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VANSTONE'S CONSTRUCTION APPLIED TO BHASKAR RAO DESIGNS

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ABSTRACT

We show how Vanstone's construction, given in his paper "A note on a construction for BIBD's", *Utilitas Mathematica*, 7(1975), 321-322, can be applied to symmetric $\text{GBRD}(v, k, \lambda; | G |, | G |, G)$ and odd, can be used to obtain $\text{GBRD}(v, \lfloor \frac{v}{2} \rfloor, \lfloor \frac{k}{2} \rfloor, \lambda; G)$ and hence many families of BIBD.

1. INTRODUCTION

Definitions of SBIBD and BIBD are standard.

Let $A = [a_{ij}]$ be a matrix of order with $a_{ij} \in \{0, 1, -1\}$. $A$ is called a *weighing matrix* of weight $p$ and order $n$, if $AA^T = A^TA = pI_n$, where $I_n$ denotes the identity matrix of order $n$. Such a matrix is denoted by $W(n, p)$. If squaring all its entries gives an incidence matrix of a SBIBD then $W$ is called a balanced weighing matrix.

An *Hadamard matrix*, $A = [a_{ij}]$, is a $W(n, n)$, that is, it is a square matrix of order $n$ with entries $a_{ij} \in \{1, -1\}$ which satisfies $AA^T = A^TA = nI_n$.

A *generalized Hadamard matrix* $GH(gh, G) = (g_{ij}) = H$ over the group $G$ of order $g$ is a $gh \times gh$ matrix such that

(i) $g_{ij} \in G$ for all $1 \leq i, j \leq gh$.

(ii) $\sum_{k=1}^{g} g_{ik}g_{jk}^{-1} = \sum_{a \in G} a$ whenever $i \neq j$ where the summation is in the group ring $R(G)$. We also write this as $Hh^* = hG$.

Suppose we have a matrix $W$ with elements from an elementary abelian group $G = \langle h_1, h_2, \ldots, h_g \rangle$, where $W = h_1A_1 + h_2A_2 + \cdots + h_gA_g$, here $A_1, \ldots, A_g$ are

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v \times b \, (0,1) \, \text{matrices, and the Hadamard product } A_i^* A_j (i \neq j) \text{ is zero. Suppose } (a_{i1}, \ldots, a_{ib}) \text{ and } (b_{j1}, \ldots, b_{jb}) \text{ are the } i \text{th and } j \text{th rows of } W; \text{ then we define } WW^* \text{ by }

(WW^*)_{ij} = (a_{i1}, \ldots, a_{ib}) \cdot (b_{j1}, \ldots, b_{jb})

\text{with } \cdot \text{ designating the scalar product. Then } W \text{ is a generalized Bhaskar Rao design or GBRD if }

(i) \quad WW^* = rl + \sum_{i=1}^{m} (c_i; G) B_i

(ii) \quad N = A_1 + \cdots + A_\ell \text{ satisfies } NN^T = rl + \sum_{i=1}^{m} \lambda_i B_i,

\text{that is, } N \text{ is the incidence matrix of a } PBIBD(m), \text{ and } (c_i; G) \text{ gives the number of times a complete copy of the group } G \text{ occurs.}

\text{Such a matrix will be denoted by } GBRD_G(v,b,r,k; \lambda_1, \ldots, \lambda_\ell; c_1, \ldots, c_\ell). \text{ In this paper we shall only be concerned with } m = 1, c = \lambda \, g, \text{ and } B_1 = J - I. \text{ In this case } N \text{ is the incidence matrix of a } PBIBD(1), \text{ that is a } BIBD. \text{ Hence, the equations become:}

(i) \quad WW^* = rl + \lambda(G)\, (J - I)

(ii) \quad NN^T = (r - \lambda)I + \lambda J.

Thus } W \text{ is a } GBRD_G(v,b,r,k; \lambda). \text{ Since } \lambda(v-1) = r(k-1) \text{ and } bk = vr, \text{ we sometimes use the notation } GBRD(v,k;\lambda,G).

2. THE CONSTRUCTION

In his 1975 paper, Vanstone gave a powerful method for constructing BIBD from SBIBDs. We show his method applies to symmetric GBRD over groups which have no elements of order 2.

