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We extend a method of Kharaghani and obtain some new constructions for weighing matrices and orthogonal designs. In particular we show that if there exists an OD$(s_1,...,s_r)$, where $w = \sum s_i$, of order $n$, then there exists an OD$(s_1w,s_2w,...,s_rw)$ of order $n(n+k)$ for $k \geq 0$ an integer. If there is an OD$(t,t,t)$ in order $n$, then there exists an OD$(12t,12t,12t,12t)$ in order $12n$. If there exists an OD$(s,s,s,s)$ in order $4s$ and an OD$(t,t,t,t)$ in order $4t$ there exists an OD$(12s^2,t,12s^2t,12s^2t,12s^2t)$ in order $48s^2t$ and an OD$(20s^2,t,20s^2t,20s^2t,20s^2t)$ in order $80s^2t$.

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A Note on Orthogonal Designs

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ABSTRACT

We extend a method of Kharaghani and obtain some new constructions for weighing matrices and orthogonal designs. In particular we show that if there exists an OD($s_1,...,s_r$), where $w = \sum s_i$, of order $n$, then there exists an OD($s_1w, s_2w, ..., s_rw$) of order $n(n+k)$ for $k \geq 0$ an integer.

If there is an OD($t,t,t,t$) in order $n$, then there exists an OD($12t,12t,12t,12t$) in order $12n$. If there exists an OD($s,s,s,s$) in order $4s$ and an OD($t,t,t,t$) in order $4t$ there exists an OD($12s^2,12s^2,12s^2,12s^2$) in order $48s^2t$ and an OD($20s^2t,20s^2t,20s^2t,20s^2t$) in order $80s^2t$.

1. Introduction.

Let $W = [w_{ij}]$ be a matrix of order $n$ with $w_{ij} \in \{0,1,-1\}$. $W$ is called a weighing matrix of weight $p$ and order $n$, if $WW^T = W^TW = pI_n$, where $I_n$ denotes the identity matrix of order $n$. Such a matrix is denoted by $W(n,p)$. If squaring all its entries gives an incidence matrix of a SBIBD then $W$ is called a balanced weighing matrix.

An orthogonal design (OD), $A$, say, of order $n$ and type $(s_1,s_2,...,s_l)$ on the commuting variables $(\pm x_1, ... , \pm x_t)$ and 0, is a square matrix of order $n$ with entries from $(\pm x_1, ... , \pm x_t)$ and 0. Each row and column of $A$ contains $s_k$ entries equal to $x_k$ in absolute value, the remaining entries in each row and column being equal to 0. Any two distinct rows of $A$ are orthogonal.

In other words

$$AA^T = (x_1x_2^2 + \cdots + x tx_t^2)I_n.$$ 

An Hadamard matrix $W = [w_{ij}]$ is a $W(n,n)$ i.e. it is a square matrix

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of order $n$ with entries $w_{ij} \in \{1,-1\}$ which satisfies
\[ WW^T = W^T W = nI_n \]

OD's have been used to construct new Hadamard matrices. For details see Geramita and Seberry (1979).

Kharaghani (1985) defined $C_k = [w_{k1}, w_{k2}]$ and with that obtained skew symmetric and symmetric $W(n^2+2n,p^2)$ from $W(n,p)$, where $s$ is any positive integer such that $n + s$ is even. Each $C_k$ is a symmetric $\{0,1,-1\}$ matrix of order $n$. We define $C_k$ by the Kronecker product and by extending Kharaghani's method we obtain some new constructions of weighing matrices and orthogonal designs.

2. Some properties of $C_k$'s.

The $C_k$'s can be defined as a Kronecker product of the $k$th row of $W$ with its transpose, in other words, if $R_k$ denotes the $k$th row of $W$, then $C_k = R_k \times R_k^T$. Similarly, we define $C_k$'s corresponding to the OD, $A$, as follows:

Let $U$ be a weighing matrix obtained from $A$ by replacing all the variables of $A$ by 1. Let $A_k$ and $U_k$ denote the $k$th rows of $A$ and $U$ respectively. Then $C_k = A_k \times U_k^T$.

Lemma 2.1. Let $V_i$ be the $i$th row of an SBIBD($v,p,\lambda$). Consider
\[ X = [V_1 \times V_1^T, \ldots , V_n \times V_n^T] \]
then $XX^T = p((p-\lambda)I + \lambda J)$.

