Chebyshev polynomials in the solution of ordinary and partial differential equations

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CHEBYSHEV POLYNOMIALS IN THE SOLUTION OF
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Abstract

Chebyshev polynomials are used to obtain accurate numerical solutions of ordinary and partial differential equations.

For functions of one or two variables, expressed in terms of Chebyshev polynomials, generalisations are obtained of formulae for finding function and derivative values.

Then, for ordinary differential equations a standard method of solution is extended for use at arbitrary points, and is applied to a number of differential equations associated with functions of mathematical physics.

Elliptic partial differential equations, similar to Laplace equations, are examined and it is shown how Chebyshev series solutions can be found, the coefficients in the solution being obtained in a manner related to that for calculating function values at grid points, when a finite difference method is used.
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Values of \( J_n(x) \), \( \frac{\partial J_n(x)}{\partial x} \), \( \frac{\partial J_n(x)}{\partial n} \),
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CHAPTER 1

INTRODUCTION

This thesis contains a report on the use of series of Chebyshev polynomials to solve certain ordinary and partial differential equations. Firstly there is a systematic approach to the solution of ordinary second order linear differential equations with simple polynomial coefficients, and then the important new work is the solution of second order elliptic partial differential equations of Laplace type, in series of Chebyshev polynomials in two variables, using a simple recurrence relation for the coefficients in the series.

The presentation is in three main parts.

Chapter 2 contains a detailed statement of formulae, and most of the required properties of Chebyshev polynomials. Emphasis is placed on evaluation schemes for both function and derivative values. In particular, for functions of two variables, formulae of Basu (1973) are extended to yield numerical values of partial derivatives at any point.

Chapters 3 and 4 contain a discussion of the solution of ordinary second order differential equations in order to present a direct method for solving such problems, and in order to provide background, and also a means of data preparation, for the work in the following chapters.

In Chapters 5 and 6, Chebyshev series in two variables are used to solve elliptic partial differential equations. A single recurrence relation is obtained for the coefficients in the series solution. The recurrence relation resembles that obtained using finite difference methods and this suggests the use of relaxation methods to solve the resulting algebraic equations. Once these are solved, the Chebyshev series solution gives excellent accuracy for very little effort, with the usual advantages that solution and derivative values are not restricted
to points on a predetermined grid, but are available at any point.

Chapter 7 contains a discussion of the relaxation method used in solving the algebraic equations obtained from the recurrence relation and the boundary conditions.

Chebyshev series have been used in the solution of differential equations by many authors, including, for ordinary differential equations Lanczos (1957), Clenshaw (1957, 1962, 1966), Fox and Parker (1972), Snell (1970), Boateng (1975), and Clenshaw and Elliott (1960), and for partial differential equations by Elliott (1961), Scraton (1965), Mason (1965), Knibb and Scraton (1971), Dew and Scraton (1973, 1975), Boateng (1975).

A Chebyshev series provides a "global" solution to a differential equation in the sense that the coefficients in the series are found, enabling numerical values of the solution to be found at any point on the solution interval. The methods fall into two main classes. One is the collocation, or selected points, method in which the coefficients in the series are found by requiring that the solution be exact at a given set of points, often the zeros of Chebyshev polynomials. A disadvantage of this method is that it generates a "full" system of linear simultaneous equations to be solved for the coefficients. The other class of methods involves finding the coefficients from recurrence relations which they satisfy. The recurrence relations may involve coefficients in further series for the derivatives of the solution, as well as the solution itself, or the derivative coefficients may be eliminated to obtain pure recurrence relations for the coefficients in the solution only.

With regard to ordinary differential equations, Fox and Parker (1972) refer to the original system, the integrated system, the mixed system and the tau (τ) method used by various authors for obtaining the recurrence relations. In these cases the solution coefficients must
still be obtained by solving simultaneous algebraic equations, but the coefficient matrices are now "sparse". These sparse systems are simple to solve by standard methods such as Gaussian elimination or sometimes by relaxation methods. The approach used here has been to find \( m \) independent series \((m \leq 4)\) whose coefficients satisfy a recurrence relation, and to form a linear combination of such series as the final solution. With the calculation based on numerical values of the solution and/or its first derivative, at data points, together with numerical values from "extra conditions" (as explained in Chapter 3), no more than four algebraic equations have to be solved to find the multipliers in the linear combination which gives the final solution.

In the algebraic equations arising in the solution of partial differential equations, there is a large number of unknowns, and even though the coefficient matrix is relatively sparse, Gaussian elimination is not attractive. A relaxation method is used and this is explained in Chapter 7.
2.1 Definitions and Standard Properties

Following Clenshaw (1962), Fox and Parker (1972), Pollard (1967), Phillips and Taylor (1973), Gerald (1970), and others, the following definitions and general results can be stated concerning Chebyshev polynomials.

The Chebyshev polynomial of the first kind of order \( k \) is written \( T_k(x) \) and is defined by

\[
T_k(x) = \cos k\theta \]

where \( x = \cos \theta \) \hspace{1cm} (2.11)

or

\[
T_k(x) = \cos(k(\cos^{-1}x)), \quad -1 \leq x \leq 1, \]

for \( k \) an integer.

From the definition it is seen that

\( T_{-k}(x) = T_k(x) \).

Also \( T_k(-x) = (-1)^k T_k(x) \),

namely \( T_k(x) \) is an odd or even function of \( x \), as \( k \) is odd or even.

We have immediately

\[
T_k(1) = 1
\]
\[
T_k(-1) = (-1)^k
\]
\[
T_{2k+1}(0) = 0
\]
\[
T_{2k}(0) = (-1)^k
\]

Writing \( z = e^{i\theta} \), (2.11) leads to an alternative definition

\[
T_k(x) = \frac{1}{2}(z^k + z^{-k})
\]

where \( x = \frac{1}{2}(z + z^{-1}) \) \hspace{1cm} (2.13)
Recalling de Moivre's theorem, and the fact that
\[ \cos k\theta = R\{\cos k\theta + i \sin k\theta\} \]
namely that \(\cos k\theta\) is the real part of \(e^{ik\theta}\), a further definition of \(T_k(x)\) is
\[ T_k(x) = R\{(x + i\sqrt{1-x^2})^k\} \quad (2.14) \]

From the trigonometrical identity
\[ 2 \cos k\theta \cos \ell\theta = \cos(k+\ell)\theta + \cos(k-\ell)\theta \]
then
\[ 2 T_k(x) T_\ell(x) = T_{k+\ell}(x) + T_{|k-\ell|}(x) \quad (2.15) \]
and in particular
\[ 2x T_k(x) = T_{k+1}(x) + T_{|k-1|}(x) \quad (2.16) \]
since \(T_1(x) = x\).

A further use of (2.16) gives
\[ 4x^2 T_k(x) = 2x T_{k+1}(x) + 2x T_{|k-1|}(x) \]
\[ = T_{k+2}(x) + 2 T_k(x) + T_{|k-2|}(x) \]
and the completely generalised result is
\[ (2x)^p T_k(x) = \sum_{j=0}^{p} \binom{p}{j} T_{|k-p+2j|}(x) \quad (2.16)' \]

With the initial conditions
\[ T_0(x) = 1, \quad T_1(x) = x \]
(2.16) can be used to generate the Chebyshev polynomials. Thus
\[ T_0(x) = 1 \]
\[ T_1(x) = x \]
\[ T_2(x) = 2x^2 - 1 \]
\[ T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]
\[ T_5(x) = 16x^5 - 20x^3 + 5x \]
\[ T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \]
The coefficient of $x^{k-2r}$ in the expression for $T_k(x)$ is given by

$$(-1)^r 2^{k-2r-1} \left\{ 2 \binom{k-r}{r} - \binom{k-r-1}{r} \right\}$$

(2.17)

Using (2.13), or otherwise, expressions for powers of $x$, as sums of Chebyshev polynomials can be found as follows:

$$
\begin{align*}
1 &= T_0(x) \\
x &= T_1(x) \\
2x^2 &= T_0(x) + T_2(x) \\
4x^3 &= 3T_1(x) + T_3(x) \\
8x^4 &= 3T_0(x) + 4T_2(x) + T_4(x) \\
16x^5 &= 10T_1(x) + 5T_3(x) + T_5(x) \\
32x^6 &= 10T_0(x) + 15T_2(x) + 6T_4(x) + T_6(x) \\
64x^7 &= 35T_1(x) + 21T_3(x) + 7T_5(x) + T_7(x) \\
128x^8 &= 35T_0(x) + 56T_2(x) + 28T_4(x) + 8T_6(x) + T_8(x)
\end{align*}
$$

The coefficient of $T_{k-2r}(x)$ in the expression for $x^{k-1}$ is

$$\binom{k}{r} \text{ when } 2r \neq k \text{ and } \frac{1}{2} \binom{k}{\frac{1}{2}k} \text{ when } 2r = k.$$  

(2.18)

The above results can be conveniently summarised in matrix form as follows, where the results explicitly stated above are represented.
\[
\begin{bmatrix}
T_0(x) \\
T_1(x) \\
T_2(x) \\
T_3(x) \\
T_4(x) \\
T_5(x) \\
T_6(x) \\
T_7(x) \\
T_8(x)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
2 & 0 & -4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & -5 & \cdots & 0 & 0 & 0 & 0 \\
-2 & 0 & 9 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -7 & 0 & 14 & \cdots & 0 & 0 & 0 & 0 \\
2 & 0 & -16 & 0 & \cdots & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
2^{-1} \\
2^0x \\
2^1x^2 \\
2^2x^3 \\
2^3x^4 \\
2^4x^5 \\
2^5x^6 \\
2^6x^7 \\
2^7x^8
\end{bmatrix}
\tag{2.19}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
6 & 0 & 4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 5 & \cdots & 0 & 0 & 0 & 0 \\
20 & 0 & 15 & 0 & 6 & 0 & 1 \\
0 & 35 & 0 & 21 & 0 & 7 & 0 & 1 \\
70 & 0 & 56 & 0 & 28 & 0 & 8 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
T_0(x) \\
T_1(x) \\
T_2(x) \\
T_3(x) \\
T_4(x) \\
T_5(x) \\
T_6(x) \\
T_7(x) \\
T_8(x)
\end{bmatrix}
\tag{2.20}
\]

In general
\[
t = Px
\tag{2.19}'
\]
and
\[
x = Ct
\tag{2.20}'
\]
where \(t, x\) are the \((n+1)\) - element vectors
\[
t^T = [T_0(x), T_1(x), T_2(x), ..., T_n(x)]
\]
\[
x^T = [t, x, 2x^2, ..., 2^{n-1}x^n]
\]
and \(P, C\) are unit lower triangular matrices such that
\[
P = [p_{ij}], C = [c_{ij}], i, j = 0, ..., n
\]
and \[ P_{oo} = 1 \]
\[ P_{10} = 0, P_{11} = 1 \]
\[ P_{20} = -2, P_{21} = 0, P_{22} = 1 \]
\[ P_{i0} = -P_{i-2,0} \]
\[ P_{ij} = P_{i-1,j} - P_{i-2,j}, \quad j = 1, \ldots, i \] \[ i = 3, \ldots, n \]
\[ c_{oo} = 1 \]
\[ c_{i0} = 2c_{i-1,1} \]
\[ c_{ij} = c_{i-1,j-1} + c_{i-1,j+1}, \quad j = 1, \ldots, i \] \[ i = 1, \ldots, n \]
Obviously \[ P^{-1} = C \] and \[ C^{-1} = P . \]
\[ P_{ij} = c_{ij} = 0 \text{ when } j > i . \]

In some circumstances it has been convenient to use these matrices to change a polynomial in \( x \) into a sum of Chebyshev polynomials, and vice versa.

2.2 The Shifted Chebyshev Polynomials

The shifted Chebyshev polynomial \( T_k^*(x) \) is defined by
\[ T_k^*(x) = T_k(2x-1) \]
\[ = \cos(k \cos^{-1}(2x-1)), \quad 0 \leq x \leq 1. \]

These shifted polynomials are not often used in the present work, as it has seemed preferable to take advantage of the many symmetry properties of the ordinary Chebyshev polynomials about \( x = 0 \).

Because \[ T_k^*(x^2) = T_{2k}(x) \]
many of the properties of the shifted Chebyshev polynomial can be obtained from the corresponding properties of the ordinary Chebyshev polynomial.
2.3 The Chebyshev Polynomial of the Second Kind

The Chebyshev polynomial of the second kind of order \( k \) is written \( U_k(x) \), and is defined by

\[
U_k(x) = \frac{\sin(k+1)\theta}{\sin \theta} \\
x = \cos \theta.
\]  

(2.31)

Now

\[
U_k(x) - U_{k-2}(x) = \frac{\sin(k+1)\theta - \sin(k-1)\theta}{\sin \theta} = 2 \cos k\theta = 2T_k(x).
\]

Repeated use of this identity yields

\[
U_k(x) = T_0(x) + 2T_2(x) + 2T_4(x) + \ldots + 2T_k(x)
\]  

(2.32)

when \( k \) is even, and

\[
U_k(x) = 2T_1(x) + 2T_3(x) + \ldots + 2T_k(x)
\]  

(2.33)

when \( k \) is odd.

Again using the definition directly, it can be shown that \( U_k(x) \) satisfies a recurrence formula identical to (2.16), namely,

\[
2x U_k(x) = U_{k+1}(x) + U_{k-1}(x).
\]  

(2.34)
2.4 Derivatives and Integrals of $T_k(x)$

Derivatives

$$
\frac{d}{dx} T_k(x) = \frac{d}{d\theta} (\cos k\theta) \frac{dx}{d\theta}
$$

$$
= k \sin (k\theta) / \sin \theta
$$

$$
= k U_{k-1}(x) \quad (k \geq 1)
$$

Hence,

$$
\frac{d}{dx} T_0(x) = 0
$$

and for $k > 0$, using (2.32) and 2.33)

$$
\frac{d}{dx} T_k(x) = 2k \{ T_0(x) + T_2(x) + \ldots + T_{k-1}(x) \} \quad , \quad k \text{ odd}
$$

$$
\frac{d}{dx} T_k(x) = 2k \{ T_1(x) + T_3(x) + \ldots + T_{k-1}(x) \}, \quad k \text{ even}
$$

Integrals

From the definition (2.11) it is easy to show that

$$
\begin{align*}
T_0(x) dx &= T_1(x) \\
T_1(x) dx &= \frac{1}{2} T_2(x) \\
T_k(x) dx &= \frac{1}{2} \left\{ \frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1} \right\} , \quad k > 1
\end{align*}
$$

(2.42)

Orthogonality Property

The Chebyshev polynomials $\{ T_k(x) \}$ form a set of polynomials orthogonal with respect to the weight function $(1-x^2)^{-1/2}$ on the interval $(-1, 1)$, namely

$$
\int_{-1}^{1} (1-x^2)^{-1/2} T_l(x) T_k(x) dx = 0 , \quad l \neq k
$$

(2.43)
The non-zero values of the integrals, when \( \ell = k \), are given by

\[
\begin{align*}
\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_o(x)^2 \, dx &= \pi \\
\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_k(x)^2 \, dx &= \frac{\pi}{2}, \quad k \neq 0
\end{align*}
\]
2.5 Functions of one variable in terms of Chebyshev polynomials.

Let the function \( f(x) \) be expressed as a uniformly convergent series of Chebyshev polynomials in the form

\[
f(x) = \sum_{i=0}^{\infty} a_i T_i(x)
\]

(2.51)

where now, and in all future occurrences, the notation \( \sum' \) indicates that \( b a \), rather than \( a \), is to be taken in the summation.

\( f'(x) \) can be expressed in a similar series,

\[
f'(x) = \sum_{i=0}^{\infty} a'_i T_i(x)
\]

\[
= b a'(0) T_0(x) + a'_1 T_1(x) + a'_2 T_2(x) + \ldots
\]

where \( a'_0, a'_1, a'_2, \ldots \) are constants and the superscripts merely indicate that the coefficients are those in the series for \( f'(x) \).

Indeed the notation can be completely generalised to give

\[
f^{(r)}(x) = \sum_{i=0}^{\infty} a_i^{(r)} T_i(x)
\]

(2.52)

as a series for the \( r \)th derivative of \( f(x) \). The notation is understood to include (2.51) when \( r = 0 \), namely \( a_i^{(0)} = a_i \).

Now from the fundamental definition of an integral,

\[
\int f^{(r)}(x) dx = f^{(r-1)}(x) + \text{constant}
\]

Hence, on substituting from (2.52), and interchanging the order of summation and integration

\[
\sum_{i=0}^{\infty} a_i^{(r)} T_i(x) dx = \sum_{i=0}^{\infty} a_i^{(r-1)} T_i(x) + \text{constant}
\]

Then, using (2.42)

\[
= b a_0^{(r)} T_0(x) + a_1^{(r)} T_1(x) + \sum_{i=2}^{\infty} a_i^{(r)} \left( \frac{T_{i+1}(x)}{i+1} - \frac{T_{i-1}(x)}{i-1} \right)
\]

\[
= \sum_{i=0}^{\infty} a_i^{(r-1)} T_i(x) + \text{constant}
\]
On rearranging the terms, and equating coefficients of like Chebyshev polynomials it is found that

\[ a_{i-1}^{(r)} - a_{i+1}^{(r)} = 2i a_i^{(r-1)}, \quad i > 0. \]  

(2.53)

With the interpretation

\[ a_{-i}^{(r)} = a_i^{(r)} \]

it is not necessary to qualify all the formulae to avoid coefficients with negative subscripts.
2.6 The coefficients in the series $\sum_{i=0}^{\infty} a_i T_i(x)$

From (2.53),

$$a_0(x) - a_2(x) = 2.1. a_1$$
$$a_2(x) - a_4(x) = 2.3. a_3$$
$$a_4(x) - a_6(x) = 2.5. a_5$$

Hence

$$a_0(x) = 2\left\{1 a_1 + 3a_3 + 5a_5 + \ldots\right\}$$
$$a_2(x) = 2\left\{3a_1 + 5a_3 + \ldots\right\}$$

assuming that $\lim_{i \to \infty} a_i = 0$,

and in general

$$a_i(x) = 2 \sum_{k=i+1}^{\infty} k a_{k}(x-1)$$  \hspace{1cm} (2.61)

where $i$ is even and the summation is taken with $k$ incremented in steps of 2, from $i + 1$.

Consideration of terms of the type $a_i(x)$ with $i$ being odd, leads again to formula (2.61).

Hence, given the coefficients in the series $\sum_{i=0}^{\infty} a_i T_i(x)$, the coefficients in the series $\sum_{i=0}^{\infty} a_i(x) T_i(x)$ can be found.

That is, given the Chebyshev series expansion for a function, the expansion for its derivative is obtained. In the present work some use is made of this result when considering boundary and initial conditions in the solution of differential equations.

However, a far more fundamental result is the one in which the coefficients $\{a_i^{(r-1)}\}$ of the integral function, are found, given the coefficients $\{a_i^{(r)}\}$. This is the basis of the Clenshaw-Curtis method of integration, and involves the direct use of (2.53) in the form

$$a_i^{(r-1)} = (\frac{r}{2i}) - a_i^{(r)}$$  \hspace{1cm} (2.62)
2.7 The numerical value of $\sum_{i=0}^{n} a_i T_i(x)$

A standard evaluation scheme for the finite sum

$$f(x) = \sum_{i=0}^{n} a_i T_i(x)$$

is that of Clenshaw, as follows:

Let $b_{n+2} = b_{n+1} = 0$

and calculate

$$b_k = a_k + 2x b_{k+1} - b_{k+2}, \quad k = n, \ldots, 1, 0$$

Then

$$f(x) = \frac{1}{2}(b_0 - b_2)$$

This result is easily proved using (2.16).

Clenshaw and Smith (1965) show how the numerical values of $f'(x)/r!$ can be evaluated using similar schemes, and Hunter (1970) points out how the method is related to the synthetic division method for evaluating a polynomial and its derivatives.

The derivative result is derived by Smith from the three term recurrence relation satisfied by orthogonal polynomials. The method adopted here is to regard the coefficients $\{b_k\}$, defined in (2.71), as functions of $x$, and to differentiate in the above formulae.

Hence

$$\frac{\partial b_{n+2}}{\partial x} = \frac{\partial b_{n+1}}{\partial x} = 0$$

$$\frac{\partial}{\partial x} b_k = 2 b_{k+1} + 2x \frac{\partial}{\partial x} b_{k+1} - \frac{\partial}{\partial x} b_{k+2}, \quad k = n, \ldots, 1, 0.$$  

$$f'(x) = \frac{1}{2}\left(\frac{\partial b_0}{\partial x} - \frac{\partial b_2}{\partial x}\right)$$
In general, for \( r > 0 \)

\[
\frac{1}{r!} \frac{\partial^r}{\partial x^r} b_{n+2} - \frac{1}{r!} \frac{\partial^r}{\partial x^r} b_{n+1} = 0
\]

\[
\frac{1}{r!} \frac{\partial^r}{\partial x^r} b_k = 2 \frac{1}{(r-1)!} \frac{\partial^{r-1}}{\partial x^{r-1}} b_{k+1} + 2x \frac{1}{r!} \frac{\partial^r}{\partial x^r} b_{k+1} - \frac{1}{r!} \frac{\partial^r}{\partial x^r} b_{k+2}
\]

\( k = n, \ldots, 1, 0 \)

It is obvious that the recurrence relation requires only two (upper) starting values, and that as \( r \) increases, \( \frac{1}{r!} \frac{\partial^r}{\partial x^r} b_k = 0 \) for more than just \( k = n+2, n+1 \). Hence the formulae can be modified as follows:

Write \( b^r_1 = \frac{1}{r!} \frac{\partial^r}{\partial x^r} b_1 \).

Then

\[
b^r_{n+2-r} = b^r_{n+1-r} = 0
\]

\[
b^r_{k-r} = 2 b^r_{k+1-r} + 2x b^r_{k+1-r} - b^r_{k+2-r}, \quad k = n-r, \ldots, 1, 0
\]

and

\[
\frac{f^{(r)}(x)}{r!} = \frac{1}{r!} \left( \frac{\partial^r}{\partial x^r} b_0 - \frac{\partial^r}{\partial x^r} b_2 \right)
\]

for \( r = 0, 1, \ldots, n \).

