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On the microwave heating of materials with temperature dependent properties

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On the microwave heating of materials with temperature dependent properties

A thesis submitted in fulfilment of the requirements for the award of the degree of Doctor of Philosophy from The University of Wollongong, by A.H. Pincombe B.Sc.(Hons) (UNSW), M.Sc., Dip.Ed.

Department of Mathematics, 1992
This thesis is submitted to The University of Wollongong, and has not been submitted for a higher degree to any other University or Institution.

Adrian Pincombe

August, 1992.
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I also wish to thank my wife and children for understanding my need to do this work.
Abstract

The time dependent microwave heating of a one-dimensional, semi-infinite body is considered. Starting from Maxwell's equations, it is shown that this heating is governed by a coupled system consisting of the damped wave equation and a forced heat equation with forcing depending on the square of the amplitude of the electric field. The dependence of the values of the electromagnetic properties on the temperature is represented by two different power laws and solutions are obtained in each case. Approximate analytical solutions are obtained for the coupled equations, using perturbation analysis, when the thermal diffusivity is constant and is smaller than the heating rate. The effect of a finite, and possibly nonlinear thermal diffusivity is also considered.
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1 Introduction

In recent years there has been a great deal of interest in the development of industrial applications of microwave heating. This interest is driven by the significant advantages which microwave heating holds over conventional heating for many heating and drying applications. But the widespread application of microwave heating has also uncovered some problems, one of which is the phenomenon of a "hot spot", which is a small region with a significantly elevated temperature. An example of hot spots and the subsequent thermal runaway is shown in Figure 1. In some cases, the temperature at the hot spot is sufficient to melt the material, which is desirable for applications such as smelting but is undesirable for applications such as sintering (see Araneta et al, 1984). Because of these problems, there has been an upsurge of interest in the mathematical modelling of microwave heating. The classical analysis of microwave heating is based on the assumption that the thermal absorptivity, the electrical permittivity, magnetic permeability and electrical conductivity are all constant, so that Maxwell’s equations reduce to the telegrapher’s equation, which can be solved without reference to the temperature. The temperature variation is then calculated by solving a forced heat equation with the forcing term dependent on the square of the amplitude of the electric field (see Metaxas and Meredith, 1983). Experimental measurements (von Hippel, 1954) however, show that the electromagnetic properties of materials, such as the thermal absorptivity, the magnetic permeability, the electrical permittivity and the electrical conductivity, are all dependent on the temperature. Thermal runaway (and hence hot spots) can occur if the thermal absorptivity increases sufficiently rapidly as the temperature rises. When
the electromagnetic properties are temperature dependent, the forced heat equation and Maxwell's equations become a highly nonlinear, coupled system, which is difficult to solve in general.

One theoretical approach (Roussy et al, 1987, Hill and Smyth, 1990, Brodwin et al, 1991, Coleman, 1991) is to use the simplifying assumption that the amplitude of the electric field is constant. This assumption can be justified for thin materials or for hot spots (where the analysis centres on a small region of the material), and the problem reduces to one of solving the forced heat equation

\[ T_t = \nu \nabla^2 T + \gamma(T) \]  \hspace{1cm} (1.1)

in isolation for the temperature \( T \), where the thermal absorptivity, \( \gamma(T) \), represents the temperature dependent absorption of energy from the microwave radiation. The thermal absorptivity \( \gamma(T) \) is not known theoretically, but it can be deduced from experimental measurements for any particular material. Various representations for \( \gamma(T) \) have been used in the literature. Hill and Smyth (1990) used the form

\[ \gamma = \gamma_0 e^{\eta T}, \]  \hspace{1cm} (1.2)

with a fixed temperature boundary condition. They found steady state solutions and identified a condition for the formation of hot spots. The exponential dependence (1.2) has been found to be valid for various ceramic materials (Barker et al, 1976, Kingery et al, 1976, Wo, 1986). Hill and Jennings (1992) used the data of von Hippel (1954) and showed that quadratic representations were also valid. Coleman (1991) used the power
Figure 1: Photographs of smelting using a microwave oven, showing the appearance of hot spots (top) followed by the subsequent thermal runaway (bottom)
\[
\gamma = \gamma_1 (1 + \gamma_2 T)^{\gamma_3},
\]

and assumed that heat diffusion could be neglected at a hot spot because it was much smaller than the rate of heat absorption. He showed that a hot spot will occur if

\[\gamma_2 (\gamma_3 - 1) > 0.\]

Roussy et al (1987) numerically solved equation (1.1) for a cylindrical body, with \(\gamma\) depending quadratically on temperature and with a convective heat loss boundary condition. An approximate condition for a hot spot to form was found. Brodwin et al (1992) found steady state solutions of equation (1.1) in a thin slab using a convective and radiative heat loss boundary condition. They found solutions corresponding to hot spots for certain types of dependencies of \(\gamma\) on temperature.

When the size of the body is not small, then the full system of Maxwell's equations and the forced heat equation must be considered. Kriegsmann et al (1990) found the steady state solution for a semi-infinite, one-dimensional body with temperature dependent electrical conductivity but with constant electrical permittivity and magnetic permeability. The boundary condition used was convective and radiative cooling, with a perturbation solution found as a series in the small electrical conductivity. Kriegsmann (1991a, 1991b, 1992) considered the microwave heating of a one-dimensional slab of arbitrary thickness in the small Biot number limit (which measures the relative effects of heat convection and radiation to heat diffusion), hence the slab is essentially thermally insulated. Steady state solutions were found as a perturbation series in the small Biot number and these solutions, as a function of incident microwave power, were
found to have an S-shaped profile. Hence for a small increase in incident microwave power the system will experience thermal runaway by jumping from the lower branch (a stable low temperature solution) to the upper branch (which corresponds to a hot spot). Smyth (1990) explored the high frequency (geometric optics) and small thermal diffusivity limit. He assumed that the electrical conductivity, electrical permittivity and magnetic permeability varied slowly with temperature in a linear manner, and found a perturbation solution of the coupled system using strained co-ordinates.

This thesis considers four aspects of the analysis of the microwave heating of a semi infinite body, which is irradiated by plane waves. Three Chapters (2, 4 and 5), look at various aspects of the high-frequency radiation and small thermal diffusivity limit. The assumption of zero diffusion makes the equations amenable to analysis. The use of this simplification is justified for microwave heating because, in general, the heat absorption will occur at a faster rate than the heat diffusion, particularly in the region of a hotspot. A transport equation valid at high frequency, which describes the evolution of the first order amplitude term of the electric field is derived. Various perturbation solutions of this amplitude evolution equation are derived and numerical solutions are presented as comparisons in each case. The case where the heat diffusion is significant is considered in Chapter 3. Here a nonlinear thermal diffusivity and thermal absorptivity are considered. The electromagnetic properties of the material are assumed constant, hence the electric field amplitude decays exponentially.

In Chapter 2, the time dependent microwave heating of a semi-infinite material with low conductivity is analysed using multiple scales, where the thermal absorptivity and the
electromagnetic properties are represented by power laws of the form (1.3). The slow time
and space equations resulting from this multiple scales analysis (the transport equation
mentioned above) will be solved exactly for special choices of the thermal absorptivity
and the material properties. In particular, analytical solutions are obtained for the case of
constant wavespeed. The temperature dependence of the material properties is found to
have a significant effect on the resulting temperature (with the occurrence of hot spots
possible). The governing equations are also solved numerically and the perturbation
solutions are found to be in good agreement with these numerical solutions.

In Chapter 3, materials with nonlinear thermal diffusivity and thermal absorptivity
are analysed. The equations are decoupled by solving Maxwell's equations for the case
where the electromagnetic properties of the material are constant, which results in the
electric field amplitude decaying exponentially with space. This form is substituted
into the forced heat equation. The thermal diffusivity and the thermal absorptivity
are assumed to have either an exponential or a power law dependence on temperature.
One-parameter group similarity transformations are applied to the resulting nonlinear
heat equation which reduces the partial differential equation to an ordinary differential
equation. This in general is solved numerically, except in some special cases where
analytical solutions are possible.

In Chapter 4, the time dependent microwave heating of a material with small constant
electrical conductivity and slowly varying electrical permittivity and magnetic permeabil-
ity is considered. The electrical permittivity and magnetic permeability are represented
by power laws of the form (1.4). Hence the effects of changes in wavespeed are con-
sidered, and due to the highly nonlinear form chosen for the thermal absorptivity, hot spot formation also. Perturbation solutions are developed using the method of strained co-ordinates (to remove the secular terms) and the method of multiple scales (to enable an explicit solution), in order to develop a uniformly valid solution. The perturbation solutions are compared with numerical solutions obtained using the numerical model developed in Chapter 2 and are found to be in good agreement.

Finally, in Chapter 5, the time dependent microwave heating of a semi-infinite material is considered, where the thermal absorptivity and the electromagnetic properties are represented by power laws of the form

\[ p = p_1 + \alpha p_2 T^{p_3}. \]  

(1.4)

The effect of reflection of radiation at the boundary is considered and dielectric losses are included, along with losses due to electrical conductivity, in a generalised loss term. For small values of this loss term, the solution of the coupled system of equations is found as a perturbation series in the loss term. This solution is shown to be in good agreement with a full numerical solution of the coupled system of equations. In Chapter 2, through the use of power laws of the form (1.3), we obtained solutions that are equivalent here to taking \( \alpha = O(1) \) in equation (1.4). The extra assumption here that \( \alpha \ll 1 \) enables more solutions to be obtained.

The remainder of this chapter is devoted to the development of the state equations which are common to all four cases above.
1.1 Governing equations

The equations governing the microwave heating of a semi-infinite space are now derived. If the material is homogeneous and isotropic and the current \( J \) and the displacement current \( D \) induced by the microwave radiation are both proportional to the electric field \( E \), and if the magnetic flux density \( B \) is proportional to the magnetic field strength \( H \), then Maxwell’s equations are

\[
\nabla \cdot D = \nabla \cdot (\varepsilon E) = 0, \\
\nabla \cdot B = \nabla \cdot (\mu H) = 0, \\
\n\nabla \times E = -(\mu H)_t, \\
\n\nabla \times H = (\varepsilon E)_t + \sigma E. 
(1.5)
\]

The electrical permittivity \( \varepsilon \), magnetic permeability \( \mu \), and electrical conductivity \( \sigma \) are, in general, functions of temperature and of the frequency of the microwave radiation (von Hippel, 1954, Metaxas and Meredith, 1983). Microwave heating uses radiation of a single frequency but the local frequency of the radiation will vary as it passes through a material, due to changes in the value of the electrical permittivity and the magnetic permeability, brought about by the changing temperature. Thus all changes in the electromagnetic properties can ultimately be attributed to changes in the temperature. The dependence of these properties on temperature means that Maxwell’s equations (1.5) must be coupled with some form of the forced heat equation. Hill (1989), Smyth (1990) and Kreigsmann et al (1990) used the equation

\[
T_t = \nu T_{xx} + g(T)|E|^2
(1.6)
\]
which will also be adopted here, except in Chapter 3, where the case of nonconstant thermal diffusivity is considered. The forced heat equation (1.6) will be derived from energy conservation principles in Chapter 2 and supported by a photon argument in Chapter 5. For a plane wave propagating in the positive x-direction, the wave, without loss of generality, can be regarded as polarised so that the electric field is in the y-direction and the magnetic field is in the z-direction. For the one-dimensional heating of a half space \( x > 0 \), the electric and magnetic fields are functions of the spatial co-ordinate and the time only. Maxwell’s equations (1.5) then become

\[
E_x = -(\mu H)_t, \tag{1.7}
\]

\[
H_x = -(\epsilon E)_t - \sigma E, \tag{1.8}
\]

where \( E \) and \( H \) are scalar quantities. Equations (1.7) and (1.8) can be combined in the usual way to obtain the damped wave equation

\[
c^2 E_{xx} = -LH - MH_t + PE + QE_t + E_{tt}, \tag{1.9}
\]

where

\[
L = \mu t_s/(\mu \epsilon), M = \mu_x/(\mu \epsilon), c = (\epsilon \mu)^{-1/2}, \tag{1.10}
\]

\[
P = \mu t \epsilon/(\mu \epsilon) + \sigma \mu/(\mu \epsilon) + (\epsilon_{tt} + \sigma)/\epsilon, \tag{1.11}
\]

and

\[
Q = \mu t/\mu + (2\epsilon_t + \sigma)/\epsilon. \tag{1.12}
\]

In each of the chapters considered in this thesis a different approach has been taken to the form of the loss term. In Chapters 2 and 4 losses are viewed as dependent on terms
which are explicitly included in Maxwell's equations, that is the electrical conductivity, electrical permittivity and magnetic permeability. In Chapter 2 a generalised electrical conductivity is defined which includes the ratio of the electrical conductivity and the magnetic permittivity as well as terms which depend on variations in the wave speed. In Chapter 4, the variations in electrical permittivity and magnetic permeability, and thus in wave speed, are kept separate from the electrical conductivity. In Chapter 5, as in Chapter 4, the apparent losses caused by variations in the wave speed are kept separate from the loss term but, in this case, an extended loss term which includes dielectric losses is used.

1.2 Summary of results

In Chapter 2 the limit of small electrical conductivity and small thermal diffusivity is considered, with analytical solutions being possible when the thermal diffusivity is much smaller than the electrical conductivity, which is normally the case in microwave heating. The power laws used for the electromagnetic and electrothermal properties of a material are of a form which allows the ambient temperature to be scaled to zero and, for this form of the power law, it is found that analytical solutions are possible only for some special cases, all of which involve the wavespeed being constant. Some of these special case solutions are presented, with further cases being presented in Chapter 5.

For the one dimensional case, of heating a semi infinite slab, with an insulated boundary at \( x = 0 \), the greatest deviation occurs at the boundary \( x = 0 \) where it is seen that the numerical and the perturbation solutions agree closely in all of the special cases, but
that the constant coefficients solution (ie the solution obtained via the standard theory) is quite different.

The effect of a greater value for the thermal diffusivity is considered in Chapter 2, where it is seen that, when the thermal diffusivity and the electrical conductivity are of the same order, the amplitude of the electrical field is virtually unchanged but the temperature at x=0 is substantially reduced. When the thermal diffusivity is $O(1)$, there is a further reduction in peak temperature and a wider, flatter shape for the spatial distribution of temperature.

For the power laws considered in Chapter 2, solutions for variable wavespeed were only possible by using an expansion just behind the wavefront. This problem was overcome in Chapters 4 and 5, where a different form of power law was used.

The formation of hotspots is investigated numerically in Chapter 2 where it is seen that hotspots occur when the energy absorption rate is much larger than the thermal energy transfer rate and when the energy absorption rate increases sufficiently rapidly with increase of temperature. Coleman(1991) and Kriegsmann et al(1990) urged caution in adopting power law models for the physical parameters. These models are only realistic within some restricted range of temperature and there is a saturation effect. When a saturation value is imposed on the energy absorption coefficient $\gamma$, it is seen that there is still a peak at the insulated boundary but the maximum temperature and the steepness of the peak depend on the value of the saturation temperature and on the steepness of the energy absorption power law near the saturation temperature.

It is shown that hotspots can be caused by inhomogeneities, but that the capacity of
the inhomogeneity to absorb energy depends on its size, and the generation of a hotspot depends on the quantity of extra energy (above that which the surrounding material would absorb) and on the value of the thermal diffusivity.

It is suggested that the continuous change in the energy absorption rate as the temperature changes, which is observed macroscopically, could be the result of a step change at the atomic level, with the probability of change increasing with temperature.

The lack of solutions for the case where the wavespeed is variable is addressed in Chapter 4 by taking the conductivity to be constant and by using the method of strained coordinates to eliminate secular terms. The form of the power laws for the physical properties is also changed to one which is more amenable to integration. In this case, it is seen that the solutions vary not only in their estimation of the temperature at a point, but also in the predicted position of the wavefront. In the physically important case where the wavespeed reduces with increasing temperature it is shown that the solution is uniformly valid for a restricted time after which temperature runaway should occur. The solutions are uniformly valid for all time in the non-physical case where the wavespeed increases with temperature. It is also seen that the temperature becomes infinite in finite time (a hotspot) as long as the power law for the thermal absorptivity has an exponent which is greater than one. This corroborates the result of Coleman (1991).

Power laws of the form (1.4), which are used in Chapter 4, are also used in Chapter 5 where it is shown that they are representative of a wide range of materials. It is seen that they lead to the complete, time dependent, perturbation solution of the coupled
equations in the limiting case of small loss factor and small thermal diffusivity. This solution also includes the effect of the changes in reflection at the boundary, \( x = 0 \), which accompany changes in the wavespeed. It is shown that the solution still remains a good predictor of microwave heating when the temperature has risen well beyond the region where the perturbation solution is uniformly valid.

When the wavespeed is an increasing function of temperature, it is seen that the decrease in wavespeed as the wave moves into cooler material contributes to an increase in the local electric field strength. When the wavespeed is a decreasing function of temperature, there is seen to be a decrease in the amplitude of the electric field at any point.

Thus it is seen, in Chapters 2, 4 and 5, that, in the absence of a thermal flux, hotspots can form if the thermal absorptivity increases sufficiently quickly with an increase in temperature. It is also shown in Chapters 2, 4 and 5, that it is possible to develop analytic solutions for the case of zero heat flux, and that the generality of these solutions depends, for example, on the form of the power laws for the electrothermal and electromagnetic properties of the material. It has also been seen, in Chapter 2, that a non-zero value of the thermal diffusivity will markedly reduce the value of the peak temperature.

In Chapter 3 we consider the effects of thermal diffusivities which are nonlinear functions of temperature and it is seen that we obtain similarity transformations for some values of the parameters in the thermal diffusivity and the thermal absorptivity functions. In this way, the nonlinear heat equation is reduced, in each case, to an ordinary differential equation. It is shown that analytical solutions are possible in some
special cases, but, in most cases, the solutions are obtained numerically. These solutions indicate the presence of moving fronts. When the thermal diffusivity and the thermal absorptivity are zero at the ambient temperature, the fronts move into a region of zero (scaled) temperature, which is a valid trivial solution of the governing equation. The solutions exhibit the waiting time phenomenon. It is also seen that hotspots and thermal runaway can occur if the absorptivity increases sufficiently rapidly with an increase in temperature.
2 When the electrical conductivity is small

The time dependent microwave heating of a semi-infinite material with low electrical conductivity is considered where the thermal absorptivity and the electromagnetic properties are represented by power laws of the form (1.3). A multiple scales analysis is used and the resulting slow time and slow space equations are solved exactly for some special cases of dependence on the temperature. A numerical model of the governing equations is also produced and the perturbation solutions are found to be in good agreement with the numerical solutions. The numerical model is also the basis for the numerical solutions used in Chapters 4 and 5. The solutions produced here show that the temperature dependence of the material properties has a significant effect on the resulting temperature (with the occurrence of hot spots possible). This result is supported by a numerical model of the effect of inhomogeneities.

2.1 Governing equations

In this chapter it is assumed that the conductivity is small so that \( \sigma = \alpha \sigma_1 \), where \( \alpha \ll 1 \). In this case, as was shown by Kriegsmann et al (1990) and as will be shown in section 2.2, the properties are all slowly varying functions of \( x \) and \( t \) so that

\[
\mu = \mu(\alpha x, \alpha t) \\
\varepsilon = \varepsilon(\alpha x, \alpha t) \\
\sigma = \sigma(\alpha x, \alpha t). 
\]

(2.1)
We then see from (2.6) and (2.7) that

\[ Q \sim O(\alpha), \frac{\mu_x}{\mu\epsilon} \sim O(\alpha) \]

\[ P \sim O(\alpha^2), \frac{\mu_x}{\mu\epsilon} \sim O(\alpha^2) \]  \hspace{1cm} (2.2)

Equations (2.2) and (2.3) hence give,

\[ E_x = -\mu H_t + O(\alpha) \]

\[ H_x = -\epsilon E_t + O(\alpha), \]  \hspace{1cm} (2.3)

showing that the behaviour to \( O(\alpha^0) \) is the same as for the case when \( \epsilon \) and \( \mu \) are constant, as expected. So at \( O(\alpha) \) we can use the intrinsic impedance relationship which has been derived for constant \( \epsilon \) and \( \mu \) (see, for example, Bleaney and Bleaney p 231), giving, in this case

\[ H = \sqrt{\frac{\epsilon}{\mu}} E + O(\alpha) \]  \hspace{1cm} (2.4)

It then follows, as the permittivity and permeability are slowly varying functions of time, that

\[ H_t = \sqrt{\frac{\epsilon}{\mu}} E_t + O(\alpha) \]

Thus equation (2.4) reduces to

\[ E_{tt} + SE_t = c^2 E_{xx} \]  \hspace{1cm} (2.5)

to \( O(\alpha) \), where

\[ S = \alpha \frac{\mu_x}{\mu} + \frac{2\epsilon_x + \sigma_1}{\epsilon} \frac{c\mu x}{\mu}, \]  \hspace{1cm} (2.6)

and \( \tau = \alpha t \) and \( X = \alpha x \). This equation is a damped wave equation with the damping due to the conductivity and the variation of the material properties. In the above
derivation we have assumed that the electrical and magnetic properties are functions of space and time. From experimental studies (von Hippel, 1954), the electrical and magnetic properties of a material vary with temperature and with the frequency of the microwave radiation. If we consider the heating of a homogeneous, isotropic material with input microwaves of a single frequency, then the attenuation coefficient $S$ and the wave speed $c$ will be functions of the temperature $T$

$$S = S(T)$$
$$c = c(T).$$

(2.7)

The propagation of energy in the body can be described in the normal way by an energy conservation equation

$$E_t + F_x = 0,$$

(2.8)

where $E$ is the energy density and $F$ is the energy flux density. The flux of energy due to electromagnetic radiation is given by the Poynting vector

$$F_e = -\varepsilon c E^2 + O(\alpha)$$

(2.9)

where $\varepsilon$ is the permittivity and $c$ is the wave speed (see Jones, 1964). The energy flux due to heat transport is

$$F_h = -\nu c_v T_x$$

(2.10)

where $c_v$ is the heat capacity of the material. The frequency time scale for the microwave radiation is much smaller than the time scale for heat diffusion. Because of this, the microwave energy flux term in (2.9) is averaged over a microwave period to give an
averaged equation for heat transport (see Metaxas and Meredith, 1983). To do this we note that when the coefficients $S$ and $c$ in equation (2.5) are constant and $S \sim O(\alpha)$, the solution is

$$E = E_0 \exp(-bX) \exp(i\theta)$$

(2.11)

where we require $E = E_0 \exp(-i\omega t)$ at $x = 0$, $b$ is a constant and the phase is $\theta = kx - \omega t$. The exponential decay is due to the damping term in equation (2.5). When $S$ and $c$ are both slowly varying functions of space and time we expect the solution for $E$ to have a form similar to equation (2.11), but with an exponent which is not necessarily proportional to $X$. Thus the electric field strength $E$ is a slowly varying damped wave of the form

$$E = a \exp(i\Theta/\alpha)$$

(2.12)

where the amplitude $a$ and the phase $\Theta$ in equation (2.12) are functions of the slow space and time variables $X$ and $t$. Using (2.12) we can evaluate the average over a wave period of the microwave energy flux term in equations (2.8) and (2.9) to obtain

$$\overline{\epsilon c E^2}_x = \alpha \overline{\epsilon c E^2}_X = \alpha(\epsilon_x c + \epsilon c_x + 2\epsilon c \frac{a_x}{a})da^2,$$

(2.13)

where $\overline{d}$ relates the average value of $E^2$ to the square of the amplitude function, $a^2$ and an overbar denotes the mean over a microwave period. Hill (1989), Smyth (1990) and Kriegsmann et al (1990) represented microwave heating by the heat equation with a source term which depends on the square of the amplitude of the electric field

$$T_t = \nu T_{xx} + g(T)\left|E\right|^2$$

(2.14)
with \( g \) being a function of the temperature. This representation of the energy supplied by the microwave radiation is widely accepted (see for example Metaxes and Meredith, 1983). If all of the energy which is removed from the electromagnetic radiation is converted into heat, then we have the relation

\[
F = F_e + F_h
\]  
(2.15)

The energy density of heat in the body can be represented by

\[
E = c_v T
\]  
(2.16)

The energy conservation relation (2.8) thus becomes,

\[
T_t = \nu T_{xx} - \frac{\alpha}{c_v} (\epsilon \chi c + \epsilon \chi c + 2 \epsilon \alpha \chi / \alpha) a^2
\]  
(2.17)

If we assume that \( a_X \) remains \( O(1) \) then we can express the energy conservation equation in a form similar to equation (2.14)

\[
T_t = \nu T_{xx} + \alpha \gamma(T) |E|^2
\]  
(2.18)

If we set

\[
\sigma = \mu^{-1} \mu + \epsilon^{-1} (\sigma_1 + 2 \epsilon \chi) - \mu^{-1} c \mu X,
\]  
(2.19)

then the damped wave equation (2.5) becomes

\[
E_{tt} + \alpha \sigma(T) E_t = c^2(T) E_{xx}.
\]  
(2.20)

The equations governing the microwave heating of a material with low conductivity are then (2.18) and (2.20).
2.2 Perturbation solution

The governing equations (2.18) and (2.20) have been derived under the assumption that the conductivity is small. Under this assumption, the electric field will have a slow decay in amplitude, and hence will consist of a fast oscillation modulated by a slow amplitude decay. Also since the source term in the heat equation (2.18) is $O(\alpha)$, the induced temperature will be $O(\alpha)$. We therefore seek an asymptotic solution of (2.20) and (2.18) of the form

$$E = a(X, \tau)e^{i\frac{k(x, \tau)}{a}} + \alpha a_1(X, \tau)e^{i\frac{k(x, \tau)}{a}} + \ldots . \quad (2.21)$$

and

$$T = \alpha T_1(t, x, X, \tau) + \alpha^2 T_2(t, x, X, \tau) + \ldots . \quad (2.22)$$

The boundary conditions to be used in the present work are

$$\frac{\partial T}{\partial x} = 0$$

$$a = E_0 at x = 0 \quad (2.23)$$

$$\theta = -\omega \tau$$

At $t = 0$, the microwave field impinges on the boundary $x = 0$ of the material and the initial condition is that the material in $x > 0$ is at a uniform temperature at $t = 0$, which by choice of temperature scale can be taken to be $T = 0$. The boundary conditions (2.23) correspond to the imposed microwave field at $x = 0$ which has amplitude $E_0$, frequency $\omega$ and wave number $k$. We shall also assume that $\nu$ is small, so that a solvable set of equations results. Since the solution is calculated to first order in the present work, it
is sufficient to take $\nu < 0(\alpha)$. The temperature boundary condition corresponds to an insulated boundary. This is a valid approximation as, due to the short time scale of microwave heating, heat does not have time to diffuse out of the boundary. Since $\nu$ is assumed to be small, there is a temperature boundary layer at $x = 0$. The boundary layer solution is not needed to calculate $E$ and $T$ outside the boundary layer, and so is not considered in the present work. Smyth (1990) contains an analysis of the boundary layer at $x = 0$. For the zero heat flux boundary condition (2.23a), the boundary layer is weak and to first order, the boundary layer solution is equal to the outer solution evaluated at $x = 0$.

