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Some orthogonal designs and complex Hadamard matrices by using two Hadamard matrices

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Abstract
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Some Orthogonal Designs and complex Hadamard matrices by using two Hadamard matrices

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Abstract

We prove that if there exist Hadamard matrices of order \(h\) and \(n\) divisible by 4 then there exist two disjoint \(W(\frac{hn}{4}, \frac{hn}{4})\), whose sum is a \((1, -1)\) matrix and a complex Hadamard matrix of order \(\frac{hn}{2}\), furthermore, if there exists an \(OD(m; s_1, s_2, \cdots, s_l)\) for even \(m\) then there exists an \(OD(\frac{hn}{4}; \frac{hn}{4}s_1, \frac{hn}{4}s_2, \cdots, \frac{hn}{4}s_l)\).

1 Introduction and Basic Definitions

A complex Hadamard matrix (see [4]), say \(C\), of order \(c\) is a matrix with elements \(1, -1, i, -i\) satisfying \(CC^* = cI\), where \(C^*\) is the Hermitian conjugate of \(C\). From [4], any complex Hadamard matrix has order 1 or order divisible by 2. Let \(C = X + iY\), where \(X, Y\) consist of \(1, -1, 0\) and \(X \wedge Y = 0\) where \(\wedge\) is the Hadamard product. Clearly, if \(C\) is an complex Hadamard matrix then \(XX^T + YY^T = cI, XY^T = YX^T\).

A weighing matrix [2] of order \(n\) with weight \(k\), denoted by \(W = W(n, k)\), is a \((1, -1, 0)\) matrix satisfying \(WW^T = kI_n\). \(W(n, n)\) is an Hadamard matrix.

Let \(A_j\) be a \((1, -1, 0)\) matrix of order \(m\) and \(A_jA_j^T = s_jI_m\). An orthogonal design \(D = x_1A_1 + x_2A_2 + \cdots + x_lA_l\) of order \(m\) and type \((s_1, s_2, \cdots, s_l)\), written \(OD(m; s_1, s_2, \cdots, s_l)\), on the commuting variables \(x_1, x_2, \cdots, x_l\) is a square matrix with entries 0, \(\pm x_1, \pm x_2, \cdots, \pm x_l\) where \(x_i\) or \(-x_i\) occurs \(s_i\) times in each row and column and distinct rows are formally orthogonal. That is

\[
DD^T = \sum_{j=1}^{l} s_j x_j^2 I_m
\]

Let $M$ be a matrix of order $tm$. Then $M$ can be expressed as

$$
M = \begin{bmatrix}
M_{11} & M_{12} & \cdots & M_{1t} \\
M_{21} & M_{22} & \cdots & M_{2t} \\
\vdots & & & \\
M_{11} & M_{12} & \cdots & M_{tt}
\end{bmatrix}
$$

where $M_{ij}$ is of order $m$ ($i,j = 1,2,\ldots,t$). Analogously with Seberry and Yamada [3], we call this a $t^2$ block $M$-structure when $M$ is an orthogonal matrix.

To emphasize the block structure, we use the notation $M(t)$, where $M(t) = M$ but in the form of $t^2$ blocks, each of which has order $m$.

Let $N$ be a matrix of order $tn$. Then, write

$$
N(t) = \begin{bmatrix}
N_{11} & N_{12} & \cdots & N_{1t} \\
N_{21} & N_{22} & \cdots & N_{2t} \\
\vdots & & & \\
N_{11} & N_{12} & \cdots & N_{tt}
\end{bmatrix}
$$

where $N_{ij}$ is of order $n$ ($i,j = 1,2,\ldots,t$).

We now define the operation $\circ$ as the following:

$$
M(t) \circ N(t) = \begin{bmatrix}
L_{11} & L_{12} & \cdots & L_{1t} \\
L_{21} & L_{22} & \cdots & L_{2t} \\
\vdots & & & \\
L_{11} & L_{12} & \cdots & L_{tt}
\end{bmatrix}
$$

where $M_{ij}, N_{ij}$ and $L_{ij}$ are of order of $m, n$ and $mn$, respectively and

$$
L_{ij} = M_{11} \times N_{ij} + M_{12} \times N_{2j} + \cdots + M_{tt} \times N_{ij},
$$

$i,j = 1,2,\ldots,t$. We call this the strong Kronecker multiplication of two matrices.

