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# Generalised Bhaskar Rao designs of block size 3 over the group Z4

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**Abstract**

We show that the necessary conditions (i) 2tv(v-l) ≡ O(mod 3) (ii) v ≥ 3 (iii) t ≡ 1,5 (mod 6) => v ≠ 3 are sufficient for the existence of a GBRD(v,3,4t;Z4) except possibly when (v,t) = (27,1) or (39,1).

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# GENERALISED BHASKAR RAO DESIGNS WITH BLOCK SIZE 3 OVER $Z_4$

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ABSTRACT: We show that the necessary conditions

(i) $2tv(v-1) \equiv 0 \pmod 3$

(ii) $v \geq 3$

(iii) $t \equiv 1,5 \pmod 6 \Rightarrow v \neq 3$

are sufficient for the existence of a $GBRD(v,3,4t;Z_4)$ except possibly when $(v,t) = (27,1)$ or $(39,1)$.

Keywords: Bhaskar Rao designs, block designs, PBD-closed sets, group divisible designs.

AMS classification: 05B20, 05B05, 05B30, 05B99, 51E05.

## 0. Introduction

Although a considerable amount of work has been done on generalised Bhaskar Rao designs, little is known about the existence of these designs over groups which are not elementary abelian. This paper considers the group $Z_4$ and finds that designs exist for $Z_4$ for parameters for which they do not exist for $Z_2 \times Z_2$ and vice versa.

Suppose we have a matrix $W$ with elements from an abelian group

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$G = \{h_1, h_2, \ldots, h_g\}$, where $W = h_1A_1 + h_2A_2 + \ldots + h_gA_g$; here $A_1, \ldots, A_g$ are $v \times b$ (0,1) matrices, and the Hadamard product $A_i * A_j$ $(i \neq j)$ is zero. Suppose $(a_{i1}, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the ith and jth rows of $W$; then we define $WW^+$ by

$$(WW^+)_{ij} = (a_{i1}, \ldots, a_{ib}^{-1}) \cdot (b_{j1}, \ldots, b_{jb}^{-1})$$

with . designating the scalar product. Then $W$ is a *generalised Bhaskar Rao design or GBRD* if

(i) $\quad WW^+ = rI + \sum_{i=1}^{m} (c_iG)B_i$

(ii) $\quad N = A_1 + \ldots + A_g$ satisfies $NN^T = rI + \sum_{i=1}^{m} {}_iB_i$,

that is, $N$ is the incidence matrix of a PBIBD(m), and $(C_iG)$ gives the number of times a complete copy of the group $G$ occurs.

Such a matrix will be denoted by $GBRD_G(v,b,r,k;\lambda_1,\ldots,\lambda_m;c_1,\ldots,c_m)$. In this paper we shall only be concerned with $m = 1$, $c = \lambda/g$, and $B_1 = J - I$. In this case $N$ is the incidence matrix of a PBIBD(1), that is, a BIBD. Hence, the equations become

(i) $\quad WW^+ = rI + \frac{\lambda G}{g}(J - I)$

(ii) $\quad NN^T = (r-\lambda)I + \lambda J.$

Thus $W$ is a $GBRD_G(v,b,r,k,\lambda)$. Since $\lambda(v-1) = r(k-1)$ and $bk = vr$, we sometimes use the notation $GBRD(v,k,\lambda;G)$.

These matrices are generalisations of generalised weighing matrices and may be used in the construction of PBIBDs.

We use the following notation for the initial blocks of a GBRD. We say $(a_\alpha, b_\beta, \ldots, c_\gamma)$ is an initial block, when the Latin letters are developed mod $n$ and the Greek subscripts are the elements of the group, which will be placed in the incidence matrix in the positions indicated by the Latin letters. Thus we place $\alpha$ in the $(i,a-1+i)$th position of the incidence matrix, $\beta$ in the $(i,b-1+i)$th position, and so on.

We form the difference table of the initial block $(a_\alpha, b_\beta, \ldots, c_\gamma)$ by placing in the position headed by $x_\alpha$ and by row $y_\beta$ the element $(x-y)_{\delta\eta^{-1}}$ where $(x-y)$ is mod $n$ and $\delta\eta^{-1}$ is in the abelian group.

A set of initial blocks will be said to form a *GBR difference set* (if there is one initial block) or *GBR supplementary difference sets* (if more than one) if in the totality of elements

$$(x-y)_{\delta\eta^{-1}} \pmod{n, G}$$

each non-zero element $a_g$, $a \pmod n$, $g \in G$, occurs $\lambda/|G|$ times.

For any other definition or notation the reader is referred to de Launey and Seberry [1].