Substituting these expressions (2.21) and (2.22) into the governing equations (2.20) and (2.18), we obtain at $O(1)$ from (2.18) and at $O(\alpha)$ from (2.20)

\[ \theta_\tau + c\theta_X = 0 \]  
(2.24)

and

\[ T_{1t} = \gamma(\alpha T_1)a^2(X, \tau) \]  
(2.25)

respectively. Equation (2.24) is the eikonal equation governing phase propagation of the microwave radiation. The heat flow equation (2.25) does not contain any diffusive term due to the assumed smallness of $\nu$. At $O(\alpha)$ we obtain from equation (2.20)

\[ \theta_{\tau\tau}a + 2\theta_\tau a_\tau + \sigma\theta_\tau a = c^2\theta_X X a + 2c^2\theta_X a_X \]  
(2.26)

This equation is the transport equation governing the development of the amplitude of the microwave radiation. Now from the eikonal equation (2.24) we have

\[ \theta_{\tau\tau} = -c_\tau \theta_X + cc_X \theta_X + c^2 \theta_{XX} \]  
(2.27)
Hence the transport equation (2.26) becomes

\[ a_t + c a_x = -\frac{1}{2}(\sigma - c x + \frac{c_\tau}{c})a \]  

(2.28)
on eliminating the phase \( \theta \). The system of equations to be considered in this section then consists of the heat flow equation (2.25) and the transport equation (2.28). These equations are not yet complete however, as \( c, \sigma \) and \( \gamma \) depend on \( X \) and \( \tau \) through their dependence on temperature. These dependencies now need to be assumed. Let us suppose that \( c, \sigma \) and \( \gamma \) have power law dependencies on \( T \), so that

\[ c = c_0(1 + c_1 T)^\beta \]

\[ \sigma = \sigma_0(1 + \sigma_1 T)^\alpha \] 

(2.29)
\[ \gamma = \gamma_0(1 + \gamma_1 T)^\eta \]

Power laws of these forms can be used to fit the temperature dependencies of material properties over restricted temperature ranges for various materials given in von Hippel (1954). The temperature equation (2.25) can be integrated to give

\[ T_1 = (\alpha \gamma_1)^{-1} \left[ \left( 1 + \alpha \gamma_0 \gamma_1 (1 - \gamma_2) a^2 \left( t - \frac{x}{c_0} \right) \right)^{\frac{1}{1-\gamma_2}} - 1 \right]. \]  

(2.30)

In deriving this expression, we have used the fact that the temperature does not begin to change until the microwave field reaches the given position. Since \( T = 0 \) prior to the arrival of the microwave field, we see from (2.29) that the microwave field arrives at the point \( X \) at time \( \tau = \frac{X}{c_0} \). Hence \( T = 0 \) for \( \tau < \frac{X}{c_0} \). If we expand (2.30) for small \( \alpha \), we obtain

\[ T_1 = \gamma_0 a^2 \left( t - \frac{x}{c_0} \right). \]  

(2.31)
The expansion for small $\alpha$ which results in (2.31) from (2.30) is valid if

$$1 + \alpha \gamma_0 \gamma_1 (1 - \gamma_2) a^2 \left( t - \frac{x}{c_0} \right)$$  \hspace{1cm} (2.32)$$

is not small. The case when this quantity is equal to zero corresponds to the formation of a hotspot. To include the possibility of hotspot formation, the expression (2.30) for $T_1$ will be used in (2.29) to give the dependence of $c$ and $\sigma$ on $a$, $X$ and $\tau$. Similarly, (2.29) can be expanded using a Taylor Series as $T = O(\alpha)$, which is valid as long as a hotspot does not form, a hotspot corresponding to $T$ becoming infinite in finite time. Again, to include the effect of a hotspot, this expansion is not done when the expressions (2.29) are substituted into (2.28).

Another viewpoint on the use of (2.30) in (2.28) and (2.29) is as follows. The temperature evolution equation (2.18) can be re-expressed as

$$T_\tau = \gamma(T)a^2(X,\tau)$$  \hspace{1cm} (2.33)$$

since $\nu$ is assumed small. Integrating this equation, we obtain

$$(1 + \gamma_1 T)^{1-\gamma_2} = 1 + \gamma_0 \gamma_1 (1 - \gamma_2) \int_{\frac{X}{c_0}}^{\tau} a^2(X,\psi) d\psi.$$  \hspace{1cm} (2.34)$$

If $\tau - \frac{X}{c_0}$ is small, the integral in (3.14) can be approximated by $a^2(\tau - \frac{X}{c_0})$, so that

$$\gamma_1 T = \left[1 + \gamma_0 \gamma_1 (1 - \gamma_2) a^2 \left( \tau - \frac{X}{c_0} \right) \right]^{1-\gamma_2} - 1.$$  \hspace{1cm} (2.35)$$

If this approximation is not made, the system (2.28) and (2.33) is difficult to solve. As a confirmation, it is found in section 2.5 that the solutions obtained using this approximation are in good agreement with numerical solutions of the full governing equations.
(2.18) and (2.20). Substituting (2.29) and (2.30) into the transport equation (2.28), and setting $\alpha x = X$ and $\alpha t = \tau$, we obtain a first order partial differential equation for $a$, which in characteristic form is

$$\frac{da}{d\tau} = \frac{-1}{2} \frac{\gamma_0 c_1 c_2 h a^3 (\gamma_1 + \gamma_1 (1 + \frac{\alpha}{\eta} (g - 1))^2) + \sigma a (\gamma_1 - c_1 + c_1 g)}{\gamma_1 - c_1 + c_1 h (1 + \gamma_0 \gamma_1 (1 - \gamma_2 + c_2) a^2 (\tau - \frac{X}{c_0}))}$$

(2.36)
on the characteristics

$$\frac{dX}{d\tau} = c \frac{\gamma_1 - c_1 + c_1 h (1 + \gamma_0 \gamma_1 (1 - \gamma_2 + c_2) a^2 (\tau - \frac{X}{c_0}))}{\gamma_1 - c_1 + c_1 h (1 + \gamma_0 \gamma_1 (1 - \gamma_2 + c_2) a^2 (\tau - \frac{X}{c_0}))}$$

(2.37)

where

$$g = (1 + \gamma_0 \gamma_1 (1 - \gamma_2) a^2 (\tau - \frac{X}{c_0}))^{\frac{1}{1 - \gamma_2}} \quad \text{and} \quad h = \frac{g}{1 + \gamma_0 \gamma_1 (1 - \gamma_2) a^2 (\tau - \frac{X}{c_0})}$$

(2.38)

The characteristic velocity (2.37) is changed from the microwave speed $c$ due to the dependence of $c$ on $a$ in (2.28), on using (2.29) and (2.30). The characteristic velocity (2.37) is then similar to a group velocity, in that it is the velocity which determines the evolution of the amplitude, while $c$ is similar to a phase velocity.

### 2.2.1 Constant wavespeed

For $c_2 = 0$, the microwave speed is independent of the temperature and equation (2.37) has the solution

$$\eta = \tau - \frac{X}{c_0}$$

(2.39)

for a wave which arrives at the boundary $X = 0$ at time $\tau = \eta$. The value of $\eta$ remains constant on a characteristic and equation (2.36) can be solved to give

$$\int_{E_0}^{s} \frac{ds}{s \left( 1 + \frac{\alpha}{\eta} \left( [1 + \gamma_0 \gamma_1 (1 - \gamma_2) s^2 \eta]^{\frac{1}{1 - \gamma_2}} - 1 \right) \right)^{\sigma_2}} = \frac{-\sigma_0}{2} (\tau - \eta)$$

(2.40)
While the integral in (2.40) cannot be evaluated for general \( \sigma_2 \), it can be evaluated for a wide range of particular values of \( \sigma_2 \). Some of these cases will be discussed below and further in section 2.5 where they are compared with numerical solutions of (2.18) and (2.20). In general, the conductivity of a material is more temperature dependent than the wavespeed, so the assumption of constant wavespeed is not as restrictive as might first appear.

2.2.2 \( \sigma_2 = 0, \ c_2 = 0 \)

In this case only the absorption of microwave energy depends on temperature, and the damped wave equation and the forced heat equation are decoupled. From (2.40) and (2.37) we find that

\[
a = E_0 e^{-\sigma_0 x / c_0} \tag{2.41}
\]

and from (2.30)

\[
T_1 = (\alpha \gamma_1)^{-1} \left[ \left( 1 + \alpha \gamma_0 \gamma_1 (1 - \gamma_2) \left( t - \frac{x}{c_0} \right) E_0^2 e^{-\sigma_0 x / c_0} \right)^{1 / \gamma_2} - 1 \right] \tag{2.42}
\]

The electric field shows the usual exponential decay in \( x \) for constant \( c \) and \( \sigma \) (see Metaxas and Meredith, 1983). For \((\gamma_2 - 1) > 0\) and \( \gamma_1 > 0 \), the temperature becomes infinite at \( X = 0 \) when

\[
\tau = [\gamma_0 \gamma_1 (\gamma_2 - 1) E_0^2]^{-1}, \tag{2.43}
\]

which corresponds to the formation of a hot spot at \( x = 0 \) at this time. The condition \((\gamma_2 - 1) > 0\) and \( \gamma_1 > 0 \) for hot spot formation agrees with Coleman (1991), which was derived for a constant amplitude electric field with zero heat diffusivity. Near the
time of hotspot formation, the present slowly varying analysis is invalid, due to the large derivatives of $T$, and the heat diffusion term $\nu T_{xx}$ becomes important. It has been shown by Hill and Smyth (1990) that for a constant amplitude electric field, sufficiently large values of $\nu$ arrest the formation of a hot spot, which is to be expected on physical grounds.

2.2.3 $c_2 = 0, \quad \sigma_1 = \gamma_1, \quad \frac{\sigma_2}{1-\gamma_2} = \frac{1}{2}$.

This case has two distinct classes of solutions, one for $1-\gamma_2 > 0$, the other for $1-\gamma_2 < 0$. When $1-\gamma_2 > 0$, then $\sigma_2 > 0$ and the damping increases with temperature. The conversion of electromagnetic energy to heat energy can increase with temperature ($0 < \gamma_2 < 1$) or it can decrease as the temperature increases ($\gamma_2 < 0$). For the physical process of microwave heating the energy absorption coefficient $\gamma$ is proportional to the damping coefficient $\sigma$, which is represented here by $\sigma_2 = \gamma_2 = \frac{1}{3}$. When $1-\gamma_2 > 0$, equation (2.40) has the solution

$$a = \frac{\cosech\{\arccosech[\{\gamma_0\gamma_1(1-\gamma_2)(\tau - \frac{X}{c_0})\frac{1}{2} E_0 + \frac{\sigma_2 X}{2c_0}\}]\}^{\frac{1}{3}}}{\{\gamma_0\gamma_1(1-\gamma_2)(\tau - \frac{X}{c_0})\}^{\frac{1}{3}}} \quad (2.44)$$

When $1-\gamma_2 < 0$, we require $\sigma_2 < 0$, so that the damping decreases as the temperature increases. Energy absorption however will increase with temperature, since $\gamma_2 > 1$. There is thus an imbalance in energy transfer which could not be maintained over an indefinite temperature range, but there could be a range of temperatures within which this case is physically valid. For $1-\gamma_2 < 0$, equation (2.40) has the solution

$$a = \frac{\sech\{\arcsech[\{\gamma_0\gamma_1(\gamma_2 - 1)(\tau - \frac{X}{c_0})\frac{1}{2} E_0 + \frac{\sigma_2 X}{2c_0}\}]\}^{\frac{1}{3}}}{\{\gamma_0\gamma_1(\gamma_2 - 1)(\tau - \frac{X}{c_0})\}^{\frac{1}{3}}} \quad (2.45)$$
This solution for $a$ is valid for $0 \leq \gamma_0 \gamma_1(\gamma_2 - 1)(\tau - \frac{X}{c_0})E_0^2 \leq 1$, the inequality first ceasing to be valid at $X = 0$. From (3.10), we see that the time at which the solution first becomes invalid, namely $\tau = [\gamma_0 \gamma_1(\gamma_2 - 1)E_0^2]^{-1}$, is the same time at which a hotspot forms. Hence (2.45) is valid until the time of hotspot formation, as expected.

2.2.4 $c_2 = 0, \quad \sigma_1 = \gamma_1, \quad \sigma_2 = 1 - \gamma_2$

This class of cases is similar to 2.2.3 and includes the physically important case where $\gamma \propto \sigma$ ($\sigma_2 = \gamma_2 = \frac{1}{2}$). Here equation (2.40) has the solution

$$a = \frac{E_0 e^{-\sigma_2 X}}{1 + \gamma_0 \gamma_1(1 - \gamma_2)E_0^2(1 - e^{-\sigma_2 X})(\tau - \frac{X}{c_0})} \left(1 - \frac{X}{c_0} \right)^{\frac{1}{2}} \quad (2.46)$$

Examining (2.46) we see that at the wavefront the amplitude will have the same value as the constant coefficients case (see (2.41)), but that behind the wavefront the value of the amplitude will be modified by the heating which has occurred. When $1 - \gamma_2 > 0$, the value of the amplitude will be less than or equal to the value with constant coefficients, while when $1 - \gamma_2 < 0$, the amplitude will be greater than its value for constant coefficients. However for $1 - \gamma_2 < 0$ we have the unphysical result that $a$ will become infinite in finite time. This is caused by the imbalance between the rate at which energy is removed from the electromagnetic waves and the rate at which electromagnetic energy is converted into heat. For the physical situation of microwave heating, we require $\sigma \propto \gamma$ when $c_2 = 0$. 

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In this case there is a balance between heat gained and electromagnetic energy lost. For these parameter values, the implicit solution to (2.40) is

\[
\ln \left( \frac{a}{E_0} \right) + \gamma_0 \gamma_1 \left( \tau - \frac{X}{c_0} \right) (E_0^2 - a^2) - \frac{[\gamma_0 \gamma_1 (\tau - \frac{X}{c_0})]^2}{4} (E_0^4 - a^4) = -\frac{\sigma_0 X}{2c_0} \quad (2.47)
\]

### 2.3 Variable wavespeed

For \( c_2 \neq 0 \), the microwave speed depends on the temperature and the characteristic velocity for the transport equation (2.37) is not the same as the microwave speed, which is the characteristic speed for the eikonal equation (2.39). The characteristic speed (2.37) plays the role of a group velocity and it is this velocity which determines the speed of energy propagation and hence the temperature of the body. For non-constant \( c \), equations (2.36) and (2.37) are nonlinearly coupled and cannot be solved in general. However, near the wavefront \( \tau - \frac{X}{c_0} \ll 1 \) and equations (2.36) and (2.37) have the solution

\[
a = \frac{E_0 e^{-\frac{\sigma_0 X}{2c_0}}}{\left[ 1 + \frac{2\gamma_0 c_1 c_2 E_0^2}{\sigma_0} \left[ 1 - e^{-\frac{\sigma_0 X}{c_0}} \right] \right]^{\frac{1}{2}}} \quad (2.48)
\]

This is the solution for constant coefficients modified by a factor which depends on the parameters \( c_1 \) and \( c_2 \) from the wave speed function (2.29). The expression (2.48) gives the development of the microwave field close to the wavefront and becomes infinite at

\[
X = -c_0 \sigma_0^{-1} \ln [1 + \sigma_0 (2\gamma_0 c_1 c_2 E_0^2)^{-1}] \quad (2.49)
\]

when \( c_1 c_2 < 0 \). From (2.29), it can be seen that, at the critical value of \( X \) defined by (2.49), \( c \) is either infinite (\( c_2 > 0 \)) or zero (\( c_2 < 0 \)), both of which are physically
unrealistic. It can also be found from (2.30) that the temperature \( T \) becomes infinite at the wavefront at this time. This physically unacceptable behaviour is due to the power law dependence for \( c \) assumed in (2.29) not being valid as the temperature becomes large.

2.4 Numerical solution of the coupled wave and heat equations

The accuracy and predictions of the asymptotic solutions of section 2.2 will be investigated by comparing them with numerical solutions of the basic differential equations (2.20) and (2.18) subject to the initial and boundary conditions (2.23). When the wave speed \( c \) is constant, equation (2.20) can be solved numerically by using a simple centred difference scheme (see for example Burden et al, 1978) which results in the following recurrence relation

\[
E_{k,j+1} = \frac{\lambda^2}{1 + \rho} E_{k+1,j} + 2 \frac{1 - \lambda^2}{1 + \rho} E_{kj} + \frac{\lambda^2}{1 + \rho} E_{k-1,j} - \frac{1 - \rho}{1 + \rho} E_{k,j-1} \tag{2.50}
\]

where

\[
\lambda = c k \Delta t / \Delta x \tag{2.51}
\]

\[
\rho = \alpha S k \Delta t / 2 \tag{2.52}
\]

and \( E_{k,j} = E(k \Delta x, j \Delta t) \) \( \tag{2.53} \)
Here \( k \leq n, j \leq m \) and \( \Delta t \) and \( \Delta x \) are the time step and the space step respectively.

The boundary conditions in (2.23) become

\[
E_{0j} = E_0(\cos(-\omega j \Delta t) + i \sin(-\omega j \Delta t)) \tag{2.54}
\]

and

\[
E_{nj} = 0, \tag{2.55}
\]

while the initial conditions in (2.23) become

\[
E_{k0} = 0 \tag{2.56}
\]

\[
E_{k,-1} = 0 \tag{2.57}
\]

The boundary \( x = n\Delta x \) of the domain of numerical integration is chosen to be large enough so that \( E \) is negligible there and so \( E_{nj} = 0, j = 1, \ldots, m \). We assume that \( E \equiv 0 \) for \( t < 0 \). Hence \( E_t = 0 \) at \( t = 0 \) and so \( E_{k,-1} = 0 \) for \( k = 1, \ldots, n \). The numerical scheme defined by equations (2.50) to (2.57) has second order accuracy, with error \( O(\Delta t^2, \Delta x^2) \); however we need to consider its stability. If we define the error in our scheme by

\[
\Delta E_{kj} = E(x_k, t_j) - E_{kj} \tag{2.58}
\]

then the difference equation (2.50) holds for \( \Delta E_{kj} \) and we can represent the error \( \Delta E_{kj} \) as the sum of Fourier components of the form \( g^j \exp(i\beta k \Delta x) \), where \( \beta \) is real. Substitution of this form into (2.50) yields a quadratic in \( g \) with the solutions

\[
g = \frac{\Gamma \pm \sqrt{\Gamma^2 - 1 + \rho}}{1 + \rho} \tag{2.59}
\]

where

\[
\Gamma = 1 - 2\lambda^2 \sin^2(\beta\Delta x/2) \tag{2.60}
\]
If the discriminant is less than zero, the size of $g$ is given by its complex modulus

$$|g| = \frac{1 - \rho}{1 + \rho}$$  \hspace{1cm} (2.61)

If $\rho < 0$ there is a region for which $|g| > 1$ and if $\rho > 1$ then equation (2.61) does not make sense. But for $0 < \rho < 1$, the scheme (2.50) is stable as $|g| < 1$. Here we have $0 < \rho << 1$ and the present analysis is similar to the well known stability analysis for the undamped wave equation. It can easily be shown that we require $\lambda \leq 1$ for the discriminant to be less than or equal to zero. The case in which the discriminant is greater than zero corresponds to $\lambda > 1$ and for $\rho << 1$ the error will be unbounded as $|g| > 1$ in this case. Thus our stability condition is simply $\lambda \leq 1$.

The numerical scheme (2.50) results in fluctuations and numerical dispersion at the wavefront due to the large $x$-derivatives there. These fluctuations are quickly damped out and are not evident a short distance behind the front. In order to obtain a more accurate representation of the solution close to the wavefront, a moving boundary scheme is used, with the wavefront being the moving boundary. To implement this moving boundary scheme, we use the centred difference scheme (2.50) to calculate $E_{k,j+1}$ as long as $E_{k+1,j} \neq 0$. When $E_{k+1,j} = 0$, we calculate $E_{k,j+1}$ (and possibly $E_{k+1,j+1}$) using a quadratic interpolation based on the points $E_{k-2,j+1}, E_{k-1,j+1}$ and $E_f$, the value of the field at the front. The value of the electric field at the wavefront can be found by using a wavefront expansion (see Whitham, 1974, page 236)

$$E = \sum_{n=0}^{\infty} \Pi_n(x,t) f_n(s),$$  \hspace{1cm} (2.62)

where $f_n(s) = s^n/n!$ and $s$ is distance measured from the wavefront. By substituting
(2.62) into the damped wave equation (2.20) and setting the coefficients of successive powers of $s$ to zero, we obtain

$$\Pi_{nt} + c_0 \Pi_{nx} = -\frac{1}{2} (\alpha \sigma_0 - c_0 c_1 c_2 T_x + c_1 c_2 \nu T_{xx} + c_1 c_2 \alpha \gamma_0 \Pi_n^2) \Pi_n$$  \hspace{1cm} (2.63)

The highest order term in the expansion (2.62) is the amplitude of the electric field ($\Pi_0 = a$), so that equation (2.63) leads to the following differential equation for the amplitude of the electric field at the wavefront

$$\frac{da}{dt} = -\frac{1}{2} (\alpha \sigma_0 - c_0 c_1 c_2 T_x + c_1 c_2 \nu T_{xx} + c_1 c_2 \alpha \gamma_0 a^2) a.$$  \hspace{1cm} (2.64)

In general equation (2.64) is solved numerically as $T$, $T_x$ and $T_{xx}$ are not known at the wavefront. However, if $\nu = 0$, then $T = 0$ at the wavefront as no diffusion of heat can occur. When $\nu = 0$, we can substitute the following wavefront expansion for the temperature

$$T = f(x,t)(t - x/c_0)^n + ...$$  \hspace{1cm} (2.65)

into the heat equation (2.18) and by balancing the resulting equation, we find $n = 1$, giving $f = \alpha \gamma a^2$. Hence $T_x$ at the wavefront can be determined and equation (2.64) becomes

$$\frac{da}{dt} = -\frac{1}{2} \alpha (\sigma_0 + 2 c_1 c_2 \gamma_0 a^2) a$$  \hspace{1cm} (2.66)

The solution to equation (2.66) is identical to (2.48), as expected, so that we can define the value of the electric field at the wavefront by

$$E_f = (a, 0),$$  \hspace{1cm} (2.67)

where $a$ is defined by (2.48).
When the wave speed $c$ is variable, the time step $\Delta t$ will need to be adjusted to ensure that $\lambda \leq 1$. For variable time step $\Delta t$, the centred difference scheme (2.50) gives an error $O(\Delta t)$. To obtain a quadratic error ($O(\Delta t^2)$) with a variable time step we need to change from a three point scheme to a four point scheme. If we take the Taylor expansions of $E_{k,j+1}, E_{k,j-1}$ and $E_{k,j-2}$ about the point $t = t_j$, we can ensure that $E_{tt}$ and $E_t$ are calculated with error $O(\Delta t^2)$ by taking

\[ E_{tt}(x_k, t_j) = \frac{2(\psi E_{k,j+1} + \phi E_{k,j-1} - (1 + \psi + \phi)E_{k,j})}{\Delta t_j^2 + \Delta t_{j-1}^2\psi + (\Delta t_{j-1} + \Delta t_{j-2})^2\phi} \quad (2.68) \]

where $\Delta t_j = t_{j+1} - t_j$

\[ \psi = \frac{\Delta t_j}{\Delta t_{j-2}} \left[ \frac{1 + 2\frac{\Delta t_{j-2}}{\Delta t_{j-1}}}{2 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}} \right] \quad (2.69) \]

\[ \phi = \frac{\frac{\Delta t_j}{\Delta t_{j-1}} \left[ \left( \frac{\Delta t_j}{\Delta t_{j-1}} \right)^2 - 1 \right]}{(1 + \frac{\Delta t_j}{\Delta t_{j-1}}) \left( 1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}} \right)^2 - 1} \quad (2.70) \]

and

\[ E_t(x_k, t_j) = \frac{E_{k,j+1} + \Omega E_{k,j-1} + \Gamma E_{k,j-2} - (1 + \Omega + \Gamma)E_{k,j}}{\Delta t_j - \Omega \Delta t_{j-1} - \Gamma (\Delta t_{j-1} + \Delta t_{j-2})} \quad (2.71) \]

where

\[ \Omega = \left( \frac{\Delta t_j}{\Delta t_{j-1}} \right)^2 \left[ \frac{\Delta t_j}{\Delta t_{j-1}} - \frac{\Delta t_{j-1}}{\Delta t_{j-2}} \left( 1 + \frac{\Delta t_j}{\Delta t_{j-1}} \right) \left( 1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}} \right) \right] \quad (2.72) \]

\[ \Gamma = \frac{\Delta t_j^2}{\Delta t_{j-1} \Delta t_{j-2}} \frac{\left( 1 + \frac{\Delta t_j}{\Delta t_{j-1}} \right)}{\left( 1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}} \right)^2} \quad (2.73) \]
Substitution of expressions (2.68) and (2.71) into the damped wave equation (2.20) and use of the usual centred difference scheme for $E_{xx}$ results in the following recurrence relation

$$(1 + \rho_1)E_{k,j+1} = \lambda_1 (E_{k+1,j} + E_{k-1,j}) + [1 + \psi + \phi + \rho_1(1 + \Omega + \Gamma) - 2\lambda_1]E_{k,j} - (\psi + \rho_1\Omega)E_{k,j-1} - (\phi + \rho_1\Gamma)E_{k,j-2}$$

(2.74)

where

$$\rho_1 = \frac{\sigma\Delta t_{j-1} \left((\frac{\Delta t_{j-1}}{\Delta t_{j-1}})^2 + \psi + (1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2\phi\right)}{2\left(\frac{\Delta t_{j-1}}{\Delta t_{j-1}} - \Omega - \Gamma(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})\right)}$$

(2.75)

and

$$\lambda_1 = \frac{c^2\Delta t_{j-1}^2 \left((\frac{\Delta t_{j-1}}{\Delta t_{j-1}})^2 + \psi + (1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2\phi\right)}{2\Delta x^2}$$

(2.76)

The heat equation is more difficult to solve. If there was no source term (i.e. $\gamma = 0$) then the equation could be solved using the Crank-Nicolson scheme (see Burden et al, 1978). When the wave speed $c$ is constant, inclusion of the source term leads to the Crank-Nicolson type recurrence relation

$$-\mu T_{i+1,j+1} + (1 + 2\mu)T_{i,j+1} = \mu T_{i-1,j+1} - g\gamma_{i,j+1} |E|_{i,j+1}^2$$

$$= \mu T_{i+1,j} + (1 - 2\mu)T_{i,j} + \mu T_{i-1,j} + g\gamma_{i,j} |E|_{i,j}^2$$

(2.77)

where $\mu = \nu\Delta t/(\Delta x)^2$. From equation (2.77) we see that in order to calculate the temperature vector $[T]_{j+1}$, we need first to evaluate $\gamma_{i,j+1}$. But $\gamma_{i,j+1}$ cannot be evaluated until $[T]_{j+1}$ is known. This is resolved by using a linearly extrapolated value of temperature to calculate the source term at time step $j + 1$ and then solving equation (2.18) using the following adaptation of the Crank-Nicolson scheme

$$A[T]_{j+1} = B[T]_{j} + g \left[\gamma |E|^2\right]_{j+1} + g \left[\gamma |E|^2\right]_{j}$$

(2.78)
In this equation $[\gamma|E|^2]_{j+1}$ is evaluated using $[T^*]_{j+1}$ which is extrapolated linearly from $[T]_j$ and $[T]_{j-1}$. The matrices $A$ and $B$ are tridiagonal and are defined by

$$A = \begin{bmatrix}
1 + 2\mu & -\mu & 0 & \ldots & \ldots & \ldots & 0 \\
-\mu & 1 + 2\mu & -\mu & 0 & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & -\mu \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 -\mu & 1 + 2\mu
\end{bmatrix}$$  (2.79)

and

$$B = \begin{bmatrix}
1 - 2\mu & \mu & 0 & \ldots & \ldots & \ldots & 0 \\
\mu & 1 - 2\mu & \mu & 0 & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \mu \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \mu & 1 - 2\mu
\end{bmatrix}$$  (2.80)

The system (2.78) is solved using a matrix decomposition method. The boundary conditions (2.23) for $T$ become

$$T_{0j} = T_{-1j} \quad j = 1, 2, \ldots$$  (2.81)

$$T_{nj} = 0 \quad j = 1, 2, \ldots$$  (2.82)

while the initial condition is

$$T_{k0} = 0 \quad k = 1, ..., n$$  (2.83)

The boundary $x = n\Delta x$ of the computational domain is taken to be large enough so that the temperature remains at the initial temperature there, which results in (2.82).
The zero flux boundary condition $T_x = 0$ at $x = 0$ gives (2.81) on using finite differences and a false boundary at $x = -\Delta x$. The implicit scheme (2.78) is solved by a two pass process as outlined in Burden et al (1978). This involves decomposing the matrix $A$ into an LU decomposition. The source terms in the difference equation (2.78) introduce a nonlinear coupling with the difference equation (2.50) and so a linear stability analysis is not possible. However when there is no source term the Crank-Nicolson scheme is unconditionally stable and the computational results here with source terms have shown no evidence of instability in the calculation of temperature. When the wave speed $c$ is variable, the recurrence relation (2.77) is unchanged because it is the result of averaging the forward difference equation at time $t_j$ with the backward difference equation at time $t_{j+1}$. The size of the time step is the same in both cases.

