1995

The effects of heat transfer on vapour, gas and vapour-gas cavity bubbles

Ali Akbar Karimi

*University of Wollongong*

---

**Recommended Citation**

NOTE

This online version of the thesis may have different page formatting and pagination from the paper copy held in the University of Wollongong Library.

UNIVERSITY OF WOLLONGONG

COPYRIGHT WARNING

You may print or download ONE copy of this document for the purpose of your own research or study. The University does not authorise you to copy, communicate or otherwise make available electronically to any other person any copyright material contained on this site. You are reminded of the following:

Copyright owners are entitled to take legal action against persons who infringe their copyright. A reproduction of material that is protected by copyright may be a copyright infringement. A court may impose penalties and award damages in relation to offences and infringements relating to copyright material. Higher penalties may apply, and higher damages may be awarded, for offences and infringements involving the conversion of material into digital or electronic form.
THE EFFECTS OF HEAT TRANSFER ON VAPOUR, GAS AND VAPOUR-GAS CAVITY BUBBLES

A thesis submitted in fulfilment of the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY

from

UNIVERSITY OF WOLLONGONG

by

Ali Akbar Karimi
B.Eng., M.Eng.

Department of Mechanical Engineering

MARCH 1995
In the name of God, the merciful and compassionate
To the people of Iran
DECLARATION

This thesis is submitted to the University of Wollongong, N.S.W., Australia and I declare that the work contained herein has not been submitted for a degree to any other university or institution.

Ali Akbar Karimi

March 1995
ACKNOWLEDGMENTS

The author is deeply indebted to his supervisor Dr. Wee-King Soh for his guidance, comments and encouragement during the course of this work.

The author also gratefully thanks the ministry of culture and higher education of the Islamic Republic of Iran for awarding him a scholarship and providing financial support to make this thesis possible.

The author would like to express his sincere appreciation to his wife and two children for their unending patience. He also expresses his heartfull appreciation to his parents and sister for their unfailing support and faith in him.

Thanks are also to the author's friend Mr. M.T. Sharvani Tabar whose discussions and comments have been invaluable. The author also thanks Mr. Geoff Cowell and Steven B. Harvey, friends that will never be forgotten, for their helpful comments.
The dynamics of cavity bubbles both in a liquid of infinite extent and near a rigid boundary is studied. A technique is developed for mathematical modelling of bubble dynamics. In the mathematical modellings heat transfer between the bubble contents and the surrounding liquid, evaporation and condensation and the buoyancy forces are considered.

The bubbles could be vapour bubbles (filled with vapour), gas bubbles (filled with a non-condensable gas) or bubbles filled with a mixture of vapour and a non-condensable gas. The properties of vapour are extracted from thermodynamic tables and those of the non-condensable gas are related by the equation of state of an ideal gas.

The surrounding liquid is assumed to be invicid and incompressible and the flow field irrotational. For isolated bubbles the flow field is simulated by the flow field of a sink or a source and for bubbles near a rigid surface the boundary element method is utilised for the solution of the flow field.

Three sets of differential equations have been derived for three types of the above mentioned bubbles. These sets of differential equations are solved by the Runge-Kutta method.

A dual reciprocity boundary element method is adopted and modified to solve the equation of energy in the liquid domain thereby heat transfer across the bubble-liquid interface is determined. The method, while retaining accuracy,
found to be more efficient than other domain methods such as finite element methods regarding computational time, computer storage and time for preparation of input data.

Different aspects of bubble dynamics such as the rate of change of volume, the pressure and temperature inside the bubbles, the velocity of liquid jets, the movement of the bubble centroid and the effect of heat transfer on each of these parameters are investigated.
TABLE OF CONTENTS

DECLARATION ................................................................. i
AKNOWLEDGMENTS ......................................................... ii
ABSTRACT .......................................................................... iii
TABLE OF CONTENTS ........................................................ v
LIST OF FIGURES ............................................................. ix
LIST OF TABLES ............................................................... xiii
LIST OF PUBLICATIONS .................................................. xiv
LIST OF SYMBOLS .......................................................... xv
CHAPTER ONE-INTRODUCTION ........................................... 1
CHAPTER TWO-LITERATURE SURVEY ...................................... 5
  2.1. INTRODUCTION ....................................................... 5
  2.2. HISTORICAL BACKGROUND ON THE STUDY OF ISOLATED
       BUBBLE DYNAMICS .................................................. 5
  2.3. A BRIEF REVIEW OF SOME EXPERIMENTAL WORKS ........ 27
  2.4. REVIEW OF SOME NUMERICAL WORKS ON NON-
       SPHERICAL BUBBLE DYNAMICS ................................ 32
CHAPTER THREE-MOVING BOUNDARY PROBLEMS ......................... 37
  3.1. INTRODUCTION ....................................................... 37
  3.2. ENTHALPY METHOD ................................................ 38
  3.3. IMMOBILISATION OF THE MOVING BOUNDARY ............... 39
  3.4. THE FINITE ELEMENT METHOD .................................. 40
  3.5. THE BOUNDARY ELEMENT METHOD ............................ 41
  3.6. CAVITATION AS A MBP ........................................... 43
CHAPTER FOUR-DYNAMIC BEHAVIOUR OF AN ISOLATED BUBBLE ....... 45
  4.1. INTRODUCTION ....................................................... 45
4.2. GOVERNING EQUATIONS FOR AN ISOLATED VAPOUR BUBBLE

4.2.1. Hydrodynamic Equation for a Vapour Bubble ................................ 46
4.2.2. The Equation of Energy for the Control Mass ................................. 50
4.2.3. Outline of Computations for a Vapour Bubble ................................. 52

4.3. GOVERNING EQUATIONS FOR AN ISOLATED GAS BUBBLE ............ 54

4.3.1. Hydrodynamic Equation for a Gas Bubble ..................................... 54
4.3.2. Radial Distribution of Temperature in the Bubble ............................ 55
4.3.3. Outline of Computations for a Gas Bubble ...................................... 58

4.4. GOVERNING EQUATIONS FOR A BUBBLE CONTAINING A MIXTURE OF VAPOUR AND A NON-CONDENSABLE GAS .......... 60

4.4.1. The Hydrodynamic Equation for a Bubble filled with Vapour and Gas .................................................. 60
4.4.2. The Energy Equation for the Proposed Control Mass ...................... 62
4.4.3. Outline of Computations for a Bubble filled with Gas and Vapour ................................................................. 63

4.5. RESULTS AND DISCUSSION ..................................................... 65

4.5.1. Results For a Vapour Bubble .................................................... 66
4.5.2. Results for a Gas Bubble ......................................................... 70
4.5.3. Results For a Bubble filled with a mixture of Vapour and a Non-condensable Gas ....................................................... 78

4.6. CONCLUDING REMARKS TO CHAPTER FOUR .............................. 87

CHAPTER FIVE-DUAL RECIPROCITY BOUNDARY ELEMENT METHOD .................................................. 89

5.1. INTRODUCTION ................................................................. 89
5.2. STATEMENT OF THE PROBLEM .............................................. 91
5.3. METHOD OF SOLUTION ......................................................... 92
5.3.1. Integral Equation form of the Energy Equation ............................................. 92
5.3.2. Mathematical Modelling of the DRM ................................................................. 94
5.3.3. Discretised Form of the Mathematical Model .................................................. 97

5.4. CALCULATION OF $\gamma_i$ ...................................................................................... 100

5.5 PROBLEMS IN WHICH $b = b(r, z, T, t)$ .................................................................. 101
5.5.1. The Case of $b = T$ ............................................................................................... 103
5.5.2. The Case of $b = \partial T/\partial r$ or $b = \partial T/\partial z$ .................................................. 104
5.5.3. The Case of $b = \frac{I \cdot DT}{\alpha \cdot Dt}$ ................................................................. 105

5.6. DIFFERENT FORMS OF $f_j$ FUNCTION .................................................................. 107
5.6.1. Definition of $f_j$ for a Bounded Axisymmetric Domain ................................. 108
5.6.2. Definition of $f_j$ for Unbounded Two Dimensional Domains ..................... 109
5.6.3. Infinite Axisymmetric Domains ......................................................................... 112

5.7. IMPLEMENTATION OF THE DRM TO BUBBLE DYNAMICS ......................... 115
5.8. CONCLUDING REMARKS TO CHAPTER FIVE ...................................................... 124

CHAPTER SIX-DYNAMICS OF BUBBLES NEAR A RIGID BOUNDARY ...... 125

6.1. INTRODUCTION ........................................................................................................ 125

6.2. THE HYDRODYNAMIC EQUATION IN THE LIQUID DOMAIN ...... 127
6.2.1. The Hydrodynamic Equation For Bubbles Near a Rigid Boundary ............ 127
6.2.2. The Boundary Integral Form of the Hydrodynamic Equation .................... 128
6.2.3. Modification of the Boundary Integral Equation for Axisymmetric Bubble Motion ................................................................. 132
6.2.4. The Unsteady Bernouli Equation .................................................................... 134

6.3. THE ENERGY EQUATION FOR A VAPOUR BUBBLE ........................................ 135
6.3.1. Solution Strategy For a Vapour Bubble ......................................................... 136
6.3.2. Effect Of Buoyancy Forces on the Bubble Motion ....................................... 138

6.4. THE ENERGY EQUATION FOR A GAS BUBBLE .............................................. 139
6.4.1. The Solution Strategy For a Gas Bubble ............................................. 140

6.5. RESULTS AND DISCUSSION ..................................................................... 140

6.5.1. Verification of the Solution of the Hydrodynamic Equation
by the BEM ...................................................................................................... 141

6.5.2. Results for a Gas Bubble Near a Rigid Boundary ............................ 142

6.5.3. Results for a Vapour Bubble Near a Rigid Boundary ....................... 160

6.5.4. Combined Effect of the Buoyancy Forces, the Rigid
Boundary and the Heat Transfer on the Dynamics of a Vapour
Bubble ........................................................................................................ 170

6.6. CONCLUDING REMARKS TO CHAPTER SIX ..................................... 175

CHAPTER SEVEN-SUMMARY AND CONCLUSION ........................................ 178

7.1. SUMMARY ............................................................................................... 178

7.2. RECOMMENDATIONS FOR FUTURE RESEARCH ............................. 181

REFERENCES .............................................................................................. 182

APPENDIX A ................................................................................................. 194
LIST OF FIGURES

Figure 3.1 Subsequent profiles of a bubble during its collapse ..............................................43

Figure 4.1 Schematic diagram of control mass.................................................................48

Figure 4.2 Flow chart depicting outline of computations for vapour bubble dynamics .................................................................53

Figure 4.3 Flow chart for a gas bubble .................................................................59

Figure 4.4 Flow chart depicting the computational procedure for a bubble filled with a mixture of non-condensable gas and vapour .................................................................64

Figure 4.5 Comparison of the results of the program for different cases ......................66

Figure 4.6 Time history of the volume of a vapour bubble ........................................68

Figure 4.7 Time history of the pressure of the vapour bubble ........................................69

Figure 4.8 Change of temperature inside the vapour bubble with respect to time .........69

Figure 4.9 Change of bubble wall velocity with respect to time ................................70

Figure 4.10 Time history of the volume of a gas bubble ............................................71

Figure 4.11 Pressure change in a gas bubble with respect to time................................72

Figure 4.12 Effect of heat transfer on the period of pulsations of a gas bubble. ............73

Figure 4.13 Comparison between the results of the program and the best fitted curve of slope 1.17 for the expansion phase of the gas bubble in Case F .........................75

Figure 4.14 Comparison between the results of the program and the best fitted curve of slope 1.21 for the collapse phase of the gas bubble in Case F .........................76

Figure 4.15 P-V Diagram of a gas bubble for one cycle of pulsations ..........................77

Figure 4.16 Radial distribution of temperature inside the bubble ..................................78
Figure 4.17 Change in volume for bubbles containing a mixture of vapour and non-condensable gas

Figure 4.18 Change in pressure for bubbles containing a mixture of vapour and non-condensable gas

Figure 4.19 Change of bubble wall velocity with respect to the bubble radius

Figure 4.20 Change in volume for a stable cavity over a long period of time

Figure 4.21 Change in pressure for a stable cavity over a long period of time

Figure 4.22 Change in vapour mass inside the bubble for a stable cavity

Figure 5.1 Discretisation of the boundary and distribution of internal nodes

Figure 5.2 Schematic form of Equation 5.19

Figure 5.3 External domain with an imaginary boundary at infinity

Figure 5.4 Flow chart depicting the procedure for comparing the results of FIDAP and those of the DRM

Figure 5.5 (a) Computational mesh used by FIDAP, (b) Distribution of nodes in the DRM

Figure 5.6 Time history of the volume of a gas bubble obtained by the DRM and by FIDAP

Figure 5.7 Time history of the volume of a vapour bubble obtained by the DRM and by FIDAP

Figure 5.8 Time history of temperature inside a vapour bubble obtained by the DRM and by FIDAP

Figure 5.9 Comparison between FIDAP and the DRM with different numbers of internal nodes

Figure 6.1 Configuration of a body of revolution

Figure 6.2 Generating plane of the body of revolution of Figure 6.1
Figure 6.3 Configuration of a bubble near a rigid boundary ........................................... 132
Figure 6.4 Configuration of bubble and its image near a rigid boundary......................... 133
Figure 6.5 Comparison between the result of the program and the analytical solution .................................................................................................................. 141
Figure 6.6 Time history of the location of the centroids for cases i and ii .................... 143
Figure 6.7 Bubbles profiles (for cases i and ii) at the non-dimensional times: .......... 145
Figure 6.8 Final shapes of the bubbles of cases i and ii ................................................ 146
Figure 6.9 Distribution of the temperature in the liquid domain for Case i ............... 147
Figure 6.10 Time history of jet velocity for cases i and ii ............................................. 148
Figure 6.11 Bubbles profiles for cases iii and iv at the non-dimensional times: ........ 150
Figure 6.12 Time history of the jet velocity for cases iii and iv .................................. 151
Figure 6.13 Bubbles' profiles for cases v and vi at the non-dimensional times: ........... 153
Figure 6.14 Movement of the bubble centroid ............................................................... 154
Figure 6.15 Change of pressure inside the bubbles ....................................................... 155
Figure 6.16 Bubbles profiles for case vii of Table 6.3 at dimensionless times: ........... 157
Figure 6.17 Bubble profiles for case viii of Table 6.3 at dimensionless times: ............ 158
Figure 6.18 Final profiles of bubbles in cases vii and viii ............................................ 159
Figure 6.19 Time history of the pressure in the bubbles in cases ix and x ................. 161
Figure 6.20 The temperature in the bubbles (cases xi and xii) ..................................... 163
Figure 6.21 The location of the bubbles' centroid in cases xi and xii ......................... 163
Figure 6.22 The velocity of the liquid jet for cases xi and xii ..................................... 164
Figure 6.23 Bubbles' profiles for cases xi and xii ......................................................... 165

xi
Figure 6.24 Time history of the volume of the bubbles in cases xiii and xiv .......... 167
Figure 6.25 Bubbles' profiles during the collapse phase (cases xiii and xiv) .......... 168
Figure 6.26 Final profiles of the bubbles in cases xiii and xiv ................................ 168
Figure 6.27 Isotherm lines in the liquid domain ....................................................... 169
Figure 6.28 Change of the bubbles' volume with respect to time for cases xv and xvi ................................................................. 171
Figure 6.29 Bubbles' profiles for cases xv and xvi .................................................... 172
Figure 6.30 The location of the centroids of the bubbles in the presence of the buoyancy forces ................................................................. 173
Figure 6.31 The effect of the buoyancy forces on bubble shapes .............................. 174
LIST OF TABLES

Table 4.1 Initial conditions for different cases of vapour and gas bubbles .......... 74
Table 4.2 Polytropic indices for bubbles of Table 4.1 ........................................ 75
Table 4.3 Radius, pressure and temperature of three bubbles at the first maximum
volume and radius of the bubble at the end of first collapse ................................. 80
Table 4.4 Initial and final conditions of the three bubbles in the previous example ..... 86
Table 5.1 Problem 1, the bubble contained perfect gas .......................................... 117
Table 5.2 Problem 2, the bubble contained vapour .................................................. 117
Table 5.3 Number of arithmetic operations performed by the DRM and FIDAP ...... 119
Table 5.4 Maximum difference in volume and temperature resulted from the
DRM compared to the results of FIDAP .................................................................. 123
Table 6.1 Initial conditions for Case i and Case ii .................................................... 142
Table 6.2 Initial conditions for bubbles close to a rigid boundary ignoring the
buoyancy forces ....................................................................................................... 152
Table 6.3 Initial conditions for bubbles close to a rigid boundary considering
buoyancy forces ....................................................................................................... 156
Table 6.4 Initial conditions of the bubbles in cases ix and x ................................. 160
Table 6.5 Initial conditions for bubbles in cases xi and xii ...................................... 162
Table 6.6 Initial conditions for the bubbles in cases xiii and xiv ............................. 166
Table 6.7 Initial conditions for the bubbles in cases xv and xvi .............................. 170
LIST OF PUBLICATIONS


LIST OF SYMBOLS

$C_l$ specific heat of the liquid

$C_{pg}$ specific heat of gas at constant pressure

$C_{rg}$ specific heat of gas at constant volume

$f_i$ interpolation functions

$H$ and $G$ influence matrices used in the boundary element method

$H$ a pressure function defined in Section 4.2.1

$h$ initial distance of the bubble centre from the rigid boundary

$KE$ kinetic energy of the liquid

$K_l$ thermal conductivity of the liquid

$K_g$ thermal conductivity of the gas

$k$ polytropic index (used in Section 4.5.2)

$l_n$ length of the $n^{th}$ boundary element

$m_l$ mass of liquid in the proposed control mass

$n$ unit outward normal vector to the boundary

$p_g$ partial pressure of the gas inside the bubble

$p_{inf}$ pressure at infinity

$p_v$ partial pressure of the vapour inside the bubble
\( p^* \) non-dimensional pressure

\( \dot{Q} \) rate of heat transfer

\( r \) radial coordinate

\( R \) radius of the bubble

\( r_{ij} \) Euclidean distance between points \( i \) and \( j \)

\( R_m \) maximum radius of the bubble

\( R^* \) non-dimensional radius of the bubble

\( \ddot{R} \) non-dimensional velocity of the bubble wall

\( T \) temperature inside the bubble

\( \ddot{T} \) particular solution of the poisson equation

\( T^* \) non-dimensional temperature

\( u \) radial velocity of gas inside the bubble

\( u_r \) radial component of the velocity

\( u_z \) vertical component of the velocity

\( u_g \) specific internal energy of the gas

\( u_v \) specific internal energy of the vapour

\( u_l \) specific internal energy of the liquid

\( y \) auxiliary function defined in Section 4.2.2

GREEK SYMBOLS

\( \alpha \) thermal diffusivity of the liquid

\( \Delta_i \) Dirac delta function
δ: a parameter which shows the strength of the buoyancy forces

Φ: velocity potential

Ω: any general 3-dimensional domain

Γ: boundary of Ω

Γ∞: imaginary boundary at infinity

ξ: local coordinate

γj: constant values (used to approximate the non-homogeneous term in the poisson equation)

θ: angular direction in the polar coordinate system

Ψ: fundamental solution (or free space Green function) for the Laplace equation

ρl: density of the liquid

ρg: density of the non-condensable gas

ρv: density of the vapour

γ: ratio of the specific heats (used in Section 4.3.2)

∀*: non-dimensional volume
CHAPTER ONE

INTRODUCTION

A distinctive feature of the hydrodynamics of liquid is cavitation. Cavitation refers to the growth and collapse of vapour or gas bubbles in a mass of liquid.

Cavitation was initially known because of the destructive force it has on fluid machinery or flow passages. Pitting damage is the most pronounced feature of cavitation and it has cost a large amount of money to fix damage.

Despite the destructive nature of cavitation, it can be employed in useful applications such as biological, medical, industrial and physical applications. As examples of the useful applications of cavitation, one can name the use of cavitation for descaling in dentistry and the use of bubbles for disintegrating kidney stones in surgery as medical applications, and the use of bubbles in detecting high energy particles as a physical application.

Although the cavitation process is involved in the formulation and pulsation of a large number of bubbles, much has been learned from the study of the dynamic behaviour of a single bubble. This study is the concern of the present work.

Different parameters such as the bubble contents, heat transfer, condensation and evaporation on the bubble wall, mass diffusion through the bubble interface, buoyancy forces, pressure and temperature of the surrounding liquid affect the
dynamic behaviour of a bubble. Much work has been carried out to show how these parameters influence the behaviour of a bubble. Most of these works deal with the study of a single isolated bubble assuming spherical symmetry for the bubble motion. Hereafter the term 'isolated bubble' refers to a bubble far from any surfaces.

Proximity to a rigid boundary is another factor which affects the bubble evolution. When the bubble evolution takes place near a rigid surface, the behaviour of the bubble is totally different from that of an isolated bubble. In this case a high speed liquid jet directed toward the rigid boundary is formed and the bubble shape will become highly distorted.

For both isolated bubbles and bubbles near a rigid boundary, the temperature inside the bubble changes over a wide range during the bubble motion while the temperature of the surrounding liquid at some distance from the bubble wall remains undisturbed. The difference between the temperature inside and outside the bubble causes the transfer of heat between the bubble contents and the surrounding liquid. Moreover, the processes of evaporation and condensation on the bubble wall are accompanied by the heat transfer. So the study of the effect of heat transfer on bubble dynamics, especially for bubbles near a rigid boundary, deserves further study.

The present work is aimed at studying the effect of heat transfer on different aspects of bubble dynamics. In this regard, for the solution of the hydrodynamic equation in the liquid domain, the boundary element method (BEM), due to its versatility, is employed. For the calculation of the heat transfer through the bubble wall a special type of the BEM, a dual reciprocity method (DRM), is exploited. This method is modified to deal with axisymmetric problems.
In the mathematical formulations, besides the heat transfer, the effects of the buoyancy forces and condensation and evaporation (for vapour bubbles) have been considered.

This thesis is organised into 7 chapters, the outline of each chapter is given below.

Chapter 1 describes the motivation and the aim of this study.

Chapter 2 has an overview of the works done so far on cavitation with an emphasis on those works that have studied the dynamics of bubbles.

Since cavitation is a moving boundary problem from the mathematical point of view, Chapter 3 is devoted to a brief study of the moving boundary problems. This chapter studies the most popular methods proposed to deal with the moving boundary problems.

Chapter 4 studies the dynamics of an isolated bubble. Governing equations of bubble dynamics are derived in this chapter. To derive these equations, the hydrodynamic equation and the energy equation in the liquid domain and the energy equation for the bubble contents are used. New formulations are proposed to link the thermodynamic processes inside bubbles with the motions of the bubble wall and the surrounding liquid. Three sets of differential equations are obtained for the three cases of vapour bubbles, gas bubbles and bubbles containing a mixture of vapour and non-condensable gas.

Chapter 5 proposes development and application of the DRM to the study of bubble dynamics. For this purpose a computer code based on the DRM is developed to deal with infinite axisymmetric domains.
Chapter 6 details the effect of heat transfer on the dynamics of a bubble near a rigid boundary. In this regard a computer program, based on the BEM, for the solution of the potential flow field in the surrounding liquid is developed. Different aspects of bubble dynamics such as the movement of bubble centroid, the velocity of liquid jets, the temperature in the bubble, the pressure in the bubble and the rate of growth and collapse of bubbles are investigated.

Chapter 7 presents concluding remarks and the achievements obtained during the course of this thesis. Suggestions for further studies are also included in this chapter.
CHAPTER TWO

LITERATURE SURVEY

2.1. INTRODUCTION

Cavitation is usually defined as the formation of bubbles of gas or liquid vapour in a medium of liquid. These bubbles in an exchange of energy with the surrounding liquid begin to pulsate and during their pulsations they interact with the flow field in the liquid. Initially cavitation was known because of its destructive action on fluid machinery or flow passages. A great deal of analytical and experimental work has been done to clarify this phenomenon. This chapter will look at a number of works in this field.

2.2. HISTORICAL BACKGROUND ON THE STUDY OF ISOLATED BUBBLE DYNAMICS

According to Young (1989) Newton was the first person to observe the creation of cavities in 1704. In his experiments on Newton's rings, he observed the creation and disappearance of white spots in a liquid but he didn't realise the cause of this phenomenon. Besant (1859) was the first person to analyse the behaviour of an empty cavity analytically. In his analysis he used the equation of motion of the fluid to find instantaneous alteration of pressure at any point in the surrounding liquid while the pressure at a great distance from the bubble was
constant. Rayleigh (1917) proceeded to solve the same problem but he considered the conservation of energy to derive the equation that governs the motion of the bubble wall. His equation has been the basis for most of the research in this field. Through his analysis he found an expression for transient motion of the bubble wall and collapse time of a bubble with a given initial radius. His equation is as follows:

\[
\frac{d^2 R}{dt^2} = \frac{2}{3} \frac{P_0}{\rho_l} \left( \frac{R_0^3}{R^3} - 1 \right),
\]

(2.1)

in which \( P_0 \) is the initial pressure in the liquid, \( R_0 \) is the initial radius of the bubble, \( \rho_l \) is the liquid density and \( R \) is the instantaneous radius of the bubble. Overdot denotes differentiation with respect to time. He also exploited the momentum equation in the liquid to find the distribution of pressure outside the bubble.

Plesset (1949) worked out another formulation for the bubble motion in which pressure in the liquid domain was allowed to change. He derived the following equation for bubble motion:

\[
\frac{P(R) - P_\infty(t)}{\rho_l} = \frac{3}{2} \frac{d^2 R}{dt^2},
\]

(2.2)

where \( P(R) \) is the pressure in the liquid at the bubble wall, \( P_\infty(t) \) is the pressure in the liquid at some distance from the bubble, \( \rho_l \) is the liquid density. This equation is believed to be the fundamental equation for bubble dynamics.

To calculate the instantaneous pressure at infinity he assumed potential flow field in the surrounding liquid with spherical symmetry with the following definition

\[
\Phi = R^2 \frac{\dot{R}}{r},
\]

(2.3)
where $\Phi$ is the velocity potential and $r$ is the radial coordinate. Then he employed the unsteady Bernoulli integral of motion as:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + \frac{P(r)}{\rho_l} \frac{P_\infty(t)}{\rho_l} = 0,$$

(2.4)

to find $P_\infty(t)$. In calculating $P_\infty(t)$ from equation (2.4) the value of $P(r)$ was needed which was calculated by the experiment.

He verified the results from his analysis by the results gained from experimental observations made in a high speed water tunnel.

Different parameters influence the dynamic behaviour of bubbles, the most important ones are, fluid viscosity, heat transfer through the bubble-liquid interface and variable ambient pressure. Zwick and Plesset (1954) and Plesset and Zwick and (1954) studied the dynamics of bubbles while heat transfer was taken into account. Because of the presence of heat transfer the governing equations were non-linear. The authors divided the growth phase of the bubble into several regimes and in each regime simplification were utilised to solve the equation. It is noteworthy that in their calculations the liquid was assumed to be superheated.

