilities were found in their computations.
Stansby & Dixon (1982) used the cloud-in-cell technique to calculate the convection of the vortices and made comparisons with direct summation methods. They mentioned that the small-scale behaviour generated by the grid appears not to contaminate large-scale structures and concluded that the inclusion of the secondary separation brings the pressure distributions and vorticity structures as subcritical and supercritical Reynolds numbers into good agreement with experiment. Basuki & Graham (1987) used the VIC method to calculate the impulsively started flow past a 11 percent-thick Joukowski airfoil at 30-degree incidence. They concluded that the method predicts too strong a roll-up, an unrealistic suction peak, and excessively large fluctuations in the lift. Smith & Stansby (1988) used random walks method for the diffusion and vortex-in-cell method for the convection of vortex elements to simulate an impulsively started flow around a circular cylinder. The streamlines patterns, which they obtained, are in good agreement with that of experiments by Bouard & Coutanceau (1980).

The cloud-in-cell method has been modified by Graham & Cozens (1988) to include viscous diffusion calculated by using finite difference on the mesh to give a mixed Eulerian-Lagrangian Navier-Stoker solver. It is used to compute impulsively started flow past sharp right-angled edges and edges with small rounding. They showed that the effect of viscosity on the impulsive edge flow is to reduce the initial rate of growth of circulation. They also pointed out that the viscous case does not approach asymptotically to the inviscid situation as Reynold number increases. This may be due to the influence of secondary separation or to the uncertainty in distinguishing between shed and boundary-layer vorticity.

Tryggvason (1989) stated that the main attraction of the VIC method is its computational efficiency and its desingularization property is similar to that of
the vortex blob methods. To carry out the comparison of the vortex-in-cell method and the vortex blob method, he used these two methods to simulate the roll-up of a vortex sheet under the influence of large amplitude Kelvin-Helmholtz instabilities. He considered that the VIC method is properly classified as a grid-based vortex blob method and possesses regularization properties. Tryggvason also compared Christiansen's original VIC method with vortex-blob method and stated that Tryggvason (1988) has an additional improvement implemented in his code. That is as the vortex sheet is stretched, the Tryggvason's code increases the number of point vortices through a redistribution process, so that the spacing between vortices will be maintained fairly even (This is the same concept as the rediscretization technique, see § 2.4 The rediscretization technique). In the VIC method the computational time depends only weakly on the number of interface points, so a very long interface causes no particular difficulties. However, in the vortex blob method, the computational time is proportional to $N^2$ and therefore it is more expensive in computer time to simulate a longer interface. Tryggvason (1989) also pointed out that the VIC is not the ideal tool to study sensitive issues such as the detailed shape of the spiral.

From the above applications of CIC or VIC method, it can be concluded that the reliability of the CIC or VIC method is dependent upon what accuracy is needed and what kind of distribution is required. For a given flow problem, comparison can be made between theory and a number of numerical results in order to acquire a quantitative understanding of how the inaccuracies are related to the length of the time integration. Evidently, the CIC or VIC method has its advantages and disadvantages. The advantages are (i) it is a rapid way to calculate the velocity field; (ii) it can use a much larger number of vortex marker, so that a large-scale structure can easily be observed; (iii) the computational time is less than that of the direct summation method (Biot-Savart law). On the other
hand, its disadvantages are (i) it introduces a pseudo-viscosity into the flow and
gives a finite width to the vortex sheet because vorticity is spread over a mesh
cell; (ii) it has a grid effect that can smooth out the appearance of the fine-scale
structures in the actual flow (this may not happen in a direct summation method);
(iii) most numerical errors arise from its anisotropic nature of the distribution
and interpolation schemes; (iv) it is sensitive to the size of the mesh, the number
of vortices and the time-step.
2.4 The rediscretization technique

It is suggested that a more accurate method is needed for the computation of vortex sheet, which will preclude the numerical instabilities found in the discrete vortex method. Fink & Soh (1974) pointed out that if the vortices were equally spaced along the sheet, it eliminates the logarithmic term and thereby reduces the numerical errors. At each time step, the vorticity density is represented by an entirely new set of equi-distant vortices whose strength is adjusted to give a good representation of vorticity distribution. This is the essence of the rediscretization method. Evidently, this procedure does not resolve all of the computational errors particularly in regions where the curvature of the sheet is large, e.g., the region close to the centre of the vortex spiral. Furthermore, the curve-fitting errors incurred in the process of interpolation at every time step may accumulate as time increases.

Sarpkaya & Shoaff (1979) used the method of Fink & Soh (1974) to rediscretize the vortices along a vortex sheet at each time step. The end points of the sheets attached to the cylinder are the separation points which are determined through boundary layer calculations. They introduced a scheme in which circulation was reduced to account for the effects of three-dimensionality of the flow and were able to obtain results in good agreement with experiments. The Strouhal number was essentially unaffected by the reduction of circulation.

Bromilow & Clements (1982) amalgamated clusters of vortices in the roll-up region into a single equivalent vortex as in the Moore's method (1974). They also subjected the remaining part of the vortex sheet to rediscretization process, which is an extension of Fink & Soh's method (1974), by using the cubic spline and a four-point Lagrangian interpolation routine to account for the curvature
effect. This prevented vortices on the evolving parts of the sheet from becoming too close or too distant.

Faltinsen and Pettersen (1982, 1987) and Pettersen and Faltinsen (1983) used a vortex tracking scheme in which sources and dipoles were distributed over boundaries and free shear layers. The boundary value problem for the velocity potential was solved at each time step. For the flow about bluff bodies, a boundary layer calculation was performed to predict the separation points. The shear layers were fed at the separation points and rediscretized, after every Eulerian-convection step, using Fink & Soh's (1974) method. The potential jump at the ends of new segments were found by a linear interpolation. In treating the problem of vortex-sheet roll-up, Faltinson and Pettersen (1982) again used Fink & Soh's (1974) rediscretization technique and Moore's (1975) core amalgamation scheme. The combination of these two schemes delays the instabilities in the roll-up process.

Higdon & Pozrikidis (1985) pointed out that, although the method of Fink & Soh demonstrated good results for their studies, it required the use of an amalgamation model to deal with the singular behaviour at the tips of the sheet. Baker (1980) proposed a test of the Fink & Soh method on a closed vortex sheet, for which the amalgamation procedure would be unnecessary. Baker corrected some minor errors in the analysis of Fink & Soh and applied the method to the roll-up of a circular vortex sheet with sinusoidal vorticity distribution. He found that the method was unreliable, leading to the sheet crossing itself, and this problem became increasingly severe as the number of points was increased. Baker concluded that the smooth results obtained by Fink & Soh were primarily attributable to the strong flow induced by the amalgamated vortex, and that the method was not suitable for treating a general class of flows.
Van de Vooren (1980) made the attempt to modify and stabilize the point vortex method. He modified the Cauchy-principal-value integral so that the singularity in the integrand was removed. He used an eighth-order finite-difference scheme for discretization and for time stepping by the Runge-Kutta method. However, irregular motion of vortices occurred when 80 points were used. Van de Vooren attributed this irregular motion to round-off errors in the computed solution of the point-vortex equations. Birkhoff & Fisher (1959) claimed that the irregular motion was a property of the vortex solution of the point-vortex equations.

The rediscretization technique of Fink & Soh (1974) together with the vortex sheet element method modified with linear distribution of vorticity (see Chapter 3) was used in the present report. The vortex sheet rolled up smoothly even without the procedure for amalgamation. It is in contrast to Baker's (1980) conclusion that the smooth results obtained by Fink & Soh (1974) were primarily attributable to the strong flow induced by the amalgamated vortex. In the study of flow around a semi-infinite plate, the rediscretization technique was also used once per ten time-steps in order to prevent the computational errors in the small-curvature region. The results show that the evolution of small-scale spirals along the vortex sheet is in good agreement with the experimental results done by Pierce (1961) (details see Chapter 5).
2.5 The other vortex methods

Later investigators have sought to repair the presumed defects in the point-vortex approximation using other schemes. In general, these can be classified into three categories: (i) The redistribution technique. (ii) The higher-order accurate discretization of the integro-differential equation. (iii) The smoothing techniques, i.e., the artificial viscosity and the filtering technique were introduced.

2.5.1 The redistribution technique

For handling the instability of the vortex sheet, Siddiqi (1987) used a redistributed scheme, without amalgamation in calculating the roll-up of a vortex sheet for an elliptically loaded wing. Starting at the tip and using the slope information, he fitted, at each time step, a cubic spline to the sheet and redistributed each line vortex into two equal vortices when the sheet segment representing the vortex stretched beyond a prescribed amount. The basic aim of the redistribution was to keep the length of the sheet-segments per line vortex approximately constant so as to ensure that each roll-up turn is represented by an adequate number of vortices even as it stretches. Note that this scheme is different from the rediscretization technique mentioned in § 2.4.

2.5.2 The higher order panel method

Graham (1985) pointed out that it is possible to carry out the integration along the sheet by considering piecewise constant (or linear) distributions of vorticity over short (usually straight) elements of sheet, e.g., Faltinsen &
Pettersen (1983), Higdon & Pozrikidis (1985), and Pullin (1978). It appears from these results that the higher the order of the representation, the smoother the resulting integral function along the sheet, and hence smaller instabilities develop in a given time interval.

Higher-order panel methods based on the slender-body approximation were developed by Hoeijmakers & Vaatstra (1983) for conical and quasi-conical flow. They used a second-order panel method to compute the velocity field induced by the multiple-segmented vortex sheet and attempted to preclude the numerical instabilities. The vortex sheet was represented by a doublet distribution. On each panel, the functions for doublet distribution and the position were approximated by a piecewise quadratic representations. To evaluate the integral in the induced velocity to second-order accuracy, Hoeijmakers & Vaatstra adopted the small-curvature expansion scheme mentioned by Hess (1975). To avoid spurious effects from the panel edges where the panelwise representations for the geometry and doublet distribution may be discontinuous, the velocity is computed at the midpoints of the panels. With this method, Hoeijmakers & Vaatstra demonstrated a set of complicated vortex sheet motion in a reliable and stable manner, i.e., the roll-up of the wake behind an elliptically loaded wing, a ring wing (nacelle), a fuselage/part-span flap/wing combination, and a delta wing with leading-edge vortex sheets.

Hoeijmakers & Rizzi (1984) applied two methods to compute the detailed flow field about wings with leading-edge vortex flow. In the first method (also called free-vortex sheet panel method which includes second-order panel method) potential flow is assumed and the free shear layers were modelled by vortex sheets. The vortex core were modelled by isolated line vortices. The position and the strength of the vortex sheets and isolated vortices were to be solved for as part of the solution. The second method was based on Euler's
equation in which rotational flow was allowed everywhere and vortical flow regions were "captured" implicitly as an integral part of the solution. Two solutions were presented and compared in detail. They found that in general the vortical flow region as computed by the Euler code occupies a larger region than enclosed by the vortex sheet computed by the panel method.

Other prominent higher-order panel methods are Boeing LEV-Model by Johnson et al. (1980); the VORSEP-Model of NLR (National Aerospace Lab of the Netherlands) by Hoeijmakers and Bennekers (1979), Hoeijmakers et al. (1983) and Hoeijmakers (1985).

2.5.3 The higher-order method

As mentioned in the above sections, the motions of the sheet may be approximated by calculating the trajectories of the marker points. Various approaches are distinguished by the method of approximation. The point-vortex method is only a crude approach which replaces the continuous sheet with an array of vortices. In a more complex level, there is a higher-order approximation such as the method of Higdon & Pozrikidis (1985) which uses a collection of circular arcs along the sheet with piecewise trigonometric polynomials to represent the circulation.

Higdon & Pozrikidis (1985) divided the calculations of vortex-sheet into two broad classes: discrete methods, including the point-vortex method and its variants; and spectral methods, such as that employed by Meiron et al. (1982). The discrete methods may be easily applied to any set of initial conditions, but have proven unreliable, resulting in chaotic motions after a finite time. The
spectral methods seem to give more reliable results, but can be implemented efficiently only for special prototype problems.

In order to solve the difficulty encountered in the discrete vortex method, Higdon & Pozrikidis (1985) applied a modified discrete vortex method to study the self-induced motion of vortex sheets. In their method, they used a higher-order discretization scheme replacing the continuous vortex sheet with a collection of circular arcs and the circulation distribution with piecewise trigonometric polynomials. It is found that after a finite time a singularity appears in the motion of the vortex sheet. This singularity takes the form of an exponential spiral with the simultaneous development of singularities in the curvature and in the circulation distribution. Higdon & Pozrikidis claimed that the principal advantage of their method is that it does not depend on any special vortex spacing for its accuracy. Thus additional marker points may be added at any time in the calculation to resolve the fine details of the motion as the sheet evolve. They pointed out that the appearance of singularities in the curvature and in the circulation are a consequence of the rapidly diminishing lengthscale and timescale in the centre of the spiral. Additionally, they conjectured that the shape of the sheet near the point of infinite curvature could be of the form of a simple inflected curve or a spiral of imperceptible size.

2.5.4 The artificial viscosity technique

In many cases of roll-up vortex sheet, strong irregularity takes place in the vortex distribution. In order to avoid this difficulty, Kuwahara & Takami (1973) proposed an improved method in which the irregularity can be suppressed by introducing an 'artificial viscosity'. In a real viscous fluid, the velocity field of a single vortex is not so strong as that of an inviscid fluid. Therefore, Kuwahara
& Takami (1973) pointed out that it is possible to get a more regular pattern if the equations can be modified to include the effect of viscosity. To do so, they changed the existing discretized equations of the motion of the ith vortex, viz.

\[
\frac{dx_i}{dt} = -\frac{1}{2\pi} \sum_{j \neq i} \kappa_j \frac{y_j - y_i}{r_{ij}^2}
\]

(2.5.1)

\[
\frac{dy_i}{dt} = \frac{1}{2\pi} \sum_{j \neq i} \kappa_j \frac{x_i - x_j}{r_{ij}^2}
\]

(2.5.2)

where \( r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 \) and \( \kappa \) is the strength of the vortex, into

\[
\frac{dx_i}{dt} = -\frac{1}{2\pi} \sum_{j \neq i} \kappa_j \left( 1 - \exp \left( -\frac{r_{ij}^2}{4\nu t} \right) \right) \frac{y_j - y_i}{r_{ij}^2}
\]

(2.5.3)

\[
\frac{dy_i}{dt} = \frac{1}{2\pi} \sum_{j \neq i} \kappa_j \left( 1 - \exp \left( -\frac{r_{ij}^2}{4\nu t} \right) \right) \frac{x_i - x_j}{r_{ij}^2}
\]

(2.5.4)

where \( t \) is the time and \( \nu \) the kinematic viscosity of the fluid. Equations (2.5.3) and (2.5.4) are not strictly exact because as Kuwahara & Takami (1973) had stated that "equations (2.5.3) and (2.5.4) are incompatible with the Navier-Stokes equation, whose nonlinearity does not permit the superposition of the vortex fields". Therefore, \( \nu \) in equations (2.5.3) and (2.5.4) should not be taken as the real viscosity, and Kuwahara & Takami (1973) called it the 'artificial viscosity'.
It must be noted that the uses of artificial viscosity reduces the circulation which is physically impossible. It is in this argument that renders this conjecture unrealistic.

Kuwahara & Takami (1973) used the discrete-vortex approximation with artificial viscosity to deal with two problems: (i) the rotation of an elliptic vortex tube with uniform vorticity and (ii) the rolling-up of a vortex sheet of finite length. From their results they concluded that the introduction of an artificial viscosity effectively avoids strong irregularity in vortex distribution in the region of high vortex concentration and this makes it possible to see the fine structure of the flow. Bloom & Jen (1974) applied the artificial viscosity method of Kuwahara & Takami to a number of practical aerodynamic configurations and obtained a well reasonable comparisons with the experimental measurements.

There are two major difficulties with the Biot-Savart approach (see equation 2.1.8). First, the vortex filaments or sheets contain singularities and, hence, unrealistically create velocities in their neighbourhood. This tends to cause numerical (as well as theoretical) instabilities. Murman & Stremel (1982) stated that such behaviour may be removed by the addition of an artificial viscosity to the model. The second difficulty with the Biot-Savart method is that the number of operations required for a velocity field calculation is of order of magnitude $N^2$ where $N$ is the number of filaments. Thus, the computer time increased significantly as more vortex markers were added.

Due to viscosity, a real vortex sheet has a finite thickness. It is perhaps "more accurate" to use finite core vortices in the potential flow model. However, with an arbitrarily specified core size the computational viscosity is also artificial. Thus it is expected that velocity field induced by a vortex filament, away from the core, is quantitatively correct. However, it is only qualitatively correct near
the core. Murman & Stremel also pointed out that the artificial viscosity does not diffuse the vortex as time increases. It only affects the velocity field at each step by removing the singular nature of the vortex. This also means that a singular problem is regularized by an artificial viscosity. However, Dalton & Wang (1988) reported that the artificial viscosity does not prevent the eventual occurrence of strong irregularity in the region of high vortex concentration.

2.5.5 The filtering technique

Krasny (1986) reported that there are two types of irregular motion which can occur in the numerical solution of the point-vortex equations. There are (i) a type which occurs at smaller time $t > 0$ as the value of $N$ increases, and (ii) a type which occurs only for times $t > t_c$ (the vortex sheet's critical time) regardless of the value of $N$.

Krasny (1986) examined Rosenhead's point-vortex approximation and made an attempt to get a better understanding on the source of the difficulty in convergent. He claimed that perturbations introduced spuriously by computer round off error are responsible for the irregular point-vortex motion that occurs at a smaller time as the number of points is increased. In order to control this source of computational error, he suggested a new filtering technique to set the computational "noise" level to zero at the end of every time step. The vortex positions will then be correspondingly adjusted by an inverse Fourier transform and the calculation can proceed to the next time step. In order for a stable mode to grow, its amplitude must jump above the noise level in a single time step and once this has happened for every mode, the filter is turned off and the computation proceeds normally. However, from their results, it shows that the filter technique suppresses the growth of the spurious roundoff-error
perturbations only for a short time. Therefore, Krasny cited that the point-vortex approximation converges as N approaches to infinity (up to the critical time of the vortex sheet).

Krasny (1986) stated that the reason for Hogdon & Pozrikidis' (1985) results in not having irregular motion is that the number of points used in their calculations is below a certain number. Krasny also stated that Hogdon & Pozrikidis only used two different sets of mesh parameters in their calculations and were unable to examine the limit as their discretization is refined. Their conclusions therefore describe two particular approximating curves whose relevance to the vortex sheet is unclear.
3. DETAILS OF THE NUMERICAL MODEL

3.1. Discretization of a vortex sheet

Consider a two-dimensional flow in the complex $z$-plane. The conjugate complex velocity $u(z) - iv(z)$ induced by a vortex sheet of strength $\gamma(s)$ situated on the curve $c$ is given by

$$\bar{q}(z) = u(z) - iv(z) = \frac{1}{2\pi i} \int_c \frac{\gamma(s) \, ds}{z - z(s)}$$

(3.1.1)

The velocity at a point on the vortex sheet is also given by equation (3.1.1) if Cauchy principal values of the integral are considered. The evaluation of the velocity begins with the discretization of the integrals in equation (3.1.1). In the process of discretization, a vortex sheet is represented by $N$ straight segments; each segment is called an element. The velocity induced by the $k$th element onto a point $z_j$ is given by $u_{jk} - iv_{jk}$.

$$u_{jk} - iv_{jk} = \frac{1}{2\pi i} \int_{-\Delta s_k - 1/2}^{\Delta s_k + 1/2} \frac{\gamma_k(s) \, ds}{z_j - z_k - s e^{i\theta_k}}$$

(3.1.2)

The integration is carried out along the $k$th element extending from $z_{k-1/2}$ to $z_{k+1/2}$. The parameter $s$ is measured along the sheet from a convenient origin to the pivotal point $z_k$, which is not necessarily a midpoint of the element. The symbols, $\Delta s_{k+1/2}$ and $\Delta s_{k-1/2}$ are the distances between the point $z_k$ and the edges of the element where it resides, so that the $k$th element has a length of $\Delta s_k = \Delta s_{k+1/2} + \Delta s_{k-1/2}$. The contribution by the all $N$ elements is given by
\[ U_j - iV_j = \sum_{k=1}^{N} u_{jk} - i v_{jk} \]  \hspace{1cm} (3.1.3)

Consider the integral in equation (3.1.2). The vorticity density \( \gamma_k \) in each element, and the angle of inclination \( \theta_k \) of the element with the real axis are assumed constant. Thus the integration in each element can be carried out to give the velocity induced by the kth element onto the point \( z_j \).

As the vorticity density \( \gamma_k \) and the angle of inclination \( \theta_k \) of the element with the real axis are assumed constant, equation (3.1.2) becomes

\[ \frac{u_{jk} - i v_{jk}}{2 \pi i} = \frac{\gamma_k}{e^{i \theta_k}} \int_{\Delta s_{k+1/2}}^{\Delta s_{k-1/2}} \frac{ds}{(z_j - z_k)e^{-i \theta_k} - s} \]  \hspace{1cm} (3.1.4)

Let

\[ y = \ln \left[ (z_j - z_k)e^{-i \theta_k} - s \right] \]  \hspace{1cm} (3.1.5)

and

\[ dy = -\frac{ds}{(z_j - z_k)e^{-i \theta_k} - s} \]  \hspace{1cm} (3.1.6)

Applying equations (3.1.5) and (3.1.6) to equation (3.1.2), we find
\[ u_{jk} - iv_{jk} = \frac{1}{2\pi i} \cdot \frac{y_k}{e^{i\theta_k}} \int y_2 \, (-dy) \]

\[ = \frac{1}{2\pi i} \cdot \frac{y_k}{e^{i\theta_k}} \left( y_2 - y_1 \right) \]

where

\[ y_1 = \ln \left[ (z_j - z_k) e^{-i\theta_k} + \Delta s_{k-1/2} \right] \]

\[ y_2 = \ln \left[ (z_j - z_k) e^{-i\theta_k} - \Delta s_{k+1/2} \right] \]

Thus, equation (3.1.7) becomes

\[ u_{jk} - iv_{jk} = \frac{-1}{2\pi i} \cdot \frac{y_k}{e^{i\theta_k}} \left\{ \ln \left[ (z_j - z_k) e^{-i\theta_k} - \Delta s_{k+1/2} \right] \right\} - \]

\[ \ln \left[ (z_j - z_k) e^{-i\theta_k} + \Delta s_{k-1/2} \right] \}

\[ = \frac{-1}{2\pi i} \cdot \frac{y_k}{e^{i\theta_k}} \ln \left[ \frac{(z_j - z_k) - \Delta s_{k+1/2} e^{i\theta_k}}{(z_j - z_k) + \Delta s_{k-1/2} e^{i\theta_k}} \right] \]

(3.1.8)

Let \( U_j \) and \( V_j \) be the horizontal and vertical components of the resultant velocity on the considered point \( z_j \) induced by the vortex sheet which is defined by \( N \) finite elements. The resultant induced velocity of the considered point \( z_j \) can be expressed as
\[ U_j - i V_j = \sum_{k=1}^{N} u_{jk} - i v_{jk} \]

\[ = \sum_{k=1}^{N} \left[ \frac{1}{2\pi i} \cdot \frac{\gamma_k}{e^{i\theta_k}} \ln \left( \frac{(z_j - z_k) - \Delta s_{k+1/2}}{(z_j - z_k) + \Delta s_{k-1/2}} e^{i\theta_k} \right) \right] \]

(3.1.9)

The concept of discretization in equation (3.1.3) is shown in detail by equation (3.1.9) when the vorticity density is assumed to be constant. The above scheme can be found in the work by Soh (1985).