THEOREM 1. Suppose there is a symmetric GBRD(v, k, \lambda; G), \mid G \mid \text{ odd}, then there is a GBRD(v, \begin{bmatrix} x \end{bmatrix}, \begin{bmatrix} k \end{bmatrix}, \lambda, \begin{bmatrix} x \end{bmatrix}; G).}

\text{Proof: } \text{We modify the construction Vanstone used to show that an SBIBD(v, k, \lambda) yields a } BIBD(v, \begin{bmatrix} x \end{bmatrix}, \begin{bmatrix} k \end{bmatrix}, \lambda, \begin{bmatrix} x \end{bmatrix}).

\text{Let } A = (a_{ij}) \text{ be the incidence matrix of the } GBRD(v, k, \lambda; G). \text{ Label the columns of a } v \times \begin{bmatrix} x \end{bmatrix} \text{ matrix } B = (b_{ij}), \text{ with the } n = \begin{bmatrix} x \end{bmatrix} \text{ pairs from the set } \{1, \ldots, v\}.\]
Consider the column labelled \( xy \), \( (b_1, \ldots, b_v)^T \), choose

\[
b_{ik} = a_{i,x}a_{j,y}, \quad i = 1, \ldots, v.
\]

Clearly, every element of \( B \) is zero or a group element, as that was true of \( A \).

To establish the inner product property, we consider the inner product of two distinct rows

\[
\sum_{k=1}^{n} b_{ik}b_{jk} = \sum_{1 \leq x < y \leq n} a_{i,x}a_{j,y}^{-1}a_{i,x}^{-1}, \quad i \neq j.
\]

We first note that, for any group \( G \) of order \( g \) with elements \( g_1, g_2, \ldots, g_v \)

\[
G^2 = (g_1 + g_2 + \cdots + g_v)^2 = gG.
\]

With \((G + \cdots + G)^t\) denoting \( t \) copies of \( G \),

\[
(G + G + \cdots + G)^t G^2 = tG^2 + 2 \frac{t}{2} G^2 = t^2 gG.
\]

Since \( g \) is odd and \( n = v = tg \), if \( g_1, \ldots, g_v \) are the elements of a row of the GBRD, \( g_1^2, \ldots, g_v^2 = tG \).

Hence, noting

\[
(\sum x_i)^2 = \sum x_i^2 + 2 \sum x_i x_j,
\]

\[
\sum_{1 \leq x < y \leq n} a_{i,x}a_{y,y}^{-1}a_{j,x}^{-1} = \frac{1}{t!} \left[ \sum_{k=1}^{n} a_{i,k}a_{j,k}^{-1} \right]^2 - \frac{1}{t!} \sum_{k=1}^{n} (a_{i,k}a_{j,k}^{-1})^2
\]

\[
= \frac{1}{t!}(G + G + \cdots + G)^t - \frac{1}{t!} G = \frac{1}{t!(t^2 g - t)G}.
\]

Now, we know from Vanstone's result that a BIBD(v,k,λ) gives a BIBD(v, [3], [3], [3]; G). Thus, we wish to show a GBRD(v, k, λ; G) gives a GBRD(v, [3], [3], [3]; G). Certainly, the underlying BIBD has these parameters. The GBRD(v, k, λ; G) has \( t = \lambda / g \) copies of the group as the inner product of each pair of rows and the constructed GBRD needs to have \( [3] / g \) copies of the group as the inner product of each pair of rows. But

\[
[3] / g = \frac{\lambda}{t}(\lambda - 1)/g = \frac{\lambda}{t}(tg - 1)
\]

as required.

Example 1. Let the group of order 3, \( Z_3 \), have generator \( \omega \). Represent \( \omega \) by 1, \( \omega^2 \) by 2 and \( \omega^3 \) by 0. Then, the GHR(6, Z_3) or GBRD(6, 6, 6; Z_3) is
yielding

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 \\
0 & 2 & 1 & 0 & 1 \\
0 & 1 & 2 & 2 & 1
\end{bmatrix}
\]
a GBRD(6,15,15,6.15;Z3).