### 2 Preliminaries

**Theorem 1** Let $A$ be an $OD(tm; p_1, \ldots, p_t)$ with entries $x_1, \ldots, x_t$ and $B$ be an $OD(tm; q_1, \ldots, q_s)$ with entries $y_1, \ldots, y_s$ then

$$(A(t) \circ B(t))(A(t) \circ B(t))^T = \left(\sum_{j=1}^{t} p_j x_j^2 \right) \left(\sum_{j=1}^{s} q_j y_j^2 \right) I_{mn}.$$

($A(t) \circ B(t)$ is not an orthogonal design but an orthogonal matrix.)
Proof.

\[
A(t) = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1t} \\
A_{21} & A_{22} & \cdots & A_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nt}
\end{bmatrix}
\]

and

\[
B(t) = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1t} \\
B_{21} & B_{22} & \cdots & B_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
B_{n1} & B_{n2} & \cdots & B_{nt}
\end{bmatrix}
\]

where \( A_{ij} \) and \( B_{ij} \) are of orders \( m \) and \( n \) respectively \((i, j = 1, 2, \ldots, t)\).

Write

\[
C = (A(t) \circ B(t))(A(t) \circ B(t))^T = \begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1t} \\
C_{21} & C_{22} & \cdots & C_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1} & C_{n2} & \cdots & C_{nt}
\end{bmatrix}
\]

where \( C_{ij} \) is of order \( mn \).

We first prove \( C_{13} = 0 \). It is easy to calculate \( C_{13} = \)

\[
= \sum_{j=1}^{t} (A_{11} \times B_{1j} + A_{12} \times B_{2j} + \cdots + A_{1t} \times B_{tj})(A_{31}^T \times B_{1j} + A_{32}^T \times B_{2j} + \cdots + A_{3n}^T \times B_{tj}^T)
\]

\[
= \sum_{j=1}^{t} (\sum_{j=1}^{t} (A_{11} A_{31}^T) \times (B_{1j} B_{1j}^T) + (A_{12} A_{32}^T) \times (B_{2j} B_{2j}^T) + \cdots + (A_{1t} A_{3t}^T) \times (B_{tj} B_{tj}^T))
\]

\[
= (A_{11} A_{31}^T + A_{12} A_{32}^T + \cdots + A_{1t} A_{3t}^T) \times \left( \sum_{j=1}^{t} q_{ij}^2 \right) I_n.
\]

But

\[
A_{11} A_{31}^T + A_{12} A_{32}^T + \cdots + A_{1t} A_{3t}^T = 0,
\]

so

\[
C_{13} = 0.
\]

Similarly,

\[
C_{ij} = 0 \text{ (} i \neq j \text{)}.
\]

We now calculate \( C_{ii} \).

\[
C_{ii} = \sum_{j=1}^{t} (A_{i1} \times B_{ij} + A_{i2} \times B_{2j} + \cdots + A_{it} \times B_{tj})(A_{i1}^T \times B_{ij} + A_{i2}^T \times B_{2j} + \cdots + A_{it}^T \times B_{tj}^T)
\]

\[
= \sum_{j=1}^{t} (\sum_{j=1}^{t} (A_{i1} A_{i1}^T) \times (B_{ij} B_{ij}^T) + (A_{i2} A_{i2}^T) \times (B_{2j} B_{2j}^T) + \cdots + (A_{it} A_{it}^T) \times (B_{tj} B_{tj}^T))
\]

\[
= (A_{i1} A_{i1}^T + A_{i2} A_{i2}^T + \cdots + A_{it} A_{it}^T) \times \left( \sum_{j=1}^{t} q_{ij}^2 \right) I_n.
\]
Thus
\[(A(t) \circ B(t))(A(t) \circ B(t))^T = \sum_{j=1}^{t} (p_{jx})^2 (q_{jy})^2 I_{mn}.\]

**Corollary 2** Let A and B be the matrices of orders \(m \times n\) respectively, consist of
1, -1, 0 satisfying \(AAT = p_{1m}t\) and \(BB^T = q_{1n}t\). Then
\[(A(t) \circ B(t))(A(t) \circ B(t))^T = pqI_{mn}.\]

