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Higher-dimensional orthogonal designs and Hadamard matrices II

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Abstract
Higher-dimensional orthogonal designs of type $(l,l)^n$ are used to obtain higher-dimensional weighing matrices of type $(q)^n$, side $q+l$ and propriety $(2,2,...,2)$ for $q = l \text{(mod 4)}$ a prime power. Next, $n$-dimensional orthogonal designs of type $(l,l,l,l)^n$, side 4 and propriety $(2,2,...,2)$ are constructed. These are then used to show that higher-dimensional Hadamard matrices of order $(4t)^t$ exist whenever $t$ is the side of 4-Williamson matrices. This establishes the existence of higher-dimensional Hadamard matrices of order $(4t)^t$ for $t$ odd, $1 \leq t \leq 33$ and several infinite families, all of propriety $(2,2,...,2)$. Finally, we establish that if there is an Hadamard matrix that can be obtained from a group difference set with parameters $(4s^2, 2s^2\pm s, s^2\pm s)$ then there is a higher-dimensional Hadamard matrix of order $(4s^2)^n$.

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Higher-dimensional orthogonal designs of type \((1,1)^n\) are used to obtain higher-dimensional weighing matrices of type \((q)^n\), side \(q+1\) and propriety \((2,2,\ldots,2)\) for \(q \equiv 1 \pmod{4}\) a prime power. Next, \(n\)-dimensional orthogonal designs of type \((1,1,1,1)^n\), side 4 and propriety \((2,2,\ldots,2)\) are constructed. These are then used to show that higher-dimensional Hadamard matrices of order \((4t)^n\) exist whenever \(t\) is the side of 4-Williamson matrices. This establishes the existence of higher-dimensional Hadamard matrices of order \((4t)^n\) for \(t\) odd, \(1 \leq t \leq 33\) and several infinite families, all of propriety \((2,2,\ldots,2)\). Finally, we establish that if there is an Hadamard matrix that can be obtained from a group difference set with parameters \((4s^2, 2s^2, s^2)\) then there is a higher-dimensional Hadamard matrix of order \((4s^2)^n\).
Introduction

In [2] it is pointed out that it is possible to define orthogonality for higher dimensional matrices in many ways.

Intuitively we see that each two-dimensional matrix within the n-dimensional matrix could have orthogonal row vectors (we call this propriety \((2,2,\ldots,2)\)); or perhaps each pair of two-dimensional layers

\[
A^1 = \begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_t 
\end{bmatrix}
\quad \text{and} \quad
b^j = \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_t 
\end{bmatrix}
\]

could have \(A \cdot B = \text{tr}(AB^T) = a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_t \cdot b_t = 0\) (note if the row vectors in this direction had been orthogonal we would have had \(a_i \cdot b_i = 0\) for each \(i\)) (we call this propriety \((\ldots,3,\ldots)\)); or perhaps each pair of three-dimensional layers

\[
a = \begin{bmatrix}
    A^1 \\
    A^2 \\
    \vdots \\
    A^n 
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
    b^1 \\
    b^2 \\
    \vdots \\
    b^n 
\end{bmatrix}
\]

could have \(a \cdot b = A^1 \cdot b^1 + \ldots + A^n \cdot b^n = 0\) (note that if the 2-dimensional matrices had been orthogonal we would have had \(A_i \cdot b_i = 0\) for each \(j\)); and so on.

We say an n-dimensional matrix is orthogonal of propriety \((d_1,\ldots,d_n)\) with \(2 \leq d_i \leq n\) where \(d_i\) indicates that in the \(i^{th}\) direction (i.e. the \(i^{th}\) coordinate) the \(d_i-1^{st}\), \(d_i^{th}\), \(d_i+1^{st}\), \ldots, \((n-1)^{th}\) dimensional layers are orthogonal but the \(d_i^{2nd}\) layer is not orthogonal. \(d_i = \) means not even the \((n-1)^{th}\) layers are orthogonal.
The Paley cube of size \((q+1)^n\) constructed in [2] for \(q \equiv 3 \pmod{4}\) a prime power has property \((e,e,\ldots,e)\) but if the 2-dimensional layer of all ones is removed in one direction the remaining \(n\)-dimensional matrix has all 2-dimensional layers in that direction orthogonal.