Example 2. Proceed as in Example 1, but represent the zero element by *. Then the GBRD(5,4,3;Z3)

\[
\begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
0 & * & 0 & 1 & 2 \\
0 & 0 & * & 2 & 1 \\
0 & 1 & 2 & * & 0 \\
0 & 2 & 1 & 0 & *
\end{bmatrix}
\]
yields the GBRD(5,10,6,3,3;Z3):

\[
\begin{bmatrix}
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 1 & 2 & * & * & 1 & 2 & 0 \\
0 & * & 2 & 1 & * & 2 & 1 & * & 0 \\
1 & 2 & * & 0 & 0 & * & 1 & * & 2 \\
2 & 1 & 0 & * & 0 & 2 & * & 1 & *
\end{bmatrix}
\]

This method is so powerful when applied to generalized Hadamard matrices that we give it as a theorem in its own right.

3. USING GENERALIZED HADAMARD MATRICES IN THE CONSTRUCTION TO FORM BIBDS

**THEOREM 2.** Suppose there is a GH(\(tg;G\), \(|G| = g\) odd. Then there is a GBRD( \(tg, \left[\frac{g}{2}\right], \left[\frac{g}{2}\right], \left[\frac{g}{2}\right]; G\)). This can be used to form a

\(GDD( g(tg + 1), g \left[\frac{g}{2}\right], \left[\frac{g}{2}\right], \left[\frac{g}{2}\right], \left[\frac{g}{2}\right]; tg + 1, \lambda_1 = 0, \lambda_2 = \frac{gt(g - 1)}{2}, m = g, n = tg + 1)\).
COMMENT. The following construction is valid for any \( GH(2 \mid G \mid; G) \) but these are presently only known for prime power orders \( |G| \). The BIBD's constructed would be multiples of biplanes \( SBIBD(2p^2 + p + 1, 2p + 1, 2) \) but these are not generally known as yet.

**Theorem 3.** Let \( p \) be any prime power. Then there exists a BIBD\((2p^2 + p + 1, p(2p^2 + p + 1), p(2p + 1), 2p + 1, 2p)\).

*Proof.* We note a \( GH(2p, EA(p)) \) exists for every prime power (Jungnickle (1979), D.J.Street (1979)). Use Theorem 2 to form a GBRD\((2p, p(2p - 1), p(2p - 1), 2p, p(2p - 1); EA(p))\). We replace each element of the GBRD by its \( p \times p \) permutation matrix representation to obtain a \((0,1)\) matrix \( B \). Let \( e \) be the \( 1 \times p\) matrix of ones. Then

\[
    A = \begin{bmatrix}
        I_p \times e \\
        B
    \end{bmatrix}
\]

is a GDD\((2p^2 + p, p^2(2p - 1), (2p + 1), 2p + 1, \lambda_1 = 0, \lambda_2 = (2p - 1))\).

Now a BIBD\((2p + 1, p(2p + 1), 2p, 2, 1) \) exists. Let \( C \) be the matrix obtained from this BIBD by replacing each 1 and 0 in its incidence matrix by the \( p \times 1 \) matrices of ones and zeros respectively. Then the matrix

\[
    \begin{bmatrix}
        C & A
    \end{bmatrix}
\]

has \( 2p^2 + p \) rows, \( 2p^3 + p^2 + p \) columns, \( 2p^2 + p \) ones per row, \( 2p \) or \( 2p + 1 \) ones per column and inner product \( 2p \). So if we let \( f \) be a \( 1 \times p(2p + 1) \) matrix of ones

\[
    f = \begin{bmatrix}
        0 \\
        C & A
    \end{bmatrix}
\]

is a PBIBD\((2p^2 + p + 1, p(2p^2 + p + 1), p(2p + 1), 2p + 1, 2p)\).

**Corollary 4.** Let \( p \) be any prime power and \( q \) any integer. Then there exists a PBIBD\((2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p-1), \lambda_2 = 2p-1+q)\).

*Proof.* As in the proof of Theorem 3, we use the \( GH(2p, EA(p)) \) to first form a GBRD\((2p, p(2p-1), p(2p-1), 2p, p(2p-1); EA(p))\). This then yields a GDD\((2p^2, p^2(2p-1), (2p-1), 2p, \lambda_1 = 0, \lambda_2 = 2p-1)\).

A. Form \( C \) as before from a BIBD\((2p, qp(2p-1), q(2p-1), 2, q)\). Then \( [C:A] \) is a PBIBD\((2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p-1), \lambda_2 = 2p-1+q)\). \( \square \)
Example 3. A GH(6,EA(3)) exists so there is a GBRD(6,10,10,6,10;EA(3)). This can be used with a BIBD(7,21,6,2,1) to form a BIBD(22,66,21,7,6).

