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On G-matrices

Christos Koukouvinos

Jennifer Seberry

University of Wollongong, jennie@uow.edu.au

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1 Introduction

Let $X_1, X_2, X_3, X_4$ be four type $\pm 1$ matrices on the same group of order $n$ (odd) with the properties:

(i) $(X_i - I)^T = -(X_i - I), \quad i = 1, 2,$

(ii) $X_i^T = X_i, \quad i = 3, 4$ and the diagonal elements are positive,

(iii) $X_iX_j = X_jX_i,$

(iv) $X_1X_2^T + X_3X_4^T + X_2X_3^T + X_4X_1^T = 4nI_n.

Call such matrices G-matrices. These were first introduced and applied to construct Hadamard matrices by Jennifer Seberry Wallis [3]. G-matrices of orders 3, 5, 7, 9, 13, 15, 19 were known previously, see [3, 5, 6]. This note constructs G-matrices of order 21, 23, 25 and 27 for the first time.

Remark 1 Multiplying both sides of (iv) by $J$ shows G-matrices can only exist for orders $n$ for which

$$4n = 1^2 + 1^2 + a^2 + b^2$$

where $a, b$ are odd integers. So, for example, they cannot exist for the following orders $\leq 10$: 11, 17, 29, 35, 39, 41.

G-matrices which are constructed using four circulants exist for at least $n = 3, 5, 7, 9, 13, 15,$ and 19, see [3, 5, 6], and for $n = 21, 23, 25,$ and 27 which are constructed in this note. This means the first unresolved case is for $n = 27$. 

2 Construction of G-matrices

The following first rows may be used to give circulant matrices which can be used in the Goethals-Seidel array to find G-matrices of order:

\[ 4n = 4 \times 21 = 84 = 1^2 + 1^2 + 1^2 + 9^2, \]

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\[ 4n = 4 \times 23 = 84 = 1^2 + 1^2 + 3^2 + 9^2, \]

\[ \{+ + + - - - - - - + + + + + + - - - - - - + + + - \} \]
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\[ 4n = 4 \times 25 = 100 = 1^2 + 1^2 + 7^2 + 7^2, \]

\[ \{+ + + - - - - - - + + + + + + - - - - + + - - - - \} \]
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and \[ 4n = 4 \times 27 = 108 = 1^2 + 1^2 + 5^2 + 9^2, \]

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3 Constructions using G-matrices

First we note that we have

Lemma 1 If there exist circulant G-matrices of order \( n \) then there exists an OD\((4n; 1, 1, 2n - 1, 2n - 1)\).

Corollary 1 Let \( n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25, \) or 27. Then an OD\((4n; 1, 1, 2n - 1, 2n - 1)\) exists.

We recall the following definitions and results from [3]. The following theorem shows how the Williamson construction (the \( D_i \)) and the Goethals-Seidel construction (the \( A_i \)) may be combined to construct Hadamard matrices.
Theorem 1 (Jennifer Seberry Wallis (1975)) Suppose $A_i$ and $B_i$, $i = 1, 2, 3, 4$ are type (±1) matrices of order $a$ and $b$, respectively, which satisfy

(i) $A_iA_j = A_jA_i$, $i, j = 1, 2, 3, 4$
(ii) $B_iB_j^T = B_jB_i^T$, $i, j = 1, 2, 3, 4$
(iii) $\sum_{i=1}^{4}(A_i \times B_i)(A_i \times B_i)^T = 4abI_{ab}$

then $H$ defined as

\[H = \begin{pmatrix} A_1 \times B_1 & A_2 \times B_2 & A_3 \times B_3 & A_4 \times B_4 \\ -A_2 \times B_2 & A_1 \times B_1 & A_4 \times B_3 & -A_3 \times B_3 \\ -A_3 \times B_3 & A_2 \times B_2 & A_1 \times B_4 & -A_4 \times B_4 \\ -A_4 \times B_4 & A_3 \times B_3 & A_2 \times B_1 & A_1 \times B_1 \end{pmatrix}\]

is an Hadamard matrix of order $4ab$.

