1990

Static and dynamic analysis of the flow of bulk materials through silos

Yong Hong Wu

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STATIC AND DYNAMIC ANALYSES OF THE FLOW OF BULK MATERIALS THROUGH SILOS

A thesis submitted in fulfilment of the requirements
for the award of the degree of

Doctor of Philosophy

from

THE UNIVERSITY OF WOLLONGONG

by

YONG HONG WU, B.E.(Hons), M.E.(Hons)

DEPARTMENT OF CIVIL AND MINING ENGINEERING
February, 1990
DECLARATION

This is to certify that the work presented in this thesis was carried out by the author in the Department of Civil and Mining Engineering, The University of Wollongong, and has not been submitted to any other university or institute for a degree except where specifically indicated.

Yong Hong Wu
ACKNOWLEDGEMENTS

This study was conducted in the Department of Civil and Mining Engineering, The University of Wollongong. The author is indebted to his supervisor Professor L.C. Schmidt, Head of the Department, for the close supervision, fruitful discussions, invaluable suggestions, generous help and the beneficial training throughout the course of this thesis.

The author is very grateful to The University of Wollongong for providing him a University of Wollongong Scholarship for his PhD study.

The help and assistance provided by Professor R.W. Upfold and the staff of the Wollongong University Computer Center in using computer facilities is gratefully Acknowledged. Acknowledgement is also made to all the staff of the department, in particular to Mrs. J. Thompson, Dr. N.I. Aziz, Dr. V.U. Nguyen, and Mr. A. Grant.

Acknowledgement is also given to Dr. A.G. McLean for his help.

The author also wishes to express his thanks to all his colleagues and friends, in particular to Ms J.P. Lu and Mr. J.K.L.Hii, for their encouragement and help.

Finally, special acknowledgement is made to his wife Hui Wang for her encouragement and all sort of help during the period of this study.
ABSTRACT

Storing and handling of bulk materials are essential aspects of grain, chemical and mining operations. Problems that commonly occur in silo operation mainly include flow blockages and structural failure of silos, which reduce the silo capacity and cause high maintenance cost. In most cases the problems occur principally due to inadequate design analysis together with a lack of knowledge of dynamic behaviour of the bulk material - containment structure system during the discharge of materials from silos.

This thesis focuses on the simulation of the filling and the discharge process of bulk materials in silos. Emphasis has been placed on the development of the numerical methods for the prediction of bulk material pressures on silo walls and the internal stresses in silo shells under both static and dynamic conditions.

The motion of bulk materials flowing through silos is modelled as the motion of a no-tension, viscous, elastic-plastic material. As large deformation occurs during discharge of the material, geometry nonlinearity is considered in the analysis. A finite element method formulated in terms of velocity as the primary variable is developed for the simulation of the flow of bulk materials and the structural response of the silo shell. The behaviour of the flowing bulk material is coupled with the behaviour of the containment structure by the contact condition of the bulk material and the silo wall. A computer program has been developed for the analysis.
Experiments have been carried out to verify the numerical model. An automatic monitoring system has been developed for the measurement of the pressure on silo walls. The data processing method based on a power spectrum analysis is employed to separate the random noise from the original signals. The experimental results have shown that the developed numerical models can predict the bulk material pressures on silo walls in a satisfactory manner.

The method of analysis has been applied to simulate the discharge of bulk material from silos. Distribution of wall pressures and internal stresses in silo shells have been computed and presented in the thesis.
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NOTATION

\( A_p \) = eq.3.21,
\( A_c \) = eq.3.28,
\( B \) = partial derivative matrix of \( N \) with respect to co-ordinates,
\( C \) = viscous matrix,
\( C_1, C_2, \) = parameters in Drucker-Prager yield function,
\( C_3 \) = constant involved in Von-Mises yield function,
\( C_C \) = plastic collapse modulus,
\( C_{pp}, C_l \) = constants in plastic-expansive work-hardening or softening function,
\( C_S, C_R, C_\tau \) = plastic potential constants,
\( C_m \) = plastic expansive yield exponent,
\( C_n \) = elastic exponent,
\( C_p \) = collapse exponent,
\( C_{A_{ij}}, C_{B_{ij}}, C_{ij} \) = boundary coefficients,
\( E \) = Young's modulus,
\( F \) = Drucker-Prager yield function,
\( G \) = relation matrix between viscous stress component and deformation rate,
\( H \) = elastic-plastic matrix,
\( H_c \) = hardening/softening function depending on plastic collapse work,
\( H_p \) = hardening/softening function depending on plastic expansive work,
$I_1, I_2, I_3 = 1$st, $2$nd and $3$rd invariants of stress tensor,

\[ I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \]

\[ I_2 = \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} - \tau_{12}^2 \]

\[ I_3 = \sigma_{11} \sigma_{22} \sigma_{33} - \tau_{12}^2 \sigma_{33} \]

$J_2$ = second invariant of deviatoric stress tensor,

$I$ = Jacobian matrix,

$K$ = stiffness matrix,

$K_{ur}$ = modulus number,

$L_r$ = eq.3.26,

$M$ = mass matrix,

$M_c$ = a matrix due to geometric nonlinearity,

$N$ = displacement shape function,

$P$ = eq.5.11,

$a$ = nodal velocity,

$c_{ij}$ = a constant,

$d$ = rate of deformation,

$d_e, d_c, d_p$ = elastic, plastic collapse and plastic expansive component of deformation rate $d$,

$f_c$ = plastic collapse yield function,

$f_p$ = plastic expansive yield function,

$f_v$ = body force,

$f_w$ = friction force or external force on boundary,

$g$ = plastic potential function,

$h_{max}$ = peak value of hardening function,

$j$ = eq.3.23,

$p_a$ = unit atmospheric pressure (0.1013 MPa),

$r$ = see eq.5.11,
\( \mathbf{u} \) = displacement vector,
\( \mathbf{v} \) = velocity vector,
\( \mathbf{w} \) = spin tensor,
\( w_p \) = plastic expansive work,
\( w_c \) = plastic collapse work,
\( x, y \) = components of global coordinate,
\( x_i, y_i \) = x, y coordinate at node i,
\( \Gamma \) = boundary of domain,
\( \Gamma_c \) = contact boundary between bulk material and silo wall,
\( \Gamma_{ca} \) = adhesion contact boundary,
\( \Gamma_{cs} \) = slip contact boundary,
\( \Gamma_f \) = free/discharge boundary,
\( \Gamma_d \) = displacement restrict boundary,
\( \psi \) = error vector of system equations,
\( \Omega \) = domain,
\( \Omega_b \) = domain of bulk materials,
\( \Omega_s \) = domain of silo shells,
\( \alpha, \beta \) = constants in plastic expansive yield function,
\( \alpha_c, \alpha_p \) = see eq.3.33,
\( \delta_{ij} \) = Kronecker delta,
\( \varepsilon \) = strain tensor,
\( \eta_1 \) = plastic expansive yield constant,
\( \phi_w \) = angle of side wall friction,
\( \lambda_1, \mu_1 \) = elastic constants,
\( \lambda, \lambda_c, \lambda_p \) = proportional constants,
\( \mu \) = viscous constant,
\( \nu \) = Poisson's ratio,
\( \rho \) = density,
\( \sigma \) = stress vector,
\( \sigma_s \) = static part of stress,
\( \sigma_v \) = viscous part of stress,
\( \sigma_3 \) = minor principal stress,
\( T \) = eq. 5.14,
\( \xi, \eta \) = local coordinate,
\( \Delta t \) = time increment,
\( \nabla \) = gradient operator,
\( 1 \) = unit matrix

**Superscripts**

\( T \) = transpose of tensors,
\( e, p \) = elastic and plastic components respectively,
\( g, s \) = quantities related to the domain of bulk granular material and silo, respectively,
\( g_{s} \) = interface of the bulk granular material - silo wall,
\( i, i+1 \) = iterative cycle,
\( n, n+1 \) = time step,
\( (\cdot) \) = partial differential with respect to time, or increment in static analysis,
\( (\circ) \) = co-rotational rate,

**Subscripts**

\( B \) = interface between bulk materials and silo wall,
\[ G \quad = \quad \text{domain of bulk granular material,} \\
ca \quad = \quad \text{adhesion status of contact,} \\
cs \quad = \quad \text{slip status of contact,} \\
d \quad = \quad \text{displacement restricted boundary,} \\
e, c, p \quad = \quad \text{elastic, plastic collapse and plastic expansive components,} \\
f \quad = \quad \text{free boundary,} \\
o \quad = \quad \text{initial,} \\
S \quad = \quad \text{silo structure,} \\
t, n \quad = \quad \text{directions tangential and normal to the boundary respectively,} \\
( )_{ij} \quad = \quad \text{tensor indices,} \\
( )_{ij} \quad = \quad \text{partial differentiation with respect to } x_j. \\

\textbf{Tensors}

<table>
<thead>
<tr>
<th>Matrix expression</th>
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| \( \sigma \) | \[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 \\
\sigma_{21} & \sigma_{22} & 0 \\
0 & 0 & \sigma_{33}
\end{pmatrix}
\] | \[
\begin{pmatrix}
\sigma_{11} & \sigma_{22} & \sigma_{12} & \sigma_{33}
\end{pmatrix}^T
\] |
| \( d \) | \[
\begin{pmatrix}
d_{11} & d_{12} & 0 \\
d_{21} & d_{22} & 0 \\
0 & 0 & d_{33}
\end{pmatrix}
\] | \[
\begin{pmatrix}
d_{11} & d_{22} & 2d_{12} & d_{33}
\end{pmatrix}^T
\] |
| \( w \) | \[
\begin{pmatrix}
0 & w_{12} & 0 \\
-w_{12} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 & w_{12} & 0
\end{pmatrix}^T
\] |
\[
\sigma w - w s = w_{12} \begin{pmatrix} -2\sigma_{12} & \sigma_{11} - \sigma_{22} & 0 \\ \sigma_{11} - \sigma_{22} & 2\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} w_{12} \begin{pmatrix} -2\sigma_{12} & 2\sigma_{12} & \sigma_{11} - \sigma_{22} & 0 \end{pmatrix}^T
\]

\[
d w - w d = w_{12} \begin{pmatrix} -2d_{12} & d_{11} - d_{22} & 0 \\ d_{11} - d_{22} & 2d_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} w_{12} \begin{pmatrix} -2d_{12} & 2d_{12} & d_{11} - d_{22} & 0 \end{pmatrix}^T
\]

\[
\nabla \sigma = \begin{pmatrix} \sigma_{11,1} & \sigma_{11,2} \\ \sigma_{22,1} & \sigma_{22,2} \\ \sigma_{12,1} & \sigma_{12,2} \\ \sigma_{33,1} & \sigma_{33,2} \end{pmatrix}
\]

\[
\nabla d = \begin{pmatrix} d_{11,1} & d_{11,2} \\ d_{22,1} & d_{22,2} \\ d_{12,1} & d_{12,2} \\ d_{33,1} & d_{33,2} \end{pmatrix}
\]

\[
\nabla v = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}
\]

\[
H = \begin{pmatrix} H_{1111} & H_{1122} & H_{1112} & H_{1133} \\ H_{2211} & H_{2222} & H_{2212} & H_{2233} \\ H_{1211} & H_{1222} & H_{1212} & H_{1233} \\ H_{3311} & H_{3322} & H_{3312} & H_{3333} \end{pmatrix}
\]
\[ G = \begin{pmatrix}
G_{1111} & G_{1122} & G_{1122} & G_{1133} \\
G_{2211} & G_{2222} & G_{2212} & G_{2233} \\
G_{1211} & G_{1222} & G_{1212} & G_{1233} \\
G_{3311} & G_{3322} & G_{3312} & G_{3333}
\end{pmatrix} \]
LIST OF PUBLICATION DURING PhD STUDY


CHAPTER ONE

INTRODUCTION

1.1 OBJECTIVE

Silos, as storage structures, have played an important part in industry. Many different materials from all sorts of agricultural and industrial sources pass through storage silos at some stage in their production or distribution.

Since the late 19th century a variety of theories have been developed and applied to the design of silos. However, in silo operation, many problems still occur. These problems mainly concern the failure of the silo shell or the blockage of the material flow.

Continuous occurrence of silo problems indicates that, in general, the usual design methods are not always on the safe side, and further research is required. This situation is brought about principally because at present the bulk material pressures on the wall of silos, the flow phenomena of the material, and the dynamic response of the silo shell still are not fully understood, and therefore cannot be predicted exactly in the process of silo design.

This thesis primary concentrates on the simulation of the flow of bulk materials from silos and the dynamic response of the silo shell.
The objective of the thesis is to develop numerical methods and corresponding computer programs for the prediction of bulk material pressures on the wall of silos under both static and dynamic conditions, and the simulation of the flow phenomena of bulk materials, as well as the dynamic response of the silo shell.

1.2 SCOPE

In reviewing previous work, which is examined in Chapter 2, the current research is mainly focussed on the following areas:

(a) establishing a simple boundary element technique and its computer program for the prediction of filling and initial discharging pressures on the wall of silos;

(b) establishing a finite element method and its computer program for the simulation of the flow of bulk materials from rigid-walled silos;

(c) establishing a numerical procedure and its computer program for the simulation of the flow of bulk materials from flexible silos;

(d) establishing a data acquisition/control system for the measurement of silo pressures, and a data processing method for the separation of noise from recorded signals;

(e) experimental investigation of the distribution of silo pressures and the corresponding flow phenomena;

(f) comparison of numerical predictions with experimental results;
(g) applying the developed numerical methods to study some fundamental problems related to silo design.

1.3 OUTLINE OF THE THESIS

The thesis is devoted to the development of the numerical methods for the static and dynamic analysis of a bulk material - silo system, and the application of numerical methods to the parametric studies for silo design. It is divided into 10 chapters.

Chapter 1 introduces the research project. It highlights the objective and the scope of the research.

Chapter 2, based on the review of previous research work, highlights the research areas in which further research is required.

Chapter 3 establishes the mathematical model for the motion of the bulk materials-silo system. The behaviour of the bulk materials during storage and discharge is described mathematically by a set of partial differential equations and algebraic equations, which are to be solved numerically in Chapters 4-6.

Chapters 4-6 are devoted to the development of numerical methods for the solution of the mathematical model described in Chapter 3.

Chapter 4 describes a boundary element method for the prediction of static pressures and initial discharge pressures. As the first attempt to use the boundary element method in predicting silo pressures, special emphasis is placed on the
establishment of the boundary integral equation for the problem analysed, and its numerical solution by the boundary element method.

Chapter 5 describes the finite element procedure for the simulation of the flow of bulk material through rigid-walled silos. The whole derivation procedure to obtain the finite element formulation is presented.

Chapter 6 describes the numerical procedure for the simulation of the flow of bulk solids from flexible silos. Special emphasis is placed on the development of the numerical method for the coupling of the motion of the bulk material with the deformation of the silo shell.

Chapter 7 describes the experimental program, which is designed to provide information for the comparison of the results computed by numerical methods with experimental results. Emphasis is placed on the description of an automated data acquisition/control system, a digital signal processing method, the experimental results and their analysis.

Chapter 8 is devoted to the comparison of numerical predictions with experimental results, analytical results from some existing pressure theories, and the IEA (Institution of Engineers, Australia) recommendation.

Chapter 9 is devoted to the application of the numerical methods developed. Several problems related to the silo design are analysed in this chapter.

Chapter 10 presents the summary and the conclusion of the research work.
CHAPTER TWO

REVIEW OF PAST WORK

2.1 GENERAL

Since the late 19th century when Janssen (1895) published his now famous theory for the evaluation of silo pressures, many research projects related to silo design have been conducted. The research principally includes the following areas: (a) evaluation of bulk material pressures on the wall of silos; (b) prediction of the flow rate of bulk materials from silos; (c) stability analysis of silo structures. Some significant contributed works related to the above areas are reviewed in this chapter.

2.2 PRESSURES ON SILO WALLS

Classical research has classified the pressures on the wall of silos as static (filling) pressures during the stage of charging a silo, and dynamic (discharge) pressures during the stage of discharging a silo.

For many years, research on the prediction of silo pressures was mainly governed by experimental and analytical studies. Recently, however, numerical methods have been employed by several researchers to evaluate the pressures on the walls of silos.
2.2.1 Analytical Studies

Since the late 19th century a variety of analytical formulae have been developed for the prediction of bulk material pressures on the wall of silos, such as those theories due to Janssen, Walker, Walters, Jenike and etc. These theories are reviewed and summarized as follows.

(a) Janssen theory

In 1895, Janssen published his theory for the prediction of pressures on the vertical walls of the silos with a circular cross section. In his analysis, Janssen made the explicit assumptions that:

(i) the distribution of vertical pressure across any cross-section is uniform;
(ii) the ratio (K) of horizontal stress to vertical stress is constant throughout a silo;
(iii) material is homogeneous and isotropic and can be treated as a continuum.

Analysing the static equilibrium of the stored solids, Janssen derived an expression to predict the pressures on the wall of silos.

Janssen's work has been examined by Arnold et al (1978) and has been extended to cover other axisymmetric cross-sections as well as long plane-flow vertical sections.

In Janssen's equation there is one factor which has not been rigidly specified, i.e. the ratio of horizontal to vertical stress. Arnold et al. (1978) have listed several
formulae for the determination of the value, such as those by Walker (1966) and Jenike (1973).

(b) Walker theory

Under a mass flow condition, i.e., a flow pattern in which all solids in a bin is in motion whenever any of it is withdrawn, Walker (1966, 1967) examined a horizontal elemental slice and adopted the assumptions that:

(i) elemental slices are simultaneously yielding within themselves and along the hopper wall;

(ii) both the bulk solid yield locus and the wall yield locus are straight lines through the origin;

(iii) the vertical stress near the wall on the slice can be linearly related to the average vertical stress on a slice by a "distribution factor" $D$;

(iv) the material adjacent to the vertical wall is in an active failure state, however, the material adjacent to the hopper wall is in a passive failure state.

Analysing the static equilibrium of the slice, Walker derived a set of formulae for the prediction of flow pressures on both vertical walls and hopper walls with or without surcharge.

In Walker’s formulae, there is one factor which cannot be exactly evaluated, that is the "distribution factor" $D$. Probably this factor is the weakness of the method. However, Walker's measurements on a small model bunker and large experimental hoppers have shown that the experimentally determined values of
pressures agree reasonably well with those predicted by the theory, if D is taken as unity.

McLean (1985) has recommended some modification to Walker's equation. He suggested a new formula for evaluating the parameter included in Walker's equation.

(c) Walters theory

Walters (1973a, 1973b) distinguished the stresses developed during the initial filling of the silo (static condition) from those developed during discharge of the materials (dynamic condition). He extended Walker's analysis to allow for the non-uniformity of the vertical stress on a horizontal slice of the stored solids, and made the following assumptions that:

(i) vertical shear stress varies linearly with radius;

(ii) an active stress state pertains on initial filling, and a passive stress state pertains during flow.

For the analysis of pressures on a vertical wall, Walters examined a horizontal cylindrical slice as Walker did. However, for pressures on a hopper, Walters used a frustum of a cone as the elemental slice, instead of a cylindrical slice which touched the hopper wall only at the lower edge, used by Walker.

Analysing the static equilibrium of the forces acting on the slice, Walters derived a series formulae for the prediction of static and dynamic pressures on the vertical wall and the hopper wall.
It has been noted (Arnold et al. 1978) that Walters's theory, when used to predict hopper pressures under static condition, is only valid for very steep hoppers. Fortunately, when Walters's theory fails Walker's theory is applicable and vice versa.

It has also been shown (Horne and Nedderman, 1978) that Walters's assumption that the shear stress on any horizontal plane increases linearly with radius is invalid if the entire mass is in a passive state of failure and the hopper angle is greater than his suggested critical value. To overcome this limitation, Horne and Nedderman suggested a method to modify the value of the distribution factor $D$.

(d) Jenike theory

Jenike and his co-worker, in their publications (Jenike and Johanson, 1969a, 1969b; Jenike et al., 1973), have described their methods for the prediction of initial (static) and flow (dynamic) pressures on both vertical walls and hopper walls.

(i) Initial pressure on vertical wall

Jenike et al. used the Janssen equation to predict the initial pressure on the vertical wall of mass-flow cylinders. A value of $K=0.4$ (the ratio of horizontal stress to vertical stress, used in Janssen's equation)) is recommended for the prediction of pressures. However, they have pointed out that Janssen's equation gives a lower bound to the maximum cylinder wall pressures and therefore is not recommended for design.
(ii) initial pressure on hopper wall

For a hopper with or without a cylindrical surcharge Jenike et al have assumed that the filling pressure can be described by a triangular distribution with a maximum value at some intermediate height. They, then, derived a formula for the determination of the distribution by considering the equilibrium of the bulk solids stored in the hopper.

When a cylindrical surcharge is present it exerts a vertical load on the solid on top of the hopper. The initial pressures perpendicular to the hopper wall are assumed to vary linearly from the apex to the transition of the bin-hopper. A set of formulae for the computation of the value of the pressure distribution has been derived by assuming the surcharge acting over the hopper wall from the top boundary to the maximum intermediate height, which is then assumed to be at the apex of the hopper.

(iii) wall pressures in vertical wall under flow condition

Jenike et al. indicated that when bulk material begin to discharge from silos, the stresses change from a static (active) to a dynamic (passive) state. This switch, defined as a region of pressure change from an active pressure field toward a passive pressure field, is initiated at the bottom of the hopper when the outlet is opened and travels rapidly upwards.

If the switch is located at the transition point of the bin-hopper, the flow pressures in vertical wall are predicted by Janssen’s equation. However, if the switch is located at some depth in the cylinder the initial Janssen stress field applies
between the top of the cylinder and the switch; below this level, between the switch and the level of the free boundary condition, a minimum strain energy stress field applies.

The minimum strain energy stress field is derived using the second law of thermodynamics which states that the internal elastic energy within the flowing mass tends to a minimum. The stress field is expressed as a function of the location of the switch and the natural boundary condition. As the occurrence of the switch is unpredictable, the evaluation of a bound enclosing all possible pressure peaks is more important. Formulae for determining this locus of strain energy stress bound have been derived.

(iv) flow pressure on hopper wall

Jenike et al. (1973) indicated that the flow pressure on a hopper wall consists of a pressure peak at the transition, then, a linear decrease to an intermediate value and another linear decrease to zero at the apex. A set of formulae for the determination of this stress field has been derived based on the assumption that the overpressure at the transition is distributed over 0.3 D (diameter of the silo) of the hopper wall from the transition point.

(e) Some other analytical studies

In addition to the above studies, many other analytical studies have been conducted for the prediction of silo pressures, such as those by Deutsch and Schmidt (1969), Deutsch (1969), Enstad (1975), Wilms (1985), Arnold and
McLean (1976). All this studies have resulted in a better understanding of the distribution of silo pressures.

2.2.2 Experimental Studies

In addition to analytical studies, many experimental studies have been conducted to investigate the distribution of pressures on the walls of silos. Most experiments are performed on model silos. However, several tests of pressures on actual silos have also been reported.

Apparently, experimental research is essential for the understanding of the pressure distribution on the walls of silos, as the experimental method is needed to investigate the characteristics of pressure distributions, and, consequently, to establish empirical equations. Also it is a valid way to verify the analytical theories and numerical models.

In addition, experimental studies are especially important for those problems which can hardly be solved by analytical methods, such as the pressure distribution on the walls of silos discharging eccentrically and on the walls of silos with a complicated geometry.

Jenike et al (1973) and Walker (1966, 1967) have conducted experiments on model silos to verify their developed pressures theories.

