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Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

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Abstract

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VANSTONE'S CONSTRUCTION APPLIED TO BHASKAR RAO DESIGNS

Jennifer Seberry
Department of Computer Science
University of Sydney
N.S.W., 2006, Australia

ABSTRACT

We show how Vanstone's construction, given in his paper "A note on a construction for BIBD's", *Utilitas Mathematica,* 7(1975), 321-322, can be applied to symmetric GBRD(v, k, λ; | G |), | G | odd, can be used to obtain GBRD(v, $\binom{v}{2}$, $\binom{k}{2}$, λ, $\binom{\lambda}{2}$;G) and hence many families of BIBD.

1. INTRODUCTION

Definitions of SBIBD and BIBD are standard.

Let $A = [a_{ij}]$ be a matrix of order with $a_{ij} \varepsilon \{0, 1, -1\}$. A is called a *weighing matrix* of weight p and order n, if $AA^T = A^TA = pI_n$, where I_n denotes the identity matrix of order n. Such a matrix is denoted by $W(n, p)$. If squaring all its entries gives an incidence matrix of a SBIBD then W is called a balanced weighing matrix.

An *Hadamard matrix,* $A = [a_{ij}]$, is a $W(n,n)$, that is, it is a square matrix of order n with entries $a_{ij} \varepsilon \{1,-1\}$ which satisfies

$$AA^T = A^TA = nI_n.$$

A *generalized Hadamard matrix* $\mathrm{GH}(gh,G) = (g_{ij}) = H$ over the group G of order g is a $gh \times gh$ matrix such that

(i)$g_{ij} \varepsilon G$ for all $1 \le i,\ j \le gh$, and

(ii) $\sum_{k=1}^{gh} g_{ik} g_{jk}^{-1} = \sum_{a \varepsilon G} ha$ whenever $i \neq j$ where the summation is in the group ring

$R(G)$. We also write this as

$$HH^* = hG.$$

Suppose we have a matrix W with elements from an elementary abelian group $G = [h_1, h_2, ..., h_g]$, where $W = h_1A_1 + h_2A_2 + \cdots + h_gA_g$; here $A_1, \ldots, A_g$ are v × b (0,1) matrices, and the Hadamard product $A_i * A_j$ $(i \neq j)$ is zero. Suppose $(a_{i1}, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the ith and jth rows of W; then we define WW^* by

$$(WW^*)_{ij} = (a_{i1},\ldots,a_{ib}) \cdot (b_{j1}^{-1},\ldots,b_{jb}^{-1})$$

with $\cdot$ designating the scalar product. Then W is a *generalized Bhaskar Rao design* or *GBRD* if

$$\text{(i)} \quad WW^* = rI + \sum_{i=1}^{m} (c_i G) B_i$$

$$\text{(ii)} \quad N = A_1 + \cdots + A_g \text{ satisfies } NN^T = rI + \sum_{i=1}^{m} \lambda_i B_i,$$

that is, N is the incidence matrix of a $PBIBD(m)$, and $(c_i G)$ gives the number of times a complete copy of the group G occurs.

Such a matrix will be denoted by $GBRD_G(\mathrm{v,b,r,k};\lambda_1, \ldots, \lambda_m; c_1, \ldots, c_m)$. In this paper we shall only be concerned with $m = 1, c = \lambda/g$, and $B_1 = J - I$. In this case N is the incidence matrix of a $PBIBD(1)$, that is a $BIBD$. Hence, the equations become:

$$\text{(i)} \quad WW^+ = rI + \lambda G/g \ (J - I)$$

$$\text{(ii)} \quad NN^T = (r - \lambda)I + \lambda J.$$

Thus W is a $GBRD_G(\mathrm{v,b,r,k},\lambda)$. Since $\lambda(\mathrm{v}-1) = r(k-1)$ and $bk = \mathrm{v}r$, we sometimes use the notation $GBRD(\mathrm{v,k},\lambda;\mathrm{G})$.

2. THE CONSTRUCTION

In his 1975 paper, Vanstone gave a powerful method for constructing BIBD from SBIBDs. We show his method applies to symmetric GBRD over groups which have no elements of order 2.

THEOREM 1. *Suppose there is a symmetric* GBRD(v, k, λ; G), | G | *odd , then there is a* GBRD(v, $\binom{\mathrm{v}}{2}$, $\binom{\mathrm{k}}{2}$, λ, $\binom{\lambda}{2}$; G).