We will call the matrices $A_i \times B_i$, $i = 1, 2, 3, 4$ of the theorem $F$-matrices and we will say $H$ is a Wallis-Whiteman like Hadamard matrix.

The $A_i$ will be called the $GS$-part and the $B_i$ the $W$-part of the $F$-matrix.

The following theorem shows how $G$-matrices may be used to construct $F$-matrices.

Theorem 2 (Jennifer Seberry Wallis (1975)) Let $X_1$, $X_2$, $X_3$, $X_4$ be $G$-matrices of order $a$. Suppose $A, B, C$ are suitable ±1 matrices of order $m$ for an odd $4n$; 1, 4n − 2) so they satisfy

(i) $A^T A$, $B^T B$, $C^T C$ are symmetric,
(ii) $A^T A + B^T B + (4n - 2)CC^T = 4nmI_m$.

Then

\[A_1 = I \times A + (X_1 - I) \times C\]
\[A_2 = I \times B + (X_2 - I) \times C\]
\[A_3 = X_3 \times C\]
\[A_4 = X_4 \times C\]

are $F$-matrices of order $mn$.

Corollary 2 Let $n = 3, 5, 7, 9, 13, 15, 19, 21, 23$ or 27. Suppose $A, B, C$ are pairwise amicable (±1) matrices of order $m$ satisfying

\[A^T A + B^T B + (4n - 2)CC^T = 4nmI_m,\]

Then there are $F$-matrices of order $mn$ and a Wallis-Whiteman like Hadamard matrix of order $4mn$.

Corollary 3 Let $n = 3, 5, 7, 9, 13, 15, 21, 23$. Set $A = A_{2n+1}$, $B = (J - I)_{2n+1}$ and $C$ the back-circulant or type 1 matrix of order $2n + 1$ obtained from the quadratic residues. Then

\[A^T A + B^T B + (4n - 2)CC^T = 4(2n+1)nI_{2n+1},\]

and hence there are $F$-matrices of order $(2n + 1)n$ and a Wallis-Whiteman like Hadamard matrix of order $4(2n + 1)n$. 

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We further extend the last theorem by observing

**Theorem 3** Let $X_1, X_2, X_3, X_4$ be $G$-matrices of order $n$. Suppose $A, B, C, D$ are suitable $\pm 1$ matrices of order $m$ for an $OD(4n; 1, 1, 2n - 1, 2n - 1)$ so they satisfy

(i) $PQ^T$ is symmetric for all $P, Q \in \{A, B, C, D\}$,

(ii) $AA^T + BB^T + (2n - 1)CC^T + (2n - 1)DD^T = 4mmnM_m$.

Then defining $Y_1 = (X_1 + X_2)/2$, $Y_2 = (X_1 - X_2)/2$, $Y_3 = (X_3 + X_4)/2$ and $Y_4 = (X_3 - X_4)/2$

$B_1 = I \times A + Y_1 \times C + Y_2 \times D$

$B_2 = I \times B + Y_3 \times D + Y_4 \times C$

$B_3 = Y_2 \times C + Y_4 \times D$

$B_4 = Y_3 \times D + Y_4 \times C$

are $F$-matrices of order $mn$.

**Corollary 4** Let $n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25$ or $27$. Suppose $A, B, C, D$ are pairwise amicable ($\pm 1$) matrices of order $n$ satisfying

$$AA^T + BB^T + (2n - 1)CC^T + (2n - 1)DD^T = 4mmnM_m.$$ 

Then there are $F$-matrices of order $mn$ and a Wallis-Whitman like Hadamard matrix of order $4mn$.

**Corollary 5** Let $n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25$ or $27$. Set $A = B = J_{2n-1}$, and $C - I = D = J_{2n-1}$ the back-circulant or type 1 matrix of order $2n - 1$ with zero diagonal obtained from the quadratic residues or the core of a symmetric conference matrix of order $2n + 2$. Then

$$AA^T + BB^T + (2n - 1)CC^T + (2n - 1)DD^T = 4(2n - 1)nM_{2n-1}.$$ 

and hence there are $F$-matrices of order $(2n - 1)n$ and a Wallis-Whitman like Hadamard matrix of order $4(2n - 1)n$.

**References**


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