For a GBRD$(v,k,\lambda;G)$ to exist $\lambda \equiv 0 \pmod g$ and there must exist a BIBD$(v,k,\lambda)$. So the parameters $v,k,\lambda$ must satisfy the constraints,

(i) $v \geq k$

(ii) $\lambda \equiv 0 \pmod g$

(iii) $\lambda(v-1) \equiv 0 \pmod{k-1}$

(iv) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

In view of these constraints a GBRD$(v,3,4t;Z_4)$ can exist only when one of the following is true,

(a) $t \equiv 0 \pmod 3$, $v \geq 3$,

(b) $t \not\equiv 0 \pmod 3$, $v \geq 0,1 \pmod 3$ and $v \geq 3$.

Moreover a theorem of Drake [2, Theorem 1.10] ensures that no GBRD$(3,3,4t;Z_4)$ exists when $t$ is odd. We show that, with the possible exception of the cases given in the abstract, these necessary conditions are also sufficient.

## §1. A Small Generating Set

In this section and the next we make extensive use of Wilson's notation [6, Sections 1 and 2] concerning PBD-closure theory. In the next section we will need a small generating set for

$$V = \{v > 3 \mid v \equiv 0,1 \pmod 3\}\ .$$

**Notation 1.1:** Let $S$ and $K$ be sets of positive integers. Define

$$[v_0]\, S \oplus K = \{v \mid v = v_0 s+k \text{ where } s \in S,\ k \in K \text{ and } s \geq k\}\ .$$

Let $a$ and $b$ be integers. Then let ${}_aS^b$ denote the set

$$\{v \mid a \leq v \leq b\} \cap S\ . \qquad \square$$

The following theorem appears in de Launey and Seberry [1, Theorem 1.2.14].

**Theorem 1.2:** *Let* $v_0 \geq 2$ *be an integer. Let* $S$ *be an increasing infinite sequence such that for all* $t \in S$ *there exists a* TD$(v_0+1,t)$. *Let* $K$ *be a set of positive integers containing* $v_0$ *and* $v_0+1$. *Let* $k_0 = \min_{k \in K} \{k\}$ *and suppose there exists a* TD$(v_0+1,t_0)$ *for some* $t_0$

*not necessarily in* S. *Then*

(i) $\mathbb{B}\left(\{t_0\} \cup {}_{t_0}S^{v_0t_0+k_0-1} \cup T \cup K\right) \supseteq [v_0]\, S \oplus K$ , *where*

$$T = \{t \in {}_{v_0t_0+k_0}S \mid t \notin [v_0]_{t_0} S \oplus K\} .$$

(ii) $\mathbb{B}\left(\{t_0\} \cup {}_{t_0}S^{v_0t_0+k_0-1} \cup U \cup K\right) \supseteq \{v \geq v_0t_0+k_0\}$ , *where*

$$U = \{t \mid t \geq v_0t_0+k_0 \text{ and } t \notin [v_0]_{t_0} S \oplus K\} . \qquad \square$$

This theorem allows us to calculate small generating sets for sets of the form $\{v \geq k \mid v \notin U\}$, where K is a finite set of integers $u \geq k$ [1, Lemma 1.2.16]. We extend the theorem so that we can calculate small generating sets for sets of the form $\{v \geq k \mid v \equiv 0,1 \pmod 3,\ v \notin U\}$.

Now slightly altering a construction appearing in Wilson's paper [6, Lemma 5.1] we have the following result.

Lemma 1.3: *Let* K *be a set of positive integers. Suppose there exists a* GDD *on* v *points with block sizes from* {4,5} *and group sizes from* K. *Then*

$$3v \in \mathbb{B}\left(\{3k \mid k \in K\}\right) \cup \{4\})$$

*and*

$$3v+1 \in \mathbb{B}\left(\{3k+1 \mid k \in K\} \cup \{4\}\right) . \qquad \square$$

But the construction of the PBD's in the proof of Theorem 1.2 [1, Theorem 1.2.14] in the first place relies on the construction of GDD's with block sizes from $\{v_0, v_0+1\}$ and group sizes from ${}_{t_0}S^{v_0t_0+k_0-1} \cup T \cup K \cup \{t_0\}$ in case (i) and from ${}_{t_0}S^{v_0t_0+k_0-1} \cup U \cup K \cup \{t_0\}$ in case (ii). So putting $v_0 = 4$ we have we have the following result.