**Proof.** In this case, \(A = OD(tm; p)\) , \(B = OD(tn; q)\) and \(x_1 = y_1 = 1.\)

In the remainder of this paper let \(H = (H_{ij})\) and \(N = (N_{ij})\) of order \(h\) and \(n\) respectively
be 16 block M-structures [3]. So
\[
H = \begin{bmatrix}
H_{11} & H_{12} & H_{13} & H_{14} \\
H_{21} & H_{22} & H_{23} & H_{24} \\
H_{31} & H_{32} & H_{33} & H_{34} \\
H_{41} & H_{42} & H_{43} & H_{44}
\end{bmatrix}
\]

where
\[
\sum_{j=1}^{4} H_{ij}H_{ij}^T = hI_h = \sum_{j=1}^{4} H_{ji}H_{ji}^T,
\]
for \(i = 1, 2, 3, 4\) and
\[
\sum_{j=1}^{4} H_{ij}H_{kj} = 0 = \sum_{j=1}^{4} H_{ji}H_{kj},
\]
for \(i \neq k, i, k = 1, 2, 3, 4.\)

Similarly, let
\[
N = \begin{bmatrix}
N_{11} & N_{12} & N_{13} & N_{14} \\
N_{21} & N_{22} & N_{23} & N_{24} \\
N_{31} & N_{32} & N_{33} & N_{34} \\
N_{41} & N_{42} & N_{43} & N_{44}
\end{bmatrix}
\]

where
\[
\sum_{j=1}^{4} N_{ij}N_{ij}^T = nI_n = \sum_{j=1}^{4} N_{ji}N_{ji}^T,
\]
for \(i = 1, 2, 3, 4\) and
\[
\sum_{j=1}^{4} N_{ij}N_{kj} = 0 = \sum_{j=1}^{4} N_{ji}N_{kj},
\]
for \(i \neq k, i, k = 1, 2, 3, 4.\)
for \( i \neq k, i, k = 1, 2, 3, 4 \).

For ease of writing we define \( X_i = \frac{1}{2}(H_{i1} + H_{i2}) \), \( Y_i = \frac{1}{2}(H_{i1} - H_{i2}) \), \( Z_i = \frac{1}{2}(H_{i3} + H_{i4}) \), \( W_i = \frac{1}{2}(H_{i3} - H_{i4}) \), where \( i = 1, 2, 3, 4 \). Then both \( X_i \pm Y_i \) and \( Z_i \pm W_i \) are \((1,-1)\)-matrices with \( X_i \wedge Y_i = 0 \) and \( Z_i \wedge W_i = 0 \) \( \wedge \) the Hadamard product.

Let

\[
S = \frac{1}{2}
\begin{bmatrix}
H_{11} + H_{12} & -H_{11} + H_{12} & H_{13} + H_{14} & -H_{13} + H_{14} \\
H_{21} + H_{22} & -H_{21} + H_{22} & H_{23} + H_{24} & -H_{23} + H_{24} \\
H_{31} + H_{32} & -H_{31} + H_{32} & H_{33} + H_{34} & -H_{33} + H_{34} \\
H_{41} + H_{42} & -H_{41} + H_{42} & H_{43} + H_{44} & -H_{43} + H_{44}
\end{bmatrix}
\]

Then \( S \) can be rewritten as

\[
S = \frac{1}{2}
\begin{bmatrix}
H_{11} & H_{12} & H_{13} & H_{14} \\
H_{21} & H_{22} & H_{23} & H_{24} \\
H_{31} & H_{32} & H_{33} & H_{34} \\
H_{41} & H_{42} & H_{43} & H_{44}
\end{bmatrix}
\begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & +1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & +1
\end{bmatrix}
\]

or

\[
S = \begin{bmatrix}
X_1 & -Y_1 & Z_1 & -W_1 \\
X_2 & -Y_2 & Z_2 & -W_2 \\
X_3 & -Y_3 & Z_3 & -W_3 \\
X_4 & -Y_4 & Z_4 & -W_4
\end{bmatrix}
\]

Obviously, \( S \) is a \((0,1,-1)\) matrix.