Occurring when \( r = 0 \), \( b^{-1}_{k+1} \) must be interpreted as \( b^{-1}_{a_k} \), \( k = 0, 1, \ldots, n \).

In a computer implementation it is convenient to proceed through the above calculations for the derivatives, using only two current vectors, for \( \begin{bmatrix} b^r_k \end{bmatrix} \), and \( \begin{bmatrix} b^{r-1}_k \end{bmatrix} \), followed by updating before proceeding to the next derivative calculation. In fact a single vector can be used in conjunction with a few auxiliary variables.
2.8 Functions of Two Variables in Terms of Chebyshev Polynomials

Let the function \( f(x, y) \) be expressed as a uniformly convergent double series of Chebyshev polynomials in the form

\[
f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} T_i(x) T_j(y)
= \sum_{i,j=0}^{\infty} a_{ij} T_i(x) T_j(y) \tag{2.81}
\]

The notation \( \sum_{i}^{'} \) now signifies that \( a_{ij} \) must be taken whenever \( i=0 \), and \( \sum_{j}^{''} \) means that \( a_{ij} \) must be taken whenever \( j = 0 \). When \( i=j=0 \), the term \( a_{00} \) is taken in the summation. For the double summation over \( i \) and \( j \), it is convenient to use the rotation \( \sum_{i,j}^{'''} \), having the combined meanings above.

The first few terms in the above expansion can be exhibited as

\[
4a_{00} T_0(x) T_0(y) + 4a_{01} T_0(x) T_1(y) + 4a_{02} T_0(x) T_2(y) + ... \\
+ 4a_{10} T_1(x) T_0(y) + 4a_{11} T_1(x) T_1(y) + 4a_{12} T_1(x) T_2(y) + ... \\
+ 4a_{20} T_2(x) T_0(y) + 4a_{21} T_2(x) T_1(y) + 4a_{22} T_2(x) T_2(y) + ... \\
+ ...
\]

Writing

\[
b_i = \sum_{j=0}^{\infty} a_{ij} T_j(y)
\]

\[
c_j = \sum_{i=0}^{\infty} a_{ij} T_i(x)
\]

then (2.81) can be interpreted in the form

\[
f(x, y) = \sum_{i=0}^{\infty} b_i T_i(x) \tag{2.82a}
\]

or \( f(x, y) = \sum_{j=0}^{\infty} c_j T_j(y) \) \tag{2.82b}

Extensions of the results of sections 2.5, 2.6, 2.7, together with the extension of the above notation to series for the partial derivatives of \( f \), are simple, but they are left to chapter 5 where partial differential equations are discussed. However, the
significant extension of (2.72) in order to provide values of partial derivatives, is given in the next section.
2.9 The numerical value of $\frac{\partial^{m+n}}{\partial x^m \partial y^n} T_i(x) T_j(y)$, and its derivatives

Basu (1973) gives the two-dimensional analogue of (2.71). Using his notation, but slightly rearranging the formulae, the results are given below.

Thus, as in (2.82a),

let

$$S = S_{mn} (x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} T_i(x) T_j(y) = \sum_{i=0}^{m} b_i T_i(x)$$

where

$$b_i = \sum_{j=0}^{n} a_{ij} T_j(y)$$

Let

$$g_{m+2} = g_{m+1} = 0$$

and

$$d_{i,n+2} = d_{i,n+1} = 0$$

$$d_{ij} = a_{ij} + 2y \; d_{i,j+1} - d_{i,j+2}, \; j = n, \ldots, 0$$

$$b_i = 4 [d_{i,0} - d_{i,2}]$$

Then

$$S = S_{mn} (x,y) = 4 [g_0 - g_2]$$

Alternatively, as in (2.82b)

let

$$S = S_{mn} = \sum_{j=0}^{n} c_j T_j(y)$$

where

$$c_j = \sum_{i=0}^{m} a_{ij} T_i(x)$$

Let

$$g_{n+2} = g_{n+1} = 0$$

and

$$d_{m+2,j} = d_{m+1,j} = 0$$

$$d_{ij} = a_{ij} + 2x \; d_{i+1,j} - d_{i+2,j}, \; i = m, \ldots, 0$$

$$c_j = 4 [d_{0,j} - d_{2,j}]$$

$$g_j = c_j + 2y \; g_{j+1} - g_{j+2}$$

(2.91b)
Then \[ S_{mn}(x,y) = \frac{\partial}{\partial x}\left[ g_o - g_2 \right] \]

In calculating \( S \), either of (2.91a) or (2.91b) is satisfactory. In the formulae below, the results are extended in order to evaluate derivatives, with the work based on (2.91a).

Write
\[
\begin{align*}
q_{p,q}^i &= \frac{1}{p!q!} \frac{\partial^{p+q} g_i}{\partial x^p \partial y^q} \\
q_{p,q}^i &= \frac{1}{p!q!} \frac{\partial^{p+q} g_i}{\partial x^p \partial y^q} \\
d_{p,q}^{i,j} &= \frac{1}{p!q!} \frac{\partial^{p+q} g_{i,j}}{\partial x^p \partial y^q}
\end{align*}
\]

Then differentiation of (2.91a) leads formally to the set of equations
\[
\begin{align*}
g_{m+2}^{p,q} &= g_{m+1}^{p,q} = 0 \\
d_{i,n+2}^{p,q} &= d_{i,n+1}^{p,q} = 0 \\
d_{i,j}^{p,q} &= 2d_{i,j+1}^{p,q} + 2y d_{i,j+1}^{p,q} - d_{i,j+2}^{p,q}, \quad j = n, \ldots, 0 \\
b_i^{p,q} &= \frac{1}{i!} \left[ d_{i,0}^{p,q} - d_{i,2}^{p,q} \right] \\
g_i^{p,q} &= b_i^{p,q} + g_{i+1}^{p,q} + 2x g_{i+1}^{p,q} - g_{i+2}^{p,q} \\
s^{p,q} &= \frac{1}{p!q!} \frac{\partial^{p+q} f(x,y)}{\partial x^p \partial y^q}
\end{align*}
\]

where \[ s^{p,q} = \frac{1}{p!q!} \frac{\partial^{p+q} f(x,y)}{\partial x^p \partial y^q} \]

However, \( \{d_{ij}\} \) and \( \{b_i\} \) are independent of \( x \), and hence, for \( p > 0 \), the above become
\[
\begin{align*}
q_{m+2}^{p,q} &= q_{m+1}^{p,q} = 0 \\
g_i^{p,q} &= g_{i+1}^{p-1,q} + 2x g_{i+1}^{p,q} - g_{i+2}^{p,q}, \quad i = m, \ldots, 0 \\
s^{p,q} &= \frac{1}{p!q!} \frac{\partial^{p+q} f(x,y)}{\partial x^p \partial y^q}
\end{align*}
\]

leading to all derivatives (including mixed derivatives) involving differentiation with respect to \( x \).
When \( p = 0, q > 0 \), the derivatives with respect to \( y \) only, can be obtained through the general scheme above, with \( g_{i+1}^{-1,q} \) being interpreted as zero. However, if a number of these derivatives is wanted, it is more efficient to start with (2.91b) and obtain results analogous to (2.92), and hence analogous to (2.93), for the case \( q = 0 \).

The results are used later for the cases \( p, q = 0,1,2 \), in connection with the solution of second order partial differential equations.

Although the above results are merely a simple extension of those in section (2.7), the candidate has not seen them stated explicitly in the literature.
CHAPTER 3
ORDINARY SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

3.1 Preliminary Remarks

The solution of the second order linear differential equation,

\[ a(x) \frac{d^2y}{dx^2} + b(x) \frac{dy}{dx} + c(x)y = g(x) , \quad -1 \leq x \leq 1 , \quad (3.11) \]

using series expressed in terms of Chebyshev polynomials, is now considered.

The coefficients \( a(x) \), \( b(x) \), \( c(x) \) are polynomials in \( x \), and for the present discussion they are restricted to be quadratic, as, in particular, this includes many of the differential equations related to the special functions of Mathematical Physics, such as Bessel functions, Airy functions, the classical orthogonal polynomials, amongst many others. The extension to the cases where \( a(x) \) is quartic and \( b(x) \) is cubic, is mentioned briefly. \( g(x) \) on the right hand side of the equation is any function which has a convergent expansion in terms of Chebyshev polynomials. A linear change of variable can be used to transform a problem on an arbitrary finite interval \( a \leq x \leq b \) to the standard interval \(-1 \leq x \leq 1 \). (See section 3.7).

Recently Morris (1973) and Morris and Horner (1977) have investigated the solution of a restricted class of fourth order homogeneous linear ordinary differential equations using Chebyshev series, and this chapter applies similar ideas, in more detail, to second order linear equations, both homogeneous and non-homogeneous. The present method of solution is by backward recurrence, using sufficient independent solutions for their linear combination to satisfy given initial/boundary conditions, together with "extra" equations arising from the form of the recurrence relations. The general method is explained in Chapter 5 of...
The recurrence method of solving differential equations in Chebyshev series is often avoided because of the tedium of establishing the recurrence relation associated with a given equation. One of the main objects here is to present Table 3.11, from which it is a simple matter to obtain a required recurrence relation by inspection. The method of solution is thus standardised and an ad-hoc approach to each new problem is avoided.

A second major point is the use of the evaluation scheme (2.73), to represent the initial/boundary conditions, rather than the expression of these directly in terms of the coefficients in a Chebyshev expansion of the solution.

Finally it is shown how a shooting technique can be used to solve eigenvalue problems (both linear and non-linear in the eigenparameter) when the differential equation is solved using Chebyshev polynomials.

Computer programmes have been written in association with the theory, so that any problem of the form (3.11) can be solved by the presentation of a simple set of data.
3.2 Solutions in Series

All of the present work is based on examining those differential equations that can be solved in uniformly convergent series. It will be shown how sums of Chebyshev polynomials can be used to approximate such series solutions. Conversely, if an approximation to some function is wanted, in terms of Chebyshev polynomials, then it will be sought as the solution to a differential equation, e.g. \( \cos x \) as the solution of \( y'' + y = 0, \ y(0) = 1, \ y'(0) = 0. \)

The homogeneous equation corresponding to (3.11) if often solved by the method of Frobenius, by assuming a series solution in the form,

\[
y(x) = (x-x_0)^r \sum_{j=0}^{\infty} a_j (x-x_0)^j.
\]

(3.21)

This method of solution is valid if \( x_0 \) is an ordinary point or a regular singular point of the homogeneous differential equation. \( x_0 \) is called an ordinary point if \( b(x)/a(x) \) and \( c(x)/a(x) \) are both analytic at \( x_0 \). If either of \( b(x)/a(x) \) or \( c(x)/a(x) \) is not analytic at \( x_0 \), then \( x_0 \) is a singular point. It is a regular singular point if it is a singular point but \( b(x)/a(x) \) and \( c(x)/a(x) \), respectively, possess poles of order 1 and 2, at most, at \( x_0 \). At a regular singular point it is thus possible to write,

\[
b(x)/a(x) = (x-x_0)^{-1} \sum_{j=0}^{\infty} b_j (x-x_0)^j \\
c(x)/a(x) = (x-x_0)^{-2} \sum_{j=0}^{\infty} c_j (x-x_0)^j
\]

(3.22)

In the method of Frobenius the expression in (3.21) for \( y(x) \) is substituted into the homogeneous differential equation (3.11), to give \( r \), and \( a_0, a_1, a_2, \ldots \), after equating coefficients of like powers of \( x-x_0 \). When \( x_0 \) is an ordinary point of the equation there is a solution as a Taylor's series, i.e. \( r = 0 \), and when \( x_0 \) is a regular singular
the more general form of solution (3.21), with \( r \) not necessarily zero, can be found. \( a_0, a_1, a_2, \ldots \) are related by recurrence relations, and values of \( r \) are given as the roots of a quadratic equation,

\[
\begin{align*}
 r^2 + (b_0-1)r + c_0 &= 0 , \\
\end{align*}
\]

called the indicial equation. Difficulties can be encountered for various properties of the roots of the indicial equation, in particular if the roots differ by an integer (including zero), and it is sometimes not possible to obtain two linearly independent solutions of (3.11) in the form (3.21). However there will always be at least one solution of the form (3.21).

Although the method of Frobenius is often used for the analytic solution of (3.11), in many cases the series converges too slowly for the method to be practical numerically, and indeed, related to a slow decrease in the magnitude of the coefficients \( a_0, a_1, a_2, \ldots \), their calculation from a recurrence relation can become numerically unstable.

To a great extent these difficulties may be overcome by replacing the series \( \sum_{j=0}^{\infty} \alpha_j (x-x_0)^j \) by a series \( \sum_{j=0}^{\infty} a_j T_j\left(\frac{x-x_0}{b}\right) \), where \( |x-x_0| \leq b \), in terms of Chebyshev polynomials, as this latter series often converges very quickly, with only a few terms necessary for accurate results.

In the following it is assumed that a solution of (3.11) can be written in the form (3.21). As a preliminary step in the solution a change of dependent variable is made so that \( y \) is replaced by \( (x-x_0)^{-r}y \), with the resultant equation having a solution expressible as a Taylor's series in \( (x-x_0) \). This equation is then actually solved by expressing the solution as a sum of Chebyshev polynomials.

Unless otherwise stated \( x_0 \) will be taken as zero, and \( b \) as 1 and the equation (3.11) will be solved on the interval \(-1 \leq x \leq 1\).
3.3 Method of Solution

Theory of the Method

In (3.11), write

\[
\begin{align*}
    a(x) &= c_0 + c_1 x + c_2 x^2 \\
    b(x) &= c_4 + c_5 x + c_6 x^2 \\
    c(x) &= c_7 + c_8 x + c_9 x^2.
\end{align*}
\]

Thus (3.11) becomes after substitution

\[
\begin{align*}
    (c_0 + c_1 x + c_2 x^2) \frac{d^2 y}{dx^2} + (c_4 + c_5 x + c_6 x^2) \frac{dy}{dx} + (c_7 + c_8 x + c_9 x^2) y &= g(x), \\
    -1 \leq x \leq 1. & \quad (3.31)
\end{align*}
\]

To solve this equation let the solution be

\[
y(x) = \sum_{k=0}^{\infty} a_k T_k(x) \\
    = a_0 + a_1 T_1(x) + a_2 T_2(x) + \ldots & \quad (3.32)
\]

Also let,

\[
g(x) = \sum_{k=0}^{\infty} g_k T_k(x). & \quad (3.33)
\]

The derivatives \( y^{(r)}(x) \), \( r = 0,1,2 \), are similarly given by,

\[
y^{(r)}(x) = \sum_{k=0}^{\infty} a_k^{(r)} T_k(x) & \quad (3.34)
\]

following the conventions of Chapter 2.

The Recurrence Relation

The general method for solving (3.31) is then to substitute (3.33) and (3.34) into (3.31), to obtain, after equating coefficients of \( T_k(x) \), followed by repeated and tedious use of (2.16)' and (2.53), the result

\[
\begin{align*}
    \sum_{i=k-m}^{k+m} \left\{ \sum_{j=1}^{9} c_{ij} w_i^{(k)} \right\} a_i &= \sum_{i=k-m}^{k+m} w_i^{(k)} g_i \\
    k &= 0,1,2,\ldots & \quad (3.35)
\end{align*}
\]

or

\[
\sum_{k=0}^{\infty} W^{(k)} T_k(x) = \sum_{k=0}^{\infty} W^{(k)} g_k & \quad (3.36)
\]

where \( W^{(k)} = (w_{ij}^{(k)}) \) is the 9x9 matrix in Table 3.11,
Table 3.11 The Matrix $W^{(k)}$

<table>
<thead>
<tr>
<th></th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_5$</th>
<th>$c_6$</th>
<th>$c_7$</th>
<th>$c_8$</th>
<th>$c_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{k-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$k+1$</td>
</tr>
<tr>
<td>$a_{k-3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$2(k-3)(k+1)$</td>
<td>$2(k+1)$</td>
</tr>
<tr>
<td>$a_{k-2}$</td>
<td></td>
<td></td>
<td></td>
<td>$4(k+1)(k-2)(k-3)$</td>
<td>$4(k-2)(k+1)$</td>
<td>$4(k+1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{k-1}$</td>
<td></td>
<td></td>
<td></td>
<td>$8(k-1)(k+1)(k-2)$</td>
<td>$8(k-1)(k+1)$</td>
<td>$2(k+3)(k-1)$</td>
<td></td>
<td>$-2(k-1)$</td>
<td></td>
</tr>
<tr>
<td>$a_k$</td>
<td>$16k(k-1)(k+1)$</td>
<td>$8k(k^2-3)$</td>
<td>$8k$</td>
<td>$-8k$</td>
<td>$-2k$</td>
<td>$-2(k+1)$</td>
<td>$2(k-3)(k+1)$</td>
<td>$-2(k+1)$</td>
<td></td>
</tr>
<tr>
<td>$a_{k+1}$</td>
<td></td>
<td></td>
<td>$-8(k+1)(k-1)$</td>
<td>$-2(k-3)(k+1)$</td>
<td>$-2(k+1)$</td>
<td>$2(k-3)(k-1)$</td>
<td>$2(k+1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{k+2}$</td>
<td></td>
<td></td>
<td>$4(k-1)(k+2)(k+3)$</td>
<td>$-4(k+2)(k-1)$</td>
<td>$4(k-1)$</td>
<td>$-2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{k+3}$</td>
<td></td>
<td></td>
<td></td>
<td>$-2(k+3)(k-1)$</td>
<td>$2(k-1)$</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$a_{k+4}$</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>$k-1$</td>
</tr>
<tr>
<td>$y''$</td>
<td></td>
<td>$xy''$</td>
<td>$x^2y''$</td>
<td></td>
<td>$y'$</td>
<td>$xy'$</td>
<td>$x^2y'$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>$y$</td>
<td>$xy$</td>
</tr>
</tbody>
</table>
\( w_j^{(k)} \) is the jth column of \( W^{(k)} \)

\[
\mathbf{z}^T = [c_1, c_2, \ldots, c_9]
\]

\[
\mathbf{a}^{(k)}^T = [a_{k-m}, \ldots,a_k, \ldots,a_{k+m}]
\]

\[
\mathbf{g}^{(k)}^T = [g_{k-m}, \ldots,g_k, \ldots,g_{k+m}]
\]

The maximum value of \( m \) is 4, but is related to the length of the vectors \( \mathbf{a}^{(k)} \) and \( \mathbf{z}^{(k)} \), each of which has \( 2m+1 \) components. Often in practice, \( m \) can be taken to be less than 4, because some of the multipliers \( \{w_{ij}^{(k)}\} \) and \( \{c_i\} \) are zero. In the following it will be understood that \( m \) is taken as small as possible, e.g. in examples 4.1, 4.3, 4.4, 4.6, \( m = 3,2,3,3 \) respectively.

Once the general form of the recurrence relation is known, a suitable truncation point in the series (3.32), is chosen, (say at \( a_n \)), so that the series solution is sufficiently well represented by the resulting polynomial. Then the appropriate equations from (3.35), together with equations representing the initial-boundary conditions, are solved for \( a_0, a_1, \ldots, a_n \). (See Clenshaw (1960, 1962), Fox and Parker (1972)). This is often done by solving an explicit set of algebraic equations using standard methods such as Gaussian elimination, or iterative methods such as successive over-relaxation.

However the method adopted here is to back-substitute through the recurrence relations after assuming starting values for \( a_n, a_{n+1}, \ldots \) the final step being to "normalise" the coefficients using the initial/boundary conditions and any "extra" conditions from the recurrence relations. (See the above references, and Morris and Horner (1977)).

Thus let \( a_i = 0 \), \( i > n \)

\[
a_n = 1, \text{ corresponding to } k = n + m. \tag{3.36}
\]

Then \( a_i \), for \( i=n-1, \ldots,1,0 \) can be found from (3.35), where \( k=n+m-1, \ldots,m+1,m \), respectively.
Extra Conditions

In the application of (3.36) those recurrence relations (3.35) corresponding to \( k = m-1, \ldots, 0 \), have not been used. Now the maximum possible value of \( m \) is 4, so that only the values \( k = 3, 2, 1, 0 \) need to be considered. In general \( k = 0, 1 \) lead to trivial results and \( k = 2, 3 \) are the only significant ones. These are stated explicitly as equations (3.37), (3.38).

\[
(12c_7 + 2c_9)a_0 + (24c_4 + 4c_6 + 4c_8)a_1 + (96c_1 + 16c_3 + 16c_5 - 16c_7 - c_9)a_2
\]
\[
+ (96c_2 - 24c_4 + 6c_6 - 6c_8)a_3 + (80c_3 - 16c_5 + 4c_7 - 2c_9)a_4
\]
\[
+ (-10c_6 + 2c_8)a_5 + c_9a_6 - (12c_0 - 16c_2 + 4c_4) = 0
\]

(3.37)

\[
4c_8a_0 + (8c_5 + 8c_7 + 6c_9)a_1 + (32c_2 + 32c_4 + 12c_6 - 2c_8)a_2
\]
\[
+ (192c_1 + 72c_3 + 12c_5 - 12c_7 - 3c_9)a_3 + (160c_2 - 32c_4 - 4c_8)a_4
\]
\[
+ (120c_3 - 20c_5 + 4c_7 - c_9)a_5 + (-12c_6 + 2c_8)a_6
\]
\[
+ c_9a_7 - (8c_1 - 12c_3 + 4c_5) = 0
\]

(3.38)

Thus when \( m = 4 \) there are in general both extra conditions (3.37), (3.38) corresponding to \( k = 2, 3 \). When \( m = 3 \) there is one extra condition (3.37) corresponding to \( k = 2 \). When \( m < 3 \) there are no extra conditions.

For homogeneous differential equations such as in examples 4.1, 4.3 Chapter 4, i.e. equations in which either \( c_1 = c_3 = c_5 = c_7 = c_9 = 0 \) or \( c_2 = c_4 = c_6 = c_8 = 0 \), there is the possibility of solutions which are purely odd or even. This is obvious from the differential equation and also from the corresponding recurrence relation in which the subscripts change in steps of 2. If such purely odd or even solutions are wanted then the number of extra conditions is reduced by 1, making the case where \( m = 4 \) the only one in which there remains an extra condition, this being chosen as the non-trivial result from (3.37) and (3.38). Thus for an even
solution with $c_1 = c_3 = c_5 = c_7 = c_9 = 0$ or an odd solution with $c_2 = c_4 = c_6 = c_8 = 0$, (3.37) is trivial, and (3.38) is used. For an odd solution with $c_1 = c_3 = c_5 = c_7 = c_9 = 0$ or an even solution with $c_2 = c_4 = c_6 = c_8 = 0$, (3.38) is trivial and (3.37) is used.