A better approximation can be made if the vorticity density \( \gamma_k \) is assumed to be a linear distribution; that is,

\[ \gamma_k(s) = A_k + B_k s \quad (3.1.10) \]

where

\[ A_k, B_k = \text{real constant}. \]

\[ s = \text{length measured along each element from the pivotal point}. \]

The angle of inclination \( \theta_k \) of the element with the real axis is still assumed constant. Substituting equation (3.1.10) into equation (3.1.2), the induced velocity by the kth element onto the considered point \( z_j \) can be written as

\[ u_{jk} - i v_{jk} = \frac{1}{2\pi i} \int_{\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{(A_k + B_k s)}{(z_j - z_k) - s e^{i\theta_k}} ds \]

(3.1.11)

For the convenience of handling the integral term in the right-hand side of equation (3.1.11), it can be divided into two parts: one contains real constant
number $A_k$ and the other contains real constant number $B_k$ and the length's variable $s$.

$$u_{jk} - iv_{jk} = \frac{1}{2\pi i} \int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \left( \frac{A_k + B_k s}{s e^{i\theta_k}} \right) ds$$

$$= \frac{1}{2\pi i} \left\{ \int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{A_k ds}{(z_j - z_k) - s e^{i\theta_k}} + \int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{B_k s ds}{(z_j - z_k) - s e^{i\theta_k}} \right\} \quad (3.1.12)$$

Define the unit vector of $z_j - z_k$ to be $E_{jk}$; that is,

$$E_{jk} = \frac{z_j - z_k}{|z_j - z_k|} \quad (3.1.13)$$

and let

$$G(s) = \frac{z_j - z_k - s e^{i\theta_k}}{|z_j - z_k|}$$

$$= E_{jk} |z_j - z_k| - s e^{i\theta_k} \quad (3.1.14)$$

The induced velocity by $k$th element onto the considered point $z_j$ can be represented as

$$u_{jk} - iv_{jk} = \frac{1}{2\pi i} \left\{ \int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{A_k ds}{G(s)} + \int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{B_k s ds}{G(s)} \right\} \quad (3.1.15)$$
Applying the same way of constant vorticity density \( \gamma_k \), the first integral term in the braces of equation (3.1.15) will become

\[
\int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{A_k}{G(s)} \, ds = - \frac{A_k}{e^{i\theta_k}} \ln \left[ \frac{G(\Delta s_{k+1/2})}{G(-\Delta s_{k-1/2})} \right]
\]

\[
= - \frac{A_k}{e^{i\theta_k}} \ln \left[ \frac{(z_j - z_k) - \Delta s_{k+1/2} \cdot e^{i\theta_k}}{(z_j - z_k) + \Delta s_{k-1/2} \cdot e^{i\theta_k}} \right]
\]

(3.1.16)

Assume

\[
G(s) = E_{jk} \mid z_j - z_k \mid - s \cdot e^{i\theta_k} = C_g + s \cdot D_g
\]

(3.1.17)

where

\[
C_g = E_{jk} \mid z_j - z_k \mid
\]

\[
D_g = - e^{i\theta_k}
\]

Substituting equation (3.1.17) into the second integral term in the braces of equation (3.1.15), we obtain

\[
\int_{-\Delta s_{k-1/2}}^{\Delta s_{k+1/2}} \frac{B_k \cdot s}{G(s)} \, ds = B_k \int \frac{s \, ds}{C_g + s \cdot D_g}
\]

\[
= B_k \int \left[ \frac{1}{D_g} \left( \frac{C_g + s \cdot D_g}{C_g + s \cdot D_g} \right) \right] ds
\]

\[
= B_k \int \left[ \frac{1}{D_g} \cdot \frac{C_g}{C_g + s \cdot D_g} \right] ds
\]
The upper limit and lower limit in all the above integral terms of equation (3.1.18) are $\Delta s_k + 1/2$ and $-\Delta s_k - 1/2$, respectively. Substitute $D_g$ of equation (3.1.18) into the first term in the square brackets of equation (3.1.18), and we can obtain

$$\frac{1}{D_g} \int ds = -\frac{\Delta s_k}{e^{i\theta_k}}$$

(3.1.19)

where

$$\Delta s_k = \Delta s_k + 1/2 + \Delta s_k - 1/2 = \text{The length of the kth element}$$

Applying the same way as for constant vorticity density $\gamma_k$ and substituting $C_g$ and $D_g$ of equation (3.1.17) into the second term in the square brackets of equation (3.1.18), we find

$$\frac{C_g}{D_g} \left[ \int \frac{ds}{G(s)} \right] = \frac{E_{jk} |z_j - z_k|}{e^{i\theta_k}} \left[ \frac{-1}{e^{i\theta_k}} \left( \ln \frac{G(\Delta s_k + 1/2)}{G(-\Delta s_k - 1/2)} \right) \right]$$

$$= -\frac{E_{jk} |z_j - z_k|}{e^{i2\theta_k}} \ln \left[ \frac{(z_j - z_k) - \Delta s_k + 1/2 \cdot e^{i\theta_k}}{(z_j - z_k) + \Delta s_k - 1/2 \cdot e^{i\theta_k}} \right]$$

(3.1.20)
Substitute equations (3.1.19) and (3.1.20) into equation (3.1.18) and the integral term containing real constant number $B_k$ becomes

$$\int_{-\Delta s_k^{1/2}}^{\Delta s_k^{1/2}} \frac{B_k \cdot s \, ds}{G(s)} = B_k \left\{ \frac{\Delta s_k}{e^{i\theta_k}} - \frac{E_{jk} \left| z_j - z_k \right|}{e^{i2\theta_k}} \ln \left[ \frac{(z_j - z_k) - \Delta s_k + 1/2 \cdot e^{i\theta_k}}{(z_j - z_k) + \Delta s_k - 1/2 \cdot e^{i\theta_k}} \right] \right\}$$

(3.1.21)

Substitute equations (3.1.21) and (3.1.16) into equation (3.1.15) and the induced velocity by the $k$th element onto the considered point $z_j$ can be expressed as

$$u_{jk} \cdot iv_{jk} = \frac{1}{2\pi i} \left\{ -\frac{A_k}{e^{i\theta_k}} \ln \left[ \frac{G(\Delta s_k^{1/2})}{G(-\Delta s_k^{1/2})} \right] - \frac{B_k \Delta s_k}{e^{i\theta_k}} \frac{E_{jk} \left| z_j - z_k \right|}{e^{i2\theta_k}} \ln \left[ \frac{G(\Delta s_k^{1/2})}{G(-\Delta s_k^{1/2})} \right] \right\}$$

$$= \frac{-1}{2\pi i} \left\{ \frac{A_k}{e^{i\theta_k}} \ln \left[ \frac{G(\Delta s_k^{1/2})}{G(-\Delta s_k^{1/2})} \right] + \frac{B_k \Delta s_k}{e^{i\theta_k}} + \frac{B_k \Delta s_k}{e^{i2\theta_k}} \ln \left[ \frac{G(\Delta s_k^{1/2})}{G(-\Delta s_k^{1/2})} \right] \right\}$$

(3.1.22)

where

$$G(\Delta s_k^{1/2}) = z_j - z_k - \Delta s_k + 1/2 \cdot e^{i\theta_k} \quad (3.1.22a)$$

$$G(-\Delta s_k^{1/2}) = z_j - z_k + \Delta s_k - 1/2 \cdot e^{i\theta_k} \quad (3.1.22b)$$
From equation (3.1.22) we may find that the case of vorticity density being constant is the only special case of linear distribution of vorticity density. When the real constant number \( B_k \) becomes to zero, equation (3.1.22) will be the same as equation (3.1.8) and vorticity density \( \gamma_k \) will be equal to \( A_k \).

Let \( U_j \) and \( V_j \) be the horizontal and vertical components of the resultant velocity on the considered point \( z_j \) induced by the vortex sheet which is defined by \( N \) finite elements. The resultant induced velocity of the considered point \( z_j \) can be expressed as

\[
U_j - iV_j = \sum_{k=1}^{N} u_{jk} - i v_{jk} = \sum_{k=1}^{N} \frac{A_k}{2\pi i} \ln \left[ \frac{G(\Delta s_{k+1/2})}{G(-\Delta s_{k-1/2})} \right] + \frac{B_k \Delta s_k}{e^{i\theta_k}}
\]

\[
+ B_k \frac{(z_j - z_k)}{e^{i2\theta_k}} \ln \left[ \frac{G(\Delta s_{k+1/2})}{G(-\Delta s_{k-1/2})} \right]
\]

(3.1.23)
3.2 Computer subprogram (SELT2) of the induced velocity by a vortex sheet

In order to apply the computer to calculate the induced velocity of the considered point \( z_j \), equation (3.1.8) has been written as a subprogram, named SELT, by Soh. For better understanding of the difference between constant vorticity density and linear distribution of vorticity density, equation (3.1.22) has been also considered for converting to a subprogram, named SELT2, in this report. Before converting equation (3.1.22) into a subprogram, the following appropriate definitions are given:

\[
X_{jk} = \text{distance to the midpoint of the } k\text{th element along the real axis to the considered point } z_j.
\]

\[
Y_{jk} = \text{distance to the midpoint of the } k\text{th element along the imaginary axis to the considered point } z_j.
\]

\[
2\Delta x = \text{the real axis length of the } k\text{th element. (see figure 3.2.1)}
\]

\[
2\Delta y = \text{the imaginary axis length of the } k\text{th element. (see figure 3.2.1)}
\]

\[
x_{k+1}, y_{k+1} = \text{the Cartesian coordinates of } (k+1)\text{th element.}
\]

\[
x_{k-1}, y_{k-1} = \text{the Cartesian coordinates of } (k-1)\text{th element.}
\]

Assume the vortex sheet to be divided into \( N \) elements of equivalent length \( \Delta s \). According to the definitions of \( \Delta x \) and \( \Delta y \), the length of \( \Delta x \) and \( \Delta y \) are

\[
\Delta x = 0.25 \left( x_{k+1} - x_{k-1} \right) \tag{3.2.1}
\]

\[
\Delta y = 0.25 \left( y_{k+1} - y_{k-1} \right) \tag{3.2.2}
\]

Define \( \Delta s_{k+1/2} \) and \( \Delta s_{k-1/2} \) to be the half length of \( k \)th element, that is,
Figure 3.2.1 Some definitions of vortex elements
\[ \Delta s_{k+1/2} = \Delta s_{k-1/2} = 0.5 \Delta s_k \]  

(3.2.3)

According to \( X_{jk} \) and \( Y_{jk} \) definitions, the \((z_j - z_k)\) vector can be expressed as

\[ z_j - z_k = X_{jk} + i Y_{jk} \]  

(3.2.4)

If we substitute equations (3.2.3) and (3.2.4) into equation (3.1.22a), the function \( G(\Delta s_{k+1/2}) \) can be written as

\[ G(\Delta s_{k+1/2}) = (X_{jk} + Y_{jk}) - 0.5 \Delta s_k e^{i\theta_k} \]  

(3.2.5)

By using vector's concept to represent each element of the vortex sheet, the \( k \)th element can be expressed as

\[ \Delta s_k \cdot e^{i\theta_k} = 0.5 (x_{k+1} - x_{k-1}) + i 0.5 (y_{k+1} - y_{k-1}) \]

\[ = 2 \Delta x + i 2 \Delta y \]  

(3.2.6)

where

\[ \theta_k = \tan^{-1}(\Delta y/\Delta x) \]  

(3.2.6a)

Substitute equation (3.2.6) into equation (3.2.5). Then, the function of \( G(\Delta s_{k+1/2}) \) can be expressed as

\[ G(\Delta s_{k+1/2}) = (X_{jk} - \Delta x) + i (Y_{jk} - \Delta y) \]  

(3.2.7)
Similarly, $G(-\Delta s_{k-1/2})$ can also be expressed as

$$G(-\Delta s_{k-1/2}) = (X_{jk} + \Delta x) + i (Y_{jk} + \Delta y) \quad (3.2.8)$$

To convert equation (3.1.23) term by term, it can be expressed as

$$U_j - i V_j = \sum_{k=1}^{N} \frac{-1}{2\pi i} \left\{ A_k (C) + B_k (D+\bar{F}) \right\} \quad (3.2.9)$$

where

$$C = \frac{1}{e^{i\theta_k}} \ln \left[ \frac{G(\Delta s_{k+1/2})}{G(-\Delta s_{k-1/2})} \right] \quad (3.2.9a)$$

$$D = \frac{\Delta s_k}{e^{i\theta_k}} \quad (3.2.9b)$$

$$F = \frac{(z_j - z_k)}{e^{i2\theta_k}} \ln \left[ \frac{G(\Delta s_{k+1/2})}{G(-\Delta s_{k-1/2})} \right] \quad (3.2.9c)$$

To substitute equations (3.2.7) and (3.2.8) into the logarithm terms in equations (3.2.9a) and (3.2.9c) the equation can be written as

$$\ln \left[ \frac{G(\Delta s_{k+1/2})}{G(-\Delta s_{k-1/2})} \right] = \ln \left[ \frac{(X_{jk} - \Delta x^2) + (Y_{jk} - \Delta y^2) + i 2(\Delta x Y_{jk} - \Delta y X_{jk})}{(X_{jk} + \Delta x)^2 + (Y_{jk} + \Delta y)^2} \right]$$

$$= \ln |Z_A| + i \tan^{-1}(Z_A) \quad (3.2.10)$$
where

\[ |Z_A| = \sqrt{\frac{[(X_{jk} - \Delta x^2) + (Y_{jk} - \Delta y^2)]^2 + [2(\Delta xY_{jk} - \Delta yX_{jk})]^2}{[(X_{jk} + \Delta x)^2 + (Y_{jk} + \Delta y)^2]^2}} \]  

(3.2.10a)

\[ \tan^{-1}(Z_A) = \tan^{-1}\left[ \frac{2(\Delta xY_{jk} - \Delta yX_{jk})}{(X_{jk} - \Delta x^2) + (Y_{jk} - \Delta y^2)} \right] \]  

(3.2.10b)

Substitute equation (3.2.10) into equation (3.2.9a). Then the C term of equation (3.2.9) will be

\[ C = e^{-i\theta_k} \cdot [\ln |Z_A| + i \tan^{-1}(Z_A)] \]

\[ = [\cos(\theta_k) - i \sin(\theta_k)] \cdot [\ln |Z_A| + i \tan^{-1}(Z_A)] \]

\[ = [\cos(\theta_k) \cdot \ln |Z_A| + \sin(\theta_k) \cdot \tan^{-1}(Z_A)] + i [\cos(\theta_k)]. \]

\[ \tan^{-1}(Z_A) - \sin(\theta_k) \cdot \ln |Z_A| \]

\[ = C_R + i C_I \]  

(3.2.11)

where

\[ C_R = \cos(\theta_k) \cdot \ln |Z_A| + \sin(\theta_k) \cdot \tan^{-1}(Z_A) \]  

(3.2.11a)
Explicitly, $C_R$ and $C_I$ are respectively the real part and the imaginary part of a complex number $C$. Substitute equation (3.2.10) into equation (3.2.9c); the $F$ term of equation (3.2.9) also can be changed as following:

$$F = e^{-i2\theta_k} \cdot (X_{jk} + iY_{jk}) \cdot \left[ \ln |Z_A| + i \tan^{-1}(Z_A) \right]$$

$$= \left[ \cos(2\theta_k) - i \sin(2\theta_k) \right] \cdot \left[ \left( X_{jk} \cdot \ln |Z_A| - Y_{jk} \cdot \tan^{-1}(Z_A) \right) + i \left( X_{jk} \cdot \tan^{-1}(Z_A) + Y_{jk} \cdot \ln |Z_A| \right) \right]$$

$$= \left[ \cos(2\theta_k) \cdot X_{jk} + \sin(2\theta_k) \cdot Y_{jk} \right] \cdot \ln |Z_A| + \left[ \cos(2\theta_k) \cdot Y_{jk} - \sin(2\theta_k) \cdot X_{jk} \right] \cdot \tan^{-1}(Z_A) + i \left[ \cos(2\theta_k) \cdot X_{jk} + \sin(2\theta_k) \cdot Y_{jk} \right] \cdot \ln |Z_A|$$

$$= F_R + i F_I$$

(3.2.12)

where

$$F_R = \left[ \cos(2\theta_k) \cdot X_{jk} + \sin(2\theta_k) \cdot Y_{jk} \right] \cdot \ln |Z_A| + \left[ \cos(2\theta_k) \cdot Y_{jk} - \sin(2\theta_k) \cdot X_{jk} \right] \cdot \tan^{-1}(Z_A)$$

(3.2.12a)

$$F_I = \left[ \cos(2\theta_k) \cdot Z_{jk} + \sin(2\theta_k) \cdot Y_{jk} \right] \cdot \tan^{-1}(Z_A) - \left[ \cos(2\theta_k) \cdot Y_{jk} - \sin(2\theta_k) \cdot X_{jk} \right] \cdot \ln |Z_A|$$

(3.2.12b)
Explicitly, $F_R$ and $F_I$ are respectively the real part and the imaginary part of a complex number $F$. The $D$ term of equation (3.2.9) is

$$D = e^{-i2\theta_k} \cdot (\Delta s_k \cdot e^{i\theta_k}) \quad (3.2.13)$$

Substitute equation (3.2.6) into equation (3.2.13). The $D$ term can be expressed as

$$D = [\cos(2\theta_k) - i \sin(2\theta_k)] \cdot [2\Delta x + i 2\Delta y]$$

$$= 2 [\cos(2\theta_k) \cdot \Delta x + \sin(2\theta_k) \cdot \Delta y] + i 2 [\cos(2\theta_k) \cdot \Delta y - \sin(2\theta_k) \cdot \Delta x]$$

$$= D_R + i D_I \quad (3.2.14)$$

where

$$D_R = 2 [\cos(2\theta_k) \cdot \Delta x + \sin(2\theta_k) \cdot \Delta y] \quad (3.2.14a)$$

$$D_I = 2 [\cos(2\theta_k) \cdot \Delta y - \sin(2\theta_k) \cdot \Delta x] \quad (3.2.14b)$$

$D_R$ and $D_I$ are respectively the real part and the imaginary part of the complex number $D$. This is similar to the representation of $F_R$ and $F_I$ with complex number $F$ (above). Substitute equations (3.2.11), (3.2.12), and (3.2.14) into $C$, $D$, and $F$ terms of equation (3.2.9) and the resultant induced velocity of the considered point $Z_j$ by the vortex sheet can be obtained.
Since $A_k$, $B_k$, $C_R$, $C_I$, $D_R$, $D_I$, $F_R$, and $F_I$ are all real numbers, the terms in the right hand side of equation (3.2.15) can be expressed to the real part terms and the imaginary part terms. By the rule of equivalent complex numbers, the horizontal component $U_j$ and the vertical component $V_j$ of the resultant induced velocity are respectively equal to the real part terms and the imaginary part terms. That is,

$$U_j = \sum_{k=1}^{N} \frac{-1}{2\pi i} \left\{ A_k (C_R + iC_I) + B_k [(D_R + iD_I) + (F_R + iF_I)] \right\} \tag{3.2.15}$$

$$V_j = \sum_{k=1}^{N} \frac{-1}{2\pi i} \left\{ A_k C_I + B_k (D_I + F_I) \right\} \tag{3.2.16}$$

$$V_j = \sum_{k=1}^{N} \frac{-1}{2\pi i} \left\{ A_k C_R + B_k (D_R + F_R) \right\} \tag{3.2.17}$$

where

$$C_R = \cos(\theta_k) \cdot \ln |Z_A| + \sin(\theta_k) \cdot \tan^{-1}(Z_A)$$

$$C_I = \cos(\theta_k) \cdot \tan^{-1}(Z_A) - \sin(\theta_k) \cdot \ln |Z_A|$$

$$D_R = 2 [ \cos(2\theta_k) \cdot \Delta x + \sin(2\theta_k) \cdot \Delta y ]$$

$$D_I = 2 [ \cos(2\theta_k) \cdot \Delta y - \sin(2\theta_k) \cdot \Delta x ]$$

$$F_R = [ \cos(2\theta_k) \cdot X_{jk} + \sin(2\theta_k) \cdot Y_{jk} ] \cdot \ln |Z_A| +$$
\[ [ \cos(2\theta_k) \cdot Y_{jk} - \sin(2\theta_k) \cdot X_{jk} ] \cdot \tan^{-1}(Z_A) \]

\[ F_I = [ \cos(2\theta_k) \cdot Z_{jk} + \sin(2\theta_k) \cdot Y_{jk} ] \cdot \tan^{-1}(Z_A) - \]

\[ [ \cos(2\theta_k) \cdot Y_{jk} - \sin(2\theta_k) \cdot X_{jk} ] \cdot \ln |Z_A| \]

Since \( C_R, C_l, D_R, D_l, F_R \), and \( F_I \) are the functions of distances \( X_{jk}, Y_{jk} \) and lengths \( \Delta x, \Delta y \), the input data of this subprogram are expressed as the function of \( z_j(x_j, y_j), z_k(x_k, y_k), z_{k-1}(x_{k-1}, y_{k-1}), \) and \( z_{k+1}(x_{k+1}, y_{k+1}) \). The input data \( z_j \) and \( z_k \) determine the values of \( X_{jk} \) and \( Y_{jk} \) by equation (3.2.4); the input data \( z_{k+1} \) and \( z_{k-1} \) determine the values of \( \Delta x \) and \( \Delta y \) by equations (3.2.1) and (3.2.2). Additionally, \( A_k \) and \( B_k \) are assumed to be known values when this subprogram is used. The effect of summation is expressed as a do-loop in this subprogram. The subprogram (SELT2) calculates the resultant induced velocity from an element which contains a linear distribution of vorticity density (See Appendix II).
3.3 Determination of A & B coefficients in the linear distribution of vorticity density

In a linear distribution of vorticity density in each vortex element as equation (3.1.11), the two unknown quantities A & B, have to be calculated before the subprogram SELT2 can be used to yield the induced velocities. The procedure for determination of the A & B coefficients will be described in this section.