Jenike et al (1973), from their experiments, found that under flow condition in the cylinder slight deviation of the bin shape from a perfect cylinder or imperfection in finish can provide support and cause thin boundary layers at the
wall. In most case the layers are unstable and are of only short duration. The formation of a boundary layer causes a change of stress from Janssen's stress field toward the minimum energy stress field described by Jenike's equation. The dissolution of the boundary layer causes a change of stress back to Janssen's stress field. These changes cause a pressure oscillation.

Perry and Jangda (1970/1971) measured the static and dynamic pressures on the wall of a model bin-hopper combination. Their experimental results show a fair agreement with the pressure theories due to Jenike, Johanson, and Walker. Based on the experimental results, Perry and Jangda made some recommendation for the design of silos.

Deutsch and Schmidt (1969) conducted measurements on a flat-bottomed model silo containing sand. The experiment showed that there were four completely different flow zones (plug flow zone, pipe feed zone, pipe zone and dead zone) during the discharge of material from a silo. In addition, overpressures were found to occur during discharge of the material. The highest overpressures - up to four times the static pressures - were found at the walls for the length of the pipe feed zone. Based on the experimental results, they recommended a more conservative design approach.

Jerzy Kmita (1985) conducted experimental study on dynamic pressure exerted by bulk materials on the walls of a model bin-hopper combination. Using dimensional analysis, he developed a formula for the pressure on the wall of a silo in terms of dimensionless parameters.
Blight (1986) conducted a series measurement of pressures in a number of full scale silos. He compared the experimentally determined results with those from some commonly accepted design theories, and then concluded that the simple Janssen arching theory provided a good estimate of the mean trend of horizontal pressure with depth in a cylindrical silo for the condition of "end of filling" and "start of discharge". He found little difference of pressure between the filling condition and the flow condition. In addition, he found that pressures are highly variable, and therefore proposed a simple containing envelope for all pressures.

Gale, Hoadley and Schmidt (1986) took pressure measurements and observed flow patterns on a model flat-bottomed circular silo discharging eccentrically to investigate the flow patterns and the associated pressure distributions. The effect of the eccentricity ratio of the outlet on pressure distribution was investigated. It was found that the peak lateral overpressure became slightly greater in magnitude as the outlet became more eccentric but decreased slightly at the extreme eccentricity. The maximum value of the envelope occurred for an eccentricity ratio \( e/D = 0.3 \), but decreased in value with further increase in eccentricity for the silo examined.

Many other experimental researches, such as those performed by Blair -Fish and Bransby (1973), Hofmeyr (1986), Tattersall (1981), Tattersall and Schmidt (1981), have also resulted in a better understanding of the distribution of pressures on the walls of silos.

2.2.3 Numerical Studies

Recently, some researchers have returned their interest to numerical methods
due to their potential capacity in solving complicated geometry problems and simulating complex dynamic behaviour. Several attempts to use the finite element method to predict bulk material pressures on the wall of silos have been reported.

Mahmoud and Abdel-Sayed (1981) developed a finite element approach accounting for the composite action of the bulk solids and the container shell by using a one-dimensional joint element. It took into consideration of the flexibility of the bin walls and the interaction characteristics between the grain pressures and the wall deformation. It was demonstrated that the relative stiffness between the wall and the stored solids affected the magnitude of the lateral pressure and its distribution. Similar techniques have also been employed by Ooi and Rotter (1986, 1987a) in a parametric study of a squat silo. However, the method developed considered the static case only. In addition, in the computational procedure, two material parameters, normal and tangential stiffnesses of linkage elements, have to be assumed initially. Then the assumed values have to be modified through an iteration procedure until the computed shear stresses for all linkage elements are within the shear strength of the solid, which may cause a computational problem. Moreover, overlap of the solids with the silo wall may occur.

Haussler and Eibl (1984) developed a numerical method for computing velocity and stress fields in plane strain silos during charge and discharge. The method incorporates an elastic-plastic constitutive law for a cohesionless granular material with a viscous part. Their computed results agree well with experimentally measured values for the case of charging, and appear physically reasonable for the more complicated process of discharge. However, the procedure developed applies to the rigid wall condition only.
Runesson and Nilsson (1986) modelled the motion of bulk granular materials subjected to gravitational forces as the motion of viscous-plastic fluids. In addition to the conventional assumption of incompressible rigid-plastic flow, linearly viscous behaviour for a small deviator stress as well as dilation at plastic yielding were assumed. Elastic strains were ignored in order to simplify the formulation. A finite element method formulated in terms of the velocity as the primary variable was developed for the transient flow including the convective acceleration terms. A time stepping algorithm was devised, which adopted a modified Newton iterative scheme for the solution of the nonlinear equation in each time step arising from the backward Euler differencing scheme.

Askari and Elwi (1988) assumed bulk solids as a no-tension Drucker-Prager elastic, perfectly-plastic material. They predicted the incipient flow pressure on a hopper-bin wall by using a technique consisting of a double iterative scheme over the friction and the bulk material nonlinearity. The procedure has been shown to be able to capture the behaviour of the high pressure concentration in the transition area of the bin-hopper. However, the method ignores the inertial behaviour, and the problem is treated as a static problem. Therefore, their procedure cannot simulate the dynamic behaviour of the discharge of materials from silos. The computed pressure may not be the critical design pressures.

In addition, a variety of other numerical analyses on the prediction of silo pressures have also been conducted and published, such as those by Chandrangsu and Bishara (1978), Bishara et al (1983), Dickenson and Jofriet (1984) and Oii and Rotter (1987a, 1987b).
2.3 DISCHARGE RATE

Many attempts to develop practical methods for the prediction of the flow rate of bulk materials from silos have been made since early this century. Initially, most efforts were made in deriving empirical equations for the flow rate. Then, research was focussed on analytical studies. Only recently have some researchers turned their interest to numerical analysis in order to predict the flow rate of bulk materials from silos in a more accurate manner. Some significant contributed work in this field is summarized as follows.

2.3.1 Empirical Equations

The initial empirical equations were developed by such authors as Franklin and Johanson (1955), Beverloo et al (1961), Rose (1959) and etc. Such correlations are important to the engineers faced with the problem of predicting flow rate. However, these equations do not lead directly to a physical understanding of the nature of the flow phenomena.

The initial empirical equations were later modified to semi-empirical relations using the fluid analogy by such authors as McDougall and Evans (1965), and Harmens (1963).

2.3.2 Theoretical Analysis

The initial significant theoretical analysis of the flow rate of flowing bulk solids was that of Jenike (1961, 1964) and Johanson (1964). They analysed the flow of cohesive bulk solids from converging channels by using the principles of
soil mechanics. In their analysis, the concept of a cohesive arch was introduced, the flow of which demands that the major principal stress in the arch must be equal to the unconfined yield strength of the material. These concepts allowed the hopper geometry to be determined using a graphical iterative procedure.

Johanson (1965) then applied the equation of motion including the effects of inertia to a cohesive arch element of uniform thickness. He derived an equation for the maximum steady state discharge rate for a bulk material as a function of the hopper shape, the opening size, and the flow property of the material.

Although Johanson's method gives good results in many cases, it involves determination of the material flow function and other frictional properties, and requires the repetitive use of charts to determined the parameters involved. The method does not lead itself readily to computerisation where analytical procedures are desired. This need had prompted a number of other investigators to develop analytical expressions for the flow rate of coarse incompressible bulk solids.

By including a term due to the momentum change, Davidson (1973) and Nedderman derived an expression for the flow rate from both plane flow and axisymmetric flow hoppers. This theory was primarily concerned with cohesionless material flowing from a channel with a small included angle and with smooth walls, but an analysis for material having a finite cohesion was also presented.

Savage (1965) also provided an approximate analysis of steady gravity flow of a cohesionless bulk solid in a vertical converging channel. Initially, Savage derived an analytical expression for the flow by coupling the velocity and stress
fields, which enforced a unique solution for the flow rate. Later, Savage extended his analysis to include the effect of small wall friction.

Williams (1977) expressed concern that no satisfactory method had been proposed for predicting the flow rate of coarse granular material from a conical hopper which took into account both the frictional properties of the material and the effect of wall friction. Consequently, he presented a theoretical treatment of the problem which accounted for both of these effects.

For fine particles where the fluid drag forces are comparable to the gravitational forces, Carleton (1972) investigated the effect of the interstitial fluid on the flow rate.

Several investigators have suggested that improvement to the flow were possible by air injection techniques (Bruff and Jenike, 1967/1968; Reed and Johanson, 1973). Many researchers have made attempts to investigate the multiphase nature of the flow. Papazoglou and Pyle (1970/1971) modelled the flow of air assisted particles using the concept of an inviscid continuum, developing a Bernoulli type equation for the particulate phase which added air flow. Papazoglou predicted a variation of the interstitial gas pressure in the hopper. He found that quite small additions of interstitial gas in the channel gave marked increases in the flow rate and altered the stress distribution throughout the whole phase.

Holland et al. (1969) conducted a two phase analysis of granular flow. He derived an equation of motion for a mixture of granules and fluid by considering the equilibrium of the forces acting upon an element of mixture. Although Holland and
his co-workers did not derive a general solution of the equation, a number of experimental observations were explained by their analysis.

Shook et al. (1972) conducted a study on the discharge of bulk solids under water. They explained their results by postulating the existence of a non-uniform voidage.

In a paper by Crewdson et al (1977), an analysis was presented for the effect of self-generated interstitial pressure gradients on the flow of granular materials from a hopper.

McLean (1979) derived three governing differential equations for the flow of bulk solids by considering the continuity of the bulk solid and the interstitial fluid, as well as the equation of motion including the effects of the interstitial gas pressure gradients. Then he derived the analytical solution for the flow rate by making assumptions for the form of the flow stress field.

2.3.3 Numerical method

Recently, several attempts to use the finite element method to compute the velocity field in flowing bulk solids have been made, such as those by Haussler and Eibl (1984) Runesson and Nilsson (1986). Apparently, based on the velocity field from finite element computation, the flow rate of bulk materials from silos can be evaluated as

\[ Q = \int_{\Gamma_0} v \, d\Gamma \]

where \( Q \) = flow rate, \( \Gamma_0 \) = area of the outlet, \( v \) = velocity.
2.4 STRUCTURAL STABILITY

The computation of internal stress resultants in silo walls and the stability analysis of the silo structure are essential in silo design. A variety of research in this field has been reported.

Ghali (1979) was concerned with the analysis of circular - cylindrical walls subjected to axisymmetrical loading. He assumed the material of silo walls as linearly elastic material. Besides, circular walls are considered as cylindrical shells for which the thickness is small compared to the radius. By using the displacement method or the force method, he developed a method by which the reactions on the edges and the internal forces in a circular-cylindrical wall can be obtained. For the sake of simplicity in practical application, a set of design tables were provided. In this method the bulk material pressures on the wall of silos are assumed as given pressures.

Mahmoud and Abdel-Sayed (1981) considered the bulk solid and the silo wall as a system. They introduced a joint element to model the interaction of the bulk solid with the silo wall. Based on the principle of continuum mechanics they derived a set of finite element formulae which can compute the internal stress resultant in a silo wall under the static (filling) condition. However, the method is not capable of analysing the dynamic stress resultants in a silo wall during discharge of materials.

Oii and Rotter (1986) have used a method similar to that by Mahmoud and Abdel-Sayed to analyse the effect of the wall friction coefficient on meridional stress in the walls of silos.
Research on the buckling failure of steel silos has been reported by several researchers (Arboez and Sechler, 1974; Blackler and Ansourian, 1983; Jumikis and Rotter, 1983 and 1986; Rotter, 1985). Rotter (1986a) summarized the information from these publications together with his own research work, and then described four principal buckling failure models. All these studies have resulted in a better understanding of the design procedures against buckling. Especially, Rotter in his publications has provided tentative design recommendation and has illustrated their assumptions.

Rotter (1986b) analysed the stress resultants in a hopper wall by using the membrane and elastic bending theory. His analysis showed that high pressures occur at the transition of hopper-bin, and therefore requires an increase in the strength of the hopper. Additionally, his analysis indicated that the static pressure distribution generally controls the design of the hopper.

2.5 SUMMARY AND CONCLUDING REMARKS

The history of the research related to silo design has been briefly reviewed. A variety of significant contributed works on the prediction of silo pressures, the flow rate of bulk material from silos, and the stability analysis of silo shells have been examined. It is clear that:

(a) Until last decade, research has been mainly governed by experimental and analytical studies, little effort has been directed to numerical analysis. Although analytical studies have resulted in a basic understanding of pressure distributions, they cannot simulate the complex dynamic behaviour of the bulk material flow, and also they cannot handle the complicated geometry that sometimes occurs.
Experimental research is reliable for applications, however as a prediction method, it is too expensive and time consuming. Different from analytical and experimental studies, numerical methods show to advantage in simulating complex geometry problems and complex transient dynamic behaviour. Therefore, current research is devoted to numerical studies.

(b) A great deal of research work on the application of the finite element method to predict bulk material pressures on the walls of silos and the velocity fields within the flowing materials as well as the stresses in silo walls have been conducted. However, it appears no attempt has been made to use the boundary element method for the prediction of silo pressures. The finite element procedures derived by previous researchers for the simulation of bulk material flow are limited to a condition of rigid-walled silos. Few attempts have been made to investigate the dynamic response of silo shells during the discharge of bulk materials from silos. Further parametric studies, such as the effect of material constitutive characteristics on the pressure distribution and internal stresses in silo shells, are required. Therefore, this research is designed and focussed in the following areas:

(i) Methodology for simulating the charge and discharge of bulk materials from silos,

(ii) Coupling of the motion of bulk materials and the deformation of silo shells,

(iii) Dynamic response of silo shells during discharge of bulk materials, such as the evaluation of the stresses in silo shells,

(iv) Effect of bulk material properties and some other parameters on the wall pressure distributions and silo stability.
CHAPTER THREE

FUNDAMENTAL EQUATIONS FOR THE MOTION OF BULK MATERIAL-SILO SYSTEM

3.1 GENERAL

This chapter is devoted to the presentation of the mathematical model for simulating the motion of bulk materials in silos, and the associated deformation of the silo shells.

Both bulk materials and the materials of silo shells are assumed as continua. In Eulerian space the fundamental equations governing the motion of bulk material-silo system consist of stress-equilibrium equations, kinematic equations, a constitutive law, conservation of mass, boundary conditions, contact conditions and initial conditions. These governing equations are recapitulated in this chapter. Emphasis is placed on the derivation of the stress-strain relation.

3.2 STRESS EQUILIBRIUM EQUATIONS OF MOTION

As large and fast deformation occurs during discharge of bulk materials from a silo, geometric nonlinearity is considered, and therefore the equations of stress equilibrium for the motion of bulk materials are formulated in an Eulerian frame of reference in an incremental form as

\[ \nabla \cdot \sigma + f_v = \rho \left( \dot{v} + \nabla v \cdot v \right) \quad \text{in } \Omega_g \]  \hspace{1cm} (3.1)
where

\[ \nabla = \text{gradient operator}; \]
\[ \nabla \cdot \sigma = \sigma_{ij,j}; \]

superposed dot = partial differential with respect to time;
\[ \mathbf{v} = \text{velocity tensor}; \]
\[ f_v = \text{body force}; \]
\[ \rho = \text{density}; \]
\[ \sigma = \text{stress tensor}; \]
\[ \Omega_g = \text{domain of bulk granular material}. \]

Compared with the large deformation of the bulk materials during discharge, an infinitesimal strain condition is assumed in modelling the deformation of the silo walls. In the domain of the silo structure \( \Omega_s \), the stress equilibrium equations for the deformation of the silo shell can be expressed as follows

\[ \nabla \sigma + f_v = \rho \dot{\mathbf{v}} \quad \text{in} \ \Omega_s \quad (3.2) \]

### 3.3 KINEMATIC EQUATIONS

In both the domain of the bulk granular materials and the domain of the silo shells the rate of deformation \( \mathbf{d} \), and the spin tensor \( \mathbf{w} \) can be related to velocity by the kinematic equations as follows:

\[ \mathbf{d} = \frac{1}{2} ( \nabla \mathbf{v} + \nabla \mathbf{v}^T ) \quad \text{or} \quad d_{ij} = \frac{1}{2} ( v_{i,j} + v_{j,i} ) \quad (3.3a) \]
\[ \mathbf{w} = \frac{1}{2} ( \nabla \mathbf{v} - \nabla \mathbf{v}^T ) \quad \text{or} \quad w_{ij} = \frac{1}{2} ( v_{i,j} - v_{j,i} ) \quad (3.3b) \]

where superscript \( T \) denotes the transpose of tensors.
3.4 CONSERVATION OF MASS

In the domain of bulk granular materials $\Omega_g$, the conservation of mass is described by the equation as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$  (3.4)

3.5 CONTACT CONDITIONS

There are two kinds of contact status between bulk granular materials and silo walls: slip status and adhesion status. Therefore, the contact boundary $\Gamma_c$, shown in Fig.3.1, can be subdivided into two parts, a slip region $\Gamma_{cs}$ and an adhesion region $\Gamma_{ca}$ by the Coulomb criterion:

$$\Gamma_{cs} \quad \text{if } |p_t| \geq |p_n| \tan \phi_w$$

$$\Gamma_{ca} \quad \text{otherwise}$$

Fig.3.1 Description of the bulk granular material-silo system
where $p = \text{the contact pressure on the interface between bulk granular materials and silo walls};$ subscripts $t$ and $n = \text{the directions tangential and normal to the boundary respectively};$ $\phi_w = \text{the angle of side wall friction};$ the first subscript $c$ denotes contact; the second subscripts $s$ and $a$ refer to the contact status of slip and adhesion respectively.

In the slip region $\Gamma_{cs},$ the relation between tangential and normal traction is determined by

$$p_t^g = (-\tan \phi_w) \frac{v_t^g - v_t^s}{v_t^g - v_t^s} (-p_n^s) = -p_t^s \quad (3.5a)$$

$$p_n^g = -p_n^s \quad (3.5b)$$

where superscripts $g$ and $s$ denote the quantities related to the domain of the bulk granular material and the domain of the silo shell, respectively, as shown in Fig.3.2. In addition, the normal displacement or velocity of the bulk material is equal to that on the silo wall at the interface, that is

$$v_n^g = v_n^s \quad (3.5c)$$
In the adhesion region $\Gamma_{ca}$ no slip occurs. Therefore the displacement or velocity of the bulk granular material is equal to that on the silo wall, that is

$$v^g_t = v^s_t \quad \text{and} \quad v^g_n = v^s_n$$  \quad (3.5d)

For a simple case, where a rigid wall condition of a silo is assumed, $v^s_t = v^s_n = 0$. Therefore, the contact conditions are simplified into

$$\begin{cases} 
  p^g_t = \left( - \tan \phi_w \frac{v^g_t}{v^s_t} \right) \left( - p^g_n \right) \\
  v^g_n = 0 \\
  v^g_t = v^g_n = 0
\end{cases} \quad \text{on } \Gamma_{cs} \quad (3.6a)$$

$$\quad \text{on } \Gamma_{ca} \quad (3.6b)$$

### 3.6 SYSTEM BOUNDARY CONDITIONS

The boundary of the bulk-material silo system includes two types — a free or surcharge boundary $\Gamma_f$ such as the top surface of the bulk materials and the outlet area during the stage of discharge, and a displacement restricted boundary $\Gamma_d$ on which external supports are exerted to the silo shell, as shown in Fig.3.1.

On $\Gamma_f$, the boundary condition is defined by the traction condition

$$p_n = p^o_n \quad \text{and} \quad p_t = p^o_t \quad \text{on } \Gamma_f \quad (3.7a)$$

where $p^o$ is the boundary traction known in advance, such as on the top surface of the bulk material $p^o = 0$, and at the outlet in the initial stage of discharge $p^o = p^f \to 0$, where $p^f$ denotes the filling pressures when the outlet is closed.
On $\Gamma_d$, displacements are constrained by the external supports. For a rigid-support condition the boundary condition can be expressed as follows

$$v_t = v_n = 0 \quad \text{on } \Gamma_d$$

(3.7b)

### 3.7 CONSTITUTIVE MODELS

Several constitutive models have been employed to describe the stress-strain relation of bulk materials, such as the elastic model by Oii and Rotter (1987b); no-tension elastic, perfectly-plastic model by Askari and Elwi (1988); and the viscoplastic model by Haussler and Eibl (1984). In viewing the different behaviours of the bulk material during storage and discharge, different constitutive models have been used in static analysis and dynamic analysis. During storage or in the static condition, the bulk material can be regarded as a solid without tensile strength, a no-tension elastic-plastic constitutive law is used herein for the analysis. However, during discharge or in a dynamic condition, bulk materials behave like a fluid, therefore a no-tension viscous elastic-plastic model is used herein for the simulation of the discharge process.

For the material from which the silo shell is fabricated, elastic-plastic models are used to describe the stress-strain relation during material deformation.

### 3.7.1 A Viscous, Elastic-Plastic Constitutive Law

#### 3.7.1-1 General form of the law

The fundamental feature of the law is a decomposition of the total stress rate into a rate independent part and a rate dependent part, that is

$$\dot{\sigma} = \dot{\sigma}_s + \dot{\sigma}_v$$

(3.8)
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where

\[
\dot{\sigma} = \frac{\partial \sigma}{\partial t} + \nabla \sigma \cdot \mathbf{v} + \sigma \mathbf{w} - \mathbf{w} \sigma
\]  

(3.9)

is the co-rotational stress rate due to Jaumann (see Masur, 1961). \( \dot{\sigma}_s \) is the time independent part of the stress rate, and \( \dot{\sigma}_v \) indicates the time dependent part of the stress rate. \( \sigma \mathbf{w} - \mathbf{w} \sigma \) is a second order tensor with components \( \sigma_{ij}w_{ij} = \sigma_{ij}w_{ij} - \sigma_{ij}w_{ij} \).

\( \dot{\sigma}_s \) is determined from the elastic-plastic constitutive law, i.e.

\[
\dot{\sigma}_s = H \dot{d} \quad \text{or} \quad \sigma_{sij} = H_{ijrs} d_{rs}
\]  

(3.10)

where \( H \) is the elastic-plastic constitutive tensor, which depends on the material assumption. In the present case, a no-tension elastic-plastic model with an associated flow rule for the plastic collapse strain, and a non-associated flow rule for the plastic expansive strain is employed. The corresponding matrix, based on the derivation of Haussler and Eibl (1984), is derived in section 3.7.1-2. It will be shown in section 3.7.1-2 that the newly derived matrix, which considers more general cases, is an extension of the matrix by Haussler and Eibl.

\( \dot{\sigma}_v \) can be determined by a rate dependent constitutive law proposed by Haussler and Eibl (1984), which is in the form of

\[
\dot{\sigma}_v = G \dot{d} \quad \text{or} \quad \sigma_{vij} = G_{ijrs} d_{rs}
\]  

(3.11)

with

\[
G_{ijrs} = 2 \mu (\delta_{ir} \delta_{js} - \frac{1}{3} \delta_{ij} \delta_{rs})
\]

\[
\dot{d} = \frac{\partial \mathbf{d}}{\partial t} + \nabla \mathbf{d} \cdot \mathbf{v} + \mathbf{d} \mathbf{w} - \mathbf{w} \mathbf{d}
\]

where \( \mu \) is a viscous constant.
Regarding eq. 3.8 - eq. 3.11, the stress rate in an Eulerian frame of reference can be related to deformation rate by

$$\frac{\partial \sigma}{\partial t} = H \frac{\partial \dot{d}}{\partial t} - (\sigma w - w \sigma + \nabla \sigma \cdot \nabla \nu) + G (\dot{d} w - w \dot{d} + \nabla d \cdot \nabla v) \quad \text{in } \Omega_g \quad (3.12)$$

Here the first term of the right hand side is associated with material elastic-plastic properties, the second term is associated with material viscous properties, and the remaining terms represent the effects due to the geometric nonlinearity.