The effect of heat diffusion on the dynamic stability of vapour and gas bubbles was investigated by Degarabedian (1953). He reported that heat diffusion has an appreciable effect on the dynamic stability of the bubble. In the same paper he also compared his analytic results with those obtained experimentally and concluded that the solution incorporating the effect of heat diffusion accords well with the experiment.
oscillations, Keller and Kolodner (1950) chose another approach. They treated water as a compressible liquid and exploited the wave equation to find the flow field in the liquid and presented a theory for the dynamics of a bubble generated by an underwater explosion. Their theory predicted damped oscillations of diminishing periods. As a result of their calculations they calculated the radius-time curve and the pressure wave emitted by the bubble.

Two mechanisms are known to be mostly responsible for cavitation damage: shock waves produced at the instant of rebound of the bubble and a high speed liquid jet generated as a result of deformation of the bubble near the boundaries. To determine whether pressure pulses emanating from the pulsations of cavitation bubbles could provide a mechanism for cavitation damage, Hickling and Plesset (1964) solved the equation of motion both in Lagrangian and in characteristic forms. They considered the compressibility of the liquid in their analysis. They found the pressure pulses produced by the pulsations of the bubble to be equivalent to a weak shock that could be strong enough to cause damage to solids in the vicinity of the bubble.

Florschuetz and Chao (1965) studied the relative importance of heat transfer and liquid inertia by ignoring surface tension and liquid viscosity. In their endeavour the equation of motion for the bubble wall was written as

\[
R R + \frac{3}{2} \dot{R} = \frac{1}{\rho_l}[P_r(T_w) - P_w(t)], \tag{2.5}
\]

in which \(T_w\) refers to the temperature of the bubble wall. The difference between this equation and equation (2.2) is the term \(P_r(T_w)\) which is the saturation pressure at \(T_w\). The following equation was used for the temperature field in the liquid:
\[
\frac{\partial T}{\partial t} + \frac{\dot{\mathbf{R}}^2}{r^2} \frac{\partial T}{\partial r} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) \quad r > R.
\] 

(2.6)

In this equation \( \alpha \) is the thermal diffusivity of the liquid. The condition at the bubble-liquid interface was written as:

\[
K \frac{\partial T}{\partial r} = \rho_c(T_u)L(\dot{R} - \dot{v}') \quad r = R,
\] 

(2.7)

where \( K \) is the thermal conductivity of the liquid, \( L \) is the latent heat and \( \dot{v}' \) is vapour velocity at the interface. To find \( \dot{v}' \) one has to solve the governing equations in the vapour region. Due to the non-linear nature of the equations and the movement of the boundary (bubble wall) the authors made several simplifying assumptions: e.g., it was assumed that the pressure and temperature inside the bubble remained constant. As a result of this analysis the authors divided the collapse of a vapour bubble into three different regimes: (i) liquid inertia control collapse in which the collapse rate is high and this rate tends to increase as the collapse proceeds; (ii) heat transfer control which has a slow collapse rate and; (iii) the intermediate case, where both effects are important.

Bornhorst and Hatsopoulos (1967b) updated their previous work (1967a) and for the first time in the cavitation research considered a non-equilibrium region near the interface of the bubble. Due to the existence of such a region, in which condensation and evaporation take place, a discontinuity in the properties of the surrounding liquid and vapour inside the bubble was considered. It was shown that if this region is neglected, large errors will occur for bubble growth rate especially at low pressures. An expression for the bubble wall velocity was derived showing its dependence on surface tension, viscosity of the liquid, inertia and non-equilibrium effects. In order to consider the effect of the non-
derived showing its dependence on surface tension, viscosity of the liquid, inertia and non-equilibrium effects. In order to consider the effect of the non-equilibrium region, they used the analysis done by Mordock (1965). Their main objective for considering discontinuity was to solve the energy equation in a thin thermal boundary layer adjacent to the bubble wall in the liquid. A single differential equation for $\frac{dR}{dR}$ was derived and the importance of discontinuities for various kinds of liquids was illustrated. The constitutive equation for heat and mass flux at the interface were obtained by methods of irreversible thermodynamics. The temperature in the thermal boundary layer was assumed to be a second-order function of radial distance and by the use of the energy integral technique the expression for the bubble wall velocity was found. They used their results to show that the asymptotic solutions suggested by Plesset and Zwick (1954), Forster and Zuber (1954) and Scriven (1959) which agreed well with the experimental results of Degarabedian (1960), were valid only when the effect of the non-equilibrium region can be neglected i.e., at high pressures and with large values of bubble radius.

Theofanous et al. (1969) worked out an elaborate formulation for bubble growth which resulted in a set of non-linear ordinary differential equations. The unknowns were radius of the bubble, thermal boundary layer thickness in the liquid, pressure and temperature in the bubble and temperature at the bubble wall. For temperature in the thermal boundary layer a quadratic distribution was assumed. In their evaluation they considered three types of liquids such as cryogens, normal liquids and liquid metals. They also made comparisons between their results and available experimental data. However, only in some
In the 1960's two methods of bubble generation were mainly used in experiments, electric spark discharge and kinetic impulse. There were some discrepancies between the results reported by several researchers: e.g., Naude and Ellis (1961) and Shutler and Mesler (1965). Because of these discrepancies a question was raised: Are the forces, which propel the inertial motion during the collapse, significantly affected by the method of generation of the bubble? Gibson (1972) in an attempt to answer this question, studied the bubble expansion generated by each of the above mentioned methods. For a bubble expansion by kinetic impulse it was assumed that the vapour and the surrounding liquid were initially at uniform temperature and pressure and that the liquid had acquired a positive radial velocity. For the bubble generated by an electric spark discharge the liquid was at rest and at uniform temperature and pressure everywhere, except at the bubble wall which was exposed to vapour at high pressure and temperature. In his work the viscosity, surface tension and compressibility of the liquid were assumed to be negligible and equation (2.5) was used as the equation for the motion of the bubble. For the calculation of the interface temperature, an approximate solution proposed by Plesset and Zwick (1952) was used. Plesset and Zwick (1952) presented an approximate solution for heat diffusion across a spherical boundary with radial motion. Their approximate solution was based on the assumption that the temperature variations were appreciable only in a thin layer adjacent to the spherical boundary. Based on this approximation, they proposed the following equation for the interface temperature

\[ T_w = T_u - \left( \frac{K}{\rho c_p \pi} \right)^{1/2} \frac{R^2(x) \left( \frac{\partial T}{\partial r} \right)_{r=R(x)}}{\int_0^{R(x)} [\int_0^y R^4(y) dy]^{1/2}} dx, \]  

(2.8)
\[ T_w = T_\infty - \left( \frac{K}{\rho_i C_p \pi} \right)^{\frac{1}{2}} \int_0^1 \frac{R^2(x) \left( \frac{\partial T}{\partial \sigma} \right)_{r=R(x)} dx}{\int_s R^4(y) dy} \] 

in which \( r \) is the radial coordinate, \( T_\infty \) is the temperature at infinity, \( T \) is the temperature, \( R \) is the radius of the bubble, \( K \), \( \rho_i \) and \( C_p \) are thermal conductivity, density and specific heat of the liquid, respectively.

Gibson concluded that expansion of bubbles, produced by electric-spark discharge and kinetic impulse, does not affect the subsequent collapse appreciably. It also was revealed that the condition of the vapour at the end of expansion has a significant influence on the subsequent collapse.

When a bubble grows from an initial state it undergoes a transient process and all properties of the vapour or gas inside it change with time. For a vapour bubble, it was believed that the initial-to-final vapour density ratio, when significantly greater than unity, influences the rate of growth of the bubble. This fact was investigated by Theofanous and Patel (1976). Their approach was mainly the same as that of Mikic et al. (1970) but in the work of Theofanous and Patel emphasis was put on the importance of the foregoing ratio. The work done by Mikic et al. (1970) was criticised because this factor was ignored. It was shown that results from the new approach, compared with the experiment, predicts the bubble growth rate better than that of Mikic et al. (1970). In both works a bubble in superheated liquid was considered.

Flynn (1975b) employed a formula developed previously in his other work (1975a) to investigate the effects of viscosity, heat conduction inside a bubble and compressibility of the liquid. Two parameters were used to measure the damping.
and also particle velocity was a linear function of radial distance from the centre of the cavity. To simplify the calculations the effect of condensation and evaporation was ignored and vapour pressure was assumed to be constant. The author used his results and criticised the unrealistic assumption of an isothermal or adiabatic processes inside the bubble. Then the effect of different initial radii of bubbles was studied and in each case the influence of heat conduction, viscosity and liquid compressibility were investigated. Cavitation thresholds were defined as a result of calculations given in this work and two definitions for a transient cavity and a stable cavity, based on these thresholds, were introduced.

The author decomposed the acceleration of a cavity interface into two quantities called acceleration functions and the total acceleration was written as

\[
\frac{dR}{dt} = IF + PF,
\]

(2.9)

where \(IF\) denoted the inertial function and \(PF\) denoted the pressure function. Both these quantities were functions of the radius of the bubble. It was shown that if these functions be plotted against radius, the intersection of such two curves is almost solely a function of initial radius of the bubble (\(R_0\)). The value of \(R\) at the intersection of \(IF\) and \(PF\) curves was called critical radius, \(R_c\).

Another value of \(R_0\) was named \(R_d\) for which the energy dissipation modulus \(\nabla w/w_m\) was a maximum. In the definition of the energy dissipation modulus \(w_m\) was the mechanical work done by the liquid on a cavity during contraction and \(\nabla w\) was the energy dissipated in a cycle.

To give alternative definitions for transient and stable cavities, a dynamical threshold radius \(R_t\) was defined. This threshold, \(R_t\), was such that any value of
was the mechanical work done by the liquid on a cavity during contraction and \( \nabla w \) was the energy dissipated in a cycle.

To give alternative definitions for transient and stable cavities, a dynamical threshold radius \( R_t \) was defined. This threshold, \( R_t \), was such that any value of \( R_o > R_t \) also lied above \( R_d \) and \( R_c \) curves. Based on this definition of \( R_t \) two definitions were proposed for the transient and the stable cavities as below:

a) a transient cavity is a model for a cavitation bubble that expands to a maximum radius greater than its dynamical threshold radius; and

b) a stable cavity is a model for a pulsating cavitation bubble whose maximum radius never exceeds its dynamical threshold radius.

One of the most important applications in which bubble dynamics plays a major role is in reactors where liquid metals are used as coolants. In this application, bubble growth rate and the rate at which heat is transferred to or from the bubble are of great importance. Donne and Ferranti (1975), in order to study bubble behaviour in this situation, derived a formulation based on mainly the same assumptions as those used by Plesset and Zwick (1954). Navier Stoke's equation in spherical symmetry and the equation of continuity of an incompressible liquid were used to derive the equation of motion of the bubble as follows:

\[
\ddot{R} = \frac{2[\sigma(T_o)R - \sigma(T)R_o]}{\rho_i R_o R^2} - \frac{3 \dot{R}^2}{2 R} - \frac{P_v(T_o) - P_v(T)}{\rho_i R} - \frac{4\mu \dot{R}}{\rho_i R^2},
\]

in which \( P_v \) is the vapour pressure, \( \sigma \) is the surface tension and \( \mu \) is kinematic viscosity. As boundary condition for the energy equation, terms due to evaporation, work done by the bubble and the energy of the bubble surface (due
to surface tension) were considered. They used energy equation in spherical coordinate system as

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) + \frac{b}{\rho C}, \quad r > R,$$

(2.11)

where $r$ is the radial coordinate, $u$ is the radial velocity, $\alpha$ is the thermal diffusivity of the liquid, $b$ is the heat produced in the liquid for unit volume and $C$ is the specific heat of the liquid. The following equation was used at the bubble boundary

$$4\pi R^2 K \left( \frac{\partial T}{\partial r} \right)_{r=R} = \frac{d}{dt} \left[ \frac{4}{3} \pi R^3 L \rho_v + \frac{4\pi R^2 \sigma}{J} + \frac{4}{3} \pi R^3 \frac{P_R - P_v}{J} \right],$$

(2.12)

in which $K$ is the thermal conductivity of the liquid, $L$ is the latent heat of evaporation, $\rho_v$ is the density of vapour, $\sigma$ is the surface tension of the liquid, $J$ is the conversion factor from work unit to heat unit (It should be noted that they did not used SI units) and $P_R$ is the static pressure at the bubble surface.

The term on the left hand side of equation (2.12) represents heat conduction, the first term of the right hand side represents heat required by evaporation, the second term is the energy of the surface and the last term represents the work done in creating a spherical hole of radius against a liquid pressure $P_R$ minus the work in filling it with vapour at pressure $P_v$. Then both equations of motion and energy were solved for $R$ and $T$ simultaneously. Since the physical properties of vapour and liquid were supposed to depend on temperature an iterative finite difference scheme was chosen to solve the above mentioned equations. It was postulated that the most important factor in determining bubble growth is heat conduction in the surrounding liquid and the combined effects of all other factors was less than 25 percent. They also confirmed results gained by the Plesset-
In an attempt to introduce a law for bubble growth in a simplified form while retaining accuracy and applicability over the range of bubble radii which are of practical interest, Prosperetti and Plesset (1978) presented a relatively simple formulation. Based on the assumption that a bubble grows in a superheated liquid, bubble growth was described by a single equation which was valid for bubbles grown by one order of magnitude beyond their initial radius where surface tension was unimportant. Their results agreed well with those of Donne and Ferranti (1975) who had solved a complete set of partial and ordinary differential equations. The difference between the analysis by Donne and Ferranti (1975) and that by Prosperetti and Plesset was that in the latter it was assumed that the thermal boundary layer in the liquid was thin compared to the radius of the bubble while this assumption was not made in the former. The authors divided the bubble growth rate into two different regions. In the first region growth rate was controlled by inertial effects, and in the second one by heat flow. To take into consideration heat flow to or from the bubble the equation of energy in the surrounding liquid was solved. The analysis given by Plesset and Zwick (1952) was exploited to solve this equation. The authors showed the validity of the thin thermal boundary layer assumption by comparing their results with those of Donne and Ferranti (1975) and it was inferred that at low superheats this assumption loses its validity. By making some approximations such as the independency of latent heat and vapour density from temperature and also assuming a linear relationship between temperature and pressure for vapour, they obtained an approximate scaling law for the bubble. They derived the following equation for the dynamic behaviour of the bubble:

\[
\frac{d^3 \gamma}{dt^2} + \frac{7}{6} \left( \frac{d\gamma}{dt} \right)^2 = \frac{3}{8} \left( 1 - \mu \right) \left( t - \theta \right) \frac{d\gamma}{d\theta} d\theta - 3^{1/3},
\]  

(2.13)
in which $\mu$ is a factor containing only physical parameters of the vapour. The left hand side of this equation represents the effect of liquid inertia. The second term of the right hand side describes the decrease in internal pressure due to thermal effects while the last term represents the effect of surface tension. The authors used this equation and gave a comprehensive discussion on the effect of each of these parameters. It is notable that the derived equation of bubble dynamics in their work is valid mainly for low superheats.

One of the most elaborate works on bubble dynamics done so far is a work by Fujikawa and Akamatsu (1980). In this work they studied the physical behaviour of a pulsating bubble both analytically and numerically. Their mathematical formulation took into consideration the effect of the compressibility of the liquid, the condensation of vapour, heat conduction and the existence of a thermal boundary layer at the bubble wall both in the gas phase and liquid phase. They studied the pressure waves radiated into the liquid as a result of bubble collapse. For the simulation of bubble dynamics with the foregoing assumptions, a complicated calculation was performed based on the use of the continuity equation, momentum equation, energy equation and equation of state for both the liquid and gas phases. It is noteworthy that both vapour and non-condensable gas were assumed to behave as perfect gases. They concluded that as the bubble collapses and rebounds, a pressure wave is radiated into the liquid with a pressure front which steepens gradually and the wave attenuation is inversely proportional to $r$, the radial distance from the centre of the bubble. Calculations were repeated for different initial conditions and different initial bubble radii and a detailed analysis of their results was given. The authors also used a water shock tube to show the generation of, and effect of shock waves produced as a result of bubble pulsation in the liquid.
to \( r \), the radial distance from the centre of the bubble. Calculations were repeated for different initial conditions and different initial bubble radii and a detailed analysis of their results was given. The authors also used a water shock tube to show the generation of, and effect of shock waves produced as a result of bubble pulsation in the liquid.

Nigmatulin et al. (1981) studied the dynamic interaction between a bubble and liquid to find out the possibility of the use of bubble screens for damping shock waves. They studied the radial motion of a bubble when subjected to a sudden pressure change in the surrounding liquid. This situation may occur when a shock wave front passes a bubble. Thermal and mass diffusion were present in their study while pressure was assumed to be uniform throughout the bubble. Temperature and density were allowed to change spatially. In the case of a mixture of vapour and non-condensable gas the thermophysical properties of the mixture were determined as a function of properties and the concentration of each component (gas and vapour). To formulate the problem, equations of continuity, state and heat diffusion were used. For the motion of the bubble an equation from the work by Nigmatulin (1978) as follows:

\[
\ddot{R} \frac{3}{2} \dot{R} + \frac{2\dot{R}}{R} = \frac{P - P_0 - \frac{2\sigma}{R}}{\rho_l} - \frac{4\mu}{\rho_l R},
\]

(2.14)

in which

\[
J = \frac{\dot{m}}{4\pi R^2},
\]

(2.15)

was adopted. In this equation \( \dot{m} \) is change of mass inside the bubble. The pressure and temperature relationship for the vapour was found by the following approximation
different from the one that had been found by Minnaert (1933). A detailed discussion on the small oscillation of bubbles was considered in which the effect of phase transition, radius of the bubble and static pressure in the liquid were taken into account.

Matsumoto and Beylich (1985) tried to find a mathematical formulation to study the influence of homogeneous condensation inside a bubble on its pressure response. Mist formation inside the bubble as well as condensation and evaporation from the bubble wall were considered. Vapour and non-condensable gas were treated as a perfect gas and a thin boundary layer, compared with the bubble radius, was assumed inside the bubble. The following equation was used as the equation of motion of the bubble:

\[
R \left( 1 - \frac{\dot{R}}{c} \right) \ddot{R} + \frac{3}{2} \left( 1 - \frac{\dot{R}}{3c} \right) \dddot{R} - \left( 1 + \frac{\dot{R}}{c} \right) R - \frac{R}{c} \left( 1 - \frac{\dot{R}}{c} \right) = 0,
\]

(2.17)

where \( H \) is enthalpy and \( c \) is the speed of sound. Besides the above equation, equations of heat transfer, mass transfer and the first law of thermodynamics were used in their analysis in the following forms:

1) the equation of energy in the liquid

\[
\frac{\partial T}{\partial t} + \left( \frac{R}{r} \right)^2 \frac{\partial T}{\partial r} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right),
\]

(2.18)

in which \( \alpha \) is the thermal diffusivity of the liquid;

2) the equation of mass diffusion

\[
\frac{\partial C}{\partial t} + \left( \frac{R}{r} \right)^2 \frac{\partial C}{\partial r} = D_a \left( \frac{\partial^2 C}{\partial r^2} + \frac{2}{r} \frac{\partial C}{\partial r} \right),
\]

(2.19)
2) the equation of mass diffusion

\[ \frac{\partial C}{\partial t} + \left( \frac{R}{r} \right)^2 \frac{R}{r} \frac{\partial C}{\partial r} = D_d \left( \frac{\partial^2 C}{\partial r^2} + \frac{2}{r} \frac{\partial C}{\partial r} \right) \]  

(2.19)

where \( C \) is the dissolved gas content and \( D_d \) is the diffusion coefficient;

3) the equation of energy inside the bubble

\[ \sum \frac{dE}{dt} + P_m \frac{dV}{dt} + \int V \cdot q dV - \int m h ds = 0.0, \] 

(2.20)

in which \( E \) is the internal energy, \( P_m \) is the pressure of mixture inside the bubble, \( q \) is the heat flux across the bubble interface, \( h \) is the specific enthalpy, \( m \) is the mass flux across the bubble wall and \( V \) and \( S \) are the volume and surface of the bubble, respectively.

The resulting set of equations was solved by the method of finite difference in combination with the Runge-Kutta method. The effects of the change of state of vapour and non-condensable gas, mass accommodation factor and initial bubble radius were plotted and discussed. The paper made the following points:

- bubble motion, considering condensation, was affected by the mass accommodation factor,
- the oscillation period of bubble was greater for larger equilibrium bubble radius.

Vokurka (1987) presented a simple model for a vapour bubble in which the speed of vaporisation was assumed to be neither infinite nor zero; two approaches which had been used by several researchers before Vokurka's model. The aim of this work was to achieve a simple model of a vapour bubble while retaining accuracy.
To calculate the properties of a vapour in a collapsing bubble two different regions were assumed. In the first one, when the bubble wall velocity was less than an average velocity, it was assumed that the rate of condensation was enough to maintain the pressure and temperature constantly equal to those of the liquid. In the second region where the bubble wall velocity was more than the foregoing average velocity, the thermodynamic process inside the bubble was assumed to be polytropic. To find the value of the average velocity several trial computations were performed and the results were compared with the experimental results of Mellen (1956). It was inferred from the calculations that, in a vapour bubble, pressure never drops below the vapour pressure at the liquid temperature. It was also suggested that the polytropic index is not constant, so a constant value for this index could be regarded as a useful correlation of simplified theory to experimental results.

From a survey of the literature on cavitation it can be seen that the assumption of a polytropic process within the bubble has been used by a number of writers such as Flynn (1964), Lautherborn (1976) and Apfel (1981). But this assumption poses some problems: e.g., the value of the polytropic exponent is not known exactly and it varies between unity (isothermal) and the ratio of specific heats (adiabatic). Another problem is that with this assumption energy dissipation from the bubble is ignored. To overcome this difficulty Prosperetti et al. (1988) proposed a formulation in which the internal pressure of the bubble was obtained numerically. Continuity and energy equations, under the assumption of spatially
uniform pressure, were combined and an expression for the velocity of the gas inside the bubble in terms of temperature gradient as shown below:

\[ u = \frac{1}{kp} \left[ (k - 1)K \frac{\partial T}{\partial r} - \frac{1}{3} r \beta \right]. \]  

(2.22)

in which \( k \) is the ratio of specific heats, \( K \) is the thermal conductivity and \( p \) is the pressure inside the bubble. The overdot denotes a derivative with respect to time.

For the motion of the bubble wall the following equation of Keller and Miksis (1980) was utilised:

\[ \left( 1 - \frac{\dot{R}}{c} \right) R \ddot{R} + \frac{3}{2} \dot{R} \left( 1 - \frac{\dot{R}}{3c} \right) = \left( 1 + \frac{\dot{R}}{c} \right) \frac{1}{\rho_l} \left[ P_B(t) - P_s \left( t + \frac{R}{c} \right) - P_\infty \right] + \frac{R}{\rho_l c} \frac{dp_B(t)}{dt}, \]  

(2.23)

where \( P_B(t) \) is the liquid pressure on the bubble wall, \( P_s \) is a component of the ambient pressure which is not constant and \( c \) is the speed of sound in the liquid.

In this analysis the temperature of the liquid was assumed to be constant and equal to the undisturbed temperature of the liquid at infinity. By using this mathematical modelling linear oscillations of the bubble was studied. A comparative study was made by the results of this formulation and the results of a polytropic process inside the bubble. Discrepancies between the outcomes of the two methods revealed unreliability of the assumption of the polytropic process.

Although the foregoing analysis circumvented the assumption of the polytropic process, since this model needed to solve a non-linear heat equation, it was computationally time consuming. To find the most efficient and accurate method in dealing with this problem Kamath and Prosperetti (1989) tried different numerical integration methods to solve the same set of equations derived in work by Prosperetti et al. (1988). Several numerical methods were applied to the problem, such as the adaptive Galerkin method with a variable number of terms,
the Galerkin method with a fixed number of terms, the collocation method and the finite difference method. The first method was found to be the most efficient and accurate.

The collapse of a bubble was studied by Okhotsimskii (1988) by considering the problem of non-linear unsteady heat and mass transfer between the liquid and vapour bubbles. The Jacob number was taken to be the only similarity criterion of this problem. The Jacob number is defined as $C\rho_i\Delta T / h\rho_v$ in which $C$ is the specific heat of liquid at constant pressure, $\rho_i$ is the liquid density, $\Delta T$ is the change in temperature of the bubble contents, $h$ is the enthalpy of vaporisation and $\rho_v$ is the density of vapour. Equations described in Zyong and Khabeev (1983) and Plesset and Zwick (1954) were used in a non-dimensional form as follows

\[
\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x} \left\{ \frac{d\alpha}{d\tau} \left[ 1 - \left( \frac{a}{x+a} \right)^2 \right] + \frac{2}{x+a} \right\},
\]

\[
\frac{d\alpha}{d\tau} = Ja \frac{\partial \theta}{\partial x} \bigg|_{x=0},
\]

\[
a(0) = 1,
\]

\[
\theta(0, \tau) = 1,
\]

\[
\theta(\infty, \tau) = 0.
\]

in order to study the collapse process. In this set of equations $\theta$ is the temperature, $\tau$ is the time, $x$ is the space coordinate, $a$ is the bubble radius and all are in non-dimensional form. It was found that the value of $Ja$ changes the form of $R(t)$, instantaneous bubble radius, significantly. For example for $Ja = 0$
a convex form of $R(t)$ was gained and when $0 < Ja < 2$ curves of $R(t)$ were S-shaped while for larger $Ja$, $(Ja > 2)$, the $R(t)$ curves were concave.

Vokurka (1988) followed the same line as his previous work (Vokurka (1987)) (with the exception that in the new work the ambient pressure was allowed to change) and studied the influence of ambient pressure on the free oscillations of bubbles in liquids. The motivation for this work was the range of ambient pressures, rather than atmospheric pressure, which were used in most experiments. The Modified Herring equation of motion (equation 2.12) was used for the motion of the bubble and for the pressure volume relationship a polytropic process was assumed as follows:

$$P = P_a \left(\frac{R_o}{R}\right)^k,$$

in which $k$ is the ratio of specific heats. For the case of a vapour bubble a simple model as described by Vokurka (1987) was utilised. It was found that by increasing the ambient pressure, very high peak pressures could be produced. Also a high damping factor, defined as the ratio of the second maximum to initial radius, resulted from choosing a high ambient pressure.

In another work Vokurka (1989) utilised the same formulation mentioned above and compared his results with the experimental results of Knapp and Hollander (1948), Chesterman (1952), Schmid (1959) and Gallant (1962). Since the model used in his work has already been explained, emphasis will only be put on the outcomes of this work. The model results were compared to experimental data from spark and laser generated bubbles and satisfactory predictions up to the time of the first collapse of the bubble were indicated. For later stages of bubble life, discrepancies appeared between the theoretical results and experimental results.
discrepancies appeared between the theoretical results and experimental results. These discrepancies, in his opinion, were due to the occurrence of large energy losses from the bubble, partially because of heat conduction. Other sources of energy loss were thought to be turbulence in the liquid, excessive gas loss from the bubble and converging shocks in the bubble interior.

Gas or vapour bubbles respond to the change of pressure in the surrounding liquid. This response depends on the bubble's condition and ambient pressure amplitude and frequency.