Let $A_k$ and $B_k$ be two unknown coefficients of the kth element. Its vorticity density can be expressed as

$$\gamma_k(s) = A_k + B_k(s) \quad (3.3.1)$$

where $s = \text{measured length along each element from the pivotal point in the kth element.}$

Let $s_1$ and $s_2$ be the end points in the kth element. Since it is possible to have positive and negative directions relative to the local origin, $s_1$ and $s_2$ can either be a positive or a negative value. The vorticity densities at the both end points of the element can be separately expressed as

$$\gamma_k(s_1) = A_k + B_k(s_1) \quad (3.3.2)$$

$$\gamma_k(s_2) = A_k + B_k(s_2) \quad (3.3.3)$$

The values of $\gamma_k(s_1)$ and $\gamma_k(s_2)$ can be calculated; this will be discussed in Section 3.3.1. From these two linear equations, i.e., equations (3.3.2) and (3.3.3), the two unknown coefficients $A_k$ and $B_k$ of the kth element can be solved and expressed as
\[ A_k = \frac{\gamma_k(s_1) s_2 - \gamma_k(s_2) s_1}{s_2 - s_1} \]  
(3.3.4)

\[ B_k = \frac{\gamma_k(s_1) - \gamma_k(s_2)}{s_1 - s_2} \]  
(3.3.5)

### 3.3.1 Determination of the vorticity density \( \gamma_k(s_1) \) and \( \gamma_k(s_2) \)

The following procedure determines the vorticity densities at the ends of the kth element, \( \gamma_k(s_1) \) and \( \gamma_k(s_2) \), from the given quantities of circulation and geometry of the vortex sheet.

1. The length parameter, \( s_k \), which is associated with \( z_k \) in the kth element is the length measured from a reference point, \( z_{1/2} \):

\[ s_k = s_1 + \sum_{j=2}^{k} |z_j - z_{j-1}| \text{ for } k > 1 \]  
(3.3.6a)

and

\[ s_1 = |z_1 - z_{1/2}|, \]  
(3.3.6b)

where \( z_{1/2} \) is the coordinates of the reference point, an end point of the first element \( z_1 \).

2. The circulation parameter, \( P_k \), of the kth element which is associated with \( z_k \), is defined as follows
\[ P_k = \int_{s_{1/2}}^{s_k} \gamma(s) \, ds = P_{1/2} + \sum_{j=2}^{k} \frac{(\Gamma_j + \Gamma_{j-1})}{2} \quad \text{for } k > 1 \]  

(3.3.7a)

\[ P_{1/2} = P_{1/2} + 0.5 \Gamma_1 \]  

(3.3.7b)

\( \Gamma_k \) is the circulation of the kth element. \( P_{1/2} \) corresponds to \( s_{1/2} \) may be set to zero. Figure 3.3.1 shows the schematic diagram of the length parameter and the circulation parameter. The dashed line from \( N \) to \( N+1/2 \) represents that the circulation parameter and the length parameter in one end of the last element are obtained by the extrapolation.

(3) The process of rediscretization of segments into equal length involves the interpolation of a new set of the circulation parameter \( P_j^{*} \) which corresponds to equal increment in the length parameter \( s_j^{*} \). For \( N^* \) number of new segments, the common length of the segment, \( \Delta s \), is given by

\[ \Delta s = \frac{s_N}{N - 1/2} \]  

(3.3.8)

The new length parameter is given by

\[ s_{j+1/2}^{*} = \Delta s \times j \]  

(3.3.9a)

\[ s_j^{*} = 0.5 \times (s_{j-1/2}^{*} + s_{j+1/2}^{*}) \]  

(3.3.9b)

Figure 3.3.2 shows the schematic diagram of the rediscretization: (a) before equally spacing and (b) after equally spacing. The dashed line from \( N^* \) to
\( N^*+1/2 \) shows that the circulation parameter and the length parameter are obtained by the extrapolation.

(4) The Lagrangian interpolation formula is used in the calculation. The new quantities \( z_j^* \) are interpolated from the data set \( z_k \) and \( s_k \) for the values at \( s_j^* \). In the same way, \( P_{j+1/2}^* \) and \( z_{j+1/2}^* \) are interpolated from the data for the values at \( s_{j+1/2}^* \). Table 3.3.1 shows the schematic relationship for this interpolation.

![Diagram of length parameter and circulation parameter](image)

Figure 3.3.1 Schematic diagram of the length parameter \( s \) and the circulation parameter \( P \)
Figure 3.3.2 Schematic diagram of the rediscretization: (a) before equally spacing and (b) after equally spacing.
Table 3.3.1 Schematic relationship of the Lagrangian interpolation

<table>
<thead>
<tr>
<th>Known old data set</th>
<th>Known new data</th>
<th>Calculated new values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_k$ and $s_k$</td>
<td>$s_i^*$</td>
<td>$z_i^*$</td>
</tr>
<tr>
<td>$z_{k+1/2}$ and $s_{k+1/2}$</td>
<td>$s_{i+1/2}^*$</td>
<td>$z_{i+1/2}^*$</td>
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<tr>
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<td>$s_{i+1/2}^*$</td>
<td>$P_{i+1/2}^*$</td>
</tr>
<tr>
<td>$P_k$ and $s_{k+1/2}$</td>
<td>$s_i^*$</td>
<td>$P_i^*$</td>
</tr>
</tbody>
</table>

(5) The vorticity densities $\gamma_{j\pm1/2}^*$ are determined by taking the derivatives of the circulation parameter with the length parameter. Since the vorticity densities are calculated after equally spacing, it is advantageous to adopt the half length of the new segment $0.5 \Delta s$ as evenly spaced while applying the following backward finite differences formulas to yield $\gamma_{j-1/2}^*$ and $\gamma_{j+1/2}^*$ which correspond to $\gamma_k(s_1)$ and $\gamma_k(s_2)$ in equations (3.3.2) - (3.3.5).

$$\gamma_{j-1/2}^* = \frac{(C_1 + C_2/2 + C_3/3 + C_4/4)(0.5\Delta s)}{(0.5\Delta s)^4} + O((0.5\Delta s)^4)$$

(3.3.10)

$$\gamma_{j+1/2}^* = \frac{(D_1 + D_2/2 + D_3/3 + D_4/4)(0.5\Delta s)}{(0.5\Delta s)^4} + O((0.5\Delta s)^4)$$

(3.3.11)

where

$$C_1 = P_{j-1/2}^* - P_{j-1}^*$$

$$C_2 = P_{j-1/2}^* - 2P_{j-1}^* + P_{j-3/2}^*$$
\[ C_3 = P_{j-1/2}^* - 3 P_{j-1}^* + 3 P_{j-3/2}^* - P_{j-2}^* \]

\[ C_4 = P_{j-1/2}^* - 4 P_{j-1}^* + 6 P_{j-3/2}^* - 4 P_{j-2}^* + P_{j-5/2}^* \]

\[ D_1 = P_{j+1/2}^* - P_j^* \]

\[ D_2 = P_{j+1/2}^* - 2 P_j^* + P_{j-1/2}^* \]

\[ D_3 = P_{j+1/2}^* - 3 P_j^* + 3 P_{j-1/2}^* - P_{j-1}^* \]

\[ D_4 = P_{j+1/2}^* - 4 P_j^* + 6 P_{j-1/2}^* - 4 P_{j-1}^* + P_{j-3/2}^* \]

The vorticity density is given correctly within the accuracy of the order of \((0.5\Delta s)^4\). For the first two points, \(k=1\) and 2, a forward finite differences formula similar to equations (3.3.10) and (3.3.11) is used.
3.4 Characteristics of a single vortex element

Consider a single vortex element located in an incompressible, irrotational, and inviscid flow field. The middle point of the vortex element is arranged on the original point (0, 0) and its length is 2. The vortex element is inclined at an angle $\theta$ to the real axis and has a constant vorticity density. For simplicity, the vorticity density $\gamma$ is assumed to be 1. All of the above conditions are shown in figure 3.4.1.

The induced complex velocity due to the considered vortex element in the flow field can be expressed as

$$u - iv = \frac{1}{2\pi i} \int_{-1}^{1} \frac{ds}{z - s e^{i\theta}}$$

$$= -\frac{e^{-i\theta}}{2\pi i} \left[ \ln \left( \frac{1 - ze^{-i\theta}}{-1 - ze^{-i\theta}} \right) \right]$$

(3.4.1)

Assume $z$ to be an arbitrary point in the flow field and its coordinates can be expressed as

$$z = x + iy$$

Let $u$ & $v$ respectively represent the horizontal and the vertical components of the induced velocity and then they can be expressed as

$$u = \frac{1}{2\pi} \left( \sin \theta \cdot \ln c - \theta c \cdot \cos \theta \right)$$

(3.4.2)
Figure 3.4.1 A simple vortex element
\[ v = \frac{-1}{2\pi} (\cos \theta \cdot \ln c + \theta_c \cdot \sin \theta) \]  
(3.4.3)

where

\[ c = \sqrt{\frac{(R^2 + S^2 - 2R)^2 + (2S)^2}{((R - 2)^2 + S^2)^2}} \]  
(3.4.4)

\[ \theta_c = \tan^{-1}\left[\frac{2S}{(R^2 + S^2 - 2R)}\right] \]  
(3.4.5)

\[ R = 1 - x \cdot \cos \theta - y \cdot \sin \theta \]  
(3.4.6)

\[ S = y \cdot \cos \theta - x \cdot \sin \theta \]  
(3.4.7)

Apply the same conditions of the simple vortex element, whose vorticity density \( \gamma \) is 1, to the SELT and SELT2 subprogram and the results of induced velocities can be obtained. Comparing the results of SELT or SELT2 with that of the analytic solution, it can be found that the results are the same except at the end points. The induced velocities of the regions approaching the vortex element are parallel to the element and equal to half of the vorticity density. The direction of the induced velocity in the upper region of the vortex element is opposite to that in the lower region of the vortex element. All these results are summarised in figures 3.4.2 and 3.4.3.
Figure 3.4.2. Different angles of the single vortex element (γ = 1)
Figure 3.4.3 Different angles of the single vortex element (γ = -1)
From the characteristics of vortex elements, it is appropriate to use them in the numerical simulation of the surfaces whose normal velocity is prescribed and the tangential velocity is the keypoint of the problems. This idea has been mentioned by Lighthill (1963) (the solid boundary is a distributed source of vorticity) and Robertson (1965).

For a closer examination on the velocity field induced by a vortex sheet element, consider a horizontal element. The given vorticity density $\gamma$ is equal to 1; the middle point of the vortex element is arranged on the original point $(0, 0)$ and the total length $L$ of the element is 2 (from $x=-1$ to $x=1$). The position of the considered point is located on the $(x, y)$, where $x$ is from -1 to 1 and $y$ is the distance from the vortex element. The induced velocity produced by the vortex element onto the point is $U_x$. According to the above discussion of the single vortex element, the magnitude of the induced velocity $U$ near the vortex element is 0.5, that is half of the vorticity density. For an isolating vortex sheet element, this is true for the most part along the sheet except very near to the end points.

Figures 3.4.4 and 3.4.5 show the relationship of the position and the induced velocity in the distances $y$ of 0.01 and 0.001, respectively. The position and the induced velocity are respectively represented by the dimensionless units $x/L$ and $U_x/U$. Figure 3.4.4 shows that the induced velocity $U_x/U$ starts from $x/L=\pm 0.36$ (approximates to 14% $L$ from the end point) and increases rapidly from $x/L=\pm 0.46$ (approximates to 8% $L$ from the end point). The values of the induced velocity $U_x/U$ reaches the maximum at the both end points. Figure 3.4.5 shows that the induced velocity starts from $x/L=\pm 0.47$ (approximates to 6% $L$ from the end point) and increases rapidly toward maximum values at the end points.
Figure 3.4.4 The relation between the position and the induced velocity for the single vortex element (distance=0.01)

Figure 3.4.5 The relation between the position and the induced velocity for the single vortex element (distance=0.001)
It must be noted that the high velocity at the both ends of the single vortex element recalls the fact of the Helmholtz vortex theorem mentioned by Robertson (1965) that "a vortex line or filament can neither begin nor end in the fluid; hence, it appears as a closed loop or ends on a boundary."
3.5 Comparison between constant vorticity density and linear distribution of vorticity density: circular distribution of vortex elements

In this section, the velocity induced by a closed and continuous uniform distribution of vortex elements is used to investigate the accuracy of the discretization mentioned in the previous subsection: discretization using elements of constant vorticity density, and that of using linear distribution of vorticity density.

Consider a circle which has its centre located at the original point (0, 0) and a unit radius (R=1). The circle is now replaced by a series of straight elements. Each element has a pivotal point $z_k$ located on the circle ($= Re^{i\theta}$); where $\theta$ is the angle formed by the horizontal axis $x$ and the normal line through the pivotal point of the vortex element. Figure 3.5.1 shows the situation of this problem.

The vorticity density $\gamma$ of each element is given as $1 + \cos \theta$, which means that the vorticity density $\gamma$ varies with the inclination of each vortex element. The induced velocity by all vortex elements onto the considered point $z_j (= x + i y)$ can be written as

$$u - iv = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma ds}{z_j - z_k}$$

(3.5.1)

where $ds = \text{the length of each vortex element}$

$= R \ d\theta$
Figure 3.5.1 Circular distribution of vortex elements
Substitute the known conditions of the vorticity density $\gamma$, the coordinates of the pivotal point $z_k$ and the length $ds$ of the vortex element into equation (3.5.1); the induced velocity of the considered point $z_j$ can be converted to

$$u - i\nu = \frac{1}{2\pi i} \oint_{C} \frac{2\pi \left(1 + \cos \theta\right) R \, d\theta}{z_j - R e^{i\theta}}$$

$$= \frac{1}{2\pi i} \left[ \oint_{C} \frac{2\pi R \, d\theta}{z_j - R e^{i\theta}} + \oint_{C} \frac{2\pi R \cos \theta \, d\theta}{z_j - R e^{i\theta}} \right]$$

$$= \frac{-1}{2\pi i} \left[ \oint_{C} \frac{2\pi d\theta}{e^{i\theta} - (z_j/R)} + \oint_{C} \frac{2\pi (e^{i\theta} + e^{-i\theta})/2}{e^{i\theta} - (z_j/R)} \, d\theta \right]$$

(3.5.2)

Define $e^{i\theta} = a$ and take the derivative on both sides, then it can obtain $d\theta = da/ia$. Substitute these results of $e^{i\theta}$ and $d\theta$ into equation (3.5.2) and obtain

$$u - i\nu = -\frac{1}{2\pi i} \left[ \oint_{C} \frac{(da/ia)}{a - (z_j/R)} + \oint_{C} \frac{(a + 1/ia)/2}{a - (z_j/R)} (da/ia) \right]$$

$$= \frac{1}{2\pi} \left[ \oint_{C} \frac{da}{a - (z_j/R)} + \frac{1}{2} \oint_{C} \frac{(a^2 + 1)}{a^2 - z_j/R} \, da \right]$$

$$= \frac{1}{2\pi} \left[ \oint_{C} \frac{da}{a - (z_j/R)} + \frac{1}{2} \oint_{C} \frac{da}{a - (z_j/R)} + \frac{1}{2} \oint_{C} \frac{da}{a^2 - (z_j/R)} \right]$$

(3.5.3)

By the Residue theorem, the analytic solution of the induced velocity on the considered point $z_j$ can be expressed as follows:
\[ u - iv = \frac{-i}{2} \left( \frac{2R}{z_j} + \frac{R^2}{z_j^2} \right) \text{ for } |z_j| > R \]  

(3.5.4)

\[ u - iv = \frac{i}{2} \text{ for } |z_j| < R \]  

(3.5.5)

$|z_j| > R$ means that the point, $z_j$, is outside the circle where vortex elements are distributed and $|z_j| < R$, the point $z_j$ is inside the circle. Substitute $z_j = x + iy$ into equation (3.5.4); the horizontal component and vertical component of the analytic induced velocity for the case of considered points outside the circle can be obtained.

\[ u = -\left[ \frac{Ry}{x^2 + y^2} + \frac{R^2xy}{(x^2 + y^2)^2} \right] \]  

(3.5.6)

\[ v = \frac{1}{2} \left[ \frac{2Rx}{x^2 + y^2} + \frac{R^2(x^2 - y^2)}{(x^2 + y^2)^2} \right] \]  

(3.5.7)

For the case of the considered points inside the circle, it is evident that the horizontal component of the analytic induced velocity is zero and the vertical component of the analytic induced velocity is 0.5.

These results can be used as reference for examining the accuracy of the discretization scheme. The contours for horizontal and vertical velocities are shown in Figure 3.5.2. Since the vorticity density of each element is $1 + \cos \theta$, it can be taken as the vorticity density of the element directly in the case of constant vorticity density. Substitute the vorticity density $\gamma_\kappa$, the coordinates $z_\kappa$ of the pivotal point of each vortex element and the coordinates of the considered point $z_j$ into equation (3.1.10); the induced velocity of the considered point $z_j$ can be obtained. The calculation of the induced velocity for the case of constant vorticity...
density is performed by using the SELT subprogram. For the simplicity, the results are briefly called "the induced velocity of SELT".

For the case of linear distribution of vorticity density, two unknown coefficients $A_k$ and $B_k$ are needed to be solved for the vorticity density of each element before calculating the induced velocity by the subprogram SELT2. The relationship of $A_k$ and $B_k$ of the kth element and its vorticity density is

$$\gamma(s) = A_k + B_k(s)$$

$$\gamma(s) = 1 + \cos \theta$$

(3.5.8)

where

$s =$ local coordinates from the pivotal point (local origin) of each vortex element

For the convenience of representing the position of the vortex element, polar coordinates are used here. The position of the pivotal point of the kth vortex element is located on the $(R,\theta_k)$ and the positions of both end points in the kth vortex element are $(R,\theta_k - \Delta \theta_k/2)$ and $(R,\theta_k + \Delta \theta_k/2)$, respectively. $s_1$ and $s_2$ are the respective local coordinates of both end points in the kth element. Then, $s_1$ and $s_2$ can be expressed as

$$s_1 = -\left(\frac{\Delta \theta_k}{2}\right) \frac{R}{180}$$

(3.5.9)

$$s_2 = \left(\frac{\Delta \theta_k}{2}\right) \frac{R}{180}$$

(3.5.10)
Substitute the corresponding angles $\theta_k-\Delta\theta_k/2$, $\theta_k+\Delta\theta_k/2$ and the respective local coordinates $s_1$ and $s_2$ of both end points into equation (3.5.8); the vorticity density of the both end points can be expressed as

$$\gamma(s_1) = A_k + B_k (s_1) = 1 + \cos (\theta_k - \Delta\theta_k/2) \quad (3.5.11)$$

$$\gamma(s_2) = A_k + B_k (s_2) = 1 + \cos (\theta_k + \Delta\theta_k/2) \quad (3.5.12)$$

Thus, $A_k$ and $B_k$ can be calculated for the values of $\gamma(s_1)$ and $\gamma(s_2)$:

$$A_k = \frac{\gamma(s_2)s_1 - \gamma(s_1)s_2}{s_1 - s_2} \quad (3.5.13)$$

$$B_k = \frac{\gamma(s_1) - \gamma(s_2)}{s_1 - s_2} \quad (3.5.14)$$

The induced velocity for the case of linear distribution of vorticity density is calculated by using the SELT2 subprogram. For the simplicity, the results are briefly called the "induced velocity of SELT2".

For 40 vortex elements, the computed velocities for constant distribution of vorticity in the element and that using linear distribution are shown in Figures 3.5.3 and 3.5.4. The contours of the horizontal velocities with constant vorticity density show some differences from that of the analytic solution during the angles $\theta$ being 70°-110° and 250°-290°. Similar differences also appear in the comparison between the results of the analytic solution and that of the linear distribution of vorticity density. The vertical component of the analytic induced velocities can be obtained by using equation (3.5.7). Figure 3.5.2 (b) shows the contours of the vertical velocities for the analytic solution. Figures 3.5.3 (b) and
3.5.4 (b) respectively show the numerical solutions of the vertical velocities for the case of constant vorticity density and the linear distribution of vorticity density. The contours of the vertical velocities with constant vorticity density show some differences from that of the analytic solution while the positions are near to the circular surface. Similar differences also appear in the comparison between the results of the analytic solution and that of linear distribution of vorticity density.

For more quantitative description of the discrepancies, a set of different distances to the circular surface is calculated. Since the horizontal velocities of the analytic solution inside the circle are zero, the absolute errors of the horizontal component of the induced velocities for the numerical cases are defined as

$$\text{Absolute errors} = |u_{\text{SEL T or SEL T2}} - u_{\text{ANALYTIC}}|$$

for the outside of a circle \hspace{1cm} (3.5.15)

$$\text{Absolute errors} = |u_{\text{SEL T or SEL T2}}|$$

for the inside of a circle \hspace{1cm} (3.5.16)
Figure 3.5.2  Analytic solution of (a) the horizontal velocities and (b) the vertical velocities for the circular distribution
Figure 3.5.3  Numerical solution of (a) the horizontal velocities and (b) the vertical velocities with constant vorticity density for the circular distribution of 40 vortex elements.
Figure 3.5.4 Numerical solution of (a) the horizontal velocities and (b) the vertical velocities with linear distribution of vorticity density for the circular distribution of 40 vortex elements.
Figures 3.5.5 (a)-(g) show the relationship of the position and the horizontal component of induced velocities for the inside of the circle. The considered distances inside the circle of radius $r = 1$, are separately 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, and 0.08. The number of vortex elements distributed on the circle is 40 for all results. It is evident that the further the distance is, the less the error will be. When the distance is 0.02, the maximum of the SELT's errors is about 0.09 and the maximum of the SELT2's errors is about 0.075. While the distance becomes 0.08, the maximum of the SELT's errors reduces to be about 0.058 and that of the SELT2's errors reduces to be about 0.006. The absolute errors around the circle show four peak points. The front two peak points are symmetrical to the rear two peak points if the angle 180° is taken to be the central axis. No matter how far the distance is in the inside of the circle, there are two large peaks separately located on the 56.25° and 303.25° and two small peaks separately located on the 146.25° and 212.25° for the case of constant vorticity density.