Proof: We modify the construction Vanstone used to show that an SBIBD(v, k, λ) yields a BIBD(v, $\binom{\mathrm{v}}{2}$, $\binom{\mathrm{k}}{2}$, λ, $\binom{\lambda}{2}$).

Let $A = (a_{ij})$ be the incidence matrix of the GBRD(v, k, λ; G). Label the columns of a v × $\binom{\mathrm{v}}{2}$ matrix $B = (b_{ij})$, with the n = $\binom{\mathrm{v}}{2}$ pairs from the set {1,...,v}.

Consider the column labelled xy, $(b_{1k}, \cdots, b_{vk})^T$, choose

$$b_{ik} = a_{ix}a_{iy}\ ,\ i = 1,...,v.$$

Clearly, every element of B is zero or a group element, as that was true of A.

To establish the inner product property, we consider the inner product of two distinct rows

$$\sum_{k=1}^{n} b_{ik}b_{jk}^{-1} = \sum_{1 \le x < y \le n} a_{ix}a_{iy}a_{jy}^{-1}a_{ix}^{-1}\ ,\ i \ne j.$$

We first note that, for any group G of order g with elements $g_1, g_2, \cdots, g_g$

$$\begin{aligned} G^2 &= (g_1 + g_2 + \cdots + g_g)^2 \\ &= gG. \end{aligned}$$

With $(G + \cdots G)$ denoting t copies of G,

$$(G+G+...+G)^2 = tG^2 + 2\binom{t}{2}G^2 = t^2gG.$$

Since g is odd and $n = v = tg$, if $g_1, \cdots, g_v$ are the elements of a row of the GBRD, $g_1^2, ..., g_v^2 = tG$.

Hence, noting

$$(\sum x_i)^2 = \sum x_i^2 + 2\sum_{i \ne j} x_i x_j,$$

$$\begin{aligned} \sum_{1 \le x < y \le n} a_{ix}a_{iy}a_{jx}^{-1}a_{jy}^{-1} &= \tfrac{1}{2}\left[\sum_{k=1}^{n} a_{ik}a_{jk}^{-1}\right]^2 - \tfrac{1}{2}\sum_{k=1}^{n}(a_{ik}a_{jk}^{-1})^2 \\ &= \tfrac{1}{2}(G+G+...+G)^2 - \tfrac{1}{2}tG \qquad (t\ copies) \\ &= \tfrac{1}{2}(t^2g - t)G. \end{aligned}$$

Now, we know from Vanstone's result that a BIBD(v,k,λ) gives a BIBD(v, $\binom{v}{2}$, $\binom{k}{2}$, λ, $\binom{\lambda}{2}$). Thus, we wish to show a GBRD(v, k, λ; G) gives a GBRD(v, $\binom{v}{2}$, $\binom{k}{2}$, λ, $\binom{\lambda}{2}$; G) . Certainly, the underlying BIBD has these parameters. The GBRD(v, k, λ; G) has $t = \lambda / g$ copies of the group as the inner product of each pair of rows and the constructed GBRD needs to have $\binom{\lambda}{2}$ / g copies of the group as the inner product of each pair of rows. But

$$\binom{\lambda}{2} / g = \tfrac{1}{2}\lambda(\lambda-1) / g = \tfrac{1}{2}t(tg-1)$$

as required. □

Example 1. Let the group of order 3, $\mathbf{Z}_3$, have generator ω. Represent ω by 1, ω^2 by 2 and ω^3 by 0. Then, the GH(6,$\mathbf{Z}_3$) or GBRD(6, 6, 6;$\mathbf{Z}_3$) is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix},$$

yielding

12	13	14	15	16	23	24	25	26	34	35	36	45	46	56
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	2	1	1	2	2	1	0	0	2	1	0	0
1	0	1	2	2	1	2	0	0	1	2	2	0	0	1
2	1	0	1	2	0	2	0	1	1	2	0	1	2	0
2	2	1	0	1	1	0	2	0	0	2	0	1	2	1
1	2	2	1	0	0	0	2	1	1	0	2	0	2	1

a GBRD(6,15,15,6,15;Z_3).