Proof.
\[
XX^T = V_1 V_1^T \times V_1 V_1^T \ldots V_n V_n^T \times V_n V_n^T
= p \sum_i V_i V_i^T
= p((p-\lambda)I + \lambda J). \quad \square
\]

Corollary 2.2. Given a balanced $W(n,p)$, based on an SBIBD($n,p,\lambda$), consider
\[ X = [C_1; C_2; \cdots ; C_n] \]
where $C_i$ is obtained from $C_i$ by squaring all its entries. Then the inner product of any two distinct rows of $X$ is $\lambda p$. 

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Proof. Observe that $C_i^T = V_i \times V_i^T$. □

3. A new construction of orthogonal designs.

Many constructions in orthogonal design theory have been expressed in terms of Kronecker products of matrices, for example see Gastineau-Hills (1983) and Gastineau-Hills and Hammer (1983). The Kronecker product of two or more designs is not in general a design since products of variables appear, for example:

$$
\begin{bmatrix}
  x_1 & x_2 \\
  -x_2 & x_1 
\end{bmatrix} \times 
\begin{bmatrix}
  y_1 & y_2 \\
  -y_2 & -y_1 
\end{bmatrix} = 
\begin{bmatrix}
  x_1 y_1 & x_1 y_2 & z_1 y_2 & z_2 y_2 \\
  -x_2 y_1 & -x_2 y_2 & z_2 y_1 & -x_2 y_1 \\
  x_1 y_2 & z_1 y_1 & z_2 y_1 & -x_2 y_1 \\
  -x_2 y_2 & -x_2 y_2 & x_1 y_1 & -x_2 y_1 
\end{bmatrix}
$$

(where $z_1 = x_1 y_1$, $z_2 = x_2 y_1$, $z_3 = x_1 y_2$, $z_4 = x_2 y_2$) is not orthogonal if we take $x_1, x_2, x_3$ and $x_4$ as independent. However it is a different matter if we take a Kronecker product of an OD with a weighing matrix.

A construction of Kharaghani can be extended to give the following result:

**Theorem 3.1.** If there exists an OD, $A$, of type $(s_1, s_2, \ldots, s_r)$, where

$$w = \sum_{k=1}^{r} s_k,$$

and order $n$ on the variables $(\pm x_1, \ldots, \pm x_r, 0)$ then there exist $n$ matrices $C_1, \ldots, C_n$ of order $n$ satisfying

$$
\sum_{i=1}^{n} C_i C_i^T = \sum_{k=1}^{n} s_k
$$

and $C_k^T C_j = 0, \ k \neq j$.

**Proof.** Let $A = (a_{ij})$ be the OD. Replace all the variables of $A$ by 1 making it a $(0, 1, -1)$ weighing matrix $U = (u_{ij})$ of order $n$ and weight $w$. Write $A_k$ and $U_k$ for the $k$th rows of $A$ and $U$ respectively. Form

$$C_k = A_k \times U_k^T.$$

Then

$$C_k C_k^T = (A_k \times U_k^T)(A_j \times U_j^T)^T$$

$$= (A_k A_j^T \times U_k^T U_j).$$

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= 0 if \( k \neq j \) because \( A \) is an orthogonal design.

Now
\[
\sum_{k=1}^{n} C_k^T C_k = \sum_{k=1}^{n} (A_k \times U_k^T)(A_k^T \times U_k)
\]
\[
= \sum A_k A_k^T \times U_k^T U_k
\]
\[
= \sum s_x x_k^T (U_k^T U_k)
\]
\[
= \sum s_x x_k^T (w_k)
\] by the properties of \( U \). □

**Example 3.2.** Let
\[
A = \begin{bmatrix}
-a & b & c & -d \\
- b & a & d & c \\
- c & - d & a & - b \\
- d & - c & b & a
\end{bmatrix}
\]
\[
U = \begin{bmatrix}
-1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1
\end{bmatrix}
\]

