On weighing matrices with square weights

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On Weighing Matrices with Square Weights

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We give a new construction for a known family of weighing matrices using the 2-adjugate method of Vartak and Patwardhan. We review the existence of $W(n, k^2)$, $k = 1, \ldots, 12$, giving new results for $k = 8, 9, 10, 11, 12$.

1. Introduction

A weighing matrix, $W = W(n,k)$, is a square matrix of order $n$, with entries 0, +1, or -1, with the property that the scalar product of distinct rows is zero. Thus

$$WW^T = km.$$ 

It has been conjectured that

Conjecture 1. (Seberry). When $n \equiv 0 \pmod{4}$ there exists a $W(n,k)$ for all $k \leq n$.

We discuss the conjecture

Conjecture 2. For given $k$ there exists an integer $n_0$, so that for every $n > n_0$, there exists a $W(n,k)$.

It is known that

THEOREM 1 (Geramita, Geramita and Seberry[1975/76]). If a $W(n,k)$ exists and $n$ is odd, then $k$ is a square and

$$(n-k)^2 - (n-k) + 2 > n.$$ 

We know that

(a) a $W(m^2 + m + 1, m^2)$ exists whenever $m$ is a prime power and only if $m$ is the order of a projective plane;
(b) given any integer $k^2$ there exists an $n$ dependent on $k^2$ so that for every $m > n$ a $W(m,k^2)$ exists;
(c) there exists a

(i) $W(m,4)$, where $m > 3$ (except $m=5,9$, which do not exist);
(ii) $W(m,9)$, where $m > 21$ ($m = 11$ does not exist and $m = 17, 21$ are undecided; $m = 15$ exists, Gibbons and Mathon[1986] found a $W(19,9)$);
(iii) $W(m,16)$, where $m > 35$ ($m = 17, 19$ do not exist and $m = 23, 25, 27, 29, 33, 35$ are undecided; $m = 31$ exists);
(iv) $W(m,25)$, where $m > 81$ ($m = 27, 29$ do not exist-- there are many undecided cases);
(v) $W(m,36)$, where $m > 137$; ($m = 37, \ldots, 41$ do not exist)
(vi) \(W(m,49)\), where \(m > 199\); \((m = 49, \ldots, 58\) do not exist). The proof of these results is in Geramita and Seberry [1979, p.324-325].

We present a construction for weighing matrices using the 2-adjugation method of Vartak and Patwardhan and present results for \(W(n,64), W(n,81), W(n,100), W(n,121), W(n,144)\). This allows us to say

(vii) \(W(n,64)\) exist for \(n > 135\) and \(W(2n,64)\) exist for \(2n > 31\) (Lemma 8); \(W(n,64)\) for \(n = 65, \ldots, 71\) do not exist;

(viii) \(W(n,81)\) for \(n > 377\), \(W(4n,81)\) for \(n > 39\), \(W(130,81)\) and \(W(91,81)\) exist (Section 1 (a)); \(W(n,81)\) for \(n = 81, \ldots, 89\) do not exist;

(ix) \(W(n,100)\) exist for all \(n > 336\) and \(W(2n,100)\) exist for all \(n > 59\). (Lemma 10); \(W(n,100)\) do not exist for \(n = 101, \ldots, 109\) do not exist;

(xi) \(W(n,121)\) exist for all \(n > 513\), \(W(2n,121)\) exist for \(2n > 259\), \(W(8n,121)\) exist for \(8n > 127\). (Lemma 11); \(W(n,121)\) do not exist for \(n = 121, \ldots, 131\);

(xii) \(W(n,144)\) exists for all \(n > 649\), \(W(2n,144)\) exist for all \(2n > 377\). (Lemma 12); \(W(n,144)\) do not exist for \(n = 145, \ldots, 155\).

2. Known results

We use the following results

THEOREM 2. (Dandawate). If \(4t\) is the order of an Hadamard matrix then there exists a \(W(2t(4t-1), 4t^2)\).

THEOREM 3. (Geramita and Seberry) Let \(r\) be any number of the form \(2^a, 10^b, 26^c, 5^d, 13^e, \ldots\), and let \(n\) be any integer \(> 2^a10^b26^c5^d13^e \ldots\) such that \(a, b, c, d, e\) nonnegative integers. Then there exist (a) orthogonal designs of order \(4n\) and types \((1,1,r,r)\) and \((1,4,r,r)\); (b) an orthogonal design of order \(2n\) and type \((r,r)\) or a \(W(2n,2r)\).

COROLLARY 4. Let \(2n\) be any integer \(> 119\) then there exists a \(W(2n,100)\).

We use the fact that if there is an orthogonal design of type \((1,k)\) in order \(n\) there is an orthogonal design of type \((1,1,k,k)\) in order \(2n\). Hence a \(W(2n,2k+1)\) exists.