The pressure response of a small air bubble in water was calculated numerically by Matsumoto and Watanabe (1989). To investigate the relationship between non-linear oscillations of a bubble subjected to an oscillatory ambient pressure, the pressure far from the bubble was allowed to change periodically. In this study, mist formation inside the bubble due to homogeneous condensation, evaporation and deposition on the bubble wall and diffusion between the vapour and non-condensable gas were taken into account (It should be noted that in this analysis the non-condensable gas retarded condensation of vapour on the bubble wall). It was concluded that

- mist formation as well as its disappearance inside the bubble changes the internal condition of the bubble such that the bubble oscillates synchronously with the driving pressure field;
- the frequency response curve of the bubble shifted toward the lower frequencies;
- when the driving pressure amplitude became large the bubble motion showed deterministically chaotic behaviour.
Most of the information available on the theoretical calculation of pressure inside a gas bubble is based on inadequate models. In particular, the polytropic approximation has been used widely. Prosperetti (1991) investigated the chaotic regime of oscillating bubbles in which he criticised the polytropic approximation. His calculation was based on the formulation originally proposed by Nigmatulin and Khabeev (1974), Nigmatulin et al. (1981) and Nigmatulin and Khabeev (1977). The aims of the work were: (i) the study of the transient behaviour of an oscillating bubble as an initial-value problem; (ii) the study of non-linear bubble response in the limiting cases of nearly isothermal (in which the polytropic index is unity) and adiabatic behaviour (in which the polytropic index is equal to the ratio of specific heats); (iii) the verification of the uniformity of pressure inside the bubble and; (iv) integration of the results from a single bubble to get a simple model of a bubbly liquid. It was shown, in comparison with the formulation without the assumption of a polytropic process, that this assumption can not be justified for the whole period of bubble pulsation. By using a perturbation approach the approximation of the uniformity of pressure inside the bubble was found to be adequate with negligible error arising from this approximation.

Kamath et al. (1993) employed the same formulation as Kamath and Prosperetti (1989) and studied the phenomenon of sonoluminescence theoretically. In their endeavour heat transfer between the liquid and bubble content and also production of $OH$ radicals, as a result of dissociation of vapour, were taken into account. Their formulation allowed the authors to study the temperature field and chemical kinetics within the bubble and an estimation of the production rate of hydroxyl radicals was obtained. It was also concluded that heat transfer inside the bubble and across its interface play a major role in the whole process.
2.3. A BRIEF REVIEW OF SOME EXPERIMENTAL WORKS

This section is concerned with a brief review of some experimental works on bubble dynamics. These experiments had two main goals:

a) the study of bubble dynamics; and

b) the study of damage resulting from cavitation.

One of the pioneering works belongs to Ellis (1956) who utilised a high speed photographic technique to investigate the behaviour of a bubble subjected to an acoustic field. A rotating-mirror camera was used to capture the evolution of the bubble with a framing rate of up to 1,000,000 fps (frame per second). Compared with existing theory, he found the growth process to be more stable than the collapse process.

Shutler and Mesler (1965) studied the mechanism of damage caused by spark-induced bubbles. They concluded that the micro liquid jet has the capability to damage and that the main cause of damage is the pressure pulses.

Benjamin and Ellis (1966) set up an experiment by which the liquid jet was studied in more detail. Their conclusion was in contradiction to that of Shutler and Mesler (1965). They proposed the hypothesis that the liquid jet is the main cause of cavitation damage.

The results of Benjamin and Ellis (1966) were confirmed by the experimental work of Kling and Hammitt (1972). They produced vapour cavity in a flowing stream. To observe the collapse and migration of the bubble towards the rigid boundary, high speed photography, of up to 1,000,000 fps, was used. Very high pressures (up to 445 atmospheres) were detected within the bubble.
Lauterborn and Bolle (1975) used a Q-switched ruby laser to produce a vapour bubble in distilled water. The main feature of this method of generating bubbles was its ability to produce highly controlled conditions. Perfect spherical bubbles free of mechanical noise were produced. Bubble evolution was filmed at a framing rate of 300,000 fps. The resulting shape-time diagram showed remarkable agreement with the calculation of Plesset and Chapman (1971). A jet speed of more than 120 m/s was observed in their experiment.

Chahine and Bovis (1979) ran an experiment to investigate the interaction between a bubble and an interface of two fluids. Bubbles were produced by the spark-discharge method and a framing rate of 20,000 fps was used. Some non-dimensional parameters were defined and the results of the experiment were interpreted using these parameters. It was shown that the bubble is repelled by a liquid-air interface but repulsion or attraction of the bubble by a liquid-liquid interface depends upon the distance of the bubble from the interface.

Fujikawa and Akamatsu (1980) pioneered the use of liquid shock tubes in the study of cavitation. To record the bubble motion, a high speed camera in conjunction with the Fraunhofer holography technique was used. To show the stress waves induced in solids, a photoelastic material (epoxy resin) was used. Stress waves were induced in the solid as a result of the collapse and rebound of the bubble. In their experiment, a small hydrogen nucleus was generated by electrolysis on a platinum electrode. This nucleus was then subjected to expansion and decompression waves to generate the bubble and its subsequent pulsation. They inferred that the strength of the liquid jet is much less than that of the shock intensity.
To study the sonoluminescence produced by stable cavitation Crum and Reynolds (1985) utilised an ultrasonic horn driver to generate bubble motion. In their work standing waves were produced in distilled water saturated with air. After achieving the desired level of sonoluminescence the images were intensified by an image intensifier tube and then recorded by a video camera. It was found that more light was emitted by a field of stable cavities than by a field of transient ones. Also they observed that gas bubbles pulsating for several cycles were able to generate intense pressures and temperatures during their collapse phases. These intense pressures and temperatures produced a considerable number of free radical combinations.

Tomita and Shima (1986) ran an experiment to clarify the mechanism of impulsive pressure generated by the collapse of a single bubble in a stationary fluid. In their experiment, a bubble was generated by a spark induced method. A high speed photographic technique (up to 100,000 fps) was used to investigate the effect of cavitation on three kinds of materials. These materials were: (i) a photoelastic material (epoxy resin); (ii) a soft material (indium specimen); and (iii) an indium specimen with a pressure transducer attached. The experimental setup enabled them to study the temporal correlation between shock wave impact and stress-fringe initiation. Their findings can be summarised as follows:

a) the impulse pressure which causes plastic deformation was closely related to the behaviour of the liquid jet;

b) the interaction of tiny bubbles with a shock wave or pressure wave affected the production of local high pressure appreciably;

c) pitting damage was resulted from the impact pressure of the liquid micro jet; and
d) four kinds of impulse pressure occurred within a very short time detailed as follows:

- pressure pulse during bubble collapse;
- impact pressure from the liquid jet;
- impulse pressure caused by the collapse of tiny bubbles; and
- impact pressure from a shock wave as a result of torus-like bubble rebound.

In another experiment Matsumoto (1986) used a hydro-shock tube to generate cavitation bubble. The tube comprised a high pressure chamber and a low pressure chamber separated by a rupturing diaphragm. A light scattering method was utilised to measure the bubble radius and the response of a small gas bubble to ambient pressure reduction was investigated by the experimental setup. He found a good correlation between the results of the experiment and those of numerical calculations except for the rebound. He also concluded that when the ambient pressure varies, the bubble grows resonantly with it.

The collapse of a single bubble in water over a wide range of temperatures was experimentally studied by Shima et al. (1988). The bubble was produced by a spark-discharge method and its motion was filmed by a high speed camera. The results can be described briefly as follows:

- an increase in water temperature makes the collapse weaker;
- the water temperature has no appreciable effect on the translational movement of the bubble; and
induced impulsive pressure depends on water temperature as well as bubble size and its initial distance from a solid wall.

Wong et al. (1989) utilised a spark-discharge technique to generate a bubble in the vicinity of a vertical solid wall and a free surface. Experimental results were compared with the predictions obtained by considering the Kelvin impulse and good agreement was reported. Up to four bubble pulsations were captured and the change of volume was illustrated with respect to time. It was found that energy dissipation mainly occurs before the second pulsation. The locations of the bubble centroid during pulsations were detected and the bubble was observed to migrate toward the rigid wall and away from the free surface.

The Kelvin impulse is defined as

\[ I = \rho_i \frac{\delta}{\delta_t} \phinds \]

in which \( \rho_i \) is the liquid density, \( \phi \) is the velocity potential, \( n \) is the outward normal to the bubble surface and \( s \) is the surface of the bubble. Briefly the Kelvin impulse for a flow field corresponds to an impulsive force needed to create the existing flow field from rest. In the case of bubble motion the Kelvin impulse shows the impulsive force acting on the nearby boundary.

Tomita and Shima (1990) conducted an elaborate experiment in which the behaviour of a pulsating bubble near a rigid boundary, free surface, concave wall, convex wall and elastic surface was analysed. Also the behaviour of two interacting bubbles, the shock waves emanated by bubbles and the effect of a large gas bubble on a small bubble were studied. They used a Q-switched laser to produce bubbles in highly controlled conditions. Bubble evolution was filmed by high speed cinematography in conjunction with streak photography. The
Schlieren method was utilised to observe shocks emanated by the bubble. The new observations in this experiment were the evolution of a bubble in the vicinity of a concave surface and also near a convex one. Their observations showed that the bubble near a convex surface grew until a part of its surface touched the tip of the cone. Thereafter it began to collapse into a conical shape. When near a concave surface the bubble shrank into a disk shape. When a small bubble interacted with a large gas bubble they found that the small bubble was significantly influenced by the large one. The main effect of the large bubble was the repulsion of the small bubble during its collapse due to the superposition of flow fields generated by the two bubbles.

2.4. REVIEW OF SOME NUMERICAL WORKS ON NON-SPHERICAL BUBBLE DYNAMICS

This section is concerned with numerical works on non-spherical bubble dynamics. Although cavitation research has a long history (more than 200 years) analytical and numerical works on non-spherical bubble motion are relatively new.

The earliest work in this field belongs to Plesset and Chapman (1971) who studied bubble evolution near a rigid boundary by the finite difference method. They simulated bubble shapes until the latest stages of collapse when a liquid jet hit the opposite side of the bubble.

Mitchell and Hammitt (1973), considering the effect of fluid viscosity, pressure gradient and velocity, studied the effect of a nearby solid boundary on bubble evolution. They utilised a modified marker and cell method in their calculations.
In the method used by Plesset and Chapman (1971), potentials on the boundary and in the liquid domain needed to be calculated at each time step. Hence, it was computationally time consuming. To overcome this difficulty Bevir and Fielding (1974) used an approximate integral equation which was faster than the finite difference method. They solved Laplace's equation (which governs the potential flow field outside the bubble) by distributing singularities such as sources and sinks on the bubble surface and on the axis of symmetry, and vortex rings inside the cavity. Their work was mainly concerned with the accuracy of the chosen model and it was reported that their proposed method enabled them to capture the bubble motion up to the early stages of jet formation.

Bubble evolution near a free surface is also non-symmetric. This problem was numerically studied by Blake and Gibson (1981) by using an approximate integral equation. This method enabled them to predict the bubble and free surface interaction. Bubble shapes in subsequent pulsation showed a good correlation with experiment. The developed formulation was sufficient to describe the growth and collapse of a vapour bubble when the distance between the bubble and the free surface was not less than the maximum radius. In order to enable the method to predict the evolution of the bubble and free surface, it was necessary to apply the complete non-linear Bernoulli pressure condition at both the free surface and the bubble surface.

Both experimental and numerical works have revealed that the direction of the liquid jet is toward a rigid boundary and away from the free surface. A free surface can be regarded as a rigid surface with very low rigidity. These facts promoted researchers to study bubble dynamics in the near a boundary with a rigidity between the rigidity of a free surface and the rigidity of a solid wall.
To study the interaction between a pulsating bubble and a compliant surface Soh (1992) chose an energy approach. In this study a bubble was simulated by a source and for representing the rigid wall an image of the source was placed such that the position of the rigid wall was halfway between the source and its image. A compliant surface, an elastic layer, was assumed to be attached to the rigid wall. Then the energy equation was applied to a system consisting of the fluid and the elastic layer. In this way the interaction of the bubble and the layer was formulated as below:

\[ m(t)^2 \left( \frac{1}{r(t)} + \frac{1}{2\gamma} \right) + A_e \left[ \frac{\dot{m}(t)}{m(t)} \right]^2 + C_e \dot{m}(t) \frac{32}{3} [1 - r(t)^3] = 0, \]  

in which \( m(t) \) is the non-dimensional source (sink) strength, \( r(t) \) is the non-dimensional radius, \( \gamma \) is the geometric parameter defined as \( \gamma = h/R_{\text{max}} \) (\( h \) is the distance between the bubble centre and the rigid wall and \( R_{\text{max}} \) is maximum bubble radius), \( A_e \) and \( C_e \) are two scaling parameters accounting for the properties of the compliant surface. These parameters account for the physical properties of layer and geometrical parameters. It was found that the relative magnitude of \( A_e \) and \( C_e \) determines the characteristics of bubble motion.

The boundary element method has become a popular means of studying cavitation when the flow field outside the bubble is considered to be a potential flow field. Soh and Shervani Tabar (1992) utilised this method and investigated the unsteady flow around the cavity. Two cases of a constant pressure cavity and a cavity filled with variable pressure vapour were studied both in an infinite liquid and in the vicinity of a rigid wall. The results revealed the rebound of a collapsing vapour bubble, unlike the constant pressure cavity which collapsed.
into zero volume. Also the formation of a liquid jet was reported to be slower in a vapour bubble than in a constant pressure cavity.

In most cases the liquid jet generated during the collapse penetrates the bubble completely and a toroidal bubble is formed. In some circumstances rebound occurs before the liquid jet touches the opposite side of the bubble. Best and Kucera (1992) employed the boundary element method and studied the non-spherical rebound of bubbles. They also investigated the concept of the Kelvin impulse and its dependence on the buoyant force and the distance between the rigid wall and the point at which the inception of motion occurs.

Most of the works on the interaction of a bubble and nearby boundaries are concerned with a stationary infinite boundary. Harris (1993) took one step further and considered the motion of a bubble close to a moving finite rigid structure. He used the boundary element method in his work. As the flow field was assumed to be invicid and irrotational he solved the Laplace's equation in the liquid domain. Boundaries of the liquid domain consisted of both the bubble surface and the surface of the moving structure. He also considered the motion of a bubble below a moving cylinder. His findings can be summarised as follows:

- during the growth phase the bubble remained almost spherical and a high pressure region was developed on the surface of the cylinder above the bubble; and

- during the collapse phase a high pressure region was again developed and a jet generated towards the cylinder.

The interaction of two cavitation bubbles with a rigid boundary was investigated by Blake et al. (1993), both numerically and experimentally. For the numerical
part the boundary integral technique was used. A modified boundary integral technique was developed which allowed the authors to study the interaction between a multiple bubble system and a rigid boundary. In the experimental part laser-generated bubbles were filmed by high speed photography. A wide range of bubble motions and jet formations were observed due to the mutual effects of the bubbles and the interaction between the bubbles and the rigid wall.

There is relatively abundant literature on the behaviour of an isolated bubble in an oscillatory pressure field. Some of which were mentioned in Section 2.2 but, to the knowledge of the author, there is little information in the literature regarding the bubble response in a variable pressure field in the proximity of a boundary. Pioneering work in this field was carried out by Sato et al. (1994) who performed a numerical analysis to investigate the effects of parameters associated with an oscillatory pressure field as well as the influence of the boundary proximity on the bubble motion. They used an image method with hydrodynamic sources and the boundary element method in their analysis. Agreement between the two methods was good and they suggested the following facts:

- in the presence of an oscillatory pressure field a liquid jet was formed both in collapse and rebound phases;
- bubble migration towards or away from the rigid boundary depended on the frequency of oscillations of the pressure field,
- increasing the frequency of the pressure field weakened the effect of the nearby boundary on the bubble motion.
CHAPTER THREE

MOVING BOUNDARY PROBLEMS

3.1. INTRODUCTION

The study of boundary-initial value problems has been a major consideration for engineers, starting from the early 1750's. Although a great deal of research work has been done in this field, it is still an area attractive to modern engineering and science. In the classical type of boundary-initial value problems, the physical domain is either fixed in space or known in advance. However, in a great number of engineering applications, the boundary of the domain is neither fixed in space nor in time. Determination of the location of the boundary is an important part of the calculation procedure and is generally involved in the use of iterative or time-marching algorithms. In this case two conditions are needed on the moving boundary: first, determination of the location of the boundary, and second, satisfaction of boundary conditions. These problems are called Moving Boundary Problems (MBP) or sometimes the Stefan problem, after J. Stefan (1889a,b), and they arise in a number of important engineering applications such as melting and solidification, crystal growth, welding, metal casting, space vehicle design, preservation of foodstuffs, cryosurgery, astrophysics, meteorology, geophysics, plasma physics and the problem of cavitation which is the subject of this research.
Dealing with the classical types of initial-boundary value problems is a straightforward and relatively easy task as they can be analysed using standard analytical or numerical methods. In the case of MBPs, despite their importance and wide range of applications, only for very simple problems have exact solutions been established, and most of the investigators until recently have confined themselves to one dimensional problems. Generally the inherent non-linearity associated with this class of problem significantly complicates their analysis. Some of the developed techniques can not be extended to multi dimensional problems although these are of more technological importance. However, a few methods of solution for multi-dimensional problems have been developed recently. The following is a brief review of some of the most important and popular methods applied to the solution of MBPs.

3.2. ENTHALPY METHOD

This method is mathematically equivalent to the use of the conventional conservation equation in the domain and at the interface. In this method the interface is eliminated from consideration but the governing equation becomes non-linear. To formulate the problem an arbitrary control volume is considered. In problems with two different phases the moving interface may or may not pass through the control volume. Conservation of energy requires that the rate of change of enthalpy in the control volume be equal to the net rate of energy transferred into the control volume or, in mathematical form:

$$\frac{d}{dt} \int_V \rho u dV = \int_A K \nabla T \cdot \bar{n} dA,$$

in which $u$ is the specific internal energy, $\rho$ is the density, $K$ is the thermal conductivity, $T$ is the temperature and $\bar{n}$ is the unit outward normal vector to the
boundary, \( A \) is the surface of the control volume and \( \mathcal{V} \) is the volume. If we denote enthalpy by \( h \) and write \( \rho u \) as \( \rho h - p \) the enthalpy equation is derived (Shamsundar (1975)) as:

\[
\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho h d\mathcal{V} + \int_{A} \rho h V dA = \int_{A} K \nabla T dA.
\]

This equation is replaced by its finite difference counterpart by discretising the domain into a number of small elements. By discretising the domain the enthalpy equation is converted into a set of algebraic equations which is solved for the field variable. To determine in which phase a specific element is located, the total enthalpy of that element is calculated from the following relation:

\[
H = \int_{\mathcal{V}} \rho h d\mathcal{V},
\]

in which \( \mathcal{V} \) is the volume of that element. By comparing this value with the total enthalpy of the same element if it was in each phase, the phase of this element can be determined. Knowing the phase of each element, the location of interface can be determined. It is noteworthy that in this formulation no work should cross the boundaries of the control volume (Shamsundar and Sparrow (1975)) and due to the use of the finite difference method the whole domain should be discretised. More information on enthalpy based methods can be found in works by Atthey (1974), Jerome (1977), Morgan (1981) and White (1982a,b).

3.3. IMMOBILISATION OF THE MOVING BOUNDARY

The basic philosophy of this method is to simplify the numerical analysis by fixing the moving boundary at the expense of complicating the partial differential equations. This is done because generally most numerical methods are able to handle complex partial differential equations but have difficulty in coping with
moving boundaries. The immobilisation of the boundary is done by a transformation of the spatial variable from the physical domain into the computational domain in which the moving boundary will be transformed to a stationary boundary. After transformation, the problem takes a form which lends itself to solution by classical numerical methods such as the finite difference method (see Duda et al. (1975) and Rizza (1981)) or the finite element method (see Ettouney and Brown (1983)). This method needs some analytical work before programming and also extra computer time is needed to return to the physical domain when the solution is obtained. Furthermore, this method has difficulty in dealing problems having complicated geometry.

Numerical methods of engineering analysis can be broadly grouped into two main classes: domain (volume) and boundary (surface) methods. In both cases the geometry of the analysed object is divided into subregions of various forms, i.e. the elements. This is done because the elements enable the definition of the complicated geometry and numerical integration to be performed accurately on each element due to their small size. As an example of the domain methods one can name the finite element method and for the boundary methods, the boundary element method. In the next two sections applications of these methods to MBP are described.

3.4. THE FINITE ELEMENT METHOD

A well established method to deal with MBPs is the well known method of finite elements. This method, like the finite difference method, is a domain method (i.e. the whole domain under consideration has to be discretised which makes the method computationally time consuming).
There are two approaches in the implementation of this method to a MBP. In the first approach the whole domain, which includes the moving boundary, is discretised at once and the grid system is spatially fixed. For those elements located at either side of the moving boundary, ordinary interpolation functions are used but for those elements through which the moving boundary passes, discontinuous interpolation functions are incorporated. In other words, in the elements containing the moving boundary, the values of field variable on different sides of the boundary are approximated by different interpolation functions.

In the second approach discretisation of the domain is repeated after each time step. Because the new nodal points do not coincide with the nodal points from the previous time step, the values of the field variable at new nodal points are determined by interpolation from the values of the field variable at previous nodal points.

The first approach gives the location of the moving boundary with less accuracy compared with the second approach but as far as computation time is concerned it is more efficient. The second approach gives the moving boundary location more accurately at the expense of more computation time. On the whole, the finite element method has difficulty in dealing with singular problems and exterior problems. For detailed description of these methods the reader is referred to works by Wellford and Oden (1975) and Bonnerot and Jamet (1974,1977).

3.5. THE BOUNDARY ELEMENT METHOD

In the conventional finite element method elements are placed inside the body and all together form the geometry of the body. In the boundary element method the governing differential equation along with boundary and initial conditions are converted to an integral equation. Then by using the concept of free space Green's
function (fundamental solution) the integral equation becomes a boundary integral
equation in which only integral terms on the boundary are present. Having in
hand the boundary integral equation one has to consider only the surface of the
body for analysis; hence the dimensions of the problem will be reduced by one. In
other words for a 2D problem the elements are line segments of the boundary and
in a 3D problem the elements are 2D surfaces that cover the body.

Briefly, the boundary element method is a technique which offers important
advantages over domain methods such as the finite element method and finite
difference method.

The most interesting features of this method are:

• the numerical accuracy of the method is generally greater than that of domain
  methods,

• it is well suited for the solution of problems with infinite domains;

• the dimensions of problem are reduced by one;

• input data preparation is less time consuming compared with domain methods;
  and

• boundary element codes are easier to use with solid modellers and mesh
  generators.

These features of the boundary element method have led to the use of this method
by many researchers in the analysis of MBPs (e.g. see Taib (1985) and a book
edited by Wroble and Brebbia (1993)).
3.6. CAVITATION AS A MBP

The study of the behaviour of a single bubble, either isolated or near boundaries, is an important step towards the understanding of the cavitation phenomenon. The bubble, in an exchange of energy with the surrounding liquid, pulsates and during these pulsations the surface of the bubble changes abruptly. When the bubble is in an infinite liquid it remains spherical in shape for most of its pulsation time and only in the later stages of collapse does its shape deviate slightly from a spherical shape. However, when the bubble is near the boundaries its shape is highly distorted. Figure 3.1 depicts a schematic diagram of bubble surfaces during its evolution near a rigid boundary.

![Figure 3.1 Subsequent profiles of a bubble during its collapse](image)

To study the dynamic behaviour of the bubble, the hydrodynamic equation in the liquid domain must be solved. This problem is referred to as an exterior problem since the liquid domain is restricted only by an inner boundary (bubble surface). In other words the solution of the hydrodynamic equation in this case is involved in solving an exterior problem with a moving boundary.
Since the equation that governs the hydrodynamics of cavitation can be converted into a boundary integral equation and due to the reasons mentioned in Section 3.5, the boundary element method seems to be a good choice among other numerical methods for the study of bubble dynamics. In this research for the solution of the energy equation in the surrounding liquid and also for the analysis of the flow field, when the bubble is near a rigid boundary, the boundary element method is used. More detail is given in chapters 4 and 6.
CHAPTER FOUR

DYNAMIC BEHAVIOUR OF AN ISOLATED BUBBLE

4.1. INTRODUCTION

The phenomenon of the pulsation of bubbles, as a result of energy exchange with surrounding liquid, has been postulated by both experimental and theoretical works on bubble dynamics. A bubble in a large mass of liquid and not close to any surface or object is called an isolated bubble. The shape of an isolated bubble remains almost spherical during its subsequent pulsations. Only during the later stages of collapse may its shape may deviate from spherical, but this deviation is small compared to the bubble size. Therefore, one would not expect any large effects due to relatively small deviations from a spherical shape.

In this chapter the dynamics of an isolated bubble will be studied in the cases listed below

- when the bubble contains only pure liquid vapour (vapour bubble);
- when the bubble contains only non-condensable gas (gas bubble); and
- when the bubble contains a mixture of vapour and non-condensable gas.

In all three cases heat transfer across the bubble interface is taken into account. For the case of a vapour bubble, the properties of the vapour are extracted from thermodynamic tables. The properties of a non-condensable gas, for the case of a gas bubble, are related by the equation of state of an ideal gas. When the bubble is
filled with a mixture of vapour and gas, the pressure inside the bubble is the sum of
the partial pressures of the vapour and the gas.
Flow in the liquid domain is assumed to be incompressible, irrotational and invicid.
When the bubble collapses (or expands) the generated flow field is simulated by the
flow field of a sink (or a source).
In the following sections governing equations for three types of bubbles will be
derived.

4.2. GOVERNING EQUATIONS FOR AN ISOLATED VAPOUR
BUBBLE
To study the dynamic behaviour of a vapour bubble when heat transfer across the
bubble interface is present, one has to solve the energy equation in the vapour
domain and the hydrodynamic equation in the liquid domain. These equations are
coupled and should be solved simultaneously. Derivation of the hydrodynamic
equation is described in the next section.