For later reference let $q$ be the number of extra conditions. $q$ will be one of $0, 1, 2$ as discussed here.
3.4 Initial/boundary Conditions

A second order differential equation is usually associated with two initial/boundary conditions. In many implementations of the Chebyshev series method the problem is written so that these conditions are associated with the points 0, ±1, for which the (truncated) infinite series results are relatively simple. In order to apply initial/boundary conditions at an arbitrary point, the present method uses numerical values of \( y(x) \), \( y'(x) \) obtained from the generalised version of the evaluation scheme for Chebyshev polynomials in (2.73).

To allow for an appropriate amount of generality the two initial/boundary conditions in the associated computer programme allow linear combinations of function and first derivative values at three points \( x_1, x_2, x_3 \) in the form,

\[
\frac{1}{3} \sum_{i=1}^{3} \{a_{i\ell}y(x_i) + \beta_{i\ell}y'(x_i)\} = \gamma_\ell, \quad \ell = 1, 2 \tag{3.41}
\]

For purely odd or even solutions of homogeneous (3.31) (i.e. \( g(x) = 0 \)), a single initial/boundary condition can be taken. To allow for the various possibilities, let \( p \) equal the number of initial/boundary conditions, and \( p \) will usually be 2, but may be 1 for a purely odd or even solution.

In general a set \( \{a_0, a_1, \ldots, a_n\} \) found from (3.35), (3.36) will not be directly consistent with the \( q \) extra conditions and the \( p \) initial/boundary conditions. In fact, a linear combination of these types of coefficients must be taken. It is convenient to treat separately each of the homogeneous and the non-homogeneous cases.
3.5 The Homogeneous Equation

Let the general solution of homogeneous (3.31) \( g(x) = 0 \), be written,

\[
y(x) = \sum_{j=1}^{q+p} \theta_j y_j(x)
\]

\((3.51)\)

where

\[
y_j(x) = \sum_{k=0}^{n_j} a_k^j T_k(x)
\]

or

\[
a_k^j = \sum_{j=1}^{q+p} \theta_j a_k^j , \quad k = 0,1,\ldots
\]

\((3.52)\)

Now, if each set \( \{a_0^j, a_1^j, \ldots, a_n^j\} \), \( j = 1,\ldots,q+p \), is an independent set satisfying (3.35), (3.36), except that \( a_k^j \) now replaces \( a_k \), \( k = 0,1,2,\ldots, \), then \( \{a_0, a_1, \ldots, a_n\} \), as defined in (3.52) also satisfy (3.35), (3.36). Then \( \theta_1, \ldots, \theta_{q+p} \) can be found so that the special conditions are also satisfied.

Thus from (3.41)

\[
\sum_{j=1}^{q+p} \left[ \sum_{i=1}^{3} \{\alpha_{i,j} y_j(x_i) + \beta_{i,j} y_j'(x_i)\} \right] \theta_j = \gamma_\ell , \ell = 1,p
\]

\((3.53)\)

and from (3.37), (3.38)

\[
\sum_{j=1}^{q+p} F_{\ell,j} \theta_j = 0 , \quad \ell = 1,q
\]

\((3.54)\)

where \( F_{\ell,j} \) represents the left-hand side(s) of the appropriate equation(s) (3.37), (3.38), with \( a_k \) replaced by \( a_k^j \).

(3.53), (3.54) can then be solved for \( \theta_1, \ldots, \theta_{q+p} \).

Now the maximum value of each of \( p \) and \( q \) is 2, and hence the maximum number of linear simultaneous equations to be solved is 4.
3.6 The Non-homogeneous Equation

When \( g(x) \neq 0 \), the set of equations (3.35) and (3.37), (3.38) are not homogeneous and the general solution must be written in the form,

\[
y(x) = y_0(x) + \sum_{j=1}^{q+p} \theta_j y_j(x)
\]

with \( y_j(x) \) defined as in (3.51).

The sets \( \{a_{o,j}, a_{1,j}, \ldots, a_{n,j}\} \), \( j = 1, \ldots, q+p \), are found from the same homogeneous equations (3.35) as above with \( g_k = 0 \) but the "particular solution" \( \{a_{o,0}, a_{1,0}, \ldots, a_{n,0}\} \) is found from the corresponding non-homogeneous equations (with \( g_k \neq 0 \)) with the backward recurrence of (3.36) being modified so that \( a_{n} = a_{n,0} = 0 \).

The equations (3.53), (3.54) become,

\[
\sum_{j=1}^{q+p} \left[ \sum_{i=1}^{3} \{a_{i,l} y_j(x_i) + \beta_{i,l} y'_j(x_i)\} \right] \theta_j = \gamma_l - \sum_{i=1}^{3} \{\alpha_{i,l} y_0(x_i) + \beta_{i,l} y'_0(x_i)\},
\]

\( \ell = 1, p \) \hfill (3.62)

and

\[
\sum_{j=1}^{q+p} F_{\ell,j} \theta_j = -F_{\ell,0}, \quad \ell = 1, q \hfill (3.63)
\]
3.7 Change of Interval

For a problem of the form,

\[ (d_1 + d_2 t + d_3 t^2) \frac{d^2 y}{dt^2} + (d_4 + d_5 t + d_6 t^2) \frac{dy}{dt} + (d_7 + d_8 t + d_9 t^2)y = G(t) , \]

\[ a \leq t \leq b \quad (3.71) \]

with initial/boundary conditions,

\[ \sum_{i=1}^{3} \{ \alpha'_{i'} y(t_i) + \beta'_{i'} y'(t_i) \} = y_{i'} , \quad i = 1, \ldots, p \quad (3.72) \]

the linear change of variable,

\[ t = \frac{1}{2}(b-a)x + \frac{1}{2}(b+a) \quad (3.73) \]

moves the problem onto the standard interval \(-1 \leq x \leq 1\), in the form (3.31), where

\[ c_1 = \frac{4d_1 + 2d_2 (b+a) + d_3 (b+a)^2}{(b-a)^2} \]
\[ c_2 = 2(d_2 + d_3 (b+a)) / (b-a) \]
\[ c_3 = d_3 \]
\[ c_4 = \frac{4d_4 + 2d_5 (b+a) + d_6 (b+a)^2}{(b-a)} \]
\[ c_5 = (d_5 + d_6 (b+a)) \]
\[ c_6 = \frac{1}{2}(b-a)d_6 \]
\[ c_7 = \frac{4d_7 + 2d_8 (b+a) + d_9 (b+a)^2}{(b-a)} \]
\[ c_8 = \frac{1}{2}(b-a)(d_8 + d_9 (b+a)) \]
\[ c_9 = \frac{1}{2}(b-a)^2d_9 . \]

The same change of variable (3.73) is of course applied to \( G(t) \) to obtain a corresponding function \( g(x) \) on \(-1 \leq x \leq 1\).

The conditions (3.72) become (3.41) by writing

\[ \alpha_{i'l'} = \alpha'_{i'l'} \quad i = 1, 2, 3 \]
\[ \beta_{i'l'} = \beta'_{i'l'}/\frac{1}{2}(b-a) \quad \ell = 1, \ldots, p \quad (3.75) \]

and particular care must be taken to evaluate \( y(x_i) \), \( y'(x_i) \) at \( x_i \) given by

\[ x_i = \frac{1}{2}(2t_i - (b+a))/(b-a) , \quad i = 1, 2, 3 \]

in (3.41) and when implementing (3.53).
3.8 The Eigenvalue Problem. A Shooting Method

Consider the problem of finding non-trivial \( \lambda \) (eigenvalues) such that,

\[
\begin{align*}
\frac{d^2 y}{dx^2} + a(x, \lambda) \frac{dy}{dx} + b(x, \lambda) y + c(x, \lambda) &= 0 \\
\alpha_1 y(x_1, \lambda) + \beta_1 y'(x_1, \lambda) &= 0 \\
\alpha_2 y(x_2, \lambda) + \beta_2 y'(x_2, \lambda) &= 0
\end{align*}
\tag{3.81}
\]

where

\[
\begin{align*}
a(x, \lambda) &= c_1 + c_2 x + c_3 x^2 \\
b(x, \lambda) &= c_4 + c_5 x + c_6 x^2 \\
c(x, \lambda) &= c_7 + c_8 x + c_9 x^2
\end{align*}
\]

each \( c_i \) being a function of the parameter \( \lambda \).

\( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are constants, or explicit functions of \( \lambda \).

The solution of the differential equation for some value of \( \lambda \) with initial conditions \( \alpha_1 y(x_1, \lambda) + \beta_1 y'(x_1, \lambda) = 0 \) and one of \( y(x_1, \lambda) \), \( y'(x_1, \lambda) \) not equal to zero (consistent with the data of a particular example) is accomplished using the method for solving initial value problems.

However, the resulting solution will almost certainly not satisfy the condition,

\[
h(\lambda) \equiv \alpha_2 y(x_2, \lambda) + \beta_2 y'(x_2, \lambda) = 0 .
\]

But, although \( h(\lambda) \neq 0 \) for arbitrary choice of \( \lambda \), the equation \( h(\lambda) = 0 \) can be solved for \( \lambda \) using, for example, the iterative scheme,

\[
\lambda_{s+2} = \lambda_{s+1} - \frac{(\lambda_{s+1} - \lambda_s)}{h(\lambda_{s+1}) - h(\lambda_s)} h(\lambda_{s+1}), \quad s = 0, 1, 2, \ldots \tag{3.83}
\]

Thus repeated solving of the initial value problem for the point \( x_1 \), with values of \( \lambda_s (= \lambda) \) given by (3.83) will lead to the required eigenvalue.
As with all shooting methods, the eigenvalues are only found one at a time, and starting values are wanted in each case. If the parameter $\lambda$ occurs linearly in the coefficients $c_i$, then a method which uses Table 3.11 to establish a standard algebraic eigenvalue problem may be preferred. However, for non-linear occurrence of $\lambda$ a shooting method may be the only possible method.

3.9 Extensions

The form of $a(x)$ can be extended to include cubic and quartic terms, as it is fairly obvious from Table 1 that the recurrence relation (3.35) would still only involve terms from $a_{k-4}$ to $a_{k+4}$. In the same way, $b(x)$ could be cubic. Of course all of the polynomials $a(x)$, $b(x)$, $c(x)$ could be chosen of higher degree if we were prepared to work with larger vectors $\mathbf{a}$ and $\mathbf{g}$, and a consequently "wider" general recurrence relation involving $a_{k-m}$ to $a_{k+m}$, $m > 4$. In fact the recurrence relation will involve the terms from $a_{k-m}$ to $a_{k+m}$, if $a(x)$ is of degree $m$, $b(x)$ is of degree $m - 1$, and $c(x)$ is of degree $m - 2$, at most. Particular cases of such extensions have been considered, but are not reported here.
An appreciation of the results of Chapter 3 can be obtained by considering some examples, as on the following pages.

In all examples, the evaluation formulae (2.73) were used to evaluate \( y(x), y'(x), y''(x), \) and \( g(x), \) and hence the values of an "error measure" equal to

\[
(c_1 + c_2 x + c_3 x^2)y'' + (c_4 + c_5 x + c_6 x^2)y' +
(c_7 + c_8 x + c_9 x^2)y - g(x)
\]

for 21 values of \( x, \) evenly spaced on the solution interval \( a \leq x \leq b. \) No automatic check was built into the computer programme but the calculated values of the "error" were printed, and inspected for their deviation from 0. Most of the numerical values of \( y(x) \) were of order 1, and in practice the "error" was often "exactly" zero, or of the order \( 10^{-8} \) to \( 10^{-10}, \) so that this test indicated a satisfactory solution of the differential equation.

In this chapter, \( N \) is written instead of \( n \) as used in Chapter 3, with regard to the starting term in the backward recurrence method.

4.1 Bessel Equation

Consider the Bessel equation of order \( n \)

\[
x^2 y'' + xy' + (x^2 - n^2)y = 0.
\]

If the equation is solved in a series (3.21), the indicial equation (3.23) is

\[
x^2 - n^2 = 0.
\]

Thus writing \( y = x^n z, \) but finally re-writing \( y \) for \( z \) in the resulting differential equation, the original equation has solution \( x^n y \) where

\[
xy'' + (2n+1)y' + xy = 0
\]

and where at least one solution can be written in a Taylor's series in \( x, \)
and hence approximated by a sum of (Chebyshev) polynomials in $x$.

Comparing with the equations of Chapter 3, it is seen that

$$c_1 = c_3 = c_5 = c_6 = c_7 = c_9 = 0, \quad c_2 = c_8 = 1, \quad c_4 = 2n + 1.$$  

Thus (3.35) becomes

$$\sum_{i=k-3}^{k+3} \left\{ w_{i2}^{(k)} + (2n+1)w_{i4}^{(k)} + w_{i8}^{(k)} \right\} a_i = 0$$

giving (3.36) as,

$$a_i = 0, \quad i > N$$

$$a_N = 1$$

$$(k+1)a_{k-3} + (k-1)(4(k+1)(k+2n-1) - 1)a_{k-1}$$

$$+ (k+1)(4(k-1)(k-2n+1) - 1)a_{k+1} + (k-1)a_{k+3} = 0$$

$$k = N+3, \ldots, 3.$$  \hspace{1cm} (4.11)

Since $m = 3$, then in general, $q = 1$, $p = 2$.

For an even solution, $q = 0$, $p = 1$ and if the single initial condition is $y(0) = 1/2^F(n+1)$ the equation for the normalising factor $0_1$ becomes,

$$y(0)0_1 = 1/2^F(n+1)$$  \hspace{1cm} (4.12)

where the coefficients $a_{0,1}, a_{1,1}, \ldots, a_{N,1}$ of $y_1(x)$ (cf. (3.51) and (3.52)) are obtained from (4.11) with $N$ an even (positive) integer, and where $y_1(0)$ is calculated using (2.73).

With the above conditions, the solution of the original problem is the Bessel function $J_n(x)$.

The coefficients in the Chebyshev expansions of $x^{-n}J_n(x)$, for $n = 0, \pm \frac{1}{3}, \pm \frac{2}{3}, 1, \quad -1 \leq x \leq 1$, have been calculated in this manner and are shown in Table 4.11.
TABLE 4.11

Coefficients \{a_{2k}\}, \ k = 0,1,\ldots,10, \ in \ Chebyshev \ expansions \ of

\[ x^{-n_j}(x) = \sum_{k=0}^{10} a_{2k}T_{2k}(x), \quad n = 0, \pm \frac{1}{3}, \pm \frac{2}{3}, 1, \ -1 \leq x \leq 1. \]

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4.2 Bessel functions on $-8 \leq x \leq 8$

Approximations to various Bessel functions were calculated on the interval $-8 \leq x \leq 8$, to obtain $J_n(x)$ as given by Clenshaw and Picken (1966). The formulae of section 3.7 for change of interval were used to yield the equation,

$$xy'' + (2n+1)y' + 64xy = 0,$$

together with the initial condition,

$$y(0) = 1/2^n \Gamma(n+1).$$

The Bessel functions $J_n(x)$ were then given by

$$J_n(x) = x^n \frac{r^n}{\Gamma(n+1)}.$$

Solutions for $n = 0, \pm \frac{1}{3}, \pm \frac{1}{2}, \pm \frac{2}{3}, 1$, amongst others, were examined. In all cases the results agreed with those in Clenshaw and Picken.

4.3 The equation $y'' + \lambda y = 0$

For the equation

$$y'' + \lambda y = 0 \quad -1 \leq x \leq 1$$

the recurrence relation is

$$\lambda(k+1) a_{k-2} + 2k(2k-1)(k+1) - \lambda) a_k + \lambda(k-1)a_{k+2} = 0.$$

With $\lambda = -1$, the functions $ae^x + be^{-x}$ can be found. With the following initial conditions, the nominated functions were calculated,

$$y(0) = y'(0) = 1 : e^x$$
$$y(0) = -y'(0) = 1 : e^{-x}$$
$$y(0) = 1, y'(0) = 0 : \cosh x$$
$$y(0) = 0, y'(0) = 1 : \sinh x.$$

The coefficients in the expansion of $e^x$ are given in Table 4.31. The coefficients in the other expansions are found from these as follows:
\( e^{-x} \): change the signs of the "odd" coefficients
\( \cosh x \): omit the "odd" coefficients
\( \sinh x \): omit the "even" coefficients.

With \( \lambda = \alpha^2 \), the expansion for the even function \( \cos \alpha x \) can be found with initial condition as \( y(0) = 1 \). Similarly the expansion for the odd function \( \sin \alpha x \) can be found with initial condition \( y'(0) = \alpha \).

With \( \alpha = \frac{\pi}{2}, \pi \) the expansions of Clenshaw and Picken have been reproduced, and many other cases have been tabulated.

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4.4 The equation $y'' + \lambda xy = 0$. The Airy Function.

For the equation,

$$y'' + \lambda xy = 0, \quad -1 \leq x \leq 1$$

the recurrence relation is

$$\lambda (k+1)a_{k-3} - \lambda (k-1)a_{k-1} + 8k(k-1)(k+1)a_k$$

$$- \lambda (k+1)a_{k+1} + \lambda (k-1)a_{k+3} = 0.$$

Here there are $2 (= p)$ initial/boundary conditions, and $1 (= q)$ extra condition corresponding to $k = 2$. Thus there are 3 simultaneous equations (3.53), (3.54) to be solved to find $\theta_1, \theta_2, \theta_3$ in the solution (3.51).

With $\lambda = -1$, $y(0) = 3^{-2/3} \Gamma(2/3)$, $y'(0) = -3^{-1/3} \Gamma(1/3)$, the Airy function $Ai(x)$ is calculated, $-1 \leq x \leq 1$, in agreement with values in Abramowitz and Stegun (1965). The results are given in Table 4.41.

Also in Table 4.41 are similar results corresponding to $Ai(x)$, $0 \leq x \leq 8$. These were obtained, following a change of variable, by solving the equation,

$$y'' - 64y' - 64xy = 0, \quad -1 \leq x \leq 1$$

$$y(0) = 3^{-2/3} \Gamma(2/3), \quad y'(0) = -4 \times 3^{-1/3} \times \Gamma(1/3)$$

and then re-writing in terms of the original variable.
TABLE 4.41 - Airy Function

(a) Coefficients \( \{a_k\}, \ k = 0,1,...,20 \) in

Chebyshev expansion, \( \text{Ai}(x) = \sum_{k=0}^{20} a_k T_k(x), \ -1 \leq x \leq 1 \)

(b) \( \text{Ai}(x), \ x = -1(0.1)1 \)

(c) \( \{a_k\}, \ k = 0,...,20, \ \text{Ai}(x) = \sum_{k=0}^{20} a_k T_k(kx-1), \ 0 \leq x \leq 8 \)

(d) \( \text{Ai}(x), \ x = 0(0.4)8 \)

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<th>k</th>
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<th>( x )</th>
<th>( \text{Ai}(x) )</th>
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4.5 The non-homogeneous equation

As one example of the solution of a non-homogeneous equation, the problem,
\[ y'' + y = \cos x, \quad y(0) = y'(0) = 1, \quad -1 \leq x \leq 1 \] (4.51)
was considered. The expansion corresponding to the expected solution
\[ y = \cos x + \sin x + \frac{1}{2}x \sin x \]
was found. The Chebyshev expansion for the right hand side had previously been obtained as the even solution of
\[ y'' + y = 0, \quad y(0) = 1, \quad -1 \leq x \leq 1. \]

The problem,
\[
\begin{align*}
y'' + xy' + xy &= 1 + x + x^2, \quad -1 \leq x \leq 1 \\
y(0) &= 1, \quad y'(0) + 2y(1) - y(-1) = -1,
\end{align*}
\] (4.52)
is considered by Fox and Parker (1972), page 104.

This is a further example of a non-homogeneous equation, and in this case there is again an "extra condition" to be satisfied. The second initial/boundary condition, involving the three points 0, ±1, uses the full flexibility allowed in the equation (3.41). A Chebyshev expansion of the solution, with the corresponding values for \( x = -1(0.1)1 \) is given in Table 4.51.
TABLE 4.51

(a) Coefficients \( \{a_k\} \), \( k = 0, 1, \ldots, 15 \) in

\[
\sum_{k=0}^{15} a_k T_k(x), \quad \text{where } f(x) \text{ is the}
\]

solution of the example in 4.52.

(b) \( f(x) \), \( x = -1(0.1)1 \).

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<th>( a_k )</th>
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4.6 Orthogonal polynomials

The Legendre, Chebyshev, Hermite and Laguerre polynomials, $P_n(x)$, $T_n(x)$, $H_n(x)$, $L_n(x)$ are solutions of (in order), the differential equations,

\[
(1-x^2)y'' - 2xy' + n(n+1)y = 0 ,
\]

\[
(1-x^2)y'' - xy' + n^2y = 0 ,
\]

\[
y'' - 2xy' + 2ny = 0 ,
\]

\[
xy'' + (1-x)y' + ny = 0 .
\]

The respective recurrence relations are,

\[
(k+1)\{(k-2)(k-1) - n(n+1)\}a_{k-2} - 2k\{(k+1)(k-1) - n(n+1)\}a_k
\]
\[+ (k-1)\{(k+2)(k+1) - n(n+1)\}a_{k+2} = 0 ,
\]

\[
(k+1)\{(k-2)^2 - n^2\}a_{k-2} - 2k\{k^2-n^2\}a_k + (k-1)\{(k+2)^2 - n^2\}a_{k+2} = 0 ,
\]

\[
(k+1)\{(k-2-n)\}a_{k-2} - 2k\{k^2-2-n\}a_k + (k-1)\{(k+2-n)\}a_{k+2} = 0 ,
\]

\[
(k+1)\{(k-2-n)\}a_{k-2} - 2(k-1)^2(k+1)a_{k-1} + 2(n+1)k a_k
\]
\[ - 2(k+1)^2(k-1)a_{k+1} - (k-1)\{(k+2+n)\}a_{k+2} = 0 .
\]

The polynomials $P_n(x)$, $T_n(x)$, $H_n(x)$ are odd or even as $n$ is odd or even, and this is indicated by the recurrence relations.

