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Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

Mieko Yamada

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Tables are given for the existence of amicable and skew-Hadamard matrices of orders \( 2^t q, t \geq 2 \) an integer, \( q(\text{odd}) \leq 2000 \). This gives further evidence to support the conjecture that "for every odd integer \( q \) there exists an integer \( t \) (dependent on \( q \)) so that there is a skew-Hadamard matrix of order \( 2^t q \)."

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Amicable Hadamard matrices and amicable orthogonal designs

Jennifer Seberry
Department of Computer Science
University College
The University of New South Wales
Australian Defence Force Academy
Canberra ACT 2600
Australia

Mieko Yamada
Department of Applied Mathematics
Faculty of Science
Konan University
Higashinada
Kobe 658
Japan

Abstract

New constructions for amicable orthogonal designs are given. These new designs then give new amicable Hadamard matrices and new skew-Hadamard matrices. In particular we show that if \( p \) is the order of normalized amicable Hadamard matrices there are normalized amicable Hadamard matrices of order \((p - 1)^u + 1, u > 0 \) an odd integer.

Tables are given for the existence of amicable and skew-Hadamard matrices of orders \( 2^q, t \geq 2 \) an integer, \( q(\text{odd}) \leq 2000 \). This gives further evidence to support the conjecture that "for every odd integer \( q \) there exists an integer \( t \) (dependent on \( q \)) so that there is a skew-Hadamard matrix of order \( 2^q \)."

1 Introduction

An orthogonal design of order \( n \) and type \( OD(n; u_1, ..., u_s) \) \( (u_i > 0) \) on the commuting variables \( x_1, ..., x_n \) is an \( n \times n \) matrix \( A \) with entries from \( \{0, x_1, ..., x_n\} \) such that

\[
AA^T = \sum_{i=1}^{s}(u_i x_i^2)I_n.
\]

In [2], where this was first defined and many examples and properties of such designs were investigated, it was shown that the numbers of variables, \( s \), satisfies \( s \leq p(n) \), where \( p(n) \) (Radon's function) is defined as follows:

- if \( n = 2^a b \), where \( b \) is odd and \( a=4c+d \), where \( 0 \leq d \leq 4 \),
  - then \( p(n) = 8c + 2^d \).

A powerful construction for Hadamard matrices in [17] showed that the existence of orthogonal designs in powers of two was of great import. W. Wolfe and D. Shapiro showed that the cases of the problem in powers of two are crucial to the understanding of the algebraic structure involved (see [16]).

All possible designs exist in order 2, 4, and 8. The existence problem for order 16 was solved in [7] and in [6] many designs were constructed for order 32.

\( M \) and \( N \) of order \( n \) are said to be amicable orthogonal designs of type \( AO\overline{D}(n; m_1, ..., m_p); (n_1, ..., n_q) \) if \( M \) is an \( OD(n; m_1, ..., m_p) \), \( N \) is an orthogonal design \( OD(n; n_1, ..., n_q) \) and \( MN^T = NM^T \). If \( M \) comprises the variables

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$x_1, \ldots, x_p$ and $N$ comprises the variables $y_1, \ldots, y_q$ then

$$MM^T = \sum_{i=1}^{p} m_i x_i^2 I_n, \quad NN^T = \sum_{j=1}^{q} n_j y_j^2 I_n$$

and

$$ZZ^T = (m_1 x_1^2 + \ldots + m_p x_p^2)(n_1 y_1^2 + \ldots + n_q y_q^2)I_n$$

where $Z = MN^T$. So amicability is linked with factorizing quadratic forms.

Wolfe and Shapiro (see [3]) have studied and solved the algebraic necessary conditions for amicable orthogonal designs but the sufficiency conditions are largely unresolved (see [6, 3, 11] for partial results).

An Hadamard matrix, $H$, is an orthogonal design of order $n$ and type $(n)$ or alternatively, a matrix with entries $\pm 1$ satisfying $HH^T = nI_n$. $H$ is said to be skew-Hadamard if $H + I$ or $H - I$ is skew-symmetric. Two Hadamard matrices $H = M + I$ and $N$ or order $n$ are called amicable Hadamard matrices if $M^T = -M$, $NT = N$, $HN^T = NH^T$. It is shown in [3] that amicable orthogonal designs $AOD(n; (1, n - 1), (n))$ give amicable Hadamard matrices (they are not the same since the orthogonal designs have no symmetry or skew symmetry conditions). Normalized amicable Hadamard matrices of order $h$ can be written in the form

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & I + S & \cdots & 1 \\ - & \cdots & - & - \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & P + R & \cdots & 1 \\ - & \cdots & - & - \end{bmatrix}$$

where

$$S^T = -S, \quad P^T = P, \quad R^T = R, \quad PR^T + RP^T = 0, \quad RR^T = I, \quad SJ = PJ = 0$$

$$RJ = -J, \quad SP^T = PST, \quad SR^T = RST, \quad SS^T = PP^T = (h - 1)I - J$$

A weighing matrix $W(n, n - 1)$ is an orthogonal design of order $n$ and type $(n - 1)$.