Then
\[
C_1 = \begin{bmatrix}
 a & -a & -a & a \\
- b & b & b & -b \\
- c & c & - c & c \\
- d & - d & - d & d
\end{bmatrix}^T
\]
\[
C_2 = \begin{bmatrix}
b & b & b & b \\
 a & a & a & a \\
d & d & d & d \\
c & c & c & c
\end{bmatrix}^T
\]
\[
C_3 = \begin{bmatrix}
 c & -c & c & - c \\
- d & d & - d & d \\
- a & a & - a & a \\
- b & b & - b & b
\end{bmatrix}
\]
\[
C_4 = \begin{bmatrix}
d & d & - d & - d \\
 c & c & - c & - c \\
- b & - b & b & b \\
- a & - a & a & a
\end{bmatrix}
\]

Thus we have:

**Theorem 3.3.** Suppose there exists an \( OD(s_1, \ldots, s_r) \), where \( w = \sum s_i \), of order \( n \). Then there exists an \( OD(s_1 w, s_2 w, \ldots, s_r w) \) of order \( n(n+k) \) for \( k \geq 0 \) an integer.

**Proof.** Form \( C_1, \ldots, C_n \) as in the previous theorem. Form a latin square of order \( n + k \) and replace \( n \) of its elements by \( C_1, \ldots, C_n \) and the other elements by the \( n \times n \) zero matrix. □

For instance, using Theorem 3.3 we can construct an \( OD(4,4,4,4) \) of order \( 4n \), for \( n \geq 4 \). Using inequivalent Latin squares in Theorem 3.3 will

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yield inequivalent ODs.

**Corollary 3.4.** If there is an OD\((t,t,t,t)\) in order \(4t\), then there is an OD\((4t^2,4t^2,4t^2,4t^2)\) in every order \(4t(4t+k), \ k \geq 0\) an integer.

But this construction can be used in other ways.

**Example 3.5.** Write \(1,2,3,4\) for \(0_1,0_2,\ldots,0_4\). Define

\[
A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 4 & 3 \\ 4 & 3 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix}.
\]

Then \(A_4A_1^T = A_3A_2^T\). Thus \(A_1, A_2, A_3, A_4\) can be used to replace the variables of any OD\((t,t,t,t)\).

Hence we have

**Theorem 3.6.** Suppose there is an OD\((t,t,t,t)\) in order \(n\). Then there exists an OD\((12t,12t,12t,12t)\) in order \(12n\).

**Proof.** Use the OD\((1,1,1,1)\) in order 4 to form \(C_1,\ldots,C_4\) of order 4. Substitute these in \(A_1,\ldots,A_4\) of Example 3.5 to obtain Williamson-type matrices of order 12, on 4 variables each repeated 12 times. Use these to replace the variables of the OD\((t,t,t,t)\) to get the result. □

Now if we had started to construct \(C_1,\ldots,C_{4s}\) of order \(4s\) from an OD\((s,s,s,s)\) in order \(4s\) we would have each of 4 variables occurring \(4s^2\) times in each row of \([C_1: C_2: \ldots : C_{4s}]\). But we can use these to form Williamson type matrices in a number of ways:

Let \(A_i\) be a circulant matrix with first row \((i+1,i+2,\ldots,i+s)\), \(i = 0,s,2s,\text{ and } 3s\). These four matrices can be substituted in an OD\((t,t,t,t)\). Hence we have:

**Theorem 3.7.** If there exists an OD\((s,s,s,s)\) in order \(4s\) and an OD\((t,t,t,t)\) in order \(4t\), then there exists an OD\((4s^2t,4s^2t,4s^2t,4s^2t)\) in order \(16s^2t\).

Now if we write \(i\) for \(B_i\) we can proceed exactly as in Example 3.5 so we have:

**Theorem 3.8.** If there exists an OD\((s,s,s,s)\) in order \(4s\) and an OD\((t,t,t,t)\) in order \(4t\), then there exists an OD\((12s^2t,12s^2t,12s^2t,12s^2t)\) in order \(48s^2t\). □

Consider the OD\((5,5,5,5)\) in order 20. The construction gives \(C_1, C_2,\ldots,C_{20}\) of order 20 and hence an OD\((300,300,300,300)\) in order 1200.
Example 3.10. We suppose as before that 1,2,3,4 are matrices of order \( n \) such that \( ij^T = 0 \) and \( \sum i^T = \sum n^2 I_{n} \).