It is important to note, that when $k = n + 2$, in each equation, the coefficient of $a_{k-2}(=a_n)$ is zero. Although this appears disastrous for the backward substitution process, it is in fact an indication that, in these cases, $k$ cannot be arbitrarily large initially, and that the first (working backwards) non-zero coefficient should be $a_n$, i.e. that $N$ should be chosen as $n$. With this understanding, Chebyshev expansions of these polynomials are readily found.
4.7 \( \sin^{-1} x \) and \( \tan^{-1} x \)

\( \sin^{-1} x \) and \( \tan^{-1} x \) satisfy the respective differential equations,

\[
(1-x^2)y'' - xy' = 0, \quad y(0) = 0, \quad y'(0) = 1,
\]

\[
(1+x^2)y'' + 2xy' = 0, \quad y(0) = 0, \quad y'(0) = 1
\]

leading to recurrence relations,

\[
(k+1)(2k-1)a_{k-2} - 2k^3 a_k + (k-1)(k+2)^2 a_{k+2} = 0,
\]

\[
(k-2)a_{k-2} + 6k a_k + (k+2)a_{k+2} = 0.
\]

For each \( k = 2 \) leads to zero coefficient of \( a_0 \), but as both \( \sin^{-1} x \) and \( \tan^{-1} x \) are odd functions, this causes no bother in practice. The choice of the "first" non-zero coefficient as \( a_0 \) actually corresponds to another independent solution, \( y = \text{constant} \).

A change of interval from \(-1 \leq x \leq 1\), to \(-a \leq x \leq a, \quad a < 1\), for \( \sin^{-1} x \) in terms of \( x/a \), is necessary in practice, to obtain a series which converges in a reasonable number of terms. The choice, \( a = 1/\sqrt{2} \), leads to results corresponding to those of Clenshaw and Picken.

For \( \tan^{-1} x \), the results of Clenshaw and Picken are also reproduced.

4.8 \( \ln(1+x) \)

The equation,

\[
(1+x)y'' + y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad 0 \leq x \leq 1
\]

is transformed to

\[
(6+2x)y'' + 2y' = 0, \quad y(-1) = 0, \quad y'(-1) = 1/2, \quad -1 \leq x \leq 1
\]

with recurrence relation,

\[
(k-1)a_{k-1} + 6k a_k + (k+1)a_{k+1} = 0
\]

to eventually give \( \ln(1+x) \) as a sum \( \sum a_k T_k(2x-1) \) or \( \sum a_k T^*_k(x) \),

\( 0 \leq x \leq 1 \) where \( T^*_k(x) \) is the shifted Chebyshev polynomial. The results are the same as those in Clenshaw and Picken.
In this problem \( p = 2 \), \( q = 0 \) and there are two subsidiary solutions.

Corresponding to the case of zero coefficient of \( a_{k-1}(=a_0) \) when \( k = 1 \), \( a_{0,1} \) is taken as the only non-zero coefficient in subsidiary solution \( y_1(x) \), and \( a_{0,2} \) is taken as zero in subsidiary solution \( y_2(x) \). The calculation of \( \theta_1, \theta_2 \) using the initial conditions then gives the required solution \( y(x) = \theta_1 y_1(x) + \theta_2 y_2(x) \).

4.9 Eigenvalue Problems

Using the shooting method of section 3.8, a separate programme has been used to solve a number of eigenvalue problems. With a good original estimate, the average number of iterations to obtain 8-figure accuracy is about 5.

The following problems represent a sample of those which have been attempted. Unless otherwise indicated (in brackets) the results are correct to 8 significant figures. Eigenfunctions have also been found in each case.

(a) \( y'' + \lambda y = 0 \), \( y(-1) = y(1) = 0 \)

Theoretical : \( \lambda = (4\pi n)^2 \), \( n = 1, 2, ... \)

Calculated : 2.4674011, 9.8696044, 22.206610, 39.478418, ...

(b) \( y'' + \lambda y = 0 \), \( 3y(-1) + 4y'(-1) = 0 \), \( 4y(1) + 3y'(1) = 0 \)

Theoretical : \( \sqrt{\lambda} = t \) is a solution of \( \tan 2t = \frac{7t}{12(1+t^2)} \)

Calculated : 2.8755581, 10.390277, 22.759458, 40.044000.

(c) \( y'' + \lambda xy = 0 \), \( y(0) = y(1) = 0 \)

Theoretical : Eigenvalues are roots of \( J_{1/3} \left( \frac{2 \lambda^{1/3}}{3} \right) = 0 \)

Calculated : 18.956266(+1), 81.886583(+2), 189.22093(-1), ...

The solutions are incorrect in the last figure as indicated.
(d) $8y'' + \lambda (1+x)y = 0$, $y(-1) = y(1) = 0$

This is the same problem as in (c), shifted to (-1,1).

The same eigenvalues are obtained. In this problem, $\lambda$ occurs in more than one coefficient in the differential equation.

(e) $x(1-x)y'' + \lambda y = 0$, $y(0) = y(1) = 0$

Theoretical: (Legendre), $\lambda = n(n+1)$, $n = 1,2,3,...$

The first five values, theoretically 2, 6, 12, 20, 30, are found correctly to eight significant figures.

(f) $(1+x)^2 y'' + \lambda y = 0$, $y(0) = y(1) = 0$

Theoretical: $\lambda = \left(\frac{n\pi}{\ln 2}\right)^2 + \frac{1}{4}$, $n = 1,2,...$

When $n = 1$, $\lambda = 20.792289$.

The calculated value is 20.792310.

The theoretical eigenfunction is $(1+x)^{\frac{1}{4}} \sin \left(\frac{n\pi \ln(1+x)}{\ln 2}\right)$ with a slowly converging Chebyshev series.

With the change of variable $t = \ln(1+x)$ the problem becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \lambda y = 0$$

and the above eigenvalue is calculated exactly. The eigenfunction is now $e^{\frac{1}{4}t} \sin\left(n\pi t/\ln 2\right)$.

Thus when an eigenfunction is accurately represented by a small number of terms in a Chebyshev series, the corresponding eigenvalue is obtained very accurately. Even when the series representation is poor, the eigenvalue is obtained to fair accuracy, as this example illustrates, the answer being correct to 6 significant figures (20.7923) from a slowly converging series.
4.10 Derivative of $J_n(x)$ with respect to order

An important practical application of the idea of obtaining "the functions of mathematical physics" from the solutions of second order differential equations, has been the tabulation of $\frac{\partial J_n}{\partial n}$, the derivative of the Bessel function $J_n(x)$, with respect to the order, $n$. A need for such tables arose in work by Worthy and Clarke (1977), in ocean dynamics research. The derivation of the method is given in the following paragraphs.

When $n = 0$, it is known that $\frac{\partial J_n}{\partial n} = \frac{1}{2} \pi Y_0(x)$, and also, when $n = \frac{1}{2}$,

$$\frac{\partial J_n}{\partial n} = \sqrt{\frac{2}{\pi x}} \left\{ C_i(2x) \sin x - S_i(2x) \cos x \right\} .$$

The results obtained here are in agreement with these formulae, and also in agreement with the recurrence formula

$$\frac{\partial}{\partial n} J_{n-1} + \frac{\partial}{\partial n} J_{n+1} = \frac{2n}{x} \frac{\partial}{\partial n} J_n + \frac{2}{x} J_n$$

obtained from the standard Bessel function recurrence formula

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) .$$

Let $J_n(x) = (\frac{\partial}{\partial x})^n u_n(x)$.

Then

$$\frac{\partial}{\partial n} J_n = (\frac{\partial}{\partial x})^n \left\{ u_n(x) \ln(\frac{\partial}{\partial x}) + \frac{3u_n}{\partial n} \right\}$$

$$= J_n(x) \ln(\frac{\partial}{\partial x}) + (\frac{\partial}{\partial x})^n \frac{3u_n}{\partial n}$$

(4.10.1)

(4.10.2)

Now

$$\frac{\partial J_n}{\partial n} = J_n(x) \ln(\frac{\partial}{\partial x}) - (\frac{\partial}{\partial x})^n \sum_{k=0}^{\infty} (-1)^k \frac{\psi(n+k+1)}{\Gamma(n+k+1)} \frac{(\frac{\partial}{\partial x})^k}{k!}$$

where $\Gamma(n)$, $\psi(n)$ are the gamma, and digamma functions, respectively.

(Abramowitz and Stegun, equation 9.1.64).
\[
\frac{\partial u_n}{\partial n} = - \sum_{k=0}^{\infty} (-1)^k \frac{\psi(n+k+1)}{\Gamma(n+k+1)} \frac{(hx)^2}{k!} \quad (4.10.3)
\]

In particular, when \( x = 0 \), \( \frac{\partial u_n}{\partial n} = - \frac{\psi(n+1)}{\Gamma(n+1)} \), and \( \frac{d}{dx} \left( \frac{\partial u_n}{\partial n} \right) = 0 \).

Thus, writing \( u \equiv u_n \), \( w \equiv \frac{\partial u_n}{\partial n} \), it is known that \( u \) satisfies the differential equation
\[
xu'' + (2n+1)u' + xu = 0
\]
with initial conditions
\[
u(0) = 1/\Gamma(n+1) \quad u'(0) = 0
\]
and it is easily shown that \( w \) satisfies
\[
xw'' + (2n+1)w' + xw = -2 \frac{du}{dx}
\]
with initial conditions
\[
w(0) = -\psi(n+1)/\Gamma(n+1) \quad w'(0) = 0
\]

Now a series for \( v \equiv v_n(x) = \frac{du}{dx} \), can be found from
\[
xv'' + 2(n+1)v' + xv = -u
\]
with initial conditions
\[
v(0) = 0 \quad v'(0) = -\psi/\Gamma(n+2)
\]

Thus solving (4.10.4) for \( u_n(x) \), and putting the series solution into (4.10.6), to solve for \( v_n(x) \), and finally putting this series in (4.10.5), to obtain \( w_n(x) \), all the information is available to use (4.10.2) to obtain values of \( \partial J_n/\partial n \).

Table 4.10.1 contains a sample of results which can be obtained, and a more extensive set of results is in the appendix, in Table 4.10.2.

For the data in the above problems, use can be made of tabulated values of the gamma and digamma functions. For example, in Abramowitz and Stegun, \( \Gamma(x) \), \( \psi(x) \) are tabulated for \( x = 1(0.005)2 \), and \( x = 1(1)101 \). However in some applications, other values were wanted,
and use was made of a Chebyshev series for \( \frac{1}{\Gamma(1+x)} \), \( 0 \leq x \leq 1 \), given by Clenshaw (1962). Noting that

\[
\frac{d}{dx} \frac{1}{\Gamma(1+x)} = - \frac{\psi(1+x)}{\Gamma(1+x)}
\]

the use of (2.72), together with the recurrence relations

\[
\Gamma(x+1) = x\Gamma(x)
\]

\[
\psi(x+1) = \psi(x) + \frac{1}{x}
\]

enabled the gamma and digamma functions to be evaluated for any values.
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**TABLE 4.10.1** A selection of calculated values of $\frac{\partial}{\partial n} J_n(x)$.

A further set of results is in the Appendix.
The important new work in this thesis is the use of a single recurrence relation to obtain the coefficients of a Chebyshev series in two variables as the solution to certain elliptic partial differential equations, on rectangular regions.

It is of interest to consider briefly other derivations of Chebyshev solutions of partial differential equations. In solving parabolic partial differential equations, Dew, Knibb and Scraton have found solutions by taking fixed values of one variable, and finding solutions in the other variable for each value of the first variable. This leads to a set of solutions for the different values of the first variable. For the solution of elliptic equations, Fox (1968) reports the work of Mason who uses a selected points method to obtain the coefficients from a full set of algebraic equations. In this thesis the approach to elliptic equations is to obtain a single solution over the whole region, after using a recurrence relation to obtain a sparse set of algebraic equations, which can be solved by an iterative method.

In section 5.1 the series solution is posed in the form (5.12) for an equation of Laplace type, and equation (5.17) is derived as the general recurrence relation for the coefficients. In section 5.2, "boundary equations" for the Dirichlet problem, are derived by matching series obtained from the data with the special form of (5.12) on each part of the boundary. In following sections the problem is modified, including consideration of boundary conditions of Neumann type, and also the existence of discontinuities at the vertices of the rectangular solution region.
5.1 General Method of Solution.

Firstly, the equation
\[ \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} = 0, \quad \mu > 0 \] (5.11)
is considered on a region \( R \) subject to given boundary conditions on \( \partial R \), the boundary of \( R \). When \( \mu = 1 \), (5.11) is the two dimensional Laplace equation, but at present it is more convenient to consider the general form above.

Let the solution of (5.11) be expressed
\[ u(x,y) = \sum_{i,j=0}^{\infty} a_{ij} T_i(x) T_j(y) \] (5.12)
using the notation already explained in Chapter 2.

Series for the partial derivatives of \( u(x,y) \), with respect to \( x \), can be written
\[ \frac{\partial u}{\partial x} = \sum_{i,j} a_{ij} T_i(x) T_j(y) \] (5.13)
\[ \frac{\partial^2 u}{\partial x^2} = \sum_{i,j} a_{iij} T_i(x) T_j(y) \] (5.14)
with similar series (5.13a), (5.14a), for \( \frac{\partial u}{\partial y} \) and \( \frac{\partial^2 u}{\partial y^2} \).

The sets of coefficients \( \{a_{ij}\}, \{a_{iij}\}, \{a_{ij}\}, \{a_{iij}\} \) are constants and have the same significance in their respective series as the sets of coefficients \( \{a_i\} \) in the series of Chapter 2.

Substitution of the series for the partial derivatives, into equation (5.11), yields
\[ \sum_{i,j} a_{iij} T_i(x) T_j(y) + \mu \sum_{i,j} a_{ij} T_i(x) T_j(y) = 0 \]
and hence

\[
\begin{align*}
a_{ij}^{(xx)} + \mu a_{ij}^{(yy)} &= 0 \\
i, j &= 0, 1, 2, \ldots
\end{align*}
\]

(5.15)
on equating coefficients of like pairs of Chebyshev polynomials.

Then, applying the ideas of Chapter 2,

\[
a_{i-1, j}^{(xx)} - a_{i+1, j}^{(xx)} + \mu \left\{ a_{i-1, j}^{(yy)} - a_{i+1, j}^{(yy)} \right\} = 0
\]
giving

\[
2i a_{ij}^{(x)} + \mu \left\{ a_{i-1, j}^{(yy)} - a_{i+1, j}^{(yy)} \right\} = 0 .
\]

Then

\[
2 \left\{ a_{i-1, j}^{(x)} - a_{i+1, j}^{(x)} \right\} + \mu \left\{ a_{i-2, j}^{(y)} - a_{ij}^{(y)} - a_{ij}^{(y)} - a_{i+2, j}^{(y)} \right\} = 0
\]

and

\[
4i(i+1)(i-1)a_{ij} + \mu \left\{ (i+1)a_{i-2, j}^{(yy)} - 2i a_{ij}^{(yy)} + (i-1)a_{i+2, j}^{(yy)} \right\} = 0. \quad (5.16)
\]

Now, the above manipulations need not be performed in the detail shown, because they are exactly the same as those used in deriving the table for \( W^{(k)} \) in Chapter 3. Indeed, examination of columns 1 and 7 of Table 3.11 shows how equation (5.16) can be written immediately.

Further use of Table 3.11, now with regard to the derivatives in \( y \), yields the final result

\[
i(i+1)(i-1)\{(j+1)a_{i, j-2} - 2j a_{ij} + (j-1)a_{i, j+2}\}
\]

\[
+ \mu j(j+1)(j-1)\{(i+1)a_{i-2, j}^{(yy)} - 2i a_{ij}^{(yy)} + (i-1)a_{i+2, j}^{(yy)}\} = 0 . \quad (5.17)
\]

This result can be conveniently written in terms of the "molecule" below.
There is an obvious resemblance to the equation

\[
\begin{align*}
\mu j(j+1)(j-l)(i-1) & \quad \text{a}_{i-2,j} \\
-2\mu j(j+1)(j-l)i & \\
i(i+1)(i-l)(j+1) & \quad -2i(i+1)(i-l)j \\
\text{a}_{i,j} & \\
\text{a}_{i,j+2} & \\
\mu j(j+1)(j-l)(i-1) & \quad \text{a}_{i+2,j}
\end{align*}
\]

\[
\begin{align*}
\mu j(j+1)(j-l)(i-1) & \quad \text{a}_{i-2,j} \\
-2\mu j(j+1)(j-l)i & \\
i(i+1)(i-l)(j+1) & \quad -2i(i+1)(i-l)j \\
\text{a}_{i,j} & \\
\text{a}_{i,j+2} & \\
\mu j(j+1)(j-l)(i-1) & \quad \text{a}_{i+2,j}
\end{align*}
\]

and the associated molecule

\[
\begin{align*}
\mu \frac{h^2}{k^2} & \quad u(x,y+k) \\
1 & -2 - 2\mu \frac{h^2}{k^2} \\
u(x-h,y) & \\
u(x,y) & \\
u(x,y) & \\
u(x+h,y) \\
\mu \frac{h^2}{k^2} & \quad u(x,y-k)
\end{align*}
\]

obtained using central difference approximations for the partial derivatives. However, the important contrast is that the unknowns in the present method are the coefficients \{a_{ij}\} in a series expansion, while in the finite difference method they are function values \{u(x,y)\} at the points \{(x,y)\} on a grid in the region \(R\). When the coefficients in the present series are known, the global solution can then be used to obtain function values, and derivatives, (using (2.93)) at any point of the region \(R\). In many cases the Chebyshev coefficients \{a_{ij}\} decrease.
rapidly in magnitude, and a smaller number of them is wanted in a Chebyshev polynomial approximation to the solution \( u(x,y) \), than the number of points required for similar accuracy using finite difference methods.

If suitably large values of \( m, n \) are chosen for the polynomial

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} T_i(x)T_j(y)
\]


to equation (5.11), then the problem becomes the algebraic one of finding the coefficients of the set \( \{a_{ij}\} \).

Thus it is assumed that

\[
a_{ij} = 0, \quad i > m, \quad j > n
\]  \hspace{1cm} (5.18)

Then letting \( i \) range from 2 to \( m \) and \( j \) from 2 to \( n \), a total of \( (m-1)(n-1) \) equations is obtained from (5.17).

Putting \( i = 0, 1 \) and \( j = 0, 1 \), merely leads to results consistent with the convention that

\[
a_{0i,0j} = a_{ij}.
\]

The remaining \( (m+1)(n+1) - (m-1)(n-1) = (2m+2n) \), equations required for finding the coefficients \( \{a_{ij}\}, \quad i = 0, \ldots, m, \quad j = 0, \ldots, n \) are obtained from boundary conditions. Various cases will be considered in subsequent sections.

The coefficients \( \{a_{ij}\}, \quad i = 0, \ldots, m, \quad j = 0, \ldots, n \), can be pictured in the matrix \( A = \{a_{ij}\} \), of order \((m+1) \times (n+1)\) in the form

\[
\begin{bmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & \cdots \cdots & a_{0n} \\
a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & \cdots \cdots & a_{1n} \\
a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \cdots & a_{2n} \\
a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & \cdots \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \cdots & \vdots \\
a_{m0} & a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots \cdots & a_{mn}
\end{bmatrix}
\]  \hspace{1cm} (5.19)
It can be seen from equation (5.17) that calculation of the coefficients involves both the i- and j-subscripts being incremented in steps of 2. Thus with regard to the use of (5.17), there are four independent sets of calculations, before boundary conditions are applied. These calculations can be visualised as starting from the top left hand corner of the matrix A with the four initial possibilities being as in Table 5.11.

\[
\begin{array}{ccc}
 a_{02} & & a_{03} \\
 a_{20} & a_{22} & a_{24} \\
 a_{42} & & \\
 a_{21} & a_{23} & a_{25} \\
 a_{43} & & \\
\end{array}
\]

Set 1 

\[
\begin{array}{ccc}
 a_{12} & & a_{13} \\
 a_{30} & a_{32} & a_{34} \\
 a_{52} & & \\
 a_{31} & a_{33} & a_{35} \\
 a_{53} & & \\
\end{array}
\]

Set 2

\[
\begin{array}{ccc}
 a_{2} & & a_{3} \\
 a_{42} & & a_{43} \\
 a_{62} & & \\
 a_{52} & & a_{53} \\
 a_{72} & & \\
\end{array}
\]

Set 3

\[
\begin{array}{ccc}
 a_{1} & & a_{2} \\
 a_{3} & & a_{4} \\
 a_{5} & & a_{6} \\
 a_{7} & & a_{8} \\
 a_{9} & & a_{10} \\
\end{array}
\]

Set 4

Table 5.11 Initial points for the four sets of calculations associated with equation (5.17).

The complete sets of calculations correspond, respectively to

\[
\begin{align*}
\text{Set 1:} & \quad \begin{cases} 
 i = 2 \text{ to } m \text{ in steps of } 2 \\
 j = 2 \quad \text{" n " \quad \quad 2} \end{cases} \\
\end{align*}
\]
In a general problem the various sets need not be considered separately, and it is sufficient to let $i = 2$ to $m$ in steps of 1 and $j = 2$ to $n$.

However, it is sometimes convenient to consider the separate cases, in order to limit the number of calculations to those for non-zero coefficients. The ideas correspond to those in Chapters 3 and 4 with regard to purely odd or purely even functions.