For the case of the linear distribution of vorticity density, there are four peaks in the results but the absolute errors of these four peaks are almost the same. The absolute errors for these peaks in the linear distribution of vorticity are less than those of the value of the largest peak in the constant vorticity density. The distributions of these four peaks are respectively in the regions of 48-56°, 115-120°, 240-244° and 300-308°.
Figure 3.5.5 (a) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.02)

Figure 3.5.5 (b) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.03)
Distance (inside the circle) = 0.04

Distance (inside the circle) = 0.05

Figure 3.5.5 (c) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.04)

Figure 3.5.5 (d) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.05)
Distance (inside the circle) = 0.06

Figure 3.5.5 (e) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.06)

Distance (inside the circle) = 0.07

Figure 3.5.5 (f) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.07)
Figure 3.5.5 (g) The relation between the position (inside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.08)
Figures 3.5.6 (a)~(f) show the relationship of the position and the horizontal component of induced velocities for the outside of the circle. The considered distances outside the circle are respectively to be 0.002, 0.01, 0.02, 0.03, 0.05 and 0.1. It is evident that the further the distance is, the less the error. When the distance is 0.002, the maximum of the SELT's errors is about 0.12 and the maximum of the SELT2's errors is about 0.15. If the distance becomes 0.1, the maximum of the SELT's errors reduces to be about 0.05 and that of the SELT2's errors reduces to be about 0.01. As in the case of the calculations for inside the circle, the absolute errors from 0° to 180° are symmetrical to those of the angles from 180° to 360° for both cases of vorticity density. Figures 3.5.6 (c) to (f) show that the SELT's errors around the circle have a tendency to appear in four peaks, separately located on the 56.25°, 145°, 215° and 303.75°, while the distance increases from 0.02 to 0.1. It is noted that the locations of the four peak points for the results outside the circle are almost the same as those inside the circle in the case of constant vorticity density. The absolute errors for the case of SELT2 are symmetrical to the angle of 180° but it is difficult to predict a distinct peak point as in the case of SELT. It is evident that the rate at which errors are reduced in the case of linear distribution of vorticity density is faster than that of constant vorticity density.
Figure 3.5.6 (a) The relation between the position (outside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.002)

Figure 3.5.6 (b) The relation between the position (outside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.01)
Distance (outside the circle) = 0.02

Figure 3.5.6 (c) The relation between the position (outside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.02)

Distance (outside the circle) = 0.03

Figure 3.5.6 (d) The relation between the position (outside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.03)
Distance (outside the circle) = 0.05

Figure 3.5.6 (e) The relation between the position (outside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.05)

Distance (outside the circle) = 0.1

Figure 3.5.6 (f) The relation between the position (outside the circle) and the horizontal component of induced velocities for the circular distribution of vortex elements (distance=0.1)
Tables 3.5.1 and 3.5.2 show the comparisons of maximum errors for both cases in the same distance outside and inside the circle, respectively. It can be concluded that the convergent situation of the linear distribution of vorticity density is faster than that of constant vorticity density.
Table 3.5.1 The maximum errors comparison between SELT & SELT2 (outside the circle)

<table>
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<td>1.0</td>
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Table 3.5.2 The maximum errors comparison between SELT & SELT2 (inside the circle)

<table>
<thead>
<tr>
<th>Distances</th>
<th>SELT max. error $\times 100$</th>
<th>SELT2 max. error $\times 100$</th>
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<td>1.0</td>
</tr>
<tr>
<td>0.08</td>
<td>5.7</td>
<td>0.6</td>
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4. A NUMERICAL EXPERIMENT ON THE ROLL-UP OF A VORTEX SHEET

4.1 Introduction

The simulation of roll up of the trailing vortex sheet well behind an elliptically loaded lifting surface has been challenged by many authors for many years. Studies of this type are primarily motivated by the hazards to which light aircraft are exposed when flying in the wake of a heavier aircraft - a wake that may contain strong vortices in a small core. Following aircraft may be subjected to a very high rolling moment when they encounter one of these vortices. Westwater (1935) was the first to use the Rosenhead (1931) method for this problem. He confined his attention to points distributed in the downstream of his wing far enough to neglect the influence of the equivalent bound vortex, and thus converted the problem into a two-dimensional one, namely a Trefftz plane calculation. He used only ten discrete vortices of equal strength to investigate the evolution of the vortex sheet. Fortuitously, a spiral structure for the vortex sheet near the tips was found and the rate of the rolling up of the vortex sheet could be estimated. Attempts to reproduce Westwater's results by Takami (1964) and Moore (1971) were not successful. Takami represented the trailing vortex sheet by rather more discrete vortices than the number used by Westwater and employed a Runge-Kutta integration with much smaller time steps. He found that the vortices moved in complicated orbits around each other so that the sheet would have had to cross itself repeatedly. Therefore, Takami ascribed this to the rapid circling motions arising from the strong interaction among vortices which had come into close proximity. Moore (1971) recalculated Takami's work using even larger numbers of elemental vortices to represent the vortex sheet for the Westwater case and further reduced the time steps. However, he found no improvement in the randomness which developed in Takami's calculations.
Chorin and Bernard (1973) postulated that a point vortex approximation to a vortex sheet cannot be taken too literally, since a point vortex induces a velocity field which becomes unbounded, and cannot approximate a bounded field in any reasonable norm. Therefore, they conjectured that as soon as the velocity field of the point vortices is smoothed out and made bounded, the approximation becomes reasonable. Also they showed that Rosenhead's point vortex approximation is valid and useful, provided the singular character of all the vortices is obviated by some smoothing.

Moore (1974) described a numerical method which introduced a tip vortex to represent the tightly rolled portion of the vortex sheet and successfully eliminated the chaotic motion which was a feature of some earlier studies. He recognized that there are two possibilities for the earlier failure of obtaining the spiral structure. The first is that the earlier failure is due to a failure of the numerical method to integrate the discrete system correctly. The second is that the failure occurs because the exact solution of the discrete system does not converge to the solution of the equations governing the continuous sheet as the number of point vortices tends to infinity. Fink and Soh (1978) carefully discretized a two-dimensional vortex sheet and subsequently reset pivotal points onto the midpoint of its segment. They found that their results were contrary to those of the multi-vortex method and their calculations converged as the discretization was refined. They therefore thought that the success of Moore's method was partly due to the weak interaction of neighbouring elements associated with the low vorticity density of portions of the sheet remote from the cores, and partly due to the use of a large enough subdivision to prevent the logarithmic term from playing a significant role.
Pullin (1978) used similarity solutions to transform the time-dependent problems for the vortex-sheet motion into an integro-differential equation, involving the roll-up of an initially plane semi-infinite vortex sheet. Pullin & Phillips (1981) further studied similarity solutions for the rolling-up spiral that forms at the tip of an elliptically loaded wing using numerical methods, obtaining good agreement with Kaden's (1931) asymptotic spiral solution.

Following the error analysis by Fink and Soh (1978), Baker (1979) calculated the time evolution of the vortex sheet by adapting the "Cloud In Cell" technique and produced a well defined rolled up structure. His results indicated that the error decreases when more points are included. Baker (1980) proposed to provide more careful error estimates for the vortex-sheet motion. He thought that the error of the open vortex sheet, e.g. the sheet rolling up behind an elliptical loaded wing, becomes large in the central region of the spiral even if infinitely many evenly spaced points could be chosen to represent the sheet; it is difficult to assess the errors of the open vortex sheet. On the other hand, using the Euler-MacLaurin summation, he presented another errors estimate for the vortex sheet forming a closed curve, i.e. the vortex sheet shed by a ring wing. Although Baker discussed several possibilities, he did not obtain a successful and reliable result as shown by his computed vortex sheet which crosses itself. He cannot find a definitive reason for the distortion of the vortex sheet.

Hoeijmakers and Vaatstra (1983) utilized a second-order panel method to treat highly rolled-up portions of the vortex sheet. For the purpose of controlling the length of the stretching vortex sheet, they employed the concept of cutting the sheet to a specified length in order to prevent the angular extent of the vortex sheet to exceed a specified amount. The open end of the vortex sheet is represented by a vortex. If the sheet is cut, the vorticity contained in the cutoff piece is dumped into a vortex, which is subsequently placed at the centre of
vorticity of the vortex at the open end and the cutoff section of the sheet. In their results, they found that there is no sign of developing instabilities, which, presumably, is related to the fact that the sheet is stretching everywhere during the computation.

Krasny (1987) adopted a refinement of the idea expressed by Chorin and Bernard (1973) and desingularized the Cauchy principal value integral which defines the sheet's velocity. He found that the evolution of the vortex sheet converges with respect to the refinement in the mesh-size and the smoothing parameter. The computed roll-up of the vortex sheet from an elliptically loaded wing is in good agreement with Kaden's asymptotic spiral at early times. He cited that the computational difficulty in the roll-up vortex sheet's problem is that spurious perturbations introduced by computer round-off error can be amplified in regions where the vortex sheet is unstable.

However, the earlier studies concerning roll up of the vortex sheet were almost concentrated on the very special case of constant vorticity density distributed on the discrete segment and few used different distribution of vorticity density to simulate the evolution of the vortex sheet. In this chapter, the new method using linear distribution of vorticity density as described in § 3.1 is applied to the Westwater problem - the roll-up trailing vortex sheet of elliptically loaded wing. Since the length of the vortex is infinite, a single vortex of constant vorticity density is adopted to represent the vortex core as described in § 4.3. Distribution of vorticity along the vortex sheet for various instant times is used to describe the evolution of the vortex sheet. Numerical methods for the motion of the vortex sheet are described in § 4.4. This includes the sequential rearrangement of vortex elements, which is similar to the method of Fink and Soh, and the determination of the coefficients A and B for the linear distribution of vorticity density. The comparisons between the results of the present method
and that of the earlier studies are shown in § 4.5. The vorticity of a spiral core, which was represented by a discrete element of finite strength in the earlier studies, is considered to convect to the adjacent other vortex elements. Results for this sample calculation are discussed in § 4.6. Finally, the concluding remarks of the present numerical experiments are shown in § 4.7.
4.2 Problem statement

Consider the Trefftz plane of Westwater's roll-up problem to be \( x-y \) plane. Initially, the distribution of vorticity density is given by

\[
\gamma(x) = 2Ux \left( a^2 - x^2 \right)^{-1/2}
\]  

(4.2.1)

which is situated on the \( x \)-axis in the interval \(-a \leq x \leq a\). The velocity, \( U \), is the constant for the elliptically loaded wing with which the sheet is instantaneously moving in \( y \)-decreasing direction. Since the flow field is a two-dimensional, incompressible and inviscid fluid, the problem of determining the evolution of the vortex sheet appears to be straightforward. It can be represented by subdividing the strip and replacing each subdivision by a vortex element whose vorticity density is linear distribution. Thus, the first derivative of vorticity density \( \gamma^{(1)} \) terms in equation (7a) of Fink and Soh (1978), which were ignored in the constant vorticity density, are necessarily to be taken into account.

Figure 4.2.1 shows the schematic diagram of the problem, where \( a \) is the span of wing, \( \Gamma(x) \) is the spanwise distribution of bound circulation which is elliptical in this problem, and \( \gamma(x) \) is the distribution of vorticity of the vortex sheet.
Figure 4.2.1  Schematic diagram of relationship between span loading $\Gamma(x)$, vortex sheet, $\gamma(x)$, Trefftz plane, and rolled-up vortex.
4.3 Representation of the vortex core

The vortex core was taken to be the last vortex element on the free end of the vortex sheet which initially was assumed to be represented by a set of straight vortex elements. With the inviscid theory, the vortex core can be represented by a tightly wound spirals in which vorticity is concentrated. Therefore, it is essential to apply an appropriate approximation to represent this very highly curved portion of the vortex sheet.

Moore (1974) described a numerical method which amalgamated the tightly wound inner portion into a single tip vortex during the calculation of shape of the outer portion of the vortex sheet. Fink and Soh (1978) represented an arbitrarily small region of the core by an equivalent discrete core and emphasized that the representative procedure became exact only for the constant radius of curvature and uniform vorticity. If the error due to departure from such conditions is thought to be too large, finer segmentation must be adopted. However, since a spiral with an infinite number of turns cannot be represented by a finite number of vortices, the method applied here was to represent the vortex core with a single vortex whose strength remains constant during the calculation of the shape of the outer portion of the sheet in order to confirm that there is constant radius of curvature and uniform vorticity in the core. It is noted that the method used in this report is very similar to that of Fink and Soh (1978) but there is not any amalgamation procedure applied during the calculation of shape of the vortex sheet. The effect of non-uniform stretching of the vortex sheet only increases the number of the spiral turns in the roll-up portion. The strong rotation of the vortex core, namely the power source of the rolling-up sheet, would not be influenced by the non-uniform stretching of the vortex sheet.
4.4 Numerical methods for the motion of the vortex sheet

Since any non-uniform stretching of the vortex sheet would move the pivotal points away from the mid-points of their elements, it is considered to be useful to rearrange the vortex elements to equally spaced and to have the pivotal points placed at the centres of the elements which they represent. Fink and Soh (1978) had shown that the logarithmic term will vanish and does not become arbitrarily large if the pivotal points are arranged on the middle of the vortex element. In this section, numerical methods for the motion of the vortex sheet are described. This includes rearranging each vortex element to an equivalent length, resetting the pivotal point onto the mid-point of each element, and determining two unknown coefficients $A$ and $B$ of the linear distribution of vorticity density. The geometry of the sheet and the vorticity density are given correctly within the combined accuracies of the linear interpolation and finite differences formulas which are used here since the input data are not equally spaced.

The motion of the vortex sheet was simulated by the following procedures:

(1) The vortex sheet was initially represented by a straight line along which the vorticity originally concentrated. The distribution of vorticity density is given by equation (4.2.1). Figure 4.4.1 shows the initial position of the vortex sheet. N elements are used to represent the whole vortex sheet. The Nth element is an isolated element which represents the vortex core. The initial two unknown coefficients $A_k$ and $B_k$ for the kth element can be obtained by substituting the vorticity densities at both ends of kth element into equations (3.3.4) and (3.3.5). The initial induced velocity $u_j - iv_j$ of the jth element in the vortex sheet is determined by the initial position and A and B coefficients of vortex elements. This induced velocity is computed by the subprogram SELT2 (See Appendix II).
Figure 4.4.1 Initial position of the vortex sheet
(2) The vortex sheet moves to the new position over a time step $\Delta t$, by the Euler integration formula: $z_j(t+\Delta t) = z_j(t) + (u_j - iv_j) \Delta t$.

(3) Since the motion of the vortex sheet would cause the pivotal point of elements to move away from the mid-point of elements, the process of rediscretization of segments into equal length and resetting the pivotal point onto the mid-point of each element, suggested by Fink and Soh (1978), were adopted.

(4) The vorticity densities $\gamma_k(s_1)$ and $\gamma_k(s_2)$ at both ends of each element are determined after equally spacing. The determination of the vorticity densities $\gamma_k(s_1)$ and $\gamma_k(s_2)$ and the process of rediscretization are described in detail in § 3.3.1.

(5) The two unknown coefficients $A_k$ and $B_k$ of the $k$th element are calculated by substituting the vorticity densities $\gamma_k(s_1)$ and $\gamma_k(s_2)$ into equations (3.3.4) and (3.3.5) (Details are shown in § 3.3).

(6) The position and the coefficients $A$ and $B$ of vortex elements determine the induced velocity $u_j - iv_j$ of the $j$th element in the vortex sheet. This induced velocity is computed by the subprogram SELT2. Then, return to step (2).
4.5 Results of the numerical experiments

In the first place the Euler method was used to integrate the velocities of pivotal points by using equation (3.1.23). Multiplication of the induced velocities of pivotal points by non-dimensional time steps $\Delta t^* = \Delta t U/a$, determined the new positions of pivotal points of original elements. The sequential resetting of the pivotal points onto mid-points of new equivalent elements was used to prevent the occurrence of the logarithmic terms becoming arbitrarily large in the computation. The sequential resetting procedure means that the stretching vortex sheet was discretized into equivalent vortex elements once for one time step and the motion of a given set of equivalent vortices was not traced for more than one time step. $A$ and $B$ coefficients in the linear distribution of vorticity density for new elements are completely dependent upon the vorticity densities of both end points of elements.

The re-calculations for the number of initial elements $N=25, 30, 40, \text{and } 50$ pivotal points per symmetrical half of vortex sheet have been performed by using time steps $\Delta t^* = 0.02$. Figure 4.5.1 shows the results for the degree of roll-up and the position of centroid of vorticity at non-dimensional time $t^* = t U/a = 0.5$. $N$ shown in the figure 4.5.1 is the number of elements at time $t^* = 0.5$. In the present method, the number of pivotal points does not need to be reduced as the calculation progresses; indeed, it may be increased if desired. The length of elements after initial calculation will be different from that of the initial elements and can be chosen as desired. For the convenience of tracing the evolution of the vortex sheet, the length of elements after initial calculation will be as far as possible approaching to that of initial elements in this report. It is evident that the number of turns increases as the number of initial elements increases.
Figure 4.5.1 (a) & (b) For caption see facing page
Figure 4.5.1 Roll-up of trailing vorticity behind elliptically loaded wing, $t^*=0.5$, time step $\Delta t^*=0.02$; (a) initial elemental number=25; (b) initial elemental number=30; (c) initial elemental number=40; and (d) initial elemental number=50.
It is interesting to note that some vortex elements of exterior turn tend to be absorbed by those of inner turn in spiral shapes for N=25 and 50. A check of the relationship of the vorticity distribution and the position of vortex elements in the vortex sheet as figure 4.5.2 shows that there is a sharp change in the vorticity distribution of the spiral portion of the vortex sheet for N=25 and 50. This situation can be interpreted as follows. From the physical viewpoint, it may be due to the strong compression or mutual induction among vortices which has come into close proximity. Accordingly these distortions make the vorticity around this portion change sharply. This explanation is similar to Tryggvason's (1989) report. He said that as the sheet is compressed, the vorticity becomes more "peaked" in the horizontal direction. Once the interface has rolled up, the "ellipticity" of the vortex increases as it is stretched in the horizontal direction. This reduces the peak of the distribution. From the mathematical point, it may be due to the accumulation of a continuous round-off error in the computational procedure. It is worthwhile to note that use of the trapezoidal rule for the velocity induced by a sheet element at points on or off the element which are near but not at the midpoint can lead to serious inaccuracies owing to the neglect of logarithmic terms which vanish at the midpoint (see Fink and Soh (1974)). Pullin (1978) mentioned that the most likely erroneous source is the use of the trapezoidal rule for the velocity-inducing effect of the segmented part of the sheet. For a particular segment the major part of this error will come from the immediately adjacent sheet segments. For the overall solution, the cumulative error will depend on the relative magnitude of local and far-field contributions to the local velocity, and may be expected to depend on the ratio of arc-length step size along the sheet to sheet spacing, particularly in the spiral region.

However, it is evident that the spiral portion of the vortex sheet became distorted mainly due to this sharp change in the vorticity density. On the
contrary to this situation, smooth and homogeneous turns are obtained in the spiral portion of the vortex sheet for N=30 and 40. Figures 4.5.2 (a) and (d) also show that there is not a sharply changed situation in the relationship of the vorticity distribution and the position of vortex elements in the vortex sheet.

Figure 4.5.2 (a) (b) For caption see facing page
Figure 4.5.2 Distribution of vorticity along the vortex sheet for $t^*=0.5$, time step $\Delta t^*=0.02$ and the number of initial elements (a) 25, (b) 30, (c) 40, and (d) 50.
The whole vortex sheet in this problem can be divided into two portions: (1) non-spiral portion and (2) spiral portion. The value \( s/a = 1 \) is defined to be the separate point of both portions. There is a crest in the vorticity distribution of the non-spiral region, which was considered as the location of the Kelvin-Helmholtz instability by Moore (1974). After the separate point, namely roll-up region or spiral portion, there is the same number of "smooth" peak points in the vorticity distribution as that of turns in the spiral shape. That the vorticity changes abruptly or sharply due to the mutual compression or induction among the close elements of different turns will introduce the spiral shape to become distorted, although there is a crest in the vorticity distribution.

Since the present method rearranges pivotal points at each time step, similar to Fink and Soh's method, it is useful to follow in their steps to check the deviation of the lateral coordinate \( x_c \) of the centre of vorticity from the theoretical value \( x_c/a = \pi/4 \). Figure 4.5.3 shows the x-coordinate of the centre of vorticity for a series of sample calculations: 40.02, 40.01, 50.01, 60.01, and 100.005, where the numbers before dot point means the number of initial elements and the numbers after dot point means the used time step \( \Delta t^* \). When the time \( t^* \) is zero, the x-coordinates of the centre of vorticity for 40.02, 40.01, 50.01, 60.01 and 100.005 are respectively 0.7842, 0.7843, 0.7844, 0.7847 and 0.7850. When \( t^* \) changes from 0 to 0.1, the x-coordinate for the centre of vorticity drops for all samples. It is evident that the smaller the time step is, the less the percentage of errors will be. From \( t^* = 0.2 \) to \( t^* = 1.0 \), the x-coordinates of the centre of vorticity for all sample calculations approach to converge to the theoretical value \( \pi/4 \). Among these samples the case of 100.005 approaches closest to the theoretical value. Evidently, a faster convergent result can be obtained if the number of initial elements increases while the time step decreases. The price to be paid for this refinement is that C.P.U. time will
increase with the square of the number of elements and a smaller time step, e.g. the C.P.U. time for the case of 100.005 is about 36 hours to reach $t^* = 0.5$.

![Graph of x-coordinate of the centre of vorticity](image)

Figure 4.5.3 X-coordinate of the centre of vorticity for different samples

Table 4.5.1 shows the vertical component of initial induced velocities on $x=0.5$ for various numbers of initial elements. Theoretically, the induced velocity is a constant and independent of the spanwise coordinate. It is evident that the percentage of errors of the initial vertical induced velocities decreases as the number of initial elements increases. The percentage of errors of the initial vertical induced velocities can be reduced to less than 1% if the number of initial elements of the trailing vortex sheet is 160.
Table 4.5.1 Initial vertical induced velocities on x=0.5 for various numbers of initial elements

<table>
<thead>
<tr>
<th>Number of initial elements</th>
<th>Initial vertical induced velocities on x=0.5</th>
<th>Percentage error * (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-0.9535398</td>
<td>4.6</td>
</tr>
<tr>
<td>20</td>
<td>-0.9461220</td>
<td>5.3</td>
</tr>
<tr>
<td>30</td>
<td>-0.9567038</td>
<td>4.3</td>
</tr>
<tr>
<td>40</td>
<td>-0.9657328</td>
<td>3.5</td>
</tr>
<tr>
<td>50</td>
<td>-0.9717576</td>
<td>2.8</td>
</tr>
<tr>
<td>60</td>
<td>-0.9762252</td>
<td>2.3</td>
</tr>
<tr>
<td>70</td>
<td>-0.9794097</td>
<td>2.0</td>
</tr>
<tr>
<td>80</td>
<td>-0.9815865</td>
<td>1.84</td>
</tr>
<tr>
<td>100</td>
<td>-0.9847157</td>
<td>1.52</td>
</tr>
<tr>
<td>160</td>
<td>-0.9902957</td>
<td>0.97</td>
</tr>
</tbody>
</table>

* The theoretical value of initial vertical induced velocity on x=0.5 is -1

Figures 4.5.4 and 4.5.5 show the comparisons of the results of linear distribution of vorticity density with that of earlier studies. The number of initial elements and the time step used in the present studies are respectively 40 and Δt* = 0.02. The results of the present studies are in good agreement with those of Westwater's (1935) calculation except the location of the Kelvin-Helmholtz instability mentioned by Moore (1974).
Figure 4.5.4 Comparison of the results of linear distribution of vorticity density with that of Westwater's calculation. (a) $t^*(\text{Westwater})=0.212$ & $t^*(\text{present method})=0.22$; and (b) $t^*=0.34$. 