Example 4. A GH(18,EA(9)) exists, so there is a GBRD(18,153,153,18,153;EA(9)). This is used with a BIBD(19,171,18,2,1) to form a BIBD(172,9·172,171,19,18).

All the following constructions can be obtained by a similar, slightly modified technique.

**Theorem 5.** Suppose there exists a GH(tg,G), $g = |G| > 3$ odd. Further suppose that there exists a BIBD$(tg + 1, s(tg + 1), t, t, \lambda)$. Then there exists a BIBD$(tg^2 + g + 1, s(tg^2 + g + 1), \alpha s(tg + 1), tg + 1, \alpha \lambda t)$ where $s = \lambda g / (t-1)$ is an integer and $2\alpha \lambda(tg-t+1) = \beta t(tg-1)(t-1)$ for some $\alpha$ and $\beta$. In particular, if $\alpha = \left[ \frac{g}{2} \right]$ and $\beta = tg - t + 1$, there is a

$$\text{BIBD}(tg^2 + g + 1, s(tg^2 + g + 1) \left[ \frac{g}{2} \right], s(tg + 1) \left[ \frac{g}{2} \right], tg + 1, st \left[ \frac{g}{2} \right]).$$

**Proof:** From theorem 1, there exists a GBRD$(tg, \left[ \frac{g}{2} \right], \left[ \frac{g}{2} \right], \left[ \frac{g}{2} \right], G)$. We replace each element of $G$ by its $p \times p$ permutation matrix to form a $(0,1)$ matrix $E$. Further, let $e$ be the $1 \times \left[ \frac{g}{2} \right]$ matrix of ones. Then,

$$B = \left[ \begin{array}{c} I \times e \\ E \end{array} \right]$$

is a GDD$(g(tg + 1), g \left[ \frac{g}{2} \right], \left[ \frac{g}{2} \right], tg + 1, \lambda_1 = 0, \lambda_2 = \lambda g / (t-1), m = g, n = tg + 1)$.

We now replace each 0 and 1 of the

$$\text{BIBD}(tg + 1, \lambda g(tg + 1) / (t-1), \lambda tg / (t-1), t, \lambda)$$

by the $g \times 1$ matrix of zeros and ones respectively to form a GDD$(g(tg + 1), \lambda g(tg + 1) / (t-1), \lambda tg / (t-1), tg, \lambda_1 = \lambda g / (t-1), \lambda_2 = \lambda, m = g, n = tg + 1)$.

A.

We now form the following $(0,1)$ matrix:

$$C = \left[ \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \alpha \text{copies} A & \beta \text{copies} B \end{array} \right]$$

The first row of $C$ has $\alpha \lambda g(tg + 1) / (t-1)$ ones and has intersection $\alpha \lambda tg / (t-1)$ with the other rows of $C$.

Every other row of $C$ has $\alpha \lambda tg / (t-1) + \beta \lambda g(tg - 1)$ ones. So we require

$$\alpha \lambda g(tg + 1) / (t-1) = \alpha \lambda tg / (t-1) + \beta \lambda g(tg - 1)$$

or

$$\alpha \lambda = \beta \lambda g(tg - 1)(t-1) / (tg - t + 1) \quad (1)$$
The intersection numbers for the rows are required to be equal, so we need
\[ \alpha \lambda g + (t - 1) = \alpha \lambda g + (t - 1) + \beta \cdot 0 = \alpha \lambda + \beta \lambda (tg - 1) \]
or, as in (1)
\[ \alpha \lambda = \beta \lambda / (tg - 1)(t - 1) / (tg - t + 1). \]

Thus \( C \) is a BIBD((tg^2 + g + 1, \alpha \lambda g (tg^2 + g + 1)/(t-1), \alpha \lambda g (tg+1)/(t-1), \alpha \lambda g ((t+1)/t))

where \( \lambda g / (t - 1) = s \), an integer, and a possible solution for \( \alpha \) and \( \beta \) is
\[ \alpha = \left\lceil \frac{tg}{2} \right\rceil, \quad \beta = s(tg + t + 1). \]

Thus \( C \) is a BIBD((tg^2 + g + 1, s(tg + g + 1), s(tg + g + 1)/t, tg + 1, st \left\lceil \frac{tg}{2} \right\rceil).