Thus when the coefficients $\{a_{ij}\}$ belong to only one of the sets, we have:

Set 1

\[
\begin{align*}
    u(-x,y) &= u(x,y) \quad \text{(even in } x) \\
    u(x,-y) &= u(x,y) \quad \text{(even in } y)
\end{align*}
\]

Set 2

\[
\begin{align*}
    u(-x,y) &= u(x,y) \quad \text{(even in } x) \\
    u(x,-y) &= -u(x,y) \quad \text{(odd in } y)
\end{align*}
\]

Set 3

\[
\begin{align*}
    u(-x,y) &= -u(x,y) \quad \text{(odd in } x) \\
    u(x,-y) &= u(x,y) \quad \text{(even in } y)
\end{align*}
\]

Set 4

\[
\begin{align*}
    u(-x,y) &= -u(x,y) \quad \text{(odd in } x) \\
    u(x,-y) &= -u(x,y) \quad \text{(odd in } y)
\end{align*}
\]

Similarly when the above symmetry and anti-symmetry conditions occur singly, a reduction in calculation can be obtained.

For example when $u(-x,y) = u(x,y)$, the odd terms in $x$ will be missing and the calculation of the coefficients $\{a_{ij}\}$ will be based on sets 1, 2 only.
5.2 The Dirichlet Problem on the Square.

The solution of (5.11) corresponding to known values of $u(x,y)$ on the boundary of the square $-1 \leq x \leq 1, -1 \leq y \leq 1$, is considered.

Let $u(x,1) = \alpha(x) \equiv \sum_{i=0}^{m} a_i T_i(x)$, $-1 < x < 1$

\[
\begin{align*}
\alpha(x) &= \sum_{i=0}^{m} a_i T_i(x) , & -1 < x < 1 \\
\beta(y) &= \sum_{j=0}^{n} \beta_j T_j(y) , & -1 < y < 1 \\
\gamma(x) &= \sum_{i=0}^{m} \gamma_i T_i(x) , & -1 < x < 1 \\
\delta(y) &= \sum_{j=0}^{n} \delta_j T_j(y) , & -1 < y < 1 \\
\end{align*}
\]

with $\alpha(x)$, $\beta(y)$, $\gamma(x)$, $\delta(y)$ being known functions.

Now $u(x,1) = \sum_{i=0}^{m} \left( \sum_{j=0}^{n} a_{ij} \right) T_i(x)$

and $u(x,-1) = \sum_{i=0}^{m} \left( \sum_{j=0}^{n} (-1)^j a_{ij} \right) T_i(x)$

and equating coefficients of like Chebyshev polynomials gives

\[
\begin{align*}
\alpha_i &= \sum_{j=0}^{n} a_{ij} = \frac{1}{2} a_{i0} + a_{il} + \ldots + a_{in} \quad i=0, \ldots, m \\
\gamma_i &= \sum_{j=0}^{n} (-1)^j a_{ij} = \frac{1}{2} a_{i0} - a_{il} + \ldots + (-1)^n a_{in} \\
\end{align*}
\]

Similarly

\[
\begin{align*}
u(1,y) &= \sum_{j=0}^{n} \left( \sum_{i=0}^{m} a_{ij} \right) \\
u(-1,y) &= \sum_{j=0}^{n} \left( \sum_{i=0}^{m} (-1)^i a_{ij} \right)
\end{align*}
\]
giving

\[ \begin{align*}
\beta_j &= \sum_{i=0}^{m} a_{ij} = a_{0j} + a_{1j} + \ldots + a_{mj} \\
\delta_j &= \sum_{i=0}^{m} (-1)^j a_{ij} = a_{0j} - a_{1j} + \ldots + (-1)^m a_{mj} 
\end{align*} \]

These equations can be re-written in the alternative forms

\[ \begin{align*}
\begin{cases}
\frac{1}{2}(\alpha_i + \gamma_i) = \frac{1}{2} a_{i0} + a_{i2} + \ldots \\
\frac{1}{2}(\alpha_i - \gamma_i) = a_{i1} + a_{i3} + \ldots
\end{cases} & \quad i=0, \ldots, m \tag{5.24} \\
\begin{cases}
\frac{1}{2}(\beta_j + \delta_j) = \frac{1}{2} a_{0j} + a_{2j} + \ldots \\
\frac{1}{2}(\beta_j - \delta_j) = a_{1j} + a_{3j} + \ldots
\end{cases} & \quad j=0, \ldots, n \tag{5.25}
\end{align*} \]

In either case there are \(2m + 2n + 4\) "boundary equations" to be satisfied by the coefficients. However only \(2m + 2n\) equations are independent. For example with \(i = 0, 1\) and \(j = 0, 1\) in (5.24) and (5.25), then

\[ \begin{align*}
\frac{1}{2}(\alpha_0 + \gamma_0) &= \frac{1}{2} a_{00} + a_{02} + \ldots \\
\frac{1}{2}(\alpha_0 - \gamma_0) &= a_{01} + a_{03} + \ldots \\
\frac{1}{2}(\alpha_1 + \gamma_1) &= \frac{1}{2} a_{10} + a_{12} + \ldots \\
\frac{1}{2}(\alpha_1 - \gamma_1) &= a_{11} + a_{13} + \ldots \\
\frac{1}{2}(\beta_0 + \delta_0) &= \frac{1}{2} a_{00} + a_{20} + \ldots \\
\frac{1}{2}(\beta_0 - \delta_0) &= a_{01} + a_{21} + \ldots \\
\frac{1}{2}(\beta_1 + \delta_1) &= \frac{1}{2} a_{01} + a_{21} + \ldots \\
\frac{1}{2}(\beta_1 - \delta_1) &= a_{11} + a_{31} + \ldots
\end{align*} \]

If the complete sets of equations (5.22) and (5.23) are regarded as a means of calculating \(a_{i0}, a_{i1}, a_{0j}, a_{1j}\) when \(i=0, \ldots, m\) and \(a_{0j}, a_{1j}\) when \(j=0, \ldots, n\), then (5.26) and (5.27) show how \(a_{00}, a_{01}, a_{10}, a_{11}\) can each be calculated in two different ways.
The values of $a_{00}, a_{01}, a_{10}, a_{11}$ as calculated by (5.26) must be consistent with those calculated by (5.27). This is ensured if the boundary conditions are continuous at the vertices $(±1, ±1)$, namely

$$\alpha(1) = \beta(1), \quad \beta(-1) = \gamma(1)$$

$$\gamma(-1) = \delta(-1), \quad \delta(1) = \alpha(-1).$$

When this condition exists then the number of independent boundary equations is reduced by 4 to $2m + 2n$ and the equations (5.17), (5.22), (5.23) taken together or (5.17), (5.24), (5.25) are sufficient to calculate the $(m + 1)(n + 1)$ coefficients in the set \(\{a_{i,j}\}\).

It will be assumed for the present that the boundary conditions are continuous at the vertices. In section 5.7 discontinuities at the vertices are considered.
Writing equations (5.17), (5.24), (5.25) together leads to the set of \((m + 1) \times (n + 1)\) equations

\[
\begin{align*}
\frac{1}{2} a_{10} + a_{12} + \ldots &= \frac{1}{2} (a_i + y_i), \quad i = 0, \ldots, m \\
a_{1i} + a_{13} + \ldots &= \frac{1}{2} (a_i - y_i), \quad i = 0, \ldots, m \\
\frac{1}{2} a_{0j} + a_{2j} + a_{4j} + \ldots &= \frac{1}{2} (\beta_j + \delta_j) \\
a_{1j} + a_{3j} + a_{5j} + \ldots &= \frac{1}{2} (\beta_j - \delta_j) \\
\end{align*}
\]

\[
\begin{align*}
\mu f(j,i) a_{i-2,j} - g(i,j) a_{ij} + \mu f(-j,-i) a_{i+2,j} + f(i,j) a_{i,j-2} + f(-i,-j) a_{i,j+2} &= 0 \\
& \quad i = 2, \ldots, m \\
& \quad j = 2, \ldots, n
\end{align*}
\]

(5.28)

where \(f(i,j) = i(i+1)(i-1)(j+1)\)

and \(g(i,j) = \{f(i,j) + f(-i,-j)\} + \mu\{f(j,i) + f(-j,-i)\}\),

\(a_{ij} = 0\) if \(i > m\) or \(j > n\).

As a simple example, when \(m = 5\), \(n = 3\) the form of these equations can be seen in Table 5.21.

Alternatively the equations can be re-ordered as

\[
\begin{align*}
\frac{1}{2} a_{0j} + a_{2j} + a_{4j} + \ldots &= \frac{1}{2} (\beta_j + \delta_j), \quad j = 0, \ldots, n \\
a_{ij} + a_{3j} + a_{5j} + \ldots &= \frac{1}{2} (\beta_j - \delta_j), \quad j = 0, \ldots, n \\
\frac{1}{2} a_{10} + a_{12} + \ldots &= \frac{1}{2} (a_i + y_i) \\
a_{1i} + a_{13} + \ldots &= \frac{1}{2} (a_i - y_i) \\
\end{align*}
\]

(5.29)

\[
\begin{align*}
f(i,j) a_{i,j-2} - g(i,j) a_{ij} + f(-i,-j) a_{i,j+2} + \mu f(j,i) a_{i-2,j} + \mu f(-j,-i) a_{i+2,j} &= 0 \\
& \quad i = 2, \ldots, m \\
& \quad j = 2, \ldots, n
\end{align*}
\]

(5.29)

When \(m = 5\), \(n = 3\), the form of these equations is exhibited in Table 5.22.
\[
\begin{bmatrix}
I & 0 & I & 0 \\
0 & I & 0 & I \\
A_2 & 0 & B_2 & 0 \\
0 & A_3 & 0 & B_3
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

\[A_j = \text{diag} \{0, 0, f(2,j), f(3,j), f(4,j), f(5,j)\}, \quad j = 2,3\]

\[
B_j = 
\begin{bmatrix}
I & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\mu f(j,2) & 0 & -g(2,j) & 0 & \mu f(-j,-2) & 0 \\
0 & \mu f(j,3) & 0 & -g(3,j) & 0 & \mu f(-j,-3) \\
0 & 0 & \mu f(j,4) & 0 & -g(4,j) & 0 \\
0 & 0 & 0 & \mu f(j,5) & 0 & -g(5,j)
\end{bmatrix}
\]

\[a_j^T = [a_{o_j}, a_{1j}, a_{2j}, a_{3j}, a_{4j}, a_{5j}], \quad j = 0,1,2,3\]

\[b_j^T = [\mu (a_o + \gamma_o), \mu (a_1 + \gamma_1), \ldots, \mu (a_5 + \gamma_5)]\]

\[b_1^T = [\mu (a_o - \gamma_o), \mu (a_1 - \gamma_1), \ldots, \mu (a_5 - \gamma_5)]\]

\[b_j^T = [\mu (\beta_j + \delta_j), \mu (\beta_j - \delta_j), 0, 0, 0, 0], \quad j = 2,3\]

\[I\] is the identity matrix of order 6.

\textbf{Table 5.21.} Representation of equations (5.28), when \(m = 5, \ n = 3\).
\[
\begin{bmatrix}
0 & I & 0 & I & 0 & 0 \\
0 & I & 0 & I & 0 & 1 \\
A_2 & 0 & B_2 & 0 & C_2 & 0 \\
0 & A_3 & 0 & B_3 & 0 & C_3 \\
0 & 0 & A_4 & 0 & B_4 & 0 \\
0 & 0 & 0 & A_5 & 0 & B_5
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}
= 
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5
\end{bmatrix}
\]

\[A_i = \text{diag} \{0, 0, \mu f(2,i), \mu f(3,i)\}, \quad i = 2,\ldots,5\]

\[
B_i = 
\begin{bmatrix}
h_i & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
f(i,2) & 0 & -g(i,2) & 0 \\
0 & f(i,3) & 0 & -g(i,3)
\end{bmatrix}, \quad i = 0,1,\ldots,5
\]

\[C_i = \text{diag} \{0, 0, \mu f(-2,-i), \mu f(-3,-i)\}, \quad i = 2,\ldots,5\]

\[A_i^T = [a_{i0}, a_{i1}, a_{i2}, a_{i3}], \quad i = 0,1,\ldots,5\]

\[P_i^T = \left[ \begin{array}{c}
h(\beta_o + \delta_o), h(\beta_1 + \delta_1), h(\beta_2 + \delta_2), h(\beta_3 + \delta_3) \\
h(\beta_o - \delta_o), h(\beta_1 - \delta_1), h(\beta_2 - \delta_2), h(\beta_3 - \delta_3) \\
h(a_i + \gamma_i), h(a_i - \gamma_i), 0, 0 
\end{array} \right], \quad i = 2,3,4,5\]

\[I\] is the identity matrix of order 4.

Table 5.22. Representation of equations (5.29) when \(m = 5, \ n = 3\).
5.3 Poisson's Equation.

The equation
\[ \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} = g(x,y) \]  
\[ (5.31) \]
with Dirichlet type boundary conditions on the square \( x = \pm 1, y = \pm 1 \), can be solved using the ideas already developed.

Thus with the previous notation, let
\[ g(x,y) = \sum_{i,j=0}^{\infty} b_{ij} T_i(x) T_j(y) \]  
\[ (5.32) \]

Then after substituting (5.13) and (5.32) into the differential equation (5.31), and equating coefficients,
\[ a_{ij}^{(xx)} + \mu a_{ij}^{(yy)} = b_{ij} \]
and use of columns 1 and 7 of Table 3.11, eventually yields
\[ i(i+1)(i-1) \left\{ (j+1)a_{i,j-2} - 2j a_{ij} + (j-1)a_{i,j+2} \right\} + \mu j(j+1)(j-1) \left\{ (i+1)a_{i-2,j} - 2i a_{ij} + (i-1)a_{i+2,j} \right\} = \frac{1}{i} \left[ (j+1)(j+2) - 2j b_{i-2,j} + (j-1)b_{i-2,j+2} \right] \]
\[ - \frac{1}{i} \left[ (j+1)b_{i,j-2} - 2j b_{ij} + (j-1)b_{i,j+2} \right] + \frac{1}{j} \left[ (j+1)b_{i+2,j-2} - 2j b_{i+2,j} + (j-1)b_{i+2,j+2} \right] \]  
\[ (5.33) \]
5.4 A more general equation.

The more general elliptic equation

\[
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \lambda u = g(x,y),
\]

(5.41)

can also be solved.

The recurrence relation is obtained as a simple modification of (5.33), by adding to its left hand side, terms of the same form as on the right hand side, with each \( b_{ij} \) being replaced by \( \lambda a_{ij} \). Thus

\[
\begin{align*}
&i(i+1)(i-1) \left\{ (j+1)a_{i,j-2} - 2j a_{ij} + (j-1)a_{i,j+2} \right\} \\
&+ \mu j(j+1)(j-1) \left\{ (i+1)a_{i-2,j} - 2i a_{ij} + (i-1)a_{i+2,j} \right\} \\
&+ \frac{\lambda}{4} \left[ (i+1) \left\{ (j+1)a_{i-2,j-2} - 2j a_{i-2,j} + (j-1)a_{i-2,j+2} \right\} \\
&- 2i \left\{ (j+1)a_{i,j-2} - 2j a_{ij} + (j-1)a_{i,j+2} \right\} \\
&+ (i-1) \left\{ (j+1)a_{i+2,j-2} - 2j a_{i+2,j} + (j-1)a_{i+2,j+2} \right\} \right]
\end{align*}
\]

(5.42)
5.5 The Neumann Problem.

Write
\[ u(x,y) = \sum_i \left\{ \sum_j a_{ij} T_j(y) \right\} T_i(x) \]
where \( \lambda_i = \lambda_i(y) = \sum_j a_{ij} T_j(y) \).

Now
\[ \frac{\partial u}{\partial x} = 2 \sum_{i=0}^{\infty} \left\{ (i+1)\lambda_{i+1} + (i+3)\lambda_{i+3} + \ldots \right\} T_i(x) \]
using the result (2.61) for differentiating a Chebyshev polynomial.

Hence when \( x = 1 \)
\[ \frac{\partial u}{\partial x} \bigg|_{x=1} = 2 \sum_{i=1}^{\infty} \left\{ \begin{array}{l} 1 \lambda_1(y) + 3 \lambda_3(y) + 5 \lambda_5(y) + \ldots \\ + 2 \left\{ \begin{array}{l} 2 \lambda_2(y) + 4 \lambda_4(y) + \ldots \\ + 2 \left\{ \begin{array}{l} 3 \lambda_3(y) + 5 \lambda_5(y) + \ldots \\ + \ldots \end{array} \right. \end{array} \right. \end{array} \right. \]
\[ = 1^2 \lambda_1(y) + 2^2 \lambda_2(y) + 3^2 \lambda_3(y) + \ldots \]
\[ = \sum_{i=1}^{\infty} i^2 \lambda_i(y) \]
\[ = \sum_{i=1}^{\infty} i^2 \left\{ \sum_{j=0}^{\infty} a_{ij} T_j(y) \right\} \]
\[ \frac{\partial u}{\partial x} \bigg|_{x=1} = \sum_{i=0}^{\infty} \left\{ \sum_{j=0}^{\infty} i^2 a_{ij} \right\} T_j(y) \] \hspace{1cm} (5.51)

Similarly, on \( x = -1 \)
\[ \frac{\partial u}{\partial x} \bigg|_{x=-1} = \sum_{j=0}^{\infty} \left\{ \sum_{i=1}^{\infty} (-1)^i i^2 a_{ij} \right\} T_j(y) \] \hspace{1cm} (5.52)

On \( y = 1 \)
\[ \frac{\partial u}{\partial y} \bigg|_{y=1} = \sum_{i=0}^{\infty} \left\{ \sum_{j=1}^{\infty} j^2 a_{ij} \right\} T_i(x) \] \hspace{1cm} (5.51a)
On $y = -1$

$$\left(\frac{\partial u}{\partial y}\right)_{y=-1} = \sum_{i=0}^{\infty} \left\{ \sum_{j=1}^{\infty} (-1)^{j} j^2 a_{ij} \right\} T_i(x). \quad (5.52a)$$

Suppose part of a boundary condition is given as

$$p u(l,y) + q \frac{\partial u}{\partial x}_{x=1} = \delta(y) \equiv \sum_{j=0}^{\infty} \beta_j T_j(y).$$

Then

$$\sum_{j=0}^{\infty} \left[ p \left( \sum_{i=0}^{\infty} a_{ij} \right) T_j(y) + q \left( \sum_{i=1}^{\infty} i^2 a_{ij} \right) T_j(y) \right] = \sum_{j=0}^{\infty} \beta_j T_j(y).$$

Equating coefficients

$$p \sum_{i=0}^{\infty} a_{ij} + q \sum_{i=1}^{\infty} i^2 a_{ij} = \beta_j. \quad (5.53)$$

Similarly, given that

$$r u(-1,y) + s \frac{\partial u}{\partial x}_{x=-1} = \delta(y) = \sum_{j=0}^{\infty} \delta_j T_j(y),$$

then

$$r \sum_{i=0}^{\infty} (-1)^i a_{ij} + s \sum_{i=1}^{\infty} (-1)^i i^2 a_{ij} = \delta_j. \quad (5.53a)$$

These equations (5.53) and (5.53a), together with similar equations from boundary conditions on $y = \pm 1$, can replace (5.22) and (5.23) (or (5.24) and (5.25)) for the solution of the Neumann problem, and mixed Neumann problem, if derivative conditions are given on the boundary.
5.6 Problems on a rectangular region.

The solution of
\[
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \lambda u = g(x,y) \tag{5.41}
\]
on any region \(a \leq x \leq b, \quad c \leq y \leq d\), can be reduced to the corresponding problem on the square with vertices \((\pm 1, \pm 1)\), by linear changes of variable
\[
x = \frac{1}{4}(b-a)x' + (b+a)
\]
\[
y = \frac{1}{4}(d-c)y' + (d+c)
\]
giving
\[
\frac{4}{(b-a)^2} \frac{\partial^2 u}{\partial x'^2} + \mu \frac{u}{(c-d)^2} \frac{\partial^2 u}{\partial y'^2} + \lambda u = G(x',y') \tag{5.61}
\]
or
\[
\frac{\partial^2 u}{\partial x'^2} + \frac{\mu}{\bar{\mu}} \frac{\partial^2 u}{\partial y'^2} + \bar{\lambda} u = \bar{G}(x',y') \tag{5.62}
\]
where
\[
\bar{\mu} = \left(\frac{b-a}{c-d}\right)^2 \mu, \quad \bar{\lambda} = \frac{(b-a)^2 \lambda}{4}
\]
and
\[
\bar{G}(x',y') = \frac{(b-a)^2}{4} G(x',y')
\]
where \(G(x',y')\) is obtained from \(g(x,y)\) following the substitution (5.61).
5.7 Discontinuities at the Vertices

In some circumstances it is possible to cope with discontinuities at the vertices \((\pm 1, \pm 1)\), by observing that for any solution \(u\) of the homogeneous equation
\[
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} = 0
\]  
(5.71)
derivatives of \(u\) are also solutions, namely
\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial^{r+s} u}{\partial x^r \partial y^s} \right) + \mu \frac{\partial^2}{\partial y^2} \left( \frac{\partial^{r+s} u}{\partial x^r \partial y^s} \right) = 0 .
\]

In particular when \(r=0, s=2\) or \(r=2, s=0\), then the equation (5.71) can be used to modify boundary conditions involving \(u\), to give conditions involving \(\frac{\partial^2 u}{\partial x^2}\) or \(\frac{\partial^2 u}{\partial y^2}\).

These ideas have been used by Knibb and Scraton (1971) in the solution of parabolic partial differential equations in Chebyshev series.

Consider the problem
\[
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for which } \begin{cases} u(x, l) = 0 \\ u(l, y) = 0 \\ u(x, -l) = \gamma(x) \\ u(-l, y) = 0 \end{cases}
\]  
(5.72)
and \(\gamma(\pm 1) \neq 0\).

Then introduce the function \(\tilde{\gamma}(x)\), such that
\[
\frac{d^2 \tilde{\gamma}}{dx^2} + \mu \gamma = 0 \quad \text{and } \tilde{\gamma}(\pm 1) = 0 .
\]  
(5.73)
Then \(u = \frac{\partial^2 \gamma}{\partial y^2} = -\frac{1}{\mu} \frac{\partial^2 \gamma}{\partial x^2}\)

(5.74)
where $\bar{u}(x,y)$ is the solution of the problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} &= 0 \\
\bar{u}(x,1) &= 0 \\
\bar{u}(l,y) &= 0 \\
\bar{u}(x,-1) &= \bar{\gamma}(x) \\
\bar{u}(1,y) &= 0
\end{align*}
\] (5.75)

in which there are no discontinuities at the vertices. Of course

\[
\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\mu} \frac{\partial^2 u}{\partial x^2}
\]

will be discontinuous at the vertices.
6.1 Introduction

The two main points of interest with regard to the present Chebyshev method of solution are firstly, the method for solving the approximating set of algebraic equations, leading secondly to an examination of the final solution of the differential equation. In this Chapter the final solutions of various problems are examined, and in Chapter 7 there is a discussion of the method used for solving the algebraic equations.