In (a), the initial position of the vortex sheet is marked, with a linear distribution of vorticity density. The comparison with Westwater's calculation shows slight differences in the initial positions and the subsequent development of the vortex sheet. In (b), a similar comparison is made, highlighting further differences in the results.
In order to simulate the correct vorticity density, Westwater replaced the sheet by 20 unevenly spaced discrete vortices of equal strength. When the time is $t^* = 0.34$, $y/a$ for the result of Westwater's calculation and that of the present studies are respectively -0.352 and -0.362 at the centre line. The number of elements for the present results is 98. It can be concluded that the present results at $t^* = 0.34$ have a good agreement with the results of Westwater's calculation.

Figure 4.5.5 (a) shows the comparative positions with the result of Fink and Soh (1978). The vorticity density of each vortex element in the vortex sheet is considered to be constant by Fink and Soh. The number of initial elements and the time steps for Fink & Soh's result are respectively 40 and 0.005. The time $t^* = 0.5$ and the integration method, which is Euler integration, are the same for both studies. It is evident that the number of turns in the spiral of the present results is more than that of Fink and Soh's result.

Figure 4.5.5 (b) shows the comparative positions with the result of Takami (1964). He used 20 vortices of identical strength to represent the half span of the vortex sheet and employed the Runge-Kutta integration method. The black spot in figure 4.5.5 (b) is the centroid of vorticity of Takami's result. It is evident that the downward speed of the Takami's vortex sheet is very slow if compared with that of the present results. The value of $y/a$ of the vortex sheet at the centre line for Takami's result is about -0.284, which is only half of that of the present results.

Figure 4.5.5 (c) shows the comparative positions with the result of Moore (1974). He applied the discrete vortex element method to the motion of the vortex sheet and used a finite discrete element to represent the core of the spiral. The number of vortices and the time step for Moore's result are respectively 60
Figure 4.5.5 (a) & (b) For caption see facing page
Figure 4.5.5 Comparisons of the results of linear distribution of vorticity density with that of earlier studies at $t^*=0.5$. (a) Fink & Soh (1978); (b) Takami (1964); (c) Moore (1974); and (d) Baker (1979).
and 0.002. The vortices are initially equispaced. The time $t^*$ for both studies is the same. The number of turns in the spiral of the present results is larger than that of Moore's result. The inner turns are very similar for both results.

Figure 4.5.5 (d) shows the comparative positions with the result of Baker (1979). He applied the "Cloud in Cell" technique, which was first used by Christiansen (1973), to the roll up of vortex sheets. Obviously, the vorticity distribution of roll-up vortex sheet for Baker's result is still within that of the present results. Besides, it can be clearly seen that the advantage of the present method over the "Cloud in Cell" method is that the spiral nature of the vortex sheet can be far more described in detail!

With the linear distribution of vorticity density, the comparative positions of roll-up vortex sheet calculated by Westwater, Fink & Soh, Takami, Moore, and Baker are clearly shown in figures 4.5.4 and 4.5.5. The values of $y/a$ at the centre line of these roll-up vortex sheets, from large to small, are respectively -0.284 (Takami), -0.437 (Fink and Soh), -0.461 (Moore), -0.469 (Baker), -0.500 (Westwater) and -0.563 (linear distribution of vorticity density). It is evident that the roll-up region of the present results is always greater than that of the others. The number of turns in the spiral of the present results is more than that of the others. Obviously, the length of the whole vortex sheet in the present results is the longest. The reasons for this longest stretching length are: (1) a linear distribution of vorticity density is adopted in each vortex element rather than taken it as a constant; this can increase the induced velocity of elements and subsequently the whole vortex sheet moves faster and stretches longer; (2) rediscretization technique as used by Fink and Soh (1978) is applied at each time step; (3) the number of elements of equivalent length in the vortex sheet randomly increases with the time; and (4) there is no consideration of equal strength of vortex elements, which was used by the earlier authors, i.e.
Westwater, Moore, and Takami, etc. It is a special feature of the present studies that the amalgamation procedure, used in order to obtain an identical strength, is not used.

Figure 4.5.6 shows the spiral shape of the case 100.005 at $t^*$=0.5. The number of turns at this instant for 100.005 is about 12, which is greater than that of 40.02. The y/a value of 100.005 is -0.600, which is also much more downward than that of 40.02. Evidently, the roll-up vortex sheet has a higher number of turns and descends faster if a larger number of initial elements and a smaller time step are used in the numerical experiments.

Figure 4.5.7 shows the relationship of the number of initial elements, time step, and the displacement y/a at $t^*$=0.5. Evidently, for the same number of initial elements, the smaller time step has a smaller value of the displacement y/a when it reaches the same instantaneous time $t^*$. For the same time step, the larger number of initial elements has a larger value of the displacement y/a while it reaches the same instantaneous time $t^*$. Obviously, it has raised a question necessary for further investigation. It is that what is the optimum for the time step and the number of initial elements used to simulate the roll up vortex sheet.
Figure 4.5.6 Roll-up of trailing vorticity behind elliptically loaded wing, $t^*=0.5$, time step $\Delta t^*=0.005$ and initial elemental number=100.
Figure 4.5.7 The relationship of the number of the initial elements, time step, and $y/a$ at $t^* = 0.5$
4.6 Analyses of the spiral core

The number of turns of the spiral generated by the elliptically loaded wing increases with the time if the core is represented by a fixed value of vortex element. Birkhoff (1962) argued that it would seem to be very difficult to simulate smooth rolling up calculations with discrete vortices. The smooth rolling up which is observed physically seems to depend on the influence of viscosity. The observed "rolling up" of vortex sheets near centres of concentrated vorticity is also strongly influenced by the molecular diffusion of vorticity. Tryggvason (1989) reported that the core contains a large fraction of the circulation; approximately half of the total circulation is within 1/8 of the wavelength from the centre. The core of the vortex, that is, the region mostly unaffected by the evolution, is somewhat smaller.

Previous works using point vortex to represent the core is inappropriate because of the unrealistic nature of the singularity of a vortex. However, the spiral evolution of the present study is very similar to the spirals of some phenomena, i.e. hurricane, tornados or typhoons etc.. In nature, vortex cores are recognized as the centres of tornados. Fett (1964) mentioned that the spiral formation of clouds can be explained with the initial condition of the storm: the warm humid, cloud-covered tongue in the tropical trough roll up during the development of the storm. The eye of the storm is a strange phenomenon as the eye of the typhoon is calm and still. This phenomenon has raised the question of the appropriateness of the assumption, using a fixed vortex element to represent the vortex core. Initially, a discrete vortex element may be used to represent the highly wound core of the spiral. However, after a fixed time, a certain mechanism is required to allow the vorticity in the tightly wound core to be convected.
Consider that the vorticity of the core starts to convect at $t^*=0.6$ and let the strength of a discrete vortex element used to represent the core of the vortex sheet be split equally to the adjacent vortex elements. If there are 20 vortex elements of circulation $\Gamma_k$ adjacent to the core of circulation $\Gamma_c$, the circulation of these adjacent vortex elements will become $\Gamma_k + \Gamma_c / 20$ when the core disappears. After splitting the core, the roll-up vortex sheet continues to change with time. Figures 4.6.1 (a) and (b) respectively show the shapes of the roll-up vortex sheet for the non-split and the split core of 60.01 sample when $t^*$ reaches 1.00. The stretching shape in the roll-up vortex sheet of the split core has already twisted together while the number of turns for the roll-up vortex sheet of the non-split core continues to increase smoothly with time, which is about 11 turns.

For the convenience of further examining the circulation density (or circulation / area) of the roll-up regions in the cases of a non-split and a split core, the average circulation per unit area along a radius vector is considered. Figure 4.6.2 shows the schematic diagram of the circulation density in the torus. Let $(X_g, Y_g)$ be the position of the centre of concentric circles and $(x_i,y_i)$ be the coordinates of vortex elements within the spiral. Then $(X_g,Y_g)$ can be obtained by the following formulae.

$$X_g = \frac{\sum_{i=N_1}^{N_2} x_i}{(N_2-N_1+1)}$$

(4.6.1)
Figure 4.6.1 Roll-up vortex sheet for (a) the non-split ($N=964$), and (b) the split core ($N=860$) of 60.01 at $t^*=1.00$. 
Figure 4.6.2 Schematic diagram of the circulation density in the torus

Band radius = 0.5\((r_1 + r_2)\)
\[ Y_g = \frac{\sum_{i=N1}^{N2} y_i}{(N2 - N1 + 1)} \] (4.6.2)

where

\( N1 \) = number of the first vortex element within the spiral
\( N2 \) = number of the last vortex element within the spiral.

In another words, \((X_g, Y_g)\) are the coordinates of the geometric centre of the vortex spiral. The representative radius of the torus, namely the band radius, is half the sum of the inside radius \( r_1 \) and the outside radius \( r_2 \) of the torus. \( \Gamma_1 \) and \( \Gamma_2 \) are respectively the circulations in the circle areas of radii \( r_1 \) and \( r_2 \). Then the circulation density of the torus can be expressed as

\[ \text{circulation density} = \frac{\Gamma_2 - \Gamma_1}{[\pi(r_2^2 - r_1^2)]} \] (4.6.3)

Figures 4.6.3 (a)–(e) show the comparative curves of the circulation density between a non-split and a split core of the roll-up vortex sheet at \( t^* = 0.7, 0.8, 0.9, 1.0 \) and 1.1. The black diamond and the white diamond represent the situation of the non-split core and split core, respectively.

At \( t^* = 0.7 \), the circulation density of the spiral centre starts to convect to the outer turns and drops very fast in the case of the split core. But, in the case of the non-split core, the maximum of the circulation density still remains in the spiral centre because we employ a fixed value of the discrete element to represent the core. The maximum circulation density for the case of the split core is about 45 and exists in the band radius of 0.075. The circulation density in the centre of the spiral core is about 35. When \( t^* \) is 0.8, the circulation
density in the centre of the spiral core continues to drop and is about 30. Figure 4.6.3 (c) shows that the circulation density in the centre of the spiral core drops to zero at $t^*=0.9$. It means that there is not any circulation within the 0.05 band radius. This result could be compared to the nature of a typhoon's eye, which is calm and still.

When $t^*$ reaches 1.0, the circulation density in the centre of the spiral core recovers to a non-zero value, which is the maximum of the spiral core again. It is about 38. Following the maximum, the circulation density in the centre of spiral core drops and then has another peak value in the 0.125 band radius (about 13% spiral radius from the centre of the core), which is about 32. Obviously, it can be cited that the instability of the spiral core is the power source of the circulation density of the core which diffuses to the outside turns of the spiral. Additionally, it is evident that with increasing time, the radius of the spiral increases and the maximum value of the circulation density in the spiral core gradually becomes small.

It is interesting to note that the curves of the circulation density of the split core are similar to the experimental results done by Piercy (1923). Piercy measured the vortex velocity distribution and vorticity behind the wing tip and found that the vorticity in the centre of the vortex core behind the wing is not the maximum. The maximum vorticity is located between the centre of the core and the position of the peak velocity. In another words, the position of the maximum circulation density is not always at the centre of the spiral (namely the core of the spiral). The vorticity of the core convects with time. The maximum circulation density decays with time. These situations cannot be found by representing the core with a fixed value of the discrete element. Obviously, the results of the split-core calculations are a more realistic approach
to model the actual situation of the roll-up vortex sheet behind the elliptically loaded wing.

Figure 4.6.3 (a) Comparative curves of circulation density between split and non-split core of the roll-up vortex sheet at $t^* = 0.7$
Figure 4.6.3 (b) Comparative curves of circulation density between split and non-split core of the roll-up vortex sheet $t^* = 0.8$

Figure 4.6.3 (c) Comparative curves of circulation density between split and non-split core of the roll-up vortex sheet at $t^* = 0.9$
Figure 4.6.3 (d) Comparative curves of circulation density between split and non-split core of the roll-up vortex sheet at $t^* = 1.0$

Figure 4.6.3 (e) Comparative curves of circulation density between split and non-split core of the roll-up vortex sheet $t^* = 1.1$
For the convenience of further examining the evolution of the spiral core, the centre of gravity for each band radius at non-dimensional time $t^*$ is calculated. The position of the centre of gravity $(X_{c.g.}, Y_{c.g.})$ is defined by the following mathematical formula.

$$X_{c.g.} = \frac{\int_{r_1}^{r_2} \gamma_i x_i \, dA}{\int_{r_1}^{r_2} \gamma_i \, dA}$$  \hspace{1cm} (4.6.4)$$

$$Y_{c.g.} = \frac{\int_{r_1}^{r_2} \gamma_i y_i \, dA}{\int_{r_1}^{r_2} \gamma_i \, dA}$$  \hspace{1cm} (4.6.5)$$

where

$\gamma_i$ = the vorticity density of the $i$th vortex element within the area of torus whose band radius is $0.5 (r_1 + r_2)$

$x_i$ & $y_i$ = the $x$ and $y$ coordinates of the pivotal point of the $i$th vortex element

d$A = d(\pi r^2)$, $r$ is within the range from $r_1$ to $r_2$

$r_1$ = inside radius of the torus

$r_2$ = outside radius of the torus.

Take the band radius as the horizontal axis and the $x$ or $y$ coordinates of the centre of gravity as the longitudinal axis. At the same time, put the $x$ or $y$ coordinates of the geometrical centre, which is constant for all band radius, on the same diagram. Then, in the diagram take the $x$ or $y$ coordinates of the
geometrical centre as a reference axis and check the distribution of the centre of gravity.

Figures 4.6.4 and 4.6.5 respectively show the distribution of the centre of gravity at $t^*=0.7$ and 1.1 for the case of the non-split core, where the black diamonds are coordinates of the geometrical centre and the white blocks are the coordinates of the centre of gravity. It is evident that at $t^*=0.7$ the $y$-coordinate of the centre of gravity in the spiral core is still unstable but the $x$-coordinate of the centre of gravity is already coincident with that of the geometrical centre. At $t^*=1.1$, the coordinates of the centre of gravity are almost the same as the coordinates of the geometrical centre in the spiral core.

Figures 4.6.6, 4.6.7 and 4.6.8 respectively show the distribution of the centre of gravity at $t^*=0.7$, 1.1 and 1.4 for the case of split core. It is evident that at $t^*=0.7$ and 1.1 the coordinates of the centre of gravity are unstable, but they happen to coincide with the geometrical centre at $t^*=1.4$. It can be concluded that the evolution of the spiral core gradually reaches a stable situation after a fixed time. It is expected that, the non-split core, having an artificial constant core vortex, stabilizes at $t^*=1.1$ against $t^*=1.4$ for the core of a split-core.
Figure 4.6.4 (a) X-coordinate distribution of the centre of gravity for non-split case, $t^*=0.7$.

Figure 4.6.4 (b) Y-coordinate distribution of the centre of gravity for non-split case, $t^*=0.7$. 
Figure 4.6.5 (a) X-coordinate distribution of the centre of gravity for non-split case, $t^*=1.1$.

Figure 4.6.5 (b) Y-coordinate distribution of the centre of gravity for non-split case, $t^*=1.1$. 
Figure 4.6.6 (a) X-coordinate distribution of the centre of gravity for split case, \( t^* = 0.7 \).

Figure 4.6.6 (b) Y-coordinate distribution of the centre of gravity for split case, \( t^* = 0.7 \).
Figure 4.6.7 (a) X-coordinate distribution of the centre of gravity for split case, $t^*=1.1$.

Figure 4.6.7 (b) Y-coordinate distribution of the centre of gravity for split case, $t^*=1.1$. 
Figure 4.6.8 (a) X-coordinate distribution of the centre of gravity for split case, $t^*=1.4$.

Figure 4.6.8 (b) Y-coordinate distribution of the centre of gravity for split case, $t^*=1.4$. 
4.7 Concluding remarks

The roll-up of a vortex sheet has been successfully computed from a vortex sheet with a linear distribution of vorticity density. The present numerical technique which eliminates singularities of the type $1/r$ in the discretized formula allows eleven turns to be calculated by using $10^4$3 elements. It is sufficient for this closely wound spiral to give a two-dimensional distribution of vorticity in terms of circulation per unit area. It is evident from the comparisons with earlier studies that the spiral region is best represented by the present method.

According to the results of the spiral core calculations in § 4.6, the vorticity distribution in the inner core after a perturbation (split) seems to approach a constant value and thus can be considered as a Rankine vortex. In the computation of the motion of the vortex sheet, the circulation in the core vortex, which was represented by a singular vortex in the earlier studies, was equally distributed to the adjacent elements after an evolving period. This process of dispersing the core vortex allows the convection of the vorticity in the core region to be simulated in the absence of viscous diffusion. The results show that there is a tendency for the vorticity density in the core to reduce in magnitude. It also indicates that the velocity in the central region of the spiral is relatively small. From these results it can be concluded that the instability in convection alone is sufficient to cause the dispersion of vorticity in the core region.
5. APPLICATION OF THE METHOD ON THE SHEDDING OF VORTICITY FROM A SEMI-INFINITE PLATE

5.1 Introduction

The problem introduced here is of the transient flow in the separated and reattached regions along a body which is characterized by a separation bubble situated between the separation point and the point of reattachment. This kind of flow has been recognized to be very important in engineering. There have been many investigations of a wide variety of flow configurations. Examples include the flow of water around a control rod inserted into a nuclear reactor; the flow around a high speed bullet; the flow around a rocket as it ascends to the sky; the air flow around a heated steel plate which has just moved out from a rolling mill; the flow around a solar collector and the flow around rough elements of various shapes attached to a surface, such as the fins of a heater exchanger; and the heat transfer from a flat surface in an air fluidized bed and the blades of a gas turbine. The vortex, which is generated by the flow separation, has a strong perturbing effect on the boundary-layer structure and consequently it plays a significant role in fluid engineering systems.

In order to investigate the problem of two-dimensional vortex shedding behind a square-based section, Clements (1973) approximated the free shear layer by a discrete-vortex model and predicted the dominant feature of the flow behind a bluff-based section. Sarpkaya (1975) used a discrete-vortex technique to study two-dimensional vortex shedding behind an inclined plate and predicted the Strouhal number and kinematic features of the flow. Kiya, Sasaki and Arie (1982) used a discrete-vortex model to simulate the separation bubble over a
two-dimensional blunt flat plate with a finite thickness and obtained reasonable predictions of the time-mean velocity, the root mean square velocity and the surface-pressure fluctuations. Nagano, Natto and Takata (1982) used a discrete vortex model to analyse a two-dimensional flow past a rectangular prism and emphasized that one of the most important process in the discrete vortex model is the determination of the strength of the nascent vortex and the location of the nascent vortex to be introduced into the flow field. Thompson and Hourigan (1986) combined finite-difference and discrete-vortex techniques to investigate an acoustically perturbed two-dimensional separated flow around a heated plate and predicted a number of features which were observed in experiments: an increase in the intensity of the sound field causes an increases in the coherence of the flow, a reduction of the reattachment length, and an increase in both the peak local time-mean Nusselt number and the spatially-averaged time-mean Nusselt number at the plate surface.

The above discrete vortex methods, with a variety of numerical approaches, give steady state results which are in good agreement with experiments. The aim of this work is to investigate the mode of transient flow in the area between the points of flow separation and reattachment with the vortex sheet method suggested by Soh, Hourigan and Thompson (1988). The mathematical model, which includes the Schwarz-Christoffel transformation, discretization of a vortex sheet, determination of the initial distribution of vorticity, calculation of the generated circulation from a corner and application of the rediscretization technique first used by Fink and Soh (1978), is described in § 5.2. The results and discussion for the four cases, including (i) flow in constant free stream velocity, (ii) flow with perturbed shedding of vorticity at resonance frequency, (iii) flow with forced oscillation of the free stream, and (iv) flow in consideration
of the decay of vorticity, are shown in § 5.3. Finally the concluding remarks of the above studies are presented in § 5.4.
5.2 Mathematical description of the model

Consider a semi-infinite plate which has a finite thickness, a square leading edge and is aligned parallel to a uniform approaching stream. The shear layers shed from the front corners of the plate are approximated by arrays of vortex sheet elements. The motion of the free shear layers is represented by the evolution of these elements in time. The velocities of a vortex sheet element consist of a component from the two-dimensional irrotational potential flow around the plate and a component which is induced by the all vortex sheet elements. These velocities can be calculated with the use of the Schwarz-Christoffel transformation.

5.2.1 Schwarz-Christoffel transformation

In the physical plane, there is a semi-infinite plate of thickness 2h which extends to infinity along the positive real axis. Figure 5.2.1 shows the schematic diagram for the Schwarz-Christoffel transformation of a semi-infinite plate which maps the region outside of the semi-infinite plate in the physical plane (z-plane) onto the upper half of the transformed plane (λ-plane) as given by equation (5.2.1). Hence, the corners on the plate B and C, at z=ih & -ih, are mapped into λ=1 & -1, respectively.

\[
z = \frac{2h}{\pi} \left[ \sin^{-1}(\lambda) + \lambda \left( 1 - \lambda^2 \right)^{1/2} \right] \\
\] (5.2.1)
Figure 5.2.1 Schwarz-Christoffel transformation of a semi-infinite plate
5.2.2 Discretization of a vortex sheet

Computations are carried out in the transformed plane (λ-plane). The image of the vortex sheet, in the λ-plane, is discretized into N number of small straight vortex sheet elements such that the jth element has a vorticity γ_j distributed over its length Δs_j. The other parameters associated with each element are: the angle of inclination with the real axis, θ_j; and the pivotal point being the mid-point of the element - denoted λ_j, which is z_j in the physical plane. The flow separation from the right-angled corner of the leading edge is computed by applying the Kutta condition to the separated shear layer, which is represented by a vortex sheet. For simplificity, the vorticity density is assumed to be uniformly distributed in each vortex element.

