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The Predominance of Procedural Knowledge in Fractions

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Teachers play a crucial role in the mathematical learning outcomes of their students. The quality of teachers' mathematical knowledge has been of interest to key stakeholders and several lines of inquiry have been running in an effort to better understand the kinds of knowledge that mathematics teachers need to acquire and use to drive their lessons. Despite a decade of research in this area, the interconnections amongst the various strands of knowledge required by mathematics teachers is still unclear. In this report we attempt to investigate this issue by focusing on procedural and conceptual knowledge utilised in the assessment responses of a cohort of prospective teachers.

Background

Two decades ago Shulman (1986), in examining the kinds of knowledge that are essential to teachers' work, identified two categories of knowledge that were deemed to be necessary for effective practice: subject-matter knowledge and pedagogical content knowledge. This seminal work has spawned a number of studies in various domains including mathematics. Ball, Hill and Bass (2005), in taking up this issue, developed a number of new strands in this knowledge cluster, including the now well-established dimensions of Content Knowledge (CK) and Pedagogical Content Knowledge (PCK). There is an emerging consensus that effective mathematics classroom practices need to be anchored by a robust body of CK and PCK. CK refers to knowledge of the concepts, principles, procedures and conventions of mathematics, and PCK indicates the translation of CK into understandings to which learners could relate. While the connections between CK and PCK have received substantial attention, the two major components of CK, namely concepts and procedures, have not been examined with sufficient rigour, particularly in the context of specific strands of primary mathematics. This is a major aim of this study.

The importance of conceptual and procedural knowledge in mathematical understanding and subsequent performance continues to be a parallel issue for mathematics teachers, the research community and other stakeholders (Council of Australian Government (2008). Skemp (1976) took the early steps in highlighting the relative roles of the two in characterising high levels of mathematical performance by focussing on instrumental and relational understanding. He argued that the former was driven by procedural knowledge, the latter by conceptual knowledge. More recently, Chinnappan and Chandler (in press) elucidated the role of mathematics teachers' conceptual knowledge in reducing information processing loads that could be associated with problem solving. In a similar vein, the decoding of structures underlying complex mathematics concepts, which is necessary for deep mathematical understanding, was deemed to be buttressed by robust conceptual knowledge, both by teachers and learners (Mason, Stephens & Watson, 2009).

Broadly speaking, procedural knowledge involves understanding the rules and routines of mathematics while conceptual knowledge involves an understanding of mathematical relationships. The relationship between procedural and conceptual knowledge, and the dependency of one on the other, continues to be a legitimate concern for mathematics teachers and researchers alike (Schneider & Stern, 2010; Schoenfield, 1985). This

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knowledge is not static and there is a need to examine the trajectory of this knowledge, throughout pre-service education and beyond, if we are to better understand teachers' practices and their professional needs (Ball et al., 2005).

The research reported here is the first of two phases. In Phase 1 (reported here), we attempt to describe the quality of knowledge of fractions that was evident in a cohort of pre-service (PS) teachers in their first year of teacher education and prior to commencement of their professional experience. In Phase 2 of the study the PS teachers will respond to the similar fraction tasks in the third year of their pre-service course, following further studies and professional experience. Results of Phase 2 are expected to provide insights into changes in both the quality and quantity of PS teachers' CK and PCK.

Conceptual Framework

Our research questions and the interpretation of relevance were guided by a model of mathematical understanding, developed by Bambry, Harries, Higgins, and Suggate (2009), in which representations played a central role. Their representational model of understanding emphasises two facets: connections between internal representations of a concept and the articulation of links among the representations via robust reasoning processes. For example, the concept of multiplications of whole numbers can be represented as a) repeated addition, b) rows and columns in a rectangular array and c) operations in a lattice algorithm. All three constitute defensible representations of the focus concept (multiplication). However, the first two representations are conceptually rich while the last one can be explained purely from a procedural angle. Our decision to use this model was based on our desire to better understand the depth of student teachers' conceptual and procedural understanding.