4.2.1. Hydrodynamic Equation for a Vapour Bubble
To derive the hydrodynamic equation in the liquid domain, it is assumed that the
movements of the bubble and the surrounding liquid maintain spherical symmetry.
Also the pressure and temperature inside the bubble are assumed to be spatially
uniform.
Work done by the bubble on the surrounding liquid is \( \int P_v dV \), where \( P_v \) is the
vapour pressure inside the bubble and \( V \) is the bubble volume. The rise of the free
surface (at some distance from the bubble) due to the growth of the bubble results
in the work done, \( \int P_{atm} dV \), by the liquid on the atmosphere which has a constant
pressure \( P_{atm} \). The liquid is put into motion by the bubble pulsations, so there is an
increase of kinetic energy, \( KE \), in the liquid. Under the assumption of potential
flow, the pulsation of the bubble can be regarded as a source with a time varying strength \( \frac{d\mathcal{V}}{dt} \). It can be shown that:

\[
KE = \frac{\rho_l}{2} \frac{\mathcal{V}^2}{4\pi R},
\]

(4.1)

where \( \rho_l \) is the liquid density and \( R \) and \( \mathcal{V} \) are the bubble radius and volume, respectively. The overdot denotes differentiation with respect to time. These three terms (work done by the bubble, work done on atmosphere and change in the kinetic energy of the liquid) are combined into the hydrodynamic equation as:

\[
\frac{\rho_l}{2} \frac{\mathcal{V}^2}{4\pi R} - \int_{\mathcal{V}_0}^{\mathcal{V}} P_v d\mathcal{V} + \int_{\mathcal{V}_o}^{\mathcal{V}} P_{am} d\mathcal{V} = 0,
\]

(4.2)

in which \( \mathcal{V}_0 \) is the initial volume of the bubble. By assuming \( P_v = \text{Const.} \) the Rayleigh equation could be derived from the equation (4.2). To cast equation (4.2) in a form suitable for our calculations the first law of thermodynamics is applied to a control mass which consists of the vapour and a portion of surrounding liquid which is going to be vaporised during a time increment \( dt \). Figure 4.1 shows a schematic diagram of the chosen control mass.
The components of energy and work included in the control mass are as follows:

- work done by the bubble, \( P_v \, dV \), \( P_v \) is vapour pressure);
- change of internal energy of vapour, \( \rho_v \, du_v \), \( \rho_v \) is vapour density);
- change in internal energy due to condensation or evaporation, \( dm_l \, (u_v - u_l) \), where \( m_l \) is mass of liquid in control mass, \( u_v \) and \( u_l \) are the specific internal energies of the vapour and liquid, respectively; and
- rate of heat transfer across the bubble interface, \( \dot{Q} \).

Due to conservation of energy we have:

\[
P_v \, dV + \rho_v \, du_v - dm_l \, [u_v - u_l] = \dot{Q} \, dt. \tag{4.3}
\]

Considering the definition of control mass, conservation of mass yields:
\[ dm_t + d(\rho_v \varphi) = 0, \]  
and substituting \( dm_t \) from (4.4) in (4.3) results in:

\[ P_v d\varphi + d\int \{ \rho_v \varphi \left( u_v - u_l \right) \} = \dot{Q} dt - \rho_v \varphi du_l. \]  

(4.5)

Introducing the function \( H \) as \( H = \rho_v (u_v - u_l) \) and substituting in (4.5) gives a new form of the energy equation:

\[ P_v d\varphi + d(H) = \dot{Q} dt - \rho_v \varphi du_l. \]  

(4.6)

It should be noted that the function \( H \) has the dimensions of pressure and can be regarded as a property of vapour. Using Equation (4.6) one can write:

\[ \int_{V_0}^{V} P_v d\varphi = -\varphi H + \varphi H_0 + \int_{t=0}^{t} \left( \dot{Q} - \rho_v \varphi C_l \frac{dT}{dt} \right) dt, \]  

(4.7)

in which \( H_0 \) is initial value of \( H \), \( C_l \) is the specific heat of the liquid and \( T \) is the temperature of the bubble contents. Substituting \( \int_{V_0}^{V} P_v d\varphi \) in (4.2) from (4.7) gives:

\[ \frac{\rho_l}{2} \frac{\dot{\varphi}^2}{4\pi R^4} - \int_{t=0}^{t} \left( \dot{Q} - \rho_v \varphi C_l \frac{dT}{dt} \right) dt + \varphi H - \varphi H_0 + \int_{V_0}^{V} P_{am} d\varphi = 0. \]  

(4.8)

To eliminate the integrals from Equation (4.8), it is differentiated with respect to time. After some further manipulations the following equation will be obtained:

\[ \frac{\rho_l}{8\pi} \left[ \frac{8\pi \varphi^2 R^3 - \dot{\varphi}^3}{4\pi R^4} \right] + H \dot{\varphi} + \varphi \ddot{H} + P_{am} \dot{\varphi} = \dot{Q} - \rho_v \varphi C_l \frac{dT}{dt}. \]  

(4.9)

Equation (4.9) is rearranged to give:

\[ \dot{\varphi} = \frac{R}{2\varphi} \left[ \frac{\varphi^3}{4\pi R^4} + \frac{8\pi}{\rho_l} \left[ \dot{Q} - \rho_v \varphi C_l \frac{dT}{dt} - (P_{am} + H) \dot{\varphi} - \varphi \frac{dH}{dt} \right] \right]. \]  

(4.10)

Dividing Equation (4.6) by \( dt \) gives:
If the first two terms in square brackets in Equation (4.10) are replaced by the left hand side of Equation (4.11) then:

\[
\dot{\Psi} = \frac{R}{2} \left\{ \frac{\dot{\Psi}^2}{4\pi R^4} + \frac{8\pi}{\rho_l} \left[ (P_v - P_{am}) \right] \right\},
\]

or

\[
\dot{\Psi} = \frac{1}{2} \left\{ \frac{\dot{\Psi}^2}{3\Psi} + \left( \frac{8\pi}{\rho_l} \right) \left( \frac{3}{4\pi} \right)^{\frac{3}{2}} \left[ (P_v - P_{am}) \right] \right\}.
\]

This equation constitutes the hydrodynamic equation. Solution of this equation yields \( \dot{\Psi} \).

The flow field in the liquid domain is defined by the velocity potential \( \Phi \) which is given by:

\[
\Phi = \frac{\dot{\Psi}}{4\pi r},
\]

where \( r \) is radial distance from the centre of the bubble and \( r \geq R \).

As Equation (4.13) depends on the pressure inside the bubble, \( P_v \), the solution of this equation requires prior knowledge of the pressure inside the bubble. In this work the pressure inside the bubble is not constant during the bubble evolution. To find \( P_v \), the equation of energy in a control mass is used which is described in the next section.

4.2.2. The Equation of Energy for the Control Mass

Experimental results have revealed that the later stages of collapse occur so rapidly that all of the vapour inside the bubble can not completely condense and the bubble rebounds. Different approaches have been chosen by researchers to relate the
pressure inside the bubble with other relevant parameters such as bubble volume. Some have used a polytropic relation between volume and pressure. In this work the equation of conservation of energy has been used to relate pressure with other factors affecting the bubble motion. To do this the control mass defined in Section 4.2.1 is used and the first law of thermodynamics is applied to this control mass. For the sake of brevity we start from Equation (4.11) which is rewritten here:

\[ P_v \dot{\vartheta} + \vartheta \dot{H} + H \dot{\vartheta} = \dot{Q} - \rho_v \vartheta C_1 \frac{dT}{dt}, \]  

Equation (4.11) becomes:

\[ P_v \dot{\vartheta} + \left( \frac{\partial H}{\partial T} \right) \dot{T} \vartheta + H \dot{\vartheta} = \dot{Q} - \rho_v \vartheta C_1 \dot{T}. \]  

Solving this equation for \( \dot{T} \) gives:

\[ \dot{T} = \frac{\dot{Q} - (H + P_v) \vartheta}{\left( \frac{\partial H}{\partial T} + \rho_v C_1 \right) \vartheta}. \]  

By introducing an auxiliary function \( y = \vartheta \), the following system of differential equations describing the dynamics of a vapour bubble will be resulted in which condensation (or evaporation) and heat transfer are considered:

\[
\begin{align*}
\dot{y} &= \frac{1}{2} \left\{ \frac{\vartheta^2}{3 \vartheta} + \left( \frac{8\pi}{\rho_i} \right) \left( \frac{3}{4\pi} \right)^{1/3} \left( P_v - P_{am} \right) \vartheta^{1/3} \right\}, \\
\dot{T} &= \frac{\dot{Q} - (H + P_v) \vartheta}{\left( \frac{\partial H}{\partial T} + \rho_v C_1 \right) \vartheta}, \\
\dot{\vartheta} &= y
\end{align*}
\]

Equation (4.17)

This system of equations is solved by the 4th order Runge-Kutta method for unknowns \( \vartheta \), \( \vartheta \) and \( T \). There are several points worthy of consideration here:
• in the derivation of this system of equations the assumption of spherical motion of the bubble is used only for the hydrodynamic equation and this assumption is not needed for the energy equation;

• from the second equation in (4.17) it is seen that the function $H$ is added to $P_v$, the vapour pressure. Recalling the definition $H = \rho_v(u_v - u_i)$, it can be said that this function is a correction to the pressure inside the bubble due to condensation and evaporation; and

• in the solution of (4.17) the value of $\dot{Q}$ is needed. This value is calculated by the solution of the energy equation in the liquid domain. This equation is:

$$\nabla^2 T = \frac{1}{\alpha} \frac{D T}{D t},$$

in which $T$ is the temperature in the liquid domain, $\alpha$ is the liquid thermal diffusivity and $\frac{D}{D t}$ has the meaning of total derivative in Lagrangian sense. To solve this equation the method of dual reciprocity boundary elements is used which will be described in detail in Chapter 5.

4.2.3. Outline of Computations for a Vapour Bubble

Figure 4.2 shows the flow chart of the computations for vapour bubble dynamics in this work:
In this figure $T_{\text{inf}}$ and $P_{\text{inf}}$ are the temperature and pressure at some distance from the bubble, respectively and $T_0$ and $R_0$ are initial temperature and radius of the bubble.
4.3. GOVERNING EQUATIONS FOR AN ISOLATED GAS BUBBLE

In this section equations which govern gas bubble dynamics are derived. Assumptions made in this case are as follows:

- the bubble wall and surrounding liquid move in a spherically symmetrical manner;
- the bubble contains only non-condensable gas;
- the pressure inside the bubble is spatially uniform;
- the temperature inside the bubble changes radially;
- evaporation and condensation are not considered; and
- properties of the gas are related by the equation of state of an ideal gas.

4.3.1. Hydrodynamic Equation for a Gas Bubble

The hydrodynamic equation for a gas bubble is almost the same as that for a vapour bubble (Equation 4.2), except that $P_v$ is replaced by $P_g$ which is the pressure of the non-condensable gas inside the bubble. This equation has the form:

$$\frac{\rho_1}{2} \frac{\nabla^2}{4\pi R} - \int_{\nabla} P_g d\nabla + \int_{\nabla} P_{am} d\nabla = 0. \tag{4.18}$$

To convert the form of this equation to a form suitable for our calculations, the first law of thermodynamics is applied to a control mass which consists of the non-condensable gas inside the bubble:

$$P_g d\nabla + \rho_g \nabla du_g = \dot{Q} dt, \tag{4.19}$$
in which \( \rho_g \) and \( u_e \) are the density and the specific internal energy of gas, respectively. Equations (4.18) and (4.19) are equivalent to equations (4.2) and (4.3) for a vapour bubble. Following the same approach of Section (4.2) one can derive the hydrodynamic equation for a gas bubble as follows:

\[
\frac{\ddot{V}}{2} = \frac{1}{3} \frac{\dddot{V}}{\dot{V}} + \left( \frac{8\pi}{\rho_l} \right) \left( \frac{3}{4\pi} \right)^{\frac{1}{3}} \left( P_s - P_{am} \right) \frac{\dot{V}}{\dot{V}^{\frac{1}{3}}}.
\]  

(4.20)

By introducing an auxiliary function \( y = \dot{V} \), the following system of equations will be obtained:

\[
\begin{align*}
\ddot{y} &= \frac{1}{2} \left( \frac{\dddot{V}}{\dot{V}} + \left( \frac{8\pi}{\rho_l} \right) \left( \frac{3}{4\pi} \right)^{\frac{1}{3}} \left( P_s - P_{am} \right) \right) \frac{\dot{V}}{\dot{V}^{\frac{1}{3}}} \\
\dot{V} &= y
\end{align*}
\]

(4.21)

In solving (4.21) the value of \( P_s \) is needed. Calculation of this parameter will be described in the next section.

4.3.2. Radial Distribution of Temperature in the Bubble

As was mentioned in Section 4.3, for the case of a gas bubble the temperature inside the bubble is allowed to change radially. To calculate the radial distribution of temperature the same formulation derived by Prosperetti et al. (1988) is adopted here. In their work heat transfer between the gas inside the bubble and the surrounding liquid was ignored and for the hydrodynamic equation, the equation of Keller and Miksis (1980) was used.

In this work for the hydrodynamic part of the problem the system of equations (4.21) is used and also heat transfer across the bubble wall is considered. In what follows, by using the assumption of uniformity of pressure, a non-linear partial
differential equation for the temperature field and an ordinary differential equation for pressure will be derived. For more details of the derivation the reader is referred to the work of Prosperetti et al. (1988).

The equation for the conservation of mass inside the bubble has the form:

$$\frac{d\rho_s}{dt} + \rho_s \nabla \cdot u = 0,$$  

(4.22)

in which $\rho_s$ and $u$ are the density and the radial velocity of gas within the bubble respectively and $\frac{d}{dt}$ has the meaning of convective derivative or in mathematical form $\frac{\partial}{dt} + u \frac{\partial}{\partial r}$. The equation for the conservation of energy in the gas is:

$$\rho_s C_{ps} \frac{dT}{dt} + \frac{T}{\rho_s} \left( \frac{\partial \rho_s}{\partial T} \right)_{P_s} \frac{dP_s}{dt} = K_s \nabla^2 T,$$  

(4.23)

where $C_{ps}$, $K_s$ and $P_s$ are the specific heat, the thermal conductivity and the pressure of the gas, respectively. In our calculations $K_s$ is assumed to be constant.

If Equation (4.22) is multiplied by $C_{ps} T$ and added to Equation (4.23) the following equation will be obtained:

$$\frac{d}{dt} \left( \rho_s C_{ps} T \right) + \frac{T}{\rho_s} \left( \frac{\partial \rho_s}{\partial T} \right)_{P_s} \frac{dP_s}{dt} + \rho_s C_{ps} T \nabla \cdot u = K_s \nabla^2 T.$$

(4.24)

For a perfect gas with constant specific heats one can write $\rho_s C_{ps} T = \frac{kP_s}{k-1}$ and

$$\frac{T}{\rho_s} \left( \frac{\partial \rho_s}{\partial T} \right)_{P_s} = -1$$  

(see appendix A) where $k$ is the ratio of specific heats.

Employing these two relations, using Equation (4.22) and recalling that $P_s = P_s(t)$, Equation (4.24) can be written as:

$$\frac{1}{kP_s} \frac{d}{dt} P_s + \nabla \cdot \left\{ u - \left[ (k-1)/kP_s \right] K_s \nabla T \right\} = 0.$$

(4.25)
Considering the assumption of spherical symmetry, integration of Equation (4.25) will give an expression for the radial velocity of the gas as:

\[ u = \frac{1}{k P_g} \left[ (k - 1) K_g \frac{\partial T}{\partial r} - \frac{1}{3} r \dot{P_g} \right]. \]  

(4.26)

Applying this equation to the bubble interface and using the boundary condition for the velocity as \( u = \dot{R} \) as \( r = R \), an ordinary differential equation for \( \dot{P}_g \) can be derived in the form of:

\[ \dot{P}_g = \frac{3}{R} \left[ (k - 1) K_g \frac{\partial T}{\partial r} \right] - k P_g \dot{R}. \]  

(4.27)

Using the relations \( \rho_g C_{pg} T = \frac{k P_g}{k - 1} \) and \( \frac{T}{\rho_g} \left( \frac{\partial \rho_g}{\partial T} \right)_{\rho_g} = -1 \), Equation (4.23) can be rewritten as:

\[ \frac{k}{k - 1} \frac{P_g}{T} \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} \right] - \dot{P}_g = K_g \nabla^2 T. \]  

(4.28)

By combining equations (4.26) and (4.28), a non-linear partial differential equation for \( T \) will be obtained as follows:

\[ \frac{k}{k - 1} \frac{P_g}{T} \left[ \frac{\partial T}{\partial t} + \frac{1}{k P_g} \left[ (k - 1) K_g \frac{\partial T}{\partial r} - \frac{1}{3} r \dot{P_g} \right] \frac{\partial T}{\partial r} \right] - \dot{P}_g = K_g \nabla^2 T. \]  

(4.29)

Equations (4.21) and (4.27) form the following set of differential equations:

\[ \dot{\nabla} = y \]

\[ \dot{y} = \frac{1}{2} \left( \frac{\nabla^2}{3 \nabla} + \left( \frac{8 \pi}{P_i} \right) \left( \frac{3}{4 \pi} \right) \frac{\frac{1}{2}}{\left( P_g - P_{am} \right)} \right) \nabla^{1/3} \left( \left( P_g - P_{am} \right) \nabla^{1/3} \right) \]

\[ \dot{\dot{\nabla}} = (36 \pi)^{1/3} (k - 1) K_g \frac{\partial T}{\partial r} \left[ \nabla^{1/3} - k P_g \frac{\nabla}{\nabla} \right]. \]  

(4.30)
\[ P_i = (36\pi)^{\frac{1}{3}}(k - 1)K_i \frac{\partial T}{\partial r} |_{r=R} \nabla T |_{r=R} - k P_i \frac{\dot{V}}{\nabla}. \]  

(4.30)

This set of equations is solved simultaneously for \( V, \dot{V}, \dot{P}_i \). The radial distribution of temperature inside the bubble is then found by the solution of Equation (4.29). To solve this equation, the boundary conditions are set as:

\[ \frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = 0, \]  

(4.31)

\[ K_i \frac{\partial T}{\partial r} = K_i \frac{\partial T}{\partial r} \quad \text{at} \quad r = R, \]  

(4.32)

in which \( K_i \) is the thermal conductivity of surrounding liquid. The right hand side of the second boundary condition is the heat transfer across the bubble wall which is calculated by the method of dual reciprocity boundary elements. This equation is solved by an implicit finite difference method.

An outline of computations for the gas bubble is illustrated in the flow chart in the next section.

4.3.3. Outline of Computations for a Gas Bubble

The following flow chart depicts the outline of computations for gas bubble dynamics.
Set $R_0 = R_1$, $P_0 = P_1$ and $T_0 = T_1$

Find $\dot{Q}$ by the method of Dual Reciprocity Boundary Elements

Solve system of equations (4.30) to find $V_i$ (or $R_i$), $P_1$ and radial distribution of temperature inside the bubble

Print $V_i$ (or $R_i$) and bubble wall velocity

$t = \text{finaltime}$

If $t = \text{finaltime}$ then Stop, otherwise repeat.
4.4. GOVERNING EQUATIONS FOR A BUBBLE CONTAINING A MIXTURE OF VAPOUR AND A NON-CONDENSABLE GAS

In this section the governing equations describing the dynamics of a bubble filled with a mixture of vapour and non-condensable gas will be derived. Derivation of these equations is based on the following assumptions

- the bubble shape remains spherical at all times;
- the bubble is filled with a mixture of vapour and non-condensable gas;
- the pressure inside the bubble is the sum of partial pressures of vapour and gas;
- since the properties of the vapour inside the bubble are found from thermodynamic tables, the pressure and temperature inside the bubble are uniform;
- the properties of the gas are related by the equation of state of an ideal gas;
- evaporation and condensation are taken into account;
- heat transfer across the bubble wall is considered; and
- the bubble wall is assumed to be impervious to gas.

4.4.1. The Hydrodynamic Equation for a Bubble filled with Vapour and Gas

The components of work done by the bubble on the surrounding liquid, work done on the atmosphere and change in the kinetic energy of the liquid are combined to give the hydrodynamic equation:
\[
\frac{\rho_i \frac{V^2}{2}}{4\pi R} - \int_{V_e} (P_g + P_v) dV + \int_{V_e} P_{am} dV = 0, \quad (4.33)
\]

where \( P_g \) and \( P_v \) are the partial pressures of gas and vapour, respectively. We follow the same procedure of Section 4.2.1 to convert Equation (4.33) into a form suitable for our calculations. To do this the first law of thermodynamics is applied to a control mass which consists of gas, vapour and a part of the liquid which is going to be vaporised in a time increment \( dt \). This equation has the form:

\[
(P_g + P_v) dV + \rho_s \frac{d}{dt} (u_e + u_i) + \rho_v \frac{d}{dt} (u_e - u_i) = \dot{Q} dt,
\]

(4.34)

All notations are the same as those of sections (4.2) and (4.3). Since the bubble wall is assumed to be impervious to the gas, the equation of conservation of mass is the same as Equation (4.4) which is repeated here:

\[
dm_l + d(\rho_v V) = 0.
\]

(4.4)

Substitution of \( dm_l \) from Equation (4.4) into Equation (4.34) gives:

\[
(P_g + P_v) dV + \rho_s \frac{d}{dt} (u_e + u_i) + \rho_v \frac{d}{dt} (u_e - u_i) + d(\rho_v V) = \dot{Q} dt,
\]

(4.35)

Defining the function \( H \) as \( H = \rho_v (u_e - u_i) \), Equation (4.35) is changed to:

\[
(P_g + P_v) dV + \rho_s \frac{d}{dt} u_e + \rho_v \frac{d}{dt} u_e + d(\forall H) = \dot{Q} dt,
\]

(4.36)

and from this equation the work done by the bubble can be written as:

\[
\int_{V_e} (P_g + P_v) dV = -\forall H + \forall H_o + \int_0^t (\dot{Q} - \rho_s \forall C_s \frac{dT}{dt} - \rho_v \forall C_v \frac{dT}{dt}) dt.
\]

(4.37)

Substitution of the second term of Equation (4.33) from (4.37) results in:
Differentiating this equation with respect to time and following the same approach as Section (4.2.1) an expression for \( \ddot{V} \) will be obtained which is:

\[
\ddot{V} = \frac{1}{2} \left\{ \frac{\dot{V}^2}{3V^3} + \left( \frac{8\pi}{\rho_1} \right) \left( \frac{3}{4\pi} \right) \left( P_s + P_e - P_{\text{am}} \right) \dot{V} \right\}. 
\] 

(4.39)

This equation constitutes the hydrodynamic equation.

**4.4.2. The Energy Equation for the Proposed Control Mass**

Equation (4.36) is the equation of conservation of energy within the control mass. If this equation is differentiated with respect to time then:

\[
(P_s + P_e) \dot{V} + \rho_s \dot{C}_v \dot{T} + \rho_s \dot{C}_v \dot{V} + \dot{V} H + H \dot{V} = \dot{Q}. 
\] 

(4.40)

Writing \( \dot{H} = (\partial H/\partial T) \dot{T} \) and solving for \( \dot{T} \) we obtain:

\[
\dot{T} = \frac{\dot{Q} - (H + P_s + P_e) \dot{V}}{\left( \frac{\partial H}{\partial T} + \rho_s C_v + \rho_s C_v \right) \dot{V}}. 
\] 

(4.41)

In this equation, as in Equation (4.16), \( H \) appears as a correction to pressure inside the bubble due to condensation or evaporation. Considering equations (4.39) and (4.41) the following system of differential equations governs the motion of the bubble:
\[ \mathbf{V} = y \]

\[ y = \frac{1}{2} \left\{ \frac{y^4}{3V} + \left( \frac{8\pi}{\rho_l} \right) \left( \frac{3}{4\pi} \right)^{\frac{1}{3}} \left[ (P_s + P_v - P_{\text{am}}) \right]^{\frac{1}{3}} \right\} \]

\[ T = \frac{\dot{Q} - (H + P_v + P_s) \dot{V}}{\left( \frac{\partial H}{\partial T} + \rho_v C_v + \rho_s C_s \right) V} \]

This system of differential equations is solved by the fourth order Runge-Kutta method for the unknowns \( V, \dot{V} \) and \( T \). The whole calculation process is depicted in the flow chart in the next section.

4.4.3. Outline of Computations for a Bubble filled with Gas and Vapour

Figure 4.4 shows the whole computational procedure for the dynamics of a bubble containing a mixture of non-condensable gas and vapour.
Figure 4.4 Flow chart depicting the computational procedure for a bubble filled with a mixture of non-condensable gas and vapour.
4.5. RESULTS AND DISCUSSION

Based on the formulations mentioned in the previous sections a computer program has been developed to study the dynamic behaviour of a single bubble. This section is concerned with some numerical results of the program.

All values in figures are in non-dimensional form. The dimensionless parameters are defined as follows:

- non-dimensional pressure is defined as $P^* = P / P_0$, where $P_0$ is the initial value of pressure in the bubble;
- non-dimensional temperature is defined as $T^* = T / T_0$, where $T_0$ is the initial temperature of the bubble contents;
- non-dimensional time is defined as $\tau = (t / R_0) \sqrt{\Delta P / \rho}$, in which $\Delta P$ is the difference between the liquid pressure at infinity and the initial pressure of the bubble, $R_0$ is the initial radius of the bubble and $\rho$ is the liquid density;
- non-dimensional radius is defined as $R^* = R / R_0$;
- non-dimensional volume is defined as $V^* = V / V_0$ where $V_0$ is the initial volume of the bubble; and
- non-dimensional bubble wall velocity is defined as $\dot{R}^* = \dot{R} / \sqrt{\Delta P / \rho}$.
4.5.1. Results For a Vapour Bubble

In this section some numerical results for a vapour bubble will be presented. A comparison is made between the results of the program for different cases. Details of each case are mentioned in Figure 4.5. In all of these cases the initial radius of the bubble is 0.02 m, initial pressure inside the bubble is 8 kPa, the initial temperature in the bubble is 60°C, and the pressure at infinity is 10 kPa.

![Graph showing the comparison of the results of the program for different cases](image)

**Figure 4.5** Comparison of the results of the program for different cases

The solid line with open circles shows the bubble radius resulting from when the pressure inside the bubble remains constant and heat transfer is not present. In this
this case the bubble collapses to a very small volume. The dashed line shows the results of program while the pressure is allowed to change but heat transfer is ignored (Case a). In this case system of equations (4.17) is solved while $\dot{Q}$ is set equal to zero ($\dot{Q} = 0$). The dashed line with triangles illustrates the results of the program considering heat transfer (Case b). Case c represents the collapse of a Rayleigh bubble. Also the Figure shows that the Case b matches with Case c for a longer period of time than Case a. This phenomenon is described as below.

When the bubble collapses the temperature and pressure inside the bubble increase. This pressure increase makes the bubble behave differently from that of a Rayleigh bubble (in which the pressure is assumed to be constant). The temperature increase in Case b, due to the presence of heat transfer, is slower than that of Case a. This slow increase in temperature is accompanied by slower increase in pressure inside the bubble in Case b compared with Case a. This causes the bubble in Case b to act more like a Rayleigh bubble.

Figures 4.6, 4.7, and 4.8 depict the time histories of a vapour bubble's volume, pressure inside the bubble and temperature inside the bubble, respectively. The bubble has the same initial radius as the bubble in the previous example, pressure at infinity is 5 kPa and the initial temperature inside the bubble is 60°C with the corresponding initial pressure of 20 kPa (from the thermodynamic tables). All these figures show a reduction in the amplitude of changes. These reductions are due to the loss of energy through heat transfer from the bubble. It should be noted that if in the solution of system of differential equations (4.17) heat transfer be ignored, $\dot{Q} = 0$, then the amplitude of the pulsations remains constant.
The vapour bubble not only loses energy during its pulsations but also loses its vapour content due to the deposition of vapour on the bubble wall through condensation. That is why the volume of the vapour bubble approaches zero.
Figure 4.7 Time history of the pressure of the vapour bubble

Figure 4.8 Change of temperature inside the vapour bubble with respect to time
Two mechanisms are known to be responsible for cavitation damage. The first one is the shock wave emitted into the liquid domain by the pulsations of the bubble and the second one is a high speed liquid jet which is produced during the bubble collapse in the vicinity of a solid wall. The change in bubble wall velocity with respect to time is shown in Figure 4.9. At the instant of rebound the interface of the bubble, which changes its direction of motion in a very short period of time, will most probably generate a weak shock wave in the liquid. This periodic emission of shock waves may excite resonance in structures and may subsequently lead to damage.