5.2.3 Velocity

The configuration of the flow consists of a free stream and a vortex sheet attached to the leading corner B of the plate. The free stream has a magnitude of U_0 and is parallel to the real axis. The complex velocity for a point X which has a complex potential, ω, is given by

\[ u(λ_i) - iv(λ_i) = \frac{dω(λ_i)}{dλ} \frac{|dλ|}{dz} \lambda_i^2 \]  

(5.2.2)

with

\[ \frac{dλ}{dz} \bigg|_{λ_i} = -\frac{iπ}{4h} \frac{1}{(1 - λ_i^2)^{1/2}} \]  

(5.2.3)
\[
\frac{d\omega(\lambda_i)}{d\lambda} = \frac{4hU_0\lambda_i}{\pi} + \left[ \frac{1}{2\pi i} \sum_{j=1}^{N} \left( \int_{-\Delta s_j-1/2}^{\Delta s_j+1/2} \frac{Y_j(s) \, ds}{\lambda_i - \lambda_j - s e^{i\theta_j}} \right) \right]_I
\]
+ \left[ \frac{1}{2\pi i} \sum_{j=1}^{N} \left( \int_{-\Delta s_j-1/2}^{\Delta s_j+1/2} \frac{Y_j(s) \, ds}{\lambda_i - \lambda_j - s e^{i\theta_j}} \right) \right]_{II}
+ \left[ \frac{1}{2\pi i} \sum_{j=1}^{N} \left( \int_{-\Delta s_j-1/2}^{\Delta s_j+1/2} \frac{Y_j(s) \, ds}{\lambda_i - \lambda_j - s e^{i\theta_j}} \right) \right]_{III}
+ \left[ \frac{1}{2\pi i} \sum_{j=1}^{N} \left( \int_{-\Delta s_j-1/2}^{\Delta s_j+1/2} \frac{Y_j(s) \, ds}{\lambda_i - \lambda_j - s e^{i\theta_j}} \right) \right]_{IV}
\]

(5.2.4)

where

\begin{align*}
2h & = \text{the thickness of the semi-infinite plate} \\
U_0 & = \text{the velocity of free stream at infinite} \\
\gamma_j & = \text{the strength of vorticity in the jth vortex element} \\
\lambda_j & = \text{the coordinates of the jth vortex element in the transformed plane} \\
\lambda_i & = \text{the coordinates of the considered point in the transformed plane.}
\end{align*}

The integration is along the length of the elements, s, with local origin on each pivotal point. The limits of the integration \(\Delta s_j+1/2\) and \(\Delta s_j-1/2\) are the distances between the pivotal point and the edges of the element, so that the jth element has a length of \(\Delta s_j = \Delta s_{j+1/2} + \Delta s_{j-1/2}\). Subscripts I, II, III and IV represent the vortex sheet in the first quadrant of the transformed plane and the appropriate images in the second, third, and fourth quadrants. It means that \(\lambda_i\) in the right-hand terms of equation (5.2.4) is the same value as that of the first quadrant for the subscripts I, II, III, or IV. But, \(\lambda_j\) and \(\gamma_j\) are different among the quadrants;
i.e. if \([\lambda_j]^1 = \eta_j + i \xi_j\) and \([\gamma_j]^1 = \gamma_j\) are in the first quadrant, \([\lambda_j]^I = - \eta_j + i \xi_j\) and \([\gamma_j]^I = - \gamma_j\) in the second quadrant; \([\lambda_j]^II = - \eta_j - i \xi_j\) and \([\gamma_j]^II = \gamma_j\) in the third quadrant; and \([\lambda_j]^IV = \eta_j - i \xi_j\) and \([\gamma_j]^IV = - \gamma_j\) in the fourth quadrant. In this way the real and imaginary axes will become streamlines and thus satisfy the boundary condition in the physical plane.

It can be found that \(d\omega(\lambda)/d\lambda\) is not the complex velocity in the transformed plane. In fact, the complex velocity in the transformed plane is \(d\omega(\lambda)/d\lambda\) times \(|d\lambda/dz|^2\) and the complex velocity in the physical plane is \(d\omega(\lambda)/d\lambda\) times \(d\lambda/dz\).

The first term in the right-hand side of equation (5.2.4) is contributed by the irrotational free-stream flow velocity, the other terms are contributed by the free shear layers separated from both front corners of the plate and the images of the free shear layer.

Figure 5.2.2 shows the schematic diagram of the "medium complex velocity" in the transformed plane. It consists of three components: (i) two-dimensional irrotational potential flow around the corners, \(4\pi U_0 \lambda / \pi\), (ii) the velocities induced by the free shear layers separated from both corners, shown as solid lines from \(\pm 1\) in figure 5.2.2, and (iii) the velocities induced by the images of the free shear layers, shown as dashed lines from \(\pm 1\) in figure 5.2.2.

The free shear layer separated from one front corner of the semi-infinite plate can be approximated by a vortex sheet. For simplifying the problem, we only consider the free shear layer in the first quadrant shown in the figure 5.2.2. When the others are to be considered, they can be easily obtained by substituting the corresponding coordinates of the vortex elements in the considered quadrant and the corresponding vorticity densities into equation (5.2.4).
Figure 5.2.2 Schematic diagram of the "medium complex velocity".
5.2.4 Determination of the initial distribution of vorticity

The vortex sheet method, unlike the point vortex method which usually sheds one vortex, needs to begin with an array of vortex elements. It becomes necessary to prescribe the shape and distribution of an initial vortex sheet just to initiate the computation.

A short vertical vortex sheet with one end attached to the separation point, a corner of the plate, is considered. The distribution of vorticity density is given by

\[ \gamma^*(s^*) = 2Ks^* (1 - s^*^2)^{-1/2} \]

where \( s^* \) is the distance of the vortex element from the separation point and \( \gamma^* \) is the given strength of the vortex elements. Since the Kutta condition requires that the flow in the \( \lambda \)-plane has stagnation points at \( \lambda = 1 \) and -1, this is expressed as

\[ \frac{d\omega}{d\lambda} = 0 \quad \text{at } \lambda = 1 \text{ and } -1 \]

Equation (5.2.6) can be satisfied by an appropriate value for \( K \) and this determines the function \( \gamma^*(s^*) \) in equation (5.2.5).
5.2.5 The Kutta condition: calculation of the generated circulation from a corner

Two approaches have often been used in the vortex methods for determining the strength of a nascent vortex and its position. In one method, the nascent vortex is placed at a certain fixed point and its strength is determined by satisfying the Kutta condition. In the other, the strength of the nascent vortex is determined by the boundary layer theory and its position is found by satisfying the Kutta condition.

Clements (1973) considered that the velocity at separation points could be zero to satisfy the no-slip condition. Hence the vortex which leaves the separation point is determined by the velocity \( U_s' \) at the outer edge of the boundary layer. The rate of vorticity shedding into the shear layers is determined by the relationship \( dT/dt = 0.5U_s'^2 \). On the other hand, Sarpkaya (1975) stated that the rate at which vorticity is shed into the wake is given by

\[
\int_0^\delta \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) u \, dy
\]  

(5.2.7)

where \( \delta \) is the boundary-layer thickness. This may be closely approximated by

\[
\frac{d\Gamma}{dt} = \frac{1}{2} \left( \frac{V_1^2 + V_2^2}{2} \right) \approx \frac{1}{2} V_1^2
\]

(5.2.8)

where \( V_1 \) and \( V_2 \) respectively represent the velocities at the outer and inner edges of the shear layer. He introduced a "nascent vortices", also called "Kutta
vortices", near the edge of the plate so that at the start of each time step the Kutta condition is satisfied at the edge.

Nagano, Natto and Takata (1982) applied a discrete vortex model to analyse a two-dimensional flow past a rectangular prism and emphasized that one of the most important points in the discrete vortex model is how to determine the strength of the nascent vortices and the location for them to be introduced. The vortices are shed from the edges into the shear layers as a result of the separation of the boundary layers along the faces of the plate. Therefore, the rate of vorticity shedding should be determined through the relationship \( \frac{d\Gamma}{dt} = 0.5 U_s^2 \), where \( U_s \) is the velocity at the outer edge of the boundary layer. It is true, as the flow is unsteady, that the outer edge of the boundary layer will also change with time.

Thompson and Hourigan (1986) combined finite-difference and discrete-vortex techniques to investigate an acoustically perturbed two-dimensional separated flow around a heated plate. They reported that the velocity \( U_s \) at the outer edge of a hypothetical boundary layer at the leading corner is used in the kinematic condition \( \frac{dT}{dt} = 0.5 U_s^2 \) to determine the rate of vorticity shedding. The location of a nascent vortex is obtained by using the Kutta condition. This requires zero velocity at the point in the transformed plane corresponding to the leading edge corner, together with the assumption that the boundary layer separates tangentially to the leading edge.

The present method uses the concept stated in equation (5.2.7). Writing \( \text{VIN} \) as the numerical approximation of \( V_1 \) in equation (5.2.8) and \( G_N \) as the circulation in the nascent element, the result is

\[
G_N = -0.5 \text{VIN}^2 \Delta t
\] (5.2.9)
The negative sign is related to the direction of the generated circulation which is positive for counter-clockwise and $\Delta t$ is the time increment. The no-slip condition is satisfied exactly only at the instant of the introduction of each new vortex. It ceases to be valid during the remainder of the time step by the very nature of discretization.

5.2.6 Determination of the time step

The time step $\Delta t$ is determined by the length of the element $\Delta s$ and the velocity $\text{VIN}$ of the nascent vortex element near the front corner of the semi-infinite plate in the transformed plane. It can be expressed as

$$\Delta t = \frac{\Delta s}{\text{VIN}}$$

(5.2.10)

where $\text{VIN}$ can be obtained by using equation (5.2.2). According to the method used here, the nascent vortex element appears near the edge with one of its ends attached to the shedding edge as shown in figure 5.2.3. This placement together with equations (5.2.9) and (5.2.10) is considered as the approximation to the Kutta condition.
5.2.7 Applying the rediscretization technique to the plate's problem

The discrete vortices used to model the vortex sheet have a tendency to become random in their motion due to the high velocities induced by the other vortices in close proximity. Such problems have been overcome by a rediscretization technique which was first used by Fink and Soh (1978). The rediscretization technique allows a movement of vorticity between different parts of the sheet, without reducing the detail of its description. The transfer of vorticity was found to depend upon the position and the difference between the strengths of the developing vortical structures. The use of the redistribution technique, in regions of rapid stretching, ensures that the coherent structure of the sheet is maintained, even though the vorticity becomes weak. But, as shown by Maskew (1977), significant errors occur in calculations of this kind when the neighbouring sheets are much closer than the point-vortex separations. In order
to prevent the chaotic motion happening in the development of the vortex sheet shed from the plate, the rediscretization technique as used by Fink and Soh will be adopted. The only difference from that of Fink and Soh is that the rediscretization method is used once per 10 time steps instead of on each time step. The main reason for choosing this alternative rediscretization is as follows. The displacements of some sections of a vortex sheet are so small that they fall within the error of rediscretization. Thus the rediscretization process will inhibit the development of these sections of the sheet. After a series of sample calculations, it is found that when the length of the element is within 0.01-0.035, the results from adopting the rediscretization once per 10 time steps are still reasonable, although there is some chaotic motion in the small-scale separation bubble.

5.2.8 Outline of computation

The computational steps are summarized as follows:

1. Initially a straight vortex sheet, consists of 10 elements with $\Delta s$ equals to 0.01, is placed at 60° to the real axis for the $\lambda$-plane. The vorticity distribution is given by equation (5.2.5).

2. The velocity $V_{IN}$ is the image velocity in the $\lambda$-plane. It is calculated by the formula: $V_{IN} = \int_0^1 \frac{d\omega(\lambda)}{d\lambda} \lambda d\lambda/dz^2$ at $\lambda = 1$. The time step is calculated by the formula: $\Delta t = \Delta s/V_{IN}$.

3. The pivotal points in the $\lambda$-plane are displaced to a new position over a time step $\Delta t$, by the Euler integration formula: $\lambda(t+\Delta t) = \lambda(t) + (u(\lambda) - iv(\lambda))\Delta t$. The edges of each element are transported in the same manner except that their velocities take the form of the average between the two adjacent pivotal points.
The edge of the element at the free end of the vortex sheet moves with the pivotal point of the element.

(4) The circulation, $\Delta \Gamma$, of the nascent vortex element is $\Delta \Gamma = -0.5VIN^2 \Delta t$.

(5) The process of redistribution as given by Fink and Soh (1978), which readjusts all elements into equal length, is used once per 10 time steps.

(6) A new element of length, $\Delta s$, is generated from the separation point and becomes part of the vortex sheet.

(7) The steps from (2) to (6) are repeated as many times as needed.
5.3 Results and discussion

The flow is symmetrical about the real axis. Normalization of parameters is achieved by setting $h$ and $U_0$ to unity.

5.3.1 Case 1: Flow in constant free stream velocity

Figures 5.3.1~5.3.4 show the situations of rolled-up vortex sheet and the corresponding streamlines around the plate in the flow of constant free stream velocity for $t=1.477, 2.356, 2.500$ and $2.701$. The length of the element is $0.015$ and the numbers of elements, $NE$, are $198, 447, 520$ and $614$, respectively. It is evident that the vortex sheet rolls up into a large spiral. The free end of the vortex sheet is found outside of the spiral. There are a few small-scale rolled-up structures, small separation bubbles, within the large spiral. The number of these small-scale spirals continues to increase with time. Amsden and Harlow (1964) explained these results in term of "slip instability" in the flow. These small-scale spirals also represent local concentration of vorticities as was reported by Nagata et al. (1985). In their experimental observation, Nagata et al. claimed that the turbulence in the vortex region is caused by the centrifugal instability owing to the local concentration of vorticity. The larger amount of vorticity shedding from the secondary vortex will give rise to a larger concentration of vorticity which causes the shear layer to undulate and roll up in localized section. It is suggested that this local undulation in the free shear layer is related to the transition to a turbulent flow.

The penetration problem mentioned by Kuwahara (1973) does not appear in the present method. From the diagrams of the rolled-up vortex sheet in figures 5.3.1~5.3.4 the separated shear layer, which becomes a large rolled-up spiral
structure, does not penetrate the surface of the plate. There is a region next to the free end of the shear layer which evolves and comes very near to the surface of the plate. The distance of this region to the surface of the plate is about 0.02h. There is a process of capturing of one spiral by another one in adjacent; this is depicted in the evolution of small-scale spirals as shown in figures 5.3.2-5.3.4.
Figure 5.3.1 (a) The evolution of the vortex sheet around a semi-infinite plate in a constant free stream velocity and (b) the corresponding streamlines patterns at $t=1.477$. 
Figure 5.3.2  (a) The evolution of the vortex sheet around a semi-infinite plate in a constant free stream velocity and (b) the corresponding streamlines patterns at t=2.356.
Figure 5.3.3 (a) The evolution of the vortex sheet around a semi-infinite plate in a constant free stream velocity and (b) the corresponding streamlines patterns at $t=2.500$. 
Figure 5.3.4  (a) The evolution of the vortex sheet around a semi-infinite plate in a constant free stream velocity and (b) the corresponding streamlines patterns at t=2.701.
5.3.2 Case 2: Flow with perturbed shedding of vorticity

A perturbation is achieved by introducing a sinusoidal component. Thus, the circulation of the nascent vortex element, $G_{N}^{*}$, is given by

$$G_{N}^{*} = G_{N} (1 + A \sin 2\pi ft) \quad (5.3.1)$$

where

- $G_{N} = 0.5 (V_{IN})^{2} \Delta t$, is the result of Kutta condition, see equation (5.2.9)
- $V_{IN} = \text{the velocity of the nascent vortex element}$
- $\Delta t = \text{time step}$
- $A = \text{the amplitude of the oscillation}$
- $f = \text{the frequency of the oscillation}$
- $t = \text{time}$

Figures 5.3.5~5.3.9 show a series of the evolution of the vortex sheet near the upper corner of a plate and the corresponding streamlines at $t=1.053, 1.652, 1.700, 1.797$ and $1.895$. These results are derived from $f=0.07, A=0.1$ and $\Delta s=0.02$. The corresponding Strouhal number is 0.14. It lies in the middle of the resonance range of 0.1 to 0.12 and 0.18 to 0.21 as discovered by Parker & Welsh (1983). The formation of numerous small-scale spirals throughout the whole of the rolled-up vortex sheet is a reminiscence of the observation by Prandtl (1904) and the spark shadowgraph by Pierce (1961), and the computed results by Hama (1962). Pullin and Perry (1980) regard the appearance of these patterns as essentially caused by the unstable nature of the free shear layer as well as the interference by an apparatus. In the investigation of a starting flow behind a triangular prism in uniform and stratified flow, Huhe et al. (1983)
observed small spirals within the rolled-up vortex sheet and stated that the vibration of the apparatus was the cause of this instability.

Kelvin-Helmholtz instabilities in the vortex sheet will lead to the formation of small-scale spirals as shown in Case 1. Any perturbations at resonance frequency, such as in this case, will enhance the structure of these small-scale spirals. In the perturbed shedding of vorticity, the spacing between adjacent small-scale spirals increases with respect to the arc length from the point of flow separation.

The movements of these small-scale spirals can be presented by plotting the arc length of the spiral from the point of flow separation against the index of the spiral. The index is an integer which identifies the spiral. An index of 1 represents the spiral nearest to the point of flow separation. The plots for various time, as shown in figure 5.3.10, are normalized by scaling such that the arc length to the first spiral becomes a unity (L1). These curves form an envelop which is the steady state line. It is interesting to note that the plot for Pierce's spirals falls on the envelop of the present results. This plot of spiral index is a convenient way of comparing the patterns of small-scale spirals from different sources. It is evident that the wavelength of the small-scale spirals gradually becomes larger as it approaches the centre of the large spiral structure. The number of small-scale spirals increases with time.

A check on the core of the rolled-up large spiral shows that it travels at a velocity 0.59U0. Although this is 18% higher than 0.5U0 which was measured by Pierce (1961) in the shedding of free shear layer from the edge of a plate in the impulse motion, it is still within the range (0.5-0.6)U0 reported by Kiya (1986).
Figure 5.3.5 (a) The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the generated vorticity and (b) the corresponding streamlines patterns at $t=1.053$. 

(a) 

(b)
Figure 5.3.6  (a) The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the generated vorticity and (b) the corresponding streamlines patterns at $t=1.652$. 

$T=1.652$
$N= 338$
Figure 5.3.7  (a) The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the generated vorticity and (b) the corresponding streamlines patterns at t=1.700.
Figure 5.3.8  (a) The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the generated vorticity and (b) the corresponding streamlines patterns at $t=1.797$. 
Figure 5.3.9 (a) The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the generated vorticity and (b) the corresponding streamlines patterns at $t=1.895$. 
Figure 5.3.10  Plot of the arc length of the small-scale spirals against their indices (smaller index represents spiral nearer to the point of flow separation).
Figures 5.3.11 (a)–(h) show another set of results of unsteady flow caused by the perturbed shedding of vorticity. The simulated conditions are (i) $\Delta s=0.025$, (ii) $f=0.24$, (iii) $A=0.1$, and (iv) $St=2fh/U_0=0.48$. Compared with the results of the lower frequency, it can be found that the small-scale bubbles become flattened more quickly for the case of higher frequency. It is evident that the frequency affects the evolution of the small-scale spirals. Although the growth of the small-scale spirals of the higher frequency is faster than that of the lower frequency, the large spiral structures for these different situations still remain the same. Evidently, the overall structure of the large spiral is unaffected by the Strouhal number.
Figure 5.3.11 (a) & (b) For caption see facing page
Figure 5.3.11 (c) & (d) For caption see facing page
Figure 5.3.11 (e) & (f) For caption see facing page
Figure 5.3.11 The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the generated vorticity; the frequency $f$ and the amplitude $A$ are 0.24 and 0.1, respectively.
5.3.3 Case 3: Flow with small oscillation in the free stream

A perturbation component is introduced into the free stream, $U^*$, as shown in equation (5.3.2).

$$U^* = U_0 \left( 1 + A \sin 2\pi ft \right)$$  \hspace{1cm} (5.3.2)

where $U_0$ is the mean velocity and is set to unity. Figure 5.3.12 shows the results for $A=0.1$, $f=0.07$ and $\Delta s=0.02$. The corresponding Strouhal number is 0.14. Figure 5.3.13 shows the results of a different Strouhal number, which is 0.48. The overall structure of the large vortex sheet spirals in these Strouhal numbers are similar. Obviously the overall structure of the large vortex sheet spiral is unaffected by the Strouhal number. Additionally, the conditions of Case 3 are the same as those of Case 2, except for different sources of perturbation. Case 2 is a flow with perturbed shedding of vorticity and Case 3 is a flow with small oscillation in the free stream. The forced oscillation of the velocity in the free stream has generated small-scale spirals. However, unlike the small-scale spirals in Case 2, they survive only for a short time before being engulfed by the large rolled-up vortex sheet spiral. The core is found to travel at a velocity of $0.60U_0$ as compared with $0.59U_0$ in Case 2.

The large-spiral core seems to be formed by a closed vortex sheet, but in fact there are two directions in the thin vortex sheet. One is entering the vortex core and the other is exiting the vortex core. When the vorticities of the whole vortex sheet were checked, the "zero" vorticity was found to exist between the large-spiral core and the nearest small spiral. The vorticities of the vortex elements in the whole vortex sheet are all the same sign except elements of zero vorticity; i.e. the vorticities in the vortex sheet above the plate are all negative except for the elements of zero vorticity.
Figure 5.3.12 (a) & (b) For caption see facing page
Figure 5.3.12 (c) & (d) For caption see facing page
Figure 5.3.12 The evolution of the vortex sheet around a semi-infinite plate in the oscillation of the free stream velocity; the frequency $f$ and the amplitude $A$ are 0.07 and 0.1, respectively.
Figure 5.3.13  (a) & (b) For caption see facing page
Figure 5.3.13 (c) & (d) For caption see facing page
Figure 5.3.13 The evolution of the vortex sheet around a semi-infinite plate with the oscillation of the free stream velocity; the frequency $f$ and the amplitude $A$ are 0.24 and 0.1, respectively.
5.3.4 Case 4: Flow in consideration of the decay of vorticity

A number of workers have, in the past, investigated the relationship between the amount of vorticity generated in each shear layer before separation and the amount that finally appears in the fully formed street vortices further downstream for various bluff bodies. Fage and Johansen (1927) found the latter to be between 51% and 65% of the former for a flat plate set at right angles to the free stream. Clements (1973) described in his paper that amounts varying between 87% and 94% of the shed vorticity in the shear layers appearing in the rolled-up vortices. The loss of vorticity in the computations arises from two mechanisms: (i) the destruction of vorticity at the rear face of the body and (ii) a small amount of cancellation between elementary vortices of opposite sign which enter the same rolled-up vortex core. The discrepancy between the computation and the experiments can probably be accounted for by the fact that computations are essentially inviscid and the mechanism whereby much of the vorticity is lost must be viscous in nature.

In real flows, only a fraction (say about 60%) of the circulation fed into the shear layers is found in the vortex clusters or in the concentrated vortices of the Karman vortex street (Mair and Maull 1971). Sarpkaya (1975) cited that the two mechanisms mentioned by Clements are sufficient to account for an approximately 40% loss in circulation. In his calculations he showed that the strength of the vortex clusters varies from 84% to 91% of the vorticity generated in each shear layer, the 10%-15% loss in vorticity arises from the removal of vorticity at the rear face of the plate and a small amount of cancellation between elementary vortices of opposite sign which enter the same rolled-up vortex core. Sarpkaya emphasized that in real flows, part of the vorticity is lost immediately behind the plate by viscous action and by mixture of the positive and negative
vorticity from two edges of the plate, and part in the shedding process, during which the vortex cluster to be shed attains its minimum transport velocity and draws oppositely signed vorticity across from the other side of the wake. In fact, it is conjectured that the loss in vorticity depends on the particular time in a given cycle and reaches its maximum rate just at the time of shedding. The numerical model cannot simulate such a kinematic- or eddy-viscosity dependent phenomenon. If necessary, an additional loss in vorticity will have to be artificially introduced into the model, beyond and above that produced by the two mechanisms cited above, in order to bring the strengths of the shed vortex clusters into closer agreement with those obtained experimentally. Stansby (1982) reported that the vorticity flux through a section of boundary layer $|\partial \Gamma / \partial t|=0.5U_s^2$ gave an unrealistic wake formation. In order to make the mean rate of shedding compatible with the experiments of Fage and Johnsen (1927), he used $|\partial \Gamma / \partial t|=0.4U_s^2$ rather than $0.5U_s^2$. The calculation then gave reasonable estimates of drag and lift but, hardly surprisingly, the prediction of pressure around separation was rather poor.