### 3.7.1-2 General form of the elastic-plastic matrix \( H \)

To be general, the total strain rate is decomposed into an elastic component \( \dot{d}_e \), a plastic collapse component \( \dot{d}_c \) and a plastic expansive component \( \dot{d}_p \), i.e.

$$\dot{d} = \dot{d}_e + \dot{d}_c + \dot{d}_p \quad (3.13)$$

The elastic component is easily related to stress rate by Hooke's law, i.e.,

$$\dot{d}_e = D^{-1} \dot{\sigma}_s \quad \text{or} \quad \dot{d}_{eij} = D^{-1}_{ijrs} \dot{\sigma}_{srs} \quad (3.14)$$

where

$$D^{-1}_{ijrs} = \frac{1}{E} \left( (1 + \nu) \delta_{ir} \delta_{js} - \nu \delta_{ij} \delta_{rs} \right)$$

\( E \) = Young's modulus of the material; \( \nu \) = Poisson's ratio of the material.

During material loading, plastic expansive strain is produced if the plastic expansive yield criterion is satisfied, i.e.

$$\begin{align*}
& f_p - H_p = 0 \\
& f_p > 0
\end{align*} \quad (3.15)$$
where superposed dot indicates an increment;

\[ f_p = f_p(I_{1s}, I_{2s}, I_{3s}) \]

denotes the general form of the plastic expansive yield function; \( I_{1s}, I_{2s} \) and \( I_{3s} \) are the 1st, 2nd, 3rd invariants of the rate-independent stress tensor;

\[ H_p = H_p(\sigma_s, w_p) \]

is the hardening or softening function depending on the stress level \( \sigma_s \) and the plastic expansive work. The rate of the plastic expansive work can be expressed as

\[ \dot{w}_p = \sigma_s \dot{\alpha}_p \]  

(3.16)

The rate of the plastic expansive strain may be determined by the non-associated flow rule (Mendelson, 1983) as

\[ \dot{d}_p = \lambda_p \frac{\partial g_p}{\partial \sigma} = \lambda_p \mathbf{n} \]  

(3.17)

where

\[ g_p = g_p(I_{1s}, I_{2s}, I_{3s}) \]

is the plastic potential;

\[ \lambda_p = \text{a proportional constant, which may be determined by the consistency condition} \]

\[ \dot{f}_p - \dot{H}_p = 0 \]  

(3.18a)

where

\[ \dot{f}_p = \frac{\partial f_p}{\partial I_{1s}} \dot{I}_{1s} + \frac{\partial f_p}{\partial I_{2s}} \dot{I}_{2s} + \frac{\partial f_p}{\partial I_{3s}} \dot{I}_{3s} \]  

(3.18b)

\[ \dot{I}_{1s} = I_{1s}^{\top} \dot{\sigma}_s \]  

(3.18c)

\[ \dot{I}_{2s} = I_{2s}^{\top} \dot{\sigma}_s \]  

(3.18d)
\[ I_{3s} = I_{3s}^T \]  
\[ \hat{H}_p = \frac{\partial H_p}{\partial w_p} + \left( \frac{\partial H_p}{\partial \sigma_s} \right)^T \]  
\[ \text{with} \]
\[ I_{1s} = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}^T \]
\[ I_{2s} = \begin{pmatrix} \sigma_{22} + \sigma_{33} & \sigma_{33} + \sigma_{11} & -2\sigma_{12} & \sigma_{11} + \sigma_{22} \end{pmatrix}^T \]
\[ I_{3s} = \begin{pmatrix} \sigma_{22} \sigma_{33} & \sigma_{33} \sigma_{11} & -2\sigma_{12} \sigma_{33} & \sigma_{11} \sigma_{22} - \sigma_{12}^2 \end{pmatrix}^T \]

Substitution of eqs.3.18c-e into 3.18b produces
\[ \dot{\sigma}_s = \Pi^T \, \dot{\sigma}_s \]  
(3.19)

Substitution of eq.3.16 and eq.3.17 into eq.3.18e produces
\[ \dot{H}_p = \lambda_p \frac{\partial H_p}{\partial w_p} \sigma_s^T \frac{\partial H_p}{\partial \sigma_s} \]  
\[ + \left( \frac{\partial H_p}{\partial \sigma_s} \right)^T \frac{\partial H_p}{\partial \sigma_s} \]  
(3.20)

Let
\[ A_p = \frac{\partial H_p}{\partial w_p} \sigma_s \]  
(3.21)

then
\[ \lambda_p = \frac{1}{A_p} \left( \dot{H}_p - \left( \frac{\partial H_p}{\partial \sigma_s} \right)^T \sigma_s \right) \]  
(3.22)
using eq.3.18a and eq.3.19, $\lambda_p$ can be determined as

$$\lambda_p = \frac{1}{A_p} \left( \bar{n} - i \right)^T \Omega_s$$  \hspace{1cm} (3.23)$$

where

$$i = \frac{\partial H_p}{\partial \Omega_s}$$

Plastic collapse strain is produced, if the plastic collapse yield criterion is satisfied, that is:

$$\begin{cases} f_c - H_c = 0 \\ \dot{f}_c > 0 \end{cases}$$  \hspace{1cm} (3.24)$$

where

$$f_c = f_c(I_{1s}, I_{2s}, I_{3s})$$

denotes the general form of the plastic collapse yield function;

$$H_c = H_c(w_c)$$

is the hardening or softening function depending on plastic collapse work $w_c$. The rate of the plastic collapse work can be determined by

$$\dot{w}_c = \Omega_s^T \dot{d}_c$$  \hspace{1cm} (3.25)$$

The rate of plastic collapse strain may be determined by the associated flow rule (Mendelson 1983) as follows

$$\dot{d}_c = \lambda_c \frac{\partial f_c}{\partial \Omega} = \lambda_c \dot{e}$$  \hspace{1cm} (3.26)$$
In the same way as in the derivation of $\lambda_p$, by considering the consistency condition

$$\dot{f}_c - H_c = 0 \quad (3.27)$$

$\lambda_c$ can be expressed as

$$\lambda_c = \frac{1}{A_c} L^T \Omega_s \quad (3.28)$$

where

$$A_c = \frac{\partial H_c}{\partial \omega_c} L^T \Omega_s$$

Regarding eqs. 3.13, 3.14, 3.17, 3.26, the total strain rate can be represented as

$$\dot{d} = D^{-1} \Omega_s + \lambda_p m + \lambda_c L \quad (3.29)$$

where

$$\lambda_p = \begin{cases} \text{defined by eq. 3.22} & \text{if eq. 3.15 is satisfied} \\ 0 & \text{otherwise} \end{cases} \quad (3.30)$$

$$\lambda_c = \begin{cases} \text{defined by eq. 3.28} & \text{if eq. 3.24 is satisfied} \\ 0 & \text{otherwise} \end{cases} \quad (3.31)$$

From eq. 3.29, eq. 3.28 and eq. 3.23, we can obtain

$$\begin{pmatrix} n \cdot j \end{pmatrix}^T D \dot{d} = \lambda_p A_p + \lambda_p \begin{pmatrix} n \cdot j \end{pmatrix}^T D m + \lambda_c \begin{pmatrix} n \cdot j \end{pmatrix}^T D L \quad (3.32a)$$
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\[ L_e^T D d = \lambda_c A_c + \lambda_p L_e^T D m + \lambda_c L_e^T D L \]  

(3.32b)

Therefore, \( \lambda_c, \lambda_p \) can be solved from eq.3.32 as

\[ \lambda_p = \frac{(n-j-\alpha_c)^T}{A_p + (n-j-\alpha_c)^T D m} D d \]  

(3.33a)

\[ \lambda_c = \frac{(L_e-\alpha_p)^T}{A_c + (L_e-\alpha_p)^T D L} D d \]  

(3.33b)

where

\[ \alpha_p = \begin{cases} 
\frac{L_e^T D m}{A_p + (n-j)^T D m} & \text{if eq.3.15 is satisfied.} \\
0 & \text{otherwise} 
\end{cases} \]

\[ \alpha_c = \begin{cases} 
\frac{(n-j)^T D L L_e}{A_c + L_e^T D L} & \text{if eq.3.24 is satisfied.} \\
0 & \text{otherwise} 
\end{cases} \]

Regarding eq.3.29 and eq.3.33 the rate independent stress rate - strain rate relation can be presented as

\[ \dot{\sigma}_s = H d \]  

(3.34)
where $H$ is the elastic-plastic matrix

$$H = \left( \frac{D - D_m (n - j - \alpha_c)^T D}{A_p + (n - j - \alpha_c)^T D_m} \cdot \frac{D L (L - \alpha_p)^T D}{A_c + (L - \alpha_p)^T D L} \right)$$

It is clear that, $H$ has the same form as that derived by Eibl and Haussler (1984). However, in the present formula, a new term $j$ is included in $H$. This new term $j$ can speed up the convergence, and improve the accuracy of the simulation.

3.7.1-3 A few specified forms of the elastic-plastic matrix

i) Lade elastic-plastic model with two yield surfaces

In Lade's model (1977), the total strain rate is decomposed into three parts: an elastic component $d_e$, a plastic collapse component $d_c$, and a plastic expansive component $d_p$. Fig.3.3 shows the parts of the total strain.

The elastic component is related to stress rate by Hook's law as described in eq.3.14. However, in this model the Young's modulus depends on stress level, i.e.,

$$E = K_{ur} p_a \left( \frac{\sigma_3}{p_a} \right)^{C_n}$$

where

$K_{ur}$ = modulus number of the material;

$p_a$ = atmospheric pressure;

$C_n$ = elastic exponent.
The plastic collapse strain rate $\dot{d}_c$ is determined by the associated flow rule as described in eq.3.20. The $\sigma$-type yield criterion is used to account for the plastic collapse strain occurring during isotropic compression. The equation for the yield cap has the form as

$$f_c = 1_{1s} - 2I_2$$  \hspace{1cm} (3.37)

The work hardening law has the form of

$$H_c = P_a \left( \frac{w}{C_p} \right) \frac{1}{C_p}$$  \hspace{1cm} (3.38)

where

$C_c = $ plastic collapse modulus,

$C_p = $ plastic collapse exponent.

Fig.3.3 Strain components in the Lade elastic-plastic model
From eq.3.37 and eq.3.38, the following formulae for the evaluation of the parameters included in the general form of elastic-plastic matrix $H$ can be obtained

$$L_s = -2I_{1s}\cdot I_{1s} - 2I_{2s} = 2\begin{pmatrix} \sigma_{11} & \sigma_{22} & 2\sigma_{12} & \sigma_{33} \end{pmatrix}^T$$ (3.39)

$$\alpha_s L_s = 2(2I_{2s} - I_{1s}) = 2f_c$$ (3.40)

$$\frac{\partial H_c}{\partial \nu_c} = c_c C_p \left( \frac{f_c}{p_a} \right)^{1.5}$$ (3.41)

$$A_c = \frac{\partial H_c}{\partial \nu_c} \alpha_s L_s = \frac{2p_a^3}{C_c C_p} \left( \frac{f_c}{p_a} \right)^{2.5}$$ (3.42)

The plastic expansive strain rate is related to stress rate by the non-associated flow rule as described in eq.3.17. The plastic expansive yield criterion is as follows

$$f_p = \begin{cases} \left( \frac{I_{1s}^3}{I_{1s}} - 27 \right) \left( \frac{I_{1s}}{p_a} \right)^{C_m} & \text{at failure} \\
1 & \text{else} \end{cases}$$ (3.43)

where

$C_m$ = plastic expansive yield exponent;

$\eta_1$ = plastic expansive yield constant.

The work hardening/work softening function is

$$H_p = C_e e^{-b_{wr}} \left( \frac{W_p}{p_a} \right)^{1/q}$$ (3.44)
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where

\[ C_a = \eta_1 \left( \frac{c_0 a}{w_{ppeak}} \right)^{\frac{1}{q}} \]

\[ b = \frac{1}{q w_{ppeak}} \]

\[ q = \alpha + \beta \left( \frac{\sigma_3}{p_a} \right) \]

\[ w_{ppeak} = C_{pp} p_a \left( \frac{\sigma_3}{p_a} \right)^{C_1} \]

\( \alpha, \beta, C_{pp}, C_1 \) are constants.

The plastic potential function is expressed in a form of

\[ g_p = -I_{1s}^3 + \left( 27 + \eta_2 \left( \frac{p_a}{I_{1s}} \right)^{C_m} \right) I_{3s} \]  \( (3.45) \)

where

\[ \eta_2 = C_S f_p + C_R \sqrt{\frac{\sigma_3}{p_a}} + C_t \]

\( \sigma_3 \) = minor principal stress;

\( C_S, C_R, C_t \) = plastic potential constants.

From eq.3.45, the following formulae can be obtained:

\[ m = \frac{\partial g_p}{\partial \omega} = \left( 27 + \eta_2 \left( \frac{p_a}{I_{1s}} \right)^{C_m} \right) I_{3s} - \left( 3I_{1s}^2 + I_{3s}^3 C_m \eta_2 \left( \frac{p_a}{I_{1s}} \right)^{C_m} \right) I_{1s} \]  \( (3.46) \)
$$
\alpha_s m = 3 g_p - C_m \eta_2 \left( \frac{p_a}{I_{15}} \right)^{C_m} I_{35} \tag{3.47}
$$

From eq. 3.43

$$
\eta = \frac{\partial \rho_p}{\partial \alpha} = \left( \frac{I_{15}}{p_a} \right)^{C_m} \left( \frac{2}{I_{15}} - \frac{27}{I_{15}} C_m \right) I_{15} - \frac{3}{I_{15}} \left( \frac{I_{15}}{p_a} \right)^{C_m} I_{35} \tag{3.48}
$$

From eq. 3.44

$$
\frac{\partial H_p}{\partial w_p} = f_p \left( \frac{1}{w_p} - \frac{1}{w_{ppeak}} \right) \tag{3.49}
$$

$$
A_p = \frac{\partial H_p}{\partial w_p} \Omega_s^T m = f_p \left( \frac{1}{w_p} - \frac{1}{w_{ppeak}} \right) \left( 3 g_p - C_m \eta_2 \left( \frac{p_a}{I_{15}} \right)^{C_m} I_{35} \right) \tag{3.50}
$$

$$
\frac{j}{\partial \alpha_s} = \frac{H_p}{C_a} \frac{\partial \alpha_s}{\partial \alpha_s} - w_p H_p \frac{\partial b}{\partial \alpha_s} - H_p \ln \left( \frac{w_p}{p_a} \right) \frac{1}{q^2} \frac{\partial q}{\partial \alpha_s} \tag{3.51}
$$

where

$$
\frac{\partial C_a}{\partial \alpha} = - C_a \ln \left( \frac{e p_a}{w_{ppeak}} \right) \left( \frac{1}{q^2} \right) \frac{\partial q}{\partial \alpha} - \frac{C_a}{q w_{ppeak}} \frac{\partial w_{ppeak}}{\partial \alpha} \tag{3.52}
$$

$$
\frac{\partial b}{\partial \alpha} = \left( \frac{-1}{q^2 w_{ppeak}} \right) \frac{\partial q}{\partial \alpha} + \frac{-1}{q^2 w_{ppeak}} \frac{\partial w_{ppeak}}{\partial \alpha} \tag{3.53}
$$

$$
\frac{\partial q}{\partial \alpha} = \frac{\beta}{p_a} \frac{\partial \sigma_3}{\partial \alpha} \tag{3.54}
$$
\[ \frac{\partial w_{\text{peak}}}{\partial \sigma} = - C_{pp} C_{1} \left( \frac{\sigma_{3}}{p_{a}} \right)^{c_{1}-1} \frac{\partial \sigma_{3}}{\partial \sigma} \]

with \( \sigma_{3} \) = minor principal stress.

\[ \sigma_{3} = \sigma_{x} \cos^{2} \theta + \sigma_{y} \sin^{2} \theta + 2\sigma_{xy} \sin \theta \cos \theta \]

\[ \frac{\partial \sigma_{3}}{\partial \sigma_{x}} = \cos^{2} \theta \]

\[ \frac{\partial \sigma_{3}}{\partial \sigma_{y}} = \sin^{2} \theta \]

\[ \frac{\partial \sigma_{3}}{\partial \sigma_{xy}} = 2\sin \theta \cos \theta \]

where \( \theta \) denotes the angle between the axis of \( \sigma_{3} \) with the coordinate axis \( x \).

Substitution of eqs.3.39, 3.42, 3.46, 3.48, 3.51, 3.50 into eq.3.35 can produce the elastic-plastic matrix for Lade's model.

ii) Drucker - Prager elastic, perfectly plastic model

In Drucker-Prager (1952) elastic, perfectly-plastic model, the total strain rate is decomposed into two parts, an elastic component and a plastic component. The plastic yield function has the form of

\[ F = C_{1} I_{1} + \sqrt{J_{2}} - C_{2} = 0 \]  \hspace{1cm} (3.52)

where \( C_{1} \) and \( C_{2} \) are the parameters determined by the angle of internal friction and the cohesion of the material; \( J_{2} \) denotes the second invariant of the deviatoric stress tensor.
By the application of an associated flow rule, the plastic strain rate can be determined as

\[
d_{pl} = \begin{cases} \\
\lambda \frac{\partial F}{\partial \sigma} & \text{if } F \geq 0 \\
0 & \text{otherwise}
\end{cases} \quad (3.53)
\]

Regarding section 3.7.1-2, at the present model

\[d_p = 0\]
\[f_c = F\]
\[H_c = 0\]

Therefore,

\[m = A_c = 0 \quad (3.54a)\]

\[L_s = \frac{\partial F}{\partial \sigma_s} \quad (3.54b)\]

Substitution eq.3.54 into eq.3.35, the elastic-plastic matrix corresponding to Drucker-Prager model can be expressed as

\[
H = \begin{pmatrix} \mathbf{D} - \mathbf{D} \mathbf{L} \mathbf{L}^T \mathbf{D} \\
\mathbf{L}^T \mathbf{D} \mathbf{L} \end{pmatrix} \quad (3.55)
\]
3.7.2 Elastic-Plastic Constitutive Laws

An elastic-plastic model is employed to described the stress-strain relation of bulk materials during storage or in the at-rest condition, and also is used to describe the stress-strain relation of the material from which silos are fabricated.

The feature of this model is that the strain increment is simply related to stress increment by an elastic-plastic matrix, that is

\[ \text{d} \sigma = H \text{d} \varepsilon \quad \text{in } \Omega_0 \text{ or } \Omega_s \quad (3.56) \]

where

\[ \text{d} \sigma, \text{d} \varepsilon = \text{increments of stress and strain respectively}; \]

\[ H = \text{elastic-plastic matrix, which dependents on the material assumptions.} \]

For the Lade elastic-plastic model, \( H \) is determined by eq.3.35. For the Drucker-Prager elastic, perfectly-plastic model, \( H \) is determined by eq.3.55. For the Von-Mises (see Mendelson, 1983) elastic, perfectly-plastic model, \( H \) is determined by eq.3.55. However, in this latter case

\[ L = \frac{\partial F}{\partial \sigma} \]

\[ F = J_2 - C_3 \]

where

\[ J_2 = \frac{1}{6} \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right) \]

is the second invariant of the deviatoric stress tensor, and \( C_3 \) is a constant of the material strength.
3.8 SUMMARY AND CONCLUDING REMARKS

In summary, in the domain of bulk granular material there are 19 equations (which include 3 dynamic equilibrium equations - eq.3.1, 9 kinematic equations - eq.3.3, 6 constitutive equations - eq.3.12, and 1 mass conservation equation - eq.3.4) in terms of 19 unknown functions (2*6 components of $s$, $d$; 2*3 components of $v$, $w$; and one component of $\rho$). In the domain of the silo shell, there are 15 equations (3 stress equilibrium equations - eq.3.2, 6 kinematic equations - eq.3.3a, and 6 constitutive equations eq.3.56) in terms of 15 unknown functions ($\sigma$, $v$, $d$). The number of unknown functions is equal to the number of equations. Therefore the problem is solvable theoretically.

Simulating the flow of bulk materials from rigid-walled silos requires the determination of the unknown functions in the domain of bulk granular materials. Theoretically the solution can be obtained by solving simultaneously the governing equations in the domain of the bulk materials, and making the solution satisfy the boundary conditions. However, mathematically it is impossible to obtain an analytical solution. Therefore, the boundary element method and the finite element method are used to solve the problem numerically, which is described in Chapters 4 and 5, respectively.

When the flexibility of the silo shell is considered, the problem becomes much more complicated. Simulating the dynamic behaviour of the bulk material - silo system requires the determination of all the unknown functions in both the domain of the bulk material and the domain of the silo shell. Theoretically, the solution can be obtained by solving the governing equations in both the domain of the bulk materials and the silo shell, and using the contact conditions and boundary
conditions simultaneously. However, mathematically it is also impossible to obtain an analytical solution. As an interaction problem, a numerical procedure which couple the motion of the bulk material and the silo shell is developed to obtain the numerical solution, which is described in Chapter 6.
CHAPTER FOUR

BOUNDARY ELEMENT METHOD
FOR THE PREDICTION OF FILLING PRESSURES

4.1 GENERAL

This chapter is devoted to the development and presentation of the boundary element method (BEM) for the prediction of bulk material pressures on the walls of rigid-walled silos under the filling condition.

At present, several attempts to use the finite element method (FEM) to predict bulk material pressures on the wall of silos have been reported, such as those by Bishara et al (1983), Haussler and Eibl (1984), Runesson and Nilsson (1986), Askari and Elwi (1988). However, no attempt appears to have been made to apply the boundary element method (BEM) to predict silo pressures.

Although the FEM has shown promise for predicting silo pressures in a satisfactory manner, the BEM offers the distinct advantage over the more classical displacement FEM of easily and accurately coupling the increments of normal traction with the increments of tangential friction traction in the system matrix, and of reducing the dimensionality of the problem by one. Therefore, it appears to be fruitful to use the BEM for the prediction of bulk material pressures on the wall of silos.
A considerable amount of literature on the application of the BEM to elastic-plastic problems has been published, such as those by Banerjee et al. (1979), Banerjee and Cathie (1980), Tells and Brebbia (1981), and Banerjee and Raveendra (1987). Research on the application of the BEM to the prediction of pressures on the contact surface of elastic bodies without body force has been reported by Andersson (1981) and Jin et al (1987). However, in Andersson's and Jin's procedures both body force and material nonlinearity, including plastic behaviour and non-tension behaviour, which for our case are the important factors dominating the silo pressures, have not been considered. Therefore, a numerical procedure based on the previous work mentioned above is developed for the prediction of silo pressures, in which all factors dominating silo pressures including elastic-plastic behaviour, no-tension behaviour and side wall friction as well as body force can be considered.

This chapter describes the mathematical model of the problem and demonstrates its implementation in the BEM analysis. It begins with the derivation of the governing partial differential equation of the problem and the boundary conditions in an incremental form. Then, the weighted residual technique is employed to convert the partial differential equation from a pointwise description to an integral form, that is, a boundary integral equation. The boundary integral equations are discretized by the subdivision of the boundary into a finite number of elements, and the internal domain into a number of cells. Finally, the discretized equations are solved numerically. Examples of application of the method are also presented in this chapter.

As this chapter concerns on the behaviour of bulk material only, all the quantities used are referred to the domain of the bulk material considered.
4.2 MATHEMATICAL DESCRIPTION OF THE PROBLEM

From Chapter 3, the equations governing the behaviour of bulk granular materials under static condition can be written in an incremental form as follows

\[ \nabla \cdot \sigma + f_v = 0 \quad \text{or} \quad \sigma_{ij} + f_{vi} = 0 \quad (4.1) \]

\[ \dot{\varepsilon} = \frac{1}{2} \left( \nabla \dot{u} + (\nabla \dot{u})^T \right) \quad \text{or} \quad \varepsilon_{ij} = \frac{1}{2} (\dot{u}_{ij} + \dot{u}_{ji}) \quad (4.2) \]

\[ \dot{\sigma} = \mathbf{H} \dot{\varepsilon} = (\mathbf{D} - \mathbf{D}_p) \dot{\varepsilon} \quad (4.3) \]

where \( \dot{\varepsilon} \) denotes a strain vector, \( \mathbf{D} \) and \( \mathbf{D}_p \) are the elasticity and plasticity matrices respectively, and the superposed dot indicates an increment. This definition is valid throughout this chapter.

Therefore, the problem of simulating the behaviour of bulk granular materials stored in rigid-walled silos is to find the solution of the eqs.4.1-4.3 in domain \( \Omega_g \), satisfying the boundary conditions:

\[ p_n = p_n^g \quad \text{and} \quad p_t = p_t^g \quad \text{on} \quad \Gamma_f \quad (4.4a) \]

\[ p_t = (-\tan \phi_w \frac{\dot{u}_t}{|\dot{u}_t|}) (-p_n) \quad \text{on} \quad \Gamma_{cs} \quad (4.4b,c) \]

\[ \dot{u}_n = 0 \quad \text{on} \quad \Gamma_{ca} \quad (4.4d) \]

For the implementation of the above model in the BEM, eq.4.3 is rewritten as

\[ \dot{\sigma}_{ij} = \lambda_1 \delta_{ij} \dot{\varepsilon}_{kk} + 2\mu_1 \dot{\varepsilon}_{ij} - \sigma_{ij}^o \quad (4.5) \]

where \( \lambda_1, \mu_1 \) are the elastic constants, and the superscript o denotes the plastic component of the stress.
Substitution of eq.4.5 into eq.4.1 produces
\[\lambda_1 \dot{e}_{kk,i} + 2\mu_1 \dot{e}_{ij,j} - \sigma_{ij,j} + f_{vi} = 0 \quad (4.6)\]

Substitution of eq.4.2 into eq.4.6 produces
\[\mu_1 \dot{u}_{i,jj} + (\lambda_1 + \mu_1) \dot{u}_{k,ik} - \sigma_{ij,j} + f_{vi} = 0 \quad (4.7)\]

The boundary conditions are also rewritten in an incremental form. For a typical load step, if the contact status remain in adhesion, the increments of displacement are equal to zero, that is
\[\dot{u}_t = 0, \quad \dot{u}_n = 0 \quad (4.8)\]

If the contact status is from slip to slip, the boundary condition can be represented in an incremental form as
\[
\begin{cases}
\dot{p}_t = (-\tan\phi_w \frac{\dot{u}_t}{|\dot{u}_t|}) (-\dot{p}_n) & \text{on } \Gamma_{cs} \\
\dot{u}_n = 0
\end{cases} \quad (4.9a,b)
\]

If a contact element changes the contact status from adhesion to slip, the element has a residual tangential traction \(p_{tr}\) (shown in Fig.4.1) which can be determined as,
\[p_{tr} = (-\tan\phi_w \frac{\dot{u}_t}{|\dot{u}_t|}) (-\dot{p}_n^o) - \dot{p}_t^o \quad (4.10)\]

where superscript \(o\) refers to the last load step. The first term of the right hand side of eq.4.10 is the critical tangential boundary traction at which slip can occur at the last load step, and the second term of the right hand side is the real tangential boundary traction at the last load step. Then, \(p_{tr}\) represents the increment of the tangential traction needed to complete the total traction to satisfy, at load step \(o\), the
Fig. 4.1 The relation of traction components in an element changing the status of contact from adhesion to slip

following equation,

$$ p_t = (-\tan \phi_w \frac{\dot{u}_t}{|\dot{u}_t|})(-p_n) $$

Therefore, for a slip element the increment of tangential traction and the increment of normal traction is coupled by the general form of the formulae:

$$ \dot{p}_t = (-\tan \phi_w \frac{\dot{u}_t}{|\dot{u}_t|})(-\dot{p}_n) + p_{tr} $$

$$ \dot{u}_n = 0 $$

where

$$ p_{tr} = \begin{cases} 
\text{defined by eq.4.10} & \text{if contact status changes from adhesion to slip} \\
0 & \text{if contact status is from slip to slip}
\end{cases} $$

Now the problem becomes to find the solution of eq.4.7, satisfying boundary condition eq.4.8 on $\Gamma_{ca}$ and eq.4.11 on $\Gamma_{cs}$. 
4.3 BOUNDARY INTEGRAL EQUATION

By applying the weighted residual technique to eq.4.7 and following similar procedures by Brebbia (1984), the following initial stress boundary integral formulae are obtained

\[
c_{ji} \dot{u}_i(x) = \int_{\Gamma} \left( u_{ji}^*(x,y) \dot{p}_i(y) \right) d\Gamma - \int_{\Gamma} p_{ji}^*(x,y) \dot{u}_i(y) d\Gamma + \int_{\Gamma_n} \left( u_{ji}^*(x,y) \dot{r}_v(y) \right) d\Omega + \\
\int_{\Gamma_n} \varepsilon_{jik}^*(x,y) \sigma_{ik}(y) d\Omega
\]

(4.12)

where the superposed dot denotes the increment; \( \Gamma \) is the total boundary of the domain \( \Omega \); \( \sigma^0 \) represents the residual stresses due to tensile failure or plastic yielding represented by eq.4.5; \( c_{ji} \) is the constant depending on the position of the source point \( x \) where unit load is applied (Danson, 1984); \( u_{ji}^* \) are the fundamental solutions of the condition

\[
\mu_1 u_{ji, kk} + (\lambda_1 + \mu_1) u_{jk, ik} + \delta_{ji} = 0
\]

(4.13)

and \( \varepsilon_{jik}^*, \ p_{ji}^* \) are the strain field and the boundary tractions corresponding to \( u_{ji}^* \).

Applying eq.4.12 on the points of the boundary and using eq.4.8 and eq.4.11 for the points on the contact boundary, two of the unknown variables are eliminated. The following system of boundary integral equations is obtained.

\[
c_{ji} \dot{u}_i = \int_{\Gamma_f} u_{ji}^* p_{ji}^* d\Gamma - \int_{\Gamma_f} p_{ji}^* \dot{u}_i d\Gamma + \int_{\Gamma_{ca}} u_{ji}^* \dot{p}_i d\Gamma \\
+ \int_{\Gamma_{ca}} \left( p_{ji}^* \dot{u}_i + \left( u_{jn}^* + \tan \phi_w \frac{\dot{u}_i}{\dot{u}_j} \right) \dot{p}_n \right) d\Gamma + 
\]
\[ + \int_{\Gamma_e} u_{ij}^* p_{tr} d\Gamma + \int_{\Omega} u_{ij}^* \dot{r}_{vi} d\Omega + \int_{\Omega} \varepsilon_{jik}^* \sigma_{ik} d\Omega \]  \hspace{1cm} (4.14)

### 4.4 BOUNDARY ELEMENT DISCRETIZATION

To solve equations 4.14 numerically, the boundary is discretized into elements, which are taken as straight line segments connected at the nodes. The values of \( u_i, \ p_i \) within one element are assumed as an interpolation function of the nodal values, and the interior region \( \Omega \), which is expected to yield or fail in tension as a result of loading, is then divided into a suitable number of quadrilateral cells, so that the integrals can be evaluated numerically. Fig.4.3 shows a typical subdivision of the boundary and domain for one half of a plane strain silo. Although such a discretization has the same appearance as that used in the finite element method, the cells are used here simply to evaluate the domain integral as a piecewise summation; they do not add new unknowns to the problem, i.e. the unknowns still refer only to the boundary. Therefore, the discretized system equations can be derived and represented as follows

\[ c_{ji} \ddot{u}_i + \sum_{T_{ji}} T_{ji} \dot{u}_i - \sum_{U_{ji}} U_{ji} \dot{p}_i + \sum_{\Gamma_{es}} \Gamma_{es} \left( T_{jn} \dot{u}_n + \tan \phi_w \frac{\dot{u}_t}{|u_t|} U_{jt} \right) \dot{p}_n = \]

\[ \sum_{T_{ji}} T_{ji} \dot{u}_i - \sum_{U_{ji}} U_{ji} \dot{p}_i + \int_{\Gamma_e} \dot{r}_{vi} d\Gamma + \int_{\Omega} \varepsilon_{jik}^* \sigma_{ik} d\Omega \]  \hspace{1cm} (4.15)

where

\[ T_{ji}^* = \int_{\Gamma_e} p_{ji} d\Gamma \]

\[ U_{ji}^* = \int_{\Gamma_e} u_{ji}^* d\Gamma \]
The integral over the boundary element $\Gamma_e$ can be evaluated analytically, which will be described in section 4.5. The domain integral over the cell $\Omega$ is evaluated by using a 9 point Gaussian Quadrature Scheme (Zienkiewicz, 1979) in every cell. The first domain integral on the right hand side of eq.4.15 only needed to be evaluated once in an incremental analysis scheme, and the other domain integrals only needed to be evaluated on those cells which fulfill the yield condition or tensile failure condition.

For the problem with $N$ boundary elements, the unknowns is $2N$. By applying eq.4.15 on the center of every element along tangential and normal direction respectively, we can obtain $2N$ equations. Therefore, the problem can be solved numerically.

For the convenience of application, eq.4.15 is rewritten in a matrix form as

\[
\sum_{j=1}^{N} C_{ii}^t \zeta_t^j + \sum_{j=1}^{N} C_{in}^t \zeta_n^j = b_i^t
\]

\[
\sum_{j=1}^{N} C_{nt}^i \zeta_t^j + \sum_{j=1}^{N} C_{nn}^i \zeta_n^j = b_n^i
\]

where

\[
\zeta_t = \begin{cases} 
\dot{u}_t & \text{on } \Gamma_{cf} \text{ and } \Gamma_{cs} \\
\dot{p}_t & \text{on } \Gamma_{ca} 
\end{cases}
\]

\[
\zeta_n = \begin{cases} 
\dot{u}_n & \text{on } \Gamma_{cf} \\
\dot{p}_n & \text{on } \Gamma_{ca} \text{ and } \Gamma_{cs} 
\end{cases}
\]
\[ C_{ij}^{k \ell} = \begin{cases} 
\int_{\Gamma_j} p_{k1}^i (F_k^i) \, d\Gamma + c_{k\ell} \delta_{ij} & \text{if } \Gamma_j \in \Gamma_f \text{ or } \Gamma_{cs} \\
- \int_{\Gamma_j} u_{k1}^i (F_k^i) \, d\Gamma & \text{if } \Gamma_j \in \Gamma_{ca} 
\end{cases} \]

\[ C_{kn}^{ij} = \begin{cases} 
\int_{\Gamma_j} p_{kn}^i (F_k^i) \, d\Gamma + c_{kn} \delta_{ij} & \text{if } \Gamma_j \in \Gamma_f \\
- \int_{\Gamma_j} u_{kn}^i (F_k^i) \, d\Gamma & \text{if } \Gamma_j \in \Gamma_{ca} \\
- \int_{\Gamma_j} u_{kn}^i (F_k^i) \, d\Gamma - \tan \phi \frac{u_{ki}}{|u_{ki}|} \int_{\Gamma_j} u_{k1}^i (F_k^i) \, d\Gamma & \text{if } \Gamma_j \in \Gamma_{cs} 
\end{cases} \]

\[ b_k^i = \sum_{\Gamma_e} \int_{\Gamma_e} u_{k1}^i (x_i, y) \, p_i \, d\Gamma(y) + \sum_{\Omega_e} \int_{\Omega_e} u_{k1}^i (x_i, y) \, \tilde{f}_{v_i} \, d\Omega(y) + \]

\[ \sum_{\Omega_e} \int_{\Omega_e} \epsilon_{ki1} (x_i, y) \sigma_{ik} \, d\Omega(y) + \sum_{\Gamma_e} \int_{\Gamma_e} u_{k1}^i (x_i, y) \, p_{tr} \, d\Gamma(y) \]

\[ k = t, n. \]

\( x_i \) is the coordinate of the center of ith element, which is the position of the unit load. The first subscript \( k \) is the direction of the unit load; the second subscript is the direction of displacement or boundary traction. The first superscript denotes the element at which unit load is applied; the second superscript is the element considered.
4.5 EVALUATION OF BOUNDARY COEFFICIENTS

The formation of eq.4.16 requires the evaluation of the coefficients $C_{kl}^{ij}$ and $b_k^i$, which in turn are due to the computation of the following integrals:

\[ C_{tt}^{ij} = \int_{\Gamma_j} u_t^* (F_t^i) \, d\Gamma \]

\[ C_{tn}^{ij} = \int_{\Gamma_j} u_n^* (F_t^i) \, d\Gamma \]

\[ C_{nt}^{ij} = \int_{\Gamma_j} u_t^* (F_n^i) \, d\Gamma \]

\[ C_{nn}^{ij} = \int_{\Gamma_j} u_n^* (F_n^i) \, d\Gamma \]

\[ C_{tt}^{ij} = \int_{\Gamma_j} p_t^* (F_t^i) \, d\Gamma + c_{tt} \delta_{ij} \]

\[ C_{tn}^{ij} = \int_{\Gamma_j} p_n^* (F_t^i) \, d\Gamma + c_{tn} \delta_{ij} \]

\[ C_{nt}^{ij} = \int_{\Gamma_j} p_t^* (F_n^i) \, d\Gamma + c_{nt} \delta_{ij} \]

\[ C_{nn}^{ij} = \int_{\Gamma_j} p_n^* (F_n^i) \, d\Gamma + c_{nn} \delta_{ij} \]

Formulae for the evaluation of $C_{kl}^{ij}$ and $C_{kl}^{ij}$ have been derived by Crouch and Starfield (1974), and therefore the coefficients $C_{kl}^{ij}$ in eq.4.16 can be evaluated as
\( C_{ij}^{kt} = \begin{cases} \begin{align*} CA_{kt}^{ij} & \text{on } \Gamma_t \text{ and } \Gamma_{cs} \\ -CB_{kt}^{ij} & \text{on } \Gamma_{ca} \end{align*} \end{cases} \) \( (k = t, n) \)

\( C_{kn}^{ij} = \begin{cases} \begin{align*} CA_{kn}^{ij} & \text{on } \Gamma_f \\ -CB_{kn}^{ij} & \text{on } \Gamma_{ca} \\ -CB_{kn}^{ij} \cdot \tan \phi_w \frac{\dot{u}_t}{|\dot{u}_t|} CB_{kt}^{ij} & \text{on } \Gamma_{cs} \end{align*} \end{cases} \) \( (k = t, n) \)

4.6 SOLUTION PROCEDURE AND COMPUTER PROGRAM

Eq. 4.16 is nonlinear, because \( \xi^0 \) and \( p_u \) depend on the unknown boundary values of \( \dot{u} \) and \( \ddot{u} \). Therefore, a double iterative scheme over the material nonlinearity and the side wall friction is used for the solution of the unknown boundary values.

Once the unknown boundary values are solved, the displacements and stresses at any internal point can be determined from eqs. 4.15, 4.2, and 4.5.

A computer program based on the above analysis has been developed. Fig. 4.2 shows the computation procedures for the solution of the unknown boundary values at a typical load step.
Chapter 4  Boundary element method for the prediction of filling pressures

4

Load step \( n \)

Form boundary coefficient matrix \( C \)

\[
\zeta_0^n = \zeta_0^{n-1} \quad \sigma = \sigma^{n-1}
\]

\( i = 1 \)

\[
\mathbf{b} = \int_\Omega \mathbf{u}_j^i \mathbf{f}_v^i \, d\Omega + \int_{\Gamma_{cs}} \mathbf{u}_j^i \mathbf{p}_{in}^i \, d\Gamma + \int_{\Gamma_f} \mathbf{u}_j^i \mathbf{p}_f^i \, d\Gamma
\]

Solve \( \Delta \zeta = C^{-1} \mathbf{b} \)

Calculate \( \Delta \mathbf{a} \), \( \Delta \mathbf{p} \), \( \Delta \sigma \) from \( \Delta \zeta \)

Update \( \zeta = \zeta^{n-1} + \Delta \zeta \) and \( \sigma = \sigma^{n-1} + \Delta \sigma \)

Yes

\( \Delta \sigma < \text{tol} ? \)

No

\( i = i + 1 \)

\[
\mathbf{b} = \int_\Omega \mathbf{u}_j^{i-1} \mathbf{f}_v^i \, d\Omega
\]

Contact condition violate ?

Yes

Change contact condition

Next load step

Fig. 4.2 Iterative scheme for the solution at a typical load step
4.7 EXAMPLES

Typical examples for the application of the method are shown as follows.

1) Single symmetrical plane strain silo (Fig.4.3)

This example is designed to compare the results by this method with some other analytical results. The bulk material adopted is of density \( \rho = 1600 \text{ kg/m}^3 \), Young's modulus \( E = 50 \text{ MPa} \), Poisson's ratio \( \nu = 0.3 \), the angle of internal friction \( \phi = 30^\circ \), and the angle of wall friction \( \phi_w = 20^\circ \).

Fig.4.3 shows the mesh adopted. Fig.4.4 shows the computed principal stress fields and the wall pressure distribution for the case of a closed and open hopper. The wall pressures are compared with the vertical wall pressure prediction of the classical Janssen theorem (1895) as well as the hopper wall initial and flow pressure predictions of Jenike et al (1973). It is noted that the pressure on the vertical wall corresponds with Janssen's pressure, and the filling hopper pressure is close to Jenike's predicted pressure for the closed hopper.

The initial discharging pressure, however, is smaller than Jenike's predicted pressure over a slant distance of 0.3 DW (DW is the width of the silo for the plane strain problem) below the transition of the vertical wall and hopper, and larger on the other part of the hopper walls. The reason is due to Jenike's assumption that on opening an outlet, a deficiency in the vertical wall support between that due to the initial pressures and that due to the flow pressures is provided by an additional wall pressure having a triangular distribution and acting over a slant distance of 0.3 DW below the transition. It is also noted that near the hopper outlet the initial
discharging pressure becomes smaller with the decrease of distance from the junction, instead of becoming larger as in Askari's model (1988). This reduction in pressure is due to the yield criterion adopted herein, as the deviatoric stresses cannot exceed a certain limiting value.

Fig.4.3 Discretization of boundary and domain
Fig. 4.4 Principal stress fields and wall pressures
2) Plane strain silo with double outlets (Fig.4.5)

This example is designed to show the capability of the method for solving a more general silo pressure problem. The same bulk material properties as in (1) are employed, and the mesh for one half of the structure is shown in Fig.4.5. In this example the wall pressures are computed for three different cases: a) both outlets closed, b) one outlet closed, and c) both outlets open. The results are shown in Figs.4.6 - 4.8. It is clear that:

a) for the case of a closed hopper the stress orientation in the lower regions is nearly vertical. When the hopper bottom is opened the orientation changes to a direction nearly normal to the hopper wall inclination;

b) the maximum bottom pressure occurs when one of outlets is opened and the other is closed;

c) the maximum wall pressure at the cylinder - hopper transition occurs when both outlets are opened;

d) the position of the maximum wall pressures is at the junction of the two hoppers.
Fig. 4.5 Discretization of boundary and domain
Scale: Wall pressure  \( 0 \rightarrow 100 \text{ KPa} \)
Principal stress  \( 0 \rightarrow 200 \text{ KPa} \)

Fig. 4.6 Principal stress field and wall pressure for both outlets closed
Scale:  Wall pressure  \[0 \rightarrow 100 \text{ KPa}\]

Principal stress  \[0 \rightarrow 200 \text{ KPa}\]

Fig. 4.7 Principal stress field and wall pressure for both outlets open
Fig. 4.8 Principal stress field and wall pressure for one outlet open and the other closed
4.8 SUMMARY AND CONCLUDING REMARKS

A complete set of formulae based on the boundary integral technique has been derived for solving the contact problem of bulk granular material with silo walls under static conditions. From the derived equations, a numerical procedure and an associated computer program have been developed for the prediction of bulk material pressures on the walls of silos under the filling and the initial discharging conditions. Although this work appears to be the first attempt to apply the boundary element method to predict pressures in silos, it is found that the method is useful at least for solving the filling condition and the initial discharge condition.
CHAPTER FIVE

SIMULATION OF THE FLOW OF BULK MATERIALS THROUGH RIGID-WALLED SILOS

5.1 GENERAL

This chapter focusses on the development and presentation of the finite element method (FEM) for the simulation of the flow of bulk materials from rigid-walled silos.

In Chapter 3, it has been demonstrated that the motion of bulk granular materials in rigid-walled silos is dominated by the following equations in the bulk solid domain $\Omega$:

$$\nabla \cdot \mathbf{\sigma} + \mathbf{f}_v = \rho (\dot{\mathbf{v}} + \nabla \mathbf{v} \cdot \mathbf{v})$$

(5.1)

$$d = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

(5.2a)

$$w = \frac{1}{2} (\nabla \mathbf{v} - \nabla \mathbf{v}^T)$$

(5.2b)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

(5.3)

$$\frac{\partial \mathbf{\sigma}}{\partial t} = \mathbf{H} d + \mathbf{G} \frac{\partial d}{\partial t} - (\mathbf{w} \mathbf{w} \mathbf{w} + \nabla \mathbf{\sigma} \cdot \mathbf{v}) + \mathbf{G} (d \mathbf{w} - w \mathbf{w} + \nabla d \cdot \mathbf{v})$$

(5.4)

and the boundary conditions:
\[
\left\{
\begin{array}{l}
p_t = ( - \tan \phi_w \frac{v_t^g}{|v_t^g|} ) ( - p_n^g ) \\
v_n^g = 0
\end{array}
\right.
\text{on } \Gamma_{cs} \tag{5.5a}
\]
\[
v_t^g = v_n^g = 0 \quad \text{on } \Gamma_{ca} \tag{5.5b}
\]
\[
\dot{p}_n = p_n^o \quad \text{and} \quad \dot{p}_t = p_t^o \quad \text{on } \Gamma_f \tag{5.5c}
\]
as well as initial conditions.

The derivation of the numerical solution of the problem by FEM starts from the conversion of stress equilibrium equations 5.1 from a local description (or pointwise description) to a global integral form (or virtual work equations). Then, the global integral equations are discretized by a finite number of elements both in space and in time. Finally, the discretized equations, which are highly nonlinear due to large deformation, side wall friction and etc., are solved by using an iterative scheme.

Procedures leading to the numerical solution of the problem are described in this chapter.

### 5.2 VIRTUAL WORK EQUATIONS

To convert eq.5.1 to a global integral form, multiply eq.5.1 by an arbitrary function \( \delta y^T \) satisfying the displacement boundary conditions eq.5.5, and integrate over the bulk material domain \( \Omega \), That is,
\[ \int_\Omega \delta \mathbf{v}^T \left[ \nabla \mathbf{\sigma} + \mathbf{f}_V \cdot \rho \left( \dot{\mathbf{v}} + \nabla \mathbf{\cdot v} \right) \right] d\Omega = 0 \quad (5.6) \]

Use of the divergence theorem, together with eq.5.2 and eq.5.5, gives

\[ \int_\Omega \delta \mathbf{v}^T \nabla \mathbf{\sigma} d\Omega = - \int_\Omega \delta \mathbf{d}^T \mathbf{\sigma} d\Omega + \int_{\Gamma_f + \Gamma_\alpha} \delta \mathbf{v}^T \mathbf{f}_\tau d\Gamma \quad (5.7) \]

where

\[
\mathbf{f}_\tau = \begin{cases} 
\mathbf{\tau}_f & \text{traction on } \Gamma_f \\
\sigma_n \tan \phi_w & \text{friction force on } \Gamma_{cs}
\end{cases}
\]

Substitution of eq.5.7 into eq.5.6 produces

\[
\int_\Omega \delta \mathbf{d}^T \mathbf{\sigma} d\Omega + \int_\Omega \delta \mathbf{v}^T \left( \dot{\mathbf{v}} + \nabla \mathbf{\cdot v} \right) \rho d\Omega = \int_\Omega \delta \mathbf{v}^T \mathbf{f}_V d\Omega + \int_{\Gamma_f + \Gamma_\alpha} \delta \mathbf{v}^T \mathbf{f}_\tau d\Gamma \quad (5.8)
\]

### 5.3 FINITE ELEMENT DISCRETIZATION IN SPACE

In order to solve eq.5.8 numerically, the domain \( \Omega \) is discretized into a finite number of elements connected at the nodes. In each element, the velocity field \( \mathbf{v} \) and strain rate \( \mathbf{d} \) are approximated by

\[
\mathbf{v} = \mathbf{N}(x) \mathbf{a}(t) \quad (5.9a)
\]

\[
\mathbf{d} = \mathbf{B}(x) \mathbf{a}(t) \quad (5.9b)
\]

where
\[ N = \text{matrix of specified shape functions}; \]
\[ B = \text{partial derivative matrix of } N \text{ with respect to coordinates } \mathbf{x}; \]
\[ \mathbf{a} = \text{vector of nodal velocity}. \]

Substitution of eq.5.9 into eq.5.8 produces

\[
\delta_\mathbf{a}^T \left( \int \mathbf{N}^T \mathbf{N} \rho \, d\mathbf{\Omega} \hat{\mathbf{a}} + \int \mathbf{N}^T \mathbf{\nabla}_\mathbf{v} \cdot \mathbf{N} \rho \, d\mathbf{\Omega} \mathbf{a} + \int \mathbf{B}^T \mathbf{\sigma} \, d\mathbf{\Omega} - \int \mathbf{N}^T \mathbf{f}_v \, d\mathbf{\Omega} \right)
\]

\[
= \delta_\mathbf{a}^T \left( \int_{\Gamma_f} \mathbf{N}^T \mathbf{f}_t \, d\Gamma + \int_{\Gamma_{cs}} \mathbf{N}^T \mathbf{f}_t \, d\Gamma \right)
\]

Eliminating \( \delta_\mathbf{a}^T \) from eq.5.10, and abbreviating the notation, there results

\[
\mathbf{M} \dot{\mathbf{a}} + \mathbf{M}_c \mathbf{a} + \mathbf{r} = \mathbf{P} \quad (5.11)
\]

where

\[
\mathbf{M} = \int \mathbf{N}^T \mathbf{N} \rho \, d\mathbf{\Omega} \]
\[
\mathbf{M}_c = \int \mathbf{N}^T \mathbf{\nabla}_\mathbf{v} \mathbf{N} \rho \, d\mathbf{\Omega} \]
\[
\mathbf{r} = \int \mathbf{B}^T \mathbf{\sigma} \, d\mathbf{\Omega} \]
\[
\mathbf{P} = \int \mathbf{N}^T \mathbf{f}_v \, d\mathbf{\Omega} + \int_{\Gamma_f} \mathbf{N}^T \mathbf{f}_t \, d\Gamma + \int_{\Gamma_{cs}} \mathbf{N}^T \mathbf{f}_t \, d\Gamma
\]

### 5.4 DISCRETIZATION IN TIME

For a typical time step \( t^{n+1} \), eq.5.11 may be approximated by

\[
\mathbf{M}^{n+1} \ddot{\mathbf{a}}^{n+1} + \frac{\mathbf{M}_c^{n+1} \mathbf{a}^{n+1} + \mathbf{r}^{n+1}}{\Delta t} = \mathbf{P}^{n+1} \quad (5.12)
\]
The stress tensor in the expression of $\mathbf{F}^{n+1}$ depends on the position $\mathbf{x}$ and the time $t$, that is,

$$\mathbf{\sigma} = \mathbf{\sigma}(\mathbf{x},t)$$

(5.13)

The stress rate has been determined in Chapter 3 as,

$$\frac{\partial \mathbf{\sigma}}{\partial t} = \mathbf{H} \mathbf{d} + \mathbf{G} \frac{\partial \mathbf{d}}{\partial t} + \mathbf{\tau}$$

(5.14)

where

$$\mathbf{\tau} = - (\mathbf{\sigma} \mathbf{w} - \mathbf{w} \mathbf{\sigma} + \nabla \mathbf{\sigma} \mathbf{v}) + \mathbf{G} (\mathbf{d} \mathbf{w} - \mathbf{w} \mathbf{d} + \nabla \mathbf{d} \mathbf{v})$$

For the time step $t^{n+1}$, eq.5.14 can be approximated by

$$\mathbf{\sigma}^{n+1} = \mathbf{\sigma}^{n} + \Delta t \left( \mathbf{H}^{n+1} \mathbf{d}^{n+1} + \mathbf{G} \frac{\partial \mathbf{d}^{n+1}}{\partial t} + \mathbf{\tau}^{n+1} \right)$$

(5.15)

Using the approximation

$$\frac{\partial \mathbf{d}^{n+1}}{\partial t} = \frac{\mathbf{d}^{n+1} - \mathbf{d}^{n}}{\Delta t}$$

(5.16)

eq.5.15 becomes

$$\mathbf{\sigma}^{n+1} = \mathbf{\sigma}^{n} + \Delta t \left( \mathbf{H}^{n+1} \mathbf{d}^{n+1} + \mathbf{\tau}^{n+1} \right) + \mathbf{G} (\mathbf{d}^{n+1} - \mathbf{d}^{n})$$

(5.17)

Using eq.5.17 together with eq.5.9 produces

$$\mathbf{R}^{n+1} = \int_{\Omega} \mathbf{B} \mathbf{\sigma}^{n+1} d\Omega = \int_{\Omega} \mathbf{B} \mathbf{\sigma}^{n} d\Omega + \Delta t \int_{\Omega} \mathbf{B} \mathbf{H}^{n+1} \mathbf{d}^{n+1} + \Delta t \int_{\Omega} \mathbf{B} \mathbf{\tau}^{n+1} d\Omega +$$

$$\int_{\Omega} \mathbf{B} \mathbf{G} \mathbf{d}^{n+1} (\mathbf{a}^{n+1} - \mathbf{a}^{n})$$
\begin{equation}
\tau^n + \Delta t \mathbf{K}^{n+1} \mathbf{a}^{n+1} + \Delta t \mathbf{f}_n^{n+1} + \mathbf{C} ( \mathbf{a}^{n+1} - \mathbf{a}^n ) \tag{5.18}
\end{equation}

Substitution of eq.5.18 into eq.5.12 produces

\begin{equation}
\left( \frac{1}{\Delta t} \mathbf{M}^{n+1} + \mathbf{M}_c^{n+1} + \mathbf{C} + \Delta t \mathbf{K}^{n+1} \right) \mathbf{a}^{n+1} + \Delta t \mathbf{f}_n^{n+1} = \mathbf{p}^n - \mathbf{r}_n^1 + C + \Delta t \mathbf{K}^{n+1} \mathbf{a}^n \tag{5.19}
\end{equation}

where

\begin{align*}
\mathbf{C} &= \int_\Omega \mathbf{B}^T \mathbf{G} \mathbf{B} \, d\Omega \\
\mathbf{K}^{n+1} &= \int_\Omega \mathbf{B}^T \mathbf{H}^{n+1} \mathbf{B} \, d\Omega \\
\mathbf{f}_n^{n+1} &= \int_\Omega \mathbf{B}^T \mathbf{f}^{n+1} \, d\Omega
\end{align*}

In the above equations, \( \mathbf{C} \) is a constant matrix. However, \( \mathbf{M}^{n+1} \) and \( \mathbf{M}_c^{n+1} \) depend on \( \mathbf{a}^{n+1} \), and \( \mathbf{K}^{n+1} \) depends on \( \mathbf{a}^{n+1} \). Therefore an iterative scheme is needed to solve the equation numerically.

### 5.5 ELEMENT CHARACTERISTICS

In Section 5.3, it has been demonstrated that for the solution of the problem, the domain has to be discretized into a finite number of elements. Then, within each element, the unknown functions are assumed as interpolation functions of the nodal values. This section describes the discretization of the domain and the characteristics of the interpolation function, or in other words, the element characteristics.
For a plane strain problem, the domain is discretized into a number of quadrilateral elements with four nodes.

For an axisymmetrical problem, the domain is divided into a series of conical frustums connected at their edges. As the domain and the loading are axisymmetric, the displacement of a point on the domain is uniquely determined by two components in the longitudinal tangential and normal directions respectively. Lateral tangential displacement does not exist and the internal stresses do not vary in the lateral tangential direction. Thus the element becomes two dimensional, as shown in Fig.5.1.

The element characteristics, which include the displacement shape function and the strain and stress variation within an element, are described in the following.
5.5.1 Velocity

Isoparametric elements (Akin, 1982) are used for the discretization of the bulk material domain. Each element in global coordinate is transformed into a basic element in local coordinates, and then the velocity shape functions are defined in terms of local coordinates.

Consider an element with nodes i, j, k, l in global Cartesian co-ordinates (Fig.5.2a). By the co-ordinate transformation

\[
x = \sum N_i(\xi, \eta) x_i \quad (5.20a)
\]
\[
y = \sum N_i(\xi, \eta) y_i \quad (5.20b)
\]

the distorted element in global Cartesian co-ordinates can be mapped into a basic element in local co-ordinates, as shown in Fig.5.2b (Zienkiewicz, 1979). In the above formulae, \(N_i\) denote shape functions, which are defined in terms of local co-ordinate \((\xi, \eta)\) and have the form of

\[
N_i(\xi, \eta) = \frac{1}{4} (1 + \xi \xi_i)(1 + \eta \eta_i)
\quad (5.21)
\]

Fig.5.2 Two dimensional mapping of a four-node element
Chapter 5 Simulation of the flow of bulk materials through rigid-walled silos

Using the basic element with the nodes numbered anti-clockwise, we define the nodal displacement by its two components as

\[ \mathbf{a}_1 = \begin{cases} a_{i_x} \\ a_{i_y} \end{cases} \]  \hspace{1cm} (5.22)

and the element displacements by the vector

\[ \mathbf{a}_e = \begin{cases} a_i \\ a_j \\ a_k \\ a_l \end{cases} \]  \hspace{1cm} (5.23)

The displacements within each element are defined uniquely by

\[ \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \mathbf{N} \mathbf{a}_e = \begin{pmatrix} \mathbf{N}_i & 0 & \mathbf{N}_j & 0 & \mathbf{N}_k & 0 & \mathbf{N}_l & 0 \\ 0 & \mathbf{N}_i & 0 & \mathbf{N}_j & 0 & \mathbf{N}_k & 0 & \mathbf{N}_l \end{pmatrix} \mathbf{a}_e \]  \hspace{1cm} (5.24)

5.5.2 Strain Rate

5.5.2-1 Plane strain condition

The total strain rate at any point within an element can be defined by its three components which contribute to the internal work in plane strain problems.
Using eq.5.24, we have

$$[d] = [B][a_d]$$  \hspace{1cm} (5.26)

where

$$[B] = \left( \begin{array}{cccccc}
\frac{\partial N_i}{\partial x} & 0 & 0 & \frac{\partial N_k}{\partial x} & 0 & \frac{\partial N_1}{\partial x} \\
0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_1}{\partial y} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \\
\end{array} \right)$$

$$[a_d] = \left( \begin{array}{cccccccc}
 a_{ix} & a_{iy} & a_{jx} & a_{iy} & a_{kx} & a_{ky} & a_{lx} & a_{ly} \\
\end{array} \right)^T$$

\[5.5.2-2\] Axisymmetric condition

Under axisymmetric conditions, the total strain at any point within an element can be defined by its four components, i.e.,
Substitution of eq.5.24 into eq.5.27 produces

\[ d = [B]a_e \]

where

\[
[B] = \begin{pmatrix}
\frac{\partial N_i}{\partial r} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial N_j}{\partial r} & 0 & 0 & 0 \\
0 & 0 & \frac{\partial N_i}{\partial z} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial N_j}{\partial z} & 0 \\
\frac{N_i}{r} & 0 & 0 & 0 & \frac{N_j}{r} \\
0 & \frac{N_i}{r} & 0 & 0 & \frac{N_k}{r} \\
0 & 0 & \frac{N_i}{r} & 0 & \frac{N_k}{r} \\
0 & 0 & 0 & \frac{N_j}{r} & 0 \\
0 & 0 & 0 & 0 & \frac{N_k}{r} \\
\end{pmatrix}
\]

\[ a_e = (a_r \ a_{rz} \ a_{r\theta} \ a_{r\phi} \ a_{kr} \ a_{krz} \ a_{kr\theta})^T \]

5.5.3 Stress Rate

Regarding eq.5.4, in order to calculate the stress rate at an instant of time, not
only the velocity \( \mathbf{v} \) and the deformation rate \( \mathbf{d} \), but also \( \nabla \mathbf{\sigma} \) and \( \nabla \mathbf{d} \) have to be evaluated.

From eq. 5.24, and eq. 5.2, \( w \) can be evaluated as

\[
\mathbf{w} = \begin{pmatrix}
0 & w_{xy} & 0 \\
-w_{xy} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

(5.28)

with

\[
w_{xy} = \frac{1}{2} \left( \frac{\partial N_i}{\partial y} - \frac{\partial N_j}{\partial x} - \frac{\partial N_j}{\partial y} + \frac{\partial N_k}{\partial x} \right) \mathbf{a}_e
\]

(5.29)

where subscript \( e \) denotes the quantity of an element.

For simplicity, assuming that stresses within an element have the same shape functions as those for displacement, there results

\[
\nabla \sigma_{ij} = \begin{pmatrix}
\frac{\partial \sigma_{ij}}{\partial x} & \frac{\partial \sigma_{ij}}{\partial y}
\end{pmatrix}
\]  

(5.29)

with

\[
\begin{pmatrix}
\frac{\partial \sigma_{ij}}{\partial x} \\
\frac{\partial \sigma_{ij}}{\partial y}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_l}{\partial x} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_l}{\partial y}
\end{pmatrix} \begin{pmatrix}
\sigma_{iij} \\
\sigma_{ijj} \\
\sigma_{ijk} \\
\sigma_{ijl}
\end{pmatrix}
\]

where the third subscript in the stress component \( \sigma_{ij} \) denotes the nodal number of the element.
Regarding eq.5.25 and eq.5.26

\[ \mathbf{V}_d = \begin{bmatrix} \mathbf{d}_x & \mathbf{d}_y \end{bmatrix} \]

with

\[
\mathbf{d}_x = \begin{bmatrix} \frac{\partial d_x}{\partial x} \\ \frac{\partial d_y}{\partial x} \\ 2 \frac{\partial d_{xy}}{\partial x} \end{bmatrix} = (B_{,x}) [a_d] \tag{5.31}
\]

\[
\mathbf{d}_y = \begin{bmatrix} \frac{\partial d_x}{\partial y} \\ \frac{\partial d_y}{\partial y} \\ 2 \frac{\partial d_{xy}}{\partial y} \end{bmatrix} = (B_{,y}) [a_e] \tag{5.32}
\]

where

\[
(B_{,x}) = \begin{bmatrix}
\frac{\partial^2 N_i}{\partial x^2} & 0 & 0 & \frac{\partial^2 N_j}{\partial x^2} & 0 & \frac{\partial^2 N_k}{\partial x^2} & 0 & \frac{\partial^2 N_1}{\partial x^2} & 0 \\
0 & \frac{\partial^2 N_i}{\partial x \partial y} & 0 & 0 & \frac{\partial^2 N_j}{\partial x \partial y} & 0 & \frac{\partial^2 N_k}{\partial x \partial y} & 0 & \frac{\partial^2 N_1}{\partial x \partial y} \\
\frac{\partial^2 N_i}{\partial x \partial y} & \frac{\partial^2 N_i}{\partial x^2} & \frac{\partial^2 N_j}{\partial x \partial y} & \frac{\partial^2 N_j}{\partial x^2} & \frac{\partial^2 N_k}{\partial x \partial y} & \frac{\partial^2 N_k}{\partial x^2} & \frac{\partial^2 N_1}{\partial x \partial y} & \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_1}{\partial x^2} \\
\frac{\partial^2 N_i}{\partial x \partial y} & \frac{\partial^2 N_j}{\partial x \partial y} & \frac{\partial^2 N_j}{\partial x^2} & \frac{\partial^2 N_k}{\partial x \partial y} & \frac{\partial^2 N_k}{\partial x^2} & \frac{\partial^2 N_1}{\partial x \partial y} & \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_1}{\partial x^2}
\end{bmatrix}
\]
5.5.4 Transformation of Coordinates

To evaluate stresses and strains within an element, matrices $B$, $B_x$ and $B_y$ have to be determined. It has been demonstrated in sections 5.5.2 and 5.5.3 that the matrix $B$ depends on the vector

$\left(\begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{array}\right)$

and the matrices $[B_x]$, $[B_y]$ depends on the vector

$\left(\begin{array}{c}
\frac{\partial^2 N_i}{\partial x^2} \\
\frac{\partial^2 N_i}{\partial x \partial y} \\
\frac{\partial^2 N_i}{\partial y^2}
\end{array}\right)$

(5.33)
Because \( N_i \) is defined in terms of local curvilinear co-ordinates, for the evaluation of above vectors, a transformation is needed to express the global derivatives of the type occurring in expressions 5.33 and 5.34 in terms of local derivatives.

By the usual rules of partial differentiation we can write the \( \xi, \eta \) derivatives as (Zienkiewicz, 1979)

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix} \begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{bmatrix} = \{J\} \begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{bmatrix}
\]

In the above equation, the left hand side and the Jacobian matrix \([J]\) can be evaluated directly, as the functions \( N_i, x \) and \( y \) are explicitly specified in local co-ordinates. From eqs.5.21 and 5.20 we can obtain,

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
\xi_i (1 + \eta N_i) \\
\eta_i (1 + \xi x_i)
\end{bmatrix}
\]

\[
[J] = \begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix} = \begin{pmatrix}
\sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i \\
\sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i
\end{pmatrix}
\]

To find now the global derivatives we invert \([J]\) and write
\[
\frac{\partial N_i}{\partial x} \phi \left( \frac{\partial N_i}{\partial x} \right) + \frac{\partial N_i}{\partial y} \psi = \left[ J \right]^{-1} \left( \begin{array} {l} \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} \end{array} \right)
\]

(5.38)

We also can write the second order derivative of \( N_i \), for instance, with respect to the \( \xi \), as

\[
\frac{\partial^2 N_i}{\partial \xi^2} = \left( \frac{\partial N_i}{\partial \xi} \right)^2 + 2 \frac{\partial N_i}{\partial \xi} \frac{\partial^2 N_i}{\partial \xi^2} + \left( \frac{\partial^2 N_i}{\partial \xi^2} \right)^2 + \left( \frac{\partial^2 N_i}{\partial \eta^2} \right)^2
\]

(5.39)

Performing the same differentiation with respect to \( \eta^2, \xi \eta \) and writing in matrix form we have

\[
\left( \begin{array} {ccc} \frac{\partial N_i}{\partial \xi} & \frac{\partial N_i}{\partial \xi} & \frac{\partial N_i}{\partial \xi^2} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \eta^2} \\ \frac{\partial N_i}{\partial \xi \eta} & \frac{\partial N_i}{\partial \xi \eta} & \frac{\partial N_i}{\partial \xi \eta^2} \end{array} \right) \left( \begin{array} {ccc} \frac{\partial^2 N_i}{\partial \xi^2} & \frac{\partial^2 N_i}{\partial \xi \eta} & \frac{\partial^2 N_i}{\partial \xi \eta^2} \\ \frac{\partial^2 N_i}{\partial \xi \eta} & \frac{\partial^2 N_i}{\partial \eta^2} & \frac{\partial^2 N_i}{\partial \eta \eta^2} \\ \frac{\partial^2 N_i}{\partial \xi \eta^2} & \frac{\partial^2 N_i}{\partial \eta \eta^2} & \frac{\partial^2 N_i}{\partial \eta \eta^2} \end{array} \right) = \left[ J' \right]
\]

(5.40)

where

\[
\left[ J' \right] = \left( \begin{array} {ccc} \left( \frac{\partial x}{\partial \xi} \right)^2 & 2 \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} & \left( \frac{\partial y}{\partial \xi} \right)^2 \\ \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \left( \frac{\partial x}{\partial \eta} \right)^2 & 2 \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} & \left( \frac{\partial y}{\partial \eta} \right)^2 \end{array} \right)
\]

(5.41)
In the above, the left hand side can be evaluated directly, as the function $N_i$, $x$ and $y$ are functions of local co-ordinates $(\xi, \eta)$, and the functions $\frac{\partial N_i}{\partial x}$, $\frac{\partial N_i}{\partial y}$ have also been expressed as the functions of local co-ordinates $(\xi, \eta)$ as shown in eq.5.38.

Comparing eq.5.35 with eq.5.41, the matrix $[J']$ can be represented as

$$[J'] = \begin{pmatrix} J'_{11} & J'_{12} & J'_{13} \\ J'_{21} & J'_{22} & J'_{23} \\ J'_{31} & J'_{32} & J'_{33} \end{pmatrix} = \begin{pmatrix} J_{11}^2 & 2J_{11}J_{12} & J_{12}^2 \\ J_{11}J_{21} & J_{11}J_{22} + J_{11}J_{21} & J_{12}J_{22} \\ J_{21}^2 & 2J_{21}J_{22} & J_{22}^2 \end{pmatrix}$$

(5.42)

where

$J_{ij}$ is the element of the Jacobian matrix.

To find now the global derivatives we invert $[J']$ and write

$$\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = [J']^{-1} \begin{pmatrix} \frac{\partial^2 N_i}{\partial \xi^2} & \frac{\partial N_i}{\partial \xi} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial \xi} \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial \eta} \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \eta} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial \eta} \frac{\partial N_i}{\partial y} \end{pmatrix}$$

(5.43)

For a four-node isoparametric element, the shape functions $N_i$ and the coordinate transformation functions $x$ and $y$ are linear functions of $\xi$, $\eta$ and $\xi\eta$. Therefore, eq.5.43 becomes
5.5.5 Numerical Integration and Evaluation of Element Matrices

To perform finite element analysis, the matrices defining element properties, such as stiffness, etc., have to be evaluated. The general form of the matrix can be expressed as

$$
\int_{\Omega} [GF] d\Omega \tag{5.45}
$$

in which the expression [GF] depends on the shape function $N$ or its derivatives with respect to global co-ordinates. For example, the stiffness matrix

$$
K = \int_{\Omega} B^T H B \, d\Omega \tag{5.46}
$$

depends on matrix $B$.

To evaluate such matrices, two transformations are needed. In the first place, as $N_i$ is defined in terms of local curvilinear co-ordinates it is necessary to express the global derivatives of the type occurring in expressions 5.33 and 5.34 in terms of local derivatives.
In the second place, the element of volume (or surface) over which the integration has to be carried out needs to be expressed in terms of the local coordinates ($\xi, \eta$).

The first transformation has been demonstrated in section 5.5.4 and will not be repeated here.

For the second transformation, a standard process (Pina, 1984) is used, which involves the determination of the Jacobian matrix $[J]$. Thus, for instance, a surface element

$$dxdy = |J| d\xi d\eta$$  \hspace{1cm} (5.47)

Based on the above transformation, the matrices defining the element properties can be written in the form of

$$I = \int_{-1}^{1} \int_{-1}^{1} GF(\xi, \eta) |J| d\xi d\eta$$  \hspace{1cm} (5.48)

While the limits of the integration are simple in the above case, unfortunately the explicit form of $G$ is not, except for the simplest elements such as triangular elements with three nodes. Numerical integration has to be used to evaluate the integrals.

A nine-point Gaussian quadrature formula (Zienkiewicz, 1979) is employed, i.e.,

$$I = \sum_{i=1}^{3} \sum_{j=1}^{3} G(\xi_{i}, \eta_{j}) w_{i} w_{j}$$  \hspace{1cm} (5.49)
Table 5.1 shows the values of $\xi_i$ and $\eta_j$ and its corresponding weight coefficients for a nine point Gaussian Quadrature Formula. Fig. 5.3 shows the location of the integration points $(\xi_i, \eta_j)$ in a nine-point Gaussian Quadrature Integration Scheme.

Table 5.1

Positions and Weight Coefficients of the Gaussian Quadrature Formula

<table>
<thead>
<tr>
<th>$\xi_i / \eta_j$</th>
<th>$w_i / w_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>±0.7745966692</td>
<td>0.5555555556</td>
</tr>
<tr>
<td>0</td>
<td>0.8888888889</td>
</tr>
</tbody>
</table>

Fig. 5.3 Location of the integration points

Now all the matrices describing the characteristics of an element can be evaluated numerically as follows.
a) mass matrix

\[
M_e = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} N_j^T \delta_{ij} \psi \rho J |w_iw_j| \tag{5.50}
\]

\[
M_{ee} = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} N_j^T \delta_{ij} \psi \xi, \eta_j \, N_j |w_iw_j| \tag{5.51}
\]

where subscript \(e\) denotes the quantity related to an element, and the subscript \(ij\) denotes the position of the integration point \((\xi, \eta)\).

b) stiffness matrix

\[
K_e = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij}^T H B_{ij} |w_iw_j| \tag{5.52}
\]

c) viscous matrix

\[
C_e = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij}^T G B_{ij} |w_iw_j| \tag{5.53}
\]

d) load vectors

\[
I_e = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij}^T \alpha(\xi, \eta_j) |J| w_iw_j \tag{5.54}
\]

\[
P_e = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} N_j^T \xi |w_iw_j| \tag{5.55}
\]

\[
T_{re} = \chi \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij}^T \xi |w_iw_j| \tag{5.56}
\]

where

\[
\chi = \begin{cases} 
1 & \text{for a plane strain problem} \\
2\pi r_{ij} & \text{for an axisymmetric problem}
\end{cases}
\]
5.6 PROCEDURE OF SOLUTION

Assuming velocities and stresses to be known for the time step \( t = t^n \), eq.5.19 allows the computation of the velocities and stresses for the instant of time \( t = t^{n+1} \).

Eq.5.19 is nonlinear, because \( P^{n+1}, K^{n+1} \) depend on \( \sigma^{n+1}, M_c^{n+1} \) depends on \( \dot{a}^{n+1} \), and \( r^n \) depends on \( \sigma^{n+1} \) and \( \dot{a}^{n+1} \). Therefore, an iterative scheme is needed to solve the equation numerically. For this purpose, eq.5.19 is rewritten as

\[
\psi(\dot{a}^{n+1}) = P^{n+1} - \frac{1}{\Delta t} M^{n+1} (\dot{a}^{n+1} - \dot{a}^n) - M_c^{n+1} \dot{a}^{n+1} \\
- \left( r^n + \Delta t K^{n+1} \dot{a}^{n+1} + \Delta t r^n + C (\dot{a}^{n+1} - \dot{a}^n) \right) = 0
\]

(5.57)

For an approximate solution of \( \dot{a}^{n+1} \), the term \( \psi(\dot{a}^{n+1}) \) represents an error vector. The solution of these equations can be found by a Newton - Raphson method

\[
\dot{a}^{n+1} = \dot{a}^n + \Delta \psi(\dot{a}^{n+1})
\]

(5.58)

where superscripts \( i \) and \( (n+1) \) denote the \( i \)th iteration cycle and \( (n+1) \)th time step respectively, and

\[
\Delta = - \left( \frac{\partial \psi}{\partial \dot{a}^{n+1}} \right) \quad \dot{a} = \dot{a}^n = \frac{1}{\Delta t} M^{n+1} + M_c^{n+1} + C + \Delta t K^{n+1}
\]

(5.59)

For convenience, \( A = \frac{1}{\Delta t} M^n + M_c^n + C + \Delta t K^n \) is chosen and kept constant through each iteration cycle until the process is stopped when \( \psi \) reaches a given small value, or incremental velocities are within acceptable tolerances. The iteration starts with \( 0 \dot{a}^{n+1} = 0 \).
Fig. 5.4 shows a double iterative scheme for solving $a^{n+1}$. The first iteration is over the material and geometry nonlinearity. The second iteration is over the side wall friction and the inertial forces.

The computation of stress is based on equation 5.4. However, it has further to be considered that no tensile stresses are allowed and that the limit condition of eq.3.43b imposed by the constitutive law must be fulfilled, i.e., the deviatoric stresses cannot exceed a certain limiting value. Two steps are employed to modify the stresses computed from eq.5.4, if they violate one of those conditions.

**Step 1:** if tensile stress components are greater than zero, the tensile stresses are set to zero.

**Step 2:** if $f_p > h_{\text{max}}$, more cycles of iteration have to be carried out over the material nonlinearity to adjust the stress components, until $f_p \leq h_{\text{max}}$.

Friction forces on silo walls are evaluated from the stress components at boundary points. Suppose, at boundary point $(x,y)$, the stress components at the $i$th iteration are $\sigma_{ij}^{n+1}$, the normal pressure can be evaluated as

$$p_n^{n+1} = \sigma_{ij}^{n+1} n_j$$

where

$n_j$ is the normal direction cosine of the boundary.

The wall friction then can be computed as

$$p_t^{n+1} = p_n^{n+1} \tan \phi_w$$
Chapter 5 Simulation of the flow of bulk materials through rigid-walled silos

5-24

To

Form and triangularize \( A^{n+1} \) (eq.5.59)

Calculate \( \psi^{n+1} \) (5.57)

Solve for \( a^{n+1} \) (5.58)

Calculate \( (d,v,w)^{n+1} \) (5.2, 5.24, 5.25)

Assume \( H = D_e \) (elasticity matrix)

Calculate \( \sigma^{n+1} \) (5.4)

No

Fail in tension ?

Yes

Modify \( \sigma^{n+1} \)

No

Plastic collapse/
Plastic expansive ?

Yes

Modify \( H \) and \( \sigma^{n+1} \)

No

Convergence of stress ?

Yes

Update stress, wall friction

No

Convergence of velocity ?

Yes

Update material properties

Next time step

Fig.5.4 Flow chart of the iterative scheme
5.7 EXAMPLE

A typical example is presented to show the complex behaviour of the bulk material discharge process.

Consider a plane strain hopper symmetrical about a vertical axis. Fig. 5.5 shows the finite element mesh adopted for half of the hopper.

Fig. 5.5 Finite element mesh
The bulk material adopted is considered as a no-tension viscous elastic-plastic material. Its properties, including the Lade elastic-plastic properties (Lade, 1978), are listed as follows:

density \( p = 1600 \text{ kg/m}^3 \),

elastic parameters

- modulus number \( K_{ur} = 960 \),
- Poisson's ratio (after adjusted) \( \nu = 0.25 \),
- elastic exponent \( C_n = 0.57 \);

plastic collapse parameters

- collapse modulus \( C_C = 0.00028 \),
- collapse exponent \( C_p = 0.94 \);

plastic expansive parameters

- yield constant \( \eta_1 = 28 \),
- yield exponent \( C_m = 0.093 \),
- plastic potential constants \( C_R = -1 \), \( C_S = 0.430 \), \( C_t = 0 \),
- work hardening/softening constants \( \alpha = 3 \), \( \beta = -0.076 \), \( C_{pp} = 0.24 \), \( C_i = 1.25 \).

viscous constant \( \mu = 0.001 \text{ secMPa} \);

The Young's modulus and the angle of internal friction are equal to 50 MPa and 35° respectively; and the friction angle of the silo walls for this example is taken as 15°.

Fig.5.6 shows the principal stress fields at three different instants. Among them, the stress field at \( t = 0 \) is the stress at the end of the filling condition, and also is the initial stress at the start of the discharge process. During the process of discharge, the orientation of the maximum principal stress in the hopper area changes from a vertical direction to a direction nearly normal to the hopper wall inclination. Subsequently, the stresses decrease above the outlet, and increase at the transition between the bin and the hopper.
Fig. 5.6 Principal stress distributions for (a) $t = 0$ sec; (b) $t = 0.025$ sec; (c) $t = 0.35$ sec.
Fig. 5.7 shows the wall pressure distribution during the discharge process. It is found that there are three different stages of pressure response to the material discharge. In the first stage the pressure near the outlet increases more quickly than that in the transition area. In the second stage the pressure in the transition area is nearly constant. However, the position of the peak pressure on the hopper wall moves slowly from the outlet to the transition point, and the pressure near the outlet decreases. During the third stage, there is a strong increase of wall pressure at the transition and a decrease near the outlet.

Fig. 5.7 Pressure distributions at several instants of time during the discharge of materials
Fig. 5.8 shows the pressure distributions at some fixed spatial points (see Fig. 5.5). It is clear that the discharge process starts with a wall pressure increase just above the outlet, then a decrease of pressure near the outlet occurs followed by a significant increase of pressure in the transition area.

Fig. 5.8  Wall pressures versus time at several fixed spatial points (see Fig. 5.5)
Fig. 5.9 shows the vertical velocity distribution on the center line of the silo at some instants of time. It is clear that there is a uniform increase of velocity in the bin area. However, larger increases occur in the region from the outlet to the transition point, the mechanism of which has been explained by the development of a pipe zone in the hopper area (Gale, Hoadley and Schmidt, 1986).
5.8 SUMMARY AND CONCLUDING REMARKS

A finite element procedure and its computer program have been developed for the simulation of the flow of bulk materials from rigid-walled silos. The method can provide transient velocity and stress fields in the domain of the bulk material and the pressure distributions on the walls of silos for plane strain and axisymmetric silos. The results look reasonable in the sense of capturing the pressure concentration in the transition area of bin-hopper. The comparison of numerical prediction with experimental results will be conducted in Chapter 8.

The illustrative example shows that discharge is a complex process. No obvious switch from static to dynamic stress field is found. The pressure distribution changes from a static field to a dynamic field gradually, although over a short interval of time. Further research is required to investigate the structure response during the whole process of the material discharge to determine the most severe condition of loading. Therefore, Chapter 6 is designed to consider the interaction between silo walls and bulk materials.
CHAPTER SIX

FINITE ELEMENT SIMULATION
OF THE FLOW OF BULK MATERIALS THROUGH
FLEXIBLE SILOS

6.1 GENERAL

This chapter focusses on the development of a numerical method for the simulation of the flow of bulk granular materials through flexible silos.

Although several attempts to use finite element method to predict bulk material pressures on the walls of silos (Mahmoud and Abdel-Sayed, 1981; Askari and Elwi, 1988; Haussler and Eibl, 1984; Runesson and Nilsson, 1986) have been made, few attempts have been made to simulate the interaction of the flowing granular solids and the flexible wall of silos under dynamic conditions; a rigid wall condition is usually assumed. Moreover, the dynamic structural behaviour of silos, which is important in the design process of silos, has not been considered to date. Mahmoud and Abdel-Sayed (1981) have employed a one-dimensional linkage element to simulate the interaction of granular solids and the silo wall. However, their procedure considered the static case only. In addition, in the procedure two material parameters, normal and tangential stiffnesses of linkage element have to be assumed initially, and then the assumed values have to be modified through an iterative procedure until the computed shear stresses for all linkage elements are
within the shear strength, which may cause a computational problem. Moreover, overlap of the solids with the silo wall may occur.

This chapter starts from the description of the formulae for determining the behaviour of bulk materials and silo structures, respectively, derived by using the procedure described in Chapter 5. Then the formulae for the motion of the bulk materials are coupled with those for the deformation of the silo shells by the compatibility requirements that the normal displacements at the interface of the bulk material and the silo wall are equal, and that the contact pressures have the same magnitude but opposite directions. Finally, the solution is obtained by using a double iterative scheme over the material non-linearity and the boundary friction.

6.2 FINITE ELEMENT FORMULAE FOR THE MOTION OF BULK MATERIALS

The bulk material under consideration is assumed as a viscous, non-tension, elastic-plastic continuum. Due to large deformation occurring during discharge of bulk materials, geometric nonlinearity is considered as well as material nonlinearity in the models for the motion of the bulk granular materials. Based on the derivation in Chapter 5, the finite element formulae for the dynamic equilibrium of bulk granular materials at an instant of time $t$ can be written as

$$M^g \ddot{a} + M^c \dot{a} + C (\dot{a} - \dot{a}^0) + \Delta t K^g \dot{a} = F^g + P^g$$

where

$$M^g = \int_{\Omega_g} N^T N \rho^g \, d\Omega$$

$$M^c = \int_{\Omega_g} N^T V_N N \rho^g \, d\Omega$$
\[ C = \int_{\Omega_g} B^T G B \, d\Omega \]

\[ K^g = \int_{\Omega_g} B^T H^g B \, d\Omega \]

\[ E^g = \int_{\Omega_g} N^T T^g \, d\Omega + \int_{\Gamma_g} N^T p^g \, d\Gamma + \int_{\Gamma_g} N_t p^g_t \, d\Gamma - \int_{\Omega_g} B^T \sigma^0 \, d\Omega - \Delta t \int_{\Omega_g} B^T \tau \, d\Omega \]

\[ P^g_{gs} = \int_{\Gamma_g} N_n p^g_n \, d\Gamma \]

where

the superscript \( g \) denotes the quantity in the domain of the bulk granular material,
subscript \( gs \) denotes the interface of the bulk granular material and the structure,
and the superscript \( o \) denotes the instant of time \( t-\Delta t \).

### 6.3 FINITE ELEMENT FORMULAE FOR THE MOTION OF SILO STRUCTURES

The material from which the silo is fabricated is considered as a Von-Mises elastic, perfectly plastic material (Mendelson, 1983), which satisfies the following constitutive equation

\[ \ddot{\sigma} = \dot{\sigma} - \dot{\sigma}_0 = \dot{H} \, d \quad \text{or} \quad \sigma^{n+1} = \sigma^n + \Delta t \, H \, d \quad (6.2) \]

In addition, an infinitesimal strain condition is assumed in the dynamic analysis.

The equation of dynamic equilibrium for the structure subjected to bulk material pressures can be represented in terms of the virtual work equation as follows.
\[
\int_{\Omega_s} \delta \mathbf{q}^T \mathbf{g} \, d\Omega + \int_{\Omega_s} \delta \mathbf{v}^T (\gamma^s - \mathbf{f}^s) \, d\Omega - \int_{\Gamma_s} \delta \mathbf{v}^T \mathbf{p}^s \, d\Gamma - \int_{\Gamma_g} \delta \mathbf{v}^T \mathbf{p}_n^s \, d\Gamma \\
- \int_{\Gamma_g} \delta \mathbf{n}^s \mathbf{p}_n^s \, d\Gamma = 0
\]  

(6.3)

where superscript \( s \) denotes the silo structure.

Using standard procedures for the finite element discretization of the structure domain \( \Omega_s \), the above equation can be represented in terms of velocity as the primary variable:

\[
M^s \ddot{\mathbf{u}}^s + \Delta t K^s \dot{\mathbf{u}}^s = \mathbf{F}^s + \mathbf{P}_{gs}^s
\]

(6.4)

with

\[
M^s = \int_{\Omega_s} \mathbf{N}^T \mathbf{N}^s \, d\Omega \\
K^s = \int_{\Omega_s} \mathbf{B}^T \mathbf{H}^s \mathbf{B} \, d\Omega
\]

where \( M^s \) and \( K^s \) denote the mass matrix and stiffness matrix of the structure, respectively, at time instant \( t \), and

\[
\mathbf{F}^s = \int_{\Omega_s} \mathbf{N}^T \mathbf{t}^s \, d\Omega + \int_{\Gamma_s} \mathbf{N}^T \mathbf{p}^s \, d\Gamma + \int_{\Gamma_g} \mathbf{N}_t \mathbf{p}_t^s \, d\Gamma - \int_{\Omega_s} \mathbf{B}^T \mathbf{\sigma}^0 \, d\Omega
\]

\[
\mathbf{P}_{gs}^s = \int_{\Gamma_g} \mathbf{N}_n \mathbf{p}_n^s \, d\Gamma
\]

where superscript \( s \) denotes the domain of the silo structure, and superscript \( o \) denotes the instant of time \( t-\Delta t \).

### 6.4 COUPLING OF THE BULK GRANULAR MATERIAL AND THE STRUCTURE

The bulk granular material and the structure are coupled by the compatibility
requirements that the contact pressures have the same magnitude but opposite
direction and the normal displacements at the interface $\Gamma_{gs}$ are equal. However,
sliding along the silo wall is allowed. The friction force on the slip surface is
determined by the Coulomb friction model, which has been formulated and
presented in eq.(3.5a). To implement the compatibility condition of displacements,
the equations of motion for the domain of the flowing granular solids are partitioned
into boundary velocity components normal to $\Gamma_{gs}$, denoted by subscript B, and all
other components denoted by subscript G, i.e:

$$
\begin{align*}
\begin{bmatrix}
M_{GG}^e & M_{GB}^e \\
M_{BG}^e & M_{BB}^e
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_G \\
\dot{\eta}_B
\end{bmatrix}
+ 
\begin{bmatrix}
M_{GG}^s & M_{GB}^s \\
M_{BG}^s & M_{BB}^s
\end{bmatrix}
\begin{bmatrix}
\eta_G \\
\eta_B
\end{bmatrix}
+ 
\begin{bmatrix}
C_{GG}^e & C_{GB}^e \\
C_{BG}^e & C_{BB}^e
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_G - \dot{\eta}_G \\
\dot{\eta}_B - \dot{\eta}_B
\end{bmatrix}
& = 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\end{align*}
$$

In the same way, the equations of motion for the domain of the silo structure
are partitioned into displacement components normal to $\Gamma_{gs}$ denoted by B and all
other components denoted by S, i.e.,

$$
\begin{align*}
\begin{bmatrix}
M_{BB}^e & M_{BS}^e \\
M_{SB}^e & M_{SS}^e
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_B \\
\dot{\eta}_S
\end{bmatrix}
+ 
\begin{bmatrix}
K_{BB}^e & K_{BS}^e \\
K_{SB}^e & K_{SS}^e
\end{bmatrix}
\begin{bmatrix}
\eta_B \\
\eta_S
\end{bmatrix}
& = 
\begin{bmatrix}
F_B^e + P_{BGS}^e \\
F_S^e
\end{bmatrix}
\end{align*}
$$

For compatibility of the normal displacements along the interface, the shape
functions for the granular material elements and the structure elements on the
interface are chosen to be identical.

By using the compatibility condition, the combination of the partitioned
equations (6.5) and (6.6) furnishes the coupled equations of motion for the granular
solids - structure system:
\[ M \dot{a} + M_c a + C (a - a^0) + \Delta t \dot{K} a = F \]  \hspace{1cm} (6.7)

where

\[ a = (a_G, a_B, a_s)^T \]

\[
M = \begin{pmatrix}
M_{GG}^g & M_{GB}^g & 0 \\
M_{BG}^g & M_{BB}^g + M_{BB}^s & M_{BS}^s \\
0 & M_{BS}^s & M_{SS}^s
\end{pmatrix}
\]

\[
K = \begin{pmatrix}
K_{GG}^g & K_{GB}^g & 0 \\
K_{BG}^g & K_{BB}^g + K_{BB}^s & K_{BS}^s \\
0 & K_{BS}^s & K_{SS}^s
\end{pmatrix}
\]

\[
M_c = \begin{pmatrix}
M_{cGG}^g & M_{cGB}^g & 0 \\
M_{cBG}^g & M_{cBB}^g & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
C_{GG}^g & C_{GB}^g & 0 \\
C_{BG}^g & C_{BB}^g & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
E = \begin{pmatrix}
E_G^g \\
E_B^g + P_B^g + P_B^s \\
E_S^s
\end{pmatrix}
\]

As the shape functions adopted for the granular solids elements and the structure elements at the interface are identical, the forces due to bulk material pressures at the interface are automatically in equilibrium, that is

\[ P_{BGS} + P_{BS}^s = 0 \]  \hspace{1cm} (6.8)
Therefore, the dynamic response of the granular solids and the structure is represented by eq.(6.7). The mass and stiffness contributions, respectively, of the granular solids and the structure are simply added at the common normal degrees of freedom along the interface $\Gamma_{gs}$.

6.5 PROCEDURE OF SOLUTION

At a typical time step $t_{n+1}$, the equations of motion for the granular solids-structure system eq.(6.7) can be approximated by

$$\frac{1}{\Delta t} M^{n+1} \ddot{a}^{n+1} + M_c^{n+1} \dot{a}^{n+1} + C (a^{n+1} - a^n) + \Delta t K^{n+1} \dot{a}^{n+1} = F^{n+1}$$

(6.9)

which are nonlinear equations, because $E^{n+1}, K^{n+1}$ depend on $\dot{a}^{n+1}$, and $M_c^{n+1}$ depends on $a^{n+1}$. Therefore, an iterative scheme is needed to solve the equation numerically. For this purpose, eq.(6.9) is rewritten as

$$\psi(a^{n+1}) = F^{n+1} - \frac{1}{\Delta t} M^{n+1} (a^{n+1} - a^n) - M_c^{n+1} \dot{a}^{n+1} - C (a^{n+1} - a^n) - \Delta t K^{n+1} \dot{a}^{n+1}$$

(6.10)

The solution of these equations can be found by a Newton-Raphson method

$$a^{i+1,n+1} = a^{i,n+1} + \Delta^{-1} \psi(a^{i,n+1})$$

(6.11)

where superscripts $i$ and $(n+1)$ denote the $i$th iteration cycle and the $(n+1)$th time step respectively, and

$$\Delta = -\left( \frac{\partial \psi}{\partial a^{n+1}} \right)_{a^n = \hat{a}} = \frac{1}{\Delta t} M^{n+1} + M_c^{n+1} + C + \Delta t K^{n+1}$$

(6.12)
For convenience, \( A = \frac{1}{\Delta t} M^n + M^n_0 + C^n + K^n \) is chosen and kept constant through each iterative cycle until the process is stopped when \( \psi \) reaches a given small value, or incremental velocities are within acceptable tolerances. The iteration starts with \( a^{n+1} = 0 \).

Once the nodal velocities are obtained, the stresses in both the bulk solid domain and the structure domain can be computed by the strain-rate velocity relations and the constitutive laws adopted. Further, in computing stresses it has to be considered that no tensile stress is allowed. If tensile stress occurs, the computed stresses have to be modified.

6.6 EXAMPLE

A typical example is presented to show the complex dynamic behaviour of the bulk solids and the silo structure during the process of material discharge.

Consider an axisymmetric hopper-silo combination. Fig.6.1 shows the finite element mesh adopted for half of the hopper-silo cross section through the symmetrical axis. The thickness of the silo wall is 4 mm, the Young's modulus of the silo material \( E = 210000 \) MPa, and the Poisson's ratio \( \nu = 0.25 \). The friction angle of the silo wall is \( \phi_w = 15^\circ \). The condition of support is also shown in the figure.

The bulk material adopted has the properties of density \( \rho = 1600 \) kg/m\(^3\), elastic-plastic properties as those in section 5.7, and viscous constant \( \mu = 0.001 \) secMPa.
Fig. 6.1 Finite element mesh
Fig. 6.2 shows the pressure distributions on the wall of the silo at several instants of time during discharge. There is an increase of hopper pressure near the outlet during the initial phase of discharging, that is followed by a decrease near the outlet and a strong increase at the transition area between the hopper and the bin wall.

Fig. 6.2 Distributions of wall pressures at several instants
Fig. 6.3 shows the distributions of the circumferential membrane stresses in the silo wall at several instants of time during the discharge process. Except at the transition area between the hopper and the bin, the whole wall of the silo is subjected to a tensile circumferential stress field. In the hopper area, at the beginning of discharge, the stresses increase. However, with the transfer of the bulk material pressure from the outlet area to the transition area, the circumferential stresses decrease dramatically near the outlet and increase slightly at the transition.

![Diagram of circumferential membrane stresses in silo wall at several instants](image)

Fig. 6.3 Circumferential membrane stresses in silo wall at several instants
Fig. 6.4 shows the distributions of the meridional stresses in the silo wall at six different instants of discharge. It is found that hopper wall is subjected to tensile stress, however, the bin wall is subjected to compressive stress. In addition, it is noted that the filling condition is the critical condition for the meridional stress in the hopper.

---

**Fig. 6.4 Meridional stresses in silo wall at several instants**
Fig. 6.5 - Fig. 6.7 show the variation of the wall pressures, the circumferential stresses and the meridional stresses in the silo versus time at several fixed spatial points (refer to Fig. 6.1), respectively. It is noted that the discharge process starts with an increase of wall pressure and circumferential stresses over the hopper wall, especially above the outlet. Then a decrease of pressure near the outlet occurs followed by a strong increase of pressure in the transition area, but only a slight increase of circumferential stress in part of the hopper wall. During the process of discharge, the meridional stress is found to decrease at the beginning, and then tend to a constant value. It is interesting to note that, although the pressure at the transition area increases dramatically after the initial stage of discharge, the stresses in the silo shell do not increase as dramatically.

![Graph showing wall pressures versus time at several fixed spatial points](image)

**Fig. 6.5** Wall pressures versus time at several fixed spatial points (see Fig. 6.1)
Fig. 6.6 Circumferential membrane stresses versus time at several fixed spatial points (see Fig. 6.1)
Fig. 6.7 Meridional stresses versus time at several fixed spatial points (see Fig. 6.1)
6.7 SUMMARY AND CONCLUDING REMARKS

A numerical procedure has been developed to simulate the flow of bulk granular materials from flexible silos, and the dynamic response of the silo structures during material discharge. The method can provide the transient pressure on the wall of silos, and the transient stresses in silo structures during discharge of bulk granular materials. The maximum values of the pressure and the stresses in a silo structure can be determined, and used as a critical condition in silo design.

An example of the numerical procedure applied to the simulation of the discharge of bulk materials from an axisymmetric silo has been presented. The maximum meridional stress is found to correspond to the filling condition. However, the circumferential stress in the hopper reaches its maximum value after a short period of discharge.
CHAPTER SEVEN

EXPERIMENTAL ANALYSIS OF FLOW MECHANISMS AND PRESSURE DISTRIBUTIONS

7.1 GENERAL

This chapter focuses on the experimental analysis of the discharge of bulk materials from plane flow silos.

The aim of the experiments is to provide information for the comparison of the numerically predicted wall pressures with the experimental observations, and to observe the actual complex flow phenomena in order to understand better the flow mechanisms.

Experiments were conducted in three plane-flow model silos with different geometries. The experiments consisted of measurements of wall pressures and the observation of flow patterns during the discharge of bulk material from silos.

To measure the dynamic wall pressures in a satisfactory manner an automatic data acquisition and control system was developed. In order to minimize the random noise contained in the measured data, a method based on a power spectrum analysis was developed to separate the noise from the measured signals.

A photographic method was introduced to analyse the flow pattern and the velocity field in the flowing bulk materials.
The emphasis of this chapter is placed on the description of the automatic data acquisition/control system, the data processing method, the experimental results and their analysis.

7.2 EXPERIMENTAL PROGRAM

7.2.1 The Model Silos

Three models of plane-flow silos with different hopper outlets, constructed from perspex materials, were used in the experiments to investigate the flow phenomena and the pressure distributions on the silo walls under different geometric conditions. Fig.7.1 shows the geometries and the dimensions of the model silos.

7.2.2 The Bulk Material

The bulk material used in experiments was sand. The angle of internal friction of the material was measured by using the Jenike shear cell (Arnold et al, 1978). The wall friction angle was measured on the actual wall material, by simply using the surface of the material as the friction surface in the Jenike wall shear test (Arnold et al 1978). Th properties of the materials used are tabulated in Table 7.1.

<table>
<thead>
<tr>
<th>density kg/m³</th>
<th>angle of internal friction (°)</th>
<th>angle of wall friction (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1600</td>
<td>35</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 7.1
Properties of the sand used
Fig. 7.1 Location of pressure transducers in model silos

(a) model 1

(b) model 2

(c) model 3

All dimensions in mm
7.2.3 Experimental Procedures

In experiments, the model silos were filled with sand up to the top. During the filling process, at intervals, a line of contrasting blue stone dust was placed against the end wall where photographs were to be taken. Then, the model silos were discharged. The flow of the bulk granular materials from the model silos was used to obtain two types of information. Firstly, measurements were made of the pressures exerted by the flowing material on the side walls. Secondly, photographs were taken of the flow patterns at the vertical end wall of the models. These photographs provided information on the flow mechanism.

The instruments for the measurement of the pressures and a technique for the processing of the recorded pressure signals are described in sections 7.3 and 7.4, respectively. The measured pressure distributions and the observed flow patterns are summarized in sections 7.5 and 7.6 respectively.

7.3 DATA ACQUISITION AND CONTROL SYSTEM

The data acquisition and control system used for the automatic pressure monitoring consisted of a microcomputer HP 9826, a data acquisition control system HP 3054A, and a set of pressure transducers. The block diagram of the system is shown in Fig.7.2.

Transducers were located on the wall of the model silos as shown in Fig.7.1. The transducers were used to convert pressures to voltage signals. Up to 100 transducers can be included in the system.
The HP 3054A includes a voltmeter which is interfaced via a HP-IB cable to the computer, and a system mainframe which connects the voltmeter to the transducers.

The microcomputer is the controller of the whole system. Its physical form is both simple and reliable, consisting of a few general purpose elements which can be programmed to make the system function as required. A computer program "ExpControl" in HP Basic code was developed to control the measurement and data acquisition procedure. The flow chart of the program is shown in Fig.7.3. Once the program was run, the voltmeter was controlled to take readings in a step by step manner continually, i.e., each transducer sampled 30 readings and the average was stored before switching to the next transducer. The recorded voltage signals were transferred to the computer in a packed format, which were then transformed to binary values, and subsequently converted to pressure values according to the calibrated formulae between voltage and pressure. Finally, a data file was created to store the data on a disk.
Chapter 7 Experimental analysis of flow mechanisms and pressure distributions

7.4 SIGNAL PROCESSING TECHNIQUE

Every measurement is bound to contain random errors which arise from sources intrinsically associated with the quantity being measured and the instrument characteristics. In general, the recorded signals consist of two components: (1) legitimate signals giving the time variation of the parameter being measured, (2) random errors causing fluctuations in the output signals which are superimposed on the legitimate signal.
To determine the true measurement, it is necessary to eliminate the noise from the signal. However, care must be exercised to assure that the process of noise elimination properly preserves the legitimate signal component. Usually, only qualitative information regarding the spectral content of the noise and the signal is known a priori. Therefore, an optimization procedure of the data processing based on a power spectral analysis has been developed to separate the noise from the signals. The procedure includes the power spectral analysis and the design of a digital filter.

7.4.1 Power spectrum analysis

The aim of the power spectrum analysis is to determine the filtering requirement, which is central in the optimization of filter parameters. The power spectrum depicts the frequency content of the data which, in turn, is related to the basic characteristics of the physical system being monitored. In digital signal processing, the power spectrum may be calculated by several methods (Bendat and Piersol, 1971; Champeney, 1973; Geckinli and Yavuz, 1983; Otnes and Enochson, 1972; and Chakrabarti et al, 1986). One of the methods was the Discrete Fast Fourier Transform (DFFT) technique. The procedure is briefly described as follows:

(a) Extend the data by zero to a number N, such that N is a power of 2 so that the DFFT technique may be applied;

(b) A cosine taper is used over 1/10 of the record length at each end of the data to reduce interaction between adjacent spectral components;

(c) Using a DFFT routine, the time domain record \( x_i \) is transformed into a frequency domain sequence,

\[
X_k = \sum_{i=0}^{N-1} x_i \exp\left(-j \frac{2\pi ik}{N}\right) \quad k = 0, 1, 2, \ldots, N-1
\]  

(7.1)
(d) A power spectrum is then calculated as

\[ H_k = 2 \left( X_k^* X_k \right) \]  \hspace{1cm} (7.2)

where \( X_k^* \) = complex conjugate of \( X_k \);

(e) The values of \( H_k \) are then adjusted for the scale factor 0.875 due to tapering, i.e., \( H_k \) divided by 0.875;

(f) The adjusted raw estimates of the power spectrum are then smoothed over \( l \) continuous raw estimates,

\[ H_{ks} = \frac{1}{l} \sum_{j=1}^{l} H_{k+j} \]  \hspace{1cm} (7.3)

Fig. 7.4 shows a plot of the power spectrum of a set of experiment data chosen for the investigation. The power spectrum of the data indicates that the primary frequency components are generally concentrated in the low frequency region. The maximum frequency of the primary components in the spectrum is judged to be in the region of 0.1 Hz by visual examination. The frequency components above 0.1 Hz are judged to be mostly random noise.

Fig. 7.4 Power spectrum versus frequency
7.4.2 Design of digital filter

The design of digital filters is based on the filtering requirement from the spectrum analysis. There are two types of digital filters: nonrecursive filters and recursive filters. A simple form of the recursive filter (Chakrabarti et al, 1986) is adopted, which is in the form of

\[ y_i = (1 - A) x_i + Ay_{i-1} \quad (7.4) \]

where \( x_i \) = measured data as recorded; \( y_i \) = output of the filter; and \( A \) is a coefficient of the filter.

The frequency response function for the filter is given by

\[ H(f) = \frac{1-A}{1-Ae^{-j2\pi fh}} \quad (7.5) \]

where \( f \) = frequency in Hz, \( h \) = sampling time interval, and \( j = \sqrt{-1} \). The square gain function \( |H(f)|^2 \) and phase shift \( \theta(f) \) are given, respectively, by

\[ |H(f)|^2 = \frac{(1 - A)^2}{(1 - 2A\cos2\pi fh + A)^2} \quad (7.6) \]

\[ \theta(f) = -\arctan \left( \frac{A\sin2\pi fh}{1 - A\cos2\pi fh} \right) \quad (7.7) \]

Fig.7.5 shows the squared gain function of the filter. It is clear that the filtered frequency range can be selected by varying \( A \). Therefore, the optimization of the filter design can be accomplished by varying a single coefficient \( A \). By comparing Fig.7.4 and Fig.7.5, a value of \( A = 0.85 \) was selected through judgement. The raw measurement data are then filtered. Fig.7.6 shows the plot of the raw data and the smoothed data using the filter parameter \( A = 0.85 \).
Fig. 7.5 Gain characteristics of filter

Fig. 7.6 Raw data and smoothed data
7.4.3 Suitability evaluation of filter parameter

From Fig. 7.6, it is clear that the filter adopted is useful in the data processing of silo wall pressures. However, the value of $A = 0.85$ needs further evaluation as to its suitability to separate noise from the signals. The evaluation is performed by examining the Wiener-Hopf ratio (see Champeney, 1973), which indicates how well the signal and the noise are separated.

The Wiener-Hopf ratio is defined by

$$W(f) = \frac{P_s(f)}{P_s(f) + P_n(f)} \quad (7.8)$$

Where $P_s$ and $P_n$ are the smoothed power spectra of signal $s_i$ and noise $n_i$ respectively. With an optimized filter the resulting $W(f)$ should be close to one for the signal frequencies and close to zero for the noise frequencies. Fig. 7.7 shows the Wiener-Hopf ratio for the filter with different values of $A$. It is noted that the filter with $A = 0.85$ appears to provide an optimal separation of noise from the signal, because at low frequencies $W(f)$ is close to one, while at high frequencies $W(f)$ is nearly zero.

Fig. 7.7 Wiener-Hopf ratio for different values of $A$
7.5. DISTRIBUTION OF PRESSURES

Based on the data processing method mentioned above, the original pressure data are smoothed by removing the higher frequency components, and are presented in the following.

7.5.1 Pressures on the Flat-Bottomed Silo (Model 1)

Figs. 7.8(a) and (b) show the variation of wall pressures at the stations on the bottom and the vertical wall of the silo, respectively. It can be noted that the discharge begins with a decrease of pressure on the bottom and the lower part of the vertical wall. Then, a strong increase of pressure occurs at stations 8-10. However, pressures at the other stations remain constant.

Fig. 7.9 shows the pressure distribution along part of the silo wall, where transducers are located, at several instants of time during the initial stage of material discharge for the flat-bottomed silo. Among them, the pressure at time t=0 is the pressure at the end of filling condition. It is clear that the filling pressure increases with the depth. During the discharge of bulk material from the silo, redistribution of wall pressures occurs. Reduction of pressures occurs at the bottom and the wall near the bottom. However, a strong increase of pressure occurs at some height above the bottom.
Fig. 7.8 Pressures versus time — Model 1
Fig. 7.9 Distribution of wall pressures — Model 1
7.5.2 Pressures on the Silo-Hopper Combination (Model 2)

Figs. 7.10 (a) and (b) show the variation of wall pressures at points on the hopper wall and the vertical wall, respectively, as functions of the discharging time. It is clear that in the initial stage of discharge the pressures on the hopper wall and the lower part of the vertical wall decrease. Then, a strong increase of pressure occurs at station 8-10. However, pressures at the other stations remain constant.

Fig. 7.11 shows the pressure distribution along part of the silo wall, where transducers are located, at several instants of time during discharge of the material. It is clear that the filling pressure increases with the depth. During discharge of the bulk material, redistribution of wall pressures occurs. A reduction of pressures occurs at the bottom and on the wall near the bottom. However, a strong increase of pressure occurs at some height above the bottom.

7.5.3 Pressures on the Silo-Hopper Combination (Model 3)

Fig. 7.12 shows the variation of wall pressures on the side wall as functions of discharging time. In the initial stage of discharge the pressures on the outlet area of the hopper wall decrease. However, a strong increase of pressure occurs at the transition area of the hopper-bin.

Fig. 7.13 shows the pressure distribution along part of the silo wall at several instants of time during discharge of the material. The filling pressure increases with the depth. During discharge of the bulk material, redistribution of wall pressures occurs with a reduction of pressure at the bottom and on the wall near the bottom. However, a strong increase of pressure occurs at the transition area of the hopper-bin.
Fig. 7.10 Pressures versus time — Model 2
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Fig. 7.11  Distribution of wall pressures — Model 2
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Fig. 7.12 Pressures versus time — Model 3
Fig. 7.13 Distribution of wall pressures — Model 3
7.6 FLOW PATTERNS AND FLOW MECHANISMS

During the flow experiments, the bulk material has a number of contrasting layers of bluestone dust inserted at intervals before discharge. On initiating discharge a series of photographs are taken of the flow patterns set up against the wall of the silo. From the successive photographs, the velocity field in bulk material can be determined including the direction and the magnitude. The observed flow patterns and the processed velocity fields are summarized in the following.

7.6.1 Flow Behaviour in the Flat-Bottomed Silo (Model 1)

Fig.7.14 shows eight selected photographs of the flow pattern of the bulk material during discharge from the flat-bottomed silo.

Fig.7.15 shows the direction and magnitude of the velocity, determined by examining the relative position of some particular points in the successive photographs taken, in the flowing bulk material at a particular instants of time. Various zones of flow, which exhibited quite different characteristics, can be distinguished from the figures.

The flow process can be divided into three stages. Stage 1: loosening of bulk material above the outlet. Once the discharge starts, the reduction of pressures on the outlet causes the material above to move, and then a loose zone is formed. Due to the side wall friction, motion of the material occurs initially only on the lower part of the silo. Therefore, the pressure on the wall near the loose zone decreases at this stage.
Fig. 7.14 Flow pattern — Model 1
Stage 2: formation of various flow zones. Upon development of the loose zone, three different flow zones are formed progressively, i.e., pipe zone, pipe feed zone and plug flow zone. In the pipe feed zone, the direction of the velocity changes from the vertical to a direction tangential to the inclined surface of the dead zone as shown in Fig. 7.16. From the equilibrium of momentum in the horizontal direction, it can be concluded that the horizontal stress will increase from $H_r$ to $H_r'$. 

Stage 3: steady flow. Once the flow zones are formed, the material discharge steadily, and the pressure will remain constant.
7.6.2 Flow Behaviour in the Silo-Hopper Combination (Model 2)

Fig. 7.17 shows eight selected photographs of the flow pattern of the bulk material during discharge from the model 2 silo.

Fig. 7.18 shows the direction and magnitude of the velocity in the flowing bulk material at a particular instant of time. Various zones of flow which exhibited quite different characteristics can be distinguished from the figures. Apparently, the effective transition of flow zones from the plug flow zone to the pipe feed flow zone occurs on the vertical wall. In this transition area, the direction of velocity changes from vertical to a direction tangential to the inclined surface of the dead zone, and therefore the pressures on this part increase with time during the initial stage of the material discharge.
Fig. 7.17 Flow pattern — Model 2
7.6.3 Flow Behaviour in the Silo-Hopper Combination (Model 3)

Fig.7.19 shows eight selected photographs of the flow pattern of the bulk material during discharge from the model 3 silo.

Fig.7.20 shows the direction and magnitude of the velocity in the flowing bulk material at a particular instants of time. It can be seen that mass flow occurs in the whole silo. Three flow zones including the pipe zone, the pipe feed zone and the plug flow zone are formed during discharge of bulk material from the silo. No dead zone is found in this case. At the transition of the hopper-bin, the direction of velocity changes from vertical direction to a direction parallel to the hopper wall. Therefore, a strong increase of pressure occurs at this transition area.
Fig. 7.19 Flow pattern — Model 3
An experimental study has been carried out to investigate the flow mechanisms for various silo-hopper combination.

An automatic monitoring system of silo wall pressures has been developed, which can monitor the dynamic pressures during bulk material discharge in a qualitatively and quantitatively satisfactory manner, and which can make the processing of a large volume of monitored data a reality. To measure the true pressures, a data processing technique has been proposed for the separation of noise from the original signals.

Significant experimental results including the pressure distributions and the
flow patterns have been presented.

Discharge starts from the loosening of the material in the area above outlet when the outlet is opened, then follows a procedure of the formation of different flow zones as shown in Fig.7.15, Fig.7.18, and Fig.7.20. The material in the plug flow zone moves as a rigid body parallel to the wall, and feeds into the pipe feed zone, that, in turn, feeds into the pipe or region of radial flow. During this stage, an increase of pressure occurs at the region of the effective transition between the plug flow zone and the pipe feed zone.

In the flat-bottomed silo (Model 1) and the model 2 silo with a big hopper convergence angle, a dead zone exists during the discharge of bulk material from the silos. The effective transition between the plug flow zone and the pipe feed zone occurs in the vertical wall, and, therefore, a strong increase of pressure occurs at this part of the vertical wall. However, in the model 3 silo with a small hopper convergence angle no dead zone is found. The effective transition between the plug flow zone and the pipe feed zone occurs at the structural transition of the hopper and bin, and, therefore, a strong increase of pressures occurs at the transition area of the hopper-bin.
CHAPTER EIGHT

COMPARISON OF NUMERICAL PREDICTIONS WITH EXPERIMENTAL RESULTS AND OTHER PREDICTIONS

8.1 GENERAL

This chapter focusses on the validation of the numerical models for the computation of bulk material pressures on the walls of silos under both filling and discharge conditions.

In general, in order to develop a numerical model for the analysis of a type of particular problem, three phases of research are required. That is, experimental research, numerical simulation, and the verification of the model by comparing the results of simulation and of experiments or other theories. If the agreement between the computed results and measured results is not sufficiently close, then the computational model must be refined. However, if the agreement is satisfactory then a sophisticated tool has been developed for the study of the problem.

Fig.8.1 shows the three phases of research in the development of numerical models for the study of the flow and stress behaviour of bulk granular materials in silos. The details of the development of numerical models and the experimental program have been described in Chapters 4-6 and Chapter 7 respectively. The following sections of this chapter are designed to compare the numerical predictions with the author's experimental results, some other published experimental results, predictions from other published pressure theories, and the IEA (The Institution of Engineers, Australia) recommendation (Gorenc et al, 1986).
8.2 COMPARISON OF PRESSURES IN A SILO-HOPPER COMBINATION

Pressures on the wall of a model silo under filling and discharge condition are calculated and compared with experimental results in this section. Fig.8.2 shows the dimension of the model and the finite element mesh adopted. The properties of the stored material have been presented in section 5.7. The angle of side wall friction of the model silo is equal to $20^\circ$.

Fig.8.2 Finite element mesh
8.2.1 Filling Pressures

Fig. 8.3 shows a comparison of numerical predictions of filling pressures with experimental results, the IEA recommendation, and some other predictions by pressure theories. On the vertical wall, both the finite element prediction and the boundary element prediction are very close to Jenike's prediction and the experimental results.

However, the experimental results are significantly smaller than both the numerical predictions and the predictions by pressure theories on the converging hopper walls. This situation is probably due to the existence of the friction of the end walls of the silo used in the experimental investigation, which effect is ignored in the plane strain model used for the numerical analysis. The numerical predictions of pressures on the hopper wall are in good agreement with Jenike's prediction.
8.2.2 Discharge Pressures

Fig.8.4 shows a comparison of numerical predictions of the steady-state discharge pressures with experimental results, the IEA recommendation, and the results computed from some other published pressure theories.

On the vertical wall, pressures are distributed in a form similar to Janssen's prediction and the IEA recommendation. However, the values of the pressures are higher than Janssen's prediction, but lower than the IEA recommendation. In addition, a strong increase of pressures occurs in the area of the transition of the bin and the hopper.

On the hopper wall, the finite element prediction lies between the Walker and Walters flow pressures. The IEA recommendation is very close to the numerical prediction. The Jenike model overestimates the flow pressure at the transition area.

![Fig.8.4 Comparison of numerical prediction of discharge pressures with experimental results and analytical predictions](image_url)
It also can be seen that the measured pressures have a similar form of distribution as that predicted by the finite element model, although the values of the pressures are smaller than the finite element prediction due to the effect of end wall friction. To consider the end wall friction, a three-dimensional model has to be developed.

8.3 COMPARISON OF PRESSURES IN A HOPPER

A small model hopper is chosen for the analysis, because experimental results are available from Tattersall and Schmidt (1981). Fig. 8.5 shows the finite element mesh adopted for the analysis. Table 8.1 shows the properties of the stored material and the friction angle of the side wall.

<table>
<thead>
<tr>
<th>Table 8.1</th>
<th>Properties of the sand used</th>
</tr>
</thead>
<tbody>
<tr>
<td>density kg/m³</td>
<td>angle of internal friction (°)</td>
</tr>
<tr>
<td>1640</td>
<td>39</td>
</tr>
</tbody>
</table>

Fig. 8.5 Finite element mesh
8.3.1 Filling Pressures

Fig. 8.6 shows a comparison of numerical predictions of filling pressures on the hopper wall with the experimental results, the IEA recommendation, and Walker's prediction of filling pressure.

The finite element prediction of filling pressure is very close to that predicted by the boundary element model. Both of the predictions indicate that the distribution of filling pressures on the hopper wall is not linear. A nonlinear distribution occurs due to the side wall friction. The maximum pressure does not occur at the lowest point of the hopper wall, which is also indicated by the experimental results. Both the Walker prediction and the IEA recommendation do not agree with the numerical predictions.

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Fig. 8.6 Comparison of numerical prediction of filling pressures with experimental results and analytical predictions
8.3.2 Discharge Pressures

Fig. 8.7 shows a comparison of numerical predictions of the steady-state discharge pressures with experimental results, the IEA recommendation, and analytical results computed from some other pressure theories. It is clear that the finite element prediction is generally in agreement with the experimental results. It also can be seen that the finite element prediction is between the Walker and Walters flow pressures. The IEA code estimates a pressure much higher than both the numerical prediction and the experimental results.

Fig. 8.7 Comparison of numerical prediction of discharge pressures with experimental results and analytical predictions
### 8.4 SUMMARY AND CONCLUDING REMARKS

By comparing the numerical predictions with the experimental results, the IEA recommendation, and the analytical results computed from pressure theories, it has been shown that the developed numerical methods can be used for the computation of bulk material pressures on the walls of silos in a satisfactory manner.

Under filling conditions both the finite element method and the boundary element method can provide information of pressure distributions on the wall of silos. On vertical walls the numerical predictions of filling pressures are very close to Jenike's prediction and the IEA recommendation. On the converging hopper walls the numerical predictions are in good agreement with Jenike's prediction. However, the predictions of pressures on hopper walls due to Walker and IEA seem to be incorrect.

Under the dynamic condition of discharge, the finite element method can be applied for the computation of the transient pressures on the walls of silos and hoppers with reasonable accuracy. On vertical walls the numerical prediction of flow pressures is between Janssen's prediction and the IEA recommendation. Janssen's equation gives a lower value of the pressures.

On the wall of a plane-flow hopper with surcharge the numerical prediction of flow pressures lies between the Walker and Walters flow pressures. The IEA recommendation agrees quite well with the numerical results, whereas, Jenike's equation overestimates the flow pressures at the transition of the hopper-bin.
On the wall of a plane flow hopper without surcharge, the finite element prediction is in good agreement with the experimental results. The computed values are between the Walker and Walters flow pressures. However, the IEA recommendation does not agree well with the numerical prediction and the experimental results.
CHAPTER NINE

PARAMETRIC STUDY FOR SILO DESIGN

9.1 GENERAL

This chapter focuses on the application of the developed numerical method to a parametric study for silo design. The effects of varying the hopper half angle, the wall friction, the wall flexibility, and using different material constitutive laws on the distribution of wall pressures and internal stresses in the walls of silos are investigated.

In all these investigations, the following assumptions are made unless redeclaration is made,

(1) the bulk material adopted is cohesionless material with the same properties as those described in section 5.7;

(2) the flexibility of the silo walls is significant, and has to be considered in modelling. The material from which the silo shell is fabricated is steel with density 7.8 ton/m³, Young's modulus \( E = 210000 \) MPa, Poisson's ratio \( \nu = 0.25 \).

The details of all above mentioned investigations are presented in the following sections.
9.2 EFFECT OF VARYING HOPPER HALF ANGLE

Distributions of normal wall pressures, circumferential membrane stresses and meridional stresses in the walls of steel silos for a range of hopper half angles under dynamic condition have been computed. Fig.9.1 shows the finite element meshes adopted for the silos examined.

Fig.9.2 shows the distributions of pressures on the walls of the examined silos. For all cases, the pressures change from static fields to dynamic fields progressively. No obvious sudden switch of pressure occurs. Similar forms of pressure distribution have been computed. However, by comparing case (a) and case (c) it can be found that in the case of a high hopper half angle (i.e. low $\alpha$ value, case c) a stronger increase of pressures occurs in the outlet and transition area of the hopper wall during the initial stage of material discharge.

Fig.9.3 shows the distributions of the meridional stresses in the walls of the silo shells. It can be noted that in all cases the hopper wall is subjected to tensile stress, however, the bin wall is subjected to compressive stress. It is also seen that the maximum meridional stress occurs at the transition area of the hopper-bin combination during the whole process of material discharge. The filling condition is found to be the critical condition for the meridional stresses in the hopper.

Fig.9.4 shows the distribution of the circumferential membrane stresses in the walls of the silo shells. For all cases the maximum circumferential membrane stresses occur at the initial stage of material discharge, instead of in the filling condition or the steady-flow condition. Also, it can be seen that in the case of high hopper half angle (case c) a higher value of circumferential membrane stress has been computed in the hopper wall.
Fig. 9.1 Finite element meshes

- (a) \( \alpha = 70^\circ \)
- (b) \( \alpha = 60^\circ \)
- (c) \( \alpha = 50^\circ \)
Fig. 9.2 Distributions of normal wall pressures
Fig. 9-3 Distinctive stresses in silo walls

(a) $\alpha = 50^\circ$

(b) $\alpha = 60^\circ$

(c) $\alpha = 70^\circ$
Fig. 9.4 Distributions of Circumferential membrane stresses in silo walls

(a) $\alpha = 70^\circ$

(b) $\alpha = 60^\circ$

(c) $\alpha = 50^\circ$
9.3 EFFECT OF WALL FRICTION COEFFICIENT

The angle of friction between the bulk granular material and the silo wall was varied from 5° to 30°, to investigate the effect of wall friction on the distribution and magnitude of wall pressures and internal stresses in the silo walls under filling conditions. The dimensions and the finite element mesh adopted for the examined silo is shown in Fig.9.1(b). Fig.9.5 shows the effect of wall friction on the distribution and magnitude of the normal filling pressures on the wall of the silo examined. It is clear that with the increase of wall friction coefficient, the normal pressures on the walls of silos decrease.

Fig.9.6 shows the effect of wall friction on the distribution and magnitude of the meridional stresses in the silo walls. It is interesting to note that with the increase of wall friction coefficient, the tensile meridional stresses in the wall of the hopper decrease, however, the compressive meridional stresses in the vertical wall increase.

Fig.9.7 shows the effect of wall friction on the distribution and magnitude of the circumferential membrane stresses in the silo wall. It is noted that with the increase of wall friction coefficient, the circumferential stresses both in the hopper wall and the vertical wall decrease.
Chapter 9 Parametric study for silo design

(a) Distribution of normal wall pressures

(b) Pressures versus angle of side wall friction

Fig. 9.5 Effect of wall friction on normal wall pressures
Chapter 9 Parametric study for silo design

(a) Distribution of meridional stresses

(b) Meridional stresses versus angle of side wall friction

Fig. 9.5 Effect of wall friction on Meridional stresses in silo wall
(a) Distribution of circumferential membrane stresses

(b) Circumferential membrane stresses versus angle of side wall friction

Fig. 9.5 Effect of wall friction on Circumferential membrane stresses in silo wall
9.4 EFFECT OF WALL FLEXIBILITY ON THE DISTRIBUTION OF WALL PRESSURES

As more and more steel silos are used in industry, it becomes more and more important to investigate the effect of wall flexibility on the distribution of wall pressures. However, until now few attempts have been made to simulate the interaction of flowing granular solids with flexible-wall of silos; a rigid wall condition is usually assumed.

Mahmoud and Abdel-Sayed (1981) and Ooi and Rotter (1986, 1987a) have analysed the effect of wall flexibility on the pressures in the flat-bottomed ground-supported storage silos. Their research indicates that the pressure distribution on the wall of flat-bottomed storage silos is significantly affected by the wall stiffness. This section is designed to investigate the effect of wall flexibility on the wall pressures on a bin-hopper combination with external support at the transition of the hopper-bin.

Fig.9.8 shows the effect of varying thickness (t) of the silo wall on the distribution of filling pressures on silo walls. A reduction in wall thickness increases the wall deformation, and this leads to a increase of pressure in the transition area of the hopper-bin where deformation of the silo wall is restricted by external support. However, in the middle part of the hopper wall, the pressures decrease with the reduction of the thickness of the silo wall.
Chapter 9 Parametric study for silo design

(a) Distribution of normal pressures

(b) Pressures versus the thickness of silo wall

Fig. 9.8 Effect of the thickness of silo wall on the distribution of normal pressures
9.5 EFFECT OF USING DIFFERENT CONSTITUTIVE LAWS

In recent years, many attempts have been made to use the finite element method to predict bulk material pressures on the walls of silos. In particular, emphasis has often been placed on the development of complex material constitutive models and their implementation in analysis (Bishara et al, 1983; Haussler and Eibl, 1984). However, no attempt appears to have been made to show the significance of using complex material constitutive laws, and to investigate the effect of bulk material non-linear deformation and yield characteristics on pressures in silos. Although Ooi and Rotter (1987b), based on their analysis, indicated that pressures in silos are dominated by equilibrium, the effect of material deformation and yield characteristics on predicted results have not been studied.

In this section, the effect of using different constitutive models on the prediction of wall pressures is investigated by applying three commonly used constitutive models in finite element analyses for the prediction of the pressures in an axisymmetric silo. The three different materials have the same density, but follow different constitutive laws.

For the analysis, three cases of filling materials are considered:

Case 1, isotropic elastic continuum with parameters

density $\rho = 1600 \text{ kg/m}^3$.

Young's modulus $E = 50 \text{ MPa}$ for case 1(a), and $E = 10 \text{ MPa}$ for case 1(b),

Poisson's ratio $\nu = 0.25$.

Case 2, Drucker-Prager (1952) elastic, perfectly plastic continuum with parameters

density $\rho = 1600 \text{ kg/m}^3$. 
Young’s modulus $E = 50$ MPa,

Poisson’s ratio $\nu = 0.25$,

cohesive strength $c = 0$,

angle of internal friction $\phi = 30^\circ$

Case 3, Lade elastic, plastic continuum with two yield surfaces. The properties of the material adopted are the same as those described in section 5.7.

The geometry and the finite element mesh adopted for half of the silos are presented in Fig. 9.1b. The friction angle of the silo walls is taken as $15^\circ$. The computed initial filling pressures in silos, which have previously been shown to be the critical loading case causing the maximum meridional stresses in the silo shell (see Figs. 6.4 and 6.7), are presented in the following.

9.5.1 Pressures in a Rigid-Walled Silo

Fig. 9.9(a) shows the pressure distributions in the rigid-walled silo for the three different materials mentioned above. It is noted that the change in the assumed elastic modulus has no effect on the prediction of the wall pressure. However, the material constitutive models have a significant effect on the value of the pressure and the form of pressure distributions. Elastic analysis gives a low value of the predicted pressures in the vertical bin wall and in the part of the hopper wall near the outlet.

9.5.2 Pressures in a Silo with Flexible Wall

Fig. 9.9(b) shows the pressure distributions on the flexible wall of an axisymmetric silo (thickness of the wall = 4 mm, $E = 210000$ MPa, $\nu = 0.25$) for
the three different filling materials described above. It is noted that pressure in the bin area is dominated not only by the material Young's modulus, but also by the yield characteristics of the materials. The prediction based on the elastic model gives a low value of pressure, and the analysis based on the Lade elastic-plastic model gives a high value of the pressure. In the wall of the hopper, the form of pressure distribution varies with the different constitutive models. In the transition area of the hopper - bin, the elastic analysis gives a high value, and the Lade model gives a low value. However in the outlet area the situation is reversed.

9.5.3 Concluding Remarks

The effect of material deformation and yield characteristics on pressures in silos has been investigated by the finite element method. The analysis shows that change in material elastic modulus has no effect on the pressure in rigid-walled silos, however, it has significant effect on pressures in flexible silos.

It has also been found that predicted pressures in silos are dominated not only by equilibrium, but also by the assumption of the material constitutive model. Predictions based on an elastic model give a low value of pressure in the vertical bin wall, but a high value in the transition area of the hopper - bin. Predictions based on the Lade model give a high value of pressure in the vertical bin wall, but a low value in the transition area.

The analyses suggest that research on the assumptions of the material constitutive model is essential in the prediction of pressures in silos.
Fig. 9.9 Effect of material assumption on the distribution of normal pressures
9.6 PREDICTION OF DYNAMIC PRESSURES EXERTED BY BROKEN ORE IN UNDERGROUND STORAGE STRUCTURES

Storage and handling of broken ore are important aspects of underground caving mining operations. Fig.9.10 shows the principal steps of the mining method, which include:

(1) development of the stope,
(2) undercutting of the ore block,
(3) caving of the block and handling of broken ores.

The optimal design of the geometry of mining structures is a key for economical and efficient mining. Jeremic (1987) has pointed out that incorrect spacing and width of drawpoints may cause heavy ore loss and dilution as well as structural failure. From consideration of the flow mechanism, if the spacing is too large, heavy ore loss may occur. This situation is shown in Fig.9.11(a). Improvement can be achieved by reducing the spacing or increasing the width of the drawpoints to allow the drawn out bodies of broken ore to overlap while flow proceeds, as shown in Fig.9.11(b).

![Fig.9.10 Principal mining stages of the block-caving method](image)

(a) (b) (c)
However, reduction of the spacing may cause serious structural failure. Therefore, not only the flow mechanism but also the stability of the structures during mining operations have to be considered in the design of the geometry of the structures and the design of the support systems.

In addition to the rock properties and the virgin stress field, the broken-ore pressure is the important factor in controlling the stability of the structures. The rock properties and the virgin stress field can be determined from rock testing and in-situ stress measurement, respectively. However, the determination of the broken-ore pressure acting on mining structures is difficult due to the complicated geometry of mining structures and the time dependence of mining operations.

The broken-ore pressure is conventionally evaluated by the static theory or classical silo pressure theories, such as the silo pressure theory by Jenike and Johanson (1973). These theories have resulted in a basic understanding of pressure distribution. However, a quantitatively satisfactory solution cannot be obtained for the mining problem because the complex dynamic procedure is not considered in those theories, and also because the geometry of mining structures is complicated.
The finite element method developed herein is applied for the analysis of the pressures on a typical underground storage structure, shown in Fig.9.12. For simplicity, a plane strain case is considered.

Fig.9.13 shows the finite element mesh adopted for part of the structure. The broken ore adopted is assumed as a continuum with the properties of:

- Young's modulus $E = 100$ MPa,
- Poisson's ratio $\nu = 0.3$,
- cohesive strength $C = 0.0$,
- angle of internal friction $\phi = 35^0$,
- tensile strength $= 0.0$ MPa,
- density $\rho = 2500$ kg/m$^3$, and
- viscous constant \( \mu = 0.001 \) sec MPa.

The angle of friction between the broken ore and the rock mass is taken as \( \phi_w = 20^0 \).

Fig.9.14 shows the principal stress distributions in the broken ore for two instants of time. In this Figure the crosses' magnitude and orientation indicate the magnitude and the direction of the stress respectively. At \( t = 0 \) sec, the outlets of both drawpoints are closed, but, at \( t > 0 \) sec, the outlet of the drawpoint on the right hand side is opened. This scheme is designed to simulate the withdrawal of broken ore from one of the drawpoints. It is found that the discharge starts with the change of the orientation of the maximum principal stress, in the opened raise area, from a nearly vertical direction to a direction nearly normal to the side wall inclination. Subsequently, the stresses decrease in the open raise, but increase at the closed raise and the hopper area.

Fig.9.15 shows the distributions of pressures exerted by the broken ore on the analysed underground structure for several instants during discharge of the material. It is clear that during the discharge of the broken ore from one of the drawpoints, the pressures in the outlet area at C decrease. However, high pressure concentrations in the wall of the closed raise and the hopper wall occur. For example at the wall DE, the pressure at point D at the instant of \( t=275 \) msec is 1.4 times larger than the filling pressure (1.27 MPa at \( t=0 \)); at point E, the pressure at \( t=275 \) msec is 4 times larger than the static pressure (0.23 MPa at \( t=0 \)).
Fig. 9.13 Finite element mesh

Fig. 9.14 Principal stress distributions for
(a) $t = 0$ sec; (b) $t = 275$ msec.
Fig. 9.15 Pressure distributions for several instants of time
9.7 SUMMARY AND CONCLUDING REMARKS

The parametric study shows that the side wall friction, the stiffness of the silo wall and the material constitutive relationships have significant effects on the distribution and magnitude of the bulk material pressures on silo walls and also on the internal stresses in silo shells.
CHAPTER TEN

CONCLUSIONS

From the theoretical and experimental research, a boundary element model and a finite element model have been developed for the study of the flow of bulk granular materials through silos, including the prediction of bulk material pressures on the wall of silos and the internal membrane stresses in the silo shells under both static and dynamic conditions. The research has resulted in a better understanding of the distribution of bulk material pressures on silo walls and the internal stresses in the wall of silos. The achievements and the conclusions reached are summarized as follows.

10.1 METHODOLOGY FOR THE PREDICTION OF BULK MATERIAL PRESSURES AND INTERNAL STRESSES IN THE WALLS OF SILOS

A complete set of formulae based on the boundary integral technique has been derived for solving the contact problem of bulk granular material with silo walls under static condition. From the derived equations, a numerical procedure and an associated computer program have been developed for the prediction of bulk material pressures on the walls of silos under filling and initial discharging conditions. Although it is the first attempt to apply the boundary element method to predict pressures in silos, the experimental results have shown that the method is useful at least for solving the filling condition and the initial discharge condition.
A finite element procedure and its computer program have been developed for the simulation of the flow of bulk materials from rigid-walled silos. The method can provide transient velocity and stress fields in the bulk material domain, and pressure distributions on silo walls for plane-flow and axisymmetric silos in a satisfactory manner.

By using a coupling technique, a numerical procedure is developed that can analyse the interaction behaviour between the bulk solids and the silo shells, and simulate the dynamic structural behaviour of flexible silo shells during discharge of materials. The method can provide the transient pressure on the wall of silos, and the transient membrane stresses in silo structures during discharge of bulk granular solids. The maximum values of pressures and stresses in silo shells can be determined and used as a critical condition in the design of silos.

10.2 DISTRIBUTIONS OF PRESSURES IN SILO WALLS

Discharge of bulk materials from silos is a dynamic process. During the period of discharge from the static to the steady-flow condition, no obvious switch of stress from static stress field to dynamic stress field is found. Pressure distributions on the walls of silos change from a static stress field to a dynamic stress field gradually in space, although quickly in time.

It is found that the discharge process starts with a wall pressure increase just above the outlet, then a decrease of pressure near the outlet occurs followed by a significant increase of pressure in the transition area. The pressure response to the material discharge can be divided into three different stages. In the first stage the pressure near the outlet increases more quickly than that in the transition area. In the second stage the pressure in the transition area is nearly constant. However, the
position of the peak pressure on the hopper wall moves slowly from the outlet to the transition point, and the pressure near the outlet decreases. During the third stage, there is a strong increase of wall pressure at the transition area and a decrease near the outlet.

By comparing the numerical predictions with the experimental results, the IEA recommendation, and the analytical results computed from pressure theories, it has been shown that the developed numerical methods can be used for the computation of bulk material pressures on the walls of silos in a satisfactory manner.

On vertical walls, the numerical predictions of filling pressures are very close to Jenike’s prediction and the IEA recommendation. The prediction of flow pressures on vertical walls lies between Janssen’s prediction and the IEA recommendation. Janssen’s equation gives a lower value of the pressures.

On converging hopper walls, the numerical prediction of filling pressures is in good agreement with Jenike’s prediction, whereas, the predictions of pressures due to Walker and IEA seem to be incorrect.

It has also been found that the numerical predictions of flow pressures on converging hopper walls lie between the Walker and Walters flow pressures. The IEA recommendation, in one case, agrees well with the numerical prediction, however, in another case, it does not agree well. This situation indicates that the IEA code for the prediction of flow pressures on hopper walls has to be modified. It is also noted that Jenike’s predictions of flow pressures at the transition between plane-flow hoppers and bins are higher than the real situation.
10.3 STRUCTURAL RESPONSE OF SILO SHELLS

Responding to the storage and discharge of bulk materials, the silo shell deforms and internal stresses are caused.

It is found that, except at the transition area between the hopper and the vertical bin wall, the whole wall of the silos is subjected to a tensile circumferential membrane stress field. In the hopper area, at the beginning of discharge, the stresses increase. However, with the transfer of bulk material pressure from the outlet area to the transition area, the circumferential stresses decrease dramatically near the outlet and increase slightly at the transition.

It is also found that hopper wall is subjected to a tensile meridional stress, however, the bin wall is subjected to a compressive stress. In addition, it is noted that the filling condition is the critical condition for the meridional stress in the hopper.

It is noted that the discharge process starts with an increase of circumferential membrane stresses over the hopper wall, especially above the outlet. During the process of discharge, the meridional stresses are found to decrease at the beginning, and then tend to a constant value. It is interesting to note that, although the pressure at the transition area increases dramatically after the initial stage of discharge, the stresses in the silo are not increased dramatically.

The numerical investigation presented in Chapter 6 has shown that the maximum meridional stresses in the silo shells occur at the filling condition. However, the circumferential membrane stresses reach their maximum values after a short period of discharge. Therefore, in the sense of capturing the maximum
circumferential membrane stresses for the silo design, it is important to conduct dynamic analyses.

Static analysis requires much less computational effort than the dynamic analysis, and also provides the critical values of the meridional stresses in silo shells.

10.4 FLOW MECHANISM OF BULK MATERIALS FROM HOPPER-BIN COMBINATIONS

Discharge of bulk materials from silos is a dynamic process. The discharge starts from the loosening of the material in the area above outlet, then follows a procedure of the formation of different flow zones as shown in Fig.7.15, Fig.7.18, and Fig.7.20. The material in the plug flow zone moves as a rigid body parallel to the wall, and feeds to the pipe feed zone, that, in turn, feeds into the pipe or region of radial flow. During this stage an increase of pressure occurs at the area of effective transition between the plug flow zone and the pipe feed zone as the direction of velocity changes from the vertical to a direction tangential to the inclined surface of dead zone or the hopper wall.

10.5 PARAMETERS AFFECTING THE PRESSURES AND THE INTERNAL STRESSES IN SILO WALLS

The parametric study has shown that the wall friction, the wall flexibility, and the material constitutive relationship have significant effects on the distribution of wall pressures and internal stresses in the walls of silos.

With the increase of wall friction, the normal wall pressures and the circumferential membrane stresses decrease. The tensile meridional stresses in the
wall of the hopper decrease. However, the compressive meridional stresses in the vertical wall of silos increase.

The distribution of wall pressures is affected by the thickness of the silo wall. A reduction in wall thickness increases the wall deformation, and this leads to an increase of pressure in the transition area of the hopper-bin combination where deformation of the silo wall is restricted. However, in the middle part of the hopper wall, the pressures decrease with the reduction of the thickness of the silo wall.

The change in material elastic modulus has no effect on the pressure in stiff-walled silos, however, it has significant effect on pressures in flexible silos.

It has also been found that predicted pressures in silos are dominated not only by equilibrium, but also by the assumption of the material constitutive model. Predictions based on an elastic model give low values of pressures in vertical bin walls, but high values in the transition area of the hopper-bin combination. Predictions based on the Lade constitutive model give high values of pressures in bin walls, however, these predictions give low values in the transition area.

10.6 RECOMMENDATIONS FOR FUTURE WORK

The work completed has been restricted to two-dimensional and axisymmetric problems. For handling the general bulk material flow problem, it is recommended to extend the work to general three-dimensional analysis. This extension causes no difficulty in mathematics. However, a much larger amount of computation is required.
As a dynamic contact problem with material and geometric nonlinearity, the simulation of the filling and discharge process by the finite element method requires a large computational effort, especially for three-dimensional problems, which limits the application of the method. Therefore, it is recommended to use the parallel processing technique (Adeli, 1988) in finite element programming for efficient computation.

A limited number of experiments on the dynamic behaviour of bulk material-silo system during the discharge of materials has been carried out. Further, detailed, experiments are needed, especially experiments on material constitutive relationships and on the corresponding bulk material pressures on silo walls, as well as on the measurement of the internal stresses in silo shells for full scale silos.

It is also recommended to use the methods developed herein for the assessment and modification of existing silo design codes.
REFERENCES


References