Example 2. Proceed as in Example 1, but represent the zero element by *. Then the GBRD(5,4,3;Z_3)

*	0	0	0	0
0	*	0	1	2
0	0	*	2	1
0	1	2	*	0
0	2	1	0	*

yields the GBRD(5,10,6,3,3;Z_3) :

12	13	14	15	23	24	25	34	35	45
*	*	*	*	0	0	0	0	0	0
*	0	1	2	*	*	*	1	2	0
0	*	2	1	*	2	1	*	*	0
1	2	*	0	0	*	1	*	2	*
2	1	0	*	0	2	*	1	*	*

This method is so powerful when applied to generalized Hadamard matrices that we give it as a theorem in its own right.

3. USING GENERALIZED HADAMARD MATRICES IN THE CONSTRUCTION TO FORM BIBDS

THEOREM 2. *Suppose there is a* GH(tg;G), $|G| = g$ *odd. Then there is a* GBRD(tg, $\left[\frac{tg}{2}\right]$, $\left[\frac{tg}{2}\right]$, tg, $\left[\frac{tg}{2}\right]$; G). *This can be used to form a*

$$\text{GDD}(\, g(tg+1),\ g\left[\tfrac{tg}{2}\right],\ \left[\tfrac{tg}{2}\right],\ tg+1,\ \lambda_1 = 0,\ \lambda_2 = \tfrac{1}{2}t(tg-1),\ m = g,\ n = tg+1).$$

COMMENT. The following construction is valid for any GH$(2|G|;G)$ but these are presently only known for prime power orders $|G|$. The BIBD's constructed would be multiples of biplanes SBIBD$(2p^2+p+1, 2p+1, 2)$ but these are not generally known as yet.

THEOREM 3. *Let p be any prime power. Then there exists a* BIBD$(2p^2+p+1, p(2p^2+p+1), p(2p+1), 2p+1, 2p)$.

Proof. We note a GH$(2p, EA(p))$ exists for every prime power (Jungnickle (1979), D.J.Street (1979)). Use Theorem 2 to form a GBRD$(2p, p(2p-1), p(2p-1), 2p, p(2p-1); EA(p))$. We replace each element of the GBRD by its $p \times p$ permutation matrix representation to obtain a (0,1) matrix B. Let e be the $1 \times p(2p-1)$ matrix of ones. Then

$$A = \begin{bmatrix} I_p \times e \\ B \end{bmatrix}$$

is a GDD$(2p^2+p, p^2(2p-1), p(2p-1), 2p+1, \lambda_1 = 0, \lambda_2 = (2p-1))$.

Now a BIBD$(2p+1, p(2p+1), 2p, 2, 1)$ exists. Let C be the matrix obtained from this BIBD by replacing each 1 and 0 in its incidence matrix by the $p \times 1$ matrices of ones and zeros respectively. Then the matrix

$$[C{:}A]$$

has $2p^2+p$ rows, $2p^3+p^2+p$ columns, $2p^2+p$ ones per row, $2p$ or $2p+1$ ones per column and inner product $2p$. So if we let f be a $1 \times p(2p+1)$ matrix of ones

$$\begin{bmatrix} f & 0 \\ C & A \end{bmatrix}$$

is a BIBD$(2p^2+p+1, p(2p^2+p+1), p(2p+1), 2p+1, 2p)$. □

COROLLARY 4. *Let p be any prime power and q any integer. Then there exists a* PBIBD$(2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p-1), \lambda_2 = 2p-1+q)$.

Proof: As in the proof of Theorem 3, we use the GH(2p, EA(p)) to first form a

GBRD(2p, p(2p-1), p(2p-1), 2p, p(2p-1); EA(p)).

This then yields a

GDD$(2p^2, p^2(2p-1), p(2p-1), 2p, \lambda_1 = 0, \lambda_2 = 2p-1)$,

A. Form C as before from a

BIBD(2p, qp(2p-1), q(2p-1), 2, q).

Then [C:A] is a

PBIBD$(2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p-1), \lambda_2 = 2p-1+q)$. □

Example 3. A GH(6,EA(3)) exists so there is a GBRD(6,10,10,6,10;EA(3)). This can be used with a BIBD(7,21,6,2,1) to form a BIBD(22,66,21,7,6).