Nagano, Naito and Takata (1982) suggested that there seem to be three main causes of the loss in vorticity. They are (i) the destruction of vorticity near the wall due to the interaction with the boundary layers along the body, (ii) the cancellation of opposite-signed vortices due to the mixing and (iii) the dissipation of vorticity due to three-dimensional turbulence. In the discrete vortex representations of the flows tried by various research workers, some mechanisms for the loss in vorticity due to (i) and (ii) have usually been included, but the loss in vorticity in these studies has been much less than that observed in the real fluid. Therefore, Nagano et al. presumed that the extra loss in vorticity in the real fluid is due mainly to (iii), whose precise mechanism in the separated flow, however, is far from being known as yet. They incorporated into their model an artificial loss of vorticity in a phenomenological way and
Kiya and Arie (1980) mentioned that the absence of sufficient cancellation of vorticity in the formation region of the rolled-up vortices is the most serious shortcomings of the discrete-vortex model. In order to simulate the vorticity cancellation, they introduced a method to reduce the strength of the elemental vortices and assumed that the fraction 1-\(\lambda\) of the total vorticity is canceled in the formation region of rolled-up vortices. After trial and error, they found that \(\lambda=0.8\) is an optimum value for their vortex patterns. The Strouhal numbers for the cases of their numerical simulation are approximately 0.2-0.25, which is larger than the average experimental value of 0.15. The fraction of the circulation in the vortex cluster amounts to 0.36 on average when they are in the region \(x/(2a)=10-16\) downstream of the plate. The values of fractions in the other experiments are 0.6 (Fage and Johansen 1927), 0.43 (Roshko 1954), 0.30 (Bloor and Gerrard 1966) and 0.26 (Davies 1976). Therefore, Kiya and Arie argued that, unless the circulation reduction is introduced, good agreement is not obtained.

Kiya, Sasaki and Arie (1982) utilized the discrete-vortex model to simulate the separation zone over a two-dimensional blunt flat plate with finite thickness and right-angled corners, which is aligned parallel to a uniform approaching stream. They introduced a reduction in the circulation of elemental vortices as a function of their ages in order to represent partly the viscous and/or turbulent

estimated variation in the strength of vortices with the streamwise distance. They found that the strength of vortices decays only minimally just after the front edge separation (-0.5\(\leq x\leq 0\)); it decays to about 60\% by \(x/2b=5.5\), where \(2b=\)thickness of the prism, and then continues to decrease slowly with the distance. They also pointed out that the decay in the strength of vortices begins near the rear edge of the rectangular plate and most of the loss in vorticity takes place during the sharp rolling up process of the vortices just behind the body.
dissipation of vorticity, and partly the three-dimensional deformation of the vortex filaments, which leads to a decrease in the vorticity component normal to the plane of flow. The circulation of every vortex was reduced according to the law

\[ \frac{\Gamma(t)}{\Gamma_0} = 1 - \exp \left[ -\frac{a^2 \text{Re}}{4(U_\infty t / H)} \right] \]  

(5.3.3)

where \( \Gamma_0 \) is the initial strength, \( \Gamma(t) \) is the circulation at time \( t \) (age of the vortex), \( a \) is a constant, and \( \text{Re} \) denotes the Reynolds number \( U_\infty H / \nu \), \( \nu \) being the kinematic viscosity of fluid. Kiya, Sasaki and Arie pointed out that although the circulation reduction is inconsistent with the requirement that angular momentum be conserved in inviscid flows, this can hardly be avoided to simulate adequately the overall structure of essentially dissipated flows. From the practical point of view, they also felt that the circulation reduction is permissible if the resulting model can reveal some fundamental features of flow in the separation zone, which cannot easily be obtained by experimental means.

In the present report, the circulation of each vortex is assumed to decay according to the exponential law; that is,

\[ \Gamma_i(t) = \Gamma_i(0) \cdot e^{-\lambda t} \]  

(5.3.4)

where \( \Gamma_i(t) \) is the circulation at time \( t \), \( \Gamma_i(0) \) is the initial strength, and \( \lambda \) is a positive constant which is related to the half life of the vorticity. When the time approaches infinite, the circulation of the vortex will become zero. It means that the generated circulation from the corners of the plate reduces to zero in the downstream through the following possible mechanisms: (i) the destruction of vorticity near the wall due to the interaction with the boundary layers along the
plate, (ii) the destruction of vorticity among the fluid particles due to the collision of the fluid particles, and (iii) the non-destructive decay of vorticity, which involves the three-dimensional effects (the effects of spanwise and streamwise vortices) and changes to the other potential forms of vorticity. From equation (5.3.4), the half life of the vorticity of each vortex can be expressed as

\[ t_{1/2} = \frac{0.6932}{\lambda} \] (5.3.5)

It is obvious that if \( \lambda \) is 0.6932, the half life \( t_{1/2} \) of the vorticity for each vortex will be 1. In the present study, two different half life sets of the vorticity of vortex elements are studied.

Figures 5.3.13-5.3.22 represent the evolution of the vortex sheet with the loss of vorticity and the corresponding streamlines patterns around a semi-infinite plate. It is called LV2. The simulated conditions are (i) \( \lambda=0.8 \), which means that the half life of vorticity \( t_{1/2} \) is 0.866; (ii) \( \Delta s=0.1 \); (iii) \( f=0.24 \); (iv) the flow perturbed by the shedding of vorticity; and (v) \( \Delta s=0.01 \). In order to prevent the number of elements increasing randomly and making the CPU time become too large, the length of elements alternatively increases with time.

There is an amalgamation process in the evolution of the vortex sheet. The amalgamated large-scale rolled-up spiral gradually decays with time and expands downstream. The velocity of the large-scale rolled-up spiral is about \( 0.67U_0 \). Evidently the velocity of the large-scale rolled-up spiral with the loss of vorticity is faster than that without consideration of the loss of vorticity. In another words, the concentration of the vorticity has the effect of slowing down the downstream velocity. As time increases, the amalgamated large-scale rolled-up spiral diffuses completely and no spiral form appears.
Figure 5.3.14 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=0.031$. 
Figure 5.3.15  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=0.356$. 
Figure 5.3.16 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=0.691$. 
Figure 5.3.17  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at t=1.063.
Figure 5.3.18  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=2.064$. 
Figure 5.3.19  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at \( t = 2.994 \).
Figure 5.3.20  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=4.045$. 
Figure 5.3.21 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=5.015$. 
Figure 5.3.22  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=6.027$. 
Figure 5.3.23  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=7.456$. 
It is interesting to note the angle formed by the separated shear layer and the upper surface of the plate, also named "the separated angle". Initially, the separated angle is about 90°. As the vorticity sheds from the corner and decays with the time, the separated angle varies within the range of 54°-58°.

Figures 5.3.23-5.3.32 show another set of results in consideration of the loss of vorticity, called LV3. The simulated conditions are the same as those of LV2 except for the half life of vorticity, \( t_{1/2} = 1 \). The comparison between LV2 and LV3 shows that the longer half life of vorticity can enhance the rolled-up vortex sheet to grow faster and preserve longer. A check on these two numerical results shows that when the half life of the vorticity is longer, the total circulation and the total length of the vortex sheet at the same instant in time are larger.

The evolution of the first small-scale bubble is described as follows. Initially it decays to a smaller bubble; later it becomes an overlapped thin layer and finally decays to an overlapped line layer. The rolled-up situation of the vortex sheet is not only dependent upon the total circulation of the vortex sheet but also dependent upon the decayed situation of the vorticity, i.e. the half life of the vorticity.

Similar to the results of LV2, an amalgamation process and an amalgamated large-scale spiral structure also appear in the evolution of the vortex sheet of LV3. There are some wave-shape curves in the corresponding streamlines patterns of the amalgamated spiral region. It is evident that the large vertically rising spiral is to induce these wave-shape curves of the streamlines. Finally, the amalgamated spiral gradually becomes a blunt curve and the corresponding streamlines become parallel to the plate, with no wave-shaped curves. Evidently there is no rolled-up vortex sheet existing further downstream. From the above
results it can be concluded that the separation bubble appears only near the plate's corner and does not remain far downstream. A check on the separated angles of these results shows that the separated angle from the semi-infinite plate of 90° leading edge, varies within a fixed range from 53° to 60°.
Figure 5.3.24 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=0.031$. 
Figure 5.3.25  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=0.256$. 
Figure 5.3.26 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=0.568$. 
Figure 5.3.27 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at t=1.042.
Figure 5.3.28  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at t=2.128.
Figure 5.3.29  (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=3.013$. 
Figure 5.3.30 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=4.030$. 
Figure 5.3.31 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at t=5.034.
Figure 5.3.32 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=6.416$. 
Figure 5.3.33 (a) The evolution of the vortex sheet around a semi-infinite plate in consideration of the loss of vorticity and (b) the corresponding streamlines patterns at $t=7.837$. 
5.4 Concluding remarks

The transient flow over a semi-infinite plate has been numerically studied by a vortex sheet technique. This includes four cases of unsteady flow: (i) flow in constant free stream velocity; (ii) flow with perturbed shedding of vorticity at resonance frequency; (iii) flow with forced oscillation of the free stream; and (iv) flow in consideration of the decay of vorticity.

Unlike the vortex sheet from an elliptically loaded wing which has a strong tendency to roll up, Fink and Soh (1978), the present flow system restricts a vortex sheet to a region near the surface of a plate and this restriction gives rise to a slower rolling up motion. The consequence is the formation of small-scale spirals within the large rolled-up vortex sheet. These structures of small-scale spirals, which are suspected to be caused by Kelvin-Helmholtz's instability, become very prominent when the flow is perturbed under resonance frequency. Forced periodical perturbation may also give rise to these small-scale spirals but they soon disappear into the large rolled-up vortex sheet.

A calculation using the perturbation as in Case 2 with Strouhal number 0.14 & 0.48, shows results very similar to those of Case 2. It can be concluded that the frequency of the perturbation is by far more influential on the mode of flow separation than its sources. This has been demonstrated by Stokes, Welsh and Hourigan (1986) in their simulation of acoustic excitation using the correct frequency but with a different source of excitation.

Evidently, the small-scale spirals always appear in the vortex sheet, whether they are in the flow of steady free stream velocity or in the perturbed unsteady flow. According to the comparison between the experimental results done by Pierce (1961) and the present numerical results, it can be confirmed that the
undulation in the vortex sheet is due to the perturbed shedding of vorticity, which
was predicted by many earlier researchers (Pullin and Perry, 1980; Huhe et al.,
1983). The main sources of this perturbed shedding of vorticity may be caused
by the perturbation of the natural sound wave, the formation of the pressure
gradient near the separation point of the corner or both.

Additionally, the exponential law similar to Kiya, Sasaki and Arie (1982), is
introduced to simulate the decay of the vorticity. It is evident that the separated
angle varies within the range of $53^\circ$-$60^\circ$ while the decay of vorticity is
considered. The large-scale separation bubble appears only near the plate’s
corner and does not remain in the far downstream. The velocity of the large-
scale rolled-up spiral with the decay of the vorticity is 11\% higher than that of
experimental results measured by Kiya (1986). Evidently the decay of the
vorticity caused by the turbulent diffusion of vorticity and three-dimensional
effects, plays an important role in the behaviour of the separation bubble.

Numerical accuracy is important for these results. The size of the element,
$\Delta s$, should be less than 0.05 to give a reasonable resolution for the simulation of
small-scale spirals. Results shown here are calculated for $\Delta s$ in the range from
0.010 to 0.035.
6. NUMERICAL STUDY OF TRANSIENT HEAT TRANSFER AROUND A HEATED PLATE WITH FLOW SEPARATION

6.1 Introduction

Flow in the separated and reattached regions along a body is characterized by a separation bubble just between the separation point and the point of reattachment. This kind of flow has been seen to be very important in engineering, as in the vortex generated for the flow separation which has a strong perturbing effect on the boundary-layer structure and it may modify the heat transfer characteristics. Eibeck and Eaton (1987) reported that a longitudinal vortex can possibly affect heat transfer rates in three ways. First, the large-scale vortical motion locally changes the mean velocity field and mean temperature field. Secondly, the vortex modifies the turbulence properties in a complex three-dimensional way. This changes the effective diffusivity to heat and therefore affects heat transfer rates. In many situations the vortex modifies the flow properties (such as the near-wall layers of the boundary layer, the sublayer, and the buffer layer) and contributes to a large change of the total resistance to heat transfer.

Although experimental techniques have been constantly improved, it is difficult for measurements to reveal the fine details of the flow; for example, in regions where the flow field separates from a body, such as the behaviour of a separation bubble near the separation point, and in the case where heat transfer involves the measurement of the local Nusselt number. In this regard, numerical techniques are able to complement experimentation as their solutions will provide information in areas which are found to be very difficult to measure.
Thompson and Hourigan (1985) combined finite difference and discrete vortex techniques to solve the problem of a two-dimensional inviscid incompressible separated flow over a heated semi-infinite plate with acoustic excitation. The solution of energy equations for unsteady separated flow around a heated plate is proceeded by setting up a mesh to solve the finite-difference approximations to these equations. The related horizontal and vertical velocities are obtained from a discrete vortex technique which also solves for the induced velocity of vortex elements. With this model, they predicted a number of features which were observed in experiments: an increase in the intensity of the sound field causes an increase in the coherence of the flow, a reduction of the reattachment length, and an increase in both the peak local time-mean Nusselt number and the spatially-averaged time-mean Nusselt number at the plate surface.

By integrating the vorticity transport equation and energy equation, Paolino, Kinney, and Cerutti (1986) carried out a numerical investigation in the unsteady, two-dimensional viscous flow of an incompressible, constant property fluid flowing over a cylinder. The velocity distribution is obtained from the vorticity distribution by integrating the velocity induction law. In their paper they described the transient behaviour of the average Nusselt number over time, (NuD)avg. After the impulsive start of the flow, the value of (NuD)avg around the surface of a circular cylinder follows a decrease and increase path, and finally decreases asymptotically to a steady-state value. They also reported that the minimum values in the local NuD occur near the separation points.

Karniadakis (1988) used the spectral element method to investigate numerically the forced convection heat transfer from an isolated cylinder in crossflow (laminar two-dimensional flow). The numerical results predicted the size of the wake, the temporal and spatial structure of the von Karman vortex
street, the unsteady lift and drag coefficient, and the unsteady local heat transfer coefficient; these are all in agreement with available experimental data. He also found that the boundary layer thickness is approximately 8% of the cylinder diameter at Reynolds number equals 200.

By using the vortex sheet technique described in Chapter 3 and a finite differences scheme, the problem similar to that of Thompson & Hourigan is calculated. However, the present study is concentrated on the transient temperature distribution and heat transfer coefficient near the separated region from the corner. The temperature distribution further downstream from the separation region, will be neglected in the present study. The mathematical model which includes the hydrodynamic equation, energy equation, finite differences and vortex sheet techniques, is described in § 6.2. The outline of computation is listed in § 6.2.5. The flow field is described in terms of three regions: (i) the rolled-up region, (ii) the transition region, and (iii) the parallel flow region. The transient results are shown as the rolled-up vortex sheet, isotherms and streamlines and presented in § 6.3. Finally the concluding remarks of the studies are listed in § 6.4.
6.2 Formulation and numerical technique

Before solving the heat transfer problem, the velocity vector in the energy equation is considered. Since the hydrodynamic problem is the same as that in Chapter 5, the flow separated from the leading edge can be calculated by a vortex sheet technique originated by Fink and Soh (1974). The mathematical description of the hydrodynamic model, such as the Schwarz-Christoffel transformation, the discretization of a vortex sheet, the complex velocity, the determination of the initial distribution of vorticity, the calculation of the generated circulation from a corner (the Kutta condition) and the determination of a time step, has shown in § 5.2. The decay of vorticity as described in § 5.3.4 is allowed in this work in order to model the reduction in the contribution of induced velocity caused by diffusion of vorticity.

6.2.1 The energy equation

Consider a semi-infinite horizontal plate of the thickness 2h in a horizontal free stream $U_0$. The plate is being heated and maintained at a temperature $T_1$. The ambient temperature of the plate is $T_0$. All variables are non-dimensionalized using $U_0$, h, and $(T_1-T_0)$ as the basic parameters. Thus, the non-dimensional velocities are $(u*/U_0)$ and $(v*/U_0)$. The non-dimensional displacements are $(x*/h)$ and $(y*/h)$. The non-dimensional time is $t*(U_0/h)$, the non-dimensional temperature is $(T*-T_0)/(T_1-T_0)$, and the non-dimensional thermal diffusivity $\beta^*$, is $(k/pcp)/(U_0h)$. The plate temperature and the ambient temperature after non-dimensionalized are 1 and 0, respectively. The energy equation is given by

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \beta^* \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

(6.2.1)
where

\[ T = \text{the local temperature} \]
\[ t = \text{time} \]
\[ u = \text{the horizontal velocity component} \]
\[ v = \text{the vertical velocity component} \]
\[ \beta^* = \text{the thermal diffusivity} = (k/\rho c_p)/(U_0 h) \]
\[ k = \text{the thermal conductivity of the fluid} \]
\[ \rho = \text{the density of the fluid} \]
\[ c_p = \text{the specific heat at constant pressure} \]

The vortex sheet technique yields the velocity components \( u \) & \( v \) for equation (6.2.1). A finite difference method is used to solve for the temperature field and the related grid is set up above the plate. The corresponding finite difference approximation to equation (6.2.1) is given by

\[
T_{i,j}^{n+1} = T_{i,j}^n - \Delta t \left[ u_{i,j} \left( \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x} \right) + v_{i,j} \left( \frac{T_{i,j+1} - T_{i,j-1}}{2\Delta y} \right) \right] + \Delta t \beta^* \left[ \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{(\Delta y)^2} \right]
\]

(6.2.2)

Here, \( x \) is the distance along the plate from the leading edge and \( y \) is the vertical distance above the plate centerline. The superscript, \( n \), refers to the \( n \)th time-step and \( \Delta t \) is the time increment.

Since the flow around the plate is symmetrical, we consider only the upper region around the plate. The grid of size 51x22 for the equal grid system occupies the region defined by
and the boundary conditions for the temperature are

\[ T(x,1)=1, \]
\[ T(x,4.2)=0, \]
\[ T(0,y)=0, \]
and
\[ \frac{\partial T}{\partial x}(10,y)=0. \]

For further investigating the situation of the region around the surface of the plate, an unequal grid system was used instead of the one described in figure 6.2.1. In this case the spacing of the horizontal grid \( \Delta y \) is made finer near the plate so that the resolution is higher in this region. Figure 6.2.2 shows the schematic diagram of the unequal grid system.
It is assumed that $\Delta x=0.2$ in both equal and unequal grid systems is the same; then the temperature of the nodal points in the unequal grid system can be calculated by using the area-weight rule and the temperature of the nodal points in the equal grid system. Figure 6.2.3 shows the overlapped diagram of the equal and unequal grid systems, where the solid lines are the equal grid system and the dash lines are the unequal grid system. Assume the nodal point $(i, J)$ of the unequal grid system lies between the nodal points $(i,j+1)$ and $(i,j+2)$ of the equal grid system. $\Delta y$ is the spacing of nodal points $(i,j+1)$ and $(i,j+2)$ and $\Delta y_j$ is the spacing of the nodal points $(i,j+1)$ and $(i,J)$. The temperature of the nodal point $(i,J)$ is

$$T(i,J) = \frac{A_1 T(i+1,j+1) + A_2 T(i-1,j+1) + A_3 T(i+1,j+2) + A_4 T(i-1,j+2)}{A_1 + A_2 + A_3 + A_4}$$

(6.2.3)
where $T(i+1,j+1)$, $T(i-1,j+1)$, $T(i+1,j+2)$ and $T(i-1,j+2)$ are the temperature of the nodal points $(i+1,j+1)$, $(i-1,j+1)$, $(i+1,j+2)$ and $(i-1,j+2)$, respectively, and $A_n$ ($n=1, 2, 3, \& 4$) are the areas shown as figure 6.2.3.

Figure 6.2.3 The overlapped diagram of the equal & unequal grid systems, where the solid lines is the equal grid system and the dotted lines is the unequal grid system.

6.2.2 Outline of computation

The computational steps are summarized as followings:

(1) Initially a straight vortex sheet, consists of 10 elements with $\Delta s$ equals to 0.01, is placed at $60^\circ$ to the real axis for the $\lambda$-plane. The vorticity distribution is given by equation (5.2.5). The plate temperature is maintained at $T_1=1$ and the initial ambient temperature of the plate is $T(0)=0$. 
(2) The velocity \( V_{IN} \) is the image velocity in the \( \lambda \)-plane at \( \lambda=1 \). It is calculated from equation (5.2.2) and resulted in the formula: \( V_{IN} = \frac{d\omega(\lambda)}{d\lambda} \cdot |d\lambda/dz|^2 \). The time step is calculated by the formula: \( \Delta t = \Delta s / V_{IN} \).

(3) The pivotal points in the \( \lambda \)-plane are displaced to a new position over a time step \( \Delta t \), by the Euler integration formula: \( \lambda_j(t+\Delta t) = \lambda_j(t) + (u_j(\lambda) - iv_j(\lambda))\Delta t \). The edges of each element are transported in the same manner except that their velocities take the form of the average between the two adjacent pivotal points. The edge of the element at the free end of the vortex sheet moves with the pivotal point of the element.

(4) The circulation \( \Gamma(t) \) of each element of vortex sheet in consideration of decay is \( \Gamma(t) = \Gamma(0) \cdot e^{-\alpha t} \), where \( \alpha \) is a positive constant which is related to the half life of the vorticity.

(5) The circulation \( \Delta \Gamma \), of the nascent vortex element is \( \Delta \Gamma = -0.5V_{IN}^2 \Delta t \).

(6) The process of rediscretization as given by Fink and Soh (1978), which readjusts all elements into equal length, is used once per 10 time steps.

(7) A new element of length, \( \Delta s \), is generated from the separation point and becomes part of the vortex sheet.

(8) The induced velocity of the grid system can be calculated by equation (5.2.2).

(9) The temperature field of the grid system is calculated by equation (6.2.2). If the temperature field nearer the surface of the plate is needed, it can be approximated by equation (6.2.3).

(10) The steps from (2) to (9) are repeated as many times as needed.
6.3 Results and discussion

Figures 6.3.1 are the results from the study of a non-dimensional thermal diffusivity $\beta^*=2.206\times10^{-5}$ which corresponds to

\[\begin{align*}
k &= 0.0261 \text{ W/m}^0\text{k} \\
\rho &= 1.177 \text{ kg/m}^3 \\
c_p &= 1005 \text{ J/kg}^0\text{k}
\end{align*}\]

and

\[\alpha = 0.2772;\] so that the half life $t_{1/2}$ is 2.5.