COROLLARY 6. Let \( g \) and \( g - 1 \) be prime powers, \( g \) odd. If there exists a BIBD((g^2 - g + 1, g(g^2 - g + 1), g - 1, g - 2) then there exists a BIBD((g^3 - g^2 + g + 1, og(g^2 - g + 1) + g + 1) = \alpha g(g^2 - g + 1) + g + 1, \alpha g(g - 1)),

where \( 2\alpha(g^2 - 2g + 2) = \beta(g - 1)(g^2 - g - 1) \) has an integer solution.

Proof: By a Theorem of Rajkundlia (1978) and Seberry (1981), a GH(g(g-1);EA(g)) always exists in these cases. \( \square \)

Remark. The BIBD obtained would be a multiple of an SBIBD((g^2 - g + 1, g(g^2 - g + 1), g - 1, g - 2) which theoretically can never exist, as \( g^3 - g^2 + g + 1 \) is even and \( k - \lambda = g^2 - g + 2 \) is not a square.

Example 5. Let \( g = 5 \). There exists a BIBD(21,105,20,4,3). Hence, there exists a BIBD(106, 38·5·106, 38·5·21, 21, 38·5·4), \( \alpha = 38 \). This is a multiple of the SBIBD(106,21,4) which is non-existent.

COROLLARY 7. Let \( g \) be an odd prime power. Let \( \alpha = 2(4g-1) \). Then there is a BIBD((4g^2 + g + 1, 2g^2(4g^2 + g + 1)(4g - 1), 2g^2(4g - 1)(4g - 1), 4g + 1, 4\alpha g / 3))

Proof: Dawson (1985) has shown a GH((4g,EA(g))) always exists. Also, the required BIBD((4g + 1, g(4g + 1), 4g - 1, 4g, 4, 3) always exists and so, with \( \alpha = 2(4g - 1) \), \( \beta = 4g - 3 \) in Theorem 5, we get the result. \( \square \)

Remark. This would be a multiple of the SBIBD((4g^2 + g + 1, 4g + 1, 4) but this can only exist (since \( 4g^2 + g + 1 \) is even) if \( k - \lambda = 4g - 3 \) is a square.

Example 6. Let \( g = 9 \). Then \( \alpha = 70 \) and a BIBD(334,70·9·334,70·9·37,37,36;70) exists.

COROLLARY 8. Let \( g = 3h \). Then there exists
\[ \text{BIBD}(4g^2 + g + 1, \alpha \lambda g(4g^2 + g + 1)/3, \alpha \lambda g(4g + 1)/3, 4g + 1, 4\alpha \lambda g / 2) \]

where \( 2\alpha \lambda (4g - 3) = 12\beta(4g - 1) \) for some \( \alpha \) and \( \beta \). In particular, if \( \alpha \lambda = 2(4g - 1) \)
and \( \beta = (4g - 3)/3 \), there is a
\[ \text{BIBD}(4g^2 + g + 1, 2g(4g - 1)(4g^2 + g + 1)/3, 2g(4g + 1)(4g - 1)/3, 4g + 1, 8g(4g - 1)/3). \]

**Proof:** We again use the GH(4g, EA(g)) found by Dawson (1985). We note that a BIBD(4g + 1, \( \lambda g(4g + 1)/3, 4Ag/3, 4, \lambda \)) exists for all \( \lambda \). We use these in Theorem 5 to get the result.

**Remark.** The constructed designs are also multiples of an SBIBD(4g^2 + g + 1, 4g + 1, 4) which never exists as \( 4g^2 + g + 1 \) is even and \( k - \lambda = 4g - 3 \) is never a square for \( g = 2h, h > 1 \).

**COROLLARY 9.** Let \( p \) be an odd prime power. Suppose there exists a BIBD(p^i + 1, aqp^i(p^i + 1), aqp^i, p^i - 1, q(p^i - 1)) where \( i \geq j \), and \( q \) are integers. Then there exists a
\[ \text{BIBD}(p^i + 1 + p^j + 1, aqp^i(p^i + 1 + p^j + 1), aqp^i(p^i + 1), p^i + 1, aqp^i). \]

where \( 2aq(p^i - p^j - 1) = 6q^i(p^i - 1), \) where there is a BIBD(p^i + 1 + p^j + 1, p^i(p^i - 1)(p^j + p^j + 1), p^j(p^2j - 1), p^i + 1, p^2j - 1(p^j - 1)).