The numerical solutions are simplified by non-dimensionalising the differential equations. If the electric field at the boundary $x = 0$ is given by

$$E = E_0 e^{-\omega t}$$  \hspace{1cm} (2.84)

then we can define

$$\varphi = \omega t, \xi = \omega x/c_0$$  \hspace{1cm} (2.85)

$$\tilde{E} = E/E_0, T = T\omega/(E_0^2 \gamma_3)$$  \hspace{1cm} (2.86)

where $\gamma_0 = \alpha \gamma_3$ so that expressions (2.29) become

$$\sigma = \sigma_0 (1 + \sigma_1 E_0^2 \gamma_3 T/\omega)^{\sigma_2}$$  \hspace{1cm} (2.87)

$$c = c_0 (1 + c_1 E_0^2 \gamma_3 T/\omega)^{c_2}$$  \hspace{1cm} (2.88)
\[ \gamma = \gamma_0 (1 + \gamma_1 E_0^2 \frac{\gamma_3 T}{\omega})^{\gamma_2} \] (2.89)

while the wave equation (2.20) and the heat equation (2.18) become

\[ E_{\varphi \varphi} + \frac{c}{\omega} E_{\varphi} = \left[ \frac{c}{c_0} \right]^2 E_{\xi \xi} \] (2.90)

\[ T_{\varphi} = \frac{\nu \omega}{c_0^2} T_{\xi \xi} + \frac{\gamma}{\gamma_3} |E|^2 \] (2.91)

The non-dimensional thermal diffusivity is then

\[ \nu' = \nu \omega / c_0^2 \] (2.92)

### 2.5 Numerical results and discussion

In this section numerical solutions of the coupled damped wave equation (2.20) and forced heat equation (2.18) will be presented. Where appropriate, these results will be compared with the predictions of the perturbation theory which was developed in section 2.2. In section 2.2 analytical solutions were derived for some special cases. In each of these cases, the amplitude of the electric field changes as the wave travels through the material. By choice of parameters, this change can be either an attenuation or an amplification. In most cases of physical significance the electric field will be attenuated and so, in order to simplify the present discussion, the change in the wavetrain will be referred to as attenuation; however the remarks apply equally well to amplification. Each of the analytical solutions of section 2.2 show that the amplitude is attenuated as the wave travels through the material and that the attenuation rate changes with time. The well known solution for constant coefficients (2.41) also predicts that the amplitude will...
be attenuated as the wave travels through the material, but the attenuation rate, at any point in the material, remains constant. The time dependence of the attenuation rate is the difference between the solutions for constant and temperature dependent material properties.

In Figure 2 the perturbation solution (2.44), the constant coefficients solution (2.41) and the numerical solution are compared for the case where $\alpha = 0.1$, $c_1 = c_2 = 0$, $\sigma_1 = \gamma_1 = 1$, $\sigma_2 = \gamma_2 = 1/3$, $\nu = 0$. The numerical and perturbation solutions (lines A and C in Figure 2) show very good agreement and they differ significantly from the solution for constant coefficients (line B in Figure 2). The comparison is for time $t = 15$. Figure 2(a) shows the amplitude of the electric field as a function of $x$, while Figure 2(b) shows the spatial distribution of temperature. The numerical and perturbation
Figure 3: Electric field amplitude and temperature far behind the wavefront for the case $\alpha = 0.1, c_1 = c_2 = 0, \sigma_1 = \gamma_1 = 1, \sigma_2 = \gamma_2 = 1/3. A = \text{numerical solution}, B = \text{constant coefficients solution}, C = \text{perturbation solution}$. Solutions both predict a higher temperature at the insulated boundary ($x = 0$) than does the constant coefficients solution. Similar agreement between the numerical solution and the perturbation solution was found at other times $t$, while the difference between these results and the result for constant coefficients increases with time. The same agreement between the numerical and perturbation solutions is evident in the value of the peak temperature, at the boundary $x = 0$. However, for $x > 0$, the perturbation and numerical solutions for temperature move progressively apart with increasing time of heating. In Figure 3 the solutions for the same case as Figure 2 are shown, but at a later time, $t = 50$, and it is seen that the agreement between the amplitude of the numerical and perturbation solutions is still excellent, while the perturbation solution underestimates the temperature throughout much of the heated region. This is due to
the perturbation solutions of section 2.2 not being uniformly valid as \( t \to \infty \). The perturbation series (2.22) for \( T \) assumes that \( T_1 = O(1) \). However, it can be seen from (2.30) that when \( t = O(\alpha^{-2}) \), \( T_1 = O(\alpha^{-1}) \) for \( \gamma_2 < 1 \). The non-uniform validity of the perturbation solution is not a great drawback in the case of microwave heating due to the short timescale over which heating occurs. In the development of the analytical solutions in section 2.2, the thermal diffusion term was assumed small. In the numerical solutions presented in the present section, we take \( \nu = 0 \), but the agreement between the numerical and perturbation solutions for the amplitude of the electric field is substantially unchanged for \( \nu \sim O(0.1) \) and there is still good agreement for \( \nu \sim O(1) \). However, the choice of \( \nu > 0 \) does change the solution for temperature, particularly the peak value at \( x = 0 \), this being due to the boundary layer at \( x = 0 \), whose presence was noted in section 2.2. As seen in Figure 4, taking \( \nu = 0.1 \) causes a substantial reduction in the peak value of the temperature, due to the high values of \( T_{xx} \) near \( x = 0 \). Higher values of \( \nu \) cause a further reduction in the peak temperature and a wider, flatter, bell shape for the spatial distribution of temperature.

In Figure 5 the perturbation solution (2.46), the constant coefficients solution (2.41) and the numerical solution are compared for the case where \( \alpha = 0.1 \), \( c_1 = c_2 = 0 \), \( \sigma_1 = \gamma_1 = 1 \), \( \sigma_2 = \gamma_2 = \frac{1}{2} \). The results are similar to those in Figure 2. The numerical and perturbation solutions show very good agreement and differ significantly from the constant coefficients solution. Both Figure 2 and Figure 5 are for the same heating time \( (t = 15) \). Figure 5 shows a higher temperature at the insulated boundary. This is consistent with the power of temperature in the energy absorption coefficient (2.29)
Figure 4: Temperature for the case $\alpha = 0.1, c_1 = c_2 = 0, \sigma_1 = \gamma_1 = 1, \sigma_2 = \gamma_2 = 1/3, \nu = 0.1$. $A =$ numerical solution, $B =$ constant coefficients solution, $C =$ perturbation solution.

Figure 5: Electric field amplitude and temperature for the case $\alpha = 0.1, c_1 = c_2 = 0, \sigma_1 = \gamma_1 = 1, \sigma_2 = \gamma_2 = 1/2$. $A =$ numerical solution, $B =$ constant coefficients solution, $C =$ perturbation solution.
Figure 6: Electric field amplitude and temperature for the case $\alpha = 0.1, c_1 = c_2 = 0, \sigma_1 = \gamma_1 = 0.1, \sigma_2 = \gamma_2 = 2$. $A =$ numerical solution, $B =$ constant coefficients solution, $C =$ perturbation solution increasing from $1/3$ to $1/2$. The comparison of the perturbation solution (2.47), the constant coefficients solution (2.41) and the numerical solution for the case $\alpha = 0.1, c_1 = c_2 = 0, \sigma_2 = \gamma_2 = 2, \sigma_1 = \gamma_1$ is not so straightforward. The choice of $\sigma_1 = \gamma_1 = 1$, as above, caused the temperature to increase rapidly, resulting in numerical overflow long before time $t = 15$ was reached. This behaviour is due to a hot spot forming at $x = 0$.

In Figure 6 the solutions are compared for $\sigma_1 = \gamma_1 = 0.1$ at time $t = 15$. Again the agreement between the perturbation solution and the numerical solution is very good with the difference between these solutions and (2.41) increasing with time. When the wave speed is not constant the perturbation solution (2.48) for the region just behind the wavefront, the constant coefficients solution and the numerical solution can be compared for $c_1 = \sigma_1 = \gamma_1 = 0.2, c_2 = \sigma_2 = \gamma_2 = 0.5$. This comparison is shown in Figure 7 and
Figure 7: Electric field amplitude and temperature for the case $\alpha = 0.1\ldots$ $\sigma_1 = \gamma_1 = 0.2\ldots\sigma_2 = \gamma_2 = 0.5$. $A$ = numerical solution, $B$ = constant coefficients solution, C = perturbation solution.

It can be seen that there is good agreement between the perturbation and numerical solutions.

The phenomena of hot spots and temperature runaway occur when the rate of absorption of energy exceeds the heat transport rate. This occurs when the energy absorption increases with temperature at a sufficiently fast rate (see for example Roussy et al, 1987, or Brodwin et al, 1991). A hot spot is formed when the temperature rises rapidly to a high level within some small isolated region of the material, while temperature runaway occurs when the rapid temperature rise occurs over a substantial region of the material. In the forced heat equation (2.18) the heat transport rate is governed by the coefficient $\nu$ and the numerical solutions for various values of $\nu$ show that even small values (eg $O(0.1)$ for the nondimensionalised coefficient) can significantly lower the peak tempera-
ture. This indicates that the energy absorption rate must be much larger than the heat transport rate in order for hot spots or temperature runaway to occur. Roussy et al (1987) modelled this by using an energy absorption coefficient which is a quadratic function of the temperature. In the present work power laws in temperature have been used to describe the properties of the material. Coleman (1991) predicted that temperature runaway will occur for $\gamma_2 > 1$ in equation (2.29). The rapid blow-up of the numerical solution for $\gamma_2 = 2$ indicated above supports this prediction. The temperature distribution shows a rapid steepening at the insulated boundary. Numerical solutions show that this temperature peak still occurs when the heat transport coefficient $\nu$ is $O(1)$. Coleman (1991) and Kriegsmann et al (1990) have expressed a need for caution in adopting exponential or power law models for the physical parameters. These models will only be realistic within some restricted range of temperature and there will be a saturation effect, with each physical parameter approaching a constant value for high enough temperature. When a saturation value is imposed on the absorption coefficient $\gamma$, there is still a temperature peak at the insulated boundary, but the maximum temperature and the steepness of the peak depend on the value of the saturation temperature. The rapid growth of the absorption coefficient $\gamma$ can also be modelled by choosing $\gamma_1, \gamma_2 < 0$ in (2.29). As the temperature approaches the critical value $T = -1/\gamma_1$, the value of $\gamma$ becomes infinite. For a saturation temperature less than this critical temperature but sufficiently close to it, the temperature distribution shows a steep peak. The insulated boundary condition, which was adopted for the analysis, ensures that the peak temperature occurs at the boundary, whereas a radiative and convective boundary condition
gives a temperature peak in the interior of the body. As the heat transport coefficient \( \nu \) is increased, the temperature distribution becomes flatter and, in the case of a non-insulated boundary, the peak moves further into the body. Thus for a homogeneous, isotropic, semi-infinite body being irradiated by plane waves, and for appropriate choice of values of the heat transport coefficient \( \nu \) and the saturation temperature, there will be a single hot spot (strictly a hot plane) which produces temperature runaway as the heat spreads through the body. If the body had a finite size and was heated in a conventional microwave oven then boundary effects and inequalities in the spatial distribution of microwave power could produce hot spots in various locations in the body prior to the occurrence of temperature runaway.

Hotspots can also be produced by introducing inhomogeneities. When the contaminant has a greater energy absorption capacity than the rest of the body, a temperature peak forms immediately at its location. The formation of a hot spot at an inhomogeneity depends on a large imbalance between the rate of heat generation in the region of the inhomogeneity and the rate of heat transport to the surrounding matrix. The rate of heat generation in the region of the inhomogeneity depends on the size of the inhomogeneity and the conductivity in the region. Figure 8 depicts the development of hotspots at each of two inhomogeneities. In this case the conductivity of the inhomogeneous region is twenty times the conductivity in the surrounding matrix, which produces a difference of 0.5 degrees within \( t = 10^{-8} \). Hotspots will be formed, over longer periods, by much smaller differences in conductivity. The nondimensionalised diffusivity is \( \nu = 0.1 \), ensuring that the heat generated at the hotspots can spread to the surrounding material and
Figure 8: Early stages in the evolution of the spatial distribution of temperature around two equal inhomogeneities, one at $x = 3$, the other at $x = 9$. $A$ = earliest time, $\ldots$, $E$ = latest time.
this heat spread would eventually cause temperature runaway. There is a shadow effect at the inner hotspot \((x = 9)\) caused by the absorption of microwave energy at the outer hotspot \((x = 3)\).

Let us again return to the homogeneous case. It has been seen that temperature runaway occurs when the value of the energy absorption coefficient \(\gamma\) rises from some small base value to a saturated value within some small range of temperature. It is possible that, at the atomic or molecular level, this change from base to saturated value could occur as a step change, and that the step changes could occur randomly with a probability which increases with temperature. This would, in effect, introduce inhomogeneities at random locations once the temperature had risen above some threshold value. As has been shown above, this would lead to the formation of hot spots followed by temperature runaway.
3 Effect of nonlinear thermal diffusivity

3.1 Introduction

Hill (1989) examined the simplest types of exact solutions which apply when all the physical properties of the material have power law dependencies of the type (1.3). In this case, under appropriate restrictions, the one-dimensional equations remain invariant under a stretching group of transformations and accordingly admit certain similarity solutions.

The simple model utilized in both Hill and Smyth (1990) and Coleman (1991) is improved by assuming that the electric field amplitude decays exponentially rather than assuming that it remains constant. The model is consistent with that proposed by Coleman (1990) in examining the Stefan problem for microwave heating. In Chapter 2 solutions were obtained for the case where heat absorption dominates the process of heat diffusion. In that case hotspots were caused by variation in the thermal absorptivity as the temperature increased. Here the electromagnetic properties are assumed constant with both the thermal diffusivity and absorptivity having a nonlinear dependence on temperature. For constant electromagnetic properties, Maxwell’s equations are decoupled from the heat equation and have a well known solution with the electric field amplitude decaying exponentially with the penetration depth $x$. This spatial dependence is incorporated into the heat source term of the forced heat equation, hence the equations are decoupled. The two cases considered are where the thermal diffusivity and the thermal absorptivity have either an exponential or a power law dependence on
temperature. The resulting nonlinear heat equations are reduced to ordinary differential equations using one-parameter group similarity transformations. For some special cases, these ordinary differential equations are solved analytically; in all other cases they are solved numerically.

The nonlinear heat equation for a linear conducting medium of density \( \rho \) becomes

\[
\rho c(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k(T) \frac{\partial T}{\partial x} \right) + q(T)|E|^2, \tag{3.1}
\]

where \( c(T) \), \( k(T) \) and \( q(T) \) denote the temperature dependent specific heat, thermal conductivity and thermal absorptivity respectively while \( \mu(T) \), \( \epsilon(T) \) and \( \sigma(T) \) denote the temperature dependent magnetic permeability, electric permittivity and electrical conductivity of the medium respectively and \( |E|^2 \) denotes the square of the modulus of the complex electric intensity. In order to present a simplified account of microwave heating Coleman (1990) is followed and it is assumed here that \( |E|^2 \) decays exponentially with distance

\[
|E|^2 = E_0^2 e^{-\kappa x}, \tag{3.2}
\]

for certain constants \( E_0 \) and \( \kappa \). This assumption is made because firstly, it is a well known result in the case when the permeability, permittivity and conductivity are known to be constants, say \( \mu_0 \), \( \epsilon_0 \) and \( \sigma_0 \) respectively, in which case \( \kappa \) is given by

\[
\kappa = \omega (2\mu_0\epsilon_0)^{\frac{1}{2}} \left[ \left(1 + \left( \frac{\sigma_0}{\epsilon_0\omega} \right)^2 \right)^{\frac{1}{2}} - 1 \right]^{\frac{1}{2}}, \tag{3.3}
\]

where \( \omega \) denotes the wave frequency (see Tralli (1963) or Metaxas and Meredith (1983)). Secondly, the assumption (3.2) will be locally valid within a limited region, depending on the variation in \( \mu(T) \), \( \epsilon(T) \) and \( \sigma(T) \). Thirdly, it is worth emphasizing that the
assumption (3.2) pertains to the modulus of $E$ rather than $E$ itself so that so long as $E(x,t)$ takes the form

$$E(x,t) = E_0e^{-\kappa x/2}e^{i\Theta(x,t)},$$

(3.4)

for some real function $\Theta(x,t)$, the assumption (3.2) remains valid. Thus (3.2) represents the simplest possible spatial dependence which has some physical basis and which enables the heating aspects of the problem to be isolated from the electrical and magnetic fields.

Metaxas and Meredith (1983) present experimental evidence which indicates that the physical properties of the material have a power law dependence on temperature. In particular if it is assumed that $c(T)$, $k(T)$ and $q(T)$ have a power law dependence then on introducing a new temperature variable $T^*$ defined by

$$T^* = \rho \int c(T)dT,$$

(3.5)

and rescaling $x$, then on dropping the asterisk, it is not difficult to show that the heat equation becomes

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( T^m \frac{\partial T}{\partial x} \right) + \alpha e^{-\beta x} T^n,$$

(3.6)

for certain constants $m$, $n$, $\alpha$ and $\beta$. Physical requirements indicate that the source term due to microwave heating decreases spatially and increases with temperature so that $n > 0$, $\alpha > 0$ and $\beta > 0$. Similarly, assuming constant specific heat $c(T)$ and that $k(T)$ and $q(T)$ have an exponential dependence on temperature then a similar procedure yields

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( e^{\gamma T} \frac{\partial T}{\partial x} \right) + \alpha e^{-\beta x} e^{\delta T},$$

(3.7)

for certain constants $\gamma$ and $\delta$ where $\delta > 0$ for a heat source which is an increasing
function of temperature. Equations (3.6) and (3.7) constitute the two basic models which we consider in this paper.

The next three sections deal with equations (3.6) while the final three sections of the paper deal with (3.7). In the following section a similarity solution of (3.6) valid for \( m \neq 0 \) and \( n \neq m + 1 \) is presented. Solutions appropriate to the special cases \( n = m + 1 \) and \( m = 0 \) are discussed separately in Sections 3.3 and 3.4 respectively. Similarly in Section 3.5 a similarity solution of (3.7) is presented assuming \( \gamma \neq 0 \) and \( \gamma \neq \delta \) and then solutions applying to the special cases \( \gamma = \delta \) and \( \gamma = 0 \) are presented in Sections 3.6 and 3.7 respectively.

Finally in this section various results are briefly presented for time independent solutions of these equations. For steady solutions of (3.6) and (3.7), with \( T = T(x) \) we may readily deduce,

\[
\frac{d^2 y}{dx^2} + \alpha (m + 1) e^{-\beta x} y^{n/(m+1)} = 0, \quad y = T^{m+1},
\]

\[
\frac{d^2 z}{dx^2} + \alpha \gamma e^{-\beta z} z^{k/\gamma} = 0, \quad z = e^{\gamma T},
\]

from which it is apparent that the special cases \( n = m + 1 \) and \( \gamma = \delta \) give rise to linear equations and in both cases the transformation \( \xi = e^{-\beta x/2} \) produces equations which have Bessel function solutions, thus

\[
\frac{d^2 y}{d\xi^2} + \frac{1}{\xi} \frac{dy}{d\xi} + \frac{4\alpha (m + 1)}{\beta^2} y = 0, \quad (3.10)
\]

\[
\frac{d^2 z}{d\xi^2} + \frac{1}{\xi} \frac{dz}{d\xi} + \frac{4\alpha \gamma}{\beta^2} z = 0. \quad (3.11)
\]

These equations have solutions of the form \( C_1 J_0(k\xi) + C_2 Y_0(k\xi) \) where \( C_1 \) and \( C_2 \) denote arbitrary constants and \( k = 2[\alpha(m + 1)]^{1/2}/\beta \) or \( k = 2(\alpha \gamma)^{1/2}/\beta \). If \( n \neq m + 1 \) or \( \gamma \neq \delta \)
then possibly both of (3.9) admit simple solutions of the form $Ae^{Bx}$ where $A$ and $B$ are constants such that

$$B = \frac{\beta(m + 1)}{(n - m - 1)} , \quad B^2 + \alpha(m + 1)A^{(n-m-1)/(m+1)} = 0, \quad (3.12)$$

$$B = \frac{\beta\gamma}{(\delta - \gamma)} , \quad B^2 + \alpha\gamma A^{(\delta-\gamma)/\gamma} = 0, \quad (3.13)$$

which certainly have solutions with $A < 0$ when $(n-m-1)/(m+1)$ or $(\delta-\gamma)/\gamma$ is an odd integer and assuming $\alpha(m + 1)$ and $\alpha\gamma$ are both positive. However, in general it appears to be difficult to obtain further simple analytical expressions for solutions of (3.9) (see Murphy (1960), page 387). A careful examination of each of the above special steady solutions indicates that there are no steady state solutions of the microwave heating problem such that $T > 0$ at all times and $T$ tends to zero as $x$ tends to infinity. In subsequent sections a number of transient solutions of (3.6) and (3.7) are examined which arise from the invariance of these equations under simple one-parameter transformation groups. Since it is not possible to impose arbitrary boundary and initial conditions for such solutions, these solutions are illustrated by assuming that both $T(x,t)$ and $\frac{\partial T}{\partial x}(x,t)$ are prescribed on the boundary $x = 0$ at some time $t = t_1$ and the temperature profile at time $t_1$ is displayed. These numerical results indicate the presence of moving fronts.

### 3.2 Power law thermal conductivity and heat source

In this section a similarity solution of (3.6) is examined which applies for $m \neq 0$ and $n \neq m + 1$. The appropriate version of the solution for these special cases is detailed in the subsequent two sections. It is a simple matter to show that (3.6) remains invariant
under the one-parameter group of transformations

\[ x_1 = x + ae, \quad t_1 = e^t, \quad T_1 = e^{bt}T, \]

(3.14)

provided the constants \( a \) and \( b \) are given by

\[ a = \frac{(m + 1 - n)}{\beta m}, \quad b = -\frac{1}{m}, \]

(3.15)

in which case two invariants of (3.14) are \( Te^{x/ma} \) and \( te^{-x/a} \) and therefore the functional form of the solution corresponding to (3.14) is given by

\[ T(x, t) = e^{-x/ma}\phi(\xi), \quad \xi = te^{-x/a}, \]

(3.16)

where \( \phi \) denotes some function which is determined by substitution of (3.16) into the partial differential equation (3.6). From (3.15) and (3.16) it can be observed that this solution is only meaningful for \( m \neq 0 \) and \( n \neq m + 1 \).

In addition from (3.16) it is seen that the solution must satisfy an initial condition of the form

\[ T(x, 0) = e^{-x/ma}\phi(0), \]

(3.17)

and therefore we can if necessary accommodate the usual zero initial condition \( T(x, 0) = 0 \) by simply taking \( \phi(0) = 0 \). Further, at \( x = 0 \) the solution (3.16) would satisfy any one of the time dependent boundary conditions

\[ T(0, t) = \phi(t), \quad \frac{\partial T}{\partial x}(0, t) = -\frac{mt\phi'(t) + \phi(t)}{ma}, \]

(3.18)

\[ \frac{\partial T}{\partial x}(0, t) + \frac{T(0, t)}{ma} = -\frac{t\phi'(t)}{a}, \]

(3.19)

where here primes denote differentiation with respect to \( t \) and the function \( \phi(t) \) is not arbitrary but is that determined by solving the ordinary differential equation (3.20). If
both $a$ and $m$ are positive the solution and its partial derivatives all tend to zero as $x$ tends to infinity. If however $a$ is negative, zero temperature at infinity might be achieved by the condition $\phi(\infty) = 0$.

On substituting (3.16) into (3.6) it is possible to deduce the second order ordinary differential equation

$$\frac{d\phi}{d\xi} = \frac{1}{ma^2} \left\{ \frac{d}{d\xi} \left[ m\xi^2 \phi^m \frac{d\phi}{d\xi} + \left( \frac{m+2}{m+1} \right) \xi^{m+1} \phi^{m+1} \right] + \frac{\phi^{m+1}}{m(m+1)} + \alpha ma^2 \phi^n \right\}, \quad (3.20)$$

which appears not to admit any simple first integrals (unless $\alpha(m+1) < 0$ and $n = m+1$ which is not possible here) and accordingly must be solved numerically. Note however that in terms of $\psi = \phi^{m+1}(\phi) = \psi^{1/(m+1)}$ equation (3.20) takes on the alternative compact form,

$$a^2 \psi^{-(m+1)} \left\{ \frac{d\psi}{d\xi} - \alpha \psi \frac{\xi^{m+1}}{m+1} \right\} = \left( \xi \frac{d}{d\xi} + \frac{(m+1)}{m} \right) \left( \xi \frac{d\psi}{d\xi} + \frac{(m+1)}{m} \psi \right). \quad (3.21)$$

In order to illustrate the behaviour of solutions of (3.20) we write the equation as a pair of first order ordinary differential equations,

$$\frac{d\phi}{d\xi} = \omega, \quad \frac{d\omega}{d\xi} = \left[ \frac{a^2}{\phi^m \xi^2} - \frac{(3m+2)}{m\xi} \right] \omega - \frac{(m+1)\phi}{m^2 \xi^2} - \frac{\alpha a^2 \phi^{-m}}{\xi^2}, \quad (3.22)$$

and assume that both $T(x,t)$ and $\frac{\partial T}{\partial x}(x,t)$ are prescribed on the boundary $x = 0$ at some fixed time $t = t_1$ so that from (3.19)$_1$ and (3.19)$_2$ we may deduce $\phi(t_1)$ and $\omega(t_1)$ which can be used as starting conditions in a Runge-Kutta scheme to determine $\phi(\xi)$ and hence the complete temperature distribution at time $t = t_1$. In the numerical results shown in Figures 9-12 we adopt throughout the values

$$\beta = 1, \quad t_1 = 10, \quad T(0,t_1) = 2.3, \quad \frac{\partial T}{\partial x}(0,t_1) = 0, \quad (3.23)$$
Figure 9: Variation of $\phi(\xi)$ and $T(x,t)$ for (124) for the case $a$ positive and $m$ negative ($m = -1, n = 2$)

and the A, B and C shown on the curves refers to the three values of $\alpha$ considered, namely $\alpha = 1/2$ (A), $\alpha = 1$ (B) and $\alpha = 3/2$ (C). We consider the two cases $a > 0$ and $a < 0$ separately.

The case $a > 0$ arises if either $m < 0$ and $n > m + 1$ or if $m > 0$ and $n < m + 1$. Further in this case $\xi$ defined by (2.3) maps the $x$ interval $(0, \infty)$ into the $\xi$ interval $(0, t)$. The two sub-cases $m < 0$ and $m > 0$ give qualitatively different behaviour for $\phi(\xi)$.

If $m < 0$ then both $\phi(\xi)$ and $\phi'(\xi)$ approach zero as $\xi$ tends to zero (that is, as $x$ tends to infinity) while for $m > 0$ there is a value $\xi = \xi_0$ ($0 < \xi_0 < t_1$) such that $\phi(\xi_0) = 0$ and $\lim_{\xi \to \xi_0} \phi'(\xi_0) = \infty$ and with $\phi(\xi)$ not defined by (3.22) for $\xi < \xi_0$. The latter situation corresponds to a moving front and for a zero initial condition it is appropriate to take $\phi(\xi) = 0$ for $\xi < \xi_0$. Moreover, this qualitative behaviour of solutions coincides with the well-known behaviour of solutions of the nonlinear diffusion equation. Typical curves
corresponding to these two sub-cases are shown in Figures 9 and 10.

The case \( a < 0 \) arises if either \( m < 0 \) and \( n < m + 1 \) or if \( m > 0 \) and \( n > m + 1 \) and in this case \( \xi \) defined by (3.16) maps the \( x \) interval \((0, \infty)\) into the \( \xi \) interval \((t, \infty)\). If \( m < 0 \) the solutions for \( \phi(\xi) \) are monotonically increasing with \( \xi \) and give rise to a variety of temperature distributions as indicated in Figure 11(b). For \( m > 0 \) the solutions for \( \phi(\xi) \) initially have the appearance of an exponential but then move steeply to zero at \( \xi = \xi_0 \) with \( \phi(\xi) = 0 \) for \( \xi > \xi_0 \) and again a moving front is exhibited. Typical curves corresponding to \( m < 0 \) and \( m > 0 \) are shown in Figures 11 and 12 respectively.
Figure 11: Variation of $\phi(\xi)$ and $T(x, t)$ for (124) for the case $a$ and $m$ negative ($m = n = -1$)

Figure 12: Variation of $\phi(\xi)$ and $T(x, t)$ for (124) for the case $a$ negative and $m$ positive ($m = 1, n = 3$)
3.3 Power law dependence with \( n = m + 1 \)

If \( n = m + 1 \) then the one-parameter group (3.14) becomes

\[
x_1 = x, \quad t_1 = e^{\epsilon t}, \quad T_1 = e^{-\epsilon/m} T,
\]

so that in this case two invariants are \( x \) and \( T_1^{1/m} \) and accordingly the functional form of the solution is

\[
T(x, t) = t^{-1/m} \phi(x),
\]

which is simply a separable solution and it is not difficult to show that \( n = m + 1 \) is an essential condition for the existence of solutions of the form \( T(x, t) = f(x)g(t) \). On substituting (3.25) into (3.6) it can be deduced that

\[
\frac{d}{dx} \left( \phi^m \frac{d\phi}{dx} \right) + \alpha e^{-\beta x} \phi^{m+1} = -\frac{\phi}{m},
\]

and the substitution \( y = \phi^{m+1} \) yields,

\[
\frac{d^2y}{dx^2} + \alpha(m + 1)e^{-\beta x} y = -\frac{(m + 1)}{m} y^{1/(m+1)}.
\]

By rearranging this equation as a pair of first order ordinary differential equations, as described in the previous section, a numerical solution can readily be generated for \( y(x) \) for given values of \( y(0) \) and \( \frac{dy}{dx}(0) \). The temperature profiles at time \( t_1 \) shown in Figure 13(a) apply for \( m = -1/2 \) with the values

\[
\beta = 1, \quad t_1 = 10, \quad T(0,t_1) = 0, \quad \frac{\partial T}{\partial x}(0,t_1) = 0,
\]

while those shown in Figure 13(b) apply for \( m = 1/2 \) with

\[
\beta = 1, \quad t_1 = 10, \quad T(0,t_1) = 100, \quad \frac{\partial T}{\partial x}(0,t_1) = 0,
\]
Figure 13: Temperature variation for (133) for the cases of \( m \) negative \( (m = -1/2) \) and \( m \) positive \( (m = 1/2) \)

and again A, B and C denotes the temperature profile appropriate to the three values of \( \alpha; 1/2, 1 \) and \( 3/2 \) respectively. Since the corresponding curves for both \( y(x) \) and \( \phi(x) \)
are similar to those shown in the figures, the former curves are not presented.