In later sections it will be seen that the Chebyshev series method leads to accurate answers for problems on rectangular regions, and that quite often a small number of terms in the series solution leads to a sufficiently accurate result. If derivatives of a solution are wanted then more terms of the series have to be taken, but even so it is significant that accurate values of the derivatives can certainly be obtained although this is not so with most other methods for solving such problems.

It is again emphasised that whereas the finite difference method calculates function values at points of a grid, with intermediate values being found by interpolation, and with a fine grid being necessary for good accuracy, the Chebyshev method calculates coefficients in a series then enabling function values to be calculated at any point, and that the number of coefficients calculated is far fewer than the number of grid points of the finite difference method, for comparable accuracy. In this Chapter none of the polynomial solutions is of degree greater than 20 in each of $x$ and $y$, so that a maximum of $21 \times 21$
coefficients is calculated, and this is reduced to 11 x 21 if there is symmetry in one of x or y, (or to 10 x 21 if there is skew-symmetry in one of x or y) and to 11 x 11 (or 10 x 10) if there is symmetry (skew-symmetry) in both x and y.

The behaviour of derivatives is as expected in the interiors of solution regions, but particularly when the derivatives are discontinuous, their values on the boundaries are not always accurate at all points. The errors are those to be expected from a polynomial approximation, and they appear to be minimal in the usual Chebyshev sense. The actual solutions (as distinct from the derivatives) also exhibit similar behaviour, but the errors are usually too small to be detected.

Any function with a convergent Chebyshev series can occur in the boundary conditions, and problems in sections 6.2 and 6.4 particularly emphasise the point.

The examples in section 6.9 are concerned with discontinuities at vertices of a square region, as explained in section 5.7.
6.2 Laplace’s Equation (1)

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

Solution of \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ u(x, \pm 1) = \cos(\pi x) \]
\[ u(\pm 1, y) = 0 . \]

(6.21)

This problem is symmetric about both axes, so that in the solution (5.12), only even values of \( i, j \) occur, corresponding to coefficients from Set 1 (see (5.17) and Table 5.11).

The analytic solution of the problem is

\[ u(x, y) = \cos(\pi x) \cosh(\pi y) / \cosh(\pi) \]

(6.22)

being the real part of \( \cos(\pi z) / \cosh(\pi) \), \( z = x + iy \).

In Table 6.20, values of this analytic solution are tabulated for \( x = 0(0.2)1, y = 0(0.2)1 \), corresponding to points in the first quadrant.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.00000 .95106 .80902 .58779 .30902 .00000</td>
</tr>
<tr>
<td>0.8</td>
<td>.75686 .71982 .61231 .44487 .23388 .00000</td>
</tr>
<tr>
<td>0.6</td>
<td>.58904 .56021 .47654 .34623 .18202 .00000</td>
</tr>
<tr>
<td>0.4</td>
<td>.47983 .45634 .38819 .28204 .14827 .00000</td>
</tr>
<tr>
<td>0.2</td>
<td>.41837 .39789 .33847 .24591 .12928 .00000</td>
</tr>
<tr>
<td>0.0</td>
<td>.39854 .37903 .32242 .23425 .12315 .00000</td>
</tr>
</tbody>
</table>

TABLE 6.20 - Analytic solution of (6.21)

With the notation of section 5.2,

\[ a(x) = \gamma(x) = \cos(\pi x) \]
\[ \beta(y) = \delta(y) = 0 . \]

The methods of Chapters 3, 4 are used to find the coefficients in the Chebyshev expansion of \( \cos(\pi x) \) on \((-1,1)\), thus yielding the values for \( \{a_i\}, \{\gamma_i\} \), in the data for the present problem.
Using (5.17), (5.24), (5.25), the approximate solution (5.12), with \( m = n = 6 \) in (5.18), is then

\[
u(x,y) = \sum_{i,j=0}^{6} a_{ij} T_i(x) T_j(y)
\]

(6.23)

and the calculated non-zero values of \( a_{ij} \) given in Table 6.21.

(2.91) now leads to values of \( u(x,y) \) at points \( (x,y) \), \( x = 0(0.2)1, y = 0(0.2)1 \), as shown in Table 6.22. Comparison with the corresponding values from the analytic solution (Table 6.20), shows that the maximum deviation of the approximate solution from the analytic solution is \( \sim 10^{-5} \).

(2.92) can now be used to find the values of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) at the same points. The maximum deviation from zero is \( \sim 10^{-2} \) at \( (1,1) \), and the smallest deviation is \( \sim 10^{-4} \) at \( (0,0) \). These values are calculated in order to provide some measure of the accuracy of the solutions, but the remarks in the previous paragraph indicate that the solution is very much better than this latter measure might indicate.

With \( m \) and \( n \) increased to 10, the non-zero coefficients in the approximate solution \( u(x,y) = \sum_{i,j=0}^{10} a_{ij} T_i(x) T_j(y) \) are shown in Table 6.23. Values of \( u(x,y) \) at the points previously used, show exact agreement with the values from the analytic solution, to the number of figures printed, and in fact the agreement is to more figures than shown, as might be expected from the rapid rate of decrease in the coefficients \( \{a_{ij}\} \).

The maximum deviation of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) from zero is now less than \( 5 \times 10^{-8} \).

Thus in this introductory example, the methods of Chapter 5 are shown to produce an extremely satisfactory numerical solution. It can be noted that although the greater number of terms produces a better
result, the solution from fewer terms is also adequate even if the values of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) are not very encouraging. It appears that accurate solutions may be obtained with a small number of terms but that if corresponding accuracy is wanted for derivative values, then a larger number of terms will be required in the solution. This behaviour is characteristic of all polynomial approximations of continuous functions.
TABLE 6.2
Non-zero coefficients in the approximate solution

\[ u(x,y) = \sum_{i,j=0}^{6} a_{ij} T_i(x) T_j(y) . \]

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.293257</td>
<td>.283627</td>
<td>.013479</td>
<td>.000268</td>
</tr>
<tr>
<td>2</td>
<td>-.684173</td>
<td>-.150045</td>
<td>-.007131</td>
<td>-.000142</td>
</tr>
<tr>
<td>4</td>
<td>.038348</td>
<td>.008410</td>
<td>.000400</td>
<td>.000008</td>
</tr>
<tr>
<td>6</td>
<td>-.000817</td>
<td>-.000179</td>
<td>-.000009</td>
<td>-.000000</td>
</tr>
</tbody>
</table>

TABLE 6.21
Non-zero coefficients in the approximate solution

\[ u(x,y) = \sum_{i,j=0}^{6} a_{ij} T_i(x) T_j(y) . \]

<table>
<thead>
<tr>
<th>( y/ )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>.99999</td>
<td>.95106</td>
<td>.80902</td>
<td>.58778</td>
<td>.30901</td>
<td>-.00001</td>
</tr>
<tr>
<td>0.8</td>
<td>.75686</td>
<td>.71982</td>
<td>.61232</td>
<td>.44487</td>
<td>.23388</td>
<td>-.00001</td>
</tr>
<tr>
<td>0.6</td>
<td>.58903</td>
<td>.56021</td>
<td>.47654</td>
<td>.34622</td>
<td>.18202</td>
<td>-.00001</td>
</tr>
<tr>
<td>0.4</td>
<td>.47983</td>
<td>.45635</td>
<td>.38819</td>
<td>.28203</td>
<td>.14827</td>
<td>-.00001</td>
</tr>
<tr>
<td>0.2</td>
<td>.41836</td>
<td>.39789</td>
<td>.33847</td>
<td>.24591</td>
<td>.12928</td>
<td>-.00001</td>
</tr>
<tr>
<td>0.0</td>
<td>.39853</td>
<td>.37903</td>
<td>.32242</td>
<td>.23425</td>
<td>.12315</td>
<td>-.00001</td>
</tr>
</tbody>
</table>

TABLE 6.22
Values of \( u(x,y) \), \( x,y = 0(0.2)1 \), from the above solution (\( m = n = 6 \)).

<table>
<thead>
<tr>
<th>( y/ )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
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<td>.000000</td>
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<td>.000000</td>
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<td>.000000</td>
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<tr>
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<td>.000000</td>
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<td>.000000</td>
</tr>
<tr>
<td>0</td>
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<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
</tr>
</tbody>
</table>

TABLE 6.23
Non-zero coefficients in the approximate solution

\[ u(x,y) = \sum_{i,j=0}^{6} a_{ij} T_i(x) T_j(y) . \]
Once the series solution
\[ u(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} T_i(x)T_j(y) \]
has been obtained, then all information concerning derivatives is available, using (2.92). Although it is not the intention to present such information for every example, the tables of values of \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial y} \), \( \frac{\partial^2 u}{\partial x^2} \), \( \frac{\partial^2 u}{\partial y^2} \) are given for this introductory example, corresponding to the solution when \( m = n = 10 \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( x )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.00000</td>
<td>-0.48540</td>
<td>-0.92329</td>
<td>-1.27080</td>
<td>-1.49392</td>
<td>-1.57080</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.00000</td>
<td>-0.36738</td>
<td>-0.69880</td>
<td>-0.96182</td>
<td>-1.13069</td>
<td>-1.18887</td>
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</tr>
<tr>
<td>0.6</td>
<td>0.00000</td>
<td>-0.28592</td>
<td>-0.54385</td>
<td>-0.74855</td>
<td>-0.87997</td>
<td>-0.92526</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.00000</td>
<td>-0.23291</td>
<td>-0.44302</td>
<td>-0.60976</td>
<td>-0.71682</td>
<td>-0.75371</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
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<td>-0.20308</td>
<td>-0.38627</td>
<td>-0.53166</td>
<td>-0.62500</td>
<td>-0.65717</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.00000</td>
<td>-0.19345</td>
<td>-0.36797</td>
<td>-0.50646</td>
<td>-0.59538</td>
<td>-0.62602</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 6.24** Values of \( \frac{\partial u}{\partial x} \). \( x, y = 0(0.2)1 \). \( m = n = 10 \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( x )</th>
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<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.37015</td>
<td>1.16552</td>
<td>0.84680</td>
<td>0.44519</td>
<td>0.00000</td>
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<tr>
<td>0.8</td>
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<td>0.96123</td>
<td>0.81768</td>
<td>0.59408</td>
<td>0.31232</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.68132</td>
<td>0.64797</td>
<td>0.55120</td>
<td>0.40047</td>
<td>0.21054</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.41974</td>
<td>0.39919</td>
<td>0.33957</td>
<td>0.24671</td>
<td>0.12971</td>
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</tr>
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<td>0.2</td>
<td>0.19992</td>
<td>0.19014</td>
<td>0.16174</td>
<td>0.11751</td>
<td>0.06178</td>
<td>0.00000</td>
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</tr>
<tr>
<td>0.0</td>
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<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 6.25** Values of \( \frac{\partial u}{\partial y} \). \( x, y = 0(0.2)1 \). \( m = n = 10 \).
Finally it should be explained that the format for presenting the results has been chosen for ease of reading. In practice the calculations have been performed on a UNIVAC 1106 computer, retaining approximately 8 decimal digit accuracy, and no truncation has been performed during the calculations.
6.3 Laplace's Equation (2)

Solution of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)

\[
\begin{align*}
    u(x, 1) &= 0 \\
    u(x, -1) &= \frac{1}{4}(1 - x^2) \\
    u(\pm 1, y) &= 0.
\end{align*}
\] (6.31)

This problem is symmetric about the y-axis, so that only even terms in \( x \) occur in the series solution. Again the analytic solution can be found, this time as

\[
    u(x, y) = \frac{2}{\pi} \sum_{r=1}^{\infty} (-1)^{r-1} \cos(x - \frac{1}{2}) \pi x \sinh[(x - \frac{1}{2})(1 - y) \pi] (r - \frac{1}{2})^{3} \sinh(2r - 1) \pi
\] (6.32)

For the numerical solution, the boundary conditions are

\[
\begin{align*}
    \alpha(x) &= \beta(y) = \delta(y) = 0 \\
    \gamma(x) &= \frac{1}{4}(1 - x^2).
\end{align*}
\] (6.33)

The method of Chapter 5 yields the approximate solution

\[
    u(x, y) = \sum_{i,j=0}^{20} a_{ij} T_i(x) T_j(y)
\] (6.34)

with the coefficients \( \{a_{ij}\} \) as in Table 6.31.

The values of \( u(x, y) \) for \( x = 0(0.2)1, \ y = -1(0.2)1 \), are given in Table 6.32.

When the analytic solution (6.32) is used to calculate values of the solution at the above points \((x, y)\), then in all cases there is exact agreement with the results in Table 6.32, to the given number of figures.

When \( \left| \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right| \) is calculated, to 5 decimal places, for the same points \((x, y)\), the values are zero at all internal points, but on the boundaries the values differ from zero, and are given in Table 6.33.

Nevertheless, as seen from Table 6.32, the actual solution \( u(x, y) \) does fit the boundary conditions, and indeed, from the remarks in the previous
paragraph, the solution using the present method is correct at all points.

The choice of \( m = n = 20 \) in the above calculations ensures that the results are extremely accurate, as stated. However, it is of interest to note that if \( m = n = 10 \), then the coefficients \( \{a_{ij}\} \) in this solution differ only slightly from those obtained with larger \( m \) and \( n \), and the values of \( u(x,y) \) obtained from this economised solution and almost the same as those obtained above. See Table 6.34.
<table>
<thead>
<tr>
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<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 6.3** Non-zero coefficients ($\times 10^6$) in the approximate solution. $m = n = 20$. 
TABLE 6.32 Values of \( u(x,y) \times 10 \). \( x = 0(0.2)1 \). \( y = -1(0.2)1 \). \( m = n = 20 \). 

<table>
<thead>
<tr>
<th>( y )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
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<tr>
<td>( x )</td>
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<td>+0.2</td>
<td>+0.4</td>
<td>+0.6</td>
<td>+0.8</td>
<td>±1</td>
</tr>
<tr>
<td>Values on ( y = 1 )</td>
<td>0.00000</td>
<td>0.14266</td>
<td>0.29949</td>
<td>0.48603</td>
<td>0.72073</td>
<td>1.02657</td>
</tr>
<tr>
<td>Values on ( y = -1 )</td>
<td>0.00000</td>
<td>0.13569</td>
<td>0.28487</td>
<td>0.46234</td>
<td>0.68569</td>
<td>0.97694</td>
</tr>
</tbody>
</table>

TABLE 6.33 Values of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) on boundaries. \( m = n = 20 \).
<table>
<thead>
<tr>
<th>x</th>
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<th>0.0000</th>
<th>0.0000</th>
<th>0.0000</th>
<th>0.0000</th>
<th>0.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.14266</td>
<td>0.13569</td>
<td>0.11545</td>
<td>0.08390</td>
<td>0.04412</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.29949</td>
<td>0.28487</td>
<td>0.24240</td>
<td>0.17619</td>
<td>0.09266</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.48604</td>
<td>0.46234</td>
<td>0.39350</td>
<td>0.28608</td>
<td>0.15049</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.72072</td>
<td>0.68569</td>
<td>0.58382</td>
<td>0.42466</td>
<td>0.22346</td>
<td>0.0000</td>
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<td>0.97694</td>
<td>0.83240</td>
<td>0.60604</td>
<td>0.31916</td>
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<td>1.16423</td>
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<td>0.44785</td>
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<td>1.61286</td>
<td>1.18028</td>
<td>0.62440</td>
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<td>3.05943</td>
<td>2.28088</td>
<td>1.23304</td>
<td>0.0000</td>
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<tr>
<td>-1</td>
<td>5.00000</td>
<td>4.80000</td>
<td>4.20000</td>
<td>3.20000</td>
<td>1.80000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

\[ \begin{array}{c|ccccccc} 
\text{y} & x & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\hline 
0 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.2 & 0.14266 & 0.13569 & 0.11545 & 0.08390 & 0.04412 & 0.0000 & 0.0000 \\
0.6 & 0.29949 & 0.28487 & 0.24240 & 0.17619 & 0.09266 & 0.0000 & 0.0000 \\
0.4 & 0.48604 & 0.46234 & 0.39350 & 0.28608 & 0.15049 & 0.0000 & 0.0000 \\
0.2 & 0.72072 & 0.68569 & 0.58382 & 0.42466 & 0.22346 & 0.0000 & 0.0000 \\
0 & 1.02657 & 0.97694 & 0.83240 & 0.60604 & 0.31916 & 0.0000 & 0.0000 \\
-0.2 & 1.43315 & 1.36454 & 1.16423 & 0.84911 & 0.44785 & 0.0000 & 0.0000 \\
-0.4 & 1.97877 & 1.88571 & 1.61286 & 1.18028 & 0.62440 & 0.0000 & 0.0000 \\
-0.6 & 2.71259 & 2.58891 & 2.22395 & 1.63793 & 0.87212 & 0.0000 & 0.0000 \\
-0.8 & 3.69566 & 3.53562 & 3.05943 & 2.28088 & 1.23304 & 0.0000 & 0.0000 \\
-1 & 5.00000 & 4.80000 & 4.20000 & 3.20000 & 1.80000 & 0.0000 & 0.0000 \\
\end{array} \]

**TABLE 6.34** Values of \( u(x,y) \times 10 \)

\( x = 0(0.2)1, \ y = -1(0.2)1. \ m = n = 10. \)
6.4 Laplace's Equation (3)

Solution of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)

\[
\begin{align*}
  u(x,1) &= J_0(x) - J_0(1) + 1 \\
  u(1,y) &= 2y^2 - 1 \quad (= T_2(y)) \\
  u(x,-1) &= x \quad (= T_1(x)) \\
  u(-1,y) &= y \quad (= T_1(y)).
\end{align*}
\] (6.41)

There is no symmetry in this problem, and no terms can be omitted from the numerical solution.

The boundary conditions are

\[
\begin{align*}
  \beta(y) &= T_2(y) , \quad \gamma(x) = T_1(x) , \quad \delta(y) = T_1(y) \\
  \alpha(x) &= J_0(x) - J_0(1) + 1.
\end{align*}
\] (6.42)

The coefficients in the expansion of \( \alpha(x) \) are obtained using the results of Chapters 3, 4 and the method of Chapter 5 then yields the approximate solution

\[
  u(x,y) = \sum_{i,j=0}^{20} a_{ij} T_i(x)T_j(y)
\] (6.43)

The coefficients \( \{a_{ij}\} \) are given in Table 6.41 and the values of \( u(x,y) \) for \( x,y = -1(0.2)1 \), are in Table 6.42.

Once again \( \left| \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right| \) is zero at all internal points, with some non-zero values on the boundary.

This example is included to illustrate the general way in which the boundary conditions can be chosen, provided they are expressible as sums of Chebyshev polynomials.
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**TABLE 6.4** Non-zero coefficients (x 10^6) in the approximate solution. m = n = 20.
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**TABLE 6.42** Values of $u(x,y)$, $x,y = -1(0.2)1$. $m = n = 20.$
6.5 Poisson's Equation

Solution of \[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2
\]
\[
\begin{align*}
u(x, \pm 1) &= 0 \\
u(\pm 1, y) &= 0.
\end{align*}
\] (6.51)

For the numerical solution
\[
u(x,y) = \sum_{i,j=0}^{m,n} a_{ij} T_i(x)T_j(y)
\] (6.52)

the data involves
\[
\alpha(x) = \beta(y) = \gamma(x) = \delta(y) = 0
\] (6.53)

and \(g(x,y) = -2\) (see Section 5.3).

With \(m = n = 20\), the non-zero coefficients in the solution (6.52) are given in Table 6.51, and the values of \(u(x,y), x,y = 0(0.1)1\) are given in Table 6.52.

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TABLE 6.51 Non-zero coefficients \((x \times 10^6)\) in approximate solution. \(m = n = 20\).
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**TABLE 6.52** Values of $u(x,y)$, $x, y = 0(0.1)1$, $m = n = 20$.  


6.6 General Problem (1)

Solution of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3u = 1 \) \综合性问题 (1)

\[
\begin{align*}
\text{u}(x, \pm 1) &= 0 \\
\text{u}(\pm 1, y) &= 0
\end{align*}
\]  

(6.61)

For the numerical solution

\[ u(x, y) = \sum_{i,j=0}^{m,n} a_{ij} T_i(x) T_j(y) \]  

(6.62)

the data involves

\[
\begin{align*}
\alpha(x) &= \beta(y) = \gamma(x) = \delta(y) = 0 \\
\lambda &= -3 \\
g(x, y) &= 1
\end{align*}
\]  

(6.63)

With \( m = n = 20 \), the non-zero coefficients in the solution (6.62) are given in Table 6.61, and the values of \( u(x, y) \), \( x, y = 0(0.1)1 \) are given in Table 6.62.

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**TABLE 6.61**  Non-zero coefficients \( (x \times 10^6) \) in approximate solution. \( m = n = 20 \).
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**TABLE 6.62** Values of $u(x,y)$, $x, y = 0(0.2)1$. $m = n = 20$.

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**TABLE 6.63** Values of $u''(x) + \frac{u''(y)}{y''}$, $x, y = 0(0.2)1$. $m = n = 20$. 
6.7 General Problem (2)

Solution of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3u = 1 \) \hspace{1cm} (6.71)

\[
\begin{align*}
&u(x, \pm\frac{1}{2}) = 0 \\
&u(\pm1, y) = 0 .
\end{align*}
\]

The change of variable \( y = \frac{1}{2}Y \), so that \( y = \pm\frac{1}{2} \) corresponds to \( Y = \pm1 \), transforms the problem into

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial Y^2} - 3u = 1 \\
&u(x, \pm1) = 0 \\
&u(\pm1, Y) = 0 .
\end{align*}
\] \hspace{1cm} (6.72)

Thus the solution of (6.72) can be found in the usual way, leading to the solution of (6.71) as

\[
\begin{align*}
u(x,y) = \sum_{i,j=0}^{m,n} a_{ij} T_i(x) T_j(2y) .
\end{align*}
\] \hspace{1cm} (6.73)

With \( m = n = 20 \), the non-zero values of \( a_{ij} \) are given in Table 6.71, and the values of \( u(x,y) \), \( x = 0(0.1)1.0 \), \( y = 0(0.1)0.5 \) in Table 6.72.