**Proof:** Use the GH(p, EA(p), i > j given by Drake (1979) or Butson (1963).

**Remark.** This would be a multiple of the SBIBD(p^i + 1 + p^j + 1, p^i + 1, p^j - 1). Since \( p^i + p^j - 1 \) is odd, for this to exist, the diophantine equation
\[ x^2 = (p^i - p^j - 1) + (-1)^{\lambda(p^i + 1)}(p^i - 1)^2 \]
must have a solution in the integers for \( x, y, z \) not all zero.

**Example 7.** Let \( i = 2, j = 1, q = 1 \) and \( p = 5 \). A BIBD(26, 130, 25, 5, 4) exists. Hence a BIBD(131, 600, 131, 600, 26, 26, 600, 5) exists.

4. USING GENERALIZED WEIGHING MATRICES IN THE CONSTRUCTION

As noted in Seberry (1979), and Geramita and Seberry (1979), infinite families of GW matrices are known.

**THEOREM 10.** Let \( p' \) be a prime power and \( q \mid p'^2 - 1, q \text{ odd}. \) Then there exists a
\[ \text{GGRD}(p'^2 + 1, \lambda p', (p'^2 + 1)/p', p'^2 - 1, \lambda(p'^2 - 1)(p'^2 - 2)/q, \lambda_1 = 0, \lambda_2 = \lambda(p'^2 - 1)(p'^2 - 2)/q, m = q, n = p'^2 + 1). \]

Hence if, \( q = p' - 1 \), there exists a
\[ \text{BIBD}(p'^2 - 1, \lambda(p'^2 - 2)(p'^2 - 1), \lambda(p'^2 - 2), p'^2 - 1, \lambda(p'^2 - 2)). \]

If \( q \mid p' - 1, q 
eq p' - 1 \) and there exists a BIBD(p', 1, b, (p' - 1)/q, \( \lambda \)), A, where

\[ \text{272} \]
\[\lambda_qp^r = p(p^r - q - 1).\] Using \(A\) to form a

\[GDD(q(p^r + 1), qb, p, p^r - 1, \lambda_1 = r, \lambda_2 = \lambda.\]

then

\([\alpha\ copies\ of\ A : \beta\ copies\ of\ B]\)

where \(2q\alpha(p - \lambda) = \beta(p^r - 1)(p^r - 2)\) gives a BIBD \((q(p^r + 1), B, R, p^r - 1, \alpha p).\)

Proof: We note first that a GW \((p^r + 1, p^r, p^r - 1; Z_q)\) exists for all \(p^r.\) The proof, then, is identical to the first part of the proof of Theorem 3.

Example 8. A GW(17, 16, 15; \(Z_q\), \(q = 15, 5\) and 3 exists. This gives a BIBD(15, 17, 127, 17, 12, 15, 7). Also, we have GDD(3, 17, 127, 12, 15, 7, \(m = 5, n = 17\)) and a GDD(2, 17, 24, 12, 15, \(m = 2, n = 17\)). Since BIBD(17, 8, 12, 23, 3, 3) and BIBD(17, 4, 17, 20, 5, 5) exist, we have a BIBD(85, 34, 24, 24, 15, 24) with \(q = 5, r = 24, \lambda = 3, \alpha = \beta = 1\) and a BIBD(51, 1700, 500, 15, 140) with \(q = 3, r = 20, \lambda = 5, \alpha = 7, \text{and } \beta = 3.\)

Example 9.

We note that there exists a GW \((p^r + 1, p^r, p^r - 1; Z_q)\) for all \(q\) and \(p.\) So we can choose \(q\) odd and proceed as in the previous theorem. We do not give full results but note some examples: the GW(21, 16; \(Z_3\)) gives a GBRD(21, 12, 66; \(Z_3\)) and a BIBD(63, 22, 60), the GW(31, 25; \(Z_5\)) gives a GBRD(31, 20, 190; \(Z_5\)) and a BIBD(155, 20, 19, 20), and the GW(85, 64; \(Z_3\)) gives a GBRD(85, 48, 24, 47; \(Z_3\)) and a BIBD(255, 24, 94, 336).

5. REFERENCES


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