Theorem 1.4: *Let* S *be an increasing infinite sequence such that for all* $t \in S$ *there exists a* TD(5,t). *Let* K *be a set of positive integers. Let* $k_0 = \min_{k \in K}\{k\}$ *and suppose there exists a* $TD(5,t_0)$ *for some* $t_0$ *not necessarily in* S. *Then*

(i) $\{3v \mid v \in [4]\, S \oplus K\} \subseteq \mathbb{B}\left(\{4\} \cup \{3v \mid v \in {}_{t_0}S^{4t_0+k_0-1} \cup T \cup K \cup \{t_0\}\}\right)$

*and*

$$\{3v+1 \mid v \in [4] S \oplus K\} \subseteq \mathbb{B}\left(\{4\} \cup \{3v+1 \mid v \in {}_{t_0}S^{4t_0+k_0-1} \cup T \cup K \cup \{t_0\}\}\right)$$

*where*

$$T = {}_{t_0+k_0}S \setminus ([4]_{t_0} S + K) ,$$

(ii) $\{3v \mid v \geq 4t_0+k_0\} \subseteq \mathbb{B}\left(\{4\} \cup \{3v \mid 3v \mid v \in {}_{t_0}S^{4t_0+k_0-1} \cup U \cup K \cup \{t_0\}\}\right)$

*and*

$$\{3v+1 \mid v \geq 4t_0+k_0\} \subseteq \mathbb{B}\left(\{4\} \cup \{3v+1 \mid v \in {}_{t_0}S^{4t_0+k_0-1} \cup U \cup K \cup \{t_0\}\}\right)$$

*where*

$$U = \{v \geq 4t_0+k_0\} \setminus ([4]_{t_0} S \oplus K) . \qquad \square$$

We apply this theorem to prove the following result.

**Theorem 1.5:** *The following set inequalities hold*

(i) $\{v \mid v \equiv 0 \pmod 3,\ v>3\} \subseteq \mathbb{B}\left(\{4,10\} \cup \{3v \mid v = 2,3,\ldots,11,13,17\}\right)$

(ii) $\{v \mid v \equiv 0,1 \pmod 3 \text{ and } v>3\} \subseteq \mathbb{B}\left(\{4,6,7,9,10,12,15,18,19,24,27,30,39,51\}\right)$

**Proof.** We apply Theorem 1.4 with

$S = \{4,5,7,8,9,11,12,13,16,17\} \cup \{v \equiv \pm 1 \pmod 6 \mid v \geq 17\}$

$K = \{2,3,4,5,6,7,\ldots,17\}$

$t_0 = 4$ .

When $v \geq 70$, $v - 4t \in \{2,3,\ldots,17\}$ for some, $t \geq 17$, $t \in S$. So $\{v \mid v \geq 70\} \subseteq [4]_t S \oplus K$. It is then a simple matter to check that

$$[4]_{t_0} S \oplus K = \{v \geq 18 \mid v \neq 21,26,27,28,29\} .$$

So $U = \{21,26,27,28,29\}$ and hence

$\{3v \mid v \geq 18\} \subseteq \mathbb{B}\left(\{4\} \cup \{3v \mid v = 2,3,\ldots,17,21,26,27,28,29\}\right)$ .

Now $3v \in \mathbb{B}(\{4,6,9\})$ for $v \in \{12,14,15,16,21,26,27,28\}$ (see Appendix A), while $87 \in \mathbb{B}(\{6,9,10\})$ (use TD(10,9)). Thus

$\{3v \mid v \equiv 0 \pmod 3,\ v>3\} \subseteq \mathbb{B}(\{4,10\} \cup \{3v \mid v = 2,3,\ldots,11,13,17\})$

Now $\{3v+1 \mid v \geq 4\} = \mathbb{B}\{4,7,10,19\}$ and $21, 33 \in \mathbb{B}(\{4,6,9\})$ (add suitable blocks and points to TD(4,5) and TD(4,8) respectively), so

$\{v \mid v \equiv 0,1 \pmod 3,\ v>3\} \subseteq \mathbb{B}(\{4,6,7,9,10,12,15,18,19,24,27,30,39,51\})$ . $\square$

Because we do not as yet have designs for $v \in \{27,39\}$ we prove the following theorem.

**Theorem 1.6:** *The following set inequality holds*

$\{v \mid v \equiv 0,1 \pmod 3,\ v>3\} \setminus \{27,39\} \subseteq \mathbb{B}(\{4,6,7,9,10,12,15,18,19,24,30,51\})$ .

**Proof.** Apply Theorem 1.4 with

$S = \{4,5,7,8,11,12,16,17\} \cup \{v \equiv \pm \pmod 6 \mid v \geq 17\}$

$K = \{2,3,\ldots,8,10,11,12,14,\ldots,17,21,25\}$

$t_0 = 4$ .