### 3.4 Power law dependence with \( m \) zero

When \( m = 0 \) equation (3.6) remains invariant under the one-parameter group of transformations

\[
x_1 = x + a \epsilon, \quad t_1 = t + b \epsilon, \quad T_1 = e^{\epsilon T},
\]

where \( a = (n - 1) / \beta \) and \( b \) is arbitrary. Two invariants of this group are \( at - bx \) and \( Te^{\alpha x} \) so that in this case the functional form of the solution becomes

\[
T(x, t) = e^{-\alpha x} \phi(x - \lambda t),
\]
where for \( n \neq 1 \), \( \kappa \) and \( \lambda \) are given by

\[
\kappa = \frac{\beta}{(1-n)}, \quad \lambda = \frac{(n-1)}{\beta b}.
\]  

(3.32)

On substituting (3.31) into (3.6) with \( m \) zero it can readily be deduced that

\[
\phi'' + (\lambda - 2\kappa)\phi' + \kappa^2 \phi + \alpha \phi^n = 0,
\]  

(3.33)

where primes denote differentiation with respect to \( \xi = x - \lambda t \). This equation can be integrated in a number of special cases.

Evidently, if \( \lambda = 2\kappa \) it follows that

\[
\phi'' + \kappa^2 \phi' + \frac{2\alpha}{(n+1)} \phi^{n+1} = C,
\]  

(3.34)

where \( C \) denotes the constant of integration. A further integration gives

\[
\xi = \xi_0 \pm \int_{\phi_0}^{\phi} \left[ C - \kappa^2 \phi^2 - \frac{2\alpha \phi^{n+1}}{(n+1)} \right]^{-1/2} d\phi,
\]  

(3.35)

where \( \xi_0 \) denotes a further arbitrary constant and \( \phi_0 \) denotes \( \phi(\xi_0) \). Special cases of this integral can be evaluated in terms of elliptic functions. However, for purposes of illustrating the temperature profile it is easy to evaluate (3.35) with the aid of Simpson's rule. Now since

\[
\frac{d\phi}{d\xi} = \pm \left[ C - \kappa^2 \phi^2 - \frac{2\alpha \phi^{n+1}}{(n+1)} \right]^{1/2},
\]  

(3.36)

it can be seen that the constant \( C \) is related to the maximum value of \( \phi(\xi) \) by

\[
C = \kappa^2 \phi_{max}^2 + \frac{2\alpha \phi_{max}^{n+1}}{(n+1)}.
\]  

(3.37)

If the positive sign is taken in equation (3.35) \( \phi(\xi) < \phi_{max} \) in the interval \( \xi_0 \leq \xi < \infty \) and \( \phi(\xi) \) tends to \( \phi_{max} \) as \( \xi \) tends to infinity. If the negative sign is adopted then
\( \phi(\xi) < \phi_{\text{max}} \) in the interval \(-\infty < \xi \leq \xi_0 \) and \( \phi(\xi) \) tends to \( \phi_{\text{max}} \) as \( \xi \) tends to minus infinity. If for given \( \xi_0 \) we adopt the positive sign for \( \xi > \xi_0 \) and the negative sign for \( \xi < \xi_0 \) then the resulting temperature profile has a cusp at \( \xi = \xi_0 \) and the function \( \phi(\xi) \) is symmetrical about this point. The physical conditions applying to the microwave heating problem are such that the negative sign is the appropriate choice with \( \xi_0 \) positive. Since if \( \xi_0 \) is negative, the temperature at \( x = 0 \) will be initially positive but will fall to zero before rising to a maximum value. For microwave heating a monotonic temperature increase at \( x = 0 \) is expected and this implies \( \xi_0 \) positive. Further, an initial temperature profile which is non-zero at \( x = 0 \) but which falls to zero at some point \( x = \xi_0 \) within the material being heated is expected. To obtain such an initial temperature profile take the negative sign to define \( \phi \) on \(-\infty < \xi \leq \xi_0 \) and set \( \phi(\xi) \equiv 0 \) for \( \xi_0 < \xi < \infty \). Even though \( \phi(\xi) \) is defined for \( \xi < 0 \), it makes sense for \( T \) to be defined by (3.31) only for \( x \geq 0 \) and \( T \equiv 0 \) for \( x < 0 \). In this case

\[
T(x,t) \rightarrow e^{-\kappa x} \phi_{\text{max}},
\]

as \( t \rightarrow \infty \) so that in particular \( T(0,t) \rightarrow \phi_{\text{max}} \) as \( t \rightarrow \infty \). This solution is illustrated by choosing \( \phi_{\text{max}} = \xi_0 = 1 \) along with the values

\[
\alpha = 0.1, \quad \beta = 0.2, \quad n = 1/2,
\]

and the resulting variation in \( \phi(\xi) \) is shown in Figure 14(a) while the corresponding temperature profiles are shown in Figure 14(b) at three times \( t = 0 \) (A) \( t = 1 \) (B) and \( t = 2 \) (C).
Figure 14: Variation of $\phi(\xi)$ and $T(x,t)$ for (139) at times $t = 0$ (A), $t = 1$ (B) and $t = 2$ (C) and with $\alpha = 0.1$, $\beta = 0.2$ and $n = 1/2$

Other integrals of (3.33) can be obtained by the successive substitutions

$$\phi(\xi) = e^{\mu \xi} \psi(\xi), \quad \eta = e^{\tau \xi},$$

(3.40)

so that equation (3.33) becomes

$$\tau^2 \eta^2 \frac{d^2 \psi}{d\eta^2} + \tau \eta (\lambda - 2\kappa + \tau + 2\mu) \frac{d\psi}{d\eta} + [\mu^2 + \mu(\lambda - 2\kappa) + \kappa^2]\psi + \alpha \eta^{(n-1)/\tau} \psi^n = 0.$$  (3.41)

Thus, if $\mu$ and $\tau$ are chosen such that

$$\lambda - 2\kappa + \tau + 2\mu = 0, \quad \mu^2 + \mu(\lambda - 2\kappa) + \kappa^2 = 0, \quad (n - 1)\mu = 2\tau,$$

(3.42)

then equation (3.41) becomes simply

$$\frac{d^2 \psi}{d\eta^2} + \frac{\alpha \psi^n}{\tau^2} = 0,$$

(3.43)

which integrates once to give

$$\left( \frac{d\psi}{d\eta} \right)^2 + \frac{2\alpha \psi^{n+1}}{(n + 1)\tau^2} = C,$$

(3.44)
and a further integration yields

\[ \eta = \eta_0 \pm \int_0^\psi \left[ C - \frac{2\alpha \psi^{n+1}}{(n+1)\tau^2} \right]^{-1/2} d\psi, \quad (3.45) \]

where \( C \) and \( \eta_0 \) denote constants of integration. From (3.42)_1 and (3.42)_3 it is found that \( \mu \) and \( \tau \) are given by

\[ \mu = \frac{2(2\kappa - \lambda)}{(n+3)}, \quad \tau = \frac{(n-1)(2\kappa - \lambda)}{(n+3)}, \quad (3.46) \]

so that from (3.42)_2 this integration procedure is possible for (3.33) provided the constants \( \kappa \) and \( \lambda \) are such that

\[ \lambda = \left\{ 2 \pm \frac{(n+3)}{[2(n+1)]^{1/2}} \right\} \kappa. \quad (3.47) \]

Thus, in this case, for all \( n \) except \( n \leq -1 \) there is a real value of \( \lambda \) for which equation (3.33) can be formally integrated. From (3.32) and (3.47) it is seen that the possible values of the constant \( b \) are determined from the equation

\[ \frac{1}{b} = - \left( \frac{\beta}{n-1} \right)^2 \left\{ 2 \pm \frac{(n+3)}{[2(n+1)]^{1/2}} \right\}. \quad (3.48) \]

Again for purposes of illustration Figure 15(c) gives two temperature profiles corresponding to \( t = 0(A) \) and \( t = 10(B) \) for \( n = 2 \) and these are obtained by direct numerical integration of (3.45) employing the values \( \alpha = \beta = 1 \). The procedure adopted follows that used for (3.35) apart from the additional transformation (3.40). Since we require \( \xi \) in the range \( -\infty < \xi \leq \xi_0 \) for some positive \( \xi_0 \), this means that we require \( \eta \) in the range \( -\infty < \eta \leq e^{\tau\xi_0} \) for \( \tau > 0 \) and \( e^{\tau\xi_0} \leq \eta < \infty \) for \( \tau < 0 \). From equation (3.31) we see that \( \lambda > 0 \) represents heating while \( \lambda < 0 \) represents cooling so for heating the
Figure 15: Variation of $\phi(\xi)$, $\psi(\eta)$ and $T(x, t)$ for (139) at times $t = 0$ (A) and $t = 10$ (B) and with $\alpha = \beta = 1$ and $n = 2$
appropriate \( \lambda \) values are

\[
\lambda = \begin{cases} 
2 - \frac{(n + 3)}{[2(n + 1)]^{1/2}} \kappa, & n > 1, \\
2 + \frac{(n + 3)}{[2(n + 1)]^{1/2}} \kappa, & n < 1,
\end{cases}
\] (3.49)

and with this choice of \( \lambda \) we find that \( \tau < 0 \) for \( n > 1 \) while \( \tau > 0 \) for \( n < 1 \). Thus for \(-1 < n < 1\) we take \( \lambda \) given by (3.50) and integrate over \(-\infty < \eta \leq e^{\tau \xi_0} \) where \( \tau > 0 \) while for \( n > 1 \) we take \( \lambda \) given by (3.49) and integrate over \( e^{\tau \xi_0} \leq \eta < \infty \) where \( \tau < 0 \). From (3.32) and these values of \( \lambda \) we see that the wave speed \( \lambda \) is discontinuous at \( n = 1 \) since we have

\[
\lim_{n \to 1^-} \lambda = \infty, \quad \lim_{n \to 1^+} \lambda = 0,
\] (3.51)

and moreover for \( n > 1 \), the wave speed \( \lambda \) increases slowly (for \( \beta = 0.2, \lambda = 0.004 \) for \( n = 2 \) while \( \lambda = 0.008 \) for \( n = 5 \)). For \( n = 2 \) and \( \alpha = \beta = 1 \) Figures 15(a) and 15(b) show \( \psi(\eta) \) as determined by (3.45) and the corresponding \( \phi(\xi) \) as obtained from (3.40). In utilizing (3.45) we have taken the plus sign, \( \eta_0 = 1, \psi_{max} = 1 \) and where the constant \( C \) is given by

\[
C = \frac{2\alpha \psi_{max}^{n+1}}{(n+1)\tau^2} = \frac{2\alpha}{(n+1)\tau^2}.
\] (3.52)

In the special case of (3.30) corresponding to \( b \) zero a simple separable solution may be deduced as follows. In this case the functional form of the solution is simply

\[
T(x,t) = e^{-\kappa x} \phi(t),
\] (3.53)

where for \( n \neq 1 \) \( \kappa \) is still defined by (3.32) and from (3.6) we may readily deduce the Bernoulli equation

\[
\phi' = \kappa^2 \phi + \alpha \phi^n,
\] (3.54)
where here the prime denotes differentiation with respect to time. In the usual way we obtain

\[ \phi(t) = \{Ce^{-(n-1)\kappa^2 t} - \alpha/\kappa^2\}^{1/(1-n)}, \] (3.55)

where \( C \) denotes the constant of integration. For \( n < 1 \) \( C \) can be chosen such that initially the temperature is zero and the solution becomes

\[ T(x, t) = e^{-\kappa x} \left\{ \frac{\alpha}{\kappa^2} \left(e^{(1-n)\kappa^2 t} - 1\right) \right\}^{1/(1-n)}. \] (3.56)

If \( n > 1 \) then for \( \beta > 0, \kappa < 0 \) and \( T(x, t) \) tends to infinity with \( x \). In addition if \( C > \alpha/\kappa^2 \) then "blow-up" occurs after a finite time \( t_c \) given by

\[ t_c = \frac{1}{(n-1)\kappa^2} \log \left( \frac{C\kappa^2}{\alpha} \right). \] (3.57)

Finally in this section it is noted that the linear case \( n = 1 \) admits separable solutions of the form

\[ T(x, t) = e^{-\lambda t} y(x), \] (3.58)

where \( \lambda \) (assumed positive) denotes the separation constant and with the usual substitution \( \xi = e^{-\beta x/2} \), \( y \) satisfies

\[ \frac{d^2 y}{d\xi^2} + \frac{1}{\xi} \frac{dy}{d\xi} + \frac{4}{\beta^2} \left( \alpha + \frac{\lambda}{\xi^2} \right) y = 0, \] (3.59)

which has solutions of the form

\[ y(\xi) = C_1 J_\nu(2\alpha^{1/2}\xi/\beta) + C_2 J_{-\nu}(2\alpha^{1/2}\xi/\beta), \] (3.60)

where \( C_1 \) and \( C_2 \) denote arbitrary constants which are possibly complex and for positive \( \Lambda \) the index \( \nu = 2i\Lambda^{1/2}/\beta \) is pure imaginary.
3.5 Exponential thermal conductivity and heat source

In this section a similarity solution of (3.7) is examined which applies for $\gamma \neq 0$ and $\delta \neq \gamma$.

Solutions corresponding to these special cases are examined in the following sections.

Equation (3.7) remains invariant under the one-parameter group of transformations

$$x_1 = x + a\epsilon, \quad t_1 = e^{t}t, \quad T_1 = T + b\epsilon,$$

(3.61)

provided the constants $a$ and $b$ are given by

$$a = \frac{(\gamma - \delta)}{\beta \gamma}, \quad b = \frac{1}{\gamma}.$$

(3.62)

Two invariants of this group are $T - bx/a$ and $te^{-x/a}$ so that the functional form of the solution corresponding to (5.1) is

$$T(x, t) = \frac{bx}{a} + \phi(\xi), \quad \xi = te^{-x/a},$$

(3.63)

where as usual $\phi(\xi)$ is determined by substitution of (3.63) into (3.7). From these equations we observe that this solution is only sensible provided both $a$ and $\gamma$ are non-zero.

From (3.63) it is apparent that the above solution necessarily has an initial condition of the form

$$T(x, 0) = \frac{bx}{a} + \phi(0),$$

(3.64)

and would satisfy at $x = 0$ one of the following time dependent boundary conditions

$$T(0, t) = \phi(t), \quad \frac{\partial T}{\partial x}(0, t) = \frac{b}{a} - \frac{t}{a}\phi'(t),$$

(3.65)

where here primes denote differentiation with respect to time and as previously mentioned $\phi(t)$ is that function produced by solving the ordinary differential equation (3.66).
On substitution of (3.63) into equation (3.7), after some rearrangement it is found that

\[
\frac{d\phi}{d\xi} = \frac{1}{\alpha^2} \left\{ \frac{d}{d\xi} \left[ \xi e^{\gamma \phi} \left( \xi \frac{d\phi}{d\xi} + \frac{1}{\gamma} \right) \right] + \alpha \alpha^2 e^{\delta \phi} \right\},
\]

(3.66)

and since this equation appears not to admit any simple first integrals, it must also be solved numerically. As previously described in Section 3.2 equation (3.66) is replaced by a pair of first order ordinary differential equations and it is assumed that both \( T(x,t) \) and \( \frac{\partial T}{\partial x}(x,t) \) are prescribed on the boundary \( x = 0 \) at some fixed time \( t = t_1 \). In the numerical results we adopt precisely the values given by (3.16) and consider the usual three values of \( \alpha \), namely \( \alpha = 1/2 \) (A), \( \alpha = 1 \) (B) and \( \alpha = 3/2 \) (C). The nature of the solution depends on whether \( \alpha \) is positive or negative and in either case there are two possibilities.

For \( \alpha > 0 \) there are two possibilities \( \gamma > \delta \) and \( \gamma > 0 \) or \( \gamma < \delta \) and \( \gamma < 0 \) which
are shown in Figures 16 and 17 respectively. In the first case ($\gamma = 2$ and $\delta = 1$) $\phi(\xi)$ increases steadily as $\xi$ decreases from $\xi = t_1$. This behaviour is shown in Figure 16(a) while the corresponding temperature variation is shown in Figure 16(b). It is noted that because of the different scales employed in these two figures, they are actually consistent with each other, which is not apparent from inspection. In the second case ($\gamma = -1$ and $\delta = 1$) $\phi(\xi)$ becomes unbounded and tends to minus infinity as $\xi$ tends to zero and therefore the initial condition (3.64) is not defined in this case. The behaviour is shown in Figure 17.

For $a < 0$, the two possibilities are $\gamma < \delta$ and $\gamma > 0$ and $\gamma > \delta$ and $\gamma < 0$ which are shown in Figures 18 and 19 respectively. In the first case ($\gamma = 1$ and $\delta = 2$) there exists some $\xi_0 > t_1$ such that both $\phi(\xi)$ and $\frac{d\phi}{d\xi}(\xi)$ rapidly approach minus infinity as $\xi$ tends to $\xi_0$ indicating a moving front with $\phi(\xi) = -\infty$ for $\xi > \xi_0$. This behaviour is not
Figure 18: Variation of $\phi(\xi)$ and $T(x,t)$ for (171) for the case $\gamma < \delta$ and $\gamma > 0$ ($\gamma = 1$, $\delta = 2$)

Figure 19: Variation of $\phi(\xi)$ and $T(x,t)$ for (171) for the case $\gamma > \delta$ and $\gamma < 0$ ($\gamma = -1$, $\delta = -2$)
clear from the portion of the curves shown in the figure. In the second case ($\gamma = -1$ and $\delta = -2$) both $\phi(\xi)$ and $\frac{d\phi}{d\xi}(\xi)$ tend to infinity as $\xi$ tends to some $\xi_0$ as is clearly apparent from Figure 19. In this case the trivial solution for $\xi > \xi_0$ is $\phi(\xi) = \infty$.

### 3.6 Exponential dependence with $\delta = \gamma$

If $\delta = \gamma$ then $a = 0$ and the one-parameter group (3.61) becomes

$$
x_1 = x, \quad t_1 = e^t t, \quad T_1 = T - e/\gamma,
$$

(3.67)

in which case two invariants are $x$ and $T + \gamma^{-1} \log t$ and therefore the functional form of the solution becomes

$$
T(x, t) = -\frac{\log t}{\gamma} + \phi(x).
$$

(3.68)

This solution has the form $T(x, t) = f(x) + g(t)$ and it can be readily shown that the condition $\delta = \gamma$ is an essential condition for the existence of solutions of this form.

Substitution of (3.68) into (3.7) with $\delta = \gamma$ readily yields

$$
\frac{d}{dx} \left( e^{\gamma \phi} \frac{d\phi}{dx} \right) + \alpha e^{-\beta x} e^{\gamma \phi} = -\frac{1}{\gamma},
$$

(3.69)

and the substitution $z = e^{\gamma \phi}$ gives

$$
\frac{d^2 z}{dx^2} + \alpha e^{-\beta x} z = -1.
$$

(3.70)

This equation can be solved in a routine manner by means of the transformation $\xi = e^{-\beta x}$ which gives an inhomogeneous Bessel's equation, namely

$$
\frac{d^2 z}{d\xi^2} + \frac{1}{\xi} \frac{dz}{d\xi} + \frac{4\alpha \gamma}{\beta^2} z = -\frac{4}{\beta^2 \xi^2},
$$

(3.71)
Figure 20: Variation of $z(\xi)$ and $T(x,t)$ for (176) at time $t_1 = 1$ and with $\beta = 1.0$ and $\gamma = 2.0$

which has the general solution

$$z(\xi) = C_1 J_0(\eta) + C_2 Y_0(\eta) + \frac{2\pi}{\beta^2} \int_{\eta}^{\infty} [J_0(s)Y_0(\eta) - Y_0(s)J_0(\eta)] \frac{ds}{s},$$

(3.72)

where $C_1$ and $C_2$ denote arbitrary constants and $\eta = 2(\alpha \gamma)^{1/2} \xi / \beta$.

Figure 20 shows the typical variation of $z(\xi)$ and the temperature profile $T(x,t)$ which are obtained by numerical integration of (3.71), rather than use of (3.72), and the following values have been employed

$$\beta = 1, \ \gamma = 2, \ t_1 = 1, \ T(0,t_1) = 2.3, \ \frac{\partial T}{\partial x}(0,t_1) = 0,$$

(3.73)

and as usual $A$, $B$ and $C$ designate the three reference values of $\alpha$, namely $1/2$, $1$ and $3/2$ respectively. These results again indicate the appearance of a moving front with $T(x,t) = -\infty$ as the appropriate value of the temperature outside the front.
3.7 Exponential dependence with $\gamma$ zero

If $\gamma$ is zero then on introducing a new temperature variable $\bar{T}$ defined by

$$\bar{T}(x,t) = \delta T - \beta x,$$  \hspace{1cm} (3.74)

it can be seen that (3.7) becomes

$$\frac{\partial \bar{T}}{\partial t} = \frac{\partial^2 \bar{T}}{\partial x^2} + \alpha \delta e^T,$$  \hspace{1cm} (3.75)

which is formally identical to the simple model used by Hill and Smyth (1990) which does not incorporate spatial exponential decay of the source term. In this case equation (3.75) remains invariant under the one-parameter group of transformations

$$x_1 = e^t x, \quad t_1 = e^{2\epsilon} t, \quad \bar{T}_1 = \bar{T} - 2\epsilon,$$  \hspace{1cm} (3.76)

for which the invariants are $xt^{-1/2}$ and $\bar{T} + 2 \log x$ so that the functional form of the solution becomes

$$\bar{T}(x,t) = -2 \log x + \phi(\xi), \quad \xi = xt^{-1/2}.$$  \hspace{1cm} (3.77)

It can be observed that in terms of the original temperature variable $T$, this solution necessarily has an initial condition of the form

$$T(x,0) = \frac{1}{\delta} \{\beta x - 2 \log x + \phi(\infty)\}.$$  \hspace{1cm} (3.78)

On substituting (3.77) into (3.75) it is easy to deduce the second order nonlinear ordinary differential equation,

$$\xi^2 \phi'' + \frac{\xi^3}{2} \phi' + 2 + \alpha \delta e^\phi = 0,$$  \hspace{1cm} (3.79)

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which again needs to be solved by a numerical scheme. Observe that if we use

$$T(x, t) = -\log t + \psi(\xi), \quad \xi = xt^{-1/2}, \quad (3.80)$$

instead of (3.77) then in this case we obtain from (3.75) the slightly simpler differential equation

$$\psi'' + \frac{\xi}{2} \psi' + 1 + \alpha \delta e^\psi = 0, \quad (3.81)$$

which can be reconciled with (3.79) by means of the relation

$$\phi(\xi) = \psi(\xi) + 2 \log \xi. \quad (3.82)$$

Altogether from (3.74) and (3.77) it is seen that the temperature $T(x, t)$ is given by

$$T(x, t) = \frac{1}{\delta} \left\{ \beta x - 2 \log(x + x_0) + \phi \left( \frac{x + x_0}{(t + t_0)^{1/2}} \right) \right\}, \quad (3.83)$$

for arbitrary constants $x_0$ and $t_0$. Figure 21 shows the variation of $\phi(\xi)$ and $T(x, t)$ for
(3.83) assuming the following values

\[ t_1 = 10, \beta = \delta = x_0 = t_0 = 1, \quad T(0, t_1) = 2.3, \quad \frac{\partial T}{\partial x}(0, t_1) = 0, \]  

(3.84)

and as usual \( A \), \( B \) and \( C \) designate the three reference values of \( \alpha \).

### 3.8 Conclusion

The microwave heating of a infinite slab has been modelled by equations of the form (3.6) and (3.7) where \( \alpha \) and \( \beta \) designate positive constants. These models incorporate a heat source term which decays exponentially with distance and increases with increasing temperature. Some simple similarity temperature profiles have been examined for special cases of the models. These solutions enable the partial differential equations to be reduced to ordinary differential equations for which numerical solutions have been obtained. Since for the similarity solutions under discussion it is not possible to impose arbitrary boundary and initial conditions, the strategy of solving the various ordinary differential equations assuming that both the temperature \( T(x, t) \) and the temperature gradient \( \frac{\partial T}{\partial x}(x, t) \) are prescribed at the boundary \( x = 0 \) at some fixed time \( t = t_1 \) has been adopted, enabling the temperature profile at this fixed time \( t_1 \) to be displayed. These numerical results indicate the appearance of moving fronts. For the model (3.6) with both \( m \) and \( n \) positive, the fronts move into a region of zero temperature which is a valid trivial solution of the governing equation. This model is therefore entirely consistent with the observed characteristics of microwave heating of materials which are known to exhibit all the classical phenomena associated with nonlinear diffusion, such as
“blow-up” which is referred to as “hot-spots” and “waiting-time” phenomena. That is, materials are known to remain at the initial temperature for a finite time when subjected to microwave radiation and then suddenly the temperature starts to increase. Moreover, certain materials are known to be either completely transparent to microwave radiation or respond after the application of conventional heating and these characteristics are also embedded in the model (3.6). However, although the model (3.7) also predicts moving fronts and typical phenomena associated with nonlinear diffusion, the associated trivial solution of (3.7) for $\gamma$ and $\delta$ both positive is $T = -\infty$ while for $\gamma$ and $\delta$ both negative it is $T = \infty$. These may be physically unrealistic and in addition, because $T = 0$ is not a trivial solution of (3.7) the model may not admit the possibility that certain materials can be transparent to microwave radiation.
4 Slowly varying wavespeed.

In Chapter 2 perturbation solutions were developed, for small electrical conductivity and for small thermal diffusivity. For constant wavespeed, various exact solutions were obtained for the first order amplitude term. Marchant and Smyth (1992) considered non-Ohmic electrical conductivity and thermal absorptivity and found a perturbation solution for small constant conductivity and constant wavespeed. Thus solutions have been obtained for constant wavespeed. In this chapter, the effect of variations in the wave speed is considered.