Example 4. A GH(18,EA(9)) exists, so there is a GBRD(18, 153, 153, 18, 153; EA(9)). This is used with a BIBD(19, 171, 18, 2, 1) to form a BIBD(172, $9 \cdot 172$, 171, 19, 18).

All the following constructions can be obtained by a similar, slightly modified, technique.

THEOREM 5. *Suppose there exists a* GH(tg,G), $g = |G|$ *odd. Further suppose that there exists a* BIBD($tg+1$, $s(tg+1)$, ts, t, λ). *Then there exists a* BIBD(tg^2+g+1, $\alpha s(tg^2+g+1)$, $\alpha s(tg+1)$, $tg+1$, αst) *where* $s = \lambda g/(t-1)$ *is an integer and* $2\alpha\lambda(tg-t+1) = \beta t(tg-1)(t-1)$ *for some* α *and* β. *In particular, if* $\alpha = \binom{tg}{2}$ *and* $\beta = tg - t + 1$, *there is a*

$$\text{BIBD}(tg^2+g+1,\ s(tg^2+g+1)\tbinom{tg}{2}, s(tg+1)\tbinom{tg}{2},\ tg+1, st\tbinom{tg}{2}).$$

Proof: From theorem 1, there exists a GBRD(tg, $\binom{tg}{2}$, $\binom{tg}{2}$, tg, $\binom{tg}{2}$; G). We replace each element of G by its $p\times p$ permutation matrix to form a (0,1) matrix E. Further, let e be the $1\times\binom{tg}{2}$ matrix of ones. Then,

$$B = \begin{bmatrix} I_g \times e \\ E \end{bmatrix}$$

is a GDD($g(tg+1)$, $g\binom{tg}{2}$, $\binom{tg}{2}$, $tg+1$, $\lambda_1 = 0$, $\lambda_2 = \frac{1}{2}t(tg-1)$, $m = g$, $n = tg+1$). We now replace each 0 and 1 of the

$$\text{BIBD}(tg+1,\ \lambda g(tg+1)/(t-1),\ \lambda tg/(t-1),\ t,\ \lambda)$$

by the $g\times 1$ matrix of zeros and ones respectively to form a GDD($g(tg+1)$, $\lambda g(tg+1)/(t-1)$, $\lambda tg/(t-1)$, tg, $\lambda_1 = \lambda tg/(t-1)$, $\lambda_2 = \lambda$, $m = g$, $n = tg+1$), A.

We now form the following (0,1) matrix:

$$C = \begin{bmatrix} 11\cdots 11 & | & 00\cdots 00 \\ \alpha\, copies A & | & \beta\, copies B \end{bmatrix}$$

The first row of C has $\alpha\lambda g(tg+1)/(t-1)$ ones and has intersection $\alpha\lambda tg/(t-1)$ with the other rows of C.

Every other row of C has $\alpha\lambda tg/(t-1) + \beta\frac{1}{2}tg(tg-1)$ ones. So we require

$$\alpha\lambda g(tg+1)/(t-1) = \alpha\lambda tg/(t-1) + \beta\tfrac{1}{2}tg(tg-1)$$

or

$$\alpha\lambda = \beta\tfrac{1}{2}t(tg-1)(t-1)/(tg-t+1) \qquad (1)$$

The intersection numbers for the rows are required to be equal, so we need

$$\alpha\lambda tg \,/\, (t-1) = \alpha\lambda tg \,/\, (t-1) + \beta\cdot 0 = \alpha\lambda + \beta \tfrac{1}{2} t(tg-1)$$

or, as in (1)

$$\alpha\lambda = \beta \tfrac{1}{2} t(tg-1)(t-1) \,/\, (tg-t+1).$$

Thus C is a BIBD(tg^2+g+1, $\alpha\lambda g(tg^2+g+1)/(t-1)$, $\alpha\lambda g(tg+1)/(t-1)$, $tg+1$, $\alpha\lambda tg/(t-1)$). Where $\lambda g/(t-1) = s$ an integer, and a possible solution for α and β is $\alpha = \left[\frac{tg}{2}\right]$, $\beta = s(tg-t+1)$. That is, C is a

$$\text{BIBD}(tg^2+g+1,\, s(tg^2+g+1)\left[\tfrac{tg}{2}\right], s(tg+1)\left[\tfrac{tg}{2}\right],\, tg+1, st\left[\tfrac{tg}{2}\right]).$$ □