![Graph showing change of bubble wall velocity with respect to time](image)

**Figure 4.9** Change of bubble wall velocity with respect to time

### 4.5.2. Results for a Gas Bubble

In this section some results about the dynamic behaviour of a gas bubble will be presented. In the first example the bubble has an initial radius of 0.02 m, an initial
pressure of 300 kPa and the initial temperature of the bubble content is 300°C. The pressure at infinity is chosen to be 25 kPa.

In figures 4.10 and 4.11 time histories of the volume and pressure of a pulsating gas bubble when heat transfer is taken into account are illustrated. The volume of the bubble in Figure 4.10 (the case of the gas bubble) approaches a finite value in contrast to Figure 4.6 in which the condensation allows the volume of the bubble to approach zero.

Figure 4.10 Time history of the volume of a gas bubble

Figure 4.11 shows that the pressure in a gas bubble remains fairly constant over a large part of one complete cycle. In this period bubble evolution takes place slowly and the decrease in temperature (because of volume increase) is compensated by heat transfer to the bubble. Therefore, the pressure remains fairly constant but when the bubble is around its minimum volume (when $\tau < 1.0$ and $4.0 < \tau < 4.75$)
the bubble evolves so rapidly that heat transfer can not keep up with it so the pressure and temperature change considerably.

![Graph](image)

**Figure 4.11** Pressure change in a gas bubble with respect to time

A pulsating bubble loses energy through heat transfer. This energy loss reduces the driving force of the bubble motion. Thus, the bubble evolution will be slowed down, i.e. the period of pulsations will be increased. This fact is depicted in Figure 4.12. In this figure the dashed line shows the bubble behaviour when heat transfer is ignored and the solid line shows the bubble behaviour considering heat transfer.
Figure 4.12 Effect of heat transfer on the period of pulsations of a gas bubble.

The dynamic behaviour of a gas bubble is strongly affected by the pressure of the gas inside the bubble. To determine the bubble pressure during its evolution, one has to solve equations of energy, continuity and momentum both inside and outside the bubble which is a very complicated task. It has been usual in the bubble dynamics literature to use a polytropic relation of the form

\[ P = P_0 \left( \frac{\mathcal{V}_0}{\mathcal{V}} \right)^k, \]

(4.43)
in which \( k \) is the polytropic index. This relation has been used by several researchers such as Vokurka (1988). Although this relation offers an easy way to calculate pressure in the bubble, it poses some problems as mentioned below.

First, the value of polytropic index is not known exactly. It ranges between 1 (for isothermal process inside the bubble) and the ratio of specific heats (for an adiabatic process inside the bubble).

Secondly, by using Equation (4.43) to find \( P \), \( PdV \) is a perfect differential and its integration over one complete cycle is zero. This corresponds to no energy loss from the bubble, which is contrary to experimental observations.

Computations were carried out for several different initial conditions for both vapour and gas bubbles to give more insight into the \( P-V \) relationship of the thermodynamic process the bubble contents undergo. The initial conditions for these cases are shown in Table 4.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Bubbles</th>
<th>Initial Pressure kPa</th>
<th>Initial Temperature K</th>
<th>Pressure at infinity kPa</th>
<th>Initial Radius of the bubble m</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Vapour</td>
<td>20</td>
<td>333.0</td>
<td>10</td>
<td>0.02</td>
</tr>
<tr>
<td>B</td>
<td>Vapour</td>
<td>200</td>
<td>393.0</td>
<td>25</td>
<td>0.02</td>
</tr>
<tr>
<td>C</td>
<td>Vapour</td>
<td>20</td>
<td>333.0</td>
<td>5</td>
<td>0.02</td>
</tr>
<tr>
<td>D</td>
<td>Gas</td>
<td>20</td>
<td>333.0</td>
<td>5</td>
<td>0.02</td>
</tr>
<tr>
<td>E</td>
<td>Gas</td>
<td>300</td>
<td>573.0</td>
<td>25</td>
<td>0.02</td>
</tr>
<tr>
<td>F</td>
<td>Gas</td>
<td>250</td>
<td>573.0</td>
<td>25</td>
<td>0.02</td>
</tr>
</tbody>
</table>

**Table 4.1** Initial conditions for different cases of vapour and gas bubbles

The corresponding polytropic indices are shown in Table 4.2.
Table 4.2 Polytropic indices for bubbles of Table 4.1

As an example a $P-V$ diagram, plotted on log-log axes, is shown in Figure 4.13 for the expansion phase of the gas bubble of Case F.
It is evident that this process is not polytropic as it deviates significantly from the best fitted curve with a polytropic index of 1.17. Figure 4.14 shows the same graph for the collapse phase of the same bubble.

![Graph showing comparison between results of the program and the best fitted curve of slope 1.21 for the collapse phase of the gas bubble in Case F]

**Figure 4.14** Comparison between the results of the program and the best fitted curve of slope 1.21 for the collapse phase of the gas bubble in Case F

This figure shows that the collapse phase of the bubble agrees very well with a polytropic process of index 1.21. These characteristics of deviation from a polytropic process for expansion and good agreement with polytropic process for the collapse phase were consistently observed in all results (see Table 4.2).

A general conclusion that may be drawn from the above argument is that the assumption of a polytropic process inside the bubble is unreliable and may result in inadequate prediction of the bubble pressure.

During the bubble evolution, there is an exchange of work between the bubble and the surrounding liquid. Figure 4.15 represents a $P-V$ diagram for one complete
cycle of the bubble evolution. Initial conditions for this Figure are specified as Case E in Table 4.1. It is evident that the path of the thermodynamic process for the expanding gas bubble is different from that of the collapsing gas bubble. This difference in process paths for expansion and collapse shows a net output of work and heat transfer from the bubble.

![Figure 4.15 P-V Diagram of a gas bubble for one cycle of pulsations](image)

**Figure 4.15** $P-V$ Diagram of a gas bubble for one cycle of pulsations

As mentioned in Section 4.2, for the vapour bubble it is assumed that the temperature inside the bubble is uniform, but for the case of the gas bubble the temperature inside the bubble is allowed to change radially. Figure 4.16 shows the
radial distribution of temperature in the gas bubble oscillating as a consequence of a sudden pressure decrease in the liquid domain. The existence of a thin thermal boundary layer, compared with the bubble radius, adjacent to the bubble wall can be inferred from this figure. It can also be concluded that the gas inside the bubble has a fairly uniform temperature over a great portion of the bubble radius. This could be explained by the low thermal diffusivity of the non-condensable gas.

Figure 4.16 Radial distribution of temperature inside the bubble

4.5.3. Results For a Bubble filled with a mixture of Vapour and a Non-condensable Gas

This section is concerned with some results for the bubble when its content is a mixture of non-condensable gas and vapour.
Figure 4.17 depicts the time history of the bubble volume and the change in pressure inside the bubble with respect to time is illustrated in Figure 4.18. Initial conditions for the three chosen cases are the same except for the initial pressures which are different in each case as mentioned in the legend on the figures. Bubbles begin to pulsate with an initial radius of 0.02 m, initial temperature of 65°C and pressure and temperature at infinity are 10 kPa and 20°C, respectively.

Figure 4.17 Change in volume for bubbles containing a mixture of vapour and non-condensible gas
Figure 4.18 Change in pressure for bubbles containing a mixture of vapour and non-condensable gas

A summary of information extracted from these figures is given in Table 4.3.

<table>
<thead>
<tr>
<th></th>
<th>$P_0$=40 kPa</th>
<th>$P_0$=50 kPa</th>
<th>$P_0$=60 kPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{\text{max}}/R_0$</td>
<td>1.99</td>
<td>2.22</td>
<td>2.42</td>
</tr>
<tr>
<td>$R_{\text{min}}/R_0$</td>
<td>0.959</td>
<td>1.024</td>
<td>1.080</td>
</tr>
<tr>
<td>$P_{\text{min}}/P_0$</td>
<td>0.090</td>
<td>0.066</td>
<td>0.052</td>
</tr>
<tr>
<td>$T_{\text{min}}/T_0$</td>
<td>0.269</td>
<td>0.185</td>
<td>0.122</td>
</tr>
</tbody>
</table>

Table 4.3 Radius, pressure and temperature of three bubbles at the first maximum volume and radius of the bubble at the end of first collapse

According to figures 4.17, 4.18 and Table 4.3, as the initial pressure inside the bubble is increased the maximum radius to which the bubble grows will be
increased. Lower minimum pressure and minimum temperature are a consequence of the higher maximum radius which can be seen in Table 4.3.

In some instances of bubble evolution the temperature in the bubble falls below the surrounding liquid temperature. In this case heat is transferred from the liquid to the bubble contents. Since bubbles having higher initial pressure expand more rapidly (see Figure 4.19), heat transfer cannot keep up with the expansion (compared to bubbles with a lower initial pressure). Hence, minimum temperature for bubbles with a higher initial pressure will be lower (see last row of Table 4.3).

![Figure 4.19 Change of bubble wall velocity with respect to the bubble radius](image)

Figure 4.19 Change of bubble wall velocity with respect to the bubble radius
Figure 4.19 shows the change in bubble wall velocity vs radius of the bubble. This figure reveals that the absolute value of the bubble wall velocity is at its maximum when the radius of the bubble is about its minimum value either in the collapse phase or in the expansion phase. It should be noted that the upper half of this figure shows the expansion phase and the lower half shows the collapse phase. Also the figure shows that for a given radius the velocity of the bubble wall is higher for a bubble with a higher initial pressure.

Two kinds of cavity bubbles are defined in the literature. The first one is a stable cavity which refers to a cavity bubble pulsating around an equilibrium radius and continuing to oscillate for many cycles. The second one is a transient cavity. The life time of the transient cavity is generally less than one cycle. Transient cavities expand to at least double their original radius and then they collapse violently or disintegrate into a mass of smaller bubbles (Flynn 1975 and 1976).

By these definitions bubbles under consideration in this work are stable cavities. This fact is shown in figures 4.20, 4.21 and 4.22. These figures depict the time histories of pressure, volume and mass of the vapour inside the bubbles of the previous example with an initial pressure of 50 kPa. Calculations were repeated for the bubbles with initial pressures of 40 kPa and 60 kPa and the initial and final conditions of the bubbles are summarised in Table 4.4. All three figures show that the volume, pressure and mass of the vapour in the bubble approach equilibrium values after a relatively long time.
Figure 4.20 Change in volume for a stable cavity over a long period of time
Figure 4.21 Change in pressure for a stable cavity over a long period of time.
Figure 4.22 Change in vapour mass inside the bubble for a stable cavity.
Dynamic Behaviour of an Isolated Bubble

Chapter Four

It should be noted that, unlike the previous examples, the pressure and temperature inside the bubbles are non-dimensionalised by the pressure and temperature at infinity, respectively. The mass of the vapour is non-dimensionalised by the initial mass of vapour inside the bubble.

<table>
<thead>
<tr>
<th></th>
<th>$P_0=40\text{ kPa}$</th>
<th>$P_0=50\text{ kPa}$</th>
<th>$P_0=60\text{ kPa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial state</td>
<td>Final state</td>
<td>Initial state</td>
</tr>
<tr>
<td>Partial pressure of gas</td>
<td>1.5</td>
<td>0.74</td>
<td>2.5</td>
</tr>
<tr>
<td>Partial pressure of vapour</td>
<td>2.5</td>
<td>0.26</td>
<td>2.5</td>
</tr>
<tr>
<td>Total pressure</td>
<td>4.0</td>
<td>1.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Pressure at infinity</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Temperature inside the bubble</td>
<td>3.25</td>
<td>1.04</td>
<td>3.25</td>
</tr>
<tr>
<td>Temperature at infinity</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Bubble radius</td>
<td>1.0</td>
<td>1.21</td>
<td>1.0</td>
</tr>
<tr>
<td>Vapour mass</td>
<td>1.0</td>
<td>0.19</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 4.4 Initial and final conditions of the three bubbles in the previous example

Table 4.4 shows that at the final state (at equilibrium) when the pulsations are almost damped out the following statements can be made:
• the partial pressures of the vapour are almost the same for all bubbles;

• the partial pressures of the non-condensable gas are almost the same for all bubbles;

• the pressure and temperature inside the bubble are equal to the pressure and temperature of the surrounding liquid at infinity;

• bubbles having a higher initial pressure will gain a larger equilibrium radius; and

• the mass of the vapour will be higher for bubbles having higher initial pressure.

4.6. CONCLUDING REMARKS TO CHAPTER FOUR

In this chapter the dynamics of an isolated bubble was studied by considering heat transfer between the bubble interior and the surrounding liquid. Derivation of formulas was based on the first law of thermodynamics and the assumption of a potential flow field in the surrounding liquid.

Three different sets of partial differential equations were obtained for the investigations of bubble evolution as follows:

a) when the bubble contents consist of vapour (vapour bubble);

b) when the bubble contents consist of non-condensable gas (ideal gas bubble); and

c) when the bubble contents consist of a mixture of vapour and non-condensable gas.
For vapour bubbles and bubbles filled with a mixture of vapour and a non-condensable gas the temperature and pressure inside the bubble were assumed uniform but for gas bubble the temperature of the bubble content was allowed to change radially.

The derived formulas enable the calculation of parameters relevant to bubble dynamics such as pressure, temperature, volume, bubble wall velocity and change of vapour mass in the bubble. Also the formulations enable one to follow the thermodynamic process which occurs in the bubble.

Thermodynamic process paths were studied over one complete cycle of bubble evolution and it was shown that the process paths are different for the expansion phase and the collapse phase.

The polytropic index for the thermodynamic process inside the bubble was investigated and it was revealed that while the expansion phase deviates significantly from a polytropic process the collapse phase follows it well (although the polytropic indices are different for different initial conditions).

The time history of the evolution of a stable cavity was studied and the equilibrium states of stable cavities with different initial conditions were compared.

The next chapter is devoted to the calculation of the term $\dot{Q}$ used in the sets of differential equations 4.17, 4.30 and 4.42.
CHAPTER FIVE

DUAL RECIPROCITY BOUNDARY ELEMENT METHOD

5.1. INTRODUCTION

This chapter is concerned with the description and implementation of a special type of boundary element method, the Dual Reciprocity Method (DRM), to bubble dynamics. This method is adopted for the calculation of the temperature distribution in an infinite domain with moving boundary.

The subject of numerical analysis is to provide efficient approximate solutions to physical problems expressed in mathematical form. The efficiency of the method depends on both its accuracy and the ease with which it can be implemented.

Although the Boundary Element Method (BEM) has been established after other popular numerical methods such as finite element methods and finite difference methods, the success of this method, as a generalised numerical technique, has been widely recognised by engineers and scientists due to its easy implementation and accuracy.

The main advantage of the BEM is its ability to provide a complete solution of the problem in terms of boundary values. This feature of the BEM reduces computation time and data preparation effort to a large extent. A comparison
between computation times required by the finite element method and the BEM for a typical engineering problem was given by Fenner (1983). Substantial reduction in computation time and time for preparation of input data for the BEM was achieved in comparison with those of the finite element method.

Although the BEM has been known as a boundary only formulation, it has some restrictions in converting some of the differential equations into boundary integral forms. In such cases domain integral terms will appear in the boundary integral equation. These restrictions arise in different situations such as:

- when a fundamental solution (free space Green's function) to the original partial differential equation is not known; and

- when non-homogeneous terms accounting for the effect of initial condition in transient problems or internal sources are present.

These restrictions make the BEM lose its "boundary-only" character. Different approaches have been developed to overcome these restrictions. For a quick review of these methods one can refer to works by Azvedo and Brebbia (1988), Nardini and Brebbia (1982), Loeffler and Mansur (1988) and a book edited by Brebbia (1992).

Among the proposed methods for dealing with domain integrals the DRM which was first introduced by Nardini and Brebbia (1982) has been used extensively. This method is a generalised way of constructing particular solutions that can be used to solve those problems whose integral equations contain domain integrals.

The physical problem under consideration, mathematical modelling and implementation of the DRM to this problem are the subjects of this chapter.
5.2. STATEMENT OF THE PROBLEM

In order to find the heat exchange between a bubble and its surrounding liquid, one has to find the temperature distribution in the liquid domain. To find the temperature distribution, the energy equation should be solved in the liquid domain. This equation has the following form:

\[ \nabla^2 T = \frac{1}{\alpha} \frac{DT}{Dt}, \quad (5.1) \]

in which \( T \) is the temperature, \( \alpha \) is the thermal diffusivity of the liquid and \( \frac{D}{Dt} \) is the total derivative having the definition:

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla, \quad (5.2) \]

where \( \vec{u} \) is the velocity vector. If the motion of the bubble is assumed to be axisymmetric Equation 5.1 will take the following form:

\[ \nabla^2 T = \frac{1}{\alpha} \left( \frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z} \right), \quad (5.3) \]

where \( u_r \) and \( u_z \) are respective components of the velocity vector in the \( r \) and \( z \) directions.

Boundary conditions relevant to Equation 5.3 could be in the form of

- essential boundary condition of the type \( T = \bar{T} \) at \( r=R \); and
- natural boundary condition of the type \( \frac{\partial T}{\partial n} = q \) at \( r=R \);

where \( R \) is the radius of the bubble and \( \bar{T} \) and \( q \) are specified temperature and heat flux at the bubble interface, respectively.
The next section is devoted to introducing a solution method for Equation 5.1 and its relevant boundary conditions.

5.3. METHOD OF SOLUTION

Due to the reasons mentioned in Chapter 3 and the introduction of this chapter about the capabilities of the BEM in dealing with

- problems with moving boundaries; and
- problems with infinite domains;

for the computation of the temperature distribution in the surrounding liquid outside a bubble, a special type of BEM called the DRM has been adopted in this research.

Mathematical modelling of the DRM is presented in the next section.

5.3.1. Integral Equation form of the Energy Equation

Considering Equation 5.1, its boundary integral equation has the following form:

\[ C_i T_i + \int_r T \frac{\partial \Psi}{\partial n} d\Gamma - \int_r \Psi \frac{\partial T}{\partial n} d\Gamma = \int_\Omega \frac{1}{\alpha} \frac{DT}{Dt} \Psi d\Omega, \]  \hfill (5.4)

where \( i \) is any point in the domain or on the boundary, \( T \) is the temperature, \( n \) is the outward normal to the boundary, \( \Gamma \) represents the boundary, \( \Omega \) represents the domain, \( \Psi \) is the fundamental solution (or free space Green's function) to Laplace's equation and \( C_i \) is a constant which depends on the location of the point \( i \). If \( i \) is inside the domain then \( C_i = 1 \) and if it is on the boundary the value of \( C_i \) is a function of the solid angle at this point.
The fundamental solution $\Psi$ represents the effect of a concentrated unit source, released at point $i$, on an unbounded domain. For example the fundamental solution for the Laplacian operator has the property of:

$$\nabla^2 \Psi + \Delta_i = 0,$$ \hspace{1cm} (5.5)

where $\Delta_i$ is a Dirac delta function with the value of infinity at point $i$ and zero elsewhere.

To convert the boundary integral Equation 5.4 into a system of algebraic equations, it should be written in discretised form. If the fundamental solution for a differential equation is known then the boundary integral equation consists of only boundary terms otherwise the boundary integral equation will include some domain integral terms as well. Looking at Equation 5.4 it is seen that the right hand side of this equation is an integral over the whole domain. If the system of algebraic equations is going to be formed by the discretised form of this equation then the whole domain, $\Omega$, has to be discretised and integration over the whole domain should be performed. This integration on the domain makes the BEM lose its advantage over domain techniques.

In order to avoid this difficulty it is desirable to modify the BEM in such a way that it:

- enables the BEM to deal with problems whose fundamental solutions are not known; and

- can be applied to different problems in a similar way.

There are several techniques aimed to satisfy the above conditions. Some of these techniques are:
a) the analytical solution of domain integrals;

b) the use of Fourier expansions;

c) the multiple reciprocity method; and

d) the DRM.

The first three methods are out of the scope of this work and only the fourth one, which satisfies the two conditions mentioned above and also is the easiest of the four methods to use in practise, will be described in detail.

5.3.2. Mathematical Modelling of the DRM

For the sake of simplicity the mathematical basis of the method is described by considering the Poisson equation as follows:

$$\nabla^2 T = b,$$

(5.6)

where $b$ is a function of space only ($b = f(r, z)$). The boundary integral equation of this equation has the form:

$$C_i T_i + \int_r T \frac{\partial \Psi}{\partial n} d\Gamma - \int_r \Psi \frac{\partial T}{\partial n} d\Gamma = \int b \Psi d\Omega.$$  

(5.7)

The main objective of the DRM is to convert the term on the right hand side of this equation to a boundary integral. If a function like $\overline{T}$ can be found such that:

$$\nabla^2 \overline{T} = b,$$

(5.8)

then $\overline{T}$ is called a particular solution of Equation 5.6 and the solution for 5.6 consists of the summation of this particular solution and the general solution of the homogeneous Laplace's equation.

94
It is difficult or in most cases impossible to have a single function of $T$ to satisfy Equation 5.8. The DRM proposes the use of a series of particular solutions $T_j$ instead of a single function $T$. Suppose that the boundary is discretised into $M$ boundary elements and $K$ internal nodes are selected inside the domain or on the boundary (see Figure 5.1) then the function $b$ can be approximated by:

$$b = \sum_{j=1}^{M+K} \gamma_j f_j.$$  \hspace{1cm} (5.9)

It is customary in the literature to designate the boundary nodes as "points" and internal nodes as "poles". Also those poles located on the boundary are usually chosen to be the same as points.

![Figure 5.1 Discretisation of the boundary and distribution of internal nodes](image)

In Equation 5.9 $\gamma_j$ are constants and initially unknown and functions $f_j$ are called interpolation functions. Types of $f_j$ functions and calculation of $\gamma_j$ will be discussed later in this chapter. These interpolation functions can be compared with
shape functions, \( \Phi_i \), used on the boundary (in the BEM) or inside the domain (in finite element methods) to approximate the field variable in terms of the values of the field variable at nodal points as:

\[
T = \sum_i \Phi_i T_i. \quad (5.10)
\]

This equation is valid over each element in the finite element method or in the BEM.

In the same way if the whole domain of Figure 5.1 is considered as a super element then Equation 5.9 may be considered to be valid over the whole domain.

The functions \( f_j \) and the particular solution \( \overline{T_j} \) have the following relation:

\[
\nabla^2 \overline{T_j} = f_j. \quad (5.11)
\]

If \( f_j \) in Equation 5.11 is substituted into Equation 5.9 then:

\[
b = \sum_{j=1}^{M+K} \gamma_j (\nabla^2 \overline{T_j}). \quad (5.12)
\]

Using this equation the right hand side of Equation 5.7, the domain integral term, can be written as:

\[
\int_\Omega b \Psi d\Omega = \sum_{j=1}^{M+K} \int_\Omega \gamma_j (\nabla^2 \overline{T_j}) \Psi d\Omega. \quad (5.13)
\]

Since \( \gamma_j \) are constants, Equation 5.13 will become:

\[
\int_\Omega b \Psi d\Omega = \sum_{j=1}^{M+K} \gamma_j \int_\Omega (\nabla^2 \overline{T_j}) \Psi d\Omega. \quad (5.14)
\]

Following the usual procedure of the boundary integral formulation, Equation 5.14 is converted into a boundary integral equation having the form of:
\[
\sum_{j=1}^{M+K} \gamma_j \int_{\Omega} (\nabla T_j) \Psi \, d\Omega = \sum_{j=1}^{M+K} \gamma_j \left( C_{ij} T_j + \int_{\Gamma} \frac{\partial \Psi}{\partial n} \, d\Gamma - \int_{\Gamma} \Psi \frac{\partial T_j}{\partial n} \, d\Gamma \right).
\] (5.15)

Substitution of Equation 5.7 into Equation 5.15 will gives:

\[
C_i T_i + \int_{\Gamma} \frac{\partial \Psi}{\partial n} \, d\Gamma - \int_{\Gamma} \Psi \frac{\partial T_i}{\partial n} \, d\Gamma = \sum_{j=1}^{M+K} \gamma_j \left( C_{ij} T_j + \int_{\Gamma} \frac{\partial \Psi}{\partial n} \, d\Gamma - \int_{\Gamma} \Psi \frac{\partial T_j}{\partial n} \, d\Gamma \right).
\] (5.16)

There is no domain integral term in this equation and the domain integral term in Equation 5.7 is substituted by equivalent boundary integral terms.

This equation is the basis of the DRM. The next step, to convert this equation into a set of algebraic equations, is to write this equation in discretised form. This is the subject of the next section.

5.3.3. Discretised Form of the Mathematical Model

To write Equation 5.16 in discretised form, the boundary should be discretised into small elements and integrals on the boundary should be written as the sum of integrals over each of these small elements. If the boundary, \( \Gamma \), is divided into \( M \) boundary elements and \( \xi \) is a point which moves on each of these elements then Equation 5.16 can be written:

\[
C_i T_i + \sum_{i=1}^{M} \int_{\Gamma_i} \frac{\partial \Psi(i, \xi)}{\partial n} \, d\Gamma(i, \xi) = \sum_{i=1}^{M} \int_{\Gamma_i} \Psi(i, \xi) \frac{\partial T(i, \xi)}{\partial n} \, d\Gamma(i, \xi)
\]

\[
C_i T_i + \sum_{i=1}^{M} \int_{\Gamma_i} \frac{\partial \Psi(i, \xi)}{\partial n} \, d\Gamma(i, \xi) = \sum_{i=1}^{M} \gamma_i \left( C_{ij} T_j + \int_{\Gamma} \frac{\partial \Psi(i, \xi)}{\partial n} \, d\Gamma(i, \xi) - \int_{\Gamma} \Psi(i, \xi) \frac{\partial T(j, \xi)}{\partial n} \, d\Gamma(i, \xi) \right),
\] (5.17)

or in a simpler form:

\[
C_i T_i + \sum_{i=1}^{M} H_{ij} T_j - \sum_{i=1}^{M} G_{ij} q_j = \sum_{i=1}^{M+K} \gamma_i \left( C_{ij} T_j + \sum_{i=1}^{M} H_{ij} T_j - \sum_{i=1}^{M} G_{ij} q_j \right),
\] (5.18)
in which \( q = \frac{\partial T}{\partial n} \) and \( \bar{q} = \frac{\partial \bar{T}}{\partial n} \),

and also:

\[
H_u = \int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} \right) d\Gamma \quad \text{and} \quad G_u = \int_{\Gamma} \Psi d\Gamma.
\]

The matrix form of Equation 5.18 is:

\[
HT - GQ = (HT - G\bar{Q})\gamma, \quad (5.19)
\]

where \( T, Q \) and \( \gamma \) are vectors whose members are values of \( T_i, q_i \) and \( \gamma_j \), respectively. \( \bar{T} \) and \( \bar{Q} \) are square matrices with respective entries of \( \bar{T}_{ij} \) and \( \bar{Q}_{ij} \).