Write

\[
R = \begin{bmatrix}
Y_1 & X_1 & W_1 & Z_1 \\
Y_2 & X_2 & W_2 & Z_2 \\
Y_3 & X_3 & W_3 & Z_3 \\
Y_4 & X_4 & W_4 & Z_4
\end{bmatrix}
\]

also a \((0,1,-1)\) matrix.

We note \( S \pm R \) is a \((1,-1)\) matrix, \( R \wedge S = 0 \) and by Corollary 1

\[
SS^T = RR^T = \frac{1}{2}h_4.
\]

**Lemma 3.** If there exists an Hadamard matrix of order \( h \) divisible by 4, there exists an \( OD(h; \frac{1}{2}h, \frac{1}{2}h) \).

**Proof.** From \( S \) and \( R \) as above. Now \( H = S + R \). Note \( HH^T = SS^T + RR^T + SR^T + RS^T = hI_4 \) and \( SS^T = RR^T = \frac{1}{2}hI_4 \). Hence \( SR^T + RS^T = 0 \). Let \( x \) and \( y \) be commuting variables then \( E = xS + yR \) is the required orthogonal design.
3 Weighing Matrices

Lemma 4 If there exist Hadamard matrices of order \( h \) and \( n \) divisible by 4, there exists a \( W(\frac{1}{2}hn, \frac{1}{2}hn) \).

Proof. Let \( H \) and \( N \) as above be the Hadamard matrices of order \( h \) and \( n \) respectively. Let

\[
P = \frac{1}{2} \begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \\ X_4 & Y_4 & Z_4 & W_4 \end{bmatrix} \odot \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix}.
\]

Rewrite

\[
P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix},
\]

Consider

\[
P_{11} = \frac{1}{2}(X_1 \times N_{11} + Y_1 \times N_{21} + Z_1 \times N_{31} + W_1 \times N_{41}),
\]

where both \( X_1 \times N_{11} + Y_1 \times N_{21} \) and \( Z_1 \times N_{31} + W_1 \times N_{41} \) are \((1, -1)\) matrices. So \( P_{11} \) has entries 1, -1, 0 and similarly for other \( P_{ij} \). By Lemma 1,

\[
PP^T = \frac{1}{8}hnI_{\frac{1}{2}hn}.
\]

Then \( P \) is a \( W(\frac{1}{2}hn, \frac{1}{2}hn) \).

Corollary 5 There exists a \( W(h, \frac{1}{2}h) \) \((h > 1)\) if there exists an Hadamard matrix of order \( h \).

Proof. If \( h > 2 \) let \( n = 4 \) in Theorem 1. For the case \( h = 2 \), note \( W(2, 1) \) is the identity matrix.

We also note that if

\[
Q = \frac{1}{2} \begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \\ X_4 & Y_4 & Z_4 & W_4 \end{bmatrix} \odot \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ -N_{31} & -N_{32} & -N_{33} & -N_{34} \\ -N_{41} & -N_{42} & -N_{43} & -N_{44} \end{bmatrix}.
\]

Then \( Q \) is also a \( W(\frac{1}{2}hn, \frac{1}{2}hn) \).

Theorem 6 Suppose \( h \) and \( n \) divisible by 4, are the orders of Hadamard matrices then there exist two disjoint \( W(\frac{1}{2}hn, \frac{1}{2}hn) \), whose sum and difference are \((1, -1)\) matrices.
Rewrite

\[
Q = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} \\
Q_{41} & Q_{42} & Q_{43} & Q_{44}
\end{bmatrix}
\]

We note

\[
P_{ij} = \frac{1}{2}(X_i \times N_{1j} + Y_i \times N_{2j} + Z_i \times N_{3j} + W_i \times N_{4j}),
\]

and

\[
Q_{ij} = \frac{1}{2}(X_i \times N_{1j} + Y_i \times N_{2j} - Z_i \times N_{3j} - W_i \times N_{4j}).
\]

Since \(P_{ij} + Q_{ij} = X_i \times N_{1j} + Y_i \times N_{2j}\) and \(P_{ij} - Q_{ij} = Z_i \times N_{3j} + W_i \times N_{4j}\) we conclude that \(P_{ij} \pm Q_{ij}\) are \((1, -1)\) matrices and \(P \pm Q = 0\). Thus \(P \pm Q = 0\). \(P\) and \(Q\) are both \(W(1/\sqrt{n}, 1/\sqrt{n})\) by Corollary 1.