In the present work the technique of Smyth (1990 and 1992) is used to consider more general power law dependencies for the material properties than used in Smyth (1990). The method of strained co-ordinates is used to eliminate secular terms and ensure the solution is uniformly valid. However, due to the more general power law dependencies, an explicit solution can only be obtained by assuming that the electrical conductivity is small, and introducing a long length scale (as the electric field amplitude is now slowly decaying). The explicit solution is then obtained using the method of multiple scales. In section 4.1 the governing equations are derived. In section 4.2 the methods of multiple scales and strained co-ordinates are used to develop a uniformly valid perturbation solution. In section 4.3 comparisons are made between numerical solutions of the governing equations and the perturbation solutions developed in section 4.2. In particular, an example involving thermal runaway (a hot-spot) is considered.
4.1 Governing Equations

In this work the microwave heating of a material is considered in the case where the electrical permittivity and the magnetic permeability are both slowly-varying functions of the temperature and where the electrical conductivity is small and constant. It was shown in Chapter 2 that for materials with small electrical conductivity $\sigma$ and slowly-varying electrical permittivity $\varepsilon$ and magnetic permeability $\mu$, Maxwell's equations (1.7) and (1.8) can be reduced to the damped wave equation

$$E_{tt} + A E_t = c^2 E_{xx}, \quad (4.1)$$

where

$$A = \mu_0 / \mu + 2\varepsilon_0 / \varepsilon + \sigma / \varepsilon - c_x x / \varepsilon, \quad (4.2)$$

and $\mu$ and $\varepsilon$ are both slowly-varying and $\sigma$ is small. As (4.1) is temperature dependent, it is coupled with the forced heat equation (2.18) which describes the absorption and diffusion of heat. The damped wave equation (4.1) and the forced heat equation (2.18) are non-dimensionalised by the scalings

$$t' = \omega_0 t, \quad x' = \omega_0 x / c_0, \quad E' = E / E_0, \quad T' = T / T_0, \quad \mu' = \mu / \mu_0, \quad \varepsilon' = \varepsilon / \varepsilon_0 \quad (4.3)$$

where $\omega_0, \mu_0, \varepsilon_0, c_0 = (\mu_0 \varepsilon_0)^{-1/2}$, $E_0$ and $T_0$ are the reference frequency, magnetic permeability, electrical permittivity, wavespeed, electric field amplitude and temperature respectively. This results in the scaled thermal diffusivity, thermal absorptivity, electrical conductivity, wavespeed and frequency having the forms

$$\nu' = \nu \omega_0 / c_0^2, \quad \gamma' = \gamma E_0^2 / \omega_0 T_0, \quad \sigma' = \sigma / \varepsilon_0 \omega_0, \quad c' = c / c_0 \quad \omega' = \omega / \omega_0. \quad (4.4)$$
The governing equations in non-dimensional form are then (with the primes dropped)

\[ E_{tt} + AE_t = c^2 E_{xx}, \]
\[ T_t = \nu T_{xx} + \gamma |E|^2, \]  

with \( A \) given by (2.4). The boundary conditions adopted are

\[ E = e^{i\omega t}, \quad T_x = 0 \text{ at } x = 0, \]  

while the initial conditions are

\[ E = 0, \quad T = T_i \text{ at } t = 0. \]  

The boundary conditions indicate that microwave radiation of constant amplitude unity (as the electric field amplitude is scaled by its value at the boundary) and non-dimensional frequency \( \omega \) is incident upon a thermally insulated material (hence no heat flow occurs through the boundary). The zero heat flux boundary condition is valid in the small Biot number limit. The Biot number measures the relative effects of heat convection and radiation to heat diffusion. Hence the zero heat flux boundary condition applies if heat loss from the material is slight (for example, the Biot number \( \sim 10^{-4} \) for ceramics, see Kriegsmann (1992)). Initially, no microwave radiation is present and the material is at a uniform temperature \( T_i \).

To obtain a solution to equations (4.5), (4.6) and (4.7) the method of Smyth (1990) is adopted. The frequency of the radiation is assumed large (\( \omega \gg 1 \)) and a geometric optics (WKB) expansion is performed. The form

\[ E = \phi(x, t)e^{i\omega \theta} + \omega^{-1} \phi_1(x, t)e^{i\omega \theta} + \ldots, \quad \omega \gg 1, \]
is assumed, where the phase function $\theta$ represents the fast oscillations of the wavetrain, while the amplitude terms $\phi$ and $\phi_1$ are modulated by slow variations only (as the wavetrain properties vary slowly over the scale of an extremely small wavelength). Expansion (4.8) is substituted into the first of (4.5) and expanded in powers of $\omega$. At $0(\omega^2)$ the eikonal equation is obtained

$$\theta_t + c\theta_x = 0,$$

which indicates that the wavetrain travels at speed $c$. At $0(\omega)$ the transport equation

$$2\phi_t \theta_t + \theta_x \phi_t + A \theta_t - \phi_1 \theta_1^2 = c^2 (2\phi_x \theta_x + \phi \theta_{xx} - \phi_x \phi_x),$$

is obtained. Using (4.9) in (4.10) gives the transport equation in the form

$$\phi_t + c \phi_x = -\frac{\phi}{2}[A - c_x + c_t/c],$$

which governs the modulation of the leading order amplitude $\phi$. Equation (4.11) is the same as equation (2.14) of Smyth (1990) (except for a factor of 2 missing from the term $\epsilon_t/\epsilon$ in Smyth’s (2.14)). Equation (4.11) is also identical to (2.28) of Chapter 2, which was derived under the assumption of small electrical conductivity and slowly-varying electrical permittivity and magnetic permeability. The forced heat equation (the second of (4.5)) is considered in the limit of no diffusion ($\nu = 0$)

$$T_t = \gamma |\phi|^2.$$  

This assumption is valid as the time scale for microwave heating is small compared to the time scale over which the diffusion of heat occurs. For example, the scaled diffusivity $\nu = 1 \times 10^{-9}$ for ceramics (see Marchant and Smyth (1992)). Equations (4.11) and (4.12)
are subject to the initial and boundary conditions

\[
\phi(0,t) = 1, \quad \phi(x,0) = 0, \quad T(x,0) = T_i. \tag{4.13}
\]

## 4.2 Asymptotic solutions

A perturbation solution to (4.11) and (4.12), derived in the limit of high-frequency radiation and zero diffusion, is developed by expanding the leading order amplitude \( \phi \) and temperature \( T \) in a small parameter \( \alpha \), which measures the size of the electrical conductivity and the size of the variations in the electrical permittivity and magnetic permeability. Power-law relations are chosen for the material properties

\[
\begin{align*}
\sigma &= \alpha \sigma_0, \\
\mu &= 1 + \alpha \mu_1 T^{m_2}, \\
\epsilon &= 1 + \alpha \epsilon_1 T^{n_2}, \\
c &= (\mu \epsilon)^{-1/2} = 1 - \frac{\alpha}{2} \mu_1 T^{m_2} - \frac{\alpha}{2} \epsilon_2 T^{n_2},
\end{align*}
\tag{4.14}
\]

where \( \mu, \epsilon \) and hence \( c \) have been scaled by their values at \( T = 0 \). In addition the thermal absorptivity is chosen to have the power law relation

\[
\gamma = \gamma_0 T^m. \tag{4.15}
\]

Physically for constant wavespeed, the thermal absorptivity must be proportional to the electrical conductivity as then a balance is achieved between energy loss from the microwave radiation and heat absorption by the material.

If the ordering

\[
\omega^{-1} \ll \alpha \ll 1, \tag{4.16}
\]
of the parameters is assumed, then $\phi_1$ (which is the second-order amplitude term in the geometric optics expansion) can be ignored as a higher-order term in the calculation of the perturbation series for the leading order amplitude $\phi$ (which is a series in $\alpha$).

To simplify the analysis, (4.11) and (4.12) are written in terms of one variable only, which is facilitated by the temperature $T$ being transformed to

$$\theta = T^{1-\gamma}, \quad \gamma_1 \neq 1,$$

(4.17) so that (4.12) becomes

$$\theta_t = \gamma_0(1 - \gamma_1)|\phi|^2,$$

(4.18) and the transformed thermal absorptivity is a constant $\gamma_0(1 - \gamma_1)$. Then substituting (4.17) into (4.11) gives

$$\theta_{tt} + \left(1 - \frac{\alpha}{2} \left(\mu_1 \theta^{\frac{\mu_2}{1-\gamma}} + \epsilon_1 \theta^{\frac{\epsilon_2}{1-\gamma}}\right)\right) \theta_{tx} = a \theta_t \left(a + b_1 \theta^{\frac{\mu_2}{1-\gamma}} - 1 \theta_t + b_2 \theta^{\frac{\mu_2}{1-\gamma}} - 1 \theta_x + c_1 \theta^{\frac{\epsilon_2}{1-\gamma}} - 1 \theta_t + c_2 \theta^{\frac{\epsilon_2}{1-\gamma}} - 1 \theta_x\right),$$

(4.19) where

$$a = -\sigma_0, \quad b_1 = -b_2 = -\frac{\mu_1 \mu_2}{2(1 - \gamma_1)}, \quad c_1 = 3c_2 = -\frac{3\epsilon_1 \epsilon_2}{2(1 - \gamma_1)}.$$

Equation (4.19) is a second-order partial differential equation for $\theta$, subject to the initial condition

$$\theta = \theta_i = T_i^{1-\gamma} \quad \text{at} \quad t = 0,$$

(4.20) and the boundary condition

$$\theta = \theta_i + \gamma_0(1 - \gamma_1)t \quad \text{at} \quad x = 0.$$

(4.21)
Equation (4.21) is the integral with respect to time of (4.18) at the boundary. \( \theta \) (and hence \( T \)) is known at the boundary \((x = 0)\) because the electric field strength is constant and no heat diffusion occurs.

Smyth (1990) obtained a perturbation solution to (4.11) using the method of strained co-ordinates. This method allows non-uniformities to be eliminated by introducing a new variable, \( \tau \), which is the characteristic variable. As the power law relations used in Smyth (1990) are linear an explicit solution is obtained. Here the more general power law relations (4.14) are used and it is found that to obtain an explicit solution, a long length scale (characterised by \( \alpha \)) must be introduced (the electric field is now decaying on this long length scale) and the method of multiple scales used as well. So the partial differential equation (4.19) is solved by applying the methods of strained co-ordinates and multiple scales which, as explained above, eliminates the non-uniformities and allows an explicit solution to be obtained. A solution of the form

\[
\begin{align*}
\theta &= \theta_0(x, X, \tau) + \alpha \theta_1(x, X, \tau) + \ldots, \\
\tau &= t_0(x, X, t) + \alpha t_1(x, X, t) + \ldots,
\end{align*}
\]

where \( X = \alpha x \),

is sought, where \( \tau \) is the characteristic variable (the stretched co-ordinate) and \( X \) is the long length variable. Substituting (4.22) into equation (4.19) gives \( O(1) \)

\[
\theta_{\tau\tau}(t_0^2 + t_0 x t_0) + \theta_{\tau}(t_0 t_0 + t_0 x t_0) + \theta_{\tau x} t_0 = 0.
\]

Choosing

\[
t_0 t + t_0 x = 0,
\]

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and integrating (4.24) gives

\[ t_0 = t - x. \] (4.25)

Hence as the characteristic speed to first-order is unity, \( \tau \) is the characteristic variable as expected. So (4.23) becomes

\[ \theta_{0x\tau} = 0, \] (4.26)

which when integrated gives

\[ \theta_0 = f_0(X, \tau) + g_0(X, x). \] (4.27)

So the first-order solution is the sum of an arbitrary function of \( \tau \), the characteristic variable and \( X \), the long length variable and an arbitrary function of the two length variables. \( \theta_0 \) is determined by eliminating secular terms at second-order. At \( O(\alpha) \) the expansion is

\[
\theta_{0\tau\tau} \left( t_{1t} + t_{1x} + \frac{1}{2} \left( \mu_1 \theta_0^{\frac{\mu_2}{1-\gamma_1}} + \epsilon_1 \theta_0^{\frac{\epsilon_2}{1-\gamma_1}} \right) \right) + \theta_{0r} (t_{1tt} + t_{1xt}) + (\theta_0 X\tau + \theta_1 x\tau)
\]

\vspace{2pt}

\[ = \theta_{0\tau} \left( a + (b_1 - b_2) \theta_0^{\frac{\mu_2}{1-\gamma_1} - 1} + (c_1 - c_2) \theta_0^{\frac{\epsilon_2}{1-\gamma_1} - 1} \right). \] (4.28)

Choosing

\[ t_{1t} + t_{1x} = -\frac{1}{2} \left( \mu_1 \theta_0^{\frac{\mu_2}{1-\gamma_1}} + \epsilon_1 \theta_0^{\frac{\epsilon_2}{1-\gamma_1}} \right), \] (4.29)

in (4.28) gives the characteristic variable to second-order as

\[
t_1 = -\frac{(t - \tau)}{2} \left( \mu_1 \theta_0^{\frac{\mu_2}{1-\gamma_1}} + \epsilon_1 \theta_0^{\frac{\epsilon_2}{1-\gamma_1}} \right). \] (4.30)

Hence (4.28) becomes

\[ \theta_{0X\tau} + \theta_{1x\tau} = -\theta_{0r}^2 \left( \frac{\mu_1 \mu_2}{2(1 - \gamma_1)} \theta_0^{\frac{\mu_2}{1-\gamma_1} - 1} + \frac{\epsilon_1 \epsilon_2}{2(1 - \gamma_1)} \theta_0^{\frac{\epsilon_2}{1-\gamma_1} - 1} \right) + a \theta_{0r}. \] (4.31)
The amplitude at first-order is determined by choosing

$$\theta_{0X\tau} = a\theta_{0\tau}.$$  \hfill (4.32)

Integrating (4.32) and using the boundary condition (4.21) gives

$$\theta_0 = \theta_i + \gamma_0 (1 - \gamma_1) \tau e^{-\sigma_0 X},$$  \hfill (4.33)

as the solution to (4.32). Hence along a characteristic (which has a fixed value of $\tau$) the first-order solution displays exponential decay (over the long length scale $X$), as expected. Using (4.33) in (4.31) gives

$$\theta_1 = \frac{1}{2} x \gamma_0 (1 - \gamma_1) e^{-\sigma_0 X}$$

$$\left[ \mu_1 \left( \theta_i + \gamma_0 (1 - \gamma_1) \tau e^{-\sigma_0 X} \right)^{\frac{\mu_0}{1-\gamma_1}} - \mu_1 \theta_1^{\frac{\mu_0}{1-\gamma_1}} \right]$$

$$+ \epsilon_1 \left( \theta_i + \gamma_0 (1 - \gamma_1) \tau e^{-\sigma_0 X} \right)^{\frac{\epsilon_0}{1-\gamma_1}} - \epsilon_1 \theta_1^{\frac{\epsilon_0}{1-\gamma_1}},$$  \hfill (4.34)

as the second-order solution, where the initial condition $\theta_1 = 0$ at $\tau = 0$ and the boundary condition $\theta_1 = 0$ at $x = 0$ is used as the first-order solution already satisfies the initial condition (4.20) and the boundary condition (4.21). The solution (4.22) is

$$\theta = \theta_0 + a\theta_1 + \ldots,$$  \hfill (4.35)

$$\tau = t - x - \frac{\alpha x}{2} \left( \mu_1 \theta_0^{\frac{\mu_1}{1-\gamma_1}} + \epsilon_1 \theta_0^{\frac{\epsilon_1}{1-\gamma_1}} \right) + \ldots,$$

with $\theta_0$ and $\theta_1$ given by (4.33) and (4.34). The solution in the original variables $T$ and $\phi$ is

$$T = \theta_0^\frac{1}{1-\gamma_1},$$

$$= \theta_0^{\frac{1}{1-\gamma_1}} + \frac{\alpha}{1-\gamma_1} \theta_1 \theta_0^{\frac{\epsilon_1}{1-\gamma_1}} + \ldots$$
\[ T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \left( T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \right)^{\frac{\mu_2}{1-\gamma_1}} - \frac{\alpha}{2} x \gamma_0 e^{-\sigma_0 x} \left( T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \right)^{\frac{\mu_2}{1-\gamma_1}} \]

\[ \left[ \mu_1 \left( T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \right)^{\frac{\mu_2}{1-\gamma_1}} + \epsilon_1 \left( T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \right)^{\frac{\epsilon_2}{1-\gamma_1}} \right] - \mu_1 T_i^{\mu_2} - \epsilon_1 T_i^{\epsilon_2} \right] + \ldots , \quad (4.36) \]

and

\[ \phi = \left( \frac{\theta_i}{\gamma_0 (1 - \gamma_1)} \right)^{1/2} = e^{-\sigma_0 x/2} \left( 1 - \frac{\alpha x}{2} \gamma_0 e^{-\sigma_0 x} \left( \mu_1 \mu_2 \left( T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \right)^{\frac{\mu_2}{1-\gamma_1} - 1} + \epsilon_1 \epsilon_2 \left( T_i^{1-\gamma_1} + \gamma_0 (1 - \gamma_1) r e^{-\sigma_0 x} \right)^{\frac{\epsilon_2}{1-\gamma_1} - 1} \right) \right) \right] + \ldots . \quad (4.37) \]

The method of strained co-ordinates has been used to eliminate non-uniformities in the solution for \( \theta \) (hence in \( T \) and \( \phi \) as well). This requires \( |\alpha \theta_1 / \theta_0| \ll 1 \) as \( \tau \to \infty \), which is satisfied if

\[ \frac{\mu_2}{1 - \gamma_1} \leq 1 \text{ and } \frac{\epsilon_2}{1 - \gamma_1} \leq 1. \quad (4.38) \]

The power-laws for the material properties (4.14) remain uniformly valid as the temperature increases if both \( \mu_2 \) and \( \epsilon_2 \) are negative. If both \( \mu_2 \) and \( \epsilon_2 \) are positive then the solution remains valid for \( t \ll O\left( \alpha^{\frac{\gamma_1 - 1}{\mu_2}} \right) \) (if \( \mu_2 \geq \epsilon_2 \)). In addition, the method of multiple scales, employed to allow an explicit solution to be found, assumes that the long length scale \( X = 0(1) \). Hence the expansion (4.35) is valid for

\[ 0 \leq x, \ t \leq 0 \left( \alpha^{-1} \right) , \quad (4.39) \]

as \( x \) and \( t \) are of the same order.
For a fixed time \( t \), the electric field amplitude \( \phi \) decays exponentially on the long length scale \( X \) from the initial amplitude unity at the boundary to

\[
e^{-\sigma_0 X^2 / 2} \left( 1 - \frac{\alpha \gamma_0 x}{2} e^{-\sigma_0 X \left( \mu_1 \mu_2 T_i^{\mu_2 + \gamma_1 - 1} + \epsilon_1 \epsilon_2 T_i^{\epsilon_2 + \gamma_2 - 1} \right)} + \ldots \right),
\]

(4.40)

at the wavefront. At a fixed position \( x \), behind the wavefront, the electric field amplitude \( \phi \) can either increase or decrease over time. If the electrical permittivity and the magnetic permeability are increasing functions of temperature (and hence \( \mu_1 \mu_2 \) and \( \epsilon_1 \epsilon_2 \) are both positive) then the effective conductivity (4.2) is decreased as the material heats up, causing the electric field amplitude to increase at a fixed position over time. If the electrical permittivity and the magnetic permeability are decreasing functions of temperature (and hence \( \mu_1 \mu_2 \) and \( \epsilon_1 \epsilon_2 \) are both negative) then the reverse is true, the effective conductivity (4.2) is increased as the material heats up, causing the electric field amplitude to decrease at a fixed position over time. For a fixed time \( t \), the temperature \( T \) decays exponentially on the same scale as the amplitude \( \phi \) from (4.21) at the boundary to the ambient temperature \( T = T_i \) at the wavefront. This occurs because the heat absorption depends on the square of the electric field amplitude at any point. At a fixed position \( x \) the temperature increases like \( O(t^{-1-\gamma_1}) \) for \( \gamma_1 < 1 \), while for \( \gamma_1 > 1 \) thermal runaway (an infinite temperature in finite time) occurs, leading to a hot-spot. The hot-spot occurs at the boundary \( x = 0 \) with the temperature becoming infinite at time \( t = T_i^{1-\gamma_1} / (\gamma_0 (\gamma_1 - 1)) \).
4.3 Numerical Solutions

4.3.1 The numerical scheme

The accuracy of the asymptotic solutions obtained in section 4.2 is examined by comparison with numerical solutions of the governing equations (4.5) with boundary and initial conditions (4.6) and (4.7). The numerical scheme of Chapter 2 is used to solve the governing equations so only an overview of the numerical scheme used is given here.

The damped wave equation (the first of (4.5)) is discretized by using a four-point centred difference scheme in time and a three-point centred difference scheme in space resulting in

\[(1 + \rho_1)E_{k,j+1} = \lambda_1(E_{k+1,j} + E_{k-1,j}) + (1 + \psi + \Phi + \rho_1(1 + \Omega + \Gamma) - 2\lambda_1)E_{k,j}\]

\[-(\psi + \rho_1\Omega)E_{k,j-1} - (\Phi + \rho_1\Gamma)E_{k,j-2},\]  

where

\[
\psi = \frac{\Delta t_j}{\Delta t_{j-2}} \left[1 + 2\frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right],
\]

\[
\Phi = \frac{\Delta t_j/\Delta t_{j-1} \left[\left(\frac{\Delta t_j}{\Delta t_{j-1}}\right)^2 - 1\right]}{(1 + \Delta t_j/\Delta t_{j-1}) \left(\left(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right)^2 - 1\right)},
\]

\[
\Omega = \left(\frac{\Delta t_j}{\Delta t_{j-1}}\right)^2 \left[\frac{\Delta t_j}{\Delta t_{j-1}} - \frac{\Delta t_{j-1}}{\Delta t_{j-2}} \left(1 + \frac{\Delta t_j}{\Delta t_{j-1}}\right) \left(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right)\right],
\]

\[
\Gamma = \frac{\Delta t_j^2}{\Delta t_{j-1}\Delta t_{j-2}} \left(1 + \frac{\Delta t_j}{\Delta t_{j-1}}\right) \left(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right)^2,
\]

\[
\rho_1 = \frac{\sigma\Delta t_{j-1} \left(\left(\frac{\Delta t_j}{\Delta t_{j-1}}\right)^2 + \psi + \left(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right)^2\Phi\right)}{2 \left(\frac{\Delta t_j}{\Delta t_{j-1}} - \Omega - \Gamma \left(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right)\right)},
\]

\[
\lambda_1 = \frac{c^2\Delta t_{j-1}^2 \left[\left(\frac{\Delta t_j}{\Delta t_{j-1}}\right)^2 + \psi + \left(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}}\right)^2\Phi\right]}{2\Delta x^2},
\]
$E_{k,j} = E(k \Delta x, t_j)$ and $\Delta t_j = t_{j+1} - t_j$. The scheme uses a variable time step to ensure stability as the wavespeed $c$ changes. If the numerical scheme is used across the wavefront, unphysical oscillations appear in the solution due to the large $x$-derivatives at the wavefront. To overcome this, the space discretisation is only done up to the wavefront where the exact value for the electric field $E$ is used. The electric field at the wavefront is found by a wavefront expansion (see Whitham (1974) or Chapter 2) and is

\[
E = \frac{e^{-\sigma_0 x/2}}{[1 + \frac{\sigma_0}{\sigma_0} (\epsilon_1 \epsilon_2 T_i^{\nu_2 + \gamma - 1} + \mu_1 \mu_2 T_i^{\nu_2 + \gamma - 1})(1 - e^{-\sigma_0 x/2})]^2}
\]  

(4.48)

along the wavefront $x = c(T_i)t$.

The boundary and initial conditions (4.6) and (4.7) for the electrical field are

\[
E_{0,j} = e^{-i\omega t_j},
\]

(4.49)

\[
E_{k,0} = E_{k,-1} = E_{k,-2} = 0, \quad k \geq 0.
\]

The first of (4.49) represents the microwave radiation of constant amplitude unity and frequency $\omega$ incident upon the boundary $x = 0$, while the second of (4.49) states that $E \equiv 0$ in the material domain for $t \leq 0$.

For the forced heat equation (the second of (4.5)) a variation of the Crank-Nicolson scheme is used

\[
-\kappa T_{k+1,j+1} + (1 + 2\kappa)T_{k,j+1} - \kappa T_{k-1,j+1} - \Delta t_j \gamma_{k,j+1} |E|_{k,j+1}^2 = \kappa T_{k+1,j} + (1 - 2\kappa)T_{k,j} + \kappa T_{k-1,j} + \Delta t_j \gamma_{k,j} |E|_{k,j}^2,
\]

(4.50)

where $T_{k,j} = T(k \Delta x, t_j)$ and $\kappa = \nu \Delta t_j/\Delta x$. If the thermal absorptivity $\gamma$ is zero then (4.50) is the Crank-Nicolson scheme. For non-zero thermal absorptivity however, the calculation of the thermal absorptivity $\gamma_{k,j+1}$ requires the temperature at time $t = t_j$, 85
which is yet to be found. An approximation for the temperature $T_{k,j+1}$ is derived using a linear extrapolation from the previous two time levels. The boundary and initial conditions (4.6) and (4.7) for the temperature become

$$
T_{-1,j} = T_{1,j}, \quad j \geq 0, \quad T_{k,0} = T_i, \quad k \geq 0. \quad (4.51)
$$

The first of (4.51) results from the zero heat flux boundary condition at $x = 0$, while the second of (4.51) states that the temperature is initially $T_i$.

### 4.3.2 Comparison of the solutions

Expression (4.48) represents the exact value of the electric field at the wavefront. Expanding in the small parameter $\alpha$ (retaining terms up to $O(\alpha)$) (4.48) becomes

$$
E = e^{-\alpha \sigma x/2} \left( 1 - \frac{\alpha \gamma_0 x}{2} \left( \mu_1 \mu_2 T_i^{\mu_2 + \gamma - 1} + \epsilon_1 \epsilon_2 T_i^{\epsilon_2 + \gamma - 1} \right) \right), \quad (4.52)
$$

on $x = \left( 1 - \frac{\alpha}{2} \left( \mu_1 T_i^{\mu_2} + \epsilon_1 T_i^{\epsilon_2} \right) \right) t$, as $x = t$ to first-order at the wavefront. Expression (4.52) is the same as (4.40), which is the second-order asymptotic solution for the electric field amplitude $\phi$ at the wavefront. Figure 22 shows the electric field amplitude $|E|$ at the wavefront versus $t$ for $t$ up to 10. The parameters are $T_i = 1, \mu = \epsilon = 1 + 0.1 T^{0.7}, \sigma = 0.1$ and $\gamma = 1$. No choice for the frequency $\omega$ need be made as the electric field amplitude at the wavefront is not a function of frequency. Shown is the first-order asymptotic solution (-A-) and the second-order asymptotic solution (4.40)(-B-). Compared with this is the exact solution at the wavefront (4.48)(-C-). There is an excellent comparison between the exact solution and the second-order asymptotic solution over this range of $t$. 86
Figure 22: Electric field amplitude at the wavefront versus time, for $\sigma = 0.0, \mu = \epsilon = 1 + 0.1T^{0.7}, T_i = 1$ and $\gamma = 1$. $A =$ first order asymptotic solution, $B =$ second order asymptotic solution, $C =$ exact solution.

Figure 23: Electric field amplitude and temperature for the case $\sigma = 0.1, \mu = \epsilon = 1 + 0.1T^{0.7}, T_i = 1, \omega = 5, \gamma = 1$ and $t = 5.6. A =$ numerical solution, $B =$ first order solution, $C =$ second order asymptotic solution.
Figure 23(a) shows the electric field amplitude $|E|$ versus $x$ for $\gamma = 1, \sigma = 0.1, \mu = \epsilon = 1 + 0.1T^{0.7}, \omega = 5, T_i = 1$ and $t = 5.6$. Shown are the numerical solution (4.41) (-A-), the first-order asymptotic solution (-B-) and the second-order asymptotic solution (4.37) (-C-). The second order asymptotic solution predicts a lower electric field amplitude due to the effective conductivity (4.2) being larger at second-order. This occurs if the electrical permittivity and the magnetic permeability are increasing functions of temperature. (and hence $\mu_1\mu_2$ and $\epsilon_1\epsilon_2$ are both positive). The wavefront to first-order is at $x = t = 5.6$, while at second-order it occurs at $x = ct \sim 5.1$. The non-zero ambient temperature at the wavefront causes the second order wavefront to slow down as the wavespeed is lower at second-order. The second-order asymptotic solution compares well with the numerical solution, with the numerical solution exhibiting some frequency dependent oscillations (see §4 of Marchant and Smyth (1992)). Figure 23(b) shows the temperature $T$ versus $x$ for the same parameters as Figure 23(a). Shown are the numerical solution (4.50) (-A-), the first-order asymptotic solution (-B-) and the second-order asymptotic solution (4.36) (-C-). At the boundary $x = 0$ the numerical solution and the two asymptotic solutions coincide (and are equal to $T = T_i + \gamma_0 t = 6.6$) as there is no diffusion. At the wavefront the temperature decays to the ambient value $T_i = 1$. This occurs at $x = t = 5.6$ for the first-order asymptotic solution, while it occurs at $x = ct \sim 5.1$ for the second-order asymptotic solution as the wavespeed is lower at second-order. The comparison between the numerical solution and the second-order asymptotic solution is excellent, the temperature being lower than that predicted by the first-order asymptotic solution due to both the later time at which the wavefront arrives (and at which heating
begins) and the smaller amplitude of the electric field (due to the increased effective conductivity).