In Equation 5.19 the \( C_i \) terms are incorporated onto the principal diagonal of the matrix \( H \).

For clarity Equation 5.19 is represented in schematic form in Figure 5.2
For a better understanding of the DRM, several features of the matrix Equation 5.19 are now described regarding Figure 5.2.

1) The matrices $\mathbf{T}$ and $\mathbf{Q}$ relate the influences of points to poles, as can be seen from Equation 5.17. It should be remembered that poles can also be located on the boundary;

2) those partitions marked $bb$ in $\mathbf{H}$ and $\mathbf{G}$ show the influence of the boundary points on the boundary points, and those marked $bb$ in $\mathbf{T}$ and $\mathbf{Q}$ account for the influence of points on those poles located on the boundary;

3) partitions marked by $bi$ show the influence on boundary by internal nodes;

4) partitions marked by $0$: 
• in $H$ and $G$ are just the augmentation of the matrices to conform the operations;

• in $Q$ they account for normal fluxes at internal poles which is meaningless;

5) partitions marked by $I$ (identity matrix) account for the self-influence of the points;

6) partitions marked by $ii$ account for the influence of the internal-to-internal poles.

7) partitions marked by $b$ in matrices $T$ and $Q$ account for values of temperature and heat flux on the boundary, respectively. The partition marked by $i$ in matrix $T$ shows the values of temperature at internal points.

5.4. CALCULATION OF $\gamma_j$

The basis of the DRM is to approximate the non-homogeneous term in differential equations by Equation 5.9 which is rewritten here:

$$b \equiv \sum_{j=1}^{M+K} \gamma_j f_j.$$  \hfill (5.9)

For the sake of simplicity this equation is written in matrix form as follows:

$$b = F\gamma,$$ \hfill (5.20)

where $F$ is a square matrix of dimensions $(M+K) \times (M+K)$ and each column of this matrix consists of a vector $f_j$ containing the values of $f_j$ at all internal and boundary points and vector $b$ containing the values of $b$ at all nodal points.

Once $b$ is a known function, the matrix $\gamma$ can be found from the equation:

$$\gamma = F^{-1}b,$$ \hfill (5.21)
Those problems in which \( b \) is a function of the field variable or time will be discussed in the following sections.

5.5 PROBLEMS IN WHICH \( b = b(r, z, T, t) \)

In the previous sections \( b \) was assumed to be a function of space only. In this case because \( b \) is a known function, the vector \( \gamma \) is simply determined from equation 5.21. For this case the final matrix equation resulting from the discretised form of the boundary integral equation is:

\[
HT - GQ = (HT - GQ)F^{-1}b, \tag{5.22}
\]

or

\[
HT - GQ = Sb, \tag{5.23}
\]

in which \( S = (HT - GQ)F^{-1} \).

Applying the boundary conditions to Equation 5.23, a system of linear algebraic equations will be obtained in the form of:

\[
AX = B, \tag{5.24}
\]

in which \( X \) is a vector containing the following three sets of unknowns:

a) values of temperature at internal points;

b) values of temperature for that part of the boundary on which heat flux is specified as the boundary condition; and

c) values of heat flux for that part of the boundary on which temperature is specified as the boundary condition.
$B$ is a vector that contains:

- the values of $Sb + GQ$ on that part of the boundary for which the heat flux is specified by the boundary conditions; and

- the values of $HT - Sb$ on that part of the boundary for which the temperature is specified by the boundary conditions.

In the majority of engineering and scientific problems $b$ is a function of time, field variable or derivatives of the field variable. For example considering Equation 5.4 $b$ has the following definition:

$$b = \frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z}, \quad (5.25)$$

it should be noted that in this equation $u_r$ and $u_z$ are constant. If a linear variation is proposed for $T$ within a time increment of $\Delta t$, the time derivative can be approximated by:

$$\frac{\partial T}{\partial t} = \frac{T - T_0}{\Delta t}, \quad (5.26)$$

in which $T_0$ is the initial value of the temperature. Substitution of 5.26 in 5.25 gives:

$$b = \frac{T - T_0}{\Delta t} + u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z}. \quad (5.27)$$

It is seen that in this case $b$ consists of three parts as follows:

- a constant part, $-T_0/\Delta t$, accounting for the effect of the initial condition;

- a function of temperature, $T/\Delta t$; and
• derivatives with respect to space coordinates, \( u_r \frac{\partial T}{\partial r} \) and \( u_z \frac{\partial T}{\partial z} \), accounting for the convective terms in the total derivative.

When \( b \) is not a constant or a specified function of a space variable only, the calculation of the \( \gamma \) vector is not a straightforward task. These cases need special treatment which is the subject of the following sections.

5.5.1. The Case of \( b = T \)

In this case \( b \) is a function of the field variable. Treatment of this case is described with reference to the following equation:

\[
\nabla^2 T = T. \tag{5.28}
\]

In this case \( b = T \) and Equation 5.22 takes on the following form:

\[
HT - GQ = (HT - GQ)F^{-1}T, \tag{5.29}
\]

in which \( T \) is a vector whose members are the values of temperature at the boundary and internal points. Defining the matrix \( S \) as before, Equation 5.27 takes the form of:

\[
(H - S)T = GQ. \tag{5.30}
\]

Applying the boundary conditions, the system of linear algebraic equations 5.24 will be obtained. In this set of equations \( X \) has the same entries as in 5.24 and \( B \) is a vector that contains:

• the values of \( GQ \) on that part of the boundary for which the heat flux is specified by the boundary condition; and
• the values of \((H - S)T\) on that part of the boundary for which the temperature is specified by the boundary condition.

By solving the set of algebraic equations, unknown values on the boundary and inside the domain will be obtained simultaneously.

5.5.2. The Case of \(b = \partial T/\partial r\) or \(b = \partial T/\partial z\)

One of the main objectives in using shape functions in the finite element method or in the boundary element method is to transfer the mathematical operations such as derivatives from the field variable to the shape functions. In other words, considering Equation 5.10 one can write:

\[
\frac{\partial T}{\partial r} = \sum_i \frac{\partial \Phi_i}{\partial r} T_i \quad \text{or} \quad \frac{\partial T}{\partial z} = \sum_i \frac{\partial \Phi_i}{\partial z} T_i. \tag{5.31}
\]

For the treatment of derivatives of the field variable in the DRM the same approach is adopted. For example consider the governing equation to be:

\[
\nabla^2 T = \frac{\partial T}{\partial r}, \tag{5.32}
\]

in this case \(b = \frac{\partial T}{\partial r}\) and matrix Equation 5.22 will take the form of

\[
HT - GQ = (HT - G\bar{Q})F^{-1} \frac{\partial T}{\partial r}. \tag{5.33}
\]

If \(T\) is approximated by:

\[
T = F\gamma, \tag{5.34}
\]

then differentiating with respect to \(r\) gives:

\[
\frac{\partial T}{\partial r} = \frac{\partial F}{\partial r} \gamma. \tag{5.35}
\]
Substitution of $\gamma$ from matrix Equation 5.34 in Equation 5.35 gives:

$$\frac{dT}{dr} = \frac{\partial F}{\partial r} F^{-1} T.$$  \hspace{1cm} (5.36)

Combining this equation with Equation 5.33 results in:

$$HT - GQ = (HT - GQ)F^{-1} \frac{\partial F}{\partial r} F^{-1} T,$$  \hspace{1cm} (5.37)

and this equation can be written as:

$$HT - GQ = RT,$$  \hspace{1cm} (5.38)

in which $R = (HT - GQ)F^{-1} \frac{\partial F}{\partial r} F^{-1}$. The derivative of $T$ with respect to $z$ can be treated in the same way. After applying the boundary condition to Equation 5.38 the corresponding set of algebraic equations will be obtained.

5.5.3. The Case of $b = \frac{1}{\alpha} \frac{DT}{Dt}$

Definition of $b$ in this case is given by Equation 5.27. According to Equations 5.23, 5.29 and 5.38 for this case the matrix equation has the following form:

$$HT - GQ = \frac{1}{\alpha} \left\{ \frac{1}{\Delta t} S + u_x R_1 + u_z R_2 \right\} T - \frac{1}{\Delta t} ST_0,$$  \hspace{1cm} (5.39)

where $S$ has the same definition as before and:

- $T_0$ is a vector whose entries are the initial values of the temperature;
- $R_1 = (HT - GQ)F^{-1} \frac{\partial F}{\partial r} F^{-1}$; and
- $R_2 = (HT - GQ)F^{-1} \frac{\partial F}{\partial z} F^{-1}$. 

105
\( \mathbf{u}_r \) and \( \mathbf{u}_z \) are two diagonal matrices whose principal diagonals contain the \( r \) and \( z \) components of the velocity at the nodal points, respectively.

It is seen that all values between curly brackets in Equation 5.39 are known and this equation can be written in the simpler form:

\[
H \mathbf{T} - G \mathbf{Q} = Y \mathbf{T} + \mathbf{D}, \tag{5.40}
\]

in which

\[
Y = \frac{1}{\alpha} \left\{ \frac{1}{\Delta t} S + \mathbf{u}_r R_r + \mathbf{u}_z R_z \right\},
\]

and

\[
\mathbf{D} = -\frac{1}{\Delta t} S \mathbf{T}_0.
\]

The boundary conditions are applied to Equation 5.40 to form the system of algebraic Equations 5.24.

In this set of equations the vector \( \mathbf{X} \) has its usual definition and the vector \( \mathbf{B} \) contains:

- the values of \((H - Y)\mathbf{T} - \mathbf{D}\) on that part of the boundary for which the temperature is specified as the boundary condition; and

- the values of \(G \mathbf{Q} + \mathbf{D}\) on that part of the boundary for which the heat flux is specified as the boundary condition.
5.6. DIFFERENT FORMS OF $f_j$ FUNCTION

The form of $f_j$ is not limited mathematically except that the matrix $F$ should be non-singular. Nardini and Brebbia (1982), the originators of the method, proposed the following forms of the function $f_j$ for a bounded domain in a cartesian coordinate system:

- elements of Pascal triangle;
- trigonometric series; and
- the distance function $r_{ij}$, Euclidean distance between nodal points $i$ and $j$, used in the definition of the fundamental solution.

The third form has been adopted by many researchers as it is the simplest and the most accurate.

Once the definition of $f_j$ is known, $\bar{T}$ can be found from Equation 5.11 and $\bar{q}$ is determined from:

$$\bar{q}_j = \frac{\partial \bar{T}_j}{\partial n} = \frac{\partial \bar{T}_j}{\partial r} \frac{\partial r}{\partial n} + \frac{\partial \bar{T}_j}{\partial z} \frac{\partial z}{\partial n}. \quad (5.41)$$

The polynomial form of the distance function $r_{ij}$ has been used extensively in many recent works with satisfactory results. The polynomial form of $r_{ij}$ can be written as:

$$f_j = F_{ij} = 1 + r_{ij} + r_{ij}^2 + ... + r_{ij}^m. \quad (5.42)$$

The $\bar{T}$ and $\bar{q}$ corresponding to 5.42 will be:

$$\bar{T} = \frac{r_{ij}^2}{4} + \frac{r_{ij}^3}{9} + \frac{r_{ij}^4}{16} + ... + \frac{r_{ij}^{m+2}}{(m+2)!} \quad (5.43)$$

107
and

\[ \bar{q} = \left( r_r \frac{\partial r}{\partial n} + r_z \frac{\partial z}{\partial n} \right) \left( \frac{1}{2} + \frac{r_{ij}^2}{3} + \frac{r_{ij}^4}{4} + \ldots + \frac{r_{ij}^{2m}}{m+2} \right). \]  

(5.44)

where \( r_r \) and \( r_z \) are components of \( r_{ij} \) in the \( r \) and \( z \) directions, respectively.

Partridge and Brebbia (1990) showed that the use of \( m=1 \) in Equation 5.42 is sufficient and gives accurate results.

5.6.1. Definition of \( f_j \) for a Bounded Axisymmetric Domain

The fundamental solution for Laplace's equation in an axisymmetric domain, which is determined from the fundamental solution in a three dimensional domain, has the form of:

\[ \psi = \frac{4K(m)}{(a + b)^{1/2}}, \]  

(5.45)

in which \( K \) is the complete elliptic integral of the first kind and:

\[ a = r_i^2 + r_j^2 + (z_i - z_j)^2, \]

\[ b = 2r_i r_j, \]

\[ m = \frac{2b}{a + b}, \]

where \( r_i \) and \( r_j \) are distances from \( i^{th} \) and \( j^{th} \) nodes to the axis of symmetry, respectively. It is seen that in this case not only does the fundamental solution depend on the Euclidean distance between nodal points \( i \) and \( j \) but also on the distances from these nodes to the axis of symmetry. Considering this fact Wroble and Telles (1986) proposed the following definition for \( f_j \):
Using this definition and the definition of the Laplacian operator for the axisymmetric domain as:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},
\]

the corresponding expressions for \( \bar{T} \) and \( \bar{q} \) will be:

\[
\bar{T} = \frac{r_q^3}{12},
\]

and

\[
\bar{q} = \frac{r_q^2}{4} \frac{\partial r_q}{\partial n}.
\]

They applied their definition of \( f_j \) to solve two simple unsteady heat conduction problems. Their results were in good agreement with those of the analytic solution and also results obtained by the finite element method.

5.6.2. Definition of \( f_j \) for Unbounded Two Dimensional Domains

As mentioned before, one of the most attractive features of the BEM its ability to deal with exterior problems. The application of Equation 5.17 or the boundary integral equation of Laplace's equation which is:

\[
C_i T_i + \int_{\Gamma} T \frac{\partial \Psi}{\partial n} d\Gamma - \int_{\Gamma} \Psi \frac{\partial T}{\partial n} d\Gamma = 0,
\]

(5.50)
to infinite regions is not valid without further hypotheses on the functions involved. Such hypotheses are concerned with the behaviour of the functions on an infinitely distant surface and are referred to as regularity conditions.

Davi and Iannelli (1984) investigated the application of the BEM to exterior potential problems. In their study they approximated the exterior domain with a multiply-connected internal problem while the outer boundary approaches to infinity. Figure 5.3 shows the authors' proposed domain.

![Figure 5.3 External domain with an imaginary boundary at infinity](image)

With this approximation each integral in Equation 5.50 was written as the summation of two integrals on the inner boundary and on the outer boundary i.e.

\[
\int_{\Gamma} T \frac{\partial \Psi}{\partial n} d\Gamma = \int_{\Gamma_1} T \frac{\partial \Psi}{\partial n} d\Gamma + \int_{\Gamma_{\infty}} T \frac{\partial \Psi}{\partial n} d\Gamma,
\]

(5.51)

and
\[
\int_{\Gamma} \psi \frac{\partial T}{\partial n} \, d\Gamma = \int_{\Gamma_{+}} \psi \frac{\partial T}{\partial n} \, d\Gamma + \int_{\Gamma_{-}} \psi \frac{\partial T}{\partial n} \, d\Gamma. \tag{5.52}
\]

They proved that the integral terms on \( \Gamma_{-} \) in equations 5.51 and 5.52 approach zero. This condition, the tendency of integrals on \( \Gamma_{-} \) to approach zero, is called the regularity condition. Therefore Equation 5.50 can be applied to exterior problems without further modification. This condition restricts forms of \( f_{j} \) applicable to exterior problems when using the DRM.

Loeffler and Mansur (1988) proposed the following form of \( f_{j} \) for a two-dimensional infinite domain:

\[
f_{j} = \frac{2C - r_{ij}}{(r_{ij} + C)^{q}}, \tag{5.53}
\]

in which \( C \) is a constant for which a lower bound was defined by the authors. In order to compare the performance of the proposed definition of \( f_{j} \), the problem of transient behaviour of temperature in an infinite domain with an internal cavity was considered. Their results showed good agreement with the analytic solution.

Another attempt was made by Power and Partridge (1992) to solve the time-dependent Stocks flow for an exterior domain by the DRM. They proposed the following form of \( f_{j} \):

\[
f_{j} = -\frac{15}{(1 + r_{ij}^{2})^{\frac{3}{2}}}. \tag{5.54}
\]

By using both definitions of 5.53 and 5.54 for \( f_{j} \), the orders of \( \overline{T} \) and \( \overline{Q} \) with respect to \( r_{ij} \) will be:

\[
\overline{T} = \mathcal{O}\left(\frac{1}{r_{ij}}\right)
\]
The above behaviour of $\bar{T}$ and $\bar{Q}$ satisfies the regularity condition.

Zhang and Zhu (1993) chose a different approach to solve the exterior problems with the DRM. Before applying the DRM, they transformed the exterior problem to an equivalent interior problem. They used the following coordinate transformation with respect to the global polar coordinate $(r, \theta)$

$$
\begin{align*}
  r' &= \frac{1}{r}, \\
  \theta' &= \theta,
\end{align*}
$$

which maps the exterior domain into a bounded interior domain. With this transformation of the domain and by transforming the governing equation to the new coordinate system, they used the usual definition of $f_j$ (Equation 5.42) in the new coordinate system. They compared the results of the proposed method with those of Loeffler and Mansur (1988) and found good agreement.

### 5.6.3. Infinite Axisymmetric Domains

When a bubble pulsates in an infinite fluid or in the vicinity of a surface (rigid or free) its motion is axially symmetric. To calculate the heat transfer across the bubble wall Karimi and Soh (1994) employed the DRM. In their work equation 5.1 was solved in the surrounding liquid which is unbounded and axisymmetric. A special type of $f_j$ was proposed which is adopted in this work and is described here.
Recalling Equation 5.11, the relationship between $f_j$ and $\overline{T}_j$ in the axisymmetric case is as follows:

$$f_j = \frac{\partial^2 \overline{T}_j}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{T}_j}{\partial r} + \frac{\partial^2 \overline{T}_j}{\partial z^2}. \quad (5.55)$$

Due to the presence of partial derivatives in this equation, it is difficult to find $\overline{T}_j$ while the definition of $f_j$ is specified. To overcome this difficulty Defiguiredo (1990) chose another approach for the solution of convection-diffusion problems. In his approach Equation 5.11 is solved for $f_j$ while the definition of $\overline{T}_j$ is specified. With this approach any function can be tried on the right hand side of Equation 5.11 giving greater freedom of choice for the function $\overline{T}_j$.

In this work the same approach will be followed. The following definition of $\overline{T}_j$ is suggested for infinite axisymmetric domains:

$$\overline{T}_j = \frac{1 + r_j^2}{1 + r_j^3}. \quad (5.56)$$

Considering Equation 5.56 and the definition of $\overline{q}$ as $\overline{q} = \frac{\partial \overline{T}}{\partial n}$, the corresponding definition of $f_j$ and $\overline{q}$ will be:

$$f_j = \frac{18A_{r_j}^4}{B^3} - \frac{12A_{r_j}^3}{B^2} - \frac{3A_{r_j}}{B} - \left( \frac{18A_{r_j}^2}{B^2} - \frac{2}{B} \right) \left( 3 - \frac{r_j}{r_i} \right), \quad (5.57)$$

and

$$\overline{q} = \frac{2B_{r_j} - 3A_{r_j}^2 \frac{\partial r_j}{\partial n}}{B^2}, \quad (5.58)$$

where

$$A = 1 + r_j^2,$$
and

\[ B = 1 + r_i^3. \]

Also \( r_i \) and \( r_j \) are radial coordinates of the \( i^{th} \) and \( j^{th} \) nodes, respectively. From Equation 5.57 it is seen that not only does this definition depend on the Euclidean distance between \( i^{th} \) and \( j^{th} \) nodes but also it depends on the distances of these nodes from the axis of symmetry. These characteristics of \( f_j \) are the same as those of the definition of \( f_j \) for bounded axisymmetric regions suggested by Wroble and Telles (1986).

From Equation 5.56 and 5.58 the orders of the functions \( \bar{T} \) and \( \bar{Q} \) with respect to \( r_i \) are:

\[ \bar{T} = O \left( \frac{1}{r_i} \right), \]

and

\[ \bar{Q} = O \left( \frac{1}{r_i^2} \right). \]

These orders of \( \bar{T} \) and \( \bar{Q} \) permit the integrals of Equation 5.17 to vanish on an imaginary boundary located at infinity. This means that these definitions satisfy the regularity condition.

Several examples have been solved based on the formulations mentioned in the previous sections, and two of them will be presented in the next section.
5.7. IMPLEMENTATION OF THE DRM TO BUBBLE DYNAMICS

Two typical examples of bubble dynamics were solved using the DRM. A well established computational fluid dynamics package, FIDAP (A Fluid Dynamics Analysis Package), was used in a comparison study. This package uses the finite element method to analyse fluid flow and heat transfer. The comparison procedure is shown in Figure 5.4.
Figure 5.4 Flow chart depicting the procedure for comparing the results of FIDAP and those of the DRM
The following sample problems were solved by both methods.

Problem 1: in this case the bubble contained perfect gas and the initial conditions are shown in table 5.1.

<table>
<thead>
<tr>
<th>Initial Radius (m)</th>
<th>Initial pressure inside the bubble (kPa)</th>
<th>Initial temperature inside the bubble (K)</th>
<th>Pressure at infinity (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>300.0</td>
<td>573.0</td>
<td>25.0</td>
</tr>
</tbody>
</table>

Table 5.1 Problem 1, the bubble contained perfect gas

Problem 2: in this case the bubble contained vapour and the initial conditions are shown in table 5.2.

<table>
<thead>
<tr>
<th>Initial Radius (m)</th>
<th>Initial pressure inside the bubble (kPa)</th>
<th>Initial temperature inside the bubble (K)</th>
<th>Pressure at infinity (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>18.0</td>
<td>333.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Table 5.2 Problem 2, the bubble contained vapour

To solve the energy equation in the liquid domain by FIDAP, the whole domain had to be discretised. Because the liquid was of infinite extent, an imaginary boundary at some distance from the bubble was assumed on which change of temperature, due to heat transfer between the bubble and the liquid, was negligible. The region between the bubble wall and the imaginary boundary was discretised and the mesh system consisted of 360 bi-linear quadrilateral elements with 399 nodal points.
For the case of using the DRM, 18 boundary elements and 36 internal nodes were used and the computations were repeated for 54 and 72 internal nodes. The computational mesh used by FIDAP and the distribution of nodes used by the DRM are shown in Figures 5.5 (a) and (b), respectively.

Figure 5.5 (a) Computational mesh used by FIDAP, (b) Distribution of nodes in the DRM

In the solution procedure, both methods convert differential equations to a system of linear algebraic equations. To see the difference between computer time needed by the two methods, one can compare the number of arithmetic operations.
performed. For a system of linear algebraic equations of order \( n \) to be solved, the numbers of arithmetic operations are as follows (Burden and Fares (1993))

- Subtractions and additions \( \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \)
- Multiplications and divisions \( \frac{n^3}{3} + n^2 - \frac{n}{3} \)

The order of a system of algebraic equations in the DRM is equal to the number of boundary nodes plus the number of internal nodes and for the finite element method it is equal to the number of nodes. The following table shows the total number of arithmetic operations performed by the two methods for the solution of the abovementioned examples:

<table>
<thead>
<tr>
<th></th>
<th>DRM 36 Nodes</th>
<th>DRM 54 Nodes</th>
<th>DRM 72 Nodes</th>
<th>FIDAP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additions and Subtractions</td>
<td>53,901</td>
<td>126,948</td>
<td>173,800</td>
<td>21,253,001</td>
</tr>
<tr>
<td>Multiplications and Divisions</td>
<td>55,386</td>
<td>129,576</td>
<td>177,040</td>
<td>21,332,801</td>
</tr>
</tbody>
</table>

**Table 5.3 Number of arithmetic operations performed by the DRM and FIDAP**

A large difference between the number of operations (or as a consequence a large difference between computation times) can be seen from this table. For completeness of the comparison, in addition to the computation time for the solution of the system of algebraic equations, the computation time needed for the conversion of the differential equations to a system of algebraic equations should be considered. This part of the operation is involved in the numerical calculation of integrals. These integrations are performed over each two-dimensional element.
inside the domain in the finite element method and over each one-dimensional element on the boundary of the domain in the DRM. The number of arithmetic operations carried out during the numerical integrations in the finite element method is far greater than that of the DRM. This makes the gap between the computation time of the two methods even wider.

It should be noted that the internal nodes, in the DRM, simulate the heating up of the whole domain so the location of these nodes can be arbitrary (Partridge et al. (1992)) but their number, as shown in Figure 5.9, affects the accuracy of the results.

Figure 5.6 shows the time history of the volume of a gas bubble calculated by the two methods. Figures 5.7 and 5.8 illustrate the respective time histories of volume and temperature for a vapour bubble.

![Figure 5.6 Time history of the volume of a gas bubble obtained by the DRM and by FIDAP](image)
Figure 5.7 Time history of the volume of a vapour bubble obtained by the DRM and by FIDAP.
**Figure 5.8** Time history of temperature inside a vapour bubble obtained by the DRM and by FIDAP

All three figures show good agreement between the results of the two methods. As it is seen from these figures maximum discrepancy occurs when the volume of the bubble is at its extremums.

To show the effect of the number of internal nodes in the DRM, problem 2 was solved by different numbers of such nodes and the results are shown in Figure 5.9.

![Figure 5.9](image)

**Figure 5.9** Comparison between FIDAP and the DRM with different numbers of internal nodes
It is seen that the calculations using the DRM converge toward the results obtained using 72 internal nodes. Also the case with 72 internal nodes gives results closer to those obtained by FIDAP. Table 5.4 shows the maximum difference between the results from the DRM and those from FIDAP.

<table>
<thead>
<tr>
<th>Problem</th>
<th>No. of Internal Nodes</th>
<th>Maximum % Error in Volume</th>
<th>Maximum % Error in Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>36</td>
<td>11.90</td>
<td>7.00</td>
</tr>
<tr>
<td>1</td>
<td>54</td>
<td>9.67</td>
<td>5.65</td>
</tr>
<tr>
<td>1</td>
<td>72</td>
<td>8.67</td>
<td>4.97</td>
</tr>
<tr>
<td>2</td>
<td>36</td>
<td>16.39</td>
<td>5.60</td>
</tr>
<tr>
<td>2</td>
<td>54</td>
<td>11.69</td>
<td>4.44</td>
</tr>
<tr>
<td>2</td>
<td>72</td>
<td>7.06</td>
<td>4.02</td>
</tr>
</tbody>
</table>

Table 5.4 Maximum difference in volume and temperature resulted from the DRM compared to the results of FIDAP

Two other facts can be inferred from these figures:

- the period of pulsations are the same for both methods; and

- the errors do not increase as the time increases.

This assessed accuracy of the DRM gives confidence to further application of this method in the study of bubble dynamics.
5.8. CONCLUDING REMARKS TO CHAPTER FIVE

The dual reciprocity boundary element method (DRM) has been described and applied to the study of bubble dynamics. The method has been used for the determination of the temperature distribution in the surrounding liquid of a bubble and thereby the heat flux across the bubble-liquid interface was also determined.