COROLLARY 6. *Let g and g-1 be prime powers, g odd. If there exists a* BIBD(g^2-g+1, $g(g^2-g+1)$, $g(g-1)$, $g-1$, $g-2$) *then there exists a* BIBD(g^3-g^2+g+1, $\alpha g(g^3-g^2+g+1)$, $\alpha g(g^2-g+1)$, g^2-g+1, $\alpha g(g-1)$), *where* $2\alpha(g^2-2g+2) = \beta(g-1)(g^2-g-1)$ *has an integer solution.*

Proof: By a Theorem of Rajkundlia (1978) and Seberry (1981), a GH(g(g-1);EA(g)) always exists in these cases. □

Remark. The BIBD obtained would be a multiple of an SBIBD($g^3-g^2+g+1, g^2-g+1, g-1$) which theoretically, can never exist, as g^3-g^2+g+1 is even and $k-\lambda = g^2-2g+2$ is not a square.

Example 5. Let $g=5$. There exists a BIBD(21,105,20,4,3). Hence, there exists a BIBD(106, 38·5·106, 38·5·21, 21, 38·5·4), $\alpha = 38$. This is a multiple of the SBIBD(106,21,4) which is non-existent.

COROLLARY 7. *Let g be an odd prime power. Let* $\alpha = 2(4g-1)$. *Then there is a* BIBD($4g^2+g+1$, $2g^2(4g^2+g+1)(4g-1)$, $2g^2(4g+1)(4g-1)$, $4g+1$, $8g^2(4g-1)$).

Proof: Dawson (1985) has shown a GH(4g,(EA(g)) always exists. Also, the required BIBD($4g+1$, $g(4g+1)$, $4g$, 4, 3) always exists and so, with $\alpha = 2(4g-1)$, $\beta = 4g-3$ in Theorem 5, we get the result. □

Remark. This would be a multiple of the SBIBD($4g^2+g+1$, $4g+1$, 4) but this can only exist (since $4g^2+g+1$ is even) if $k-\lambda = 4g-3$ is a square.

Example 6. Let $g=9$. Then $\alpha = 70$ and a BIBD(334,70·9·334,70·9·37,37,36·70) exists.

COROLLARY 8. *Let* $g = 3^h$. *Then there exists* BIBD($4g^2+g+1$, $\alpha\lambda g(4g^2+g+1)/3$, $\alpha\lambda g(4g+1)/3$, $4g+1$, $4\alpha\lambda g/3$) *where* $2\alpha\lambda(4g-3) = 12\beta(4g-1)$ *for some* α *and* β. *In particular, if* $\alpha\lambda = 2(4g-1)$

and $\beta = (4g-3)/3$, *there is a*
BIBD$(4g^2+g+1,\ 2g(4g-1)(4g^2+g+1)/3, 2g(4g+1)(4g-1)/3,\ 4g+1, 8g(4g-1)/3)$.

Proof: We again use the GH(4g, EA(g)) found by Dawson (1985). We note that a BIBD$(4g+1,\ \lambda g(4g+1)/3, 4\lambda g/3,\ 4, \lambda)$ exists for all λ. We use these in Theorem 5 to get the result. □

Remark. The constructed designs are also multiples of an SBIBD$(4g^2 + g + 1, 4g + 1, 4)$ which never exists as $4g^2 + g + 1$ is even and $k - \lambda = 4g - 3$ is never a square for $g = 3^h, h>1$.

COROLLARY 9. *Let p be an odd prime power. Suppose there exists a* BIBD$(p^i + 1,\ qp^j(p^i + 1),\ qp^i,\ p^{i-j},\ q(p^{i-j} - 1))$ *where* $i \geq j$, *and q are integers. Then there exists a*

$$\text{BIBD}(p^{i+j} + p^j + 1,\ \alpha qp^j(p^{i+j} + p^j + 1),\ \alpha qp^j(p^i + 1),\ p^i + 1,\ \alpha qp^i))$$

where $2\alpha q(p^i - p^{i-j} + 1) = \beta p^{i-j}(p^i - 1)$, *there is a*

$$\text{BIBD}(p^{i+j} + p^j + 1,\ p^i(p^i - 1)(p^{i+j} + p^j + 1),\ p^i(p^{2i} - 1),\ p^i + 1,\ p^{2i-j}(p^i - 1)).$$

Proof: Use the GH$(p^i, \text{EA}(p^j))$, $i > j$ given by Drake (1979) or Butson (1963). □

Remark. This would be a multiple of the SBIBD$(p^{i+j}+p^j+1, p^i+1, p^{i-j})$. Since $p^{i+j}+p^j+1$ is odd, in order for this to exist, the diophantine equation

$$z^2 = (p^i - p^{i-j} + 1)x^2 + (-1)^{½(p^i+1)}p^{i-j}y^2$$

must have a solution in the integers for x, y, z not all zero.