By the results of Geramita and Seberry [1979, Chapter 8] and Seberry [1982] we have

THEOREM 5. Every orthogonal design of type \((1,k)\), \(k < n-1\), exists in order \(n = 2^a, 2^a+1, 2^a+3, 2^a+5, 2^a+7, 2^a+9, 2^a+15, 2^a+21\), \(a\) a positive integer. Hence \(W(2n,81)\) and \(W(2n,121)\) exist for those \(m, n\) provided \(n > 40\), \(m > 60\). Thus \(W(16a,81)\) exist for \(a > 5\) and \(W(16b,121)\) exist for all \(b > 7\) (since by Theorem 1.149 of Geramita and Seberry an OD(1,30) exists in every order \(4n, n \geq 11\) we have OD(1,1,2,60,60) and hence \(W(16a,121)\) for \(n \geq 41\)).

Also from Theorem 1.149 of Geramita and Seberry we have

LEMMA 6. Orthogonal designs of type \((1,40)\) exist in all orders \(4n, n > 10\). Thus \(W(8n,81)\) exist \(n > 10\).

Since there are Golay sequences of length 40 we have

LEMMA 7. There is an orthogonal design of type \((1,140,40)\) in all orders \(4n, n \geq 40\). Thus \(W(4n,81)\) exist for \(n \geq 40\).
3. Some $W(n, k^2)$

3.1. Case $W(n, 64)$

The following exist:

- $W(2n, 64), \ n > 31$ (Corollary 4),
- $W(120, 64)$ (Dandawate),
- $W(73, 64)$ (Section 1, (i)).

Thus we have

**LEMMA 8.** There exist $W(n, 64)$ for $n > 135$.

3.2. Case $W(n, 81)$

The following exist:

- $W(4n, 81), \ n > 39$ (Lemma 7)
- $W(91, 81)$ (complement of $PG(9, 2)$)
- $W(130, 81)$ (from $W(10, 9)$ and $W(13, 9)$)
- $W(169, 81)$ (from $W(13, 9)$).

Thus we have

**LEMMA 9.** There exist $W(n, 81)$ for $n > 337$.

3.3. Case $W(n, 100)$

$W(2n, 100)$ exist for all $n > 59$, (Corollary 4). $W(n, 25)$ and $W(n, 4)$ give a $W(mn, 100)$ so since $W(m, 25)$ exist for $m \in \{26, 28, 31, 32, 36, 40, 42, 44, 48, \text{and all even orders} > 51\}$ and $W(n, 4)$ exist for $n \in \{4, 6, 7, 8, 10, 11, \ldots\}$ we have $W(mn, 100)$ for $mn \in \{104, 112, 217\}$. Thus we have

**LEMMA 10.** There exist $W(n, 100)$ for all $n > 336$. $W(2n, 100)$ exist for $2n = 104, 112$ and all $2n > 119$.

3.4. Case $W(n, 121)$

- $W(133, 121)$ exists (i), section 1).
- $W(16t^2, 121)$ exist for all $t > 2$ since $W(4t, 11)$ exists.
- $W(16b, 121)$ exists for all $b > 7$ (from Theorem 5).
- $W(122, 121)$ exists since 121 is a prime power.

We construct a $W(132, 121)$. Let $A$ be a $(1, 11)$ design in order 12, replace the first variable by $J_{11}$, the matrix of ones, and the second variable by $B$ where $B = (b_{ij}) = (\chi(j-i))$, $\chi$ the quadratic character with $\chi(0) = 0$. Then the new matrix is the required $W(132, 121)$.

Thus we have

**LEMMA 11.** There exist $W(n, 121)$ for all $n > 313$. $W(2n, 121)$ exist for $2n > 381$. $W(4n, 121)$ exist for $4n$
3.5 Case $W(n,144)$

Since there are 4-complementary sequences of length 9 and weight 36 and Golay sequences of length 4, we have 4-complementary sequences of length 36 and weight 144. Thus we have $W(4n,144)$ exist for all $n > 35$.

$W(273,144)$ exists as it is $W(13,9) \times W(21,16)$.  
$W(234,144)$ exists as it is $W(18,16) \times W(13,9)$.

Hence

**Lemma 12.** $W(n,144)$ exists for all $n > 649$. $W(2n,144)$ exists for all $2n > 377$.

4. A new construction for weighing matrices using 2-adjugates

Vartak and Patwardhan [1971] and Patwardhan and Vartak [1980] constructed the 2-adjugate mod 2 class of designs for unreduced BIBDs and symmetrical BIBDs with $\lambda = 1$. Patwardhan, Dandawate and Vartak [1984] applied the technique of 2-adjugation to a $(0,1,-1)$ matrix and obtained a class of generalized PBIBDs with two associate classes and with triangular association scheme with unequal block sizes.

In this section we use the concept of 2-adjugates to obtain a construction for weighing matrices.