Amicable orthogonal designs, amicable Hadamard matrices, and skew-Hadamard matrices have proved difficult to find. The Kronecker produce of skew-Hadamard

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matrices is not a skew-Hadamard matrix. But if $h_1$ and $h_2$ are the orders of amicable Hadamard matrices then there are amicable Hadamard matrices of order $h_1 h_2$; further, if $g$ is the order of a skew-Hadamard matrix there are skew-Hadamard matrices of orders $h_1 g$ and $h_2 g$. We list from [3] and [12] the orders for which Hadamard matrices are known.

Summary 1

A1 $2^t$ $t$ a non-negative integer

AII $p^r + 1$ $p^r$ (prime power) $\equiv 3 \pmod{4}$.

AIII $(p - 1)^u + 1$ $p$ the order of normalized amicable Hadamard matrices, there are normalized amicable Hadamard matrices of order $(p - 1)^u + 1$, $u > 0$ an odd integer

AIV $2(q + 1)$ $2q + 1$ is a prime power, $q$ (prime) $\equiv 1 \pmod{4}$.

AV $(|t| + 1)(q + 1)$ $q$ (prime power) $\equiv 5 \pmod{8} = s^2 + 4t^2$, $s \equiv 1 \pmod{4}$, and $|t| + 1$ is the order of amicable orthogonal designs of type $AOD(1 + |t|; 1, |t|)$; $\frac{1}{2}(|t| + 1)$.

$2^r(q + 1)$ $q$ (prime power) $\equiv 5 \pmod{8} = s^2 + 4(2^r - 1)^2$, $s \equiv 1 \pmod{4}$, $r$ some integer.

AVI $S$ $S$ is a product of the above orders.

Skew-Hadamard matrices are known for the following orders (the reader should consult [18, p451] for more details):
2 New Results

First we give some definitions. Let $Y$ of order $n$ be an $OD(n; u_1, ..., u_s)$ then we say the $s$ matrices $Y_i, i = 1, ..., s$ of order $m$ are suitable $\pm 1$ matrices for the orthogonal design $Y$ or suitable matrices for $Y$ if

(i) the elements of each $Y_i$ are $\pm 1$,

(ii) the $Y_i$ are pairwise amicable so $Y_i Y_j^T = Y_j Y_i^T, i, j = 1, ..., s,$

(iii) $\sum_{i=1}^{s} Y_i Y_i^T = nmI.$

The matrices are said to be near suitable $\pm 1$ matrices for the orthogonal design $Y$ or near suitable $\pm 1$ matrices for $Y$ if (i) and (ii) hold but also

(iii)$a \sum_{i=1}^{s} Y_i Y_i^T = n(m + 1)I - nJ.$

Similarly let $X = X(x_1, ..., x_t)$ and $Y = Y(y_1, ..., y_s)$ be amicable orthogonal designs $AOD(n; u_1, ..., u_t; v_1, ..., v_s)$ of order $n$. Then we say the $t$ matrices $X_i, i = 1, ..., t$ of order $m$ and the $s$ matrices $Y_j, j = 1, ..., s$ of order $m$ are suitable $\pm 1$ matrices for the amicable orthogonal designs $X$ and $Y$ if

(i) the elements of each $X_i$ and $Y_j$ are $\pm 1$,

(ii) the $X_i$ and $Y_j$ are all pairwise amicable so

$$X_i X_k^T = X_k X_i^T, Y_j Y_j^T = Y_j Y_j^T, X_i Y_j^T = Y_j X_i^T, i, k = 1, ..., t, j, l = 1, ..., s,$$