Figure 24(a) shows the transformed temperature $\theta$ versus $x$ for $\gamma = T^{1.17}$, $\sigma = 0.1$, $\mu = \epsilon = 1 + 0.1T^{-0.051}$, $\omega = 5$, $T_i = 1$ and $t = 5.6$. Shown are the numerical solution (-A-), the first-order asymptotic solution (-B-), and the second-order asymptotic solution (4.35) (-C-). In this case $\gamma_1 > 1$, so thermal runaway (an infinite temperature in finite time) will occur at the boundary $x = 0$. With an insulated boundary condition the hot-spot occurs at the boundary because this is the point of maximum temperature (as the electric field amplitude has not decayed and there is no heat loss through the boundary). Solution (4.36) predicts the hot-spot formation at time $t = T_i^{1-\gamma_1}/(\gamma_0(\gamma_1 - 1)) \sim 5.9$. Hence Figure 24(a) represents a case rapidly approaching thermal runaway at the boundary (as $t = 5.6$). This can be seen from the small value of $\theta$ at the boundary ($\sim 0.05$) since $\theta \to 0 \Rightarrow T \to \infty$ for $\gamma_1 > 1$. There is an excellent comparison between the numerical solution and the second-order asymptotic solution. Figures 24(b) and (c) show the electric field amplitude $|E|$ and temperature $T$ respectively versus $x$ for the same parameters as Figure 24(a). Shown are the numerical solution (-A-), the first-order asymptotic solution (-B-), and the second-order asymptotic solution (-C-). Since the numerical calculations and asymptotics are performed on the transformed variable $\theta$, which remains bounded, there is an extremely good correspondence between the numerical solution and the second-order asymptotic solution even just before thermal runaway occurs (see Fig 24(c) near $x = 0$). Figure 24(b) shows the wavefront travelling more slowly at second-order as in Figure 23(a); however in this case the second order asymp-
Figure 24: Transformed temperature, electric field amplitude and temperature for the case \( \sigma = 0.1, \mu = \epsilon = 1 + 0.1 T^{-0.051}, T_i = 1, \omega = 5, \gamma = T^{1.17} \) and \( t = 5.6. \)

- \( A \) = numerical solution,
- \( B \) = first order solution,
- \( C \) = second order asymptotic solution
Totic solution predicts a higher electric field amplitude due to the effective conductivity (4.2) being reduced (as the terms $\mu_1 \mu_2$ and $\varepsilon_1 \varepsilon_2$ are negative in this case).
In this chapter the time dependent microwave heating of a semi-infinite material is considered, where the thermal absorptivity and the electromagnetic properties are represented by power laws of the form (1.4), and where the loss factor is small. The analysis uses the high frequency, geometric optics limit adopted by Smyth (1990) and the complete time dependent solution of the coupled system of equations is found as a perturbation series. These solutions are shown to be in good agreement with full numerical solutions of the coupled system of equations. The approximate equations governing the microwave heating of the material in the geometric optics limit will be found to be the same as those governing the heating in the limit of small conductivity, found in Chapter 2.

The form of the loss factor considered here is also different to the form used in earlier chapters. Note that the function $Q$ defined by (1.12) will cause changes in the amplitude of the electric field as it passes through the material but these changes do not represent absorption of energy. Rather, they represent either storage of energy (in the case of $\epsilon_t/\epsilon$ and $\mu_t/\mu$) or reflection or scattering of energy (in the case of $\sigma/\epsilon$). Some of the energy thus imparted to the electrons will be converted into heat by processes such as stresses within atoms or by collisions. Thus some proportion of the energy given to conduction electrons will be converted into heat and, similarly, some proportion of the energy which distorts atoms or reorients molecules will be converted into heat. The atomic and molecular dissipative processes have not been specifically included in the analysis of Maxwell’s equations. They are usually included via an imaginary term in the permittivity (in the case of dielectrics) or in the permeability (in the case of magnetic
materials). This results in an extra term being added to the conductivity term to obtain the total attenuation (see, for example, Metaxas and Meredith, 1983). As an illustration, consider the case where the only loss term is ohmic, where the electric field is sinusoidal with frequency $\omega$ and where the permittivity $\epsilon$ is constant. Then equation (1.8) becomes

$$H_x = -\sigma E - i\omega E = -i\epsilon_0 \epsilon^* \omega E$$

(5.1)

where

$$\epsilon^* = \epsilon/\epsilon_0 - i\sigma/(\epsilon_0 \omega)$$

(5.2)

and $\epsilon_0$ is the permittivity of free space. Thus the loss term appears as the imaginary part of the dielectric constant. All other forms of loss (e.g. due to polarisation) can also be represented by the imaginary part of the dielectric constant

$$\epsilon^* = \epsilon' - i\epsilon''$$

(5.3)

Here also that simplifying approach is adopted and it is assumed that the atomic and molecular energy absorption can be represented by an extension of the conductivity term, by replacing the expression $\sigma/(\omega \epsilon)$ by the loss term $\epsilon''$. Thus, in the case of dielectrics, where conductivity is low, our loss function $\epsilon''(T)$ might be quite large. In the work below it is assumed that

$$\epsilon'' \ll 1,$$

(5.4)

which allows the microwave heating of a wide range of materials to be modelled. However (5.4) excludes some highly polar materials, such as water, for which $\epsilon'' > 1$.

Smyth (1990) set

$$E = \phi(x, t)e^{i\omega \theta(x, t)}, \quad H = \psi(x, t)e^{i\omega \theta(x, t)}$$

(5.5)
and considered the high frequency limit \( \sigma/(\omega\epsilon) \ll 1 \) in nondimensional variables) and was thus able to separate equation (1.9) into equations of \( O(\omega^2) \), \( O(\omega) \) and \( O(1) \). A similar approach is taken here, where \( \omega \) is taken to be the constant frequency of the incident radiation, giving the damped wave equation

\[
c^2(\phi_{xx} + \omega \theta_{xx} \phi + 2i\omega \theta_x \phi_x - \omega^2 \phi_{x}^2) = -L\psi - M(\psi_t + i\omega \theta_t \psi) + P\phi \\
+Q(\phi_t + i\omega \theta_t \phi) + \phi_{tt} + i\omega \theta_{tt} \phi \\
+2i\omega \theta_t \phi_t - \omega^2 \phi_{x}^2.
\]

It was demonstrated in Chapter 1 that for slowly varying \( \epsilon \) and \( \mu \), and for small \( \sigma \), the intrinsic impedance relationship between \( E \) and \( H \), that is

\[
H = (\frac{\epsilon}{\mu})^{\frac{1}{2}} E \tag{5.7}
\]

holds to first order. This result can be extended to the case where \( \sigma \) is not small, but where \( \sigma/(\omega\epsilon) \) is small, that is where \( \omega \) is large, by using the nondimensionalising definitions

\[
t' = \omega t, \quad x' = \frac{\omega x}{c_0}, \quad E' = \frac{E}{E_0}, \quad H' = \frac{H}{H_0}, \quad e' = \frac{\epsilon}{\epsilon_0}, \quad \mu' = \frac{\mu}{\mu_0}, \quad c' = \frac{c}{c_0},
\]

with \( c_0, E_0, H_0, \epsilon_0, \mu_0 \) and \( \omega \) being the values of the wavespeed, the electric field strength, the magnetic field strength, the electrical permittivity, the magnetic permeability and the frequency respectively of the incident radiation. Thus \( c_0 \) is the speed of light in a vacuum and \( H_0 \) and \( E_0 \) satisfy the intrinsic impedance relation, giving

\[
H_0 = (\frac{\epsilon_0}{\mu_0})^{\frac{1}{2}} E_0, \quad c_0 = (\epsilon_0 \mu_0)^{-\frac{1}{2}}. \tag{5.9}
\]

On substituting equations (5.8) and (5.9) into equations (1.7) and (1.8), it is seen that

\[
H'_{xx} = -(e'E')_{x} - e''e'E' \tag{5.10}
\]
\[ E'_{x'} = -\left(\mu'H'\right)_{x'}, \quad (5.11) \]

where \( \epsilon'' = \sigma/(\omega\epsilon) \) is the dielectric loss factor.

From now on, the prime denoting a nondimensionalised variable will be dropped and all variables are understood to be non-dimensional. The dielectric loss factor \( \epsilon'' \), which is essentially a nondimensional variable, will continue to be denoted by the double prime since this is the way it is normally denoted in the literature.

When

\[ \epsilon_t + \epsilon'' = O(\delta) \text{ and } \mu_t = O(\delta), \text{ where } \delta \ll 1, \quad (5.12) \]

we have

\[ H_x = -\epsilon E_t + O(\delta) \quad (5.13) \]

\[ E_x = -\mu H_t + O(\delta), \quad (5.14) \]

from which it follows that the intrinsic impedance relation holds to first order (see Bleaney and Bleaney, 1976). Hence

\[ H = \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} E + O(\delta) \quad (5.15) \]

\[ H_t = \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} E_t + O(\delta). \quad (5.16) \]

Since it has been assumed that \( Q/\omega \ll 1 \), it follows that \( \epsilon'' \ll 1 \) and thus, if \( \epsilon_t, \mu_t \ll 1 \), then

\[ \psi = \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} \phi + O(\delta). \quad (5.17) \]

This can also be shown directly from equations (1.7) and (1.8) by substituting expressions (5.5) and then using the normal method (see Bleaney and Bleaney, 1976) which is used when \( \mu \) and \( \epsilon \) are constant, assuming that (5.12) hold.
It was convenient in the above (see (5.8)) to nondimensionalise by using values of $c$, $E$ and $H$ associated with the incident radiation. However, if the radiation is considered to be incident on the surface of the material, then we must take reflection into account. In the case we are considering, where the loss factor, the permittivity and the permeability are all functions of temperature, this reflection is also temperature dependent. Since the temperature of the material and its surface are determined by the solution of the governing equations, the reflection of the microwave radiation at the surface of the material and the microwave heating of the material are coupled. When the radiation is normally incident on the surface of the material, the ratio of the amplitude of the electric field in the transmitted beam to the amplitude of the electric field in the incident beam is given by

$$r = \frac{2}{(n + 1)},$$

(5.18)

where $n$ is the refractive index of the material (see Bleaney and Bleaney, 1976).

In general, the refractive index is complex and is given by

$$n = \sqrt{[(\mu + i\mu'')(\epsilon + i\epsilon'')]^{\frac{1}{2}}},$$

(5.19)

where $\mu''$ and $\epsilon''$ are the magnetic and electric loss factors respectively. This makes the ratio $r$ in (5.18) complex, which indicates a phase change at the boundary. The ratio $r$ is a function of the temperature through its dependence on the magnetic and electric parameters and is thus a function of time. That is,

$$r = r(T(0, t)) = R(t)e^{it\xi}. $$

(5.20)

Hence the amplitude of the electric field at $x = 0$ is given by $R(t)$. A similar result can
be obtained for the magnetic field $H$ and so we can nondimensionalise with respect to the characteristics of the incident radiation; the incident wavespeed $c_0$, frequency $\omega$ and the amplitudes $E_0$ and $H_0$.

If the nondimensionalising transformations (5.8) are applied to equations (1.10), (1.11) and (1.12), and $\sigma$ is replaced by $\omega \epsilon''$ as discussed above, then

$$L = \frac{\omega^2 \mu_{xt}}{c_0 \epsilon_0 \mu}, \quad M = \frac{\omega \mu_x}{\epsilon_0 \mu},$$

(5.21)

$$P = \omega^2 P', \quad P' = \frac{\mu_{xt}}{\mu \epsilon} + \frac{\epsilon'' \mu_t}{\mu} + \frac{\epsilon\epsilon + \epsilon'' \epsilon_t}{\epsilon},$$

(5.22)

$$Q = \omega Q', \quad Q' = \frac{\mu_t}{\mu} + \frac{2 \epsilon_t}{\epsilon} + \epsilon''.$$  

(5.23)

The nondimensionalising transformations (5.8) are also applied to the damped wave equation (5.6) to obtain, after using (5.17), (5.21), (5.22) and (5.23),

$$c^2(\omega^2 \phi_{xx} + \omega^3 \theta_{xx} \phi + 2 \omega^2 \theta_x \phi_x - \omega^4 \phi \theta_x^2)
= -\omega^2 \frac{\epsilon_{xt}}{\mu} \phi - \frac{\mu x \epsilon}{\mu} (\omega^2 \phi_t + \omega^3 \theta_t \phi) + \omega^2 P' \phi + Q'(\omega^2 \phi_t + \omega^3 \theta_t \phi) + \omega^3 \phi_{tt}$$

$$+ \omega^3 \theta_{tt} \phi + 2 \omega^3 \theta_t \phi_t - \omega^4 \phi \theta_x^2.$$  

(5.24)

Equation (5.24) can be separated into equations $\sim O(\omega^n)$, the first two equations being

$$O(\omega^4): \quad \theta_t^2 = c^2 \theta_x^2$$

(5.25)

$$O(\omega^3): \quad c^2 \theta_{xx} \phi + 2 c^2 \theta_x \phi_x = -\frac{\mu x \epsilon}{\mu} \theta_t \phi + Q' \theta_t \phi + \theta_{tt} \phi + 2 \theta_t \phi_t.$$  

(5.26)

The first of these equations, (5.25), is the eikonal equation and the second, (5.26), is the transport equation of geometric optics. The solution

$$\theta_t = -c \theta_x$$  

(5.27)
to equation (5.25) is chosen, representing a wave travelling in the direction of increasing \( x \). By substituting (5.27) into equation (5.26), it is seen that

\[
\phi_t + c\phi_x = \frac{1}{2}(c\frac{\mu}{\mu} - Q' - \frac{\mu}{\varepsilon} + c_x)\phi
\]

\[
= -\frac{1}{4}(\frac{\mu}{\mu} - c\frac{\mu}{\mu} + 3\frac{\mu}{\varepsilon} + c\frac{\mu}{\varepsilon} + 2\varepsilon')\phi.
\]

The amplitude equation (5.28) is the same as that found in Chapter 2 under the assumption that the conductivity \( \sigma \) is small. This is to be expected as the perturbation parameter in the present work is \( \sigma_0/(\omega \varepsilon_0) \), which is small if either \( \omega \) is large or \( \sigma_0 \) is small.

Thus the microwave heating of the halfspace \( x > 0 \) will be governed by equations (1.6), (5.27) and (5.28) if it is assumed that \( \varepsilon'' \ll 1 \) and that the electromagnetic properties are slowly varying. For plane waves normally incident on the surface \( x = 0 \), we adopt as our boundary conditions

\[
\phi(0, t) = R(t), \quad \theta(0, t) = -t + \xi(t).
\]  

Note that \( R(t) \) and \( \xi(t) \) are not determined at this stage since both \( R(t) \) and \( \xi(t) \) depend on \( \mu, \varepsilon, \mu'' \) and \( \varepsilon'' \), which are functions of the yet to be determined temperature. Since we are solving a system where Maxwell's equations (1.5) are coupled to the forced heat equation (1.6) and we have nondimensionalised Maxwell's equations, we also need to nondimensionalise the forced heat equation (1.6). Let

\[
T' = \frac{T}{T_0},
\]

where \( T_0 \) is some suitable reference temperature (e.g., the absolute temperature of the...
surroundings at time \( t = 0 \), so that equation (1.6) transforms to

\[
T' = \frac{\nu}{c_0^2} T_{xx} + g(T) \frac{E_0^2}{\omega T_0} \phi^2.
\] (5.31)

The coupled system (5.28) and (5.31) is difficult to solve in general, so the further assumption that the nondimensional heat diffusivity \( \nu c_0^2 \) is small will be made. For the source term to dominate over the heat diffusion term in (5.31), we require

\[
\frac{\nu \omega^2}{c_0^2} \ll \frac{E_0^2}{T_0}
\] (5.32)

and, when this condition (5.32) is satisfied, equation (5.31) will become, to first order,

\[
T' = g(T) \frac{E_0^2}{\omega T_0} \phi^2
\]

\[
= G(T') \phi^2.
\] (5.33)

The prime which indicates the nondimensional temperature is now dropped. From now on, all references to temperature are understood to refer to the nondimensional temperature unless otherwise stated.

The variation of conductivity, permittivity and permeability with temperature can, for many materials, be represented by a power law (von Hippel, 1954). The following forms are used

\[
\epsilon'' = a_1 + \alpha a_2 T^{a_3},
\] (5.34)

\[
\epsilon = \epsilon_1 + \alpha \epsilon_2 T^{\epsilon_3},
\] (5.35)

\[
\mu = \mu_1 + \alpha \mu_2 T^{\mu_3},
\] (5.36)

where \( \alpha \ll 1 \). Here, for convenience, we consider materials for which the magnetic loss factor is zero, so that we are considering dielectric materials. Note that equations
(5.34), (5.35) and (5.36) are not invariant under translational transformations of the temperature and thus the ambient temperature cannot arbitrarily be rescaled to zero. However, the form of the expressions is invariant under stretching transformations and thus it is possible to use the nondimensional temperature $T$. The material being heated will have an initial temperature given by

$$T(x, 0) = T_i(x)$$  \hspace{1cm} (5.37)

and in the important case where the material is initially at equilibrium with its surroundings, the initial temperature will be constant (i.e. $T_i(x) = T_i = constant$). If we are given a relationship of the form of (5.34), we cannot arbitrarily rescale the temperature so that the initial value is zero. However it is possible, in many cases, to find an alternative power law, with three different parameters, which can adequately represent $\epsilon''$ as a function of a temperature which is scaled to have an initial value of zero. In this case there is one important restriction. When $T_i = 0$, to ensure that the derivatives of the electromagnetic properties are finite at $t = 0$, we require $a_3, \epsilon_3, \mu_3 \geq 1$.

In the forced heat equation (1.6), the source term includes the function $g(T)$, where the temperature has not been nondimensionalised, which represents the temperature dependent absorption rate. The heat gain must be related to the loss term. For example, Metaxas and Meredith (1983) use a relationship where the rate of heat gain is proportional to the product of the frequency and the effective loss term. In our terminology, this is equivalent to

$$g(T) = k \omega \epsilon''$$,  \hspace{1cm} (5.38)

where $k$ is a constant of proportionality and $\epsilon''$, in this work refers to all possible loss
mechanisms. In general, not all losses will be converted to heat and for each loss term a different proportion of the energy lost might be converted to heat. Because of these complications, we adopt a source term in equations (5.31) and (5.33) of the form

\[ G(T) = \gamma_1 + \alpha \gamma_2 T^\alpha, \]  

while noting that (5.38) must be approximately true and that the value of \( G(T) \) is influenced by the amplitude \( E_0 \) of the incoming radiation. All of the other variables in the nondimensional equations can be expected to have initial values of \( O(1) \). In the nondimensional forced heat equation (1.6), we have

\[ T_t = \nu' T_{xx} + g' \phi^2, \]  

where

\[ \nu' = \frac{\nu \omega}{c^2}, \quad g' = \frac{g E_0^2}{T_0 \omega}, \]  

so that, if (5.38) holds, the parameters in (5.39) can be defined by

\[ \gamma_1 = \frac{k a_1 E_0^2}{T_0}, \quad \gamma_2 = \frac{k a_2 E_0^2}{T_0}, \quad \gamma_3 = a_3. \]  

In the current work, we consider the heating of a semi-infinite slab of material with an insulated boundary at \( x = 0 \), giving

\[ T_x = 0 \text{ at } x = 0, \]  

and a uniform initial temperature of

\[ T = T_i \text{ at } t = 0. \]
The thermally insulated boundary condition is a good approximation for dielectric materials, for which the Biot number, measuring the ratio of heat convection or radiation at the boundary to heat conduction, tends to be small. For example, the Biot number \( \sim 0.0001 \) for ceramics (Kingery et al, 1974, Kriegmann, 1991). Expressions (5.43) and (5.44) together with (5.29) are the initial and boundary conditions used in this work.

In general, the transmission ratio \( r \), given by (5.20), will be a function of the temperature \( T \). However, in the case considered here, where \( \mu'' = 0 \) and \( \epsilon, \mu \) and \( \epsilon'' \) are given by the relations (5.34) to (5.36), it can readily be shown that the modulus \( R \) and the phase \( \xi \) of the transmission ratio are of the form

\[
R = R_0 + \alpha R_1 + O(\alpha^2)
\]

\[
\xi = \xi_0 + \alpha \xi_1 + O(\alpha^2),
\]

where \( R_0 \) and \( \xi_0 \) are constant and \( R_1, \xi_1 \) etc are functions of the temperature. The expressions for \( R_0, R_1, \xi_0, \xi_1 \) in the expansions (5.45), (5.46) cannot be determined until the relationship between the constant \( a_1 \), from the expression (5.34) for the dielectric loss factor \( \epsilon'' \) and the ordering constant \( \alpha \) is known. When \( a_1 \sim O(\alpha^n) \), we write

\[
a_1 = \alpha^n a_{11},
\]

and, by substituting this expression (5.47) into (5.18) and (5.19), we obtain for \( n \geq \frac{1}{2} \)

\[
R_0 = \frac{2}{1 + (\mu_1 \epsilon_1)^{\frac{1}{2}}}
\]

\[
R_1 = -R_0(A_1 T^{\mu_2} + A_2 T^{\epsilon_2} + A_3),
\]
where
\[ A_1 = \frac{(\mu_1 \epsilon_1) \frac{1}{2} \mu_2}{2 \mu_1 (1 + (\mu_1 \epsilon_1) \frac{1}{2})} \]
\[ A_2 = \frac{(\mu_1 \epsilon_1) \frac{1}{2} \epsilon_2}{2 \epsilon_1 (1 + (\mu_1 \epsilon_1) \frac{1}{2})} \]
\[ A_3 = \frac{\alpha^{2n-1}(\mu_1 \epsilon_1) \frac{1}{2} (1 + 2(\mu_1 \epsilon_1) \frac{1}{2}) a_{11}^2}{2 \epsilon_1^2 (1 + (\mu_1 \epsilon_1) \frac{1}{2})} \]
\[ \xi_0 = -\frac{\alpha^n(\mu_1 \epsilon_1) \frac{1}{2} a_{11}}{2 \epsilon_1 (1 + (\mu_1 \epsilon_1) \frac{1}{2})} \]
\[ (5.50) \]
\[ \xi_1 = \frac{\xi_0}{(1 + (\mu_1 \epsilon_1) \frac{1}{2})^2} (B_1 T_{m3} + B_2 T_{\epsilon3} + B_3 T_{\alpha3} + B_4). \]

Here
\[ B_1 = \frac{\epsilon_1^2 \mu_2}{2 \mu_1 (1 + (\mu_1 \epsilon_1) \frac{1}{2})} \]
\[ B_2 = -\frac{\epsilon_1^2 \epsilon_2 (1 + 2(\mu_1 \epsilon_1) \frac{1}{2})}{2 \epsilon_1 (1 + (\mu_1 \epsilon_1) \frac{1}{2})} \]
\[ B_3 = \frac{a_{22} \epsilon_1^2}{a_{11}} \]
\[ (5.51) \]
\[ B_4 = -\frac{\alpha^{2n-1} a_{11}^2}{24 \epsilon_1^\frac{1}{2} (1 + (\mu_1 \epsilon_1) \frac{1}{2})^2} (3 + 9(\mu_1 \epsilon_1) \frac{1}{2} + 8 \mu_1 \epsilon_1) \]
and the temperature \( T \) is evaluated at the surface \( x = 0 \).

### 5.1 Perturbation Solution For \( \alpha \ll \epsilon'' \ll 1 \)

The governing equations to be solved are the forced heat equation (5.31) with a source term depending on the square of the amplitude of the electric field, the phase (eikonal) equation (5.25) and the amplitude (transport) equation (5.28). We first consider the case when \( 0 < \alpha \ll a_1 \ll 1 \). Most materials which are suited to microwave heating have low conductivities. However, in a lossy material, the \( \epsilon'' \) term is not merely a conductivity
term, but also includes the other loss mechanisms (see Metaxas and Meredith, 1983). Thus, if the material is capable of being heated, this term will not be zero. Here we shall assume that this loss factor is small enough to avoid the propagation changes which accompany large loss rates, but that it is larger than the rate at which the loss term, the permeability and the permittivity are changing. This is not difficult to justify as it is quite likely that the rate of change of the electromagnetic properties, with respect to temperature, will be smaller than the order of the loss term, and the case where \( \varepsilon'' \sim O(\alpha) \) has already been treated in Chapter 2. The case where the rate of change of the electromagnetic properties is larger than the order of the loss term is treated in the next section. The transport equation (5.28) can be written as

\[
\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = -\frac{a_1}{2} \phi - \alpha \frac{h_1}{2} \phi + O(\alpha^2),
\]

where

\[
h_1 = \frac{\mu_2 \mu_3}{2 \mu_1} T^{\mu_3-1} (T_t - c_1 T_x) + \frac{\varepsilon_2 \varepsilon_3}{2 \varepsilon_1} T^{\varepsilon_3-1} (3 T_t + c_1 T_x) + a_2 T^{a_3}
\]

(5.52)

\[
c = c_1 (1 - \alpha c_2 + O(\alpha^2))
\]

(5.53)

\[
c_1 = (\mu_1 \varepsilon_1)^{-\frac{1}{2}}
\]

\[
c_2 = \frac{1}{2} \left( \frac{\mu_2}{\mu_1} T^{\mu_3} + \frac{\varepsilon_2}{\varepsilon_1} T^{\varepsilon_3} \right)
\]

(5.54)

(5.55)

The functions \( h_1 \) and \( c_2 \) in equations (5.52) to (5.55) are temperature dependent and it will be found that they contain secular terms in time. These secular terms are present before the perturbation analysis is undertaken and cannot be removed. They are due to the temperature being an increasing function of time due to the continual energy input.
from the microwave radiation. However, it will be shown here that the perturbation solutions are uniformly valid as long as the temperature is in the range $0 < T < T_1$, where $T_1$ will be defined later in this section. At the boundary $x = 0$, we have the condition

$$
\phi(0, t) = \phi_0(0, t) + \alpha \phi_1(0, t) + O(\alpha^2)
$$

We adopt the solution form

$$
\phi(x, t) = f(x, t)e^{-\theta(x, t)}
$$

by analogy with the solution for constant material properties, for which $f = R_0$ and $g = a_1/(2c)$ (see Chapter 2). On substituting (5.57) into the amplitude equation (5.52), we obtain

$$
f_t + cf_x = (g_t + cg_x - \frac{1}{2}(a_1 + \alpha h_1 + O(\alpha^2)) )f. \tag{5.58}
$$

The simplest solution to equation (5.58) is obtained by setting

$$
g_t + cg_x = \frac{a_1}{2} + \alpha \frac{h_1}{2} + O(\alpha^2), \tag{5.59}
$$

which leaves

$$
f_t + cf_x = 0. \tag{5.60}
$$

If we combine equation (5.60) with the boundary condition (5.56) and with the expansion

$$
f = f_0 + \alpha f_1 + O(\alpha^2), \tag{5.61}
$$

we obtain

$$
f_{0t} + c_1 f_{0x} = 0 \tag{5.62}
$$
subject to \( f_0(0,t) = R_0 \) and

\[
f_{1t} + c_1f_{1x} = c_1c_2f_{0x} \quad (5.63)
\]

subject to \( f_1(0,t) = R_1(T(0,t)) \).

Equations (5.62) and (5.63) can be solved by the method of characteristics to yield

\[
f_0 = R_0 \quad (5.64)
\]

\[
f_1 = R_1(T(0,\tau)), \quad (5.65)
\]

where

\[
\tau = t - \frac{x}{c_1}. \quad (5.66)
\]

We now use a regular perturbation expansion for \( g(x,t) \) to obtain a solution of (5.59). We thus set

\[
g(x,t) = g_0(x,t) + \alpha g_1(x,t) + O(\alpha^2), \quad (5.67)
\]

while at the boundary \( x = 0 \), we have \( g_0 = g_1 = \ldots = 0 \). On substituting the expansion (5.67) into equation (5.59) we obtain

\[
\text{at } O(1): \quad g_{0t} + c_1g_{0x} = \frac{a_1}{2} \quad (5.68)
\]

subject to \( g_0(0,t) = 0 \) and

\[
\text{at } O(\alpha): \quad g_{1t} + c_1g_{1x} - c_2g_{0x} = \frac{h_1}{2} \quad (5.69)
\]

subject to \( g_1(0,t) = 0 \).