When $v \geq 70$ there exists a $k \in \{2,3,\ldots,8,10,11,12,14,\ldots,17,21,25\}$ and $t \in S$ such that

$$v = 4t + k \quad \text{and} \quad t \geq k$$

except when

(i) $v = 4t + 9$ and $t = 17$, 23 or 29

(ii) $v = 4t + 13$ and $t = 19$.

When $18 \leq v \leq 70$, $v \in [4]\, S \oplus K$ except when

$$v \in \{21,26,27,28,29,41,42,43,45,57,61,62,63,65\} .$$

So $U = \{21,26,27,28,29,41,42,43,45,57,61,62,63,65,77,89,101,125\}$ .

But using the designs given in Appendix A

$$\{3v \mid v \in U\} \subseteq \mathbb{B}(\{4,6,7,9,10,12,13,15,18,19,21,31\})$$
$$\subseteq \mathbb{B}(\{4,6,7,9,10,12,15,18,19\}) \quad \ldots\ldots\ldots\ldots (1.1)$$

Note that $21 \in \mathbb{B}(\{4,6\})$ (add a point to TD(4,5)) and that $31 \in \mathbb{B}(\{4,10\})$ [6, see the proof of Theorem 5.1(ii)].

Let $V = \{v \mid v \equiv 0 \pmod 3,\ v > 3,\ v \neq 27,39\}$ and apply Theorem 1.4(ii). Then $V \subseteq \mathbb{B}(\{4\} \cup \{3v \mid v \in \{2,3,\ldots,8,10,11,12,14,\ldots,17,21,25\}\} \cup U)$. But then, by (1.1),

$$V \subseteq \mathbb{B}(\{4,6,7,9,10,12,15,18,19,21,24,30,33,36,42,\ldots,51,63,75\})$$

Finally $\{36,42,45,48,63\} \subseteq \mathbb{B}(\{4,6,9,12,15\})$ (Table 1, Appendix A), $21, 33 \in \mathbb{B}(\{4,6,9\})$ (see the proof of Theorem 1.5), $75 \in \mathbb{B}(\{4,15\})$ (Appendix A), and $\{3v+1 \mid v \geq 1\} \subseteq \mathbb{B}(\{4,7,10,19\})$. The result then follows. □

## §2. The Constructions

**Lemma 2.1:** *There exists a* GBRD$(v,3,4;Z_4)$ *for all* $v \equiv 0,1 \pmod 3$, $v \geq 4$ *except possibly when* $v = 27,39$.

**Proof.** The necessary conditions give $v \equiv 0,1 \pmod 3$. Drake's theorem [2,Theorem 1.10] ensures that $v \geq 4$ but since the number of blocks, $2v(v-1)/3$, is divisible by 4 the Seberry, Street, Rodger theorem [theorem 1,4; or see 5] gives no new conditions.

By Theorem 1.6 we need to establish existence for

$$v \in \{4,6,7,9,10,12,15,18,19,24,30,51\} .$$

The required designs for $v = 4,6,9,10$ and $15$ are given in Appendix B. The designs for $v = 7,12,18$ and $30$ can be obtained by developing the initial blocks indicated:

$v = 7$ develop the initial blocks

$$(0_1,1_1,6_i),\ (0_1,2_{-1},5_{-i}),\ (0_1,3_1,4_i),\ (0_1,1_{-1},3_{-1})\ (\text{mod } 7,Z_4);$$

$v = 12$ develop the initial blocks

$$(\infty_1,3_1,9_i),\ (\infty_1,6_{-1},7_i),\ (1_1,3_1,4_{-i}),\ (3_{-1},5_{-1},9_1),\ (1_1,4_1,5_1),$$

$$(2_1,6_{-1},8_1),\ (6_1,7_{-1},10_{-i}),\ (2_1,8_{-i},10_{-1})(\text{mod } 11,Z_4);$$

$v = 18$ develop the initial blocks

$$(0_1,a_1,(17-a)_i),\quad a = 1,3,5,7,$$

$$(0_1,b_1,(17-b)_{-i}),\quad b = 2,4,6,8,$$

$$(0_1,2_1,6_1),\ (0_1,3_{-1},8_1),\ (\infty_1,0_1,1_{-1}),\ (\infty,0_i,7_{-i})(\text{mod } 17,\ Z_4);$$

$v = 30$ develop the initial blocks

$$(0_1,a_1,(29-a)_i),\quad a = 1,3,\ldots,13\ (\text{odd numbers}),$$

$$(0_1,b_{-1},(29-b)_{-i}),\quad b = 4,6,\ldots,14\ (\text{even numbers}),$$

$$(0_1,2_{-1},15_1),\ (0_1,2_1,10_1),\ (0_1,3_{-1},12_1),\ (0_1,4_1,11_{-1}),\ (0_1,1_{-1},6_1),$$

$$(\infty_1,0_1,2_i),\ (\infty_1,0_{-i},4_{-1})\ (\text{mod } 29,Z_4)$$

Finally $19 = 6(4-1) + 1$, $24 = 4 \times 6$ and $51 = 10(6-1) + 1$. So a composition theorem applies [1,Theorem 1.1.3] to give designs for $v = 19$, $24$ and $51$. □