The 2-adjugate of a $(0,1,-1)$ matrix of $N$ of order $(v \times b)$ is defined to be the matrix whose elements are the determinants of all possible $2 \times 2$ submatrices of $N$, arranged in lexicographic order. The 2-adjugate $N^*$ of a matrix $N$ whose entries come from a group is defined to be the matrix whose entries are formed by taking formal determinants of all possible $2 \times 2$ submatrices of $N$. Here by formal determinant we mean the following:

If the submatrix is

$$
x, y \\
z, w
$$

then the corresponding entry of $N^*$ is $xw - yz$.

Notice that in the special case when $N = GH(p, EA(p))$, the generalized Hadamard matrix of order $p$, $p$ a prime power, the $(i,j)$th entry of $N$ is $T^{[i-1][j-1]}$ and hence by replacing the group elements by the corresponding matrix representation the 2-adjugate of any arbitrary two by two submatrix

$$
T^{[i-1][j-1]} T^{[i-1][k-1]} \\
T^{[i-1][k-1]} T^{[j-1][k-1]}
$$

with $i < j$ is $T^{[i-1][j-1] + [j-1][k-1]} - T^{[i-1][k-1] + [j-1][k-1]}$.

An arbitrary row of $H^*$ obtained from the $i^{th}$ and $j^{th}$ rows of $H$ is given by

$$
T^{[i-1][i-1] + [j-1][i-1]} - T^{[i-1][j-1] + [j-1][i-1]} ; t=1,\ldots,p-1; s=t+1,\ldots,p.
$$

We prove the following

**Lemma 13.** Let $H^*$ be the 2-adjugate formed from $H = GH(p, EA(p))$, the generalized Hadamard matrix
of size \( p, p \) a prime power, then the inner product of any distinct rows of \( H^* \) obtained from the \( i^{th} \) and \( j^{th} \) rows and \( k \) and \( l^{th} \) rows of \( H \) respectively is zero and the inner product of a row obtained from the \( i^{th} \) and \( j^{th} \) rows of \( H \) with itself is \( p^2I - pJ \).

Proof:
The inner product of two rows of \( H^* \) is given by

\[
\sum_{i=1}^{(p-1)} \sum_{s=0}^{p} [I(i-1)(j-1)(s-1), r(i-1)(j-1)(s-1)]
\]

Notice that the first and second terms give

\[
\sum_{i=1}^{(p-1)} \sum_{s=0}^{p} T(i-k)(s-1) \cdot T(j-r)(s-1)
\]

Which gives us

\[
\sum_{i=1}^{(p-1)} \sum_{s=0}^{p} [I(i-k)(s-1), r(i-k)(s-1)] (s\neq 1).
\]

Now we add and subtract

\[
\sum_{i=1}^{(p-1)} T(i-k)(s-1) \cdot T(j-r)(s-1).
\]

Similarly we evaluate third and fourth terms and get:

\[
\sum_{i=1}^{(p-1)} T(i-k)(s-1) \cdot T(j-r)(s-1) - \sum_{i=1}^{(p-1)} T(i-k)(s-1) \cdot T(i-r)(s-1)
\]

Which is \( p^2I - pJ \) when \( i=k \) and \( j=r \), and zero otherwise.

**THEOREM 14.** Weighing matrices \( W(p^2(p-1), p^2) \) exist for all prime powers \( p \).

Proof: Form \( H^* \) from \( H = GH(p,EA(p)) \) the generalized Hadamard matrix. Obtain \( G \) from \( H^* \) by replacing the entries of \( H^* \) by the corresponding \( (0,1,-1) \) matrices. Form a matrix \( S = I \times J_p \). Then

\[
\begin{bmatrix}
G & S
\end{bmatrix}
\]

is the required weighing matrix.

**EXAMPLE.** Let \( H = GH(3,EA(3)) \) given by

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{bmatrix}
\]
Replace $\omega^i$ by $T^i$ and 1 by the identity matrix of order 3, where $T$ is given by

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
$$

Now form $H^*$:

$$
\begin{bmatrix}
1-\omega & 1-\omega^3 & \omega^2-\omega \\
1-\omega^2 & 1-\omega & \omega^2 \\
\omega^2-\omega & \omega-\omega^2 & \omega^2-\omega
\end{bmatrix}
$$

which gives $G$:

$$
\begin{array}{cccc}
-1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0
\end{array}
$$

Now form $S$:

$$
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

Now form $W$:

$$
\begin{bmatrix}
G & S \\
S & G
\end{bmatrix}
$$

Note that in this particular case we have $W(18, 3^2)$. We also note that in the case of order 3 we have using

$$
GW(4,3,2) =
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & - \\
1 & 0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}
$$
that

\[
\begin{array}{ccccccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & - & - & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & - & - \\
0 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & - \\
1 & 0 & 1 & - \\
1 & 0 & 1 & -
\end{array}
\]

is a W(13,9).

References