Purpose and Research Questions

The purpose of the study is to examine the quality of PS teachers' representations of fraction concepts in terms of their demonstration of procedural and conceptual knowledge. This will be addressed by generating data relevant to the following research questions:

1. What is the relative use of procedural and conceptual knowledge when PS teachers represent fraction problems that involve subtraction?
2. What is the relative use of procedural and conceptual knowledge when PS teachers represent fraction problems that involve multiplication?
3. Is there a relationship in the use of procedural/conceptual knowledge by PS teachers when they attempt to represent subtraction and multiplication fraction problems?
4. How robust are the above representations?
5. What are the categories of error committed by PS teachers in their representations?

Methodology

Participants

One hundred and eighty-six students (22 males and 164 females) completed two questions (following) as part of the final assessment task for a first year mathematics content and pedagogy unit. The unit is compulsory and generally completed in the second semester of a four-year Bachelor of Primary Education degree.

Tasks and Procedure

The following tasks were two parts of one question in a fifteen question examination. They were selected from a pool of thirty-two questions given to students in their unit outlines at the beginning of the semester. The pool of questions was designed to examine both content and pedagogical knowledge. These particular tasks were chosen to assess students' conceptual and/or procedural knowledge of fractions and fraction algorithms. While the values of the fractions were different from those given in the unit outline, students had been able to engage with similar questions throughout the session to consolidate their procedural and conceptual understandings.

Task 1: Subtraction Problem involving a mixed number and fractions with different denominators $1\frac{2}{5} - \frac{5}{6}$

There are several separate procedures involved in solving this problem: changing the mixed number to an improper fraction; identifying the lowest common denominator of the minuend and subtrahend; changing the minuend and subtrahend to equivalent fractions; performing the subtraction; checking if the answer can be simplified

Conceptual knowledge of this problem involves an understanding that: the minuend and subtrahend are related to the same size "whole"; $1\frac{2}{5}$ is the same as $\frac{7}{5}$ because one whole is the same as $\frac{5}{5}$; equivalent fractions are the same size; subtraction of the subtrahend involves removing 25 lots of $\frac{1}{6}$.

Task 2: Multiplication Problem $\frac{1}{4} \times \frac{2}{3}$

Procedurally the solution involves: multiplying the numerators together; multiplying the denominators together; simplifying the answer.

A conceptual understanding of this task involves the notion that $\frac{1}{4} \times \frac{2}{3}$ involves finding $\frac{1}{4}$ *of* $\frac{2}{3}$ or $\frac{2}{3}$ *of* $\frac{1}{4}$. This entails more than repeating the idea that $\frac{1}{4} \times \frac{2}{3}$ can be expressed as $\frac{1}{4}$ *of* $\frac{2}{3}$, or what Skemp (1976) described as "rules without reasons" (p. 20). Conceptual understanding involves partitioning $\frac{2}{3}$ into four equal parts to find $\frac{1}{4}$ *of* $\frac{2}{3}$ or cutting $\frac{1}{4}$ into three equal parts to find $\frac{2}{3}$ *of* $\frac{1}{4}$.

In undertaking the tasks, students were expected to show all steps, including any visual representations that could be used to demonstrate their thinking. To complete the calculations students could use an algorithm for carrying out a particular operation with fractions. The successful use of an appropriate algorithm would indicate that students have a procedural understanding, what Skemp (1976) defined as *instrumental knowledge* (knowing a rule and being able to use it). Conceptual or relational understanding involves knowing what to do and why (Skemp, 1976). In this case it involves a comprehension of the nature of fractions (equal parts of a whole object or group) including the meaning of the common fraction symbols (as opposed to the misconception common among children that the numerator and denominator are simply two whole numbers) (NSW Department of Education and Training, 2003). Additionally, a conceptual understanding of these tasks involves grasping what happens when multiplying and subtracting fractions.

Coding Scheme

Students' responses to each of the two problems were analysed in terms of the evidence of conceptual and procedural knowledge and coded as per the scheme below. In developing the scheme we were guided by the framework of Bambry et al. (2009) and analysis of problem representation (Goldin, 2008).