4 Complex Hadamard Matrices

Lemma 7 \(PQ^T = Q^TP\).

Proof. Write

\[
PQ^T = \begin{bmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & E_{22} & E_{23} & E_{24} \\
E_{31} & E_{32} & E_{33} & E_{34} \\
E_{41} & E_{42} & E_{43} & E_{44}
\end{bmatrix}
\]

and

\[
Q^TP = \begin{bmatrix}
F_{11} & F_{12} & F_{13} & F_{14} \\
F_{21} & F_{22} & F_{23} & F_{24} \\
F_{31} & F_{32} & F_{33} & F_{34} \\
F_{41} & F_{42} & F_{43} & F_{44}
\end{bmatrix}
\]

We first prove \(E_{13} = F_{13}\).

We note

\[
E_{13} = \frac{1}{4} \sum_{j=1}^{4}(X_1 \times N_{1j} + Y_1 \times N_{2j} + Z_1 \times N_{3j} + W_1 \times N_{4j})(X_3^T \times N_{1j}^T + Y_3^T \times N_{2j}^T - Z_3^T \times N_{3j}^T - W_3^T \times N_{4j}^T)
\]

and

\[
F_{13} = \frac{1}{4} \sum_{j=1}^{4}(X_1 \times N_{1j} + Y_1 \times N_{2j} - Z_1 \times N_{3j} - W_1 \times N_{4j})(X_3^T \times N_{1j}^T + Y_3^T \times N_{2j}^T + Z_3^T \times N_{3j}^T + W_3^T \times N_{4j}^T).
\]
Obviously, \( E_{13} = F_{13} \) if and only if

\[
\sum_{j=1}^{4} (X_i \times N_{ij} + Y_i \times N_{ij}) (Z_i^T \times N_{ij}^2 + W_i^T \times N_{ij}^2)
\]

\[(1)\]

\[
= \sum_{j=1}^{4} (Z_i \times N_{ij} + W_i \times N_{ij}) (X_i^T \times N_{ij}^2 + Y_i^T \times N_{ij}^2).
\]

\[(2)\]

To show this, note

\[
\sum_{j=1}^{4} (X_i \times N_{ij} + Y_i \times N_{ij} + Z_i \times N_{ij} + W_i \times N_{ij}) (X_i^T \times N_{ij}^2 + Y_i^T \times N_{ij}^2 - Z_i^T \times N_{ij}^2 - W_i^T \times N_{ij}^2)
\]

and similarly for other parts in (1) and (2). Thus \( E_{13} = F_{13} \). Similarly, \( E_{ij} = F_{ij} \), for other \( i \neq j \).

We now prove \( E_{ii} = F_{ii} \). We see

\[
E_{ii} = \frac{1}{4} \sum_{j=1}^{4} (X_i \times N_{ij} + Y_i \times N_{ij} + Z_i \times N_{ij} + W_i \times N_{ij}) (X_i^T \times N_{ij}^2 + Y_i^T \times N_{ij}^2 - Z_i^T \times N_{ij}^2 - W_i^T \times N_{ij}^2)
\]

and

\[
F_{ii} = \frac{1}{4} \sum_{j=1}^{4} (X_i \times N_{ij} + Y_i \times N_{ij} + Z_i \times N_{ij} + W_i \times N_{ij}) (X_i^T \times N_{ij}^2 + Y_i^T \times N_{ij}^2 + Z_i^T \times N_{ij}^2 + W_i^T \times N_{ij}^2).
\]

Obviously, \( E_{ii} = F_{ii} \) if and only if

\[
\sum_{j=1}^{4} (X_i \times N_{ij} + Y_i \times N_{ij}) (Z_i^T \times N_{ij}^2 + W_i^T \times N_{ij}^2)
\]

\[(3)\]

\[
= \sum_{j=1}^{4} (Z_i \times N_{ij} + W_i \times N_{ij}) (X_i^T \times N_{ij}^2 + Y_i^T \times N_{ij}^2).
\]

\[(4)\]

The proof is the same as in (1) and (2). Hence \( E_{ii} = F_{ii} \). Finally, we conclude \( PQ^T = QP^T \).