An \(n\)-cube orthogonal design, \(D = [d_{ij}^k\ldots]^n\), of property 
\begin{align*}
(d_1,d_2,\ldots,d_n), \quad \text{side } d \text{ and type } (s_1,s_2,\ldots,s_t)^n \text{ on the commuting variables } \\
x_1,x_2,\ldots,x_t \text{ has entries from the set } \{0,tx_1,\ldots,tx_1,\ldots,tx_t\} \text{ where } tx_i \\
\text{occurs } s_i \text{ times in each row and column of each 2-dimensional layer and in which each } e_j \text{-dimensional layer, } d_j-1 \leq e_j \leq n-1, \text{ in the } i^{th} \text{ direction is orthogonal.}
\end{align*}

Shlichta [3] found \(n\)-dimensional Hadamard matrices of size \((2^t)^n\) and property \((2,2,\ldots,2)\). In [2] the concept of higher dimensional suitable matrices was introduced to show that if \(t\) is the side of 4-Williamson matrices there is a 3-dimensional Hadamard matrix of size \((4t)^3\) and property \((2,2,2)\). In [4] it is shown that there are \(n\)-dimensional orthogonal designs of type \((2^t,2^t)^n\), side \(2^t\), \(t \geq 0\) and property \((2,2,\ldots,2)\).

The results of [4] and [2] can be combined to give

**Theorem 1.** When \(q \equiv 1 \pmod{4}\) is a prime power, there exist higher-dimensional orthogonal designs of order \((q)^{\frac{n}{2}}(q+1)^{\frac{n}{2}}\) and property \((2,2,\ldots,2)\).

The equivalence relations:

A \((1,1,1,1)^{k+1}\) design would be preserved under the following equivalence relations:

1. each variable is replaced throughout by its negative;
2. rearrangement of the parallel \(k\)-dimensional hyper-planes (for the \((1,1,1,1)^2\) design this is the rows and/or columns, for the \((1,1,1,1)^3\) design this is the parallel planes);
3. multiplication of every variable of one entire \(k\)-dimensional hyper-plane by -1.

There are exactly two inequivalent \((1,1,1,1)^2\) designs of order 4 on the variables \(a,b,c,d\). They are
A little checking shows that a \((1,1,1,1)^3\) design can be obtained with either I or II as the front face. The design in the figure at the end of the paper has faces from equivalence class II.

**The \((1,1,1,1)^n\) design**

We proceed inductively. First define

\[
\begin{align*}
\mathbf{g}_1 &= (-d, c, b, a) \\
\mathbf{g}_2 &= (c, -d, a, b) \\
\mathbf{g}_3 &= (b, a, d, c) \\
\mathbf{g}_4 &= (-a, b, c, d)
\end{align*}
\]

and note

\[
\mathbf{g}_i \cdot \mathbf{g}_j = 0 \quad \text{for all } i \neq j.
\]

Now we can describe the faces of the \((1,1,1,1)^3\) design given in Figure 4a of Hammer and Seberry [2], in one direction as:

\[
\mathbf{b}_4 = \begin{bmatrix} \mathbf{g}_4 \\ \mathbf{g}_3 \\ \mathbf{g}_2 \\ \mathbf{g}_1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -\mathbf{g}_4 \\ \mathbf{g}_3 \\ \mathbf{g}_2 \\ -\mathbf{g}_1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} \mathbf{g}_4 \\ -\mathbf{g}_3 \\ \mathbf{g}_2 \\ -\mathbf{g}_1 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} -\mathbf{g}_4 \\ -\mathbf{g}_3 \\ \mathbf{g}_2 \\ \mathbf{g}_1 \end{bmatrix}
\]