**Theorem 2.2:** *There exists a* $GBRD(v,3,8;Z_4)$ *for all* $v \geq 3$.

**Proof.** By Hanani's theorem (see Proposition 5.1 of [6]) and the construction of Theorem 2.2 of Lam and Seberry [3] we only need to establish the existence of $GBRD(v,3,8;Z_4)$ for $v = 3,4,6$. The design for $v = 3$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & i & 1 & i & - & \bar{i} & - & \bar{i} \\ i & 1 & \bar{i} & - & 1 & i & - & \bar{i} \end{bmatrix}$$

and the designs for $v = 4$ and $6$ are two copies of the suitable designs with $\lambda = 4$ given in Appendix B. Hence we have the result □

**Theorem 2.3:** *There exists a* $GBRD(v,3,12;Z_4)$ *for all* $v \geq 4$.

**Proof.** By Drake's Theorem [1, Theorem 1.10] we cannot obtain this design for $v = 3$. Now combining Hanani's Theorem (as stated in [1, Corollary 1.1.2(ii)]) with Theorem 2.2 of Lam and Seberry [3] we only need to establish existence for $v \in K_4^2 = \{4,5,6,7,8,9,10,11,12,14,15,18,19,22,23\}$. Now these designs can be obtained in the following manner:

| v | Construction |
|---|---|
| 4 | 3 copies of design for $\lambda = 4$ |
| 5 | Use $SBIBD(5,4,3)$ and $GBRD(4,3,4;Z_4)$ in Theorem 2.2 of [3] |
| 6 | 3 copies of design for $\lambda = 4$ |
| 7 | 3 copies of design for $\lambda = 4$ |
| 8 | Use $BIBD(8,4,3)$ with $GBRD(4,3,4;Z_4)$ |
| 9 | 3 copies of design for $\lambda = 4$ |
| 10 | 3 copies of design for $\lambda = 4$ |
| 11 | Use $SBIBD(11,6,3)$ and $GBRD(6,3,4;Z_4)$ |
| 12 | 3 copies of design for $\lambda = 4$ |
| 14 | Remove one row of $SBIBD(15,7,3)$ to obtain a $PBD(\{7,6\},14,3)$, use with $GBRD(u,3,4;Z_4)$, $u \in \{6,7\}$ |
| 15 | 3 copies of design for $\lambda = 4$ or use $SBIBD(15,7,3)$ and $GBRD(7,3,4;Z_4)$ |
| 18 | Use $PBD(\{6,9\},18,3)$ (found from an $SBIBD(25,12,3)$ by de Launey and Seberry [1], Lemma 1.3.7, by removing the first seven rows) with $GBRD(u,3,4;Z_4)$, $u \in \{6,9\}$. |
| 19 | $6(4-1)+1$ and so Theorem 3 of Seberry [4] applies |
| 22 | $7(4-1)+1$ and so Theorem 3 of Seberry [4] applies |
| 23 | Develop the following initial blocks $(0_1,(2t+1)_1,(22-2t)_i)$, $(0_1,(2t)_{-1},(23-2t)_{-i})$, $t = 1,\dots,5$ all thrice, $(0_1,1_{-i},2_{-i})$ thrice, $(0_1,5_{-1},7_{-1})$ three times, $(0_1,1_{-1},11_{-1})$ twice, $(0_1,3_{-1},9_{-1})$ twice, $(0_1,1_{-1},9_{-1})$, $(0_1,3_{-1},11_{-1})$, $(0_1,4_1,8_1)$, $(0_1,4_1,10_1)$ (mod $23,Z_4$). |

Hence we have the result. □

Note: A straightforward construction for $GBRD(27,3,12;Z_4)$ can be obtained by using the $PBD(\{6,9\},27,3)$ of Lemma 1.3.5 of de Launey and Seberry and $GBRD(u,3,4;Z_4)$ $u \in \{6,9\}$.

**Theorem 2.4:** *The necessary conditions*

$$2tv(v-1) \equiv 0 \pmod 3$$
$$t \equiv 1,5 \pmod 6 \Rightarrow v \neq 3$$

*are sufficient for the existence of a* $GBRD(v,3,4t;Z_4)$ *except possibly for* $(v,t) = (27,1)$ and $(39,1)$.