The method considers only the boundary of the domain under consideration. Hence, computation time, computer storage and human effort for input data preparation is markedly reduced.

For a comparison study two problems have been solved by the finite element method and the DRM. In the analysis by the finite element method the computer package FIDAP has been used. Results from the two methods showed good agreement.

In conclusion to this chapter one can say:

• since there is no limitation for the geometry of the domain under consideration, the DRM can be used for further investigations of bubble dynamics especially in non-spherical bubble motion;

• the same procedure can be used for the determination of mass diffusion across the bubble wall; and

• compared to the finite element method the DRM is more efficient regarding computer time and storage.

The next chapter is devoted to the investigation of non-spherical bubble dynamics.
CHAPTER SIX

DYNAMICS OF BUBBLES NEAR A RIGID BOUNDARY

6.1. INTRODUCTION

In this chapter the dynamics of bubbles near a rigid boundary will be investigated. Bubbles under consideration are gas bubbles or vapour bubbles. The phenomenon of heat flux through the bubble-liquid interface and the effect of buoyancy forces on the bubble motion have been taken into account in mathematical modelling.

Almost similar to the case of an isolated bubble, the energy equations on both sides of the bubble-liquid interface and the hydrodynamic equation in the liquid domain has been used in the formulations. The energy equations for the inside and outside of the bubble are almost the same as those for an isolated bubble because in their derivations the assumption of spherical motion of the bubble was not used.

Although potential flow fields are assumed for the motion of the liquid for both cases of bubbles (isolated bubbles and bubbles near a rigid boundary), the treatment of the hydrodynamic equation, \( \nabla^2 \Phi = 0 \), is totally different.

In the case of isolated bubbles the flow field was assumed to be generated by a source or a sink with a time varying strength, \( \frac{d\Phi}{dt} \). Due to the spherical motion of
the bubble explicit mathematical expressions were found for the motion of the liquid.

When a bubble pulsates near a rigid boundary its shape is highly distorted from a spherical shape and the flow field generated by the bubble motion is very difficult or almost impossible to be studied analytically. In this case the use of numerical methods is inevitable.

In this work the boundary element method (BEM), due to its accuracy and efficiency, is used to study the flow field in the liquid domain.

All calculations of vapour and gas bubbles in this chapter are based on the following assumptions:

- the bubble surface and the surrounding liquid move axially symmetric;
- the bubble is located above the rigid boundary;
- for the case of a vapour bubble, evaporation and condensation are considered;
- properties of the vapour are found from thermodynamic tables and properties of the non-condensable gas are found from the equation of state of an ideal gas;
- the bubble wall is impervious to the non-condensable gas;
- heat transfer occurs between the bubble contents and the surrounding liquid; and
- pressure and temperature are uniform inside the bubble.

In the following sections the equations which govern the motion of a bubble in the vicinity of a rigid boundary will be derived.
6.2. THE HYDRODYNAMIC EQUATION IN THE LIQUID DOMAIN

In order to study the behaviour of bubbles near a rigid boundary when heat is allowed to cross the bubble-liquid interface, the equation of energy inside the bubble and the hydrodynamic equation in the liquid domain should be solved. Moreover, to calculate the heat transfer between the bubble contents and the surrounding liquid, the energy equation in the liquid side has to be considered.

To solve the energy equation, in which the components of the velocity vector are present, the hydrodynamic equation should be solved. Furthermore, to track the location of the bubble wall during its motion, the velocity of each point on the bubble wall is needed and is found from the solution of this equation.

The next section is devoted to the description of the hydrodynamic equation.

6.2.1. The Hydrodynamic Equation For Bubbles Near a Rigid Boundary

As the surrounding liquid of the bubble is assumed to be incompressible and invicid and the flow field to be irrotational, the flow field will be a potential flow field. In other words the flow field can be described by a velocity potential which satisfies:

$$\nabla^2 \Phi = 0. \quad (6.1)$$

By this definition the components of the velocity vectors in the $r$ and $z$ directions will be:

$$u_r = \frac{\partial \Phi}{\partial r}, \quad (6.2)$$
\[ u_z = \frac{\partial \Phi}{\partial z}. \quad (6.3) \]

It is assumed that the motion of the bubble is axisymmetric. So, Equation (6.1) has to be solved in an axisymmetric exterior domain whose internal boundary is the bubble surface.

In this work the boundary element method is utilised for the solution of (6.1). The implementation of this method to our problem is described in the next section.

6.2.2. The Boundary Integral Form of the Hydrodynamic Equation

For any function which satisfies the Laplace's equation in a domain \( \Omega \) having a piecewise smooth boundary \( \Gamma \) (such as \( \Phi \) in (6.1)), the boundary integral equation has the following form:

\[ C(\xi)\Phi(\xi) + \int_\Gamma \Phi(x)\frac{\partial \Phi(\xi, x)}{\partial n}d\Gamma(x) - \int_\Gamma \Psi(\xi, x)\frac{\partial \Phi(x)}{\partial n}d\Gamma(x) = 0, \quad (6.4) \]

in which \( C(\xi) \) is a constant with the same definition of \( C_i \) in (5.4), \( \xi \) represents every point on the boundary or inside the domain (\( \xi \in \Omega \cup \Gamma \)), \( x \) shows the position of every point on the boundary (\( x \in \Gamma \)) and \( n \) is the unit outward normal vector to the boundary.

In this equation \( \Psi(\xi, x) \) is called the fundamental solution (or free space Green's function) to (6.1) and has the definition (Brebbia et al. (1980)):

\[ \Psi(\xi, x) = \frac{1}{4\pi|\xi - x|} \quad \text{in 3 dimensions,} \quad (6.5) \]

\[ \Psi(\xi, x) = \frac{1}{2\pi} \ln\left(\frac{1}{|\xi - x|}\right) \quad \text{in 2 dimensions.} \quad (6.6) \]
Since the flow field outside the bubble is axisymmetric, to convert (6.4) to an axisymmetric form, this equation is written in 3D polar coordinate system and then integrals over the polar angle are performed analytically. Equation (6.4) in polar coordinate can be written as:

\[
C(\xi)\Phi(\xi) + \int_{\Gamma} \Phi(x) \left( \frac{\partial \Psi(\xi, x)}{\partial n} \right) d\theta(x) r(x) d\Gamma(x) = \int_{\Gamma} \frac{\partial \Phi(x)}{\partial n} \left( \frac{\partial \Psi(\xi, x)}{\partial n} \right) d\theta(x) r(x) d\Gamma(x),
\]

since \( d\Gamma = r d\theta d\Gamma(x) \). Definitions of \( r(x) \), \( \Gamma \) and \( \Gamma \) are shown in Figure (6.1):

![Figure 6.1 Configuration of a body of revolution](image)

In this figure \( \Gamma \) is shown for a ring element having a circumferential area of \( \Gamma \). \( \Gamma \) for the whole domain will be the generating contour which is the intersection of the body of revolution with the \( R^+ - Z \) semi-plane (see Figure 6.2):
In Figure 6.2 the thick solid line shows the generating contour of the body of revolution. The 3D fundamental solution for the cylindrical polar coordinate system has the form:

\[
\Psi(\xi, x) = \frac{1}{4\pi} \frac{1}{\left[ r^2(\xi) + r^2(x) - 2r(\xi)r(x)\cos[\theta(\xi) - \theta(x)] + [z(\xi) - z(x)]^2 \right]^{1/2}}.
\]

Using this definition of the fundamental solution, the integrals over the polar angle in (6.7) can be performed analytically i.e.:

\[
\int_0^{2\pi} \Psi(\xi, x) d\theta(x) = \frac{4K(m)}{(a + b)^{1/2}},
\]

in which \(K(m)\) is the complete elliptic integral of the first kind and:

\[
m = \frac{2b}{a + b},
\]
\[ a = r^2(\xi) + r^2(x) + (z(\xi) - z(x))^2, \]
\[ b = 2r(\xi)r(x), \quad (6.10) \]

and

\[ \int_0^{2\pi} \frac{\partial \Phi(\xi, x)}{\partial n} \, d\theta = \frac{4}{(a+b)^{3/2}} \left\{ \frac{1}{2r(x)} \left[ \frac{r^2(\xi) - r^2(x) + [z(\xi) - z(x)]^2}{a-b} E(m) - K(m) \right] n_r(x) \right. \]
\[ + \left. \frac{z(\xi) - z(x)}{a-b} E(m)n_z(x) \right\}, \quad (6.11) \]

where \( n_r \) and \( n_z \) are the unit vectors in the \( r \) and \( z \) directions, respectively. \( E(m) \) is the complete elliptic integral of the second kind. Definitions and polynomial approximations of the complete elliptic integrals can be found in Hestings (1953).

Substitution of (6.9) and (6.11) in (6.7) yields:

\[ C(\xi)\Phi(\xi) + \int_\Gamma \Phi(x) \left\{ \frac{4}{(a+b)^{3/2}} \left[ \frac{1}{2r(x)} \left[ \frac{r^2(\xi) - r^2(x) + [z(\xi) - z(x)]^2}{a-b} E(m) - K(m) \right] n_r(x) \right. \right. \]
\[ + \left. \left. \frac{z(\xi) - z(x)}{a-b} E(m)n_z(x) \right] r(x)d\Gamma(x) = \int_\Gamma \frac{\partial \Phi(x)}{\partial n} \frac{4K(m)}{(a+b)^{3/2}} d\Gamma(x). \quad (6.12) \]

This equation constitutes the axisymmetric form of the boundary integral Equation (6.4).

By choosing axisymmetric motion for the bubble, the 3D problem in \( (r, z, \theta) \) is reduced to a 2D problem in \( (r, z) \). By choosing the boundary element method in our analysis, instead of working on the whole hatched area of Figure 6.2, only the boundary \( \Gamma \) has to be considered. In other words the 3D problem is reduced to a 2D problem.
The solution of (6.12) can be attempted by following the same procedure as for two dimensional problems. In other words, by discretising $\bar{T}$ into small boundary elements and writing integral terms of Equation (6.12) as the sum of integrals over each of these elements, (6.12) will be converted to a system of linear algebraic equations. The procedure of discretisation is the same as described in Section 5.3.2.

In all formulations cited above the presence of the rigid boundary has not been considered. In order to take the effect of a rigid boundary on the bubble motion into account, the relevant boundary integral equation needs some modifications which is described in the next section.

6.2.3. Modification of the Boundary Integral Equation for Axisymmetric Bubble Motion

When the bubble evolves in the vicinity of a rigid boundary, besides the boundary conditions on the bubble surface, the condition of no flow on the rigid boundary should be satisfied. Without the loss of generality we assume that the rigid boundary is located at $z=0$ as in Figure 6.3:

![Figure 6.3 Configuration of a bubble near a rigid boundary](image)

Figure 6.3 Configuration of a bubble near a rigid boundary
The condition of no flow at the rigid boundary can be achieved by locating an image of the bubble on the other side of the rigid boundary (see Figure 6.4).

In this case the generated flow field can be determined from the modified form of the boundary integral equation (Taib (1986)):

$$ C(\xi)\Phi(\xi) + \int_{r} \Phi(x) \frac{\partial}{\partial n} \left[ \frac{1}{|\xi - x|} + \frac{1}{|\xi - \hat{x}|} \right] d\Gamma(x) = \int_{r} \frac{\partial \Phi(x)}{\partial n} \left[ \frac{1}{|\xi - x|} + \frac{1}{|\xi - \hat{x}|} \right] d\Gamma(x) $$

(6.13)

where \( \hat{x} \) is the image of \( x \) with respect to the rigid boundary. Following the same procedure as the previous section, the axisymmetric form of this equation can be derived.
The next section is devoted to the description of the unsteady Bernoulli equation utilised in this work.

### 6.2.4. The Unsteady Bernoulli Equation

In order to link the motion of liquid outside the bubble to the change of pressure inside the bubble the unsteady Bernoulli equation is used. This equation on the bubble surface in terms of the velocity potential, \( \Phi \), is:

\[
\frac{P_* - P(t)}{\rho_l} = \frac{\partial \Phi}{\partial t} + \frac{1}{2} \| \mathbf{u} \|^2,
\]  

where \( P(t) \) is the pressure at the bubble wall (which is assumed to be equal to the uniform pressure inside the bubble), \( \rho_l \) is the density of the surrounding liquid and \( \mathbf{u} \) is the velocity vector of points at which \( \frac{\partial \Phi}{\partial t} \) is measured. Using \( (6.14) \) the rate of change of potential on the bubble surface can be calculated from:

\[
\frac{d\Phi}{dt} = \frac{P_* - P(t)}{\rho_l} + \frac{1}{2} \| \mathbf{u} \|^2.
\]  

This equation is used throughout the calculations to update the values of the velocity potential on the bubble surface.

To track the location of the bubble surface during its evolution the following equation is used:

\[
\frac{d\mathbf{x}}{dt} = \nabla \Phi,
\]  

where \( \mathbf{x} \) is the position vector of every point on the bubble surface. Once the position of every point on the boundary is calculated, the shape of the bubble, its volume and the rate of change of volume can be determined.
In this work a second order Runge-Kutta method is used to update the velocity potential and the location of the bubble surface.

6.3. THE ENERGY EQUATION FOR A VAPOUR BUBBLE

In order to update the values of the velocity potential by using Equation (6.15), the value of \( P(t) \), the pressure inside the bubble, is required. For the calculation of the pressure inside the bubble, the energy equation is applied to the control mass depicted in Figure 4.1. This will give a relation between the pressure and the rate of change of temperature inside the bubble. Since in the derivation of equation (4.16), in Section 4.2.2, the assumption of spherical motion of the bubble was not used, this equation is applicable to non-spherical bubble motion and is repeated here:

\[
\dot{T} = \frac{\dot{Q} - (H + P_v) \dot{V}}{\left( \frac{\partial H}{\partial T} + \rho C_v \right) \dot{V}}.
\] (4.16)

Definitions of parameters are the same as those in Section 4.2.2. It should be noted that \( P_v \) in this equation is the vapour pressure inside the bubble and this pressure is substituted for \( P(t) \) in the Bernouli equation. The DRM has been used for the calculation of \( \dot{Q} \) in a similar way which was done for an isolated bubble. Up to this point all the necessary equations required for the study of the bubble motion near a rigid boundary have been derived. In the next section the solution strategy for a vapour bubble will be described.
6.3.1. Solution Strategy For a Vapour Bubble

In order to study the behaviour of a vapour bubble near a rigid boundary the boundary integral Equation (6.13), the Bernouli equation and the energy equations inside and outside the bubble are employed.

At the beginning of the bubble motion the following parameters are known as initial and boundary conditions:

- the initial pressure and temperature of the bubble contents;
- the pressure at infinity;
- the initial temperature of the surrounding liquid;
- the geometry of the bubble surface; and
- the values of velocity potential on the bubble surface.

By the prior knowledge of the velocity potential on the bubble surface and the shape of the bubble, the values of $\frac{\partial \Phi}{\partial n}$ can be determined from the solution of the discretised form of (6.13). This term is the normal velocity of nodal points on the boundary. To find the velocity vector at the nodal points, values of the tangential velocity at the nodal points are needed which are approximated by:

$$\frac{\partial \Phi}{\partial s} = \frac{l_n^2 \Phi_{n+1} - [l_{n-1}^2 - l_n^2] \Phi_n - l_n^2 \Phi_{n-1}}{l_n l_{n-1} (l_n + l_{n-1})},$$

(6.17)

where $\frac{\partial \Phi}{\partial s}$ is the tangential velocity of the $n^{th}$ nodal point, $l_n$ is the length of the $n^{th}$ boundary element and $s$ represents the unit vector tangent to the bubble surface. This expression was obtained by fitting a quadratic curve to the points:
New positions of nodal points can be found by exploiting Equation (6.16), i.e.:

\[ \bar{x}(t + \Delta t) = \bar{x}(t) + \nabla \Phi \cdot \Delta t + O(\Delta t^2). \]  

(6.18)

This gives the new volume and shape of the bubble at time \( t + \Delta t \). Also the value of \( \Phi \) can be determined from the values of volume at times \( t \) and \( t + \Delta t \). Solution of (4.16) gives the new value of the temperature inside the bubble. The pressure of the bubble contents will be the corresponding vapour pressure at the new temperature which is found from thermodynamic tables.

Knowing the new bubble pressure and velocity vectors at nodal points and by exploiting the Bernoulli equation, the following equation yields the updated values of the velocity potential:

\[ \Phi(\bar{x}(t + \Delta t), t + \Delta t) = \Phi(\bar{x}(t), t) + \Delta t \frac{d\Phi}{dt} + O(\Delta t^2). \]  

(6.19)

This completes one cycle of calculations. This procedure is repeated at each time step.

One of the most pronounced features of the bubble motion in the proximity of a rigid boundary is the generation of a high speed liquid jet towards the rigid boundary. At the start of the generation of the jet, those nodal points near the tip of the jet (which is the top of the bubble in Figure 6.3) move at a higher velocity compared to the other nodal points on the bubble surface. This difference in the velocity of the nodal points has made some researchers choose a special variable time step instead of a constant one. For example Gibson and Blake (1982) proposed the following expression for the time step:
By choosing this expression for the time increment, $\Delta \Phi$ will be the maximum increment in the velocity potential that could occur at nodal points on the bubble. In the present work this criterion is used for the determination of the time increment.

Another factor which has a significant effect on the bubble motion is the buoyancy forces. In the next section it is shown how these forces are considered in the governing equations of the bubble motion.

### 6.3.2. Effect of Buoyancy Forces on the Bubble Motion

Due to the difference between the density of the gas or vapour inside the bubble and the density of the surrounding liquid, buoyancy forces push the bubble upward. It has been shown that this force has a significant effect on the bubble behaviour (Taib (1986)). To make the mathematical model of the bubble motion more realistic, these forces should be considered.

In order to incorporate these forces into the mathematical model, equations (6.15) and (6.20) are modified as shown below:

$$\frac{d\Phi}{dt} = \frac{P_\text{m} - P(t)}{\rho_i} + \frac{1}{2} |\mathbf{u}|^2 - g(z - z_0),$$  \hspace{1cm} (6.21)

and

$$\Delta t = \frac{\Delta \Phi}{\max \left( \frac{1}{1 + \frac{1}{2} |\mathbf{u}|^2 - g(z - z_0)} \right)},$$  \hspace{1cm} (6.22)
in which $z_0$ is the initial location of the bubble centre and $z$ is the vertical coordinate of the nodal points on the bubble surface.

### 6.4. THE ENERGY EQUATION FOR A GAS BUBBLE

Unlike the case of an isolated gas bubble, in the case of a gas bubble near a rigid boundary, it is assumed that the temperature inside the bubble is uniform.

To find a relation between relevant parameters to bubble motion such as pressure, temperature and volume, the first law of thermodynamics is applied to a control mass which consists of the non-condensable gas in the bubble. Conservation of energy for the bubble contents gives:

$$P_g d\mathcal{V} + \rho_g \mathcal{V} du_g = \dot{Q} dt,$$

in which $P_g$ is the pressure of the gas, $\rho_g$ is the density of the gas, $u_g$ is the specific internal energy of the gas and $\dot{Q}$ is the rate of heat transfer. The first term on the left hand side of this equation is the work done by the bubble on the surrounding liquid, the second term represents the change in the internal energy of the gas and the term on the right hand side is the heat transfer through the bubble wall. By writing $du_g = C_{vg} dT$, and after some manipulations one gets:

$$\dot{T} = \frac{\dot{Q} - P_g \mathcal{V}}{\rho_g C_{vg} \mathcal{V}},$$

where $C_{vg}$ is the specific heat of the gas at constant volume. This equation and the equation of state of an ideal gas are used to calculate the pressure and temperature of the bubble contents. The solution strategy for a gas bubble is described in the next section.
6.4.1. The Solution Strategy For a Gas Bubble

The solution strategies for vapour and gas bubbles differ only in the following ways:

a) the Equation (6.24) is used instead of (4.16) to calculate the temperature of the bubble contents; and

b) after finding the temperature, the pressure of the gas is found by using the equation of state of an ideal gas.

Otherwise the same procedure used in the study of vapour bubbles is applicable to the investigation of the motion of gas bubbles.

6.5. RESULTS AND DISCUSSION

In this section some results for both gas bubbles and vapour bubbles will be presented.

All variables in this section are in non-dimensional form. Definitions of the non-dimensional variables are the same as those in Chapter 4. Two more dimensionless variables are defined here. The first one shows the closeness of the bubble to the rigid boundary and has the definition:

\[ \gamma = \frac{h}{R_m}, \]

where \( h \) is the initial \( z \) coordinate of the bubble centre and \( R_m \) is the maximum radius to which the bubble grows. The other non-dimensional parameter which shows the strength of the buoyancy forces is defined as:
\[ \delta = \left( \frac{\rho g R_m}{p_0 - p_-} \right)^{1/2}, \]

in which \( p_0 \) is the pressure inside the bubble and \( \rho_i \) is the density of the surrounding liquid.

### 6.5.1. Verification of the Solution of the Hydrodynamic Equation by the BEM

The only difference between the mathematical modelling in this chapter and that of Chapter 4 is in the use of the BEM for the solution of the hydrodynamic equation. In order to verify the results of this method, the spherical bubble motion is solved by the proposed formulations and also by the analytic solution of the Rayleigh equation. The results of the two methods are compared in Figure 6.5:

![Figure 6.5](image-url)

**Figure 6.5** Comparison between the result of the program and the analytical solution
This figure shows the time history of a collapsing bubble radius resulting from the analytical solution of the Rayleigh equation and also from the solution of $\nabla^2 \Phi = 0$ by the BEM. In this figure the solid line represents the result of the BEM and the open circles show the results from the solution of the Rayleigh equation which shows good agreement. The analytic solution gives the non-dimensional time of 0.915 for the complete collapse of the bubble while the program gives 0.9145. It is notable that the Rayleigh equation is solved by the PC based symbolic program MATHEMATICA®.

6.5.2. Results for a Gas Bubble Near a Rigid Boundary

Several examples have been solved to show the combined effects of heat transfer, buoyancy forces and the rigid boundary on the dynamic behaviour of a gas bubble.

In the first example buoyancy is neglected. Heat transfer is considered in Case $i$ and in Case $ii$ the bubble is adiabatic. In both cases bubbles are in the vicinity of a rigid boundary. Non-dimensional initial conditions for both cases are shown in Table 6.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.125</td>
<td>0.88</td>
<td>0</td>
<td>1.5</td>
<td>Yes</td>
</tr>
<tr>
<td>$ii$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.125</td>
<td>0.88</td>
<td>0</td>
<td>1.5</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 6.1 Initial conditions for Case $i$ and Case $ii$
The zero value for $\delta$ shows that the buoyancy forces are ignored. In both cases the bubbles grow to a maximum radius and in this time the centroids of the bubbles move slightly upward. The growth phase is followed by the collapse phase in which the centroids of the bubbles move toward the rigid boundary which is located at $z = 0$. Also high speed liquid jets directed towards the rigid boundary are formed on the tops of the bubbles. In these examples after the collapse phase there is a rebound phase in which movement of liquid jets towards the far sides of the bubbles continue until they touch the bottoms of the bubbles. At this moment the shapes of the bubbles are no longer simply connected and calculations are terminated. Figure 6.6 shows the time history of the location of the centroids of the bubbles.

![Graph](image)

**Figure 6.6** Time history of the location of the centroids for cases $i$ and $ii$
It can be seen that the whole process of bubble evolution takes place in a shorter time for Case \textit{ii} in which heat transfer is ignored. Also from this figure rapid movement of the centroids of the bubbles towards the rigid boundary during the collapse phase can be seen.

Figure 6.7 depicts the subsequent shapes of the bubbles (figures a and b correspond to Case \textit{i} and Figures c and d correspond to Case \textit{ii}).
Figure 6.7 Bubbles profiles (for cases i and ii) at the non-dimensional times:

a) 0.0 (inner first), 1.6223, 3.31404, 5.5144, 8.1044 and 9.4695.


c) 0.0 (inner first), 0.9021, 2.3572, 4.3593, 6.8765 and 8.2362.


r and z in these figures are dimensionless.
According to this figure both bubbles grow an almost spherical fashion to their maximum radius (see figures 6.7a and 6.7c). Because the initial locations of bubbles are far from the rigid boundary (relative to the initial radii of the bubbles), the effect of the rigid boundary is insignificant during the growth phase. During the collapse phase the shapes of the bubbles are distorted from spherical, showing the effect of the rigid boundary on the bubbles' profiles (see figures 6.7b and 6.7d). During the collapse phase the bubbles become elongated in the $z$ direction and then the upper sides of bubbles are flattened. At this point in time is the generation of the liquid jets commences. The liquid jets continue their movement at high speed even during the subsequent rebound. Noting the dimensionless times in the caption of Figure 6.7 one can see that the bubble evolution occurs at a longer time in the presence of heat transfer.

For a better comparison, the final shapes of bubbles are depicted in Figure 6.8.

![Figure 6.8](image)

**Figure 6.8** Final shapes of the bubbles of cases $i$ and $ii$
This figure shows that in the absence of heat transfer the final volume of the bubble is larger.

Due to the transfer of heat between bubble contents and the surrounding liquid, the temperature in the liquid domain is changed from its initial uniform state. Figure 6.9 shows a contour plot of temperature in the liquid domain. This figure shows the distribution of the temperature at dimensionless time 20.8187.

Figure 6.9 Distribution of the temperature in the liquid domain for Case i
As it is seen from the figure, temperature change in the liquid domain is confined in a narrow region adjacent to the bubble-liquid interface and outside of this region the temperature remains undisturbed. Also the figure shows that the region in which temperature range is felt, is wider in the liquid jet.

Heat transfer also effects the speed of the liquid jet. This fact is illustrated in Figure 6.10. Note that a velocity vector directed towards the bubble centre is considered to be positive.

Figure 6.10 Time history of jet velocity for cases i and ii
According to this figure the maximum value of the velocity of the liquid jet is reduced due to heat transfer.

To study the effect of the buoyancy forces on the bubble evolution two cases of bubble evolution are studied. In Case \textit{iii} the heat transfer is considered and Case \textit{iv} studied an adiabatic bubble. For both cases the initial temperature and pressure of the liquid are 0.88 and 0.125, respectively. In these examples $\delta$ is 0.125 and $\gamma$ is 1.5. The bubbles' shapes for cases \textit{iii} and \textit{iv} are depicted in figure 6.11 a,b,c and d. Figures 6.11a and 6.11b refer to Case \textit{iii} and figures 6.11c and 6.11d refer to Case \textit{iv}.
Figure 6.11 Bubbles profiles for cases iii and iv at the non-dimensional times:

a) 0.0 (inner first), 0.9133, 2.7744, 4.1214, 5.6567.


c) 0.0 (inner first), 0.8892, 2.7973, 4.2628, 5.9351.


In both cases although the bubbles are in the vicinity of a rigid boundary, due to buoyancy forces the direction of the jets and also migration of the centroids of the
bubbles are away from the rigid surface. Figure 6.12 shows the time history of the jet velocities for the two cases.