Example 7. Let i = 2, j = 1, q = 1 and p = 5. A BIBD(26, 130, 25, 5, 4) exists. Hence a BIBD(131, 600·131, 600·26, 26, 600·5) exists.

4. USING GENERALIZED WEIGHING MATRICES IN THE CONSTRUCTION

As noted in Seberry (1979), and Geramita and Seberry (1979), infinite families of GW matrices are known.

THEOREM 10. *Let* p^r *be a prime power and* $q \mid p^r-1$, *q odd. Then there exists a*

$$\text{GBRD}(p^r+1, ½p^r(p^r+1), ½p^r(p^r-1), p^r-1, ½(p^r-1)(p^r-2); Z_q)$$

and B, a GDD with parameters
$(q(p^r+1), ½qp^r(p^r+1), ½p^r(p^r-1), p^r-1, \lambda_1=0, \lambda_2=½(p^r-1)(p^r-2)/q, m=q, n=p^r+1)$.
Hence if, $q = p^r - 1$, *there exists a*

$$BIBD(p^{2r}-1, ½(p^{2r}-2)(p^r+1), ½(p^{2r}-2), p^r-1, ½(p^r-2)).$$

If $q \mid p^r - 1$, $q \neq p^r - 1$ *and there exists a BIBD* $(p^r + 1, b, \rho, (p^r - 1)/q, \lambda)$, *A, where*

$\lambda q p^r = \rho(p^r - q - 1)$. Using A to form a

$$GDD\,(q\,(p^r + 1),\ qb,\ \rho,\ p^r - 1, \lambda_1 = r, \lambda_2 = \lambda)$$

then

$$[\alpha \textit{ copies of } A : \beta \textit{ copies of } B\ ]$$

where $2q\alpha(\rho - \lambda) = \beta(p^r - 1)(p^r - 2)$ gives a BIBD $(q\,(p^r + 1),\ B,\ R,\ p^r - 1, \alpha\rho)$.

Proof: We note first that a $GW(p^r{+}1, p^r, p^r{-}1; Z_q)$ exists for all p^r. The proof, then, is identical to the first part of the proof of Theorem 3. □

Example 8. A $GW(17, 16, 15; Z_q)$, $q = 15, 5$ and 3 exists. This gives a BIBD(15·17, 127·17, 127, 15, 7). Also, we have GDD(5·17, 40·17, 120, 15, $\lambda_1 = 0$, $\lambda_2 = 21$, m = 5 , n = 17) and a GDD(3·17, 24·17, 120, 15, $\lambda_1 = 0$, $\lambda_2 = 35$, m = 3, n = 17). Since BIBD(17, 8·17, 24, 3, 3) and BIBD(17, 4·17, 20, 5, 5) exist, we have a BIBD(85, 34·24, 6·24, 15, 24) with q = 5, r = 24, $\lambda = 3$, $\alpha = \beta = 1$ and a BIBD(51, 1700, 500, 15, 140) with q = 3, r = 20, $\lambda = 5$, $\alpha = 7$, and $\beta = 3$.

Example 9.

We note that there exists a

$$GW((p^{n+1} - 1)/(p - 1), p^n; Z_q)$$

for all $q \mid p^n(p - 1)$. So we can choose q odd and proceed as in the previous theorem. We do not give full results but note some examples: the GW(21, 16; Z_3) gives a GBRD(21, 12, 66; Z_3) and a BIBD(63, 12, 22·60), the GW(31, 25; Z_5) gives a GBRD(31, 20, 190; Z_5) and a BIBD(155, 20, 19·20), and the GW(85, 64; Z_3) gives a GBRD(85, 48, 24·47; Z_3) and a BIBD(255, 24, 94·336).

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