(iii) $\sum_{i=1}^{t} X_i X_i^T = \sum_{j=1}^{s} Y_j Y_j^T = mnm.$
The matrices are said to be near suitable \( \pm 1 \) matrices for the amicable orthogonal designs \( X \) and \( Y \) or near suitable \( \pm 1 \) matrices for \( X \) and \( Y \) if (i) and (ii) hold but also

\[
(iii) \sum_{i=1}^{t} X_i X_i^T = \sum_{j=1}^{s} Y_j Y_j^T = n(m + 1)I - nJ.
\]

Now we have

**Theorem 1** Let \( X = X(z_1, ..., z_t) \) and \( Y = Y(y_1, ..., y_s) \) be amicable orthogonal designs \( AOD(n; (u_1, ..., u_t); (v_1 = 1, ..., v_s)) \) of order \( n \). Suppose there are \( n \) near suitable \( 1, -1 \) matrices of order \( m, X_1, ..., X_t \) and \( Y_1, ..., Y_s \) for \( X \) and \( Y \) satisfying

(i) \( e X_i^T = e Y_i^T = e, \quad i = 1, ..., t, \quad j = 1, ..., s. \)

(ii) \( Y_1 \) is a skew-type and \( Y_2, ..., Y_s, X_1, ..., X_t \) are symmetric.

Then

\[
U = \begin{pmatrix} X(1,1,1,1) & X(e,e,e) \\ X(e^T,e^T,e^T,e^T) & -X(X_1,X_2,...,X_t) \end{pmatrix}, \quad V = \begin{pmatrix} Y(1,1,1,1) & X(e,e,e) \\ -X(e^T,e^T,e^T,e^T)^T & Y(Y_1,Y_2,...,Y_t)^T \end{pmatrix},
\]

are amicable Hadamard matrices of order \( n(m+1) \). If \( Y_1 = yI + Z \), where \( eZ = 0 \) and \( y \) is a variable and \( v_1 = 1, \)

\[
U = \begin{pmatrix} X(1,1,1,1) & X(e,e,e) \\ X(e^T,e^T,e^T,e^T) & -X(X_1,X_2,...,X_t) \end{pmatrix},
\]

\[
V = \begin{pmatrix} Y(y,1,1,1) & X(e,e,e) \\ -X(e^T,e^T,e^T,e^T)^T & Y(yI + Z,Y_2,...,Y_t)^T \end{pmatrix},
\]

are amicable orthogonal designs \( AOD(nm + n; (nm + n); (1, nm + n - 1)) \).

Proof: By straightforward verification.

Example:

\[
X = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & -z_1 & z_3 & -z_4 \\ z_3 & z_2 & -z_1 & z_4 \\ z_4 & -z_3 & z_2 & -z_1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ -y_2 & y_1 & y_3 & -y_4 \\ -y_3 & -y_2 & y_1 & y_4 \\ -y_4 & y_3 & -y_2 & y_1 \end{pmatrix}
\]

are \( AOD(4; (1,1,2); (1,1,2)) \). Then
are $AOD(4(p + 1); (4p); (1, 4p - 1))$ whenever near suitable $X_i$, $i = 1, 2, 3$, and $Y_j$, $j = 1, 2, 3$ of order $p$ exist.

This is now stated as a corollary.

**Corollary 2** If there exist near suitable matrices $X_1, X_2, X_3, Y_1, Y_2, Y_3$ of order $s$ satisfying

(i) $X_iX_j^T = X_jX_i^T$, \quad $i \neq j$, $i, j = 1, 2, 3, 4$

(ii) $Y_iY_j^T = Y_jY_i^T$, \quad $i \neq j$, $i, j = 1, 2, 3, 4$

(iii) $X_1Y_1^T + X_2X_2^T + 2X_3X_3^T + 2X_4X_4^T = Y_1Y_1^T + Y_2Y_2^T + 2Y_3Y_3^T + 2Y_4Y_4^T = 4(s + 1)I - 4J$

(or with $y$ and $z$ variables,
$Y_1Y_1^T + z^2Y_2Y_2^T + 2z^2Y_3Y_3^T + (y^2 + (4s + 3)z^2)I - 4z^2J$)

(or $Y_1 = yI + zZ$, $eZ = 0$, $Z^T = -Z$),

(iv) $eX_i^T = eY_i^T = e$, \quad $i = 1, 2, 3, 4$

(v) $X_iY_j^T = Y_jX_i^T$, \quad $i, j = 1, 2, 3, 4$

(vi) $Y_1$ is a skew-type and other matrices are symmetric.

Then there exist amicable Hadamard matrices of order $4(s + 1)$ and amicable orthogonal designs $AOD(4s + 4; (4s + 4); (1, 4s + 3))$.