Using the solution (6.73) and the formulae (2.92) for evaluating derivatives, the values of \( \frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial y^2} - 3u \) can be calculated. These values are equal to 1 at almost all interior points but again there are variations on the boundary.
TABLE 6.7  Non-zero coefficients in the approximate solution

\[
  u(x,y) = \sum_{i=0}^{20} \sum_{j=0}^{20} a_{ij} T_i(x) T_j(y), \quad -1 \leq x \leq 1, \quad -\frac{1}{2} \leq y \leq \frac{1}{2}.
\]
TABLE 6.72  Values of $u(x,y)$ .  $m = n = 20$.
6.8 Laplace's Equation on a Triangle

For the triangular region bounded by \( x = 1, y = -1, x-y = 0 \), Laplace's equation can be solved in a simple manner if \( u \) is zero, or if the normal derivative \( \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \) is zero on the hypotenuse \( x-y = 0 \).

The problems can be solved on the complete square bounded by \( x = \pm 1, y = \pm 1 \), in the former case, with boundary conditions that ensure skew-symmetry about \( x-y = 0 \), and in the latter with boundary conditions giving symmetry about this line.

6.81 \( u = 0 \) on \( x-y = 0 \)

Solution of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)

\[
\begin{align*}
\text{u} &= 0 \text{ on } x-y = 0 \\
u(1,y) &= 2 - y + y^2 \\
u(x,-1) &= x + 1.
\end{align*}
\] (6.811)

The problem is extended to the square inside \( x = \pm 1, y = \pm 1 \) with the extra conditions

\[
\begin{align*}
u(x,1) &= -(2 - x - x^2) \\
u(-1,y) &= -(y + 1).
\end{align*}
\] (6.812)

The series solution is given in Table 6.811 (Appendix) for the choice \( m = n = 20 \), and values of \( u(x,y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) for \( x,y = -1(0.2)1 \), are given in Tables 6.812, 6.813, 6.814 respectively.
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**TABLE 6.812** Values of $u(x,y)$, $x,y = -1(0.2)1$. 


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**TABLE 6.813** Values of $\frac{\partial u}{\partial x}$, $x, y = -1(0.2)1$. 
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TABLE 6.814  Values of $\frac{\partial u}{\partial y}$, $x, y = -1(0.2)1$. 
\[ \frac{\partial u}{\partial n} = 0 \text{ on } x - y = 0 \]

Solution of \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ \frac{\partial u}{\partial n} = 0 \text{ on } x - y = 0 \] \hspace{1cm} (6.821)

\[ u(1,y) = y^2 + y \]
\[ u(x,-1) = 1 - x . \]

The problem is extended to the complete square inside \( x = \pm 1 \), \( y = \pm 1 \), with the extra conditions

\[ u(x,1) = x^2 + x \]
\[ u(-1,y) = 1 - y . \] \hspace{1cm} (6.822)

The series solution is given in Table 6.821 (Appendix) for the choice \( m = n = 20 \), and values of \( u(x,y) \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) for \( x,y = -1(0.2)1 \), are given in Tables 6.822, 6.823, 6.824 respectively.

Now \[ \frac{\partial u}{\partial n} = \hat{n} \cdot \text{grad } u \]

\[ = \frac{(-1 + j)}{\sqrt{2}} \cdot \left( \frac{1}{\sqrt{2}} \frac{\partial u}{\partial x} + j \frac{\partial u}{\partial y} \right) \]
\[ = \left( - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) / \sqrt{2} . \]

Thus from Tables 6.823, 6.824, it can be seen that the derivative condition \[ \frac{\partial u}{\partial n} = 0 \] is satisfied on \( x - y = 0 \).
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**TABLE 6.822** Values of $u(x,y)$, $x,y = -1(0.2)1$
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**TABLE 6.823** Values of \( \frac{\partial u}{\partial x} \), \( x, y = -1(0.2)1 \)
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**TABLE 6.824 Values of** \( \frac{\partial u}{\partial y} \), x,y = -1(0.2)1.
6.9 **Discontinuities at the Vertices**

The ideas described in section 5.7 are applied to two problems.

### 6.9.1 Problem 1

**Solution**

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
u(x,-1) &= 1 \\
u(x,1) = u(1,y) = u(-1,y) &= 0.
\end{align*}
\]

(6.9.11)

With the notation of section 5.7, \( \gamma(x) = 1 \), so that \( \gamma(x) \) is the solution of

\[
\begin{align*}
\frac{d^2 \gamma}{dx^2} + 1 &= 0 \\
\gamma(\pm 1) &= 0.
\end{align*}
\]

Hence \( \gamma(x) = \frac{1}{4}(1-x^2) \).

(6.9.12)

Now \( \tilde{u}(x,y) \) satisfying equation (5.75) has already been found as the solution of the problem in section 6.3. For that problem the required second derivatives \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2} \) are calculated using (2.92) and are given in Tables 6.9.1, 6.9.2. Thus the solution \( u(x,y) \) of (6.9.11) is known because

\[
\begin{align*}
u &= \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}.
\end{align*}
\]

As noted in previous discussions about the values of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) on the boundary of the region, it is seen that \( \frac{\partial^2 u}{\partial y^2} \) and \( -\frac{\partial^2 u}{\partial x^2} \) differ near the boundary. An empirical rule is that the solution of (6.9.11) is given by \( \frac{\partial^2 u}{\partial y^2} \) "near" \( x = \pm 1 \) and by \( -\frac{\partial^2 u}{\partial x^2} \) "near" \( y = 1 \), and by either expression in the interior of the region.
### TABLE 6.911

Values of $\frac{\partial^2 u}{\partial x^2} = -u$. $x = 0(0.2)1$, $y = -1(0.2)1$.

$u$ is the solution of the problem in Section 6.3.

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### TABLE 6.912

Values of $\frac{\partial^2 u}{\partial y^2} = u$. $x = 0(0.2)1$, $y = -1(0.2)1$.

$u$ is the solution of the problem in Section 6.3.

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Since the Laplace operator is linear for the Dirichlet problem, the solution of more general problems can be considered as the sum of solutions of simpler problems of the above type. The following example illustrates this.

Consider the problem for which

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
u(x, 1) &= 1 \\
u(1, y) &= 2 + y \\
u(x, -1) &= 1 - x \\
u(-1, y) &= 1 + y
\end{align*}
\]  

(6.921)

with discontinuities at each vertex.

Let \( u = u_1 + u_2 \) where

\[
\begin{align*}
\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} &= 0 \\
u_1(x, 1) &= 1 \\
u_1(1, y) &= 0 \\
u_1(x, -1) &= 1 - x \\
u_1(-1, y) &= 0 \\
u_2(x, 1) &= 0 \\
u_2(1, y) &= 2 + y \\
u_2(x, -1) &= 0 \\
u_2(-1, y) &= 1 + y
\end{align*}
\]  

(6.922)

The aim is to find \( \tilde{\alpha}(x), \tilde{\beta}(x), \tilde{\gamma}(x), \tilde{\delta}(x) \) such that

\[
\begin{align*}
\frac{d^2 \tilde{\alpha}}{dx^2} + 1 &= 0 \\
\tilde{\alpha}(\pm 1) &= 0 \\
\frac{d^2 \tilde{\beta}}{dy^2} + 2 + y &= 0 \\
\tilde{\beta}(\pm 1) &= 0 \\
\frac{d^2 \tilde{\gamma}}{dx^2} + 1 - x &= 0 \\
\tilde{\gamma}(\pm 1) &= 0 \\
\frac{d^2 \tilde{\delta}}{dy^2} + 1 + y &= 0 \\
\tilde{\delta}(\pm 1) &= 0
\end{align*}
\]  

(6.923)
Then it is easily seen that
\[ \begin{align*}
\bar{a}(x) &= \gamma(1-x^2) \\
\bar{b}(y) &= (1-y^2) + \frac{1}{6}(y-y^3) \\
\bar{\gamma}(x) &= \gamma(1-x^2) - \frac{1}{6}(x-x^3) \\
\bar{\delta}(y) &= \gamma(1-y^2) + \frac{1}{6}(y-y^3).
\end{align*} \]

(6.924)

Then \( \bar{u}_1(x,y) \) and \( \bar{u}_2(x,y) \) are found as solutions of
\[ \begin{align*}
\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} &= 0 \\
\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} &= 0 \\
\bar{u}_1(x,1) &= \bar{a}(x) \\
\bar{u}_2(x,1) &= 0 \\
\bar{u}_1(1,y) &= 0 \\
\bar{u}_2(1,y) &= \bar{b}(y) \\
\bar{u}_1(x,-1) &= \bar{\gamma}(x) \\
\bar{u}_2(x,-1) &= 0 \\
\bar{u}_1(-1,y) &= 0 \\
\bar{u}_2(-1,y) &= \bar{\delta}(y)
\end{align*} \]

(6.925)

and \( u_1 = \frac{\partial^2 u_1}{\partial y^2} = -\frac{\partial^2 u_1}{\partial x^2} \), \( u_2 = \frac{\partial^2 u_2}{\partial x^2} = -\frac{\partial^2 u_2}{\partial y^2} \).

The numerical values associated with the solution when \( m = n = 20 \) are given in Tables 6.921 to 6.924.

The appropriate choice of second derivative near the boundaries is fairly obvious, and finally the solution \( u \) of (6.922) can be found as the sum of \( u_1 \) and \( u_2 \).

The series solutions for \( \bar{u}_1 \) and \( \bar{u}_2 \) are given in Tables 6.925, 6.926 in the Appendix.
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**TABLE 6.921** Values of \( \frac{\partial^2 u_1}{\partial x^2} = -u_1 \) at \( x, y = -1(0.2)1 \).
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**TABLE 6.922** Values of $\frac{\partial^2 u_1}{\partial y^2} = u_1$, $x, y = -1(0.2)1$.  


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<td>.80826</td>
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<td>1.28432</td>
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**TABLE 6.923** Values of \( \frac{\partial^2 u_2}{\partial x^2} = u_2 \), \( x, y = -1(0.2)1 \).
<table>
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**TABLE 6.924** Values of \( \frac{\partial^2 u_2}{\partial y^2} = -u_2 \), \( x, y = -1(0.2)1 \).
6.10 A Neumann Problem

To illustrate how the present method can be applied when derivative conditions are given on the boundary, the following example is chosen. It appears in the text of Smith (1975), chapter 5.

Consider the problem
\[
\frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial y^2} = -16
\]
inside the square \( x = \pm 1, y = \pm 1 \), with \( u = 0 \) on \( x = 1 \), \( \frac{\partial u}{\partial y} = -u \) on \( y = 1 \), and the solution is to be symmetric with respect to the axes.

Using the theory of section 5.5, the problem can be solved if it is rewritten in the form
\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial y^2} &= -16 \\
\frac{\partial u}{\partial y} + u &= 0 \text{ on } y = 1, \\
\frac{\partial u}{\partial y} + u &= 0 \text{ on } y = -1, \\
u(\pm 1, y) &= 0.
\end{align*}
\] (6.10.1)

The solution
\[
u(x,y) = \sum_{i,j=0}^{20} a_{ij} T_i(x) T_j(y)
\] (6.10.2)
is given in Table 6.10.1, and the values of \( u(x,y), \frac{\partial u}{\partial y}, x,y = 0(0.1)1 \), are given in Tables 6.10.2, 6.10.3, respectively. It can be seen that the boundary condition \( \frac{\partial u}{\partial y} + u = 0 \), on \( y = 1 \), is satisfied, as well as the condition \( u = 0 \) on \( x = 1 \). The result \( \frac{\partial u}{\partial y} = 0 \), on \( y = 0 \), is in agreement with the symmetry about the \( x \)-axis.

When \( \frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial y^2} + 16 \) is calculated, the values are (effectively) zero at all interior points, with the usual variations on the boundaries.
### TABLE 6.10.1  Non-zero coefficients in the solution. $m = n = 20.$

<table>
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<th>j</th>
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<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
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</tr>
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<td>0.6</td>
<td>0.7</td>
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<tr>
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<td>-----</td>
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**TABLE 6.10.2** Values of \(u(x,y)\). \(x,y = 0(0.1)1\) \(m = n = 20\).
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</thead>
<tbody>
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</table>

**TABLE 6.10.3** Values of $\frac{\partial u}{\partial y}$. $x, y = 0(0.1)1$. $m = n = 20.$
CHAPTER 7

SOLUTION OF THE ALGEBRAIC EQUATIONS

7.1 Introduction

The relatively small number of coefficients \( \{a_{ij}\} \) required in the present Chebyshev solution, compared with the larger number of grid points required for reasonable accuracy in finite difference methods, leads to the possibility of solving the algebraic equations by direct methods, such as Gaussian elimination. In fact in the original testing of the present method of solution of the Dirichlet problem for Laplace's equation on the square, with data corresponding to known solutions, \( R(z^n) \) or \( I(z^n) \), \( n = 2, 3, \ldots \), Gaussian elimination was used.

However, the analogy with the molecule of the finite difference method, together with a desire to take advantage of the sparse nature of the coefficient matrix, led to a consideration of iterative methods. Alternating direction implicit (ADI) methods, and relaxation methods, were considered. In both cases the "boundary equations" caused problems, but it was possible to establish suitable relaxation methods in a fairly simple manner, and these were used in all the major calculations. These relaxation methods are discussed in the remainder of this chapter.

7.2 The Relaxation Method for the Dirichlet Problem

The Dirichlet problem of section 5.2 is suitable for describing the general approach to the relaxation method. Consider \( \mu > 0 \).

With the algebraic equations ordered precisely as in equation (5.28) (Table 5.21 is a special case) the problem can be stated as

\[
A \mathbf{x} = \mathbf{b}
\]  

(7.21)
where

$$x^T = (a_{00}, a_{10}, \ldots, a_{m0}, a_{01}, a_{11}, \ldots, a_{m1}, a_{0n}, a_{1n}, \ldots, a_{mn}),$$

a vector with \((m+1) \times (n+1)\) components, and where the coefficient matrix \(A\) and the vector \(b\) have the form indicated in Table 5.21, with the modification here that each of the equations in (5.28) is divided by its pivot (diagonal) element, giving \(A\) with unit diagonal elements.

In those rows of (5.28) associated with boundary conditions, the pivot coefficients are \(\frac{1}{2}\) or 1, and in the other rows, each pivot is an element \(-g(i,j)\) such that

$$g(i,j) = 2i(j+1)(i-1) + 2\mu ij(j+1)(j-1), \quad \begin{cases} i = 2, \ldots, m \\ j = 2, \ldots, n \\ \mu > 0 \end{cases}$$

so that no pivot element is zero, and the division is valid.

In the first \(m + 1\) rows of \(A\) the sum of the off-diagonal elements (which are all in the upper triangle) is \(2[\frac{m}{2}]\). In a further \(n\) rows of the same type (where the original diagonal element was \([\frac{1}{2}]\)) the sum is \(2[\frac{1}{2}(m-1)]\). In rows \(m + 2\) to \(2m + 2\), the sum of the off-diagonal elements is \([\frac{1}{2}n]\), and in a further \(n\) rows of this type (where the original diagonal element was 1) the sum is \([\frac{1}{2}(m-1)]\), with \([k]\) being the integer part of \(k\). In all the remaining rows, the sum of the moduli of the off-diagonal elements is less than, or equal to 1. Thus although some rows of \(A\) exhibit diagonal dominance, there is an extreme lack of this property in a number of other rows, and neither a simple Jacobi nor Gauss-Seidel method is convergent. However a successive relaxation (SR) method, with suitable parameter \(\omega\), can be established in a simple way.

Write \(A = I - L - U\)

where \(L\) is strictly lower triangular, and

\(U\) is strictly upper triangular.
Then the successive relaxation method can be stated as

\[ x^{(l+1)} = x^{(l)} + \omega [x^{(l)} + Lx^{(l)} + Ux^{(l)} - x^{(l)}] \quad l=0,1,... \]

or

\[ x^{(l+1)} = [(1-\omega)I + \omega U]x^{(l)} + \omega Lx^{(l+1)} + \omega b. \]

The iteration matrix, \( M_\omega \), of the method is given by

\[ M_\omega = (I-\omega L)^{-1}[(1-\omega)I + \omega U] \]

and the method converges to give a solution of (7.21) if all the eigenvalues of \( M_\omega \) have moduli less than 1.

In Chapter 5 the equation (5.28) are written to correspond to a column by column order of calculation of \( \{a_{ij}\} \) as exhibited in the matrix (5.19). The implementation of the solution by the SR method merely involves the calculations

\[
\begin{align*}
a_{ij}^{(l+1)} &= a_{ij}^{(l)} + \omega \left( a_{ij}^{(l)} + \gamma_i - 2 \left( a_{i2}^{(l)} + a_{i4}^{(l)} + \ldots \right) - a_{i0}^{(l)} \right) & i=0,...,m \\
a_{ij}^{(l+1)} &= a_{ij}^{(l)} + \omega \left( b_j - \gamma_j - 2 \left( a_{j2}^{(l)} + a_{j4}^{(l)} + \ldots \right) - a_{j0}^{(l)} \right) & j=2,...,n \end{align*}
\]

The actual superscripting of the coefficients is unnecessary if the result of the calculation on the right hand side, overwrites the value of the variable on the left hand side. Successive iterates are written in the matrix (5.19) of order \((m+1)\times(n+1)\). The coefficient matrix \( A \), and the iteration matrix \( M \), of order \( (m+1)(n+1) \times (m+1)(n+1) \) are never used explicitly in the solution, but are necessary in the theory in order to discuss the convergence.
7.3 General Remarks on Relaxation Methods

Because of the form of the matrix $A$, the theory of Varga and Young for irreducible, cyclic matrices is not applicable, and a suitable value of the relaxation factor $\omega$ has to be found empirically, as indeed is often the case for better behaved matrices. The method of solution, is being called successive relaxation (SR) here and not successive over-relaxation (SOR) because it is found that convergence occurs for values of $\omega$ between 0 and 1, rather than between 1 and 2 as in standard theory.

In order to put some of the later discussion in perspective, a few aspects of standard SOR theory are discussed.

The spectral radius, $\rho(M)$, of a matrix $M$ is defined as the magnitude of the largest eigenvalue of $M$.

For an iterative method to converge it is necessary that the spectral radius of the iteration matrix $M$ should be less than unity. The iteration matrix $M$ is found by writing the iterative method in the form

$$x^{(l+1)} = Mx^{(l)} + g.$$

Thus in solving

$$Ax = b$$

with $A = I - L - U$ the Jacobi method is

$$x^{(l+1)} = b + (L+U)x^{(l)}, \quad l = 0,1,2,...$$

with iteration matrix $L + U (= B)$.

The Gauss-Seidel iteration is

$$x^{(l+1)} = b + Lx^{(l+1)} + Ux^{(l)}, \quad l = 0,1,2,...$$

with iteration matrix $(I - L)^{-1}U$. 
The SOR (and SR) iteration has already been stated in 7.2 and has iteration matrix
\[ M_\omega = (I - \omega L)^{-1} \{(1 - \omega)I + \omega U\} . \]

Thus the Gauss-Seidel method is the special case of SOR with \( \omega = 1 \).

As section 7.5 will show the matrices met in the present work have few of the properties mentioned in the literature, and empirical results concerning convergence have had to be obtained. The important cases in the literature deal with positive, cyclic, irreducible matrices with diagonally dominant elements. Such matrices occur when elliptic partial differential equations are solved by finite difference or finite element methods.

For such matrices the standard theory states that
\[ (\lambda_\omega + \omega - 1)^2 = \lambda_\omega^2 \mu^2 \]
where \( \lambda_\omega \) is an eigenvalue of \( M_\omega \) and \( \mu \) is an eigenvalue of \( B \).

In particular \( \lambda_1 = \mu^2 \) with \( \omega = 1 \), and the spectral radius of the Gauss-Seidel method is the square of the spectral radius of the Jacobi method.

The aim of the relaxation method is to find a value of \( \omega \), called the optimum value, written \( \omega_{\text{opt}} \), so that \( \rho(M_\omega) \) is a minimum.

For the matrices defined above,
\[ \omega_{\text{opt}} = \frac{2}{1 + (1 - \mu^2)^{\frac{1}{2}}} \]
and for real \( \mu \), it is seen that
\[ 1 < \omega_{\text{opt}} < 2 . \]

The general shape of the graph of the spectral radius \( \rho(M_\omega) \), against \( \omega \), is shown in
For \( \omega < \omega_{opt} \), the dominant eigenvalue is real, and for \( \omega > \omega_{opt} \) the dominant eigenvalue is complex, of modulus \( \omega - 1 \). The graph shows that in the vicinity of \( \omega_{opt} \), it is better to over-estimate, than under-estimate the value of \( \omega \).

### 7.4 Estimation of the Spectral Radius

Assuming that the dominant eigenvalue of a matrix \( M \) is real, and not repeated, let it be \( \lambda_1 \), where the full set of eigenvalues is \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[ |\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n| . \]

Let the corresponding eigenvectors be the linearly independent set \( \chi_1, \chi_2, \ldots, \chi_n \).