4 – Conceptual with explicit reasoning. Correct algorithm, model supported with language

3 – Conceptual, no evidence of reasoning. Correct algorithm, model of concept evident
2 – Procedural/conceptual. Correct algorithm and model demonstrates more than a procedural understanding of a concept involved in fraction operations
1 – Procedural. Correct algorithm and/or model only of fractions
0 – No evidence of procedural or conceptual understanding

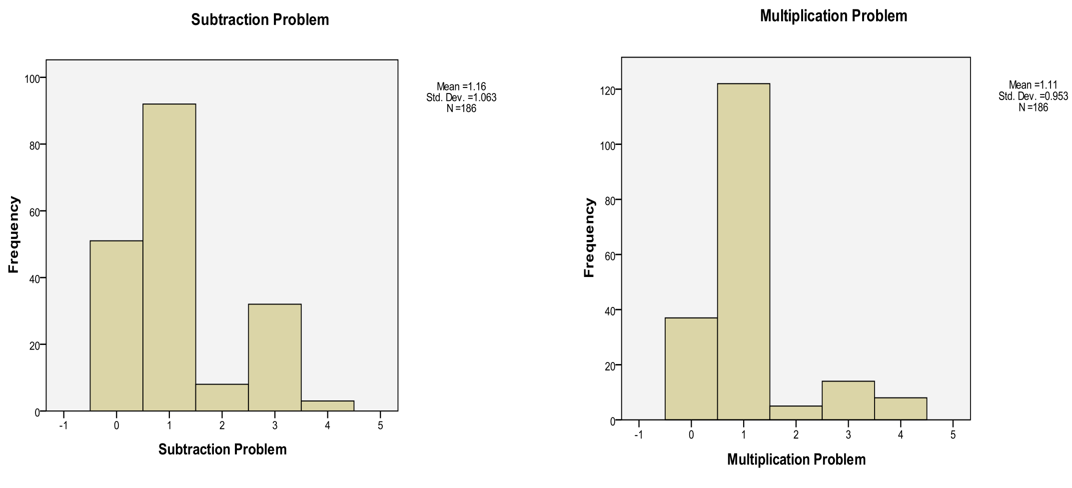
Data and Analysis

Quantitative data analyses were conducted with the aid of SPSS version 18. Our analyses focused on the above five categories of problem representation; the scale of our data was nominal.

We set out to examine the relative role of procedural and conceptual knowledge used by PS teachers as they attempted to represent two problems in the area of fractions. The general issue of PS teachers' proclivity to draw on different proportions of these knowledge components were examined in terms of five research questions. We present the data relevant to each question below.

Research Question 1 - What is the relative use of procedural and conceptual knowledge when pre-service teachers represent fraction problems that involve subtraction?

Figure 1a shows the results of the analysis of frequency of the two knowledge categories for the subtraction problem. Almost double the number of PS teachers activated procedural knowledge in comparison to those that displayed conceptual knowledge. The relatively low instances of scores of 4 indicate a tendency not to elucidate this conceptual knowledge. We also note that a high proportion of responses demonstrate neither procedural nor conceptual knowledge (scores of 0).



Figures 1a & 1b. Frequency of responses for Subtraction Problem and Multiplication Problems

Research Question 2 - What is the relative use of procedural and conceptual knowledge when pre-service teachers represent fraction problems that involve multiplication?

Almost four times the number of PS teachers activated procedural knowledge components in comparison to those that demonstrated conceptual knowledge in their solution attempts of the multiplication problems (Figure 1b). About one fifth of responses displayed neither procedural nor conceptual knowledge (scores of 0).

Research Question 3 - Is there a relationship in the use of procedural/conceptual knowledge by pre-service teachers when they attempt to represent subtraction and multiplication fraction problems?

In order to answer Research Question 3, we computed cross tabulations for the categories of scores for the subtraction and multiplication problems. The results of this analysis are presented visually in Figure 2.