Also we note

**Corollary 3** Suppose $p \equiv 5 \pmod{8} = s^2 + 36$ is a prime power then there exist $AOD(4p + 4; (2p + 2, 2p + 2), (1, 4p + 3))$.  

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Proof: From J. Wallis [19] we have four ±1 matrices $M$ and $N$ where $(M - I)^T = M - I$, $N^T = N$

$$MM^T + 3NN^T = (4p + 4)I - 4J.$$  

Choose $Y_1 = yI + z(M - I)$, $Y_2 = Y_3 = N = NR$ (the type two or back-circulant matrix from $N$). Let $U$ and $V$ be the ±1 incidence matrices of the quadratic residues and quadratic non-residues so

$$eU = eV = e, UT = U^T, VT = V,$$

$$U^T + VV^T = 2(p + 1)I - 2J.$$  

Choose $X_1 = X_2 = zUR, X_3 = wVR$ (type two or back-circulant matrices) and the OD$(4p + 4; 2p + 2, 2p + 2)$ as

\[
\begin{bmatrix}
  z & x & w & z & w & e & y & e \\
  z & -z & w & -w & z & e & y & e \\
  w & -w & -z & -z & w & e & y & e \\
  w & x & w & -w & z & e & y & e \\
  x & z & -z & -z & w & e & y & e \\
  x & -z & z & z & w & e & y & e \\
  -z & -w & z & z & w & e & y & e \\
  y & x & y & y & w & e & y & e \\
  y & -x & y & y & w & e & y & e \\
  y & -y & y & y & w & e & y & e \\
  y & y & y & y & w & e & y & e \\
  y & y & y & y & w & e & y & e \\
  y & y & y & y & w & e & y & e
\end{bmatrix}
\]

to obtain the result.  

Lemma 4 If there exist suitable matrices $X_1, X_2, X_3, Y_1, Y_2, Y_3$, of order $s$ satisfying

$$X_1X_1^T + X_2X_2^T + 2X_3X_3^T = Y_1Y_1^T + Y_2Y_2^T + 2Y_3Y_3^T = 4sI,$$

then there are amicable Hadamard matrices of order $4s$. If $Y_1 = yI + zZ$ then there exist orthogonal designs $OD(4s; (4s); (1, 4s - 1))$.

Proof: Use the matrices $X$ and $Y$ of the Example.  

If we have good matrices $X_1, X_2, X_3 = X_4$ and circulant Williamson matrices $Y_1, Y_2, Y_3 = Y_4$ then Lemma 4 will be satisfied. This is certainly true for $s = 3, 5$. For example:

$s = 3$ use the circulant matrices with first rows

$X_1 = 1 - 1, X_2 = 111, X_3 = 1 - -,$

$Y_1 = 111, Y_2 = Y_3 = 1 - -$

$s = 5$ $X_1 = 11 - 1 -, X_2 = 11 - - 1, X_3 = 1 - - - -,$

$Y_1 = 1 - 11 -, Y_2 = 11 - - 1, Y_3 = 1 - - - -.$

We now give a slightly different construction which gives useful amicable orthogonal designs.
3 A powering theorem

Theorem 5 If there exists a skew-Hadamard matrix of order \( p+1 \) with circulant or type one core there exist amicable Hadamard matrices of order \( p^u + 1 \) and \( AOD(p^u + 1;(1,p^u),(1,p^u)) \) for odd integers \( u > 0 \).

Proof: We illustrate for \( u = 3 \). Let

\[
H = \begin{bmatrix} 1 & e \\ -e^T & I + B \end{bmatrix}
\]

be the skew-Hadamard matrix of order \( p + 1 \), with \( B \) the circulant or type one core. So

\[
BB^T = pI - J, \quad JB = 0, \quad BT = B.
\]

Now let \( A = BR \) be the back-circulant or type two matrix obtained from \( H \) which satisfies

\[
AAT = pI - J, \quad JA = 0, \quad A^T = A, \quad AB^T = BA^T, \quad A \pm R \text{ is } \pm 1.
\]