**Proof.** The necessary conditions follow from the necessary conditions for block designs and the non-existence for $v = 3$, $t \equiv 1,5 \pmod 6$ from Drake's Theorem [2].

To establish existence we distinguish four cases:

1. $2\nmid t$, $3\nmid t$ then the necessary condition is $v \equiv 0,1 \pmod 3$ and the result follows, except for $v = 27$ or $39$ by taking multiple copies of the designs given in Theorem 2.1. For $v = 27$ or 39 we note $GBRD(v,3,8;Z_4)$ and $GBRD(v,3,12;Z_4)$ exist and so multiple copies give the designs for $v = 27$ or $39$ and $t > 1$;
2. $2|t$, $3\nmid t$ then the necessary condition is $v \equiv 0,1 \pmod 3$, $v \geq 3$, but this is established in Theorem 2.2;
3. $2\nmid t$, $3|t$ then the necessary condition is $v \geq 4$ (by Drake's Theorem [2, Theorem 10.1] and this is established in Theorem 2.3;
4. $2|t$, $3|t$, here there is no condition of $v$. By part 2. of this theorem we only have to consider the cases $v = 3$ and $\lambda = 12s$, $s$ even but these can be obtained using multiples of the $GBRD(3,3,8;Z_4)$ of part 3.

Hence we have the result. □

## Appendix A

**Notation.** By $TD(r,t)$ we denote a *transversal design* on $r$ groups each of size $t$. □

Table 1 gives designs needed for Theorem 1.5. In particular it lists GDD's which have been constructed to satisfy Lemma 1.3. See Street and Rodger for the construction involving GBRD's. The constructions involving point and block removals from certain designs are quite standard [6, Remarks 3.5 and 3.6]. Table 2 gives PBD designs needed in Theorem 1.6. Any references given in a table give a place where a design used in a construction can be found. The reader should note MacNeish's Theorem [6, Theorem 3.2].

Table 1. (GDD's on 3v points satisfying Lemma 1.3.)

| v | |
|---|---|
| 12 | Obtain a GDD by removing a point from SBIBD(13,4,1). |
| 14 | Use GBRD(7,4,2;$Z_2$) de Launey and Seberry [1,Theorem 4.1.1]. |
| 15 | GBRD(5,4,3;$Z_3$) [1, Lemma 5.1.1]. |
| 16 | Use TD(4,4). |
| 21 | see matrix below |

$$\begin{pmatrix} 0000 & 1111 & 0000 & 0000 & 0000 & 0000 \\ I & 0 & I & A & A^3 & A^2 \\ \hline I & A^2 & 0 & I & A & A^3 \\ \hline I & A^3 & A^2 & 0 & I & A \\ \hline I & A & A^3 & A^2 & 0 & I \\ \hline I & I & A & A^3 & A^2 & 0 \end{pmatrix}$$

where $I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $A = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix}$

| v | |
|---|---|
| 26 | GBRD(13,4,2;$Z_2$) [1, Theorem 4.1.1] |
| 27 | GBRD(9,4,3;$Z_3$) [1, Lemma 5.1.1] |
| 28 | Use TD(4,7). |

Table 2. (PB-design on v points)

| v | | | | |
|---|---|---|---|---|
| 123 | TD(10,13) | $\in$ | 123 | $\mathbb{B}(\{6,9,10,13\})$. |
| 126 | TD(10,13) | $\in$ | 126 | $\mathbb{B}(\{9,10,13\})$. |
| 129 | TD(10,13) | $\in$ | 129 | $\mathbb{B}(\{9,10,12,13\})$. |
| 171 | TD(9,19) | $\in$ | 171 | $\mathbb{B}(\{9,19\})$. |
| 183 | TD(10,19) | $\in$ | 183 | $\mathbb{B}(\{9,10,12,19\})$. |
| 186 | TD(10,19) | $\in$ | 186 | $\mathbb{B}(\{9,10,15,19\})$. |
| 189 | TD(10,19) | $\in$ | 189 | $\mathbb{B}(\{9,10,18,19\})$. |
| 195 | TD(7,31) | $\in$ | 195 | $\mathbb{B}(\{6,7,9,31\})$. |

□

Finally 75 and 135 $\in \mathbb{B}(\{4,15\})$. There exist a GBRD(4,4,5;$Z_5$) [1, Theorem 2.2(iii)(b)] and a GBRD(u,4,3;$Z_3$) for $u \in \{5,9\}$ [1, Theorem 5.1.1] so there exists a GBRD(u,4,15;$Z_{15}$) for $u \in \{5,9\}$ and hence a GDD with u groups of size 15 and with all blocks of size 4. It follows that 75 and 135 $\in \mathbb{B}(\{4,15\})$.