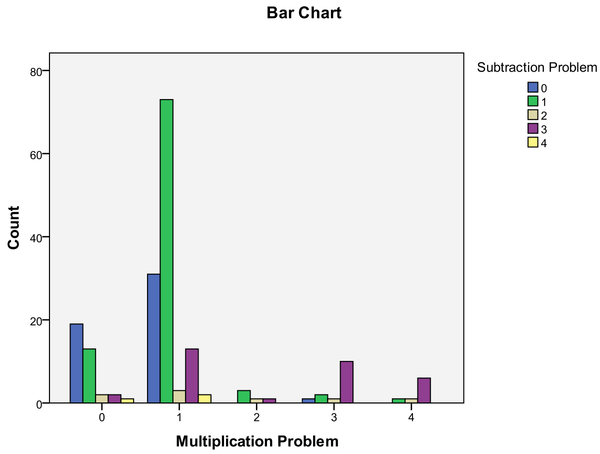
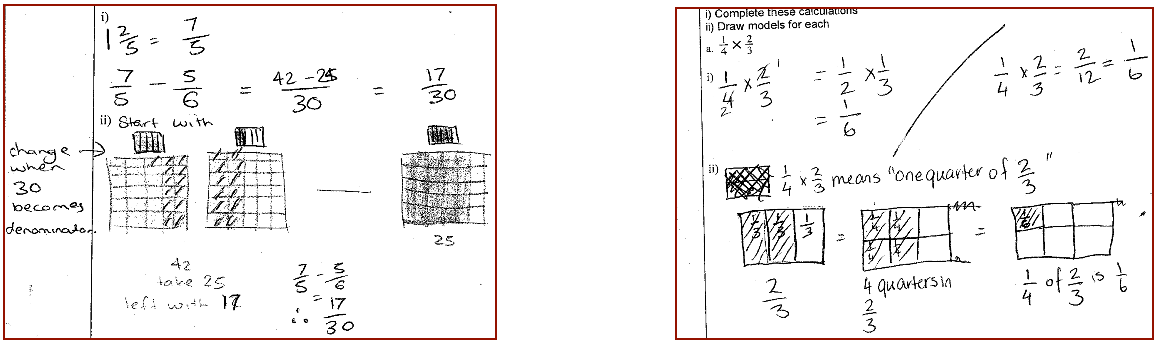


Figure 2: Clustered bar chart within Multiplication Problem

This suggests that there was a similar pattern in PS teachers' use of procedural and conceptual knowledge in both problems. A two-way contingency table analysis was conducted to evaluate whether representation categories in the subtraction problem were associated with those for the multiplication problem. The two variables were the subtraction problem with five levels of representation and the multiplication problem with five levels of representation. The quality of representations in both the problems was found to be significantly related, χ^2 (16, N=186) = 75.26, p = .00, Cramer's V = 0.32.

Research Question 4 - How robust are the above representations?

Our examination of the robustness of representations was informed by the model of Bambry et al. (2009), in that we searched for evidence not only of the construction of powerful representations but also the PS teachers' abilities to reason about the connectivity among these representations. Figures 3a and 3b provide episodes of robust representations for the subtraction and multiplication problems respectively. In Figure 3a, the student teacher determined an answer using a procedure and converts the problem into visual form, clearly demonstrating equivalence, through the drawing and explanation, before subtracting the second fraction from the first one. In Figure 3b, the PS teacher demonstrated two related procedures for completing the calculation and clearly translated the problem from a symbolic representation to a more meaningful form ("one quarter of"), which is subsequently re-represented in visual form.



Figures 3a & 3b. Examples of robust representations for each task

Research Question 5 - What are the categories of error committed by pre-service teachers in their representations?

We identified nine patterns in the type of errors committed by the PS teachers, five in the multiplication and four in the subtraction question. Here are examples of the two most common error types for each task:

Task 1: Subtraction $1\frac{2}{5}-\frac{5}{6}$

Fifty-one of the one hundred and eighty-six students who completed this examination received a rating of 0 for procedural and conceptual understanding of the subtraction question.

Error A: $\frac{7}{5}-\frac{5}{6} \rightarrow \frac{7}{30}-\frac{5}{30}=\frac{2}{30}-\frac{1}{15}$; subtracting numerators and multiplying denominators. Ten students did not change the minuend and subtrahend to equivalent fractions but subtracted the numerators and multiplied the denominators.

Error B: e.g., $\frac{7\times 6}{5\times 6}-\frac{5\times 6}{6\times 5}=\frac{42}{30}-\frac{30}{30}=\frac{12}{30}$; an error in making equivalent fractions. Thirteen students made an error in changing the minuend and subtrahend into equivalent fractions. There was a range of different mistakes within this group.

Task 2: Multiplication $\frac{1}{4}\times\frac{2}{3}$

Thirty-eight of the one hundred and eighty-six students who completed the examination received a 0 rating in this question, indicating an absence of both procedural and conceptual understanding.

Error 1: $\frac{1}{4}\times\frac{2}{3}=\frac{3}{12}$; adding numerators and multiplying denominators. Eleven students made this error, all getting an answer of $\frac{3}{12}$ with nine simplifying the answer to $\frac{1}{4}$. It was not possible to determine if the error was in adding, rather than multiplying the numerators, was unintentional, or was a misunderstanding of the correct procedure to multiply fractions.

Error 2: $\frac{1}{4}\times\frac{2}{3}=\frac{3}{12}\times\frac{8}{12}=\frac{24}{12}=2$; making equivalent fractions and multiplying the numerators. Eight students found equivalent fractions for the multiplicand and multiplier, and then multiplied the numerators together but not the denominators (a procedure which could be a transfer of the knowledge regarding the addition or subtraction fractions with different denominators).

Eighteen of the nineteen students making these two errors produced a model but only modelled the multiplicand, multiplier and/or the product but not the concept of finding $\frac{1}{4}$ *of* $\frac{2}{3}$.

The dominance of procedural over conceptual knowledge, observed in our analyses of the representations, was also evident in the type of errors made. All of the nine common error groups were related to common algorithmic procedures being applied incorrectly because they were not understood at a conceptual level, which was further evident in the PS teachers' models. When comparing responses to the multiplication and subtraction problems, more PS teachers utilised conceptual knowledge in the subtraction problem than the multiplication task. Interesting, more PS teachers also demonstrated no procedural or conceptual knowledge (receiving a rating of 0), in completing the subtraction problem. This may to be due to the multiple procedural steps needed to complete the subtraction task successfully, which allows more scope for error when approached without conceptual understanding. Additionally, none of the PS teachers made procedural or calculation errors while simultaneously demonstrating conceptual understanding in their models.

Discussions and Implications

Our main interest in the present study was to distinguish between the conceptual and procedural knowledge of a cohort of PS teachers in the context of two fraction problems. These research questions were set against the need to describe the quality of our prospective teachers' CK (Ball et al., 2005) early in their teacher education courses and with regard to their professional experience.

A close look into the quality of representations in these tasks indicates that some of the PS teachers who participated in the present study have developed a robust body of conceptual and procedural knowledge of fractions and operations involving fractions. Results indicate that these PS teachers' CK of fractions, in the context of both of these problems, was primarily procedural in nature. This was more pronounced in the multiplication problem. A possible explanation for this may be because this task was somewhat denser conceptually than the subtraction problem and thus provided a richer problem context for the PS teachers to reveal flexibility in their activation and use of conceptual knowledge. The high number of errors in both tasks also indicates that these PS teachers rely heavily on procedural knowledge which, when not supported by conceptual understanding, is difficult to utilise without error and difficult to review for possible mistakes.

From a cognitive perspective, there exists an underlying structure in each of the problems. The unpacking of this structure, we contend, calls for a greater proportion and better use of conceptual knowledge. For instance, the multiplication problem involved fractions as the multiplicand and multiplier. The multiplicative structure of this type of problem is not congruent with the structure of problems involving whole numbers and requires an understanding of the partitioning nature of the multiplication of fractions (Mack, 2001).

The cross-tabulation analyses indicated that PS teachers' patterns of activating procedural or conceptual knowledge in the subtraction and multiplication tasks were not independent. That is, regardless of the problem type, the CK of the PS teachers was mainly procedural in nature. While both strands of knowledge are necessary for teaching, the predominance of the procedural over conceptual, we suggest, is not a healthy situation. Teachers who develop CK that is predominantly procedural cannot be expected to help children develop rich conceptual connections that are necessary for modelling of problems (Lesh & Zawojewski, 2007).

Limitations

The cohort had a wide range of mathematical backgrounds including some who had not attempted mathematics at the HSC level to those with a high level of achievement in HSC mathematics. It is possible that the patterns of results evident in the present study were reflective of these differing backgrounds in mathematics. It would be interesting to examine this issue by analysing the influence of the educational backgrounds of the cohort, and we plan to take this issue up in the next phase (Phase 2) of the study.

Phase 2 of this project will also include a follow-up investigation of the development of PS teachers' CK following professional experience and further studies, including a group of approximately 50 students who will complete additional mathematics content units. We intend to further examine this cohort of PS teachers' procedural and conceptual understandings in light of their previous mathematics experience, exposure to subsequent university content and pedagogy courses, and their professional experience.

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