Define

\[
W_B = B \times B \times B \times I \times J + I \times J \times B \times J \times B \times I
\]

and

\[
W_A = A \times A \times A + A \times R \times J + R \times J \times A + J \times A \times R
\]

then

\[
W_BW_B^T = p^3I - J, \quad JW_B = 0, \quad W_B^T = -W_B, \quad W_B(R \times R \times R) = (R \times R \times R)|W_B^T,
\]

\[
W_AW_A^T = p^3I - J, \quad JW_A = 0, \quad W_A^T = W_A.
\]

and

\[
W_AW_B^T = AB^T \times AB^T \times AB^T + AB^T \times R \times J^3 + R \times J^3 \times AB^T + J^3 \times AB^T \times R = W_BW_A^T.
\]

Writing \( x, y, z, w \) as variables

\[
\begin{bmatrix} x \\ -e^Ty \end{bmatrix} \text{ and } \begin{bmatrix} -z \\ ew \end{bmatrix}
\]

are the required \( AOD \) from which the amicable Hadamard matrices are constructed by choosing \( z = y = z = w = 1 \).

The method works for other \( u \) by choosing \( W_B \) with \( B \times B \times \cdots \times B \) \( (u \text{ B's}) \) and then replacing pairs \( B \times B \) by \( I \times J \), always keeping an odd number of \( B \)'s. (This construction is discussed more fully in [18, pp309-312].)

The first example of interest is \( 15^3 + 1 \) which does not arise in other ways.

Corollary 6 If there exist normalized amicable Hadamard matrices of order \( p+1 \) there exist normalized amicable matrices of order \( p^u + 1 \), \( u \text{ odd } > 0 \).
4 Existence of AOD

In this section we further explore the existence of $AOD(2s + 2; (1, 2s + 1), (s + 1, s + 1))$.

Theorem 7 Let $p = 4m + 3$ be a prime power and $q = 2m + 1 \equiv 1 \pmod{4}$ a prime. Then there exist amicable orthogonal designs $AOD(2(q + 1); (1, 2q + 1), (q + 1, q + 1))$, and amicable Hadamard matrices of order $2q + 2$.

Proof: Form the $2 - (2m + 1; m; m - 1)$ Szekeres difference sets $X$ and $Y$ of size $m$ in the cyclic group of order $q = 2m + 1$. Now $X$ and $Y$ have the property that $z \in X \Rightarrow -z \notin X$ and $y \in Y \Rightarrow -y \notin Y$. Let $M$ be the $(0, 1, -1)$ incidence matrix of $X$ with zero diagonal, and let $N$, the $(1, -1)$ incidence matrix of $Y$, be a diagonal of ones, both of order $q$. Then

$$MM^T + NN^T = (2q + 1)I - 2J, \quad eM = 0, \quad eN = e, \quad M^T = -M, \quad N^T = N.$$

Let $Q$ be the $(0, 1, -1)$ incidence matrix of the quadratic residues of $q$. Then

$$QQ^T = qI - J, \quad eQ = 0, \quad Q^T = Q.$$

Now we let $a, b, c$ and $d$ be commuting variables and $e$ the $1 \times q$ matrix of ones. Then, with $R$ the back-diagonal matrix,

$$A = \begin{bmatrix} a & b & be & be \\ -b & a & be & -be \\ be^T & be^T & aI + bM & bNR \\ -be^T & be^T & -bNR & aI + bM \end{bmatrix}$$

and

$$C = \begin{bmatrix} d & c & ce & de \\ c & -d & -de & ce \\ ce^T & -de^T & (dI + cQ)R & (-cI + dQ)R \\ de^T & ce^T & (-cI + dQ)R & (-dI - cQ)R \end{bmatrix}$$

are two $AOD(2q + 2; (1, 2q + 1), (q + 1, q + 1))$. □

Corollary 8 $AOD(2q + 2; (1, 2q + 1), (q + 1, q + 1))$ and amicable Hadamard matrices of order $2q + 2$ exist as follows:

(i) $AOD(12; (1, 11), (6, 6))$,
(ii) $AOD(28; (1, 27), (14, 14))$,
(iii) $AOD(60; (1, 59), (30, 30))$,
(iv) $AOD(84; (1, 83), (42, 42))$,
(v) $AOD(108; (1, 107), (54, 54))$.

These give class AIV of the list.
Theorem 9 Suppose amicable Hadamard matrices of order $s$ exist. Further suppose $AOD(2p; (1, 2p - 1), (p, p))$ exist. Then $AOD(2ps; (1, 2ps - 1), (ps, ps))$ exist. In particular $AOD(2s; (1, 2s - 1), (s, s))$ exist.