Equation (5.68) can be solved, using the method of characteristics, to yield

\[
g_0 = \frac{a_1x}{2c_1}, \quad (5.70)
\]
and, upon using this result, we can simplify the $O(\alpha)$ equation (5.69) to obtain

$$g_{1t} + c_1 g_{1x} = \frac{h_1}{2} + \frac{c_2 a_1}{2c_1}.$$  \hspace{1cm} (5.71)

From (5.70) we see that

$$\phi(x, t) = R_0 e^{-\frac{a_1 x}{2c_1}} + O(\alpha).$$  \hspace{1cm} (5.72)

The forced heat equation (5.31) is difficult to integrate. However, if the heat diffusivity is zero, which is a reasonable first order approximation in the case of the microwave heating of dielectric materials, we can replace (5.31) by (5.33) which, after substitution for $\phi$, becomes

$$T_t = (\gamma_1 + \alpha \gamma_2 T^\gamma) R^2 e^{-2g_0 - 2a_1 \gamma_1 - \ldots}.$$  \hspace{1cm} (5.73)

If we expand $T$ in terms of the small parameter $\alpha$,

$$T = T_0 + \alpha T_1 + \alpha^2 T_2 + \ldots,$$  \hspace{1cm} (5.74)

and use Taylor expansions in (5.73) for the case

$$\alpha T_1 + \alpha^2 T_2 + \ldots \ll T_0,$$  \hspace{1cm} (5.75)

$$\alpha g_1 + \alpha^2 g_2 + \ldots \ll g_0,$$  \hspace{1cm} (5.76)

we obtain, after substitution of the expansion (5.74) into the reduced heat equation (5.73) and separation of the resulting equation into equations $\sim O(\alpha^m)$,

$$O(\alpha^0) : \ T_{0t} = \gamma_1 R^2_0 e^{-2g_0}$$  \hspace{1cm} (5.77)

$$O(\alpha^1) : \ T_{1t} = (\gamma_2 T_0^\gamma + 2\gamma_1 \frac{R_1}{R_0} - 2\gamma_1 g_1) R_0^2 e^{-2g_0},$$  \hspace{1cm} (5.78)
plus equations $\sim O(\alpha^n), n \geq 2$. To evaluate the exponent function $g_1$, using equation (5.71), we need to calculate $h_1$ and $c_2$, which are both functions of temperature. However, only terms of first order should be included in (5.71) as higher order terms would be taken into account in the equations for $g_2, g_3$, etc. Thus we have a simple solution path where we solve for $g_0$ (equation (5.70)), then use $g_0$ to solve (5.77) for $T_0$, then use the solutions for $g_0$ and $T_0$ to solve (5.69) for $g_1$, and complete the $O(\alpha)$ solution by using $g_0, g_1$ and $T_0$ to solve (5.78) for $T_1$.

The initial condition for the temperature,

$$T(x, 0) = T_i, \quad x \geq 0,$$

leads us to set the following initial values,

$$T_0(x, t) = T_i, \quad \text{when } t - \frac{x}{c_1} \leq 0, \quad (5.80)$$

$$T_1 = T_2 = \ldots = 0, \quad \text{when } t - \frac{x}{c_1} \leq 0, \quad (5.81)$$

where the condition $t - x/c_1 \leq 0$ is a recognition of the fact that heating cannot begin at any point until the wavefront has reached that point.

We can integrate (5.77) to obtain the following first order expression for the temperature

$$T_0 = \gamma_1 R_0^2 \tau e^{-\frac{a_1 \tau}{c_1}} + T_i, \quad (5.82)$$

where

$$\tau = t - \frac{x}{c_1}, \quad (5.83)$$

making it a simple matter to obtain

$$T_{0x} = \gamma_1 R_0^2 e^{-\frac{a_1 \tau}{c_1}} \left(-\frac{a_1}{c_1} \tau - \frac{1}{c_1}\right), \quad (5.84)$$
\[ T_{\text{diff}} = \gamma_1 R_0^2 e^{-\frac{a_1 x}{c_1}}, \quad (5.85) \]

and to evaluate the functions \( h_1 \) and \( c_2 \) as

\[
\begin{align*}
    h_1 &= \frac{\mu_2 \mu_3}{2 \mu_1} (T_i + \gamma_1 R_0^2 \tau e^{-\frac{a_1 x}{c_1}})^{\mu_3^{-1}} R_0^2 \gamma_1 e^{-\frac{a_1 x}{c_1}} (2 + a_1 \tau) \\
    &\quad + \frac{\epsilon_3}{2c_1} (T_i + \gamma_1 R_0^2 \tau e^{-\frac{a_1 x}{c_1}})^{\epsilon_3^{-1}} R_0^2 \gamma_1 e^{-\frac{a_1 x}{c_1}} (2 - a_1 \tau) \\
    &\quad + a_2 (T_i + \gamma_1 R_0^2 \tau e^{-\frac{a_1 x}{c_1}})^{a_2} \\
    c_2 &= \frac{\mu_2}{2 \mu_1} (T_i + \gamma_1 R_0^2 \tau e^{-\frac{a_1 x}{c_1}})^{\mu_3} + \frac{\epsilon_2}{2c_1} (T_i + \gamma_1 R_0^2 \tau e^{-\frac{a_1 x}{c_1}})^{\epsilon_3}. \quad (5.86)
\end{align*}
\]

We can solve (5.71) by using the method of characteristics. On the characteristic defined by

\[
\frac{dx}{dt} = c_1, \quad (5.88)
\]

we have \( t - x/c_1 = \text{constant} = \tau \), and

\[
\begin{align*}
    \frac{d\phi_1}{dt} &= \frac{h_1}{2} + \frac{c_2 a_1}{2c_1} \\
    &= \frac{\mu_3 \mu_3}{4 \mu_1} (T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 x}{c_1}})^{\mu_2^{-1}} R_0^2 \gamma_1 (2 + a_1 \tau) e^{-\frac{a_1 x}{c_1}} \\
    &\quad + \frac{\epsilon_3}{4c_1} (T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 x}{c_1}})^{\epsilon_3^{-1}} R_0^2 \gamma_1 (2 - a_1 \tau) e^{-\frac{a_1 x}{c_1}} \\
    &\quad + \frac{a_2}{2} (T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 x}{c_1}})^{a_3} + \frac{a_1 \mu_2}{4 \mu_1 c_1} (T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 x}{c_1}})^{\mu_3} \\
    &\quad + \frac{a_1 \epsilon_2}{4c_1 c_1} (T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 x}{c_1}})^{\epsilon_3}. \quad (5.89)
\end{align*}
\]

The first two terms on the right hand side of (5.89) are directly integrable, regardless of the value of the initial temperature \( T_i \), while the final three terms can be integrated for general \( a_3, \mu_3 \) and \( \epsilon_3 \) only for the case \( T_i = 0 \). However, it is possible to integrate the Taylor expansions of the final three terms, truncated when a term in the expansion is reduced to \( O(\alpha) \).
When $\gamma_1 R_0^2 \exp(-a_1 x/c_1) > T_i$, we can use the following expansion

\[
(T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}})^p = (\gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}})^p \frac{p(p-1)T_i^{p-2}(\gamma_1 R_0^2 e^{-\frac{a_1 \tau}{c_1}})^2}{2} + \ldots,
\]

truncating as soon as

\[
|\frac{p(p-1)(p-2) \ldots (p-n+1)}{n!} T_i^n (\gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}})^{p-n}| \leq \alpha,
\]

while for $T_i > \gamma_1 R_0^2 \exp(-a_1 x/c_1)$ we use

\[
(T_i + \gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}})^p = T_i^p + pT_i^{p-1} \gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}}
\]

\[
+ \frac{p(p-1)}{2} T_i^{p-2} (\gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}})^2 + \ldots
\]

truncating as soon as

\[
|\frac{p(p-1) \ldots (p-n+1)}{n!} T_i^{p-n} (\gamma_1 \tau R_0^2 e^{-\frac{a_1 \tau}{c_1}})^n| \leq \alpha.
\]

Thus, when $\gamma_1 R_0^2 \exp(-a_1 x/c_1) > T_i$,

\[
\int q(T_i + \gamma_1 \tau R_0^2 e^{a_1 \tau} e^{-a_1 \rho}) d\rho = q(\gamma_1 R_0^2 \tau a_1)^p (1 - e^{-\frac{a_1 \tau}{c_1}})
\]

\[
+ \frac{pT_i}{a_1(p-1)} (\gamma_1 R_0^2 \tau)^{p-1} (1 - e^{-\frac{a_1(p-1)\tau}{c_1}})
\]

\[
+ \frac{p(p-1)}{2a_1(p-2)} T_i^2 (\gamma_1 R_0^2 \tau)^{p-2} (1 - e^{-\frac{a_1(p-2)\tau}{c_1}})
\]

\[
+ \ldots
\]

where (5.94) is truncated in accordance with the relation (5.91).

Similarly, when $T_i > \gamma_1 R_0^2 \exp(-a_1 x/c_1)$, we have

\[
\int q(T_i + \gamma_1 \tau R_0^2 e^{a_1 \tau} e^{-a_1 \rho}) d\rho = q(T_i^p \tau + pT_i^{p-1} \gamma_1 R_0^2 \tau a_1^{-1} (1 - e^{-\frac{a_1 \tau}{c_1}})
\]

\[
+ \frac{p(p-1)}{4a_1} T_i^{p-2} (\gamma_1 R_0^2 \tau)^2 (1 - e^{-\frac{2a_1 \tau}{c_1}})
\]

\[
+ \ldots
\]

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where (5.95) is truncated in accordance with (5.93).

We thus have two approximate solutions for $g_1$, namely

$$g_1 = Z + W_1 + W_2 + W_3 \quad (5.96)$$

where

$$Z = \frac{\mu_2}{4\mu_1 c_1} \left[ (T_i + \gamma_1 R_0^2 \tau)^{t_3} - (T_i + \gamma_1 R_0^2 e^{-\frac{\alpha_1 x}{c_1}} \tau^{t_3}) \right]$$

$$+ \frac{\epsilon_{s2}(2^{-\alpha_1 x})}{4\mu_1 c_1} \left[ (T_i + \gamma_1 R_0^2 \tau)^{t_3} - (T_i + \gamma_1 R_0^2 e^{-\frac{\alpha_1 x}{c_1}} \tau^{t_3}) \right] \quad (5.97)$$

and, when $\gamma_1 R_0^2 \exp(-a_1 x/c_1) > T_i$,

$$W_1 = \frac{\alpha_2}{2a_1 a_3} \left[ (\gamma_1 R_0^2 \tau)^{t_3} \left( 1 - e^{-\frac{\alpha_1 x}{c_1}} \right) \right]$$

$$+ \frac{\alpha_2 a_3}{2a_1 (a_3 - 1)} \left[ (\gamma_1 R_0^2 \tau)^{t_3 - 1} \left( 1 - e^{-\frac{\alpha_1 (a_3 - 1) x}{c_1}} \right) \right] + \ldots \quad (5.98)$$

$$W_2 = \frac{\mu_2}{4\mu_1 c_1} \left[ \mu_3^{-1} (\gamma_1 R_0^2 \tau)^{t_3} \left( 1 - e^{-\frac{\alpha_1 x}{c_1}} \right) \right]$$

$$+ \frac{\mu_3}{(\mu_3 - 1)} \left[ T_i (\gamma_1 R_0^2 \tau)^{t_3 - 1} \left( 1 - e^{-\frac{\alpha_1 (\mu_3 - 1) x}{c_1}} \right) \right] + \ldots \quad (5.99)$$

$$W_3 = \frac{\epsilon_{s1} \epsilon_{s2}}{4c_1 c_1} \left[ (\gamma_1 R_0^2 \tau)^{t_3} \left( 1 - e^{-\frac{\alpha_1 x}{c_1}} \right) \right]$$

$$+ \frac{\epsilon_{s1} \epsilon_{s2}}{c_1 (c_3 - 1)} \left[ (\gamma_1 R_0^2 \tau)^{t_3 - 1} \left( 1 - e^{-\frac{\alpha_1 (c_3 - 1) x}{c_1}} \right) \right] + \ldots \quad (5.100)$$

while, for $\gamma_1 R_0^2 \exp(-a_1 x/c_1) < T_i$,

$$W_1 = \frac{\alpha_2}{2} \left[ T_i^{t_3 - 1} + \frac{\alpha_3}{a_1} T_i^{t_3 - 1} \gamma_1 R_0^2 \tau (1 - e^{-\frac{\alpha_1 x}{c_1}}) \right]$$

$$+ \frac{\alpha_3 (a_3 - 1)}{4a_1} T_i^{t_3 - 2} (\gamma_1 R_0^2 \tau)^2 (1 - e^{-\frac{2a_1 x}{c_1}}) + \ldots \quad (5.101)$$

$$W_2 = \frac{\alpha_3 \mu_3}{4 a_1 c_1} \left[ T_i^{t_3} \tau + \frac{\mu_3}{a_1} T_i^{t_3 - 1} \gamma_1 R_0^2 \tau (1 - e^{-\frac{\alpha_1 x}{c_1}}) \right]$$

$$+ \frac{\mu_3 (\mu_3 - 1)}{4a_1} T_i^{t_3 - 2} (\gamma_1 R_0^2 \tau)^2 (1 - e^{-\frac{2a_1 x}{c_1}}) + \ldots \quad (5.102)$$

$$W_3 = \frac{\alpha_3 \epsilon_{s1} \epsilon_{s2}}{4c_1 c_1} \left[ T_i^{t_3} \tau + \frac{\epsilon_{s1} \epsilon_{s2}}{a_1} T_i^{t_3 - 1} \gamma_1 R_0^2 \tau (1 - e^{-\frac{\alpha_1 x}{c_1}}) \right]$$

$$+ \frac{\epsilon_{s1} \epsilon_{s2} (c_3 - 1)}{4a_1} T_i^{t_3 - 2} (\gamma_1 R_0^2 \tau)^2 (1 - e^{-\frac{2a_1 x}{c_1}}) + \ldots \quad (5.103)$$
The first of these expressions for \( W_1, W_2 \) and \( W_3 \) is useful for \( \tau \) large and the second for \( \tau \) small. The solution of (5.89) is simplified greatly when \( T_i = 0 \), in which case

\[
g_1 = \frac{H_1}{2} + \frac{a_1 C_2}{2c_1} \tag{5.104}
\]

where

\[
H_1 = \frac{\mu_2}{2a_1 \mu_3} \gamma_{11} \mu_3 (R_0^2 \tau)^{\mu_3-1} R_0^2 (2 + a_1 \tau) \left( 1 - e^{-\frac{g_1 a_3}{c_1}} \right) + \frac{\epsilon_2}{a_1 c_1} \gamma_{11} \mu_3 (R_0^2 \tau)^{\mu_3-1} R_0^2 (2 - a_1 \tau) \left( 1 - e^{-\frac{g_1 a_3}{c_1}} \right) + \frac{\epsilon_2}{a_1 c_1} (R_0^2 \gamma_{11} \tau)^{a_3} \left( 1 - e^{-\frac{g_1 a_3}{c_1}} \right) \tag{5.105}
\]

and

\[
C_2 = \frac{\mu_2}{2a_1 \mu_1 \mu_3} (R_0^2 \gamma_{11} \tau)^{a_3} \left( 1 - e^{-\frac{g_1 a_3}{c_1}} \right) + \frac{\epsilon_2}{2a_1 c_1 c_3} (R_0^2 \gamma_{11} \tau)^{a_3} \left( 1 - e^{-\frac{g_1 a_3}{c_1}} \right), \tag{5.106}
\]

which is, of course, the limiting case of (5.96) as \( T_i \to 0 \).

In order for the expansion to be uniformly valid, we require

\[
\frac{|\alpha g_1|}{|g_0|} \ll 1. \tag{5.107}
\]

An inspection of (5.70) and (5.104) shows that the condition (5.107) for uniform validity can easily be satisfied for large values of \( x \). However, we need to show that (5.107) is satisfied for small values of \( x \). For \( x \ll 1, 1 - e^{-\delta x} \approx \delta x \), so that the relation (5.107) reduces to the sum of terms involving powers of \((t - x/c_1)\). These terms are either of the form \((t - x/c_1)^a\) or of the form \((t - x/c_1)^{a-1}\). When \( a \geq 2 \), then for any variable \( y \geq 0, y^a \geq y^{a-1} \), but for \( 1 \leq a \leq 2 \), we require \( y \geq 1 \) for \( y^a \geq y^{a-1} \). If an expression containing \( y^a \) is to be small for \( y > 1 \), the coefficients need to be small and therefore the expression will necessarily have a small value when \( y \leq 1 \). The expression \( t - x/c_1 \)
represents the time since the wavefront passed the point \( x \) and is thus the heating time, to first order. In order for the expansion to be uniformly valid, we require

\[
\gamma(t - \frac{x}{c_1}) < \min\left(\frac{4\mu_1}{\alpha\mu_2\mu_3}, \frac{4\epsilon_1}{\alpha\epsilon_2\epsilon_3}, \frac{2a_1}{\alpha a_2}, \frac{4c_1\mu_1}{\mu_2}, \frac{4c_1\epsilon_1}{\epsilon_2}\right). \tag{5.108}
\]

When the above condition (5.108) for uniform validity is satisfied, we can write our solution as

\[
\phi(x, t) = R e^{-\frac{q_{1x}}{c_1} - \alpha g_1}, \tag{5.109}
\]

where \( g_1 \) is defined by (5.104), and \( R \) is given by

\[
R = R_0 + \alpha R_1(T(0, \tau)) + O(\alpha^2). \tag{5.110}
\]

When \( \alpha g_1 \ll 1 \), we can write

\[
e^{-\alpha g_1} = \frac{1}{1 + \alpha g_1} \tag{5.111}
\]

and thus we can write our solution in the form

\[
\phi(x, t) = \frac{R_0 e^{-\frac{q_{1x}}{c_1}}}{1 + \alpha(g_1 - \frac{R_1}{R_0})}, \tag{5.112}
\]

which closely parallels the form of the solution obtained in Chapter 2 for the case where \( \epsilon'' \sim O(\alpha) \) and for different power law temperature dependencies for the wave speed \( c \) and the loss term \( \epsilon'' \). For example, when the permittivity and the permeability are constant and \( T_i = 0 \), (5.112) becomes

\[
\phi(x, t) = \frac{R_0 e^{-\frac{q_{1x}}{c_1}}}{1 + \frac{\alpha g_2}{2a_1 a_3}} \tag{5.113}
\]

where

\[
\Gamma = (R_0^2 \gamma_1(t - \frac{x}{c_1}))^{a_3}(1 - e^{-\frac{q_{1x}}{c_1}}). \tag{5.114}
\]
Equation (5.78) for the $O(\alpha)$ temperature component $T_1$ is readily integrable. For the case when $T_i = 0$, the solution is

$$T_i = \frac{\gamma_
u}{\gamma_i(\gamma_{i+1})} \left( R_0^2 e^{-\frac{\alpha_1 \rho}{\gamma_i}} \right)^{\gamma_{i+1}} - 2\left[ A_1(\gamma_t R_0^2 e^{-\frac{\alpha_1 \rho}{\gamma_i}})^{\gamma_{i+1}} + \frac{A_2(\gamma_t R_0^2 e^{-\frac{\alpha_1 \rho}{\gamma_i}})^{\gamma_{i+1}} + A_3 \gamma_t R_0^2 e^{-\frac{\alpha_1 \rho}{\gamma_i}}}{\gamma_{i+1}} \right]$$

$$+ \frac{R_0^2 \gamma_{i+1} e^{-\frac{\alpha_1 \rho}{\gamma_i}}}{\gamma_{i+1}} \left[ \frac{\mu_2}{\mu_3} (R_0^2 \gamma_{i+1})^{\mu_3} (1 - e^{-\frac{\alpha_1 \mu_3}{\gamma_i}}) \left( \frac{2}{\mu_3} \gamma_{i+1}^{\mu_3 + 1} + \frac{\alpha_1}{\mu_3 + 1} \gamma_{i+1}^{\mu_3 + 1} \right) \right]$$

$$+ \frac{2a_2}{a_3} (R_0^2 \gamma_{i+1})^{a_3} (1 - e^{-\frac{\alpha_1 a_3}{\gamma_i}}) \gamma_{i+1}^{a_3 + 1} \right]$$

$$+ \frac{R_0^2 \gamma_{i+1} e^{-\frac{\alpha_1 \rho}{\gamma_i}}}{\gamma_{i+1}} \left[ \frac{\mu_2}{\mu_3 (\mu_3 + 1)} (R_0^2 \gamma_{i+1})^{\mu_3} (1 - e^{-\frac{\alpha_1 \mu_3}{\gamma_i}}) \gamma_{i+1}^{\mu_3 + 1} \right]$$

$$+ \frac{R_0^2 \gamma_{i+1} e^{-\frac{\alpha_1 \rho}{\gamma_i}}}{\gamma_{i+1}} \left[ \frac{\mu_2}{\mu_3 (\mu_3 + 1)} (R_0^2 \gamma_{i+1})^{\mu_3} (1 - e^{-\frac{\alpha_1 \mu_3}{\gamma_i}}) \gamma_{i+1}^{\mu_3 + 1} \right].$$

### 5.1.1 Solution of Phase Equation

The phase equation (5.27) can also be solved using a regular perturbation expansion. If we take

$$\theta(x,t) = \theta_0(x,t) + \alpha \theta_1(x,t) + O(\alpha^2)$$

(5.116)

and substitute into (5.27), we obtain

at $O(1)$: \[\theta_{0t} + c_1 \theta_{0x} = 0\] (5.117)

at $O(\alpha)$: \[\theta_{1t} + c_1 \theta_{1x} - c_1 c_2 \theta_{0x} = 0.\] (5.118)

The $O(1)$ equation (5.117) can be solved to obtain

$$\theta_0 = t - \frac{x}{c_1} + \xi_0$$

(5.119)

and the substitution of this result into (5.118) gives

$$\theta_{1t} + c_1 \theta_{1x} = -c_2$$

(5.120)
which has the solution, after applying the boundary condition $\theta_1 = \xi_1$ at $x = 0$,

$$
\theta_1 = -\frac{1}{2} \left[ \mu_2 \mu_3 \mu_1 \right] (\gamma_1 R_0^2 \tau)^\mu_3 \left( 1 - e^{-\frac{\alpha_1 \mu_1 \tau}{\xi_1}} \right) + \frac{\xi_1 \xi_2}{\xi_1 \xi_3 \xi_2} (\gamma_1 R_0^2 \tau)^\xi_3 \left( 1 - e^{-\frac{\alpha_1 \xi_1 \tau}{\xi_1}} \right) + \xi_1 \tag{5.121}
$$

for $T_i = 0$.

It can be seen from solutions (5.119) and (5.121) that, to order $O(\alpha)$, the phase $\theta$ has an explicit dependence on the loss factor $\epsilon''$ only through the phase $\xi$ of the transmission ratio. However, it is also apparant that $\theta_i$ is not constant. In the well known case where the loss factor, the permittivity and the permeability are all constant, Maxwell’s equations (1.7) and (1.8) transform to the equations of telegraphy, and these are normally solved by assuming that the frequency of the radiation will remain constant. This assumption can be justified by a strong physical argument, for example on a conservation of waves basis. In the more general case which we consider here, where the wave speed varies with the temperature, the local frequency will not be constant, but will vary from the incident frequency because the wavespeed through the material is changing with time.

### 5.2 Perturbation Solution For $\epsilon'' \sim O(\alpha)$

When $\epsilon'' \sim O(\alpha)$, we can set $a_1 = \alpha a_n$ in (5.34) and write (5.52) as

$$
\phi_t + c \phi_x = -\alpha \left( \frac{a_n}{2} + \frac{h_1}{2} \right) \phi + O(\alpha^2). \tag{5.122}
$$

We can solve (5.122) using the same method we applied in section 3. We thus let $\phi = f e^{-s}$ and we can then show, as before, that $f = R_0 + \alpha R_1(T(0, \tau)) + O(\alpha^2)$, and
that, in this case, we have

\[ g_t + cg_x = \alpha \left( \frac{a_n}{2} + \frac{h_1}{2} \right) + O(\alpha^2). \]  

(5.123)

Here we use a multiple scales perturbation expansion to solve for \( g \),

\[ g(x,t) = g_0(x,X,t) + \alpha g_1(x,X,t) + O(\alpha^2), \]  

(5.124)

where

\[ X = \alpha x. \]  

(5.125)

This multiple scales expansion is necessary to eliminate secular terms which will occur in the subsequent analysis.

On substituting the expansion (5.124) into the equation (5.123), we obtain

at \( O(1) \) : \[ g_{tt} + c_1g_{0x} = 0 \]  

(5.126)

at \( O(\alpha) \) : \[ g_{tt} + c_1g_{1x} - c_1c_2g_{0x} + c_1g_{0x} = \frac{a_n}{2} + \frac{h_1}{2}. \]  

(5.127)

The \( O(1) \) equation (5.126) can be solved by the method of characteristics to yield

\[ g_0 = d(\tau, X), \]  

(5.128)

where \( \tau \) is the characteristic variable, that is, \( \tau \) can be considered constant along a characteristic. The characteristic variable \( \tau \) is also the value of the time when the referenced part of the wave was at the boundary \( x = 0 \). At the boundary \( x = 0 \), we have \( g_0 = d(\tau, 0) \), but we know that the amplitude of the wave at \( x = 0 \) has the value \( R(\tau) \). Thus it follows that \( d(t, 0) = 0 \) for all values of \( t \), so that

\[ g_0 = d(X) \]  

(5.129)
and (5.127) simplifies to

\[ g_{1t} + c_1 g_{1x} + c_1 d' = \frac{a_n}{2} + \frac{h_1}{2}, \quad (5.130) \]

where the prime denotes differentiation with respect to \( X \). The constant term on the right hand side, \( \frac{a_n}{2} \), giving the decay of the wave, should be a part of the first order solution for \( g \), so we set

\[ d' = \frac{a_n}{2c_1}. \quad (5.131) \]

Equation (5.131), when combined with the boundary condition \( d(0) = 0 \), has the solution

\[ d = g_0 = \frac{a_n X}{2c_1}. \quad (5.132) \]

Equation (5.130) then reduces to

\[ g_{1t} + c_1 g_{1x} = -\frac{h_1}{2}. \quad (5.133) \]

At this stage we can proceed to an \( O(\alpha) \) solution in exactly the same way as in section 3. We use the expansion (5.74) for the temperature and the subsequent equations for \( T_0 \) (5.77) and \( T_1 \) (5.78), where \( T_0 \) depends only on \( g_0 \) which is given by (5.132) and \( g_1 \) depends only on \( T_0 \) and \( g_0 \), while \( T_1 \) depends on \( g_0, T_0 \) and \( g_1 \) and on the form of the transmission ratio \( R(\tau) \). In this case, the solution of (5.77) is

\[ T_0 = \gamma_1 R_0^2 e^{-\frac{a_n \tau}{c_1}} + T_1, \quad (5.134) \]

so that, to first order,

\[ T_{0t} = \gamma_1 R_0^2 e^{-\frac{a_n \tau}{c_1}} \quad \text{and} \]

\[ T_{0x} = -\frac{\gamma_1 R_0^2}{c_1} e^{-\frac{a_n \tau}{c_1}} + O(\alpha). \quad (5.135) \]
In section 5.1 we treated the two cases $T_i = 0$ and $T_i \neq 0$. Here we deal with only the simpler case, $T_i = 0$ and demonstrate that the solution is completely analagous to the section 5.1 solution, thus showing that the solution obtained in section 5.1 is correct regardless of the size of $a_1$ in (5.34).

By substituting (5.134) and (5.135) into expressions (5.53) for $h_1$, we obtain

$$h_1 = \frac{\mu_1 \mu_3}{\mu_1} \gamma_1^{\mu_3} \left( R_0^2 \right)^{\mu_3 - 1} R_0^2 e^{-\frac{a_0 a_3 X}{c_1}} + \frac{\mu_2 \mu_3}{\mu_1} \gamma_1^{\mu_3} \left( R_0^2 \right)^{\mu_3 - 1} R_0^2 e^{-\frac{a_0 a_3 X}{c_1}}$$

$$+ a_2 \left( \gamma_1^2 R_0^2 \right)^{\mu_3} e^{-\frac{a_0 a_3 X}{c_1}}. \quad (5.136)$$

We can then integrate (5.133) to yield

$$g_1(x, t) = \frac{\mu_1 \mu_3}{2a_0 a_1} \gamma_1^{\mu_3} \left( R_0^2 \right)^{\mu_3 - 1} R_0^2 (1 - e^{-\frac{a_0 a_3 X}{c_1}})$$

$$+ \frac{\mu_2 \mu_3}{2a_0 a_1} \gamma_1^{\mu_3} \left( R_0^2 \right)^{\mu_3 - 1} R_0^2 (1 - e^{-\frac{a_0 a_3 X}{c_1}})$$

$$+ \frac{a_2}{2a_0 a_3} \left( \gamma_1 R_0^2 \right)^{\mu_3} (1 - e^{-\frac{a_0 a_3 X}{c_1}}). \quad (5.137)$$

When we consider the equivalent case from section 5.1, apply the transformations $X = \alpha x$ and $a_1 = \alpha a_n$, and drop the terms which are relatively of $O(\alpha)$, then solution (5.70) for $g_0$ transforms to solution (5.132), and solutions (5.104) to (5.106) transform to solution (5.137), which shows that the solutions of section 3 are valid regardless of the size of $a_1$.

### 5.3 Some Exact Solutions For Power Laws of the Form $p = p_1(1 + p_2 T)^{p_3}$

In sections 5.1 and 5.2 we have developed solutions of the coupled transport equation (5.28) and the forced heat equation (5.40) for the case when the electromagnetic properties (i.e. the loss factor $\epsilon''$, the permittivity $\epsilon$ and the permeability $\mu$) and the heating rate $\gamma$ are all described by power laws of the form (5.34) to (5.36). In Chapter 2, power
laws of the following form were adopted

\[ p = p_1 (1 + p_2 T)^{p_3}, \]  

(5.138)

and approximate solutions were developed for some special cases. In this section, we again use power laws of the form (5.138) and develop some solutions.