The flow fields presented in figure 6.3.1 can be described in terms of the following three regions. The first, is the rolled-up region, which is to the right of the leading edge of the plate; this region is strongly influenced by the rolling up of the vortex sheet. The second region is the parallel flow region; it is unaffected by the roll up of the vortex sheet and the streamlines are horizontal. The third region is called the transition region. It lies between the above mentioned two regions. The following measurements are used to quantify the description of the results: the size of the vortex sheet, $X_V$, which is the horizontal measurement between the leading edge and the right most end of the spiral; the size of the thermal bubble, $X_t$, which is the horizontal measurement between the leading edge and the right most end of the looped isotherm given by $T=1/40$; the reattachment of the streamline, $X_C$, which is the size of the separation bubble; and $X_r$ which is the distance between the leading edge and the point in which the thermal boundary layer is the thinnest.
Figure 6.3.1 (b) For caption see facing page
Figure 6.3.1 (c) For caption see facing page
Figure 6.3.1 (d) For caption see facing page
Figure 6.3.1 (e) For caption see facing page
Figure 6.3.1 (a)-(g) Transient results of separated flow and forced convection heat transfer around a heated plate. Top is the rolled-up vortex sheet, middle is isotherms, and low is streamlines.
Figure 6.3.2 shows the evolution of these four horizontal lengths $X_v$, $X_t$, $X_c$ and $X_r$ with time. All four values are expressed as the half plate thickness ($h$). It is evident that all values $X_v$, $X_t$, $X_c$ and $X_r$ increase with time. It means that the size of the vortex sheet, $X_v$, the size of the thermal bubble, $X_t$, and the size of the separation bubble $X_c$, increase with time. The position of the thinnest thermal boundary layer $X_r$ slowly moves downstream with time. Before the 4th unit of time, the streamwise growth rates of $X_v$, $X_t$, $X_c$ and $X_r$ are in proportion to time. After the 4th unit of time, the streamwise growth rates of $X_v$, $X_t$, $X_c$ and $X_r$, from large to small, are $X_v > X_c > X_r > X_t$. The phase of the size of the vortex sheet, $X_v$, grows faster than these of the others after the 4th unit of time. It is expected that $X_v$ is the largest as the vortex sheet is the driving force on the evolution of the separation bubble and the thermal bubble.
Since both the thickness of the plate $2h$ and the thermal conductivity, $k$, were constants, the distribution of the Nusselt numbers based on the plate thickness $2h$ is calculated by the formula $\text{Nu} = -\frac{\partial T}{\partial y} \frac{h}{(T_1 - T_0)}$. Figure 6.3.3 shows the distribution of instantaneous Nusselt numbers along the surface of the plate at different times. As time equals 1.00, 2.03, 3.00, 4.04, 5.08, 6.08, 7.06, 8.02, and 9.05, the locations of the maximum instantaneous Nusselt numbers are 0.5, 1.5, 2, 3, 3.5, 4.5, 5.5, 6.25, and 7.5, respectively. Evidently as time increases, the maximum value of the instantaneous Nusselt number decreases in magnitude and its location moves downstream. Figure 6.3.4 shows the relationship of the maximum value of the instantaneous Nusselt numbers to time.

Moreover, the maximum Nusselt number occurs upstream of the reattachment point. This results are in contrast with the earlier experimental results obtained by Ota and Kon (1974) and Cooper, Sheridan, and Flood (1986). Cooper et al. defined the reattachment point to be the position of the maximum local heat transfer coefficient. Ota and Kon reported that the heat transfer coefficient is always maximum at the position of reattachment point, and defined the reattachment point as a point of zero skin friction, its position having previously been determined in the mean by tuft probe examination (Ota and Itasaka (1976) and Ota (1975)). However, the present results are in good agreement with the experimental results obtained by Sparrow (1988). Sparrow (1988) clearly showed that for the case of the blunt-faced longitudinal cylinder, the points of reattachment and the maximum Nusselt number do not coincide. Indeed, the maximum Nusselt number consistently occurs upstream of the reattachment point. Sparrow, Kang, and Chuck (1987) suggested that the common assumption of the reattachment point to be the position of the maximum local heat transfer coefficient is, at best, a special case. They also stated that the magnitude of the wall-adjacent streamwise velocity and the direction of the
transverse velocity in the wall-adjacent flow are the two factors affecting the relationship of the reattachment point and the maximum local heat transfer coefficient.

According to the earlier experimental results (Ota (1975); Ota and Kon (1977); Kottke, Blenke and Schmidt (1977)), the maximum heat transfer occurs at a streamwise location $X_{\text{max}}$ and the reattachment point $X_r$ for the blunt-face longitudinal cylinder are within the range of 1.2-2.5 and 2-3, respectively. The present results are in agreement with their results for the time less than 4.04. For the time greater than 4.04, the maximum instantaneous Nusselt number continues to move downstream and gradually becomes smaller in magnitude.

The general distribution of instantaneous Nusselt number is characterized by an initial decrease, follows by an increase to a maximum and, further on, approaches a constant value. In regards to the lines $t=1.00$ and 2.02, it is expected that the same characteristics can be derived from a computation at higher resolution. This trend is in good agreement with the numerical results by Thompson and Hourigan (1986) for flow under acoustic perturbation and the experimental results by Ota and Kon (1974; 1977; 1979), and Cooper, Sheridan and Flood (1986).
Figure 6.3.3 Relation of instantaneous Nusselt number and x/h at different time.
Figures 6.3.5 (a) and (b) show the temperature distribution above the surface of the plate at $t=2.552$ and $4.044$, respectively. The horizontal lengths $X_v$ of the rolled-up region for these different times are $1.92$ and $3.17$, respectively. In figures 6.3.5 (a) and (b) it is evident that there are two distinctly different regions separated by the value of $X_v$. One is the region of streamwise length from the separation point less than $X_v$, which is called region I. In this region, the temperature distribution is unstable due to the effect of the rolled-up vortex sheet. The other is the region of streamwise length from the separation point greater than $X_v$, which is called region II. Since there is no rolled-up vortex sheet existing in this region, the temperature distribution of this region shows a steady state. At $t=2.552$, from figure 6.3.5 (a), the approximate value of the thermal boundary layer thickness of the region I is about $2$, which equals the thickness of the plate, and that of the region II is about $0.14$. When the time increases to $4.044$, from figure 6.3.5 (b), the approximate value of the thermal boundary layer thickness of the region I increases to $3.5$ and that of the region II
increases to 0.25. The thickness of the thermal boundary layer in the separation flow increases downstream with increasing time and the size of the rolled-up vortex sheet.
Figure 6.3.4 (b) Temperature distribution above the surface of the plate at
(a) $t=2.522$, and (b) $t=4.044$
6.4 Concluding remarks

The two-dimensional separated flow and forced convection heat transfer around a heated plate has been studied numerically by using a combined vortex sheet technique and a finite difference method. The flow separation from the right-angled corners of the leading edges is represented by a vortex sheet. Velocity vectors which are calculated by the vortex sheet technique, are incorporated with the energy equation for the solution of temperature by a finite difference technique.

The complicated aspects of the fluid flow and heat transfer characteristics under a constant wall temperature condition are shown in terms of the rolled-up vortex sheet, isotherms, and the streamlines. Furthermore, distributions of temperatures and the Nusselt number, at any time, were systematically examined in detail. The findings of this study are summarized as follows:

(i) The heat transfer coefficient and the temperature distribution near the separation point are strongly affected by the rolled-up vortex sheet.

(ii) The maximum of instantaneous Nusselt number decreases its magnitude and moves downstream with time.

(iii) The points of reattachment and the maximum Nusselt number do not coincide. The maximum Nusselt number occurs in the upstream of the reattachment point. These results are in good agreement with the experimental results obtained by Sparrow (1988).

(iv) The distributions of the instantaneous Nusselt number have a common shape characterized by an initial decrease, the attainment of a maximum, and a subsequent decrease. The trend is in good agreement with the numerical results by Thompson and Hourigan (1986) for flow under acoustic perturbation and the experimental results by Ota and Kon (1974; 1977; 1979), and Cooper, Sheridan and Flood (1986).
7. CONCLUDING REMARKS AND RECOMMENDATIONS FOR FURTHER WORKS

The vortex sheet technique which represents the free shear layer by a vortex sheet is one promising approach to the computation of unsteady flows. In this study the evolution of the vortex sheet is traced and the results can be used to derive other quantities in the flow field, such as velocity, stream function, velocity potential, and forces. In the case of forced convection, the heat transfer coefficient can be calculated by amalgamating with the energy equation.

Development has been made on the discretization of the vortex sheet by replacing the constant distribution of vorticity in each element with a linear distribution. This refinement increases accuracy and reduces the strength of the singularities at the ends of the discretized elements. Different cases of unsteady flow have been successfully studied by this vortex sheet technique. This includes (i) the roll-up of a vortex sheet and the analyses of a spiral core; (ii) the shedding of vorticity from a semi-infinite plate and (iii) the transient heat transfer around a heated plate with flow separation.

From these applications, a series of significant results of unsteady flow have been obtained.

(i) The circumferential average of the vorticity distribution in the inner core of a vortex sheet spiral approaches that of a Rankine vortex as shown in the results in which the geometrical centre of the core coincides with its centroid.

(ii) For the flow separation over a semi-infinite plate, local concentration of vorticity induces many small-scale spirals along the free shear layer which emerges from the separation point. This, to a certain extent, explains the observation by Pierce (1961). Computed results also show the process of
amalgamation between small-scale vortex spirals; this structure of small-scale spirals can be sustained only at resonant perturbation.

(iii) During the time interval following the initial shedding of vorticity, the growth rate of the separation bubble, near the leading edge of a semi-infinite plate, is a constant, and is independent of the Strouhal number.

(iv) The reattachment bubble reduces in size when the "decay" of vorticity (see § 5.3.4) is modelled in the computation. The heat transfer coefficient and the temperature distribution near the separation point are strongly affected by the rolled-up vortex sheet.

Detailed discussion of these studies can be found at the end of Chapters 4, 5 and 6.

The outcome of this study is not an end to itself. It also raises many interesting problems in unsteady flows which warrant further research:

(i) the question of whether there is any fixed pattern in the distribution of vorticity of a roll-up vortex sheet, other than a Rankine vortex, Lamb vortex (Staufenbiel (1984)) and a Kaden vortex, has to be investigated;

(ii) the time scale for a spiral core to be diffused to the surroundings and the mechanism for this need a further study;

(iii) for a body without sharp corners, the position of a flow separation as demonstrated by Soh, Hourigan & Thompson (1988) and Sarpkaya (1989) has to be studied further, so that a more rigorous approach can be found;

(iv) the roles of numerical instability and physical instability which lead to turbulence have to be indentified;

(v) the works by Kiya, Sasaki & Arie (1982); Cooper, Hourigan, Flood & Thompson (1986); Thompson & Hourigan (1986); Kiya (1986) and Thompson, Hourigan, Welsh & Soh (1987) on the flow over a plate have to be studied further and with more detailed investigation.
8. REFERENCES


APPENDIX I. SELT SUBPROGRAM

SUBROUTINE SELT(XI,YI,XJ1,YJ1,XJ,YJ,XJ2,YJ2,GJ,FK,U2,V2)

VORTICITY DISTRIBUTION IS A CONSTANT VALUE (GJ)
FOR J.EQ.1.OR.IN FK=0.5 OTHERWISE FK=0.25

COMMON/TESTDL/I,J
DXK=(XJ2-XJ1)*FK
DYK=(YJ2-YJ1)*FK
DXJK=XI-XJ
DYJK=YI-YJ
DSK2=4.0*(DXK**2+DYK**2)
DH2=DXJK**2+DYJK**2
DKD=DH2-DSK2*0.25
DKN=2.0*(DXK*DYJK-DXJK*DYK)
DK=ATAN(DKN/DKD)
IF(DKD.LT.0.0.AND.DKN.GT.0.0)DK=3.14159+DK
IF(DKD.LT.0.0.AND.DKN.LT.0.0)DK=-3.14159+DK
IF(DKD.EQ.0.0.AND.DKN.GT.0.0)DK=1.5708
IF(DKD.EQ.0.0.AND.DKN.LT.0.0)DK=-1.5708
HSK=DH2+DSK2
HSK1=(DXJK*DXK+DYJK*DYK)*2.0
DL=(HSK-HSK1)/(HSK+HSK1)
IF(DL.EQ.0.0)GO TO 10
100 FORMAT(' DL = 0.0 AT I & J = ',2I4)
DL=-0.5*ALOG(DL)

GK=-GJ/(3.14159*DSK2)

U2=U2+GK*(DXK*DK+DYK*DL)

V2=V2-GK*(-DYK*DK+DXK*DL)

RETURN

10 CONTINUE

WRITE(6,100)I,J

RETURN

END
APPENDIX II. SELT2 SUBPROGRAM

C*****************************************************************************

C

C SELT2 SUBPROGRAM

C*****************************************************************************

SUBROUTINE SELT2(XI,YI,XJ1,YJ1,XJ,YJ,XJ2,YJ2,A,B,FK,U2,V2)

C VORTICITY DISTRIBUTION IS A LINEAR DISTRIBUTION

DX=(XJ2-XJ1)*FK
DY=(YJ2-YJ1)*FK
THEJ=ATAN(DY/DX)

IF(DX.LT.0.0.AND.DY.GT.0.0)THEJ=3.14159+THEJ
IF(DX.LT.0.0.AND.DY.LT.0.0)THEJ=-3.14159+THEJ
IF(DX.EQ.0.0.AND.DY.GT.0.0)THEJ=1.5708
IF(DX.EQ.0.0.AND.DY.LT.0.0)THEJ=-1.5708

DXJK=XI-XJ
DYJK=YI-YJ

ZA1=(DXJK**2-DX**2)+(DYJK**2-DY**2)
ZA2=2*(DX*DYJK-DY*DXJK)

THEZA=ATAN(ZA2/ZA1)

IF(ZA1.LT.0.0.AND.ZA2.GT.0.0)THEZA=3.14159+THEZA
IF(ZA1.LT.0.0.AND.ZA2.LT.0.0)THEZA=-3.14159+THEZA
IF(ZA1.EQ.0.0.AND.ZA2.GT.0.0)THEZA=1.5708
IF(ZA1.EQ.0.0.AND.ZA2.LT.0.0)THEZA=-1.5708

ZA3=(DXJK+DX)**2+(DYJK+DY)**2
ZA4=(ZA1/ZA3)**2+(ZA2/ZA3)**2
ZA5=SQRT(ZA4)

IF(ZA5.EQ.0.0)GO TO 10
ZA6 = ALOG(ZA5)
AR = -(ZA6*COS(THEJ) + SIN(THEJ)*THEZA)
AI = -(COS(THEJ)*THEZA - ZA6*SIN(THEJ))
ZB1 = 2*THEJ
BR = -(COS(ZB1)*2.*DX + SIN(ZB1)*2.*DY)
BI = -(COS(ZB1)*2.*DY - SIN(ZB1)*2.*DX)
ZC1 = COS(ZB1)*DXJK + SIN(ZB1)*DYJK
ZC2 = COS(ZB1)*DYJK - SIN(ZB1)*DXJK
CR = -ZC1*ZA6 + ZC2*THEZA
CI = -ZC1*THEZA - ZC2*ZA6
U2 = U2 + 1/6.28318*(A*AI + B*(BI + CI))
V2 = V2 + 1/6.28318*(A*AR + B*(BR + CR))
RETURN
10 CONTINUE
RETURN
END
Following the examiners' suggestions, an additional explanation is given in regard to the concluding remarks in Chapters 5 and 6.

1) On the results computed with conservation of vorticity (in Chapter 5):

The evolution of the roll-up vortex sheet in cases 2 & 3 are similar to the experimental results from Pierce (1961) and Finalish, Freymush & Bank (1986). Evidently all their visualizations of flow phenomena are concentrated on the separation edge of any shape of the bodies, such as the edge of a wing, T-shape, and spoilers. In these cases, the viscous effects in the region of fluid are small enough to be neglected. This situation is just the same as the evolution of the roll-up vortex sheet simulated in the Chapter 5 that the considered region is mostly concentrated on the separation point and the nearby region. Evidently the inertial force and the generated vorticity from the corner completely dominate the evolution of the roll-up vortex sheet. The present vortex sheet method is successful in simulating this flow phenomena; it also models the small-scale spirals found in the results from Pierce (1961) (see Figures 5.3.5~5.3.9).

2) On the results computed with decay of vorticity (in Chapters 5 & 6):

For the flow over a flat plate, the viscous effects in the boundary layer cannot be neglected. In order to compensate this viscous influence, the present vortex sheet method uses the exponential decay of vorticity to simulate the flow phenomena. The numerical results as shown in Figures 5.3.16~5.3.23; 5.3.27~5.3.33 and Figures 6.3.1 (e)–(f) agree with the experiments presented by Cherry, Hillier & Latour (1984), at Reynolds number Re=5×10^3 (as shown in Figure 11). However, these results do not agree with the flow visualization obtained by Kiya (1986), at Re=680 and 580 (as shown in Figure 12). It is not surprising that the results produced by the present method give a better agreement with the case in which the Reynolds number is higher. It can be concluded that the
Figure 11. Instantaneous smoke flow visualization for Re=$5\times10^3$. Flow from right to left. (Reprinted from Cherry, Hillier & Latour (1984))

Figure 12. Flow patterns in (a) the (x,y)-plane and (b) in the (x,z)-plane, Re=680 & 580, respectively. (Reprinted from Kiya (1986))
present method may not be suitable for simulating flow when the viscous effect is too strong, for example $\text{Re}<1000$.

3) **Re-calculation of the streamfunction around a plate (in Chapters 5 & 6):**

In order to calculate the transformed coordinates of the grid points in the transformed plane by Equation (5.2.1), the region around a plate is divided into two subregions: one is the region in the area above the plate $S_2$ and the other is the region in front of the plate $S_3$. Figure 13 shows the schematic diagram of the location of grids $S_2$ & $S_3$. Since Equation (5.2.1) is not an explicit function of $\zeta$, the transformations from the grid points in the $z$-plane to the $\zeta$-plane are obtained by a searching technique as described below.

1) Since the corner of the plate $Z_A(0,h)$ in the physical plane, which corresponds to the point $\zeta_A(1,0)$ in the transformed plane, is known, it is assumed that the unknown grid point $\zeta_B(\eta,\xi)$, which corresponds to $Z_B(x,y)$ in the physical plane, is near the point $\zeta_A$.

2) Constructs a set of grid points $\zeta_C$ in a rectangular block which contains $\zeta_B$ in the transformed plane. The spacing of the mesh in the rectangular block is $\Delta \zeta_C$. The coordinates of the left-bottom and the right-uppermost in the rectangular block are $(\eta_0,\xi_0)$ and $(\eta_1,\xi_1)$, respectively. It means $\eta_0 \leq \eta \leq \eta_1$ and $\xi_0 \leq \xi \leq \xi_1$.

3) By using equation (5.2.1), the grid points $\zeta_C$ map into $Z_C$ of the physical plane. A grid point $(\zeta_{C1})$ in which $Z_B$ is nearest to $(Z_{C1})$ can be found.

4) A smaller rectangular block in the $\zeta$-plane can be constructed, and the transformed coordinates of the left-bottom and the right-uppermost in the new block are $(\zeta_{C1}) - \Delta \zeta_C = (\eta_0^*,\xi_0^*)$ and $(\zeta_{C1}) + \Delta \zeta_C = (\eta_1^*,\xi_1^*)$, respectively. Evidently, $\eta_0 \leq \eta \leq \eta_1^*$ and $\xi_0 \leq \xi \leq \xi_1^*$. Let a new set of grid points which are in the new rectangular block constraining $\zeta_B$ in the transformed plane to be $(\zeta_{C1})^*$ and the spacing of mesh is reduced to be $\Delta \zeta_C^*$.

5) Repeats the process by returning to the step (3) and continues to search until $|Z_B-Z_C|$ is smaller than a prescribed tolerance.
Figure 13  Schematic diagram of the location of grids S2 & S3 around a plate
(6) The transformation of the other unknown grid points follows the procedure of steps (1)~(5).

Figure 14 shows the distribution of S2 and S3 in the transformed and physical planes obtained by the above procedure.

Figure 15 shows the flowchart of computing and plotting the streamline patterns around a plate. According to the flowchart, the streamfunctions are calculated and shown as contours in Chapters 5 and 6. The Δx and Δy used in both grids S2 & S3 are the same; they are equal to 0.1. Since the streamfunctions of the considered grids are invariant in both physical plane and transformed planes, it is more convenient to do these calculations in the transformed plane. The streamfunction in the transformed plane can be obtained from the imaginary part of ω in Equation (5.2.4).

In order to clarify the problem of streamline patterns mentioned by the examiners, the Δx and Δy used in both grids S2 & S3 are reduced from 0.1 to 0.05 and the considered region is concentrated on the area S2 from (0,h) to (4h,2h); 2h is the plate's thickness. Figures 16 (a) ~ (f) show the re-calculated results of the streamline patterns of Figure 6.3.1 at t=1.004, 2.033, 3.004, 5.086, 6.080, and 8.024. Compared to the original results of Figure 6.3.1, it is evident that the finer grid does not improve the streamline patterns. The main reason for the difficulty of observing the structure of roll-up vortex sheet in the streamline patterns may be that the decay of vorticity downstream has become too weak to have any significant effect on the flow. However, in other situations the separation bubble may still exist in the flow after t=4.044. The value of Xc cannot be obtained directly from the present streamline patterns. The plot of Xc in Figure 6.3.2 is an extrapolation of the measured values between 0 ≤ t ≤ 6.080. The formula for this extrapolation is Xc = -0.51428 + 1.8207 t - 0.34328 t^2 + 0.038026 t^3. The extrapolation is made to show the trend of Xc so that it can be compared with the other measurements.
Figure I4. The distribution of S2 & S3 in the (a) physical plane and (b) transformed plane.
Read the data of the instantaneous vortex sheet, including the number of elements, coordinates and vorticity of vortex elements, and instant time.

Use Eq.(5.2.1) to calculate the transformed coordinates of the grid S3 in front of the plate.

Use Eq. (5.2.1) to calculate the transformed coordinates of the grid S2 in the upper region of the plate.

Read the transformed coordinates of the grid S3.

Read the transformed coordinates of the grid S2.

Use Eq.(5.2.4) to calculate the image part of the complex potential (streamfunction) of the grids S2 & S3 in the transformed plane. The streamfunction is contributed by the free stream, vortex elements and the image vortex elements.

Write the streamfunction of S2 & S3 to an output file OTP.

Use the file OTP to organize a matrix for plotting the contour of streamlines.

Figure I5. The flowchart of computing and plotting the streamline patterns.
(a) $t=1.004$;

(b) $t=2.033$;
(c) \( t = 3.004; \)

(d) \( t = 5.086; \)
Figure 16. The re-calculated streamline patterns of Figure 6.3.1. (a) $t=1.004$; (b) $t=2.033$; (c) $t=3.004$; (d) $t=5.086$; (e) $t=6.080$; and (f) $t=8.024$. 
References:


