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POSETS AND DIFFERENTIAL GRADED ALGEBRAS

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Abstract

If \( P \) is a partially ordered set and \( R \) is a commutative ring, then a certain differential graded \( R \)-algebra \( \mathcal{A}(P) \) is defined from the order relation on \( P \). The algebra \( \mathcal{A}(\emptyset) \) corresponding to the empty poset is always contained in \( \mathcal{A}(P) \) so that \( \mathcal{A}(P) \) can be regarded as an \( \mathcal{A}(\emptyset) \)-algebra. The main result of this paper shows that if \( R \) is an integral domain and \( P \) and \( P' \) are finite posets such that \( \mathcal{A}(P) \cong \mathcal{A}(P') \) as differential graded \( \mathcal{A}(\emptyset) \)-algebras, then \( P \) and \( P' \) are isomorphic.

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1. Introduction

A common way to study partially ordered sets involves associating certain algebraic objects with a poset and then trying to gain new insights by considering these associated objects. For example, the concept of a Cohen–Macaulay poset arises naturally from the study of Stanley–Reisner rings [1, 3]. On the other hand, algebraic constructions associated with partially ordered sets have also proven to have widespread applicability within algebra itself, particularly in the area of representation theory [2].

The current work, which grew out of an interest in posets that arise in group representation theory, is based upon this interplay between partially ordered sets and algebra. If \( P \) is a partially ordered set and \( R \) is an integral domain, then we define a graded \( R \)-algebra \( \mathcal{A}(P) \). The definition involves forming a new poset \( P_0 \) by adjoining a minimum element 0 to the poset \( P \). For any \( n \geq 0 \) the component \( A_n(P) \) of degree \( n \) is the free \( R \)-module on the symbols \([x_1 < \cdots < x_n]\) whenever \( x_1 < \cdots < x_n \) is a chain in \( P_0 \). Using the order relation on \( P_0 \), one can define a multiplication on \( \mathcal{A}(P) \),
and it also has an $R$-endomorphism of degree $-1$ that makes $A_\bullet(P)$ into a differential graded $R$-algebra. The algebra $A_\bullet(\emptyset)$ corresponding to the empty poset is necessarily contained in $A_\bullet(P)$ so that $A_\bullet(P)$ is in fact an $A_\bullet(\emptyset)$-algebra.

Now suppose that $P$ and $P'$ are finite posets and $f_\bullet : A_\bullet(P) \rightarrow A_\bullet(P')$ is an isomorphism of differential graded $A_\bullet(\emptyset)$-algebras. If $f_\bullet$ maps the distinguished basis of $A_\bullet(P)$ to that of $A_\bullet(P')$, then the definition of the multiplication in $A_\bullet(P)$ makes it easy to see that $P$ and $P'$ are isomorphic. The main result of this paper shows that this conclusion is valid even if $f_\bullet$ does not preserve the distinguished basis. Thus one can recover the poset $P$ from the algebra $A_\bullet(P)$ with no additional information.

Section 2 of the paper contains the definition of $A_\bullet(P)$ and a proof that it is a differential graded $A_\bullet(\emptyset)$-algebra. The proof that the algebra $A_\bullet(P)$ determines the poset $P$ is given in Section 3. Finally, Section 4 gives a description of the graded center in terms of certain annihilators in $A_\bullet(P)$. Although we have chosen to assume throughout the paper that the coefficient ring $R$ is an integral domain, it should be noted that this assumption is often not necessary. In particular, all of the results of Section 2 hold over an arbitrary commutative ring.

2. The definition and basic properties of the algebra

If $P$ is a partially ordered set and $R$ is an integral domain, then we will define a differential graded $R$-algebra $A_\bullet(P)$ from the poset $P$. The first step is to define a new poset $P_0$ in which the points consist of the points in $P$, together with one additional point called $0$. The order $<$ on $P_0$ is given by taking $x < y$ in $P_0$ if either $x = 0$ and $y \in P$ or $x, y \in P$ and $x < y$ in $P$.

For each $n \geq 0$ the component $A_n(P)$ is defined to be the free $R$-module on the symbols $[x_1 < x_2 < \cdots < x_n]$ whenever $x_1 < x_2 < \cdots < x_n$ is a strictly increasing chain in $P_0$. For convenience we will also use the symbol $[x_1 < x_2 < \cdots < x_n]$ even when $x_1, x_2, \ldots, x_n$ do not form a strictly increasing chain in $P_0$, but in this case we set $[x_1 < x_2 < \cdots < x_n]$ equal to 0 in $A_n(P)$. Note that $A_0(P)$ is a free $R$-module of rank one, generated by the symbol $[\ ]$.

Define a multiplication on the (non-zero) basis elements of $A_\bullet(P)$ by setting

$$[x_1 < \cdots < x_m][y_1 < \cdots < y_n] = \begin{cases} [x_1 < \cdots < x_m < y_1 < \cdots < y_n] & \text{if } x_m < y_1 \\ (-1)^m[-1][0 < x_1 < \cdots < x_{m-1} < y_1 < \cdots < y_n] & \text{if } x_m \not< y_1, \end{cases}$$

and extend this multiplication to all of $A_\bullet(P)$ by linearity. In the proofs of the following propositions it is important to bear in mind that the equation defining this multiplication applies only to products of non-zero generators of $A_\bullet(P)$. 
PROPOSITION 2.1. Let \( P \) be a partially ordered set. Then \( A_\star(P) \) is a graded associative algebra with 1.

PROOF. The identity element of \( A_\star(P) \) is given by \([\ ]\), and it is clear from the definition of the product that \( A_m(P)A_n(P) = A_{m+n}(P) \). Thus it is only necessary to show that \( A_\star(P) \) is associative.

Let \( a, b, c \in A_\star(P) \) be homogeneous elements. We will prove that \((ab)c = a(bc)\) by induction on \( \deg b \). The equality clearly holds if \( \deg a = 0 \), \( \deg b = 0 \), or \( \deg c = 0 \), so assume that \( \deg b = 1 \), \( \deg a \geq 1 \), and \( \deg c \geq 1 \). To prove that \((ab)c = a(bc)\), it suffices to consider the case in which \( a, b, \) and \( c \) are non-zero homogeneous generators. Suppose, then, that \( a = [x_1 < \cdots < x_m] \), \( b = [y_1] \), and \( c = [z_1 < \cdots < z_p] \). If \( x_m < y_1 < z_1 \), then it is easy to see that \((ab)c = a(bc)\), so suppose that \( x_m \neq y_1 \) but \( y_1 < z_1 \). Then

\[
(ab)c = ([x_1 < \cdots < x_m][y_1])[z_1 < \cdots < z_p] \\
= (-1)^{m-1}[0 < x_1 < \cdots < x_{m-1} < y_1][z_1 < \cdots < z_p] \\
+ (-1)^m[0 < x_1 < \cdots < x_m][z_1 < \cdots < z_p] \\
= (-1)^{m-1}[0 < x_1 < \cdots < x_{m-1} < y_1 < z_1 < \cdots < z_p] \\
+ (-1)^m[0 < x_1 < \cdots < x_m < z_1 < \cdots < z_p] \\
= [x_1 < \cdots < x_m][y_1 < z_1 < \cdots < z_p] \\
= [x_1 < \cdots < x_m]([y_1][z_1 < \cdots < z_p]) \\
= a(bc).
\]

Similar computations show that \((ab)c = a(bc)\) when \( x_m < y_1 \) and \( y_1 \neq z_1 \), and also when \( x_m \neq y_1 \) and \( y_1 \neq z_1 \).

It follows that if \( a, b, \) and \( c \) are any homogeneous elements of \( A_\star(P) \) with \( \deg b = 1 \), then \((ab)c = a(bc)\). Assume by induction that \( n \geq 1 \) and that if \( a, b, \) and \( c \) are homogeneous with \( \deg b \leq n \), then \((ab)c = a(bc)\). Then

\[
(a[y_1 < \cdots < y_{n+1}])c = (a([y_1 < \cdots < y_n][y_{n+1}]])c \\
= (a([y_1 < \cdots < y_n][y_{n+1}]c \\
= (a[y_1 < \cdots < y_n])([y_{n+1}]c) \\
= a([y_1 < \cdots < y_n]|[y_{n+1}]c) \\
= a([y_1 < \cdots < y_n][y_{n+1}]c) \\
= a([y_1 < \cdots < y_{n+1}]c).
\]

Hence \((ab)c = a(bc)\) whenever \( a, b, \) and \( c \) are homogeneous with \( \deg b \leq n + 1 \), and it follows that \( A_\star(P) \) is associative. This completes the proof.
If $1 \leq i \leq n$, then we write $[x_1 < \cdots < \hat{x}_i < \cdots < x_n]$ for $[x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_n]$. Define a sequence of $R$-linear maps $d : A_n(P) \rightarrow A_{n-1}(P)$ by setting

$$d[x_1 < \cdots < x_n] = \sum_{i=1}^{n} (-1)^{i-1} [x_1 < \cdots < \hat{x}_i < \cdots < x_n]$$

on all non-zero homogeneous generators $[x_1 < \cdots < x_n]$. It is easy to verify that $d^2 = 0$.

**Proposition 2.2.** Let $P$ be a partially ordered set, and suppose that $a \in A_m(P)$ and $b \in A_n(P)$. Then

$$d(ab) = (da)b + (-1)^m a(db),$$

and $(A_*(P), d)$ is a differential graded $R$-algebra.

**Proof.** We will prove that $d(ab) = (da)b + (-1)^m a(db)$ by induction on $m$. It is clear that the equation holds if $m = 0$ or $n = 0$, so assume that $m = 1$ and $n \geq 1$. To prove that the equation holds in this case, it suffices to consider the situation in which $a$ and $b$ are non-zero homogeneous generators. Suppose, then, that $a = [x_1]$ and $b = [y_1 < \cdots < y_n]$. If $x_1 < y_1$, then

$$(da)b + (-1)^m a(db)$$

$$= [y_1 < \cdots < y_n] - \sum_{i=1}^{n} (-1)^{i-1} [x_1 < y_1 < \cdots < \hat{y}_i < \cdots < y_n]$$

$$= d[x_1 < y_1 < \cdots < y_n] = d(ab).$$

Now suppose that $x_1 \not< y_1$. Then one can check that

$$(da)b + (-1)^m a(db)$$

$$= [y_1 < \cdots < y_n] - \sum_{i=1}^{n} (-1)^{i-1} [x_1][y_1 < \cdots < \hat{y}_i < \cdots < y_n]$$

$$= [y_1 < \cdots < y_n] - [x_1][y_2 < \cdots < y_n]$$

$$- \sum_{i=2}^{n} (-1)^{i-1} [0 < y_1 < \cdots < \hat{y}_i < \cdots < y_n]$$

$$+ (-1)^{i'}[0 < x_1 < y_2 < \cdots < \hat{y}_i < \cdots < y_n]$$
= \sum_{i=1}^{n} (-1)^i [0 < y_1 < \cdots < y_i < \cdots < y_n] \\
- [x_1] [y_2 < \cdots < y_n] + [0 < y_2 < \cdots < y_n] \\
- \sum_{i=2}^{n} (-1)^i [0 < x_1 < y_2 < \cdots < y_i < \cdots < y_n] \\
= d[0 < y_1 < \cdots < y_n] - d[0 < x_1 < y_2 < \cdots < y_n] \\
= d([x_1] [y_1 < \cdots < y_n]) = d(ab).

It now follows that if \( a \) and \( b \) are any homogeneous elements of \( A_*(P) \) with \( \deg a = 1 \), then \( d(ab) = (da)b - a(db) \). Assume by induction that \( m \geq 1 \) and that if \( a \) and \( b \) are homogeneous with \( \deg a \leq m \), then \( d(ab) = (da)b + (-1)^{\deg a} a(db) \). Then

\[
\begin{align*}
(d[x_1 < \cdots < x_{m+1}])b + (-1)^{m+1}[x_1 < \cdots < x_{m+1}]db &= d([x_1][x_2 < \cdots < x_{m+1}])b + (-1)^{m+1}[x_1 < \cdots < x_{m+1}]db \\
&= [x_2 < \cdots < x_{m+1}]b - [x_1](d[x_2 < \cdots < x_{m+1}])b \\
&\quad + (-1)^{m+1}[x_1 < \cdots < x_{m+1}]db \\
&= [x_2 < \cdots < x_{m+1}]b \\
&\quad - [x_1](d[x_2 < \cdots < x_{m+1}])b + (-1)^m[x_2 < \cdots < x_{m+1}]db \\
&= (d[x_1])[x_2 < \cdots < x_{m+1}]b - [x_1]d([x_2 < \cdots < x_{m+1}]b) \\
&= d([x_1 < x_2 < \cdots < x_{m+1}]b).
\end{align*}
\]

Hence \( d(ab) = (da)b + (-1)^{\deg a} a(db) \) whenever \( a \) and \( b \) are homogeneous with \( \deg a \leq m + 1 \), and it follows that \( A_*(P) \) is a differential graded \( R \)-algebra.

If \( P \) is any poset, then the algebra \( A_*(\emptyset) \) corresponding to the empty poset is just the subalgebra of \( A_*(P) \) spanned by \([x]\) and \([0]\). Thus \( A_*(P) \) is actually a differential graded \( A_*(\emptyset) \)-algebra.

Let \( P \) and \( P' \) be partially ordered sets, and let \( f_1 : A_1(P) \to A_1(P') \) be an \( R \)-linear map given by

\[
f_1[x] = \sum_{x' \in P'_0} c_{x',x} [x']
\]

for some elements \( c_{x',x} \in R \). We want to explore the conditions under which \( f_1 \) extends to a homomorphism \( f_2 : A_*(P) \to A_*(P') \) of differential graded \( A_*(\emptyset) \)-algebras. The matrix \( C = (c_{x',x}) \) will be referred to as the \textit{matrix of} \( f_1 \).
Let \( f_0 : A_0(P) \to A_0(P') \) be the unique \( R \)-linear map satisfying \( f_0[\ ] = [\ ] \), and for \( n \geq 2 \) let \( f_n : A_n(P) \to A_n(P') \) be the unique \( R \)-linear map defined on basis elements of \( A_n(P) \) by

\[
f_n[y_1 < \cdots < y_n] = f_1[y_1] \cdots f_1[y_n].
\]

In this way we associate an \( R \)-linear map \( f_* : \Lambda_*(P) \to \Lambda_*(P') \) to each \( R \)-linear map \( f_1 : A_1(P) \to A_1(P') \).

**Lemma 2.3.** Let \( P \) and \( P' \) be posets, and let \( f_1 : A_1(P) \to A_1(P') \) be an \( R \)-linear map with matrix \( C = (c_{x',x}) \). Then the \( R \)-linear map \( f_* : \Lambda_*(P) \to \Lambda_*(P') \) satisfies \( df_1 = f_0d \) if and only if \( \sum_{x' \in P'_0} c_{x',x} = 1 \) for all \( x \in P_0 \).

**Proof.** Let \( x \in P_0 \). Then \( df_1[x] = d \sum_{x' \in P'_0} c_{x',x}[x'] = \sum_{x' \in P'_0} c_{x',x}[\ ] \), and \( f_0d[x] = f_0[\ ] = [\ ] \). Hence \( df_1[x] = f_0d[x] \) if and only if \( \sum_{x' \in P'_0} c_{x',x} = 1 \), as desired.

**Lemma 2.4.** Let \( P \) and \( P' \) be posets, and let \( f_1 : A_1(P) \to A_1(P') \) be an \( R \)-linear map with matrix \( C = (c_{x',x}) \). Suppose that \( f_1[0] = [0] \) and that \( df_1 = f_0d \). Then the following conditions are equivalent:

1. If \( x, y \in P_0 \) and \( x \neq y \), then \( [0]f_1[x]f_1[y] = 0 \).
2. If \( a, b \in A_*(P) \), then \( f_*(ab) = f_*(a)f_*(b) \).
3. If \( x \neq y \) in \( P_0 \) and \( 0 \neq x' < y' \) in \( P'_0 \), then \( c_{x',x}c_{y',y} = 0 \).

**Proof.** Let \( x, y \in P_0 \) with \( x \neq y \). Then

\[
[0]f_1[x]f_1[y] = [0] \sum_{x' \in P'_0} c_{x',x}[x'] \sum_{y' \in P'_0} c_{y',y}[y'] = \sum_{0 \neq x' < y'} c_{x',x}c_{y',y}[0 < x' < y'],
\]

and it follows that (1) and (3) are equivalent.

Now suppose that (2) holds. If \( x, y \in P_0 \) and \( x \neq y \), then

\[
[0]f_1[x]f_1[y] = f_3([0][x][y]) = f_3([0][0 < y] - [0][0 < x]) = 0.
\]

Thus we see that (2) implies (1).

Finally, we show that (3) implies (2). To prove that \( f_*(ab) = f_*(a)f_*(b) \) for all \( a, b \in A_*(P) \), it suffices to consider the case in which \( a \) and \( b \) are homogeneous basis elements. In fact, it is enough to prove that

\[
f_{n+1}([x][y_1 < \cdots < y_n]) = f_1[x]f_n[y_1 < \cdots < y_n]
\]
whenever $x \in P_0$ and $y_1 < \cdots < y_n$ in $P_0$. The result is immediate if $n = 0$, so assume that $n \geq 1$. If $x < y_1$, then

$$f_{n+1}([x][y_1 < \cdots < y_n]) = f_{n+1}[x < y_1 < \cdots < y_n]$$

$$= f_1[x]f_1[y_1] \cdots f_1[y_n]$$

$$= f_1[x]f_n[y_1 < \cdots < y_n],$$

as desired. Thus we may assume that $x \neq y_1$.

We now prove that if $n \geq 1$ and $x \neq y_1$, then $f_{n+1}([x][y_1 < \cdots < y_n]) = f_1[x]f_n[y_1 < \cdots < y_n]$. First suppose that $n = 1$. Then (3) implies that

$$f_1[x]f_1[y_1] = \sum_{x', y' \in P_0} c_{x', y_1}[x'][y']$$

$$= \sum_{y' \in P_0} \sum_{0 \neq x' < y'} c_{x', y_1}[x' < y'] + \sum_{y' \in P_0} c_{0, y_1}[0 < y']$$

$$- \sum_{x' \in P_0} c_{x', 0}[0 < x'] + \sum_{y' \in P_0} \sum_{0 \neq x' < y'} c_{x', y_1}[0 < y'] - [0 < x']$$

$$= \sum_{y' \in P_0} \left( c_{0, y_1} - c_{x', y_1} \right) + \sum_{y' \in P_0} c_{x', y_1} - \sum_{x' \in P_0} c_{x', y_1}[0 < y']$$

$$= \sum_{y' \in P_0} \left( \sum_{0 \neq x' < y'} c_{x', y_1} - \sum_{x' \in P_0} c_{x', y_1} \right)[0 < y'].$$

Since $df_1 = f_0d$, Lemma 2.3 implies that

$$f_1[x]f_1[y_1] = \sum_{y' \in P_0} (c_{y_1} - c_{y_1})[0 < y']$$

$$= \sum_{y' \in P_0} c_{y_1}[0 < y'] - \sum_{y' \in P_0} c_{y_1}[0 < y']$$

$$= [0]f_1[y_1] - [0]f_1[x]$$

$$= f_2[0 < y_1] - f_2[0 < x]$$

$$= f_2([x][y_1]).$$

(2.5)

Now suppose that $n \geq 2$. Using (2.5) and (1), we see that

$$f_1[x]f_n[y_1 < \cdots < y_n] = f_1[x]f_1[y_1] \cdots f_1[y_n]$$

$$= [0]f_1[y_1] \cdots f_1[y_n] - [0]f_1[x]f_1[y_2] \cdots f_1[y_n]$$

$$= f_{n+1}[0 < y_1 < \cdots < y_n] - f_{n+1}[0 < x < y_2 < \cdots < y_n]$$

$$= f_{n+1}([x][y_1 < \cdots < y_n]).$$

Thus (2) follows, and this completes the proof.
PROPOSITION 2.6. Let $P$ and $P'$ be partially ordered sets, and let $f_1 : A_1(P) \to A_1(P')$ be an $R$-linear map with matrix $C = (c_{x'y})$. Then $f_1$ extends to a homomorphism $f_* : A_*(P) \to A_*(P')$ of differential graded $A_*(\emptyset)$-algebras if and only if the following conditions are satisfied.

(1) $c_{00} = 1$ and $c_{x'0} = 0$ for all $x' \in P'$.
(2) $\sum_{x' \in P_0} c_{x'x} = 1$ for all $x \in P_0$.
(3) If $x \neq y$ in $P_0$ and $0 \neq x' < y'$ in $P_0'$, then $c_{x'y}c_{x'y'} = 0$.

PROOF. Note that $f_1$ extends to a homomorphism $f_*$ of differential graded $A_*(\emptyset)$-algebras if and only if the following conditions are satisfied:

(1') $f_0[1] = [1]$ and $f_1[0] = [0]$.
(2') $df_n = f_n d$ for all $n \geq 0$.
(3') $f_* (ab) = f_*(a)f_*(b)$ for all $a, b \in A_*(P)$.

Thus it suffices to show that conditions (1), (2), and (3) are equivalent to conditions (1'), (2'), and (3'). We have defined $f_0$ so that $f_0[1] = [1]$ and $f_1[0] = [0]$ precisely when $c_{00} = 1$ and $c_{x'0} = 0$ for all $x' \in P'$. Thus (1) is equivalent to (1').

Suppose that (1'), (2'), and (3') hold. Then Lemma 2.3 implies that (2) holds, and Lemma 2.4 implies that (3) holds.

Conversely, suppose that $f_1$ satisfies (1), (2), and (3). Then $f_*$ also satisfies (1'), and Lemma 2.3 implies that $df_1 = f_0 d$. By Lemma 2.4 it follows that $f_*$ satisfies (3'), so it only remains to show that $df_{n+1} = f_n d$ for $n \geq 1$. If $[y_1 < \cdots < y_{n+1}]$ is any basis element of $A_{n+1}(P)$, then by induction it follows that

$$
\begin{align*}
df_{n+1}[y_1 < \cdots < y_{n+1}] &= d\left(f_n[y_1 < \cdots < y_n]f_1[y_{n+1}]\right) \\
&= (df_n[y_1 < \cdots < y_n])f_1[y_{n+1}] + (-1)^n f_n[y_1 < \cdots < y_n]df_1[y_{n+1}] \\
&= (f_{n-1}d[y_1 < \cdots < y_n])f_1[y_{n+1}] + (-1)^n f_n[y_1 < \cdots < y_n]f_0 d[y_{n+1}] \\
&= f_n\left((d[y_1 < \cdots < y_n])[y_{n+1}] + (-1)^n [y_1 < \cdots < y_n]d[y_{n+1}]\right) \\
&= f_n d[y_1 < \cdots < y_{n+1}].
\end{align*}
$$

This completes the proof.

COROLLARY 2.7. Let $f : P \to P'$ be a map of posets. Then the following conditions are equivalent.

(1) There is a homomorphism $f_* : A_*(P) \to A_*(P')$ of differential graded $A_*(\emptyset)$-algebras satisfying $f_1[x] = [f(x)]$ for all $x \in P$.
(2) There is a homomorphism $f_* : A_*(P) \to A_*(P')$ of differential graded $A_*(\emptyset)$-algebras such that $f_n$ satisfies

$$
f_n[x_1 < \cdots < x_n] = [f(x_1) < \cdots < f(x_n)] \text{ for all } n \geq 1.
$$
(3) If \( f(x) < f(y) \), then \( x < y \) for all \( x, y \in P \).

**Proof.** First suppose that (1) holds. We will prove by induction on \( n \) that \( f_n \) is given by

\[
f_n[x_1 < \cdots < x_n] = \left[ f(x_1) < \cdots < f(x_n) \right]
\]

for all \( n \geq 1 \). This equation is true for \( n = 1 \) by assumption. Let \( [x_1 < \cdots < x_{n+1}] \) be a non-zero homogeneous generator. Because \( x_n < x_{n+1} \) and \( f \) is a map of posets, it follows that \( f(x_n) \leq f(x_{n+1}) \). Thus

\[
\left[ f(x_1) < \cdots < f(x_n) \right] \left[ f(x_{n+1}) \right] = \left[ f(x_1) < \cdots < f(x_{n+1}) \right]
\]

even if \( f(x_n) = f(x_{n+1}) \). Hence

\[
f_{n+1}[x_1 < \cdots < x_{n+1}] = f_{n+1}([x_1 < \cdots < x_n][x_{n+1}])
= f_n[x_1 < \cdots < x_n] f_1[x_{n+1}]
= \left[ f(x_1) < \cdots < f(x_n) \right] \left[ f(x_{n+1}) \right]
= \left[ f(x_1) < \cdots < f(x_{n+1}) \right],
\]

and (2) follows.

It is trivial that (2) implies (1), so assume that (1) holds. If \( x \in P \), then the matrix \( C = (c_{x'y'}) \) of \( f_1 \) satisfies \( c_{x'y'} = 1 \) if \( x' = f(x) \) and \( c_{x'y'} = 0 \) if \( x' \neq f(x) \). Proposition 2.6 shows that if \( x \neq y \) in \( P_0 \) and \( 0 \neq x' < y' \) in \( P'_0 \), then \( c_{x'y} c_{y'y} = 0 \). But if \( x, y \in P \) are elements such that \( f(x) < f(y) \), then \( c_{f(x),x} c_{f(y),y} = 1 \), so it follows that \( x < y \). Hence (1) implies (3).

Finally, suppose that (3) holds. Extend \( f \) to a map \( f : P_0 \to P'_0 \) by defining \( f(0) = 0 \), and let \( f_1 : A_1(P) \to A_1(P') \) be the \( R \)-linear map satisfying \( f_1[x] = [f(x)] \) for all \( x \in P_0 \). Then all of the conditions of Proposition 2.6 are satisfied, and it follows that \( f_1 \) extends to a homomorphism \( f_* : A_*(P) \to A_*(P') \) of differential graded \( A_*(\emptyset) \)-algebras, as desired.

Finally, we end this section with the following simple but useful observation.

**Proposition 2.8.** If \( P \) is a poset, then \( A_*(P) \) is contractible. In fact, if \( s_* : A_*(P) \to A_*(P') \) is the map of degree one satisfying \( s_*(x) = [0]x \) for every homogeneous element \( x \in A_*(P) \), then \( s_* \) is a contracting homotopy.

**Proof.** Let \( x \in A_*(P) \) be homogeneous. Because \( d \) is a derivation, it follows that

\[
ds_*(x) + s_* d(x) = d([0]x) + [0](dx) = x,
\]
as desired.
3. Isomorphic algebras

In this section our goal is to show that if \( P \) and \( P' \) are finite posets such that \( A_\ast(P) \cong A_\ast(P') \) as differential graded \( A_\ast(\emptyset) \)-algebras, then \( P \cong P' \). While this fact is obvious if there is an isomorphism from \( A_\ast(P) \) to \( A_\ast(P') \) that maps each basis element \([x_1 < \cdots < x_n]\) of \( A_\ast(P) \) to a basis element of \( A_\ast(P') \), not all isomorphisms arise in this way. Nevertheless, it is easy to see that certain invariants associated with the posets are the same. For example, the rank of \( A_1(P) \) is just the cardinality \(|P_0| = |P| + 1\), so it follows that \(|P| = |P'|\).

Another invariant that can easily be recovered from the algebra \( A_\ast(P) \) is the height of the poset. Recall that an element \( x \in P \) is said to have height \( h_P(x) = n \) if \( n \) is the largest number such that there is a chain \( x_1 < \cdots < x_n = x \) in \( P \). The height \( h(P) \) of the poset \( P \) is defined to be the supremum of the heights of its elements. If \( P \) is finite with \( h(P) = n \), then \( h(P_0) = n + 1 \) so that \( A_{n+1}(P) \neq 0 \) but \( A_m(P) = 0 \) for all \( m > n+1 \). Thus \( h(P) = h(P') \) if \( P \) and \( P' \) are finite posets such that \( A_\ast(P) \cong A_\ast(P') \).

A connection between \( A_\ast(P) \) and the heights of individual elements in \( P \) is given by the following lemma.

**Lemma 3.1.** Let \( P \) be a poset, and let \( x \in P \). If there is an element \( a \in A_{n-1}(P) \) such that \([0]a[x] \neq 0\), then \( h_P(x) \geq n \).

**Proof.** It suffices to consider the case in which \( n \geq 2 \). Suppose that \( a \in A_{n-1}(P) \) is an element such that \([0]a[x] \neq 0\). Then there is a basis element \([y_1 < \cdots < y_{n-1}] \in A_{n-1}(P) \) such that \([0][y_1 < \cdots < y_{n-1}][x] \neq 0\), so the product \([y_1 < \cdots < y_{n-1}][x] \) does not lie in the ideal \([0]A_\ast(P) \) generated by \([0]\). Hence \( y_1 \neq 0 \) and \( y_{n-1} < x \) so that \( y_1 < \cdots < y_{n-1} < x \) is a chain in \( P \). Thus \( h_P(x) \geq n \), as desired.

**Proposition 3.2.** Suppose that \( P \) and \( P' \) are finite posets and \( f_\ast : A_\ast(P) \rightarrow A_\ast(P') \) is an isomorphism such that \( C = (c_{x'x}) \) is the matrix of \( f_1 \). Let \( H \subseteq P \) and \( H' \subseteq P' \) be the subposets consisting of all elements that are not of maximum height, and let \( x' \in P' \). Then \( x' \in H' \) if and only if \( c_{x'x} \neq 0 \) for some \( x \in H \).

**Proof.** Suppose that \( x' \in P' \) is an element such that \( c_{x'x} = 0 \) for all \( x \in H \). Because \( f_\ast \) is an isomorphism, there are distinct elements \( m_1, \ldots, m_s \in P - H \) and \( b_1, \ldots, b_s \in R - \{0\} \) such that \([x'] = b_1f_1[m_1] + \cdots + b_sf_1[m_s]\). Let \( 0 < x_1 < \cdots < x_{n-1} < m_1 \) be a chain of maximum length in \( P_0 \), and set \( a = b_1[m_1] + \cdots + b_s[m_s] \in A_1(P) \). Then \([0 < x_1 < \cdots < x_{n-1}]a \neq 0\), so

\[
0 \neq f_{n+1}([0 < x_1 < \cdots < x_{n-1}]a) = [0]f_{n-1}[x_1 < \cdots < x_{n-1}][x'].
\]

It follows by Lemma 3.1 that

\[
h_P(x') \geq n = h_P(m_1) = h(P) = h(P').
\]
Hence \( x' \notin H' \), as desired.

Conversely, suppose that \( x' \in P' - H' \) and \( x \in P \) are elements such that \( c_{x,x} \neq 0 \).

Let \( 0 < x'_1 < \cdots < x'_{n-1} < x' \) be a chain of maximum length in \( P_0' \), and let \( b \in A_{n-1}(P) \) be the element with \( f_{n-1}(b) = [x'_1 < \cdots < x'_{n-1}] \). Then

\[
    f_{n+1}([0]b[x]) = [0 < x'_1 < \cdots < x'_{n-1}] \sum_{y' \in P_0'} c_{y,x}[y']
\]

is non-zero because \( c_{x',x}[0 < x'_1 < \cdots < x'_{n-1} < x'] \neq 0 \). Hence \([0]b[x] \neq 0\), and Lemma 3.1 implies that

\[
    h_P(x) \geq n = h_P(x') = h(P') = h(P).
\]

Thus \( x \notin H \), and this completes the proof.

**Corollary 3.3.** Suppose that \( P \) and \( P' \) are finite posets and \( f_\bullet : A_\bullet(P) \to A_\bullet(P') \) is an isomorphism. Let \( H \subseteq P \) and \( H' \subseteq P' \) be the subposets consisting of all elements that are not of maximum height. Then \( f_\bullet \) restricts to an isomorphism \( h_\bullet : A_\bullet(H) \to A_\bullet(H') \).

**Proposition 3.4.** Let \( P \) and \( P' \) be finite posets, and let \( f_\bullet : A_\bullet(P) \to A_\bullet(P') \) be an \( A_\bullet(0) \)-isomorphism such that \( C = (c_{x',x}) \) is the matrix of \( f_1 \). If \( x \in P \) and \( x' \in P' \) are elements with \( c_{x',x} \neq 0 \), then \( h_P(x') \leq h_P(x) \).

**Proof.** The proof proceeds by induction on \( h(P) \). The result is obvious if \( h(P) = 1 \), so assume that \( h(P) > 1 \). Let \( H \subseteq P \) and \( H' \subseteq P' \) be the subposets consisting of all elements that are not of maximum height. Corollary 3.3 implies that if \( x \in H \) and \( x' \in P' \) are elements such that \( c_{x',x} \neq 0 \), then \( x' \in H' \). Then \( h_{H'}(x') \leq h_H(x) \) by induction, and the result follows in this case. On the other hand, if \( x \in P - H \), then

\[
    h_P(x) = h(P) = h(P') \geq h_P(x')
\]

for all \( x' \in P' \), as desired.

**Definition 3.5.** Let \( P \) be a finite poset, and let \( a \in A_1(P) \). Write \( a = \sum_{x \in P_0} a_x[x] \).

The set \( \text{supp} \ a = \{ x \in P \mid a_x \neq 0 \} \) will be called the support of \( a \) in \( P \).

Let \( P' \) be another poset, and let \( f_\bullet : A_\bullet(P) \to A_\bullet(P') \) be an \( A_\bullet(0) \)-isomorphism. Two elements \( x \in P \) and \( x' \in P' \) will be called mutually \( f_\bullet \)-supportive (or simply mutually supportive when \( f_\bullet \) is understood) provided that \( x' \in \text{supp} \ f_1[x] \) and \( x \in \text{supp} \ f_1^{-1}[x'] \).
Note that the support of an element \( a \in A_1(P) \) is defined to be a subset of \( P \), not of \( P_0 \); we do not consider 0 to lie in the support of \( a \) even if \( a_0 \neq 0 \).

It will be important to observe that if \( f_* : A_*(P) \to A_*(P') \) is an isomorphism and \( x \in P \), then there is always an element \( x' \in P' \) such that \( x \) and \( x' \) are mutually supportive. Indeed, suppose that \( C \) is the matrix of \( f_1 \) and \( D \) is the matrix of \( f_1^{-1} \). Then \( 1 = \sum_{x' \in P_0'} d_{x'x} c_{x'{x}} \), and there is an element \( x' \in P_0' \) such that \( d_{x'x} c_{x'{x}} \neq 0 \).

Because \( f_1 \) is an isomorphism with \( f_1[0] = [0] \), it is easy to see that \( x' \neq 0 \). Then \( x \in P \) and \( x' \in P' \) are mutually supportive. Moreover, any two mutually supportive elements must have the same height by Proposition 3.4.

If \( P \) is a finite partially ordered set, then it will sometimes be useful to consider total orders on \( P_0 \) in addition to the original partial order. For convenience we will generally specify a total ordering on \( P_0 \) simply by listing all of the elements \( x_0, \ldots, x_n \) of \( P_0 \) in increasing order. The symbol \( < \) will still be reserved for the partial order on \( P_0 \).

**Definition 3.6.** Let \( P \) be a partially ordered set with \( |P| = n \), and write \( P_0 = \{x_0, x_1, \ldots, x_n\} \). We will say that \( x_0, x_1, \ldots, x_n \) is a *total order* on \( P_0 \) if \( i < j \) whenever \( h_{P_0}(x_i) < h_{P_0}(x_j) \).

Suppose that \( x_0, x_1, \ldots, x_n \) is a total order on \( P_0 \), and suppose that \( x_i < x_j \) for some \( i \) and \( j \). Then \( h_{P_0}(x_i) < h_{P_0}(x_j) \), so \( i < j \). Thus the total ordering on \( P_0 \) specified by \( x_0, x_1, \ldots, x_n \) is compatible with the original partial ordering. In particular, \( x_0 = 0 \).

Now suppose that \( P \) and \( P' \) are finite partially ordered sets, and let \( f_* : A_*(P) \to A_*(P') \) be an \( A_*(\emptyset) \)-isomorphism. Suppose that \( x_0, \ldots, x_n \) is a total order on \( P_0 \) and \( x'_0, \ldots, x'_n \) is a total order on \( P_0' \). If \( C \) is the matrix of \( f_1 \), then for simplicity write \( c_{ij} \) for \( c_{x_i\{x_j} \). For any integer \( m \) with \( 1 \leq m \leq n \) let \( P(m) \) be the subposet of \( P \) given by \( P(m) = \{x_0, \ldots, x_m\} \), and let \( P'(m) \) be the subposet of \( P' \) given by \( P'(m) = \{x'_0, \ldots, x'_m\} \). Let \( f_1^{(m)} : A_1(P(m)) \to A_1(P'(m)) \) be the \( R \)-linear map satisfying

\[
{f}_1^{(m)}[x_i] = \left( 1 - \sum_{j=1}^{m} c_{ij} \right)[0] + \sum_{j=1}^{m} c_{ij}[x'_j]
\]

for \( 0 \leq i \leq m \). Then Proposition 2.6 shows that \( f_1^{(m)} \) extends to a homomorphism \( f_*^{(m)} : A_*(P(m)) \to A_*(P'(m)) \) of differential graded \( A_*(\emptyset) \)-algebras. We will say that the orderings \( x_0, \ldots, x_n \) of \( P_0 \) and \( x'_0, \ldots, x'_n \) of \( P_0' \) are \( f_* \)-compatible if \( f_*^{(m)} \) is an isomorphism such that \( x_m \) and \( x'_m \) are mutually \( f_*^{(m)} \)-supportive for \( 1 \leq m \leq n \). Note that this condition implies that \( x'_m \in \text{supp} \, f_1[x_m] \) for all \( m \).

**Proposition 3.7.** Assume that \( R \) is a field. Let \( P \) and \( P' \) be finite posets of height one, and let \( f_* : A_*(P) \to A_*(P') \) be an \( A_*(\emptyset) \)-isomorphism. Let \( 0 = x_0, x_1, \ldots, x_n \) be any ordering of \( P_0 \). Then there exists an \( f_* \)-compatible ordering \( x'_0, \ldots, x'_n \) of \( P_0' \).
PROOF. The proof proceeds by induction on \( n = |P| \). If \( n = 1 \), then \( P = \{x_1\} \). Let \( x'_0 = 0 \), and let \( x'_1 \) be the unique element of \( P' \). Because \( x_1 \) and \( x'_1 \) must be mutually \( f_* \)-supportive, the orderings \( x_0, x_1 \) and \( x'_0, x'_1 \) are \( f_* \)-compatible.

Now suppose that \( n > 1 \). Let \( x = x_n \in P \), and let \( x' \in P' \) be an element such that \( x \) and \( x' \) are mutually \( f_* \)-supportive. Let \( C \) be the matrix of \( f_1 \), and let \( D \) be the matrix of \( f_1^{-1} \) so that \( c_{x',x} \neq 0 \) and \( d_{x,x'} \neq 0 \). Set \( Q = P - \{x\} \) and \( Q' = P' - \{x'\} \), and let \( g_1 : A_1(Q) \to A_1(Q') \) be the \( R \)-linear map satisfying

\[
g_1[y] = (c_{y'x} + c_{x'y})[0] + \sum_{y'' \in Q'} c_{x'y''}[y']
\]

for all \( y \in Q_0 \). By Proposition 2.6 the map \( g_1 \) extends to an \( A_*(\emptyset) \)-homomorphism \( g_* : A_*(Q) \to A_*(Q') \), and we will show that \( g_* \) is an isomorphism.

Let \( B \) be the matrix of \( g_1 \), and let \( B_0 \) be the submatrix obtained by deleting the row and column corresponding to the basis element \( [0] \). Because \( g_1[0] = [0] \), expanding by minors along the column corresponding to \([0] \) shows that \( \det B = \det B_0 \). But \( B_0 \) is also the submatrix of \( C \) obtained by deleting the rows corresponding to \([0] \) and \([x'] \) and the columns corresponding to \([0] \) and \([x] \). Because \( D = C^{-1} \) and \( f_1[0] = [0] \), it follows that \( d_{x,x'} = \det B_0/\det C \). But \( d_{x,x'} \neq 0 \), so \( \det B = \det B_0 \neq 0 \). Hence \( g_* \) is an isomorphism.

It now follows by induction that there exists an ordering \( x'_0, \ldots, x'_{n-1} \) of \( Q_0' \) that is \( g_* \)-compatible with the ordering \( x_0, \ldots, x_{n-1} \) of \( Q_0 \). Set \( x'_n = x' \). Because \( g_* = f_*^{(n-1)} \), the orderings \( x_0, \ldots, x_n \) of \( P_0 \) and \( x'_0, \ldots, x'_n \) of \( P_0' \) are \( f_* \)-compatible. This completes the proof.

The next result is essentially a convenient restatement of Proposition 2.6(3).

**Lemma 3.8.** Suppose that \( P \) and \( P' \) are finite posets and \( f_* : A_*(P) \to A_*(P') \) is an \( A_*(\emptyset) \)-isomorphism. Let \( x, y \in P \) and \( x', y' \in P' \) be elements such that \( x' \in \text{supp } f_1[x] \) and \( y' \in \text{supp } f_1[y] \). If \( x' < y' \), then \( x < y \).

**Proof.** Let \( C \) be the matrix of \( f_1 \). Then \( c_{x',x} \neq 0 \) and \( c_{y',y} \neq 0 \), so \( c_{x',x}c_{y',y} \neq 0 \). If \( x' < y' \), then Proposition 2.6(3) implies that \( x < y \).

Suppose that \( P \) is a poset, \( S \) is a subset of \( P \), and \( y \in P \). We will write \( S < y \) if \( x < y \) for all \( x \in S \). Recall that \( P_{<y} \) denotes the subposet of \( P \) consisting of all elements \( x \) such that \( x < y \). Thus \( S < y \) if and only if \( S \subseteq P_{<y} \).

**Lemma 3.9.** Assume that \( P \) and \( P' \) are finite posets and \( f_* : A_*(P) \to A_*(P') \) is an isomorphism. Let \( H \subseteq P \) and \( H' \subseteq P' \) be the subposets consisting of all elements that are not of maximum height, and let \( h_* : A_*(H) \to A_*(H') \) be the isomorphism
obtained by restricting \( f_\bullet \) to \( A_\bullet(H) \). Suppose that there exist isomorphisms of posets
\[
\psi : H \rightarrow H' \quad \text{and} \quad \psi' : H' \rightarrow H
\]
and tall orders \( x_0, \ldots, x_m \) on \( H_0 \) and \( x'_0, \ldots, x'_m \) on \( H'_0 \) such that \( x_0, \ldots, x_m \) is \( h_\bullet \)-compatible with \( 0, \psi(x_1), \ldots, \psi(x_m) \) and \( x'_0, \ldots, x'_m \) is \( h^{-1}_\bullet \)-compatible with \( 0, \psi'(x'_1), \ldots, \psi'(x'_m) \). If \( S \subseteq H \), let \( e(S) \) denote the number of \( y \in P - H \) such that \( S = P \prec y \); if \( S' \subseteq H' \), let \( e'(S') \) denote the number of \( y' \in P' - H' \) such that \( S' = P' \prec y' \). Then \( e(S) = e'(\psi(S)) \) for all \( S \subseteq H \), and \( e'(S') = e(\psi'(S')) \) for all \( S' \subseteq H' \).

**PROOF.** If \( S \subseteq H \), let \( g(S) \) denote the number of elements \( y \in P - H \) such that \( S < y \); define \( g'(S') \) similarly for any \( S' \subseteq H' \).

Fix \( S \subseteq H \), and suppose that there is an element \( y' \in P' - H' \) such that \( \psi(S) < y' \). Let \( y \) be an element of \( P \) such that \( y' \in \text{supp } f_1[y] \). Then \( y \in P - H \) by Proposition 3.4. Let \( x \) be an element of \( S \), and let \( i \) be the index such that \( x = x_i \). Then \( x_i \) and \( \psi(x_i) \) are mutually \( h^{(i)} \)-supportive, and the definition of \( h^{(i)} \) shows that \( \psi(x_i) \in \text{supp } f_1[x_i] \). But \( \psi(x_i) < y' \), so Lemma 3.8 implies that \( x = x_i < y \) and hence \( S < y \). Because this holds for every \( y \) such that \( y' \in \text{supp } f_1[y] \), the element \( a \in A_1(P) \) such that \( f_1(a) = [y'] \) is an \( R \)-linear combination of an element of \( A_1(H) \) and elements \([y]\) such that \( S < y \). It follows that \( g(S) \geq g'(\psi(S)) \) for all \( S \subseteq H \). Similarly, \( g'(S') \geq g(\psi'(S')) \) for all \( S' \subseteq H' \). In particular, if \( S \subseteq H \), then \( g(S) \geq g'(\psi(S)) \geq g(\psi'(\psi(S))) \). By induction it follows that
\[
g(S) \geq g'(\psi(S)) \geq g((\psi'\psi)'(S))
\]
for all \( t \geq 1 \). But \( \psi'\psi : H \rightarrow H \) is a bijection, so it permutes the subsets of \( H \). Thus there is an integer \( t \geq 1 \) such that \( (\psi'\psi)'(S) = S \) for all \( S \subseteq H \), and \( g(S) = g'(\psi(S)) \) for all \( S \subseteq H \).

We now use induction on \(|H - S|\) to show that \( e(S) = e'(\psi(S)) \) for all \( S \subseteq H \). If \(|H - S| = 0 \), then \( S = H \) and \( \psi(S) = H' \). But \( e(H) = g(H) = g'(H') = e'(H') \), so the result holds in this case.

Now assume that \( S \subseteq H \) and \(|H - S| > 0 \). Let \( S_1, \ldots, S_r \) be all of the distinct subsets of \( H \) that contain \( S \) properly. Then \( \psi(S_1), \ldots, \psi(S_r) \) are all of the distinct subsets of \( H' \) that contain \( \psi(S) \) properly. By induction it follows that \( e(S_i) = e'(\psi(S_i)) \) for all \( i \), so
\[
e(S) = g(S) - \sum_{i=1}^r e(S_i) = g'(\psi(S)) - \sum_{i=1}^r e'(\psi(S_i)) = e'(\psi(S)).
\]
Similarly, \( e'(S') = e(\psi'(S')) \) for all \( S' \subseteq H' \), and this completes the proof.

**THEOREM 3.10.** Assume that \( R \) is a field. Let \( P \) and \( P' \) be finite posets, and let \( f_\bullet : A_\bullet(P) \rightarrow A_\bullet(P') \) be an isomorphism. Then there exist isomorphisms of posets
\( \phi : P \to P' \) and \( \phi' : P' \to P \) and tall orders \( x_0, \ldots, x_n \) on \( P_0 \) and \( x'_0, \ldots, x'_n \) on \( P'_0 \) such that \( x_0, \ldots, x_n \) is \( f_* \)-compatible with 0, \( \phi(x_1), \ldots, \phi(x_n) \) and \( x'_0, \ldots, x'_n \) is \( f_*^{-1} \)-compatible with 0, \( \phi'(x'_1), \ldots, \phi'(x'_n) \).

**Proof.** The proof proceeds by induction on \( h(P) \). First suppose that \( h(P) = 1 \). By Proposition 3.7 there are \( f_* \)-compatible orderings \( x_0, \ldots, x_n \) of \( P_0 \) and \( y'_0, \ldots, y'_n \) of \( P'_0 \). Define \( \phi : P \to P' \) by setting \( \phi(x_i) = y'_i \) for \( 1 \leq i \leq n \). Then \( \phi \) is an isomorphism of posets having the desired properties. The same argument applied to \( f_*^{-1} \) gives the isomorphism \( \phi' : P' \to P \).

Now suppose that \( h(P) > 1 \). Let \( H \subseteq P \) and \( H' \subseteq P' \) be the subposets consisting of all elements that are not of maximum height. Then \( h(H) = h(P) - 1 \), and \( f_* \) restricts to an isomorphism \( h_* : A_*(H) \to A_*(H') \). By induction there are isomorphisms of posets \( \psi : H \to H' \) and \( \psi' : H' \to H \) and tall orders \( x_0, \ldots, x_m \) on \( H_0 \) and \( x'_0, \ldots, x'_m \) on \( H'_0 \) such that \( x_0, \ldots, x_n \) is \( h_* \)-compatible with 0, \( \psi(x_1), \ldots, \psi(x_m) \) and \( x'_0, \ldots, x'_n \) is \( h_*^{-1} \)-compatible with 0, \( \psi'(x'_1), \ldots, \psi'(x'_m) \).

Write the power set \( \mathcal{P}(H) \) of \( H \) as \( \mathcal{P}(H) = \{ S_1, \ldots, S_{2^n} \} \), where the subsets \( S_1, \ldots, S_{2^n} \) are indexed so that \( |S_1| \leq \cdots \leq |S_{2^n}| \). For \( 1 \leq i \leq 2^m \) set

\[
T_i = \{ y \in P - H \mid S_i = P_{<y} \} \quad \text{and} \quad T'_i = \{ y' \in P' - H' \mid \psi(S_i) = P'_{<y'} \}.
\]

Then \( P - H \) is the disjoint union of \( T_1, \ldots, T_{2^n} \), and \( P' - H' \) is the disjoint union of \( T'_1, \ldots, T'_{2^n} \). Moreover, \( |T_i| = |T'_i| \) for all \( i \) by Lemma 3.9.

Choose an ordering \( x_{m+1}, \ldots, x_n \) on \( P - H \) such that if \( x_i \in T_i \), \( x_i \in T'_j \), and \( i < j \), then \( s < t \). Similarly, choose an ordering \( y'_{m+1}, \ldots, y'_n \) on \( P' - H' \) such that if \( y'_i \in T'_i \), \( y'_i \in T'_j \), and \( i < j \), then \( s < t \). Let \( C \) denote the matrix of \( f_* \), and assume that \( C \) is written with respect to the ordered bases \([x_0], \ldots, [x_n] \) of \( A_1(P) \) and \([0], [\psi(x_1)], \ldots, [\psi(x_m)] \) of \( A_1(P') \). Then \( C \) is a block upper triangular matrix: the first diagonal block \( C_1 \) has columns indexed by \([x_0], \ldots, [x_n] \) and rows indexed by \([0], [\psi(x_1)], \ldots, [\psi(x_m)] \); the other diagonal block \( C_2 \) has columns indexed by \([x_{m+1}], \ldots, [x_n] \) and rows indexed by \([y'_{m+1}], \ldots, [y'_n] \). In particular, \( \det C = (\det C_1)(\det C_2) \).

Suppose that \( y' \in T'_i \) and \( y \in T_j \) are elements with \( c_{y'y} \neq 0 \). If \( x \in S_i \), then \( \psi(x) < y' \). Because \( x_0, \ldots, x_m \) is \( h_* \)-compatible with 0, \( \psi(x_1), \ldots, \psi(x_m) \), it follows that \( \psi(x) \in \operatorname{supp} h_1[x] = \operatorname{supp} f_1[x] \) and hence \( x < y \) by Lemma 3.8. Then \( S_i < y \) so that \( S_i \subseteq P_{<y} = S_j \). Hence \( i < j \), and the submatrix \( C_2 \) is itself block upper triangular: the \( i \)th diagonal block of \( C_2 \) has columns indexed by elements in \( T_i \) and rows indexed by elements in \( T'_j \).

Let \( x \in P_0 \) and \( x' \in P' \). If \( x \in H_0 \), set \( \tilde{c}_{xx} = c_{xx} \); if \( x \in T_i \) and \( x' \in T'_i \), set \( \tilde{c}_{xx} = c_{xx} \); and if \( x \in T_i \) and \( x' \in P' - T'_i \), set \( \tilde{c}_{xx} = 0 \). Finally, set

\[
\tilde{c}_{00} = 1 - \sum_{x' \in P'} \tilde{c}_{xx}.
\]
for all \( x \in P_0 \). By Proposition 2.6 the matrix \( \tilde{C} = (\tilde{c}_{x,x}) \) determines a homomorphism \( \tilde{f}_* : A_\bullet(P) \to A_\bullet(P') \). Because \( \tilde{C} \) is a block upper triangular matrix with the same diagonal blocks as \( C \), it follows that \( \det \tilde{C} = \det C \neq 0 \). Thus \( \tilde{f}_* \) is an isomorphism. Moreover, \( \tilde{f}_* \) restricts to an isomorphism \( \tilde{f}_* : A_\bullet(T_i) \to A_\bullet(T'_i) \) for all \( i \). Let \( 0 = t_{i0}, t_{i1}, \ldots, t_{im_i} \) be the ordering on \( (T_i)_0 \) obtained by regarding \( T_i \) as a subset of the ordered set \( P - H = \{ x_{m+1}, \ldots, x_n \} \). By Proposition 3.7 there is an \( \tilde{f}_* \)-compatible ordering \( t'_{i0}, \ldots, t'_{im_i} \) of \( (T'_i)_0 \). Then the function \( \psi_i : T_i \to T'_i \) given by \( \psi_i(t_{ij}) = t'_{ij} \) for \( 1 \leq j \leq m_i \) is a bijection.

Because \( P - H \) is the disjoint union of \( T_1, \ldots, T_{2^m} \), it is possible to define a function \( \phi : P \to P' \) by setting

\[
\phi(x) = \begin{cases} 
\psi(x) & \text{if } x \in H \\
\psi_i(x) & \text{if } x \in T_i 
\end{cases}
\]

and it is clear that \( \phi \) is a bijection. Suppose that \( x < y \) in \( P \). If \( x, y \in H \), then \( \phi(x) < \phi(y) \) because \( \psi \) is an isomorphism of posets. If \( x \) and \( y \) are not both in \( H \), then \( x \in S_i \) and \( y \in T_i \) for some \( i \). Then \( \phi(y) = \psi_i(y) \in T'_i \), so \( \psi(S_i) < \phi(y) \). But \( \phi(x) = \psi(x) \in \psi_i(S_i) \), so \( \phi(x) < \phi(y) \). Hence \( \phi \) is an isomorphism of posets.

Finally, the ordering \( x_0, \ldots, x_m \) of \( H_0 \) is \( h_\bullet \)-compatible with \( 0, \phi(x_1), \ldots, \phi(x_m) \), and for each \( i \) the orderings \( t_{i0}, \ldots, t_{im_i} \) of \( (T_i)_0 \) and \( 0, \phi(t_{i1}), \ldots, \phi(t_{im_i}) \) of \( (T'_i)_0 \) are \( \tilde{f}_* \)-compatible. It follows that the ordering \( x_0, \ldots, x_n \) of \( P_0 \) is \( f_\bullet \)-compatible with the ordering \( 0, \phi(x_1), \ldots, \phi(x_n) \) of \( P'_0 \).

The same argument shows that there exist an isomorphism of posets \( \phi' : P' \to P \) and a tall order \( x'_0, \ldots, x'_n \) on \( P'_0 \) that is \( f_\bullet^{-1} \)-compatible with the ordering \( 0, \phi'(x'_1), \ldots, \phi'(x'_n) \), and this completes the proof.

**Corollary 3.11.** If \( P \) and \( P' \) are finite partially ordered sets such that \( A_\bullet(P) \cong A_\bullet(P') \), then \( P \cong P' \).

**Proof.** By working over the quotient field of \( R \), we may assume that \( R \) is itself a field. Then the result follows immediately from Theorem 3.10.

### 4. Annihilators and the graded center

The purpose of this section is to give a description of the graded center of \( A_\bullet(P) \) in terms of the elements that annihilate all homogeneous elements of positive degree in \( A_\bullet(P) \). Recall that the graded center \( Z_\bullet(P) \) is defined to be the \( R \)-submodule generated by all homogeneous elements \( z \in A_\bullet(P) \) such that \( az = (-1)^{\deg a \cdot \deg \ z} za \).
for all homogeneous elements \( a \in A_\bullet (P) \). Note that if \( z \in Z_m(P) \) and \( a \in A_n(P) \) are any two homogeneous elements, then
\[
(da)z + (-1)^{n}a(dz) = d(az)
\]
\[
= (-1)^{m}d(za)
\]
\[
= (-1)^{m}d(z)a + (-1)^{m(n-1)}z(da)
\]
\[
= (-1)^{m}d(z)a + (da)z.
\]
Hence \( a(dz) = (-1)^{m-1}d(z)a \), and it follows that \( dz \in Z_{m-1}(P) \). Thus \( Z_\bullet (P) \) is a differential graded \( A_\bullet (\emptyset) \)-subalgebra of \( A_\bullet (P) \).

If \( S \) is any subset of \( A_\bullet (P) \), then \( \text{Ann } S \) will denote the ideal consisting of all two-sided annihilators of \( S \); in other words,
\[
\text{Ann } S = \{ x \in A_\bullet (P) \mid xs = sx = 0 \text{ for all } s \in S \}.
\]
Let \( A_+ (P) \) denote the ideal of \( A_\bullet (P) \) generated by all homogeneous elements of positive degree. Then the annihilator \( \text{Ann } A_+ (P) = \text{Ann } A_1(P) \) is a homogeneous ideal of \( A_\bullet (P) \). Let \( I_\bullet (P) \) denote the differential graded ideal generated by \( \text{Ann } A_+ (P) \). The first result of this section gives an explicit description of \( \text{Ann } A_+ (P) \).

**Proposition 4.1.** Let \( P \) be a finite non-empty poset. Then \( \text{Ann } A_+ (P) \) is the span of all elements of the form \([0 < m < \cdots < M]\), where \( m \) is minimal and \( M \) is maximal in \( P \). In particular, if \( P \) contains no connected components of height one, then \( I_1(P) = 0 \).

**Proof.** If \( m \) is minimal and \( M \) is maximal in \( P \), then the definition of the multiplication in \( A_\bullet (P) \) shows that \([0 < m < \cdots < M] \in \text{Ann } A_+ (P) \). Conversely, suppose that \( x = \sum_{i=1}^{s} c_i [x_{0i} < \cdots < x_{ni}] \) is a homogeneous element of \( \text{Ann } A_+ (P) \) with \( c_i \neq 0 \) for \( 1 \leq i \leq s \). Because \([0]x = 0 \), it follows that \( x_{0i} = 0 \) for all \( i \). If \( n = 0 \), then it is easy to see that \( P \) is empty, so we may assume that \( n > 0 \). Let \( m \) be a minimal element of \( P \). Then
\[
0 = [m]x = \sum_{i=1}^{s} c_i [0 < m < x_{1i} < \cdots < x_{ni}],
\]
and it follows that \( m \neq x_{1i} \) for all \( i \). Because this relation holds for every minimal element \( m \) of \( P \), we conclude that \( x_{1i} \) is minimal for all \( i \). Similarly, if \( M \) is a maximal element of \( P \), then the fact that \( 0 = x[M] \) implies that \( x_{ni} \) is maximal for all \( i \). This proves the first statement, and the second follows easily.

**Proposition 4.2.** Let \( P \) be a finite non-empty poset. If \( a \) and \( b \) are homogeneous elements of \( I_\bullet (P) \), then \( ab = 0 \).
PROOF. Because $a, b \in I_\ast(P)$, it is possible to write $a = a' + da''$ and $b = b' + db''$ for some homogeneous elements $a', a'', b', b'' \in \text{Ann } A_+(P) \subseteq A_+(P)$. Then
\[ ab = (a' + da'')(b' + db'') = (da'')(db'') = d(a''(db'')) = 0, \]
as desired.

PROPOSITION 4.3. Let $P$ be a finite poset. Then $Z_\ast(P)$ is the differential graded $A_\ast(\emptyset)$-algebra generated by $\text{Ann } A_+(P)$. Moreover, if $P$ is non-empty, then $Z_\ast(P) = A_\ast(\emptyset) \oplus I_\ast(P)$ as graded $R$-modules.

PROOF. We begin by showing that $Z_\ast(P) = A_\ast(\emptyset) + I_\ast(P)$. It is clear that $A_\ast(\emptyset) + I_\ast(P) \subseteq Z_\ast(P)$, and we will prove that $Z_n(P) = A_n(\emptyset) + I_n(P)$ for all $n$ by downward induction on $n$. If $N$ is the largest degree such that $A_N(P) \neq 0$, then certainly $Z_n(P) = A_n(\emptyset) + I_n(P) = 0$ for all $n > N$, and $Z_N(P) = A_N(P) = A_N(\emptyset) + I_N(P)$.

Now suppose that $1 \leq n < N$ and that $Z_{n+1}(P) = A_{n+1}(\emptyset) + I_{n+1}(P)$. Let $x \in Z_n(P)$. Then $x = [0](dx) + d([0]x)$, and by induction $[0]x \in Z_{n+1}(P) = A_{n+1}(\emptyset) + I_{n+1}(P) = I_{n+1}(P)$. Hence $d([0]x) \in I_n(P)$, and it suffices to show that $[0](dx) \in A_n(\emptyset) + I_n(P)$. If $n = 1$, then $[0](dx)$ is a multiple of $[0]$, so it lies in $A_1(\emptyset)$. Thus we may assume that $2 \leq n < N$. Write $dx = \sum_{i=1}^{s} c_i[x_{1i} < \cdots < x_{n-1,i}]$, and let $y \in P_0$. Then
\[
\sum_{i=1}^{s} (-1)^{n-1} c_i[0 < x_{1i} < \cdots < x_{n-1,i} < y] = \sum_{i=1}^{s} c_i[x_{1i} < \cdots < x_{n-1,i}][0 < y] = (dx)[0][y] = [0][y](dx) \]
\[
= \sum_{i=1}^{s} c_i[0 < y][x_{1i} < \cdots < x_{n-1,i}].
\]
If any term in this last sum is non-zero, then it follows that $c_i[0 < y < x_{1j} < \cdots < x_{n-1,j}] \neq 0$ for some $j$ with $1 \leq j \leq s$. But such a term cannot occur in the sum $\sum_{i=1}^{s} (-1)^{n-1} c_i[0 < x_{1i} < \cdots < x_{n-1,i} < y]$ because $n \geq 2$. Thus $[y][0](dx) = (-1)^n[0](dx)[y] = -[0][y](dx) = 0$, and it follows that $[0](dx) \in A_n(P) \cap \text{Ann } A_1(P) \subseteq I_n(P)$. Hence $Z_n(P) = A_n(\emptyset) + I_n(P)$ for all $n \geq 1$. But $Z_0(P) = A_0(P) = A_0(\emptyset) + I_0(P)$, so $Z_\ast(P) = A_\ast(\emptyset) + I_\ast(P)$, as desired.

To show that the sum $A_\ast(\emptyset) + I_\ast(P)$ is direct when $P$ is non-empty, it suffices to show that $I_0(P) = 0$ and $R[0] \cap I_1(P) = 0$. Both of these facts follow easily from Proposition 4.1.

If $P$ is a finite non-empty poset, let $P^*$ denote the dual of $P$. By Proposition 4.1 there is an $R$-linear map $f_\ast : \text{Ann } A_+(P) \rightarrow \text{Ann } A_+(P^*)$ satisfying
\[ f_\ast[0 < m < \cdots < M] = [0 < M < \cdots < m], \]
and $f_*$ extends uniquely to an isomorphism of differential graded $A_*(\emptyset)$-algebras $f_* : Z_*(P) \to Z_*(P^*)$ by Proposition 4.3. Thus we obtain the following result.

**COROLLARY 4.4.** If $P$ is a finite poset, then $Z_*(P) \cong Z_*(P^*)$.

It often happens, however, that two posets $P$ and $Q$ satisfy $Z_*(P) \cong Z_*(Q)$ even when $Q \not\cong P$ and $Q \not\cong P^*$. Such an example is given by the following posets $P$ and $Q$:

![Diagram of posets P and Q]

Indeed, $\text{Ann } A_+(P)$ is given by the span of $\{[0 < a < b_i] \mid 1 \leq i \leq 4\}$, whereas $\text{Ann } A_+(Q)$ is given by the span of $\{[0 < u_i < v_j] \mid 1 \leq i, j \leq 2\}$. If $f$ is any bijection between these sets, then it is easy to see that $f$ extends uniquely to a differential graded $A_*(\emptyset)$-isomorphism between $Z_*(P)$ and $Z_*(Q)$.

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