## Appendix B

We use the notation - for $-1$ and $\bar{i}$ for $-i$, $e = (1,1,1)$ and

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Then the following designs exist:

GBRD$(4,3,4;Z_4)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & i & - & -i & 0 & - & 1 \\ 1 & - & 0 & i & 0 & -i & 1 & i \\ 0 & 1 & - & 0 & i & -i & i & i \end{bmatrix}$$

GBRD$(6,3,4;Z_4)$

$$\left[\begin{array}{cccccccccc|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\ 1 & 0 & - & i & 0 & \bar{i} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & - & 0 & 0 & i & \bar{i} & 0 & - & 0 & 0 & i & \bar{i} & 0 \\ 0 & 1 & 0 & 0 & - & i & 0 & 0 & 0 & \bar{i} & 1 & \bar{i} & 0 & 0 & 0 & i \\ 0 & 1 & 0 & - & 0 & 0 & i & \bar{i} & 0 & 0 & 0 & 1 & - & 0 & \bar{i} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & - & 0 & \bar{i} & i & 0 & 0 & i & \bar{i} & 0 & 1 \end{array}\right]$$

GBRD$(9,3,4;Z_4)$

$$\left[\begin{array}{ccc|ccc|cccccccccc} A & I & O & B & & & I & I & I & iI & I & I & iI & I & I & iI \\ O & A & I & & B & & I & -I & -iI & T & -T & iT & -I & T^2 & iT^2 & T^2 \\ I & O & A & & & B & I & iT & T & -T^2 & iI & -I & I & -iT^2 & -T^2 & T \end{array}\right]$$

GBRD$(10,3,4;Z_4)$

$$\left[\begin{array}{ccc|ccc|ccc|ccc|cccccccc} e & -e & ie & -e & e & -ie & & & & & & & & & & & & & & \\ I & I & & iI & iI & & A & I & O & B & & & I & I & iI & I & I & iI & I & I \\ I & & I & T^2 & & T^2 & O & A & I & & B & & -I & -iT & T & -T & iT & -I & T^2 & iT^2 \\ & I & I & & T & T & I & O & A & & & B & iT & T & -T^2 & iI & -I & I & -iT^2 & -T^2 \end{array}\right]$$

$$\text{where } A = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} i & 1 & 1 \\ 1 & i & 1 \\ 1 & 1 & i \end{bmatrix}.$$