Define

\[
A_1 = \begin{bmatrix}
3 & 1 & 2 & -2 & 1 \\
1 & 3 & 1 & 2 & -2 \\
-2 & 1 & 3 & 1 & 2 \\
2 & -2 & 1 & 3 & 1 \\
1 & 2 & -2 & 1 & 3
\end{bmatrix},
A_2 = \begin{bmatrix}
1 & 3 & 4 & -4 & 3 \\
3 & 1 & 3 & 4 & -4 \\
-4 & 3 & 1 & 3 & 4 \\
4 & -4 & 3 & 1 & 3 \\
3 & 4 & -4 & 3 & 1
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
4 & 1 & 2 & 2 & -1 \\
1 & 2 & 2 & -1 & 4 \\
2 & 2 & -1 & 4 & 1 \\
2 & -1 & 4 & 1 & 2 \\
-1 & 4 & 1 & 2 & 2
\end{bmatrix},
A_4 = \begin{bmatrix}
2 & 3 & 4 & 4 & -3 \\
3 & 4 & 4 & -3 & 2 \\
4 & 4 & -3 & 2 & 3 \\
4 & -3 & 2 & 3 & 4 \\
-3 & 2 & 3 & 4 & 4
\end{bmatrix}
\]

Then \( A_iA_j^T = A_jA_i^T \) and \( \sum A_iA_j^T = \sum 5x_i^2 I_{n} \).

Thus if \( B_i \) are as described after Theorem 3.7 we have

**Theorem 3.11.** Suppose there is an \( OD(s,s,s,s) \) in order \( 4s \) and an \( OD(t,t,t,t) \) in order \( 4t \). Then there is an \( OD(20s^2t,20s^2t,20s^2t,20s^2t) \) in order \( 80s^2 t \).

4. Method used to form inequivalent Hadamard matrices.

**Construction 4.1.** Let \( H \) be Hadamard of order \( n \). Form \( C_i, \ i = 1,2,...,n \), from \( H \) as before. Let \( L \) and \( M \) be Hadamard matrices of order \( t \). Then

\[
(L \times C_i)(M \times C_j) = 0, \quad i \neq j.
\]

So if \( H_1,...,H_n \) are Hadamard matrices of order \( t \) (inequivalent or just different equivalence operations applied to one) then the matrices

\[
H_{i_1} \times C_{i_1}, \ H_{i_2}^2 \times C_{i_2}, \ldots, H_{i_n} \times C_{i_n}, \quad i_j \in \{1,2,...,n\}
\]

can be put into a latin square of order \( n \) to form Hadamard matrices of order \( n^2 t \). The method will possibly give many inequivalent Hadamard matrices. The method can be generalized to give weighing matrices and orthogonal designs which are also possibly inequivalent.
5. Method used with coloured designs to form rectangular weighing matrices.

In a recent paper Rodger, Sarvate and Seberry (1987) have studied coloured BIBDs showing every BIBD can be coloured. By definition a coloured BIBD is the incidence matrix of the $BIBD(v,b,r,k,\lambda)$ whose nonzero entries are replaced by $r$ fixed symbols such that each row and column has no repeated symbol. Consider a coloured symmetric $BIBD(v,k,\lambda)$ and a $W(k,p)$. If we replace the $i$th symbol by $C_i$ for $i = 1,2,...,k$ and the 0 entries by the $k$ by $k$ zero matrix, we get $W(u_k,p_0)$. In general, if we consider a coloured $BIBD(v,b,r,k,\lambda)$ and there exists a weighing matrix $W(r,p)$ then we form the $C_i$, $i = 1,...,r$ and replace the $i$th colour by $C_i$ and zeros by the zero matrix of order $r$. This matrix, $B$, has size $vr \times vr$, $rp$ nonzero elements in each row and $pk$ non-zero elements in each column. Hence we have:

**Theorem 5.1.** Suppose there is a $BIBD(v,b,r,k,\lambda)$ and a $W(r,p)$. Then there is a $(0,1,-1)$ matrix $B$ with $rp$ nonzero elements in each row and $pk$ non-zero elements in each column such that

$$BB^T = rpI.$$ 

In particular, if the BIBD is symmetric then we have a $W(u_k,p^2)$. □

**Remark.** If we replace entries of an $n$-dimensional latin cube by suitable $C_i$'s then we will get $n$-dimensional orthogonal designs.
References.


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