![Graph showing time history of jet velocity for cases iii and iv](image)

**Figure 6.12** Time history of the jet velocity for cases iii and iv

This figure shows that heat transfer decreases the velocity of the jet. These two cases also show that the lifetime of the bubble is longer in the presence of heat transfer.

In the previous examples because the bubbles were initially located far from the rigid boundary (compared to the initial radii of the bubbles), the bubbles remained
spherical in shape during the growth phase. In order to investigate the effect of the rigid boundary on the growth phase, the following examples are investigated in which the distance between the bubble centre and the rigid boundary is 1.5 times the initial bubble radius. The initial conditions for these cases are summarised in Table 6.2.

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>δ</th>
<th>γ</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>ν</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.25</td>
<td>1.15</td>
<td>0.0</td>
<td>0.74</td>
<td>Yes</td>
</tr>
<tr>
<td>vi</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.25</td>
<td>1.15</td>
<td>0.0</td>
<td>1.04</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 6.2 Initial conditions for bubbles close to a rigid boundary ignoring the buoyancy forces

Figure 6.13 shows the subsequent shapes of the bubbles during growth phase (figures 6.13a and 6.13c) and during the collapse phase (figures 6.13b and 6.13d). In this case the effect of the rigid boundary on the growth phase is more evident as the bubbles do not grow spherically. For both cases the bubbles become oblate in shape and the lower parts of the bubbles become flattened.
Figure 6.13 Bubbles' profiles for cases v and vi at the non-dimensional times:

a) 0.0 (inner first), 1.0622, 2.0167, 3.1008, 4.3090, 5.5674.


c) 0.0 (inner first), 0.9424, 1.8694, 2.9505, 4.0442.

During the growth phase the centroids of the bubbles move upward (see Figure 6.14) and in the collapse phase the bubbles are attracted to the rigid boundary.

![Graph showing movement of the bubble centroid](image)

**Figure 6.14** Movement of the bubble centroid

By comparing Figure 6.6 and Figure 6.14 it can be inferred that the upward movement of the bubble centroid is more pronounced when the bubble is closer to the rigid boundary. In Figure 6.6 the upward movement is $0.12R_b$ and in Figure 6.14 this distance is about $0.3R_b$. In the collapse phase the lower parts of the bubbles remain flattened and the upper parts take oval shapes after a few time steps (see bubble profiles at dimensionless times 9.9071 and 10.4482 in Figure 6.13b and at times 8.6078 and 9.2452 in Figure 6.13d). After becoming oval in shape, the upper parts of the bubbles begin to flatten at which point in time the generation of liquid jets commences. The final profile is concluded at dimensionless time
1.6432 for case v and at 10.4458 for case vi Figure 6.15 compares the change in pressure inside the bubbles for cases v and vi.

![Graph showing the change of pressure inside the bubbles](image)

**Figure 6.15** Change of pressure inside the bubbles

According to this figure the adiabatic bubble reaches a lower minimum pressure but the maximum pressure during the final stage of collapse is lower for the non-adiabatic bubble. The maximum pressure for the adiabatic bubble occurs at dimensionless time 10.4572 while this parameter occurs at time 11.8126 in the presence of heat transfer.
The combined effects of the buoyancy forces, heat transfer and the rigid boundary on the bubbles of previous examples are depicted in Figure 6.16 (for a bubble to which heat transfer is allowed) and in Figure 6.17 (for an adiabatic bubble). Initial conditions for these cases are shown in Table 6.3.

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>δ</th>
<th>γ</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>vii</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.5</td>
<td>0.88</td>
<td>0.24</td>
<td>1.05</td>
<td>Yes</td>
</tr>
<tr>
<td>viii</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.5</td>
<td>0.88</td>
<td>0.24</td>
<td>1.05</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 6.3 Initial conditions for bubbles close to a rigid boundary considering buoyancy forces

In these cases the growth phase is almost spherical and the centroids of the bubbles move upward during the growth phase (see figures 6.16a and 6.17a). The growth phase is followed by the collapse phase (figures 6.16b and 6.17b). In the collapse phase the lower sides of the bubbles become oval in shape. The lower parts of the bubbles, which are oval in shape, begin to flatten and from this point formation of the liquid jets starts. Noting the dimensionless times corresponding to the shapes of the bubbles, it can be concluded that the formation of a jet occurs earlier in the absence of heat transfer.
Figure 6.16 Bubbles profiles for Case vii of Table 6.3 at dimensionless times:

a) 0.0 (inner first), 0.6718, 1.9429, 2.6044.

b) 3.2658 (outer first), 3.8969, 4.4765, 4.9289, 5.1507.

c) 5.3185 (lower first), 5.6989, 6.1916, 6.8109, 7.5639, 8.4404.
Figure 6.17 Bubble profiles for Case viii of Table 6.3 at dimensionless times:

a) 0.0 (inner first), 0.6612, 1.2784, 1.0120, 2.5847.

b) 3.22391 (outer first), 3.8524, 4.4128.

c) 5.1264 (lower first), 5.3116, 5.8079, 6.5013, 7.3589, 8.3351, 9.2178, 10.0621, 11.0421.
The final profile of the bubbles are depicted together in Figure 6.18.

![Figure 6.18 Final profiles of the bubbles in cases vii and viii](image)

One can deduce from this figure that:

- the final volume of the bubble is larger for the adiabatic bubble; and

- the location of centroids of the bubble is at a greater distance from the rigid boundary for the adiabatic bubble.

The next section is devoted to the investigation of dynamic behaviour of vapour bubbles in the vicinity of a rigid boundary.
6.5.3. Results for a Vapour Bubble Near a Rigid Boundary

In this section some examples regarding the behaviour of a vapour bubble near a rigid boundary will be investigated.

In the study of the dynamic behaviour of a vapour bubble, the effects of heat transfer and buoyancy forces are considered.

In this section, as in the case of isolated vapour bubbles, the evaporation and condensation are taken into account.

All parameters are in non-dimensional form as defined in Section 4.5.

The initial conditions for the first example are tabulated in Table 6.4. This example has been solved for two cases $i x$ and $x$. For Case $i x$ heat transfer is ignored and in Case $x$ it is considered.

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i x$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.92</td>
<td>0.45</td>
<td>0.0</td>
<td>1.88</td>
<td>No</td>
</tr>
<tr>
<td>$x$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.92</td>
<td>0.45</td>
<td>0.0</td>
<td>1.88</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 6.4 Initial conditions of the bubbles for cases $i x$ and $x$

Time history of the pressure inside the bubble is shown in Figure 6.19.
Figure 6.19 Time history of the pressure in the bubbles in cases ix and x

According to this figure during the growth phase, there is a slight difference between the pressures inside the two bubbles but during the collapse phase the pressure increases sharply for Case ix. Any increase in the pressure inside the bubble should be accompanied by an increase in the temperature inside the bubble. However, when the temperature inside the bubble increases due to the large difference in temperature between the inside and outside the bubble the rate of heat transfer (for Case x) will be increased. Consequently this transfer of energy from the bubble contents to the surrounding liquid controls the increase of the temperature inside the bubble. So for Case x one should not expect very high
pressure inside the bubble (compared to that of Case \(ix\)) as it is seen from the figure.

Similar to the case of a gas bubble, the rigid boundary makes vapour bubbles move in a vertical direction during their evolution. Also heat transfer to or from the bubble effects the velocity of the liquid jet and the temperature of the bubble contents.

The effect of the heat transfer on the centroid position, the velocity of the liquid jet and the temperature inside the bubble are investigated in the second example whose non-dimensional initial conditions are shown in Table 6.5.

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>(\delta)</th>
<th>(\gamma)</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>(xi)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.25</td>
<td>0.33</td>
<td>0.0</td>
<td>1.01</td>
<td>No</td>
</tr>
<tr>
<td>(xii)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.25</td>
<td>0.33</td>
<td>0.0</td>
<td>1.01</td>
<td>Yes</td>
</tr>
</tbody>
</table>

**Table 6.5** Initial conditions for bubbles in cases \(xi\) and \(xii\)

Relevant time history of the temperature inside the bubble and the position of the centroids are depicted in figures 6.20 and 6.21, respectively.
Dynamics of Bubbles Near a Rigid Boundary

Chapter Six

1.1

Figure 6.20 The temperature in the bubbles (cases xi and xii)

Figure 6.21 The location of the bubbles' centroid in cases xi and xii
As it is seen from Figure 6.20, bubble evolution is concluded (when the jet touches the lower part of the bubble) in a shorter time and with a lower temperature in Case xi.

The bubbles are repelled by the rigid boundary during the growth phase and attracted to it during the collapse phase. Heat transfer decreases the maximum value of the liquid velocity as it is illustrated in Figure 6.22.

![Figure 6.22](image)

**Figure 6.22** The velocity of the liquid jets for cases $x_i$ and $x_{ii}$

The shapes of the bubbles for the whole period of their evolution are shown in Figure 6.23.
Figure 6.23 Bubbles' profiles for cases xi and xii at dimensionless times:

a) 0.0 (inner first), 1.3083, 2.4629, 3.7967, 5.3132, 6.9750 and 8.7256.

b) 0.0 (inner first), 1.3365, 2.5614, 3.9916, 5.6108 and 8.2499.


During the growth phase due to the effect of the rigid boundary lower parts of the bubbles become flattened while the upper halves remain spherical in shape. Comparing figures (6.23a) and (6.23b) it is seen that the bubble in Case $xi$ grows to a larger volume. During the collapse phase the upper parts of the bubbles take an oval shapes and are then flattened. After flattening the liquid jet is formed. In Case $xi$ the lifetime of the bubble is concluded while the bubble is attached to the rigid boundary but the bubble in Case $xii$ is slightly above the rigid boundary at the end of its lifetime.

The next example studies the behaviour of a vapour bubble having the following initial conditions:

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$xiii$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.58</td>
<td>0.8</td>
<td>0.0</td>
<td>1.0</td>
<td>No</td>
</tr>
<tr>
<td>$xiv$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.58</td>
<td>0.8</td>
<td>0.0</td>
<td>1.0</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 6.6 Initial conditions for the bubbles in cases $xiii$ and $xiv$

Both bubbles in this example begin to shrink due to the higher pressure of the surrounding liquid. The time history of the changes of bubbles' volumes is plotted in Figure 6.24.
As the collapse phase proceeds the temperature of the bubbles' contents increases and as a consequence the pressure of the bubbles increases. Heat flow from the bubble to the surrounding liquid slows down the temperature increase in the bubble in Case xiv. As a result the pressure in the bubble in Case xiii is greater than that in Case xiv. So, the bubble will rebound in the absence of heat transfer while heat transfer causes the complete collapse of the bubble. It should be noted that the term 'complete collapse' refers to a situation in which the liquid jet touches the lower part of the bubble. Profiles of the bubbles of the above example are illustrated in Figure 6.25.
Figure 6.25 Bubbles' profiles during the collapse phase, Figure (a) refers to the Case xiii and Figure (b) is related to the Case xiv.

According to this figure both bubbles become elongated in a vertical direction during the collapse phase before the generation of the liquid jet. However, this elongation is more for the bubble in case xiv. Also the generated liquid jet is broader in case xiv. For a better comparison of the final shapes and volumes of the bubbles in this example, Figure 6.26 shows the final profiles of the bubbles.

Figure 6.26 Final profiles of the bubbles in cases xiii and xiv.
In order to find the effect of heat transfer from the bubble contents on the distribution of temperature in the liquid domain, isotherm contours in the liquid are illustrated in Figure 6.27.

This figure shows the distribution of the temperature at dimensionless time 1.1358. The innermost contour shows the bubble-liquid interface. The figure reveals the existence of a thin thermal boundary layer adjacent to the bubble-liquid interface. The thermal boundary layer is wider in the area around the liquid jet.
6.5.4. Combined Effect of the Buoyancy Forces, the Rigid Boundary and the Heat Transfer on the Dynamics of a Vapour Bubble

In this section the combined effects of the heat transfer, the rigid boundary and buoyancy forces on the dynamics of a vapour bubble will be studied.

In this regard several examples have been investigated the first of which has the following initial conditions.

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial radius</th>
<th>Initial bubble pressure</th>
<th>Initial bubble temperature</th>
<th>Initial pressure at infinity</th>
<th>Initial temperature at infinity</th>
<th>δ</th>
<th>γ</th>
<th>Heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>xv</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.58</td>
<td>0.5</td>
<td>0.32</td>
<td>1.2</td>
<td>No</td>
</tr>
<tr>
<td>xvi</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.58</td>
<td>0.5</td>
<td>0.32</td>
<td>1.2</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 6.7 Initial conditions for the bubbles in cases xv and xvi

In this example the evolution of the bubbles begins with the growth phase in which both bubbles grow to their maximum value. Figure 6.28 shows the time history of the bubbles' volumes. Since the pressure inside the bubbles is higher than that of the surrounding liquid, bubble evolution begins with the growth phase. As the volumes of the bubbles increase, the pressure of the bubbles drops. The decrease in the pressure takes place more rapidly in the bubble in Case xvi, and as a consequence, the bubble grows to a smaller maximum volume in the presence of the heat transfer.

170
During the growth phase both bubbles remain almost spherical in shape but in the collapse phase, liquid jets are formed from the lower parts of the bubbles directed away from the rigid boundary. The bubbles' profiles are illustrated in Figure 6.27. Figures 6.29a and 6.29c refer to Case xv and figures 6.29b and 6.29d show the profiles of the bubble in Case xvi.
Figure 6.29 Bubbles' profiles for cases xv and xvi at dimensionless times:

a) 0.0 (inner first), 0.4767, 0.9225, 1.3617, 1.8062, 2.0311 and 2.4851.

b) 0.0 (inner first), 0.4792, 0.7108, 0.9410, 1.1721 and 1.6399.

c) 2.7134 (outer first), 3.1623, 3.5915, 3.9790, 4.2584, 4.4372, 4.5797, 4.7147, 4.8517, 4.9934, 5.1411 and 5.2952

As figures 6.29a and 6.29b show both bubbles remain almost spherical in shape during the growth phase and the larger maximum volume of the bubble in Case xv can be seen from these two figures. The life time of the bubble in the presence of the heat transfer will be concluded in the collapse phase (see Figure 6.29d) while there is a small rebound before the jet touches the upper part of the bubble in case xv.

As it was seen in the previous sections the rigid boundary always attracts the bubble but this may be changed in the presence of buoyancy forces. The location of centroids of the bubbles during the bubbles' lifetimes are shown in Figure 6.30.

![Graph](image)

**Figure 6.30** The location of the centroids of the bubbles in the presence of the buoyancy forces

According to this figure the centroids of both bubbles have upward motion. Also it is seen that the upward motion of the bubble is more in the absence of heat transfer.
In the last example the effect of the buoyancy forces on the bubble shapes is studied. Bubble profiles for two cases, with and without buoyancy forces, are depicted in Figure 6.31.

Figure 6.31 The effect of the buoyancy forces on bubble shapes

In this figure the bubble profiles in the presence of the buoyancy forces are depicted in figures 6.31a (growth phase) and 6.31c (collapse phase). Figures 6.31b and 6.31d show the profiles of a bubble having the same initial conditions when the buoyancy forces are not considered. Despite the similar shapes of the bubbles in the growth phase, the profiles of the bubbles are completely different during the
collapse phase. Because of the upward direction of the buoyancy forces the bubble becomes elongated in a vertical direction but when the buoyancy forces are not considered due to the effect of the rigid boundary the bubble tends to be flattened. The lifetime of both bubbles will be concluded when the liquid jets touch the lower parts of the bubbles.

6.6. CONCLUDING REMARKS TO CHAPTER SIX

In this chapter the motion of gas and vapour bubbles near a rigid boundary was investigated.

In the mathematical modelling the effects of the heat transfer and the buoyancy forces on the non-spherical bubble evolution were studied. In order to calculate the heat transfer between the bubble contents and the surrounding liquid of the bubble, the dual reciprocity method, developed in Chapter 5, was used.

For a gas bubble, unlike the case of an isolated bubble, the temperature inside the bubble was assumed to be uniform. The first law of thermodynamics was exploited to correlate the thermodynamic process inside the bubble with the bubble dynamic.

In Chapter 4, due to the spherical motion of bubbles, explicit relations were found for the motion of the liquid outside the bubble. These relations are not applicable to the non-spherical motion of the bubble near a rigid surface, hence the use of numerical methods seems inevitable.

The flow field in the liquid was supposed to be a potential flow field which is governed by \( \nabla^2 \Phi = 0 \), where \( \Phi \) is the velocity potential. To solve this equation with the relevant boundary conditions the boundary element method was used.
Several parameters related to bubble dynamics such as the movement of the bubble centroid, the pressure inside the bubble, the temperature of the bubble contents and the velocity of the liquid jet were studied.

Although the effects of rigid boundary, heat transfer and buoyancy forces on bubble dynamics have been studied together one can name the effects of each of these factors separately as below.

a) The effects of the rigid boundary on the bubble motion are mentioned below:

- slight repulsion of the bubble during the growth phase and rapid attraction during the collapse phase;
- the generation of a liquid jet towards the rigid boundary; and
- non-spherical shape of the bubble in the growth phase when the maximum radius of the bubble is greater than the initial distance of the bubble centre from the rigid boundary.

b) The effects of the heat transfer are as follows:

- the liquid jet is broader in the presence of heat transfer;
- at the end of the collapse phase, when the liquid jet touches the lower part of the bubble, the non-adiabatic bubble has a smaller volume;
- the liquid jet gains a smaller velocity due to the effect of heat transfer; and
- the pressure in a non-adiabatic bubble is lower at the end of the lifetime of the bubble.
• due to heat transfer between the bubble contents and the surrounding liquid, a thin thermal boundary layer is formed in the liquid domain. This thermal boundary layer is wider in the liquid jet area.

c) The buoyancy forces cause:

• the upward motion of the centroid of the bubble;

• the elongation of the bubble in a vertical direction; and

• sometimes a change of direction of the jet.
CHAPTER SEVEN

SUMMARY AND CONCLUSION

7.1. SUMMARY

In this thesis an investigation of the dynamic behaviour of bubbles with especial focus on thermodynamic processes inside bubbles has been carried out.

Three cases of vapour bubbles, gas bubbles and bubbles containing a mixture of vapour and a non-condensable gas have been considered and for each case a set of non-linear differential equations has been derived. These sets of differential equations have been solved by the method of Runge-Kutta.

The proposed models link the thermodynamic processes inside bubbles with the motion of the bubble wall and the motion of the surrounding liquid. The models enable a detailed study of the different aspects of bubble dynamics and thermodynamic processes inside the bubbles. These aspects can be listed as the pressure and temperature of the bubble contents, the volume of the bubble, the velocity of the bubble interface, the speed of liquid jets and the effect of heat transfer on these parameters.

The radial distribution of the temperature inside gas bubbles has been determined. The processes of evaporation and condensation on the bubble wall for vapour
bubbles and bubbles filled with a mixture of vapour and a non-condensable gas have been considered.

Potential flow fields have been considered in the liquid domain where, either the bubbles pulsate in a liquid of infinite extent, or evolve close to a rigid boundary. The flow field outside an isolated bubble has been simulated by the flow field of a sink or source. When a bubble pulsates near a rigid boundary, the versatile boundary element method has been used for the solution of the flow field and a computer program has been developed in this regard.

The following reviews some findings and implications of the study.

The dual reciprocity boundary element method was adopted and modified to solve the energy equation in the surrounding liquid of a bubble. It was shown that the method is more efficient regarding computer time and memory compared with its counterparts such as the finite element method and the finite difference method. The method was verified by the finite element method and implemented to the problem of bubble dynamics for both cases of isolated bubbles and bubbles evolving close to a rigid boundary.

It is customary in the literature to make use of a polytropic relation to find the pressure inside a gas bubble. In spite of its simplicity, the assumption of the polytropic relation poses some problems and it may lead to inaccurate prediction of the pressure in the bubble as was described in Chapter 4. It was shown that the thermodynamic processes inside the bubbles can not be considered as polytropic processes. Also it was revealed that the collapse phase accords well with a polytropic process although different initial conditions of the bubble result in different polytropic indices.
The radial distribution of the temperature in gas bubbles was determined and it was revealed that the temperature changes considerably in a thin layer of the gas adjacent to the bubble wall. The thickness of this layer is around 20% of the bubble radius and the temperature out of this layer changes uniformly.

Also the existence of a thin thermal boundary layer in the liquid domain was revealed and it was shown that the thickness of this layer is more in the liquid jet area.

Two features of the bubbles have been known to be responsible for cavitation damage, shock waves emanating from the bubbles at the instant of rebound and high speed liquid jets. The study of the collapse and rebound of the bubbles showed a rapid change of direction in the motion of the bubble interface. This change of direction in a short period of time will most probably generate a weak shock wave in the liquid. As a consequence of periodic emission of such shock waves, resonance may be excited in structures which leads to cavitation damage.

In a study on the $P-V$ relationship of the thermodynamic processes in bubbles it was revealed that the thermodynamic paths for expansion and collapse are different, which implies a net output of energy from the bubbles.

For the evolution of bubbles near a rigid boundary the lifetime of a bubble was defined to be from the beginning of the bubble motion up to the time when the liquid jet touches the other side of the bubble. At this point the bubbles become toroidal. It was inferred that the heat transfer between the bubble contents and the surrounding liquid affects the liquid jet in the following two ways:

- the liquid jet is broader in the presence of heat transfer; and
Summary and Conclusion Chapter Seven

- the heat transfer slows down the liquid jet and it reaches a lower maximum speed compared to the speed of a liquid jet in an adiabatic bubble.

It was also found that the volume of a bubble is larger when the bubble wall is impervious to the heat and sometimes there is a rebound phase before the lifetime of the bubble ends.

7.2. RECOMMENDATIONS FOR FUTURE RESEARCH

Since the equation which governs mass transfer is very similar to the equation of energy in the liquid domain, the proposed method of dual reciprocity can be easily employed to study the mass transfer through the bubble-liquid interface. This will give a more realistic insight into the dynamics of bubbles.

Most of the numerical works on bubble dynamics have considered the bubble evolution up to the time when the liquid jet touches the opposite side of the bubble, but recently successful attempts have been made to study the bubble beyond this point (Best (1993)). The developed computer programs in this work have the capability of being further improved to deal with the case of toroidal bubbles.

By further modification of mathematical modelings the effects of other types of surfaces such as free surface and compliant surfaces can be investigated.

For the sake of simplicity the surrounding liquid was assumed to be non-viscous. Considering the effect of viscosity on the motion of the liquid gives a more realistic picture of the flow field outside the bubble. This could be done by the use of the boundary element method. The proposed mathematical modelling and computer programs could be used as a basis for such consideration.
REFERENCES


References


References


References


Harris P.J. 1993, "A numerical method for predicting the motion of a bubble close to a moving rigid structure", Communications in numerical methods in engineering, Vol. 9, pp 81-86.


References


References


Mordock J. 1965, Internal report in mechanical engineering department, MIT.


References


Rayleigh L. 1917, "On the pressure developed in a liquid during the collapse of a spherical cavity", Phil. Mag., 34, 94.


References


References


This appendix describes two equations used in Chapter 4 for the derivation of radial distribution of temperature inside a gas bubble. These two equations are:

\[
\left( \frac{T}{\rho_g} \right) \left( \frac{\partial \rho_g}{\partial T} \right)_{p_g} = -1, \quad (A.1)
\]

and

\[
\rho_g C_{pg} T = \frac{kP_g}{k - 1}. \quad (A.2)
\]

To derive the first relation we utilise the fact that the pressure, temperature and density are point functions, i.e. these quantities depend only on the state and are independent of the path. For these properties we can write the following mathematical relationship (Van Wylen 1985):

\[
\left( \frac{\partial T}{\partial p_g} \right)_{p_g} \left( \frac{\partial \rho_g}{\partial T} \right)_{p_g} \left( \frac{\partial \rho_g}{\partial p_g} \right)_{T} = -1. \quad (A.3)
\]

The equation of state of a gas gives:

\[
P_g = \rho_g RT, \quad (A.4)
\]

so

\[
\left( \frac{\partial T}{\partial p_g} \right)_{p_g} = \frac{1}{\rho_g R}. \quad (A.5)
\]
Substituting A.5 and A.6 in A.3 gives:

\[
\left( \frac{I}{\rho^* R} \right) \left( \frac{\partial p^*_g}{\partial T} \right)_{P_g} (RT) = -I, \tag{A.7}
\]
or

\[
\left( \frac{T}{\rho^*_g} \right) \left( \frac{\partial p^*_g}{\partial T} \right)_{P_g} = -I \text{ which is A.1.}
\]

To derive A.2 we use the definitions of \( \gamma \), the ratio of the specific heats, and \( R \), the gas constant, as:

\[
k = \frac{C_{pg}}{C_{vg}}, \tag{A.8}
\]

and

\[
R = C_{pg} - C_{vg}, \tag{A.9}
\]
in which \( C_{pg} \) and \( C_{vg} \) are the specific heats of gas at constant pressure and constant temperature, respectively.

Using A.8 one can write:

\[
\frac{k}{k-1} = \frac{C_{pg}}{C_{pg} - C_{vg}}, \tag{A10}
\]
or

\[
\frac{k}{k-1} = \frac{C_{pg}}{R}. \tag{A.11}
\]

The equation of state of a gas can be written as:
\[
\frac{P \cdot C_{pl}}{R} = \rho \cdot C_{ps} T.
\] (A.12)

Substituting for \( \frac{C_{ps}}{R} \) from A.11 results in
\[
\rho \cdot C_{ps} T = \frac{kP}{k - 1}
\] which is A.2.
Reply to examiners' report

In reply to the examiners' questions the following commands are made:

First examiner:

1) I agree with the examiner's suggestion of using variable conductivity for gas in the bubble. The emphasis of this thesis is on heat transfer across the bubble-liquid boundary. The effect of change of thermal conductivity of the gas on the whole process of bubble evolution deserves more investigation in the future research.

2) The surface tension effects in pulsating bubbles is known to be small compared with the hydrodynamic force of the liquid.

3) For a perfect gas, when the pressure is assumed uniform, the continuity and energy equations can be combined derive an exact expression for the velocity field in terms of the temperature gradient. Eliminating the velocity term in the energy equation inside the bubble by the above obtained expression will result in a differential equation for the radial distribution of the temperature inside the bubble. For a vapour bubble filled with saturated vapour a uniform pressure inside the bubble implies uniform temperature.

4) For determining the direction of liquid jet, a technique, based on the concept of Kelvin impulse, has been developed by Blake (1988). This method is applicable to bubble evolution while the pressure inside the bubble is constant with no heat transfer. I agree that the evolution of bubbles affected by buoyancy forces and other parameters such as the initial temperature and pressure inside the bubble, the initial temperature and pressure of the liquid domain, the initial radius of the bubble and closeness if the bubble to the rigid boundary requires thorough investigation in future.
Second Examiner:

i) Non-equilibrium thermodynamics governs evolution of a bubble only in the earliest stages of the bubble initiation and the very end stage of collapse which is out of context of this work. For the state of the bubbles assumed in this work the equilibrium thermodynamics concept is applicable.

ii) In the range of change in the liquid temperature used in this work, parameters such as pressure, density and kinematic viscosity are not changed significantly.