Then any arbitrary vector \( \chi \) can be written

\[ \chi = c_1 \chi_1 + c_2 \chi_2 + \ldots + c_n \chi_n \]

and \( M^l \chi = c_1 \lambda_1^l \chi_1 + c_2 \lambda_2^l \chi_2 + \ldots + c_n \lambda_n^l \chi_n \)

\[ + c_1 \lambda_1^l \chi_1 \text{ as } l \to \infty . \]

Thus the dominant eigenvalue \( \lambda_1 \) can be found as the limiting ratio of a corresponding set of components in \( \chi^{(l+1)} \) and \( \chi^{(l)} \) where

\[ \chi^{(l+1)} = M \chi^{(l)} , \quad \chi^{(0)} = \chi , \quad l = 0,1,\ldots . \]  \(7.41\)

The spectral radius \( \rho(M) \) can be found as the absolute value of \( \lambda_1 \), or as the limiting value of any ratio \( ||\chi^{(l+1)}|| / ||\chi^{(l)}|| \), of vector norms.
When the largest eigenvalue $\lambda_1$ is complex then its complex conjugate $\bar{\lambda}_1$ is also an eigenvalue, and the corresponding eigenvectors can be written $y_1$ and $\bar{y}_1$. Then

$$z^{(l)} + c_1 \lambda_1 y_1 + \frac{1}{\lambda_1} y_1.$$ 

If $\lambda_1, \bar{\lambda}_1$ are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$, then

$$z^{(l+2)} + bz^{(l+1)} + cz^{(l)} = 0, \text{ as } l \to \infty, \quad (7.42)$$

and $b, c$ can be found by considering any pair of sets of corresponding components of the vectors $z^{(l)}, z^{(l+1)}, z^{(l+2)}$.

The magnitude of $\lambda_1$ is $\sqrt{c}$ and can be calculated without explicitly finding $\lambda_1$ and $\bar{\lambda}_1$.

If $M$ is the iteration matrix of the solution process

$$x^{(l+1)} = Mx^{(l)} + z$$

then

$$x^{(l+1)} - x^{(l)} = M(x^{(l)} - x^{(l-1)})$$
or

$$z^{(l+1)} = Mz^{(l)}$$

where

$$z^{(l+1)} = x^{(l+1)} - x^{(l)}$$

and the sequence of vectors $\{z^{(l)}\}$ can be used in the preceding theory to find the dominant eigenvalue of $M$.

When the above ideas are implemented during the solution of a set of simultaneous equations, a parallel set of calculations can be made to see whether it is appropriate to use (7.41) or (7.42).

In the present context, $\left[ a_{ij}^{(l+1)} - a_{ij}^{(l)} \right]$, for suitable $i, j$ can be used to find the components of $z^{(l+1)}$. In practice these methods often give reasonable estimates only, because once the values of elements $\left[ a_{ij}^{(l)} \right]$ are near their limiting values, any rounding errors cause considerable relative error in the calculations of differences of nearly equal quantities. Nevertheless the estimates are useful in locating $\omega_{\text{opt}}$. 
7.5 Successive Relaxation for the Dirichlet Problem (ctd.)

The discussion of the convergence of the successive relaxation method for the solution of (5.28) is now continued from section 7.2.

When \( m=n=3, \mu=1 \), the coefficient matrix \( A \) can be written as

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}
\]

Now after a suitable permutation of rows and columns, corresponding to a re-ordering of the equation (5.28) so that solutions \( \{a_{ij}\} \) are obtained in the order corresponding to the elements of the sets 1, 2, 3, 4 of section 5.1 (Table 5.11) it is seen that the above matrix is similar to

\[
\begin{array}{ccc}
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
2 & 2 & 2 \\
\end{array}
\]
\[
\begin{bmatrix}
A_{00} & A_{01} \\
A_{01} & A_{10}
\end{bmatrix}
\]

namely similar to the direct sum

\[
A_{00} \otimes A_{01} \otimes A_{10} \otimes A_{11}
\]

where

\[
A_{00} = \begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & \frac{-3}{8} & \frac{-3}{8} & 1
\end{bmatrix}, \quad A_{01} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & \frac{-2}{11} & \frac{-6}{11} & 1
\end{bmatrix},
\]

\[
A_{10} = \begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1 \\
0 & \frac{-6}{11} & \frac{-2}{11} & 1
\end{bmatrix}, \quad A_{11} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & \frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}.
\]

Omitting the first row and column of \(A_{00}\), the resulting matrix \(\tilde{A}_{00}\), has Jacobi iteration matrix

\[
\begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 2 \\
\frac{-3}{8} & \frac{-3}{8} & 0
\end{bmatrix},
\]

cyclic of order 2, although not positive. Nevertheless, a simple diagonal similarity transformation shows that this iteration matrix is similar to

\[
\begin{bmatrix}
0 & 0 & \sqrt{3/2} \\
1 & 0 & 0 \\
\sqrt{3/2} & \sqrt{3/2} & 0
\end{bmatrix}.
\]

The eigenvalues are 0, \(\pm \sqrt{3/2}\) i. The eigenvalues of the corresponding Gauss-Seidel iteration matrix are 0, 0, \(-\frac{3}{2}\), as standard theory would predict.

Thus it is suggested that the formula for \(\omega_{\text{opt}}\) should be tried, giving

\[
\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - (-3/2)}} = 0.774852.
\]

The corresponding spectral radius should be \(|1-\omega| = 0.225148\).
It is a simple matter to examine the SR iteration matrix, i.e.

\[
\begin{bmatrix}
1 & . & . \\
. & 1 & . \\
-\frac{3}{8} & \omega & -\frac{3}{8} \\
\end{bmatrix}^{-1}
\begin{bmatrix}
1-\omega & 2\omega \\
. & 1-\omega & 2\omega \\
. & . & 1-\omega \\
\end{bmatrix}
\]

for various values of \( \omega \) to show that \( \omega_{\text{opt}} \) is in fact equal to 0.774852 as predicted above.

Corresponding to the other matrices \( A_{01}, A_{10}, A_{11} \) in the direct sum above, matrices \( \bar{A}_{01}, \bar{A}_{10}, \bar{A}_{11} \) can be formed as

\[
\bar{A}_{01} = \begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 1 \\
-\frac{6}{11} & -\frac{2}{11} & 0
\end{bmatrix}, \quad \bar{A}_{10} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 2 \\
-\frac{2}{11} & -\frac{6}{11} & 0
\end{bmatrix}, \quad \bar{A}_{11} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-\frac{1}{3} & -\frac{1}{3} & 0
\end{bmatrix}
\]

For these matrices:

<table>
<thead>
<tr>
<th></th>
<th>( \bar{A}<em>{01} ) and ( \bar{A}</em>{10} )</th>
<th>( \bar{A}_{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues of Jacobi iteration matrix</td>
<td>0, ( \pm \sqrt{14/11} ) i</td>
<td>0, ( \pm \sqrt{2/3} ) i</td>
</tr>
<tr>
<td>Eigenvalues of Gauss-Seidel iteration matrix</td>
<td>0, 0, -( 14/11 )</td>
<td>0, 0, -2/3</td>
</tr>
<tr>
<td>( \omega_{\text{opt}} )</td>
<td>0.797589</td>
<td>0.872983</td>
</tr>
<tr>
<td>( \rho \left( M_{\omega_{\text{opt}}} \right) )</td>
<td>0.202411</td>
<td>0.127017</td>
</tr>
</tbody>
</table>

Thus, corresponding to the submatrix \( A_{00} \), the optimum spectral radius of \( A \) is 0.225148, occurring when \( \omega = 0.774852 \). The submatrix \( A_{00} \) arises in calculating those coefficients \( \{a_{ij}\} \) in (5.28) for which both \( i, j \) are even, i.e. \( \{a_{ij}\} \) belonging to Set 1 (Table 5.11).

Unfortunately, when \( m,n > 3 \), the matrices like \( A_{00}, A_{01}, A_{10}, A_{11} \), no longer lead to irreducible, weakly cyclic Jacobi matrices, and there is no hope of using standard theory, even in the above manner.
Nevertheless, for the Dirichlet problem, it is always possible to find a direct sum of the above form $A_{00} \oplus A_{01} \oplus A_{10} \oplus A_{11}$, and a number of empirical results can be established. In all of the following, $m$ and $n$ are taken to be equal and odd. Results for either of $m$ and $n$ being even, or not equal to each other, can be found as subsets of results which will be quoted.

As a further specific example where it is still simple to find the matrices $A_{00}, A_{01}, A_{10}, A_{11}$, these are exhibited for $m = n = 5$.


\[ A_{10} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ -6/11 & -2/11 & 1 -1/11 -2/11 \\ -2/3 & -1/15 & 1 -2/9 \\ 0 & 1 & 1 \\ -5/23 & -10/23 & 1 -5/23 \\ -5/13 & -3/13 & 1 \end{bmatrix} \]

\[ A_{11} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1/3 & -1/3 & 1 -1/6 -1/6 \\ -1/2 & -3/20 & 1 -1/4 \\ 0 & 1 & 1 \\ -3/20 & -1/2 & 1 -1/4 \\ -3/10 & -3/10 & 1 \end{bmatrix} \]
TABLE 7.51 Eigenvalues of the Jacobi and Gauss-Seidel matrices when \( m = n = 5 \).
For $\tilde{A}_{00}$, $\tilde{A}_{11}$ there is a relation $\lambda_1 = |\mu|^2$ between some eigenvalues $\lambda_1$ of the Gauss-Seidel iteration matrix and some eigenvalues $\mu$ of the Jacobi iteration matrix. However this does not happen with all eigenvalues and certainly not with the dominant eigenvalues, and thus a result true for classical theory is not true here. It is seen that $A_{00}$ again leads to the largest eigenvalue, and in the following this is assumed to be so for all values of $m$ and $n$, i.e. the search for the spectral radius of $A$ is based on the assumption that the required value is the spectral radius of the submatrix $A_{00}$.

The numerical methods for finding eigenvalues have included the LR algorithm, and a simplified version of the LZ algorithm of Kaufman, and of course, the power method for dominant eigenvalues. The LR and LZ algorithms are useful for finding the complete spectrum of eigenvalues, including complex eigenvalues, but for large $m$, $n$ there is trouble because of the multiplicity of eigenvalues of equal magnitude, and unless elaborate "shifting" is performed so that these equal eigenvalues are found no more than two at a time, the methods often fail. For the times when the dominant eigenvalue is real, the power method is satisfactory, although sometimes slow to converge.
7.6 **Values of the Spectral Radii**

The spectral radii of various iteration matrices have been calculated for values of \( m, n, \omega \), where the number of iterations is not prohibitive, using the power method and the LR, LZ algorithms. Results are shown in Table 7.61.

When the spectral radius is obtained using the LR or LZ algorithms, observation of the complete spectrum of eigenvalues shows the occurrence of many sets of roots of equal magnitude, and this phenomenon leads to difficulty in obtaining reliable results in some cases. Table 7.61 contains only those results verified using two different methods, often the LZ algorithm, and inverse iteration.

The tabulated results are sufficient to indicate that, with increasing \( m, n \), there appears to be a limiting value of \( \omega_{\text{opt}} \), slightly greater than 0.75. For values of \( \omega > \omega_{\text{opt}} \), the dominant eigenvalue is real and increases quite sharply in magnitude, and for values of \( \omega < \omega_{\text{opt}} \), there are complex dominant eigenvalues, with magnitude increasing less rapidly than when \( \omega > \omega_{\text{opt}} \). These features can be seen from the graphs in Tables 7.62.

Curves of \( \rho(M_{\omega}) \) against \( \omega \) are plotted for \( m(=n) = 3, 5, 7, \) and for values \( m = 3, 5, 7, 11, 13, 15, \) estimated points corresponding to \( \omega_{\text{opt}} \) are plotted. As \( m \) becomes larger, the value of \( \omega_{\text{opt}} \) seems to increase only slowly, with the rate of increase of the corresponding spectral radius becoming less.
<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$m$ (=$n$)</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
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<tr>
<td>0.1</td>
<td>0.9</td>
<td>0.9256</td>
<td>0.9323</td>
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<tr>
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<td>0.8458</td>
<td>0.8585</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>0.7605</td>
<td>0.7780</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.6697</td>
<td>0.6906</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5731</td>
<td>0.5959</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.4707</td>
<td>0.4936</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.3</td>
<td>0.3623</td>
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<td>0.3511</td>
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</tr>
<tr>
<td>0.72</td>
<td>0.28</td>
<td>0.3399</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.73</td>
<td>0.27</td>
<td>0.3286</td>
<td>0.4912</td>
<td>0.5980</td>
<td>0.6757</td>
<td>0.7331</td>
<td></td>
<td></td>
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<tr>
<td>0.74</td>
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<td>0.3172</td>
<td>0.4848</td>
<td>0.5926</td>
<td>0.6713</td>
<td>0.7294</td>
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<tr>
<td>0.75</td>
<td>0.25</td>
<td>0.3058</td>
<td>0.3328</td>
<td>0.4783</td>
<td>0.5873</td>
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<tr>
<td>0.76</td>
<td>0.24</td>
<td>0.4220</td>
<td>0.5278</td>
<td>0.5564</td>
<td>0.5818</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.77</td>
<td>0.23</td>
<td>0.5383</td>
<td>0.6411</td>
<td>0.6728</td>
<td>0.6822</td>
<td>0.6849</td>
<td>0.7181</td>
<td></td>
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<td>0.78</td>
<td>0.3225</td>
<td>0.6306</td>
<td>0.7397</td>
<td>0.7758</td>
<td>0.7872</td>
<td>0.7906</td>
<td>0.7916</td>
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<td>0.4081</td>
<td>0.7141</td>
<td>0.8319</td>
<td>0.8729</td>
<td>0.8864</td>
<td>0.8907</td>
<td>0.8920</td>
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</tr>
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<td>0.8</td>
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<td>0.9828</td>
<td>0.9881</td>
<td>0.9898</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1.0051</td>
<td>1.5232</td>
<td>1.7828</td>
<td>1.9037</td>
<td>1.9576</td>
<td>1.9810</td>
<td>1.9910</td>
<td></td>
</tr>
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<td>2.28125</td>
<td>2.7296</td>
<td>2.9728</td>
<td>3.1002</td>
<td>3.1654</td>
<td>3.1983</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 7.61** Values of the spectral radius $\rho(M_\omega)$ for various values of the relaxation parameter $\omega$, and the degrees $m,n$ of the series solution.
TABLE 7.6 \( p(M_\omega) \) plotted against \( \omega \) for various values of \( m(=n) \).

Points corresponding to \( \omega_{opt} \) are marked with dots.
7.7 Convergence

An operating strategy has always been to choose a value of \(\omega\) approximately equal to 0.75, and this has produced satisfactory convergence.

The practical test for convergence of the present SR method has been that the relative change, from one iteration to the next, in the value of every series coefficient \(a_{ij}\), should be less than \(10^{-6}\) in magnitude. However, the behaviour of Chebyshev series expansions leads to the belief that the coefficients \(\{a_{ij}\}\) will decrease rapidly, for increasing \(i, j\), and a supplementary test of convergence has been that a coefficient is considered to be sufficiently accurate if its magnitude is less than \(10^{-9}\) times the magnitude of the largest coefficient, provided a chosen minimum number of iterations (usually 10) has occurred.

With the above criteria and a choice of \(\omega = 0.75\) in all cases, the number of iterations to obtain the given solutions to the Dirichlet problems of Chapter 6 are shown in Table 7.71.

As the previous discussion would suggest, values of \(\omega\) greater than 0.75 lead to a slower convergence rate, and ultimately to divergence.

Actually the analysis of the earlier part of this Chapter is concerned with equations of the form

\[
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} = g(x, y)
\]

with \(\mu = 1\). However the following table of results includes the more general case

\[
\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \lambda u = g(x, y)
\]

with \(\mu \neq 1\), \(\lambda \neq 0\), (sections 6.6, 6.7) and at least for \(m = n = 20\), the choice of \(\omega = 0.75\) has still led to convergence in a satisfactory
<table>
<thead>
<tr>
<th>Problem</th>
<th>m = n</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2</td>
<td>6</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>59</td>
</tr>
<tr>
<td>6.3</td>
<td>20</td>
<td>98</td>
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<td>6.4</td>
<td>20</td>
<td>97</td>
</tr>
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<td>6.5</td>
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<td>96</td>
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<td>6.6</td>
<td>20</td>
<td>99</td>
</tr>
<tr>
<td>6.7</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>6.81</td>
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<td>91</td>
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<td>6.82</td>
<td>20</td>
<td>97</td>
</tr>
<tr>
<td>6.91</td>
<td>20</td>
<td>95</td>
</tr>
<tr>
<td>6.92</td>
<td>20</td>
<td>98 &amp; 96</td>
</tr>
</tbody>
</table>

**TABLE 7.7** Number of iterations for the coefficients \( \{a_{ij}\} \) to satisfy the given convergence criteria.
number of iterations. Actually, when \( m \) and \( n \) are reduced in value, it is found necessary to choose \( \omega \approx 0.6 \) in the problems of sections 6.6, 6.7. For large \( m, n \) it seems that a choice of \( \omega = 0.75 \) is satisfactory in all cases, and a search for better values of \( \omega \) has never significantly reduced the number of iterations. The convergence criteria themselves are possibly too severe but have been chosen so that the solutions of the partial differential equations can be assumed correct with great confidence. For most examples, an examination of the coefficients after each iteration, shows that the leading coefficients in the solution, are correct after \( \sim 1/3 \) of the final number of iterations. The further iterations are needed to make the later, smaller coefficients satisfy a test which may be too severe.

The relaxation parameter for the Neumann problem differs significantly from that of the Dirichlet problem, and it is found that values near \( \omega = 0.4 \) produce satisfactory convergence. In the problem of section 6.10, a value of \( \omega = 0.39 \) leads to convergence in 160 steps.
CONCLUSION

The first part of this thesis has presented a systematic method for solving the initial and boundary value problems associated with an ordinary linear second order differential equation with quadratic coefficients. Computer implementation has led to results of at least eight figure accuracy, both in the coefficients of the series, and the values of the solution, making the method very attractive.

An important point, stated a number of times, is that once a Chebyshev series is known, it can be used directly to calculate the value of the solution, and the values of its derivatives of any order at any point on the solution interval (section 2.7). Also results of Clenshaw (section 2.6) can be used in a simple manner to obtain the Chebyshev series representing integrals and derivatives of the solution. The feature of having an "almost analytic" solution, is always a powerful argument for the Chebyshev method of solution.

As stated in the Introduction there are other techniques for finding Chebyshev series solutions of differential equations. However, this thesis has shown how a recurrence relation can be found directly for the coefficients in the solution, whenever the equation is linear with quadratic coefficients. Then from the recurrence relation, the $n + 1$ coefficients of the solution can be found by the solution of a set of, at most, 4 simultaneous equations, following backward substitution and the use of values of the solution and its derivatives at the data points.

The main emphasis has been on the solution of elliptic partial differential equations. It is believed that the technique described in Chapter 5 has not been used previously. That it leads to successful solutions of equations has been demonstrated in the problems in Chapter 6, and the simple application of an iterative method for solving the algebraic equations as described in Chapter 7.
It is hoped to extend the work in many directions, for example, to elliptic equations of more complicated type, to different shaped boundaries, to different types of discontinuities, including discontinuities within the solution region, and to eigenvalue problems on the more complicated regions. For the simple case of the square region the eigenvalue problem has in fact been treated successfully to give the expected results.

As with the single variable solutions, the double variable solutions have enabled the calculation of function values and derivatives at any point in the solution region. The formulae of sections 2.7 and 2.9 have been fundamental in such calculations.

All the numerical calculations reported in this thesis have been performed on the UNIVAC 1106 Computer at the University of Wollongong, using programmes written in ASCII FORTRAN. They have been performed in single precision, except for those of section 4.10, which were performed in double precision to avoid possible numerical errors in the use of the solution of one problem as the data for the next. An important feature of the programmes has been their generality. It has been necessary to vary the data cards only, in order to change problems. Of course, when a particular problem is to be studied extensively it is a simple manner to omit unnecessary parts of a general programme.
References


CLENSHAW, C.W. and PICKEN, Susan M. (1966). Chebyshev series for Bessel Functions of Fractional Order. Mathematical Tables, Volume 8, National Physical Laboratory.


NAG (Numerical Algorithms Group). Programs DO2 and DO2 AFF. NAG subroutine library.
APPENDIX
Appendix. Section 4.10

Tables of $J_n(x)$, $\frac{\partial J_n(x)}{\partial x}$, $\frac{\partial J_n(x)}{\partial n}$

for $n = 0(0.5)10$, 
$x = 0.2(0.2)10$. 
<table>
<thead>
<tr>
<th>X</th>
<th>JNX</th>
<th>DJN/DX</th>
<th>DJN/DN</th>
</tr>
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<td>.2</td>
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</tr>
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**TABLE 6.8.11** Coefficients \( a_{ij} \) in the approximate solution

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TABLE 6.82  Coefficients \( \{a_{ij}\} \) in the approximate solution

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u(x,y) = \sum_{i=0}^{20} \sum_{j=0}^{10} a_{ij} T_i(x) T_j(y), \quad -1 \leq x, y \leq 1.
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**TABLE 6.925** Coefficients \( \{a_{ij}\} \) in the approximate solution

\[
\tilde{u}_1(x,y) = \sum_{i=0}^{20} \sum_{j=0}^{20} a_{ij} T_i(x)T_j(y), \quad -1 \leq x, y \leq 1.
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**TABLE 6.926** Coefficients \( a_{ij} \) in the approximate solution

\[
\bar{u}_2(x,y) = \sum_{i=0}^{20} \sum_{j=0}^{20} a_{ij} T_i(x) T_j(y) \quad \text{for} \quad -1 \leq x, y \leq 1.
\]
| i   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 0   | 0.000011 | 0.000010 | 0.000004 | 0.000004 | 0.000002 | 0.000002 | 0.000001 | 0.000001 | 0.000000 | 0.000000 |
| 1   | 0.000000 | 0.000003 | 0.000000 | 0.000001 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 2   | 0.000010 | 0.000009 | 0.000004 | 0.000004 | 0.000002 | 0.000002 | 0.000001 | 0.000001 | 0.000000 | 0.000000 |
| 3   | 0.000000 | 0.000003 | 0.000000 | 0.000001 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 4   | 0.000005 | 0.000005 | 0.000003 | 0.000003 | 0.000001 | 0.000001 | 0.000001 | 0.000001 | 0.000000 | 0.000000 |
| 5   | 0.000000 | 0.000001 | 0.000000 | 0.000001 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 6   | 0.000000 | 0.000000 | 0.000001 | 0.000001 | 0.000001 | 0.000001 | 0.000001 | 0.000001 | 0.000000 | 0.000000 |
| 7   | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 8   | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
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| 10  | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 11  | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
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| 14  | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
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| 18  | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 19  | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 20  | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |

TABLE 6.926  (Continued)