GBRD(15,3,4;$Z_4$) has blocks with $Z_4 = \{1,2,3,4\}$

| | | | | |
|---|---|---|---|---|
| $1_1\ 2_1\ 3_1$ | $1_1\ 6_3\ 8_2$ | $2_1\ 3_2\ 4_1$ | $2_1\ 8_4\ 14_2$ | $3_1\ 5_2\ 11_1$ |
| $1_1\ 2_2\ 4_1$ | $1_1\ 6_4\ 9_2$ | $2_1\ 3_3\ 11_1$ | $2_1\ 9_2\ 10_1$ | $3_1\ 5_3\ 12_1$ |
| $1_1\ 2_3\ 5_1$ | $1_1\ 7_2\ 10_3$ | $2_1\ 3_4\ 12_1$ | $2_1\ 9_3\ 13_4$ | $3_1\ 5_4\ 15_3$ |
| $1_1\ 2_4\ 6_1$ | $1_1\ 7_4\ 11_2$ | $2_1\ 4_2\ 5_1$ | $2_1\ 9_4\ 15_1$ | $3_1\ 6_2\ 11_4$ |
| $1_1\ 3_2\ 7_1$ | $1_1\ 8_3\ 12_2$ | $2_1\ 4_3\ 12_2$ | $2_1\ 10_2\ 13_3$ | $3_1\ 6_3\ 14_1$ |
| $1_1\ 3_3\ 8_1$ | $1_1\ 8_4\ 13_2$ | $2_1\ 5_2\ 6_1$ | $2_1\ 10_3\ 14_4$ | $3_1\ 6_4\ 15_2$ |
| $1_1\ 3_4\ 9_1$ | $1_1\ 9_3\ 14_2$ | $2_1\ 5_4\ 15_2$ | $2_1\ 10_4\ 15_3$ | $3_1\ 7_1\ 11_2$ |
| $1_1\ 4_2\ 10_1$ | $1_1\ 9_4\ 15_2$ | $2_1\ 6_3\ 7_1$ | $2_1\ 11_2\ 12_4$ | $3_1\ 7_2\ 13_1$ |
| $1_1\ 4_3\ 11_1$ | $1_1\ 10_2\ 11_3$ | $2_1\ 6_4\ 12_3$ | $2_1\ 11_4\ 14_3$ | $3_1\ 7_3\ 14_2$ |
| $1_1\ 4_4\ 12_1$ | $1_1\ 10_4\ 12_3$ | $2_1\ 7_2\ 8_1$ | $2_1\ 11_3\ 15_4$ | $3_1\ 8_1\ 12_3$ |
| $1_1\ 5_2\ 13_1$ | $1_1\ 11_4\ 13_3$ | $2_1\ 7_3\ 13_1$ | $3_1\ 4_1\ 6_1$ | $3_1\ 8_2\ 13_3$ |
| $1_1\ 5_3\ 14_1$ | $1_1\ 12_4\ 14_3$ | $2_1\ 7_4\ 14_1$ | $3_1\ 4_3\ 10_1$ | $3_1\ 8_4\ 14_3$ |
| $1_1\ 5_4\ 15_1$ | $1_1\ 13_4\ 15_3$ | $2_1\ 8_2\ 9_1$ | $3_1\ 4_2\ 15_1$ | $3_1\ 9_1\ 10_3$ |
| $1_1\ 6_2\ 7_3$ | $1_1\ 14_4\ 15_4$ | $2_1\ 8_3\ 13_2$ | $3_1\ 5_1\ 10_2$ | $3_1\ 9_3\ 14_2$ |
| $3_1\ 9_4\ 14_4$ | $4_1\ 9_2\ 14_3$ | $5_1\ 8_2\ 11_3$ | $6_1\ 9_4\ 14_2$ | $8_1\ 10_1\ 15_1$ |
| $3_1\ 10_4\ 13_4$ | $4_1\ 9_4\ 15_3$ | $5_1\ 8_3\ 14_4$ | $6_1\ 10_2\ 13_1$ | $8_1\ 11_1\ 15_3$ |
| $3_1\ 12_4\ 15_4$ | $4_1\ 10_2\ 14_4$ | $5_1\ 9_1\ 11_2$ | $6_1\ 10_3\ 15_4$ | $9_1\ 10_1\ 11_3$ |
| $4_1\ 5_1\ 7_1$ | $4_1\ 11_4\ 14_2$ | $5_1\ 9_2\ 12_4$ | $6_1\ 13_4\ 15_2$ | $9_1\ 11_1\ 12_4$ |
| $4_1\ 5_2\ 8_1$ | $4_1\ 8_3\ 11_2$ | $5_1\ 9_4\ 13_2$ | $7_1\ 8_1\ 9_3$ | $10_1\ 11_4\ 12_1$ |
| $4_1\ 5_3\ 9_1$ | $4_1\ 12_1\ 13_2$ | $5_1\ 13_1\ 14_1$ | $7_1\ 8_2\ 12_2$ | $10_1\ 12_2\ 14_4$ |
| $4_1\ 6_2\ 10_1$ | $4_1\ 12_3\ 15_2$ | $5_1\ 14_2\ 15_1$ | $7_1\ 8_3\ 15_4$ | $10_1\ 13_3\ 14_1$ |
| $4_1\ 6_4\ 11_1$ | $5_1\ 6_1\ 10_1$ | $6_1\ 7_1\ 12_1$ | $7_1\ 9_2\ 11_1$ | $11_1\ 12_1\ 13_3$ |
| $4_1\ 6_3\ 13_1$ | $5_1\ 6_2\ 11_1$ | $6_1\ 7_4\ 14_4$ | $7_1\ 9_4\ 12_4$ | $11_1\ 13_2\ 14_1$ |
| $4_1\ 7_2\ 13_3$ | $5_1\ 6_3\ 12_1$ | $6_1\ 8_3\ 11_1$ | $7_1\ 9_1\ 15_1$ | $11_1\ 13_1\ 15_1$ |
| $4_1\ 7_3\ 14_1$ | $5_1\ 7_2\ 10_4$ | $6_1\ 8_1\ 14_1$ | $7_1\ 10_4\ 11_4$ | $11_1\ 14_2\ 15_4$ |
| $4_1\ 7_4\ 15_1$ | $5_1\ 7_4\ 12_2$ | $6_1\ 8_2\ 15_1$ | $7_1\ 10_1\ 15_3$ | $12_1\ 13_4\ 14_1$ |
| $4_1\ 8_2\ 9_3$ | $5_1\ 7_3\ 13_3$ | $6_1\ 9_1\ 12_2$ | $8_1\ 9_1\ 10_2$ | $12_1\ 13_1\ 15_2$ |
| $4_1\ 8_4\ 13_4$ | $5_1\ 8_1\ 10_3$ | $6_1\ 9_2\ 13_2$ | $8_1\ 10_4\ 12_2$ | $12_1\ 14_2\ 15_3$ |

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