When the electrical permittivity \( \varepsilon \) and the magnetic permeability \( \mu \) are both constant, the phasespeed \( c \) will also be constant. In this case, the transport equation (5.28) reduces to

\[ \phi_t + c \phi_x = -\frac{\varepsilon''}{2} \phi, \]  

(5.139)

which is analogous to the transport equation, for constant wavespeed \( c \), derived in Chapter 2 on the assumption of low conductivity and for properties which varied on a slow space scale and time scale (equation (2.28)),

\[ \phi_t + c \phi_x = -\frac{\epsilon'}{2} \phi, \]  

(5.140)

where \( \eta = \alpha t, X = \alpha x \) and \( \epsilon' = \alpha \varepsilon'' \). Note that the transmission ratio \( R \) is constant in this case and can be factored out of the equations as part of the nondimensionalising process. When the thermal diffusivity \( \nu \) is zero, the forced heat equation (5.40) reduces to

\[ T_t = \gamma(T) \phi^2, \]  

(5.141)

where, in this section, the heat coefficient \( \gamma \) and the loss factor \( \varepsilon'' \) are given by

\[ \gamma = \gamma_1 (1 + \gamma_2 T)^{\gamma_3} \]

\[ \varepsilon'' = a_1 (1 + a_2 T)^{a_3}. \]  

(5.142)
By making the transformation
\[ \theta = (1 + \gamma_2 T)^{1-\gamma_3}, \quad (5.143) \]
equation (5.141) can be reduced to
\[ \theta_t = \gamma_1 \gamma_2 (1 - \gamma_3) \phi^2, \quad (5.144) \]
for which the transformed heat coefficient has a constant value. For \( \gamma_3 < 1 \), \( \theta \) will grow with time and the associated temperature will also grow with time, but will always be finite for finite time. However, for \( \gamma_3 > 1 \), \( \theta \) will decrease with increasing heating time and will reach zero in a finite time, with the associated temperature approaching infinity as \( \theta \) approaches zero. This blow-up of the temperature in a finite time is associated with the formation of a hotspot (see Hill and Smyth, 1990, Brodwin et al, 1992, Coleman, 1991, and Chapter 3 of this work).

In general, the heating rate \( \gamma \) is proportional to the loss factor \( \varepsilon'' \), as shown in (5.38). Thus, by combining (5.38) with the power laws (5.142), we expect that
\[ \gamma_2 = a_2, \quad \gamma_3 = a_3 \quad (5.145) \]
and so the expression for the loss factor \( \varepsilon'' \) (5.142) transforms to
\[ \varepsilon'' = a_1 \theta^{(1-\gamma_3)}. \quad (5.146) \]

We can combine equations (5.139) and (5.144) to obtain the single equation
\[ \theta_{tt} + c \theta_{tx} = -\varepsilon'' \theta_t. \quad (5.147) \]

On substituting for \( \varepsilon'' \) from (5.146), we finally obtain
\[ \theta_{tt} + c \theta_{tx} = -a_1 \theta^\gamma \theta_t, \quad (5.148) \]
where
\[ n = \frac{\gamma_3}{(1 - \gamma_3)}. \] (5.149)

Equation (5.148) can be integrated once to yield the first order hyperbolic equation
\[ \theta_t + c\theta_x = -\frac{a_1}{n + 1}\theta^{n+1} + f(x). \] (5.150)

Ahead of the wavefront, \( \theta = 1 \) (as \( T = 0 \) there by scaling) and \( \theta_t = \theta_x = 0 \), giving
\[ f(x) = \frac{a_1}{n + 1}. \] (5.151)

In order to maintain continuity of \( \theta \), and hence the temperature \( T \), at the wavefront, we require (5.151) to be valid behind the wavefront, giving
\[ \theta_t + c\theta_x = \frac{a_1}{n + 1}(1 - \theta^{n+1}). \] (5.152)

In characteristic form, (5.152) becomes
\[ \frac{d\theta}{dt} = \frac{a_1}{n + 1}(1 - \theta^{n+1}) \quad \text{on} \quad \tau = t - \frac{x}{c}. \] (5.153)

From (5.144), the boundary condition is \( \theta = 1 + \gamma_1\gamma_2(1 - \gamma_3)t \) at \( x = 0 \) as \( \phi = 1 \) at \( x = 0 \).

Equation (5.153) cannot be solved in general, but exact solutions can be obtained for particular values of \( n \). Solutions are given below for the cases \( \gamma_3 = 0 \) \((n = 0)\), \( \gamma_3 = 1/2 \) \((n = 1)\), \( \gamma_3 = 2 \) \((n = -2)\) and \( \gamma_3 = 3/2 \) \((n = -3)\).

Case 1 \( n = 0 \)

In this case, the heating rate \( \gamma \) and the loss factor \( \epsilon'' \) are both constant (temperature independent) and the solution of (5.153) is
\[ \theta = \begin{cases} 1 + \gamma_1\gamma_2\tau e^{-\frac{a_1}{c} \tau}, & 0 \leq x \leq ct, \\ 1, & x > ct, \end{cases} \] (5.154)
where, as before, $\tau = t - x/c$ is the characteristic variable.

**Case 2 $n = 1$**

Here the solution is

$$\theta = \begin{cases} \coth\left(\frac{a_1 x}{2c} + \theta_c\right), & 0 \leq x \leq ct, \\ 1, & x > ct, \end{cases}$$  (5.155)

where

$$\theta_c = \coth^{-1}(1 + \frac{1}{2}\gamma_1\gamma_2\tau), \quad \tau = t - \frac{x}{c}.$$  

**Case 3 $n = -2$**

Here $\gamma_3 = 2$ and thus we have a case in which the temperature can become infinite in finite time. The solution is given implicitly by

$$\theta + \log(1 - \theta) = -\frac{a_1 x}{c} + 1 - \gamma_1\gamma_2\tau + \log(\gamma_1\gamma_2\tau), \quad 0 \leq x \leq ct,$$

$$\theta = 1, \quad x > ct,$$  (5.156)

where as before $\tau = t - x/c$. It can be seen that $\theta$ vanishing, so that $T$ becomes infinite and a hotspot forms, first occurs at $x = 0$ at the time $t = (\gamma_1\gamma_2)^{-1}$.

**Case 4 $n = -3$**

In this case, $\gamma_3 = 3/2$ and $\theta \leq 1$, so the appropriate solution is given implicitly by

$$\tanh^{-1}(\theta) - \theta = \frac{a_1 x}{2c} - 1 + \frac{1}{2}\gamma_1\gamma_2\tau + \tanh^{-1}(1 - \frac{1}{2}\gamma_1\gamma_2\tau), \quad 0 \leq x \leq ct,$$

$$\theta = 1, \quad x > ct,$$  (5.157)

where, as before $\tau = t - x/c$. In this case, $\theta$ becomes zero at $x = 0$, and a hotspot forms at $x = 0$, at time $t = 2(\gamma_1\gamma_2)^{-1}$.  

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5.4 Numerical Solution of the Coupled Wave and Heat Equations

In sections 5.1 and 5.2, we developed approximate analytical solutions for the electric field and the temperature, for the case where the electromagnetic properties vary slowly with temperature. It was shown in section 5.1 that the solution for the electric field can be transformed so that it has a similar form to the solutions derived in Chapter 2, which indicates that the solutions are qualitatively correct. However, the most critical test of the validity, both qualitatively and quantitatively, of the analytical solutions is obtained by comparing them with numerical solutions of the damped wave equation (1.9) and the forced heat equation (1.6), subject to the initial and boundary conditions (5.29), (5.43) and (5.44). The numerical scheme for the damped wave equation and the forced heat equation is summarised here and, for full details, the reader is referred to Chapter 2.

After applying the nondimensionalising transformations (5.8) to the damped wave equation (1.9), we obtain

\[ c^2 E_{xx} = AE + BE_t + E_{tt} \]  \hspace{1cm} (5.158)

where

\[
A = \frac{\mu t \epsilon t}{\mu \epsilon} + \frac{\epsilon'' \mu t}{\mu} + \frac{\epsilon t}{\epsilon} + \frac{\epsilon'' \epsilon t}{\epsilon} - \frac{c \mu x t}{\mu},
\]

\[
B = \frac{\mu t}{\mu} + \frac{2 \epsilon t}{\epsilon} + \epsilon'' - \frac{c \mu x}{\mu},
\]

where \( c, \mu \) and \( \epsilon \) are in nondimensional form.

When the wavespeed \( c \) is either a constant or a decreasing function of the temperature, equation (5.158) can be solved numerically by using a centred difference scheme (see
Burden et al, 1978), which results in the recurrence relation

\[ E_{k,j+1} = \frac{\lambda^2}{1 + \rho} E_{k+1,j} + \frac{2 - 2\lambda^2 - \eta}{1 + \rho} E_{k,j} + \frac{\lambda^2}{1 + \rho} E_{k-1,j} - \frac{1 - \rho}{1 + \rho} E_{k,j-1}, \]  

(5.161)

where

\[ \lambda = \frac{c\Delta t}{\Delta x}, \]

(5.162)

\[ \rho = \frac{B\Delta t}{2}, \]

(5.163)

\[ \eta = A(\Delta t)^2, \]

(5.164)

\[ E_{k,j} = E(k\Delta x, j\Delta t) \] and \( A, B \) are evaluated at \( x_k, t_j \). Here \( k \leq n, j \leq m \) and \( \Delta t \) and \( \Delta x \) are the time step and the space step respectively. The boundary conditions (5.20) become

\[ E_{0,j} = R(j\Delta t)[\cos(-j\Delta t + \xi(j\Delta t)) + i\sin(-j\Delta t + \xi(j\Delta t))], \]

(5.165)

and

\[ E_{n,j} = 0, \]

(5.166)

where the boundary \( x = n\Delta x \) of the domain of numerical integration is chosen to be large enough so that \( E \) is negligible there and thus \( E_{n,j} = 0, j = 1, 2, \ldots, m \). Consideration of computer storage and run time limitations places significant restrictions on the number of timesteps \( m \) which we can use. Initially, the material is in thermal equilibrium with no microwave radiation present, so that

\[ E_{k,0} = 0, \]

(5.167)

\[ E_{k,-1} = 0, \]

(5.168)
where we have assumed that $E = 0$ for $t < 0$.

The numerical scheme defined by (5.159) to (5.168) has second order accuracy, with error $\sim O(\Delta t^2, \Delta x^2)$; however we need to consider its stability. The scheme used here is almost identical to that of Chapter 2, which was found to be stable as long as $\lambda \leq 1$. If we define the error in our scheme by

$$\Delta E_{k,j} = E(x_k, t_j) - E_{k,j},$$  \hspace{1cm} (5.169)

then the difference equation (5.161) holds for $\Delta E_{k,j}$ and we can represent the error by the sum of Fourier components of the form $g^i \exp(i \beta k \Delta x)$ where $\beta$ is real. Substitution of this form into (5.161) gives a quadratic in $g$ with the solutions

$$g = \frac{\Gamma \pm (\Gamma^2 - 1 + \rho^2)^{\frac{1}{2}}}{1 + \rho}$$ \hspace{1cm} (5.170)

where

$$\Gamma = 1 - \frac{\eta}{2} - 2\lambda^2 \sin^2\left(\frac{\beta \Delta x}{2}\right).$$ \hspace{1cm} (5.171)

Note that $\eta \sim O(\Delta t^2)$ and, for $\Delta t \ll 1$ and $\Delta t^2 \ll \Delta x$, the contribution from the $\eta/2$ term can be ignored and the stability condition is thus $\lambda \leq 1$.

When the wavespeed $c$ is an increasing function of temperature, the time step $\Delta t$ must be adjusted so that the scheme remains stable. In Chapter 2 we used Taylor expansions to produce a four point scheme, which gave second order accuracy with variable wavespeed, in order to integrate the state equation (5.158) with $A = 0$. Here the coefficient $A$ is non-zero, but the four point scheme still gives second order accuracy.
The recurrence relation is

$$(1 + \rho_1)E_{k,j+1} = \lambda_1(E_{k+1,j} + E_{k-1,j}) + [1 + \varphi + \chi + \rho_1(1 + \Omega + \Gamma)]$$

$$-2\lambda_1 - \eta_1]E_{k,j} - (\varphi + \rho_1\Omega)E_{k,j-1}$$

$$- (\chi + \rho_1\Gamma)E_{k,j-2},$$

where

$$\rho_1 = \frac{B\Delta t_{j-1}[(\frac{\Delta t_j}{\Delta t_{j-1}})^2 + \varphi + (1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2]}{2[\frac{\Delta t_j}{\Delta t_{j-1}} - \Omega - \Gamma(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})]}$$

$$\lambda_1 = \frac{c^2\Delta t_{j-1}[(\frac{\Delta t_j}{\Delta t_{j-1}})^2 + \varphi + (1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2\chi]}{2\Delta x^2}$$

$$\eta_1 = \frac{1}{2}A\Delta t_{j-1}^2[(\frac{\Delta t_j}{\Delta t_{j-1}})^2 + \varphi + \chi(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2]$$

$$\Delta t_j = t_{j+1} - t_j$$

$$\varphi = \frac{\Delta t_j(1 + 2\frac{\Delta t_{j-2}}{\Delta t_{j-1}})}{\Delta t_{j-2}(2 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})}$$

$$\chi = \frac{\frac{\Delta t_j}{\Delta t_{j-1}}[(\frac{\Delta t_j}{\Delta t_{j-1}})^2 - 1]}{(1 + \frac{\Delta t_j}{\Delta t_{j-1}})[(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2 - 1]}$$

$$\Omega = (\frac{\Delta t_j}{\Delta t_{j-1}})^2[\frac{\Delta t_j}{\Delta t_{j-1}} - \frac{\Delta t_{j-1}}{\Delta t_{j-2}}(1 + \frac{\Delta t_j}{\Delta t_{j-1}})(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})]$$

$$\Gamma = \frac{\Delta t_j^2(1 + \frac{\Delta t_j}{\Delta t_{j-1}})}{\Delta t_{j-1}\Delta t_{j-2}(1 + \frac{\Delta t_{j-2}}{\Delta t_{j-1}})^2}.$$ 

To this stage, we have treated the coefficients $A$ and $B$ in (5.158) as functions which have a known value at the point $(x_k, t_j)$. However, we need to know the values of $\mu_t, \mu_x$ and $\epsilon_t$ before we can evaluate $B$ and we need $\mu_t, \mu_{xt}, \epsilon_t, \epsilon_{tt}$ and $\epsilon''_t$ before we can evaluate $A$. Since $\mu, \epsilon$ and $\epsilon''$ are functions of the temperature $T$, evaluation of $A$ and $B$ requires the numerical estimation of $T_t, T_x, T_{xt}$ and $T_{tt}$.

The numerical schemes (5.161) and (5.172) result in fluctuations and numerical dispersion at the wavefront due to the large $x$-derivatives there. In Chapter 2 we used a
moving boundary located at the wavefront to overcome this and calculated the amplitude of the electric field at the wavefront by using a wavefront expansion of the form

\[ E = a p_0(s), \]  

(5.181)

where \( s = x - c_1 t \) is the distance behind the wavefront, to first order, and \( a \) is the amplitude of the electric field at the wavefront. We take the same approach here, with the amplitude \( a \) being given by

\[ a = \frac{R(\tau)e^{-\frac{\xi^2}{2\xi^2}}}{[1 + 2\frac{\xi}{a_1}(1 - e^{-\frac{\xi^2}{2\xi^2}})]^{\frac{1}{2}}}, \]  

(5.182)

where

\[ b = \alpha_1(\frac{\mu_2\mu_3\mu_i^{\xi-1}}{\mu_i} + \frac{\epsilon_2\epsilon_3\mu_i^{\xi-1}}{\epsilon_i}), \]  

(5.183)

and

\[ \mu_i = \mu(T_i), \quad \epsilon_i = \epsilon(T_i). \]

in the case of zero diffusivity \( \nu = 0 \). We shall see, in section 7, that the estimate of the amplitude \( a \) of the electric field at the wavefront given by (5.182) enables us to obtain a smooth result from the integration of the damped wave equation (5.158) near the moving boundary.

The heat equation (1.6) can be integrated using the Crank-Nicolson scheme (see Burden et al, 1978), except for the fact that the nonlinear source term needs to be evaluated at the \((j + 1)th\) time step. This problem has been satisfactorily dealt with in Chapter 2 where the source term was evaluated using an extrapolated temperature which enabled the equation to be solved using the Crank-Nicolson scheme. We adopt
the same scheme here, using the recurrence relation

\[-\zeta T_{k+1,j+1} + (1 + 2\zeta)T_{k,j+1} - \zeta T_{k-1,j+1} - \Delta t \gamma_{k,j+1} |E|^2_{k,j+1}\]

\[= \zeta T_{k+1,j} + (1 - 2\zeta)T_{k,j} + \zeta T_{k-1,j} + \Delta t \gamma_{k,j} |E|^2_{k,j},\]  \hspace{1cm} (5.184)

where

\[\zeta = \frac{\nu \Delta t}{\Delta x^2}.\]  \hspace{1cm} (5.185)

5.5 Discussion

In sections 5.1 and 5.2, we derived solutions of the state equation (5.52). These solutions do not depend on specific values of the parameters, so that in this sense they are general solutions. However, they do rely on the assumption that the electromagnetic properties of the material being heated can be adequately represented by the power laws (5.34) to (5.36) and (5.39). It might well be assumed that a model of the form

\[p = p_1 + p_2 T^{p_3}\]  \hspace{1cm} (5.186)

with \(p_1, p_2, p_3\) being free parameters and with a free choice of a zero and a scale for the temperature \(T\), will be capable of fitting a wide range of curves for the temperature dependence of, for example, the real component of the permittivity \(\epsilon'\) and the loss factor \(\epsilon''\). This assumption can be easily verified by consulting von Hippel (1956). The graphs published by von Hippel were not specifically developed for microwave heating, but do contain examples with frequency \(\sim O(10^{10} \text{Hz})\), which is in the microwave range. Some simple curve fitting shows that a model of the form (5.186), where the zero of temperature can be chosen to help the fit, can represent any curve for which \(d^2 p/dT^2\) does not change.
sign in the temperature range of interest. The simplest of our solutions was obtained for the case in which the initial temperature $T_i$ was zero. To check the viability of this representation, we chose some typical workplace temperatures (e.g. $10^\circ C$, $25^\circ C$) as the zeroes of our scaled temperature and, after some simple curve fitting, found that, using this model, we could obtain good representations of curves as long as

(a) $p(T)$ is monotonic increasing

and (b) $\frac{d^2p}{dT^2} > 0$.

We can also use the model when $p(T)$ is monotonic decreasing and $d^2p/dT^2 < 0$, but this is of little practical interest. As an example, the compound AlSiMag A-196 (von Hippel, 1956, p377-378) can be represented by

$$
\epsilon' = 5.3 \\
\epsilon'' = 0.011 + 1.4 \times 10^{-2}T^{1.825},
$$

where the temperature $T$ is obtained from the Celsius temperature $T^*$ by

$$
T = \frac{T^* - 25}{500},
$$

that is, our scaled temperature is zero at a normal room temperature of $25^\circ C$ and each unit of temperature represents $500^\circ C$. In terms of (5.34), equation (5.188) gives

$$
a_1 = 0.011, \ a_2 = 1.4, \ a_3 = 1.825.
$$

Thus the evidence shows that the power laws (5.34) to (5.36) and (5.39) are practical models for the representation of the variation of the electromagnetic properties with temperature, even in the case where the initial temperature is constrained to be (scaled)
zero. However, we still need to check the correctness of the perturbation solutions. We do this by comparing them with the equivalent numerical solutions generated using the schemes outlined in section 5.4.

Because the analytical solutions were derived under the assumption that the temperature diffusivity coefficient $\nu$ was zero, $\nu$ will be set to zero for the numerical solutions unless otherwise stated. In the numerical scheme defined in section 5.4, we set the far boundary at a value of the space variable $x$ which will not be reached during the simulation. Use of a semi-infinite block simplifies the analytical solution, but creates some difficulties numerically. Basically, with limited computer memory available, we want the simulation to be completed before the initial wavefront reaches the limit of storage for the space variable $x$. This can be arranged by making the coefficient of heat absorption $\gamma$ sufficiently large. It turns out, after some numerical trials, that $\gamma = \epsilon''$ is large enough. In the definition (5.42) of the parameters $\gamma_1, \gamma_2, \gamma_3$, the constant $k$ is very small ($k \sim O(10^{-12})$, Metaxas and Meredith, 1983), so that our choice of $\gamma = \epsilon''$ is equivalent to requiring that the amplitude $E_0$ of the incident electric field is very large ($\sim O(10^6)$ volts/metre), which is equivalent to a power of $\sim O(10^9)$ watts. Note that we do not claim that the perturbation model is applicable at such high field strengths, merely that the numerical solution at the high field strength is a valid scaled-up version of what happens at lower field strengths.

We first consider the case where the electrical permittivity $\epsilon$ and the magnetic permeability $\mu$ are both constant, but the loss factor $\epsilon''$ and the heating coefficient $\gamma$ vary with temperature. In this case, the transmission ratio $R(\tau)$ is constant and can thus be
Figure 25: Electric field amplitude and temperature for the case $\alpha = 0.01$, $\epsilon = \mu = 1.0$, $a_1 = \gamma_1 = 0.1$, $a_2 = \gamma_2 = 1.0$, $a_3 = \gamma_3 = 1.5$ at $t = 30$. $A$ is the numerical solution, $B$ is the analytical solution to first order and $C$ is the analytical solution to $O(\alpha)$, incorporated in the $\gamma$ term. The following temperature variations are used to represent the case

$$\epsilon'' = 0.1 + \alpha T^{1.5}, \quad \gamma = 0.1 + \alpha T^{1.5}, \quad \alpha = 0.01, \quad \epsilon = \mu = 1.0.$$  \hspace{1cm} (5.191)

The results, after 600 time steps ($t=30$), are shown in Figure 25, with $A$ representing the numerical solution, $B$ representing the constant coefficients solution and $C$ representing the perturbation solution. There are some fluctuations in the numerical solution for the amplitude function $\phi$ (Figure 25a) which arise just behind the wavefront and rapidly attenuate. These may be partly caused by numerical errors. The methods outlined in section 5.4 have been used to minimise these errors, but it has not been possible to eliminate them completely. Furthermore, the finite size of $\epsilon''$ leads to oscillations in the electric field near the wavefront. This was found by Marchant and Smyth (1992), who
solved (1.9) for constant $\mu$, $\epsilon$ and $\epsilon''$ using Laplace transforms. The amplitude of these oscillations was found to go to zero as $\epsilon''$ goes to zero. It can be seen that the perturbation solution is very close to the numerical solution both for the amplitude function $\phi$ (Figure 25a) and for the temperature $T$ (Figure 25b). The greatest error comes, as expected, where $T$ is greatest, at the boundary $x = 0$. At this time, at $x = 0$, the perturbation solution predicts a temperature $T = 3.65$. If we apply the conditions for uniform validity (5.108), we note that $\alpha(3.65)^{1.5}$ is not $\ll 1$, so that we have gone beyond the region of uniform validity. However, the perturbation solution still predicts the temperature at $x = 0$ with an error of only 2.7%, compared to 20% for the constant coefficients solution.

When the same parameter values are used, but the numerical solution is run for 800 time steps ($t = 40$), the perturbation solution is still close to the numerical solution both for
Figure 27: Electric field amplitude and temperature for the case $\alpha = 0.01$, $\epsilon'' = \gamma = 0.1$, $\epsilon_1 = \mu_1 = 1.0$, $\epsilon_2 = \mu_2 = 1.0$, $\epsilon_3 = \mu_3 = 2.1$ at $t = 30$. $A$ is the numerical solution, $B$ is the analytical solution to first order and $C$ is the analytical solution to $O(\alpha)$.

the amplitude function $\phi$ (Figure 26a) and for the temperature $T$ (Figure 26b). In this case, the temperature at $x = 0$ is $T = 5.3$ and, since $\alpha(5.3)^{1.5} > 0.1$, we are obviously well outside the region of uniform validity. Yet, at $x = 0$ where the error is greatest, the error in $T$ is still only 5.1% for the perturbation solution, against 31% for the constant coefficients solution.

We next consider the case where the loss factor $\epsilon''$ and the heating coefficient $\gamma$ are both constant, but the permittivity $\epsilon$ and the permeability $\mu$ both vary with temperature. This case is represented here by

$$\epsilon'' = \gamma = 0.1, \quad \epsilon = \mu = 1.0 + \alpha T^{2.1}, \quad \alpha = 0.01.$$  \hspace{1cm} (5.192)

There is not a great deal of difference between the three solutions in this case (see Figure 27), but there is no doubt that the perturbation solution ($C$) is much closer than the
constant coefficients solution (B) to the numerical solution (A).

In order to test the solution (5.96) for the case when \( T \neq 0 \), we need to separate the space into two regions, \( 0 \leq x < x_t \) and \( x_t \leq x < \infty \), where \( \gamma_1 R_0^2 \tau e^{x - a_1 x/c_1} > T_i \) in the first region and \( \gamma_1 R_0^2 \tau e^{x - a_1 x/c_1} \leq T_i \) in the second. A comparison of (5.98) to (5.100) and (5.101) to (5.103) shows that the two solution sets are identical when \( T_0 = T_i \). The transition point \( x_t \), where we change from one solution form to the other, is the solution of

\[
\gamma_1 \left( t - \frac{x_t}{c_1} \right) e^{-\frac{a_1 x_t}{c_1}} = T_i. \tag{5.193}
\]

When the wavespeed \( c \), given by (5.54), is not constant, the evaluation of the electric field strength at any point is more complicated than it is for constant wavespeed. The amplitude \( \phi \) of the electric field, given by (5.57) coupled with (5.61), (5.64), (5.65), (5.67), (5.70), (5.104), (5.105) and (5.106), is affected both by the absorption of energy and by variations in the wavespeed. When the wavespeed \( c \) is an increasing function of temperature, then wavespeed will decrease with an increase of the space variable \( x \) and the terms containing \( \mu_2, \mu_3, \epsilon_2, \epsilon_3 \) in (5.105) and (5.106) will contribute to an increase in the local value of the electric field strength, this contribution being caused by compression of the waveform. In the opposite case, when the wavespeed is a decreasing function of the temperature, there is a stretching of the waveform and a consequent decrease in the local value of the amplitude of the electric field.

Confirmation of the amplification/reduction effect can be obtained from a physical argument. If \( n(x, t) \) is the photon density, i.e. the number of photons per unit length at
(x, t), then we can write the continuity equation for photons in the form

\[ n_t + cn_x = -(c_x + \beta)n, \quad (5.194) \]

where \( \beta \) is the proportion of photons which is absorbed by the material in unit time and \( c \) is the phasespeed, as before. The continuity equation (5.194) can be obtained in the usual way by considering the population of photons in a region of infinitesimal width or it can be obtained by considering how the difference in the wavespeed at different points causes the waves to bunch up (\( c_x < 0 \)) or to stretch out (\( c_x > 0 \)). If the energy density is given by

\[ I = \hbar \omega n, \quad (5.195) \]

where \( \hbar \) is Planck’s constant divided by \( 2\pi \), then it is a simple matter to show that

\[ I_t + cI_x = \hbar \omega(n_t + cn_x) + \hbar \omega(n + c\omega_x). \quad (5.196) \]

Using the eikonal equation (5.27), we can show that

\[ \omega_t + c\omega_x = \frac{\omega c_t}{c}. \quad (5.197) \]

Thus (5.196) gives

\[ I_t + cI_x = -(c_x + \beta - \frac{c_t}{c})I. \quad (5.198) \]

The energy density \( I \) for an electromagnetic wave is proportional to \( \varepsilon \phi^2 \), so that (5.198) leads to

\[ (\varepsilon \phi^2)_t + c(\varepsilon \phi^2)_x = \varepsilon \phi^2 \left( \frac{c_t}{c} - c_x - Q + 2 \frac{\varepsilon_t}{\varepsilon} + \frac{\mu_t}{\mu} \right), \quad (5.199) \]

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where $Q$ is the nondimensionalised coefficient defined as $Q'$ in (5.23), and we have taken $\beta = \epsilon''$. By rearranging (5.199) we obtain (5.28) after noting that

$$\frac{\epsilon_t}{\epsilon} + \frac{\mu_t}{\mu} = -2\frac{c_t}{c} \quad (5.200)$$

and

$$-c\frac{\epsilon_x}{\epsilon} = 2c_x + c\frac{\mu_x}{\mu}. \quad (5.201)$$
6 References


7 Publications of the author


Papers 1, 2, 4 and 5 are included in this thesis. Paper 1 is Chapter 3, paper 2 is Chapter 4, paper 4 is Chapter 2 and paper 5 is Chapter 5.