Proof: Let $I + S$ and $P$ be the amicable Hadamard matrices, and $zA + yB, zC + wD$ be the $OD(1, 2p - 1)$ and $OD(2p; p, p)$ which are amicable $x, y, z, w$ commuting variables. Then

$$(zI + yS) \times A + yP \times B \quad \text{and} \quad zP \times C + wP \times D$$

are the required $AOD$.

The second result follows by choosing $p = 1$.

In [3, Corollary 5.50, 5.52 and 5.56] it is shown that the following corollary holds.

Corollary 10 $AOD(2^{t+1}; (1, 2^{t+1} - 1), (2^t, 2^t))$ exist for all $t \geq 0$. $AOD(p + 1; (1, p), (1, p)), p \equiv 3 \pmod{4}$ a prime power exist. $AOD(2p; (2, 2p - 2), (2, 2p - 2)), p$ the order of a symmetric conference matrix exist.

We also have

Theorem 11 Suppose there exist $AOD(2h; (h, h), (1, 2h - 1))$. Let $p = s^2 + 4t^2 \equiv 5 \pmod{8}, 2h = |h| + 1$, be a prime power. Then there exist $AOD(2h(p + 1); (h(p + 1), h(p + 1)), (1, 2hp + 2h - 1))$.

Proof: We use Theorem 1 with $X(2h; h, h)$ the OD on the variables $x, y$ and $Y(2h; 1, 2h - 1)$ the OD on the variables $z$ and $w$. Let $X_1, X_2, Y_1, Y_2$ of Theorem 1 be $-zX_1, -yX_2, zI + w(Y_1 - I), -zY_2$ as given in Lemma 13.

Thus we have

Corollary 12 There exist

(i) $AOD(4(p + 1); (2p + 2, 2p + 2), (1, 4p + 3))$ for $p = s^2 + 36 \equiv 5 \pmod{8}$ a prime power,

(ii) $AOD(8(p + 1); (4p + 2, 4p + 2), (1, 8p + 7))$ for $p = s^2 + 188 \equiv 4 \pmod{8}$ a prime power,

(iii) $AOD(16(p + 1); (8p + 8, 8p + 8), (1, 16p + 15))$ for $p = s^2 + 900 \equiv 5 \pmod{8}$ a prime power.

5 The existence of sets of near suitable matrices

We explore the existence of four $(1, -1)$ matrices $Y_1$ (circulant or type one and skew-type), $Y_2, X_1, X_2$ (circulant or type one and symmetric) of order $q$ which satisfy

$$Y_1Y_1^T + (2u + 1)(Y_2R)(Y_2R)^T = (u + 1)(X_1R)(X_1R)^T + (u + 1)(X_2R)(X_2R)^T = (2u + 2)(q + 1)I - (2u + 2)J$$

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where $R$ is the back-diagonal matrix.

**Lemma 13** Let $p = 4f + 1 = s^2 + 4t^2 \equiv 5 \pmod{8}$ be a prime power. Then there are $|t| + 1$ near suitable matrices of order $p$.

**Proof:** Now from [19, Lemma 24] we have that if $p = 4f + 1 = s^2 + 4t^2 \equiv 5 \pmod{8}$ is a prime power then the ±1 incidence matrices of the sets

$$C_0 \cup C_1 \quad \text{and} \quad |t| \text{ copies of } C_0 \cup C_2$$

will give near suitable $(1, -1)$ matrices for $Y_1$ and $Y_2$ satisfying

$$Y_1Y_1^T + |t|Y_2Y_2^T = (|t| + 1)(p + 1)I - (|t| + 1)J.$$

Furthermore the ±1 incidence matrices of

$$\frac{1}{2}(|t| + 1) \text{ copies of } C_0 \cup C_2 \quad \text{and} \quad \frac{1}{2}(|t| + 1) \text{ copies of } C_1 \cup C_3$$

will give near suitable $(1, -1)$ matrices $X_1$ and $X_2$ satisfying

$$\frac{1}{2}(|t| + 1)[X_1X_1^T + X_2Y_2^T] = (|t| + 1)(p + 1)I - (|t| + 1)J.$$

Note $Y_2, X_1, X_2$ are circulant (type one) and symmetric but we use their back-circulant (type two) form. $Y_1$ is skew-type circulant (type one). Hence $X_1, X_2, Y_1, Y_2$ are the required matrices. \qed

**References**


### 6 Tables of Amicable Hadamard Matrices

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Orders for which amicable Hadamard matrices exist.
7 Tables of orders for which skew-Hadamard matrices exist

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Orders for which skew-Hadamard matrices exist.