**Theorem 8** If there exist Hadamard matrices of order \( h \) and \( n \) divisible by 4 then there exists a complex Hadamard matrix of order \( \frac{1}{2}hn \).

**Proof.** By the proof of Theorem 2, \( P \) and \( Q \) are the two disjoint \( W(\frac{1}{2}hn, \frac{1}{2}hn) \) i.e. \( P \land Q = 0 \) and \( P \land Q \) is a \((1,-1)\) matrix. Furthermore by Lemma 3, \( PQ^T = QP^T \). Thus \( P + iQ \) is a complex Hadamard matrix of order \( \frac{1}{4}hn \).
5 Orthogonal Designs

Theorem 9 If there exist Hadamard matrices of order $h$, $n$ divisible by 4 and an $OD(m; s_1, s_2, \ldots, s_l)$, where $m$ is even, then there exists an

$$OD\left(\frac{1}{4} hnm; \frac{1}{4} hns_1, \frac{1}{4} hns_2, \ldots, \frac{1}{4} hns_l\right).$$

Proof. Let

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

be the $OD(m; s_1, s_2, \ldots, s_l)$ on the commuting variables $x_1, \ldots, x_l$, where $D_j$ is of order $\frac{1}{2}m$. Let

$$D' = \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} \circ \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

where $P$ and $Q$, constructed above, are from the Hadamard matrices of order $h$ and $n$.

Then by Theorem 3 and Corollary 1,

$$D'D'^t = \frac{1}{4} hnm \left( \sum_j s_j x_j^2 \right) H_{\frac{1}{4} hnm}.$$  

Since $P \land Q = 0$, if $D$ consists of $0, \pm x_1, \ldots, \pm x_l$ then $D'$ also consists of $0, \pm x_1, \ldots, \pm x_l$ so $D'$ is an

$$OD\left(\frac{1}{4} hnm; \frac{1}{4} hns_1, \frac{1}{4} hns_2, \ldots, \frac{1}{4} hns_l\right).$$

Corollary 10 If there exist Hadamard matrices of order $h$ and $n$ divisible by 4 then there exists an $OD\left(\frac{1}{4} hnm; \frac{1}{4} hns, \frac{1}{4} hns\right)$.

Proof. Let

$$D = \begin{bmatrix} z & y \\ -y & z \end{bmatrix}$$

in the proof of Theorem 4, where $x$ and $y$ are commuting variables, put $m = l = 2$ and $s_1 = s_2 = 1$.

6 Remark

Theorem 1 cannot be replaced by Corollary 1 because the existence of Hadamard matrices of order $h$ and $n$ does not imply the existence of an Hadamard matrix of order $\frac{1}{2}h$. For example, there exist Hadamard matrices of order 4 · 3 and 4 · 71 but no Hadamard matrix of order 4 · 213 has been found [1], however, by Theorem 1, we have a $W(4 \cdot 213, 2 \cdot 213)$. 

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By the same result, there exists a $W(4k, 2k)$ and a complex Hadamard matrix of order $4k$, where $k$ is

$$
781 \ 789 \ 917 \ 1315 \ 1349 \ 1441 \ 1633 \ 1703 \ 2059 \ 2227 \ 2489 \ 2515 \\
2627 \ 2733 \ 3013 \ 3273 \ 3453 \ 3479 \ 3715 \ 4061 \ 4331 \ 4435 \ 4737 \ 4781 \\
4899 \ 4979 \ 4997 \ 5001 \ 5109 \ 5371 \ 5433 \ 5467 \ 5515 \ 5533 \ 5609 \ 5755 \\
5767 \ 5793 \ 5893 \ 6009 \ 6059 \ 6177 \ 6209 \ 6333 \ 6377 \ 6497 \ 6539 \ 6575 \\
6501 \ 6833 \ 6887 \ 6943 \ 7233 \ 7277 \ 7387 \ 7513 \ 7555 \ 7663 \ 7739 \ 7811 \\
7999 \ 8023 \ 8057 \ 8189 \ 8549 \ 8591 \ 8611 \ 8809 \ 8879 \ 8927 \ 9055 \\
9097 \ 9167 \ 9557 \ 9563 \ 9573 \ 9659 \ 9727 \ 9753 \ 9757 \ 9869 \ 9913 \ 9991
$$

References