or

\[
\begin{bmatrix} \mathbf{b}_4 \\ \mathbf{b}_3 \\ \mathbf{b}_2 \\ \mathbf{b}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_4 & \mathbf{g}_3 & \mathbf{g}_2 & \mathbf{g}_1 \\ \mathbf{g}_3 & -\mathbf{g}_4 & \mathbf{g}_2 & \mathbf{g}_1 \\ \mathbf{g}_2 & -\mathbf{g}_3 & -\mathbf{g}_4 & \mathbf{g}_1 \\ \mathbf{g}_1 & -\mathbf{g}_2 & -\mathbf{g}_3 & -\mathbf{g}_4 \end{bmatrix}.
\]
Now there are two inequivalent 2-dimensional \((1,1,1,1)\) orthogonal designs, but both of these can be completed to give a \((1,1,1,1)^3\) design. Thus we have a \((1,1,1,1)^3\) design on the commuting variables \(a_1, a_2, a_3, a_4\) and hence a \((1,1,1,1)^4\) design on the variables \(a, b, c, d\).

The orthogonality within the \((1,1,1,1)^3\) design is established by construction. The orthogonality of the \((1,1,1,1)^4\) design is obtained by using the extra property that \(a_i \cdot a_j = 0\) for all \(i \neq j\).

To obtain the \((1,1,1,1)^{k+1}\) design, we assume the existence of the \((1,1,1,1)^{k}\) design for every \(k \geq 1\) made by the construction. Now we have by construction a \((1,1,1,1)^k\) design whose hyper-rows, \(c_i, i = 1, 2, 3, 4\), comprise objects which are the hyper-rows of the \((1,1,1,1)^{k-1}\) design. We now write down the hyper-rows of each of the four hyper-planes containing these rows as the columns of a \(4 \times 4\) matrix, \(D\). By the construction \(D\) is an orthogonal design of type \((1,1,1,1)\) whose objects are the \(c_i\). We now complete \(D\) to form a \((1,1,1,1)^3\) design, \(E\), with objects \(c_i\). Now \(E\) is a \((1,1,1,1)^{k+1}\) design whose orthogonality is guaranteed by the existence assumption.

In fact at all stages of the construction property was completely preserved. Hence we have shown

**Theorem 2.** There exist higher-dimensional orthogonal designs of type \((1,1,1,1)^n\) and side 4 for every dimension \(n \geq 0\) with property \((2, 2, \ldots, 2)\).

Using the results of [2], we can now say

**Corollary 3.** Let \(t\) be the order of 4-Williamson matrices. Then there is a higher-dimensional Hadamard matrix of order \((4t)^t\). In particular there are higher-dimensional Hadamard matrices of order \((4t)^t\) for all odd \(t < 100\) except possibly 35, 39, 47, 53, 59, 67, 71, 73, 77, 83, 89.

*All these matrices have property \((2, 2, \ldots, 2)\)*

**n-dimensional Hadamard matrices from difference sets**

Let \(H = (h_{ij})\) be any Hadamard matrix of side \(h\) which can be defined by a function of the form

\[
h_{ij} = \gamma(a(i) + a(j)),
\]

where \(a\) is 1-1 and onto, so that
Example: Any Hadamard matrix that can be obtained from abelian group difference set is of this form.

**Theorem 4.** Let \( H \) be a Hadamard matrix of the type described above. Define \( G = (g_{pq\ldots s}) \) of side \( n \) by

\[
\sum_{1}^{h} h_{pq\ldots s} g_{pq\ldots s} = h_{p+q+r+\ldots+i+s}.
\]

Then \( G \) is a proper Hadamard matrix of side \( h \) and dimension \( n \).

**Proof.** We need to show any 2-dimensional face is an Hadamard matrix. We let the \( c \)-th coordinate have two fixed values \( a \) and \( b \) and let the \( d \)-th coordinate take the values \( 1, \ldots, h \). We now fix all other coordinates and consider

\[
\sum_{1}^{h} x_{pq\ldots s} = x_{p+q+r+\ldots+i+s}.
\]
REFERENCES


The orthogonal design of type $(1, 1, 1, 1)^3$