A survey of orthogonal designs

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The known results are given and unsolved problems indicated.

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A SURVEY OF ORTHOGONAL DESIGNS

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Abstract

This paper surveys orthogonal designs which are an overview of Baumert-Hall arrays, Hadamard matrices and weighing matrices. The known results are given and unsolved problems indicated.

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§1 INTRODUCTION AND DEFINITIONS

This paper is intended to survey orthogonal designs and highlight the unsolved problems in the area. No proofs are given but the source of the result is indicated.

Definition. An orthogonal design of order $n$ and type $(s_1, \ldots, s_q)$ on the commuting variable $x_1, \ldots, x_q$ is an $n \times n$ matrix, $A$, with entries chosen from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_q\}$ such that

$$AA^\top = (s_1 x_1^2 + \cdots + s_q x_q^2) I_n.$$

Alternatively, the rows (and hence columns) of $A$ are formally orthogonal and every row (column) contains $s_i$ entries of the type $\pm x_i$.

If $A$ is as above, we may write

$$A = x_1 A_1 + x_2 A_2 + \cdots + x_q A_q,$$

where

i) $A_i A_i^\top = s_i I_n$

ii) $A_i A_j^\top + A_j A_i^\top = 0$, $i \neq j$

iii) the $A_i$ are $(0, 1, -1)$ matrices,

iv) $A_i^* A_j = 0$ for $i \neq j$ (* is the Hadamard product).
It was shown in [7] that $\rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b, \quad b \text{ odd, } a = 4c + d, \quad 0 \leq d < 4.$$

**Definition** A weighing matrix of weight $k$ and order $n$, is a square $(0, 1, -1)$ matrix, $A$, of order $n$ satisfying

$$AA^T = kI_n$$

In [7] it was shown that the existence of an orthogonal design of order $n$ and type $(s_1, \ldots, s_t)$ is equivalent to the existence of weighing matrices $A_1, \ldots, A_t$, of order $n$, where $A_i$ has weight $s_i$ and the matrices, $\{A_i\}_{i=1}^t$, satisfy the matrix equations

$$XY^T + YX^T = 0 \quad \text{and} \quad A_i^\otimes A_j = 0, \quad \text{if} \quad i \neq j.$$

In pairs. In particular, the existence of an orthogonal design of order $n$ and type $(1, k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight $k$ and order $n$.

Weighing matrices are a generalization of Hadamard matrices, $H$, which are $(1, -1)$-matrices satisfying

$$HH^T = nI_n,$$

where $n$ is the order of the matrix. It is conjectured that Hadamard matrices exist for every order $n \equiv 0 \pmod{4}$. Further it is conjectured that for these orders there exists an Hadamard matrix (called a skew-Hadamard matrix) of the form $H = I_n + S$ where $S^T = -S$. 

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Let $R$ be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be constructed from two circulant matrices $A$ and $B$ if it is of the form

$$\begin{bmatrix}
A & B \\
B^t & -A^t
\end{bmatrix}$$

and to be of Goethals-Seidel type if it is of the form

$$\begin{bmatrix}
A & BR & CR & DR \\
-BR & A & D^t R & -C^t R \\
-CR & -D^t R & A & B^t R \\
-DR & C^t R & -B^t R & A
\end{bmatrix}$$

where $A, B, C, D$ are circulant matrices.

Two weighing matrices $W$ and $N$ of order $n$ and weights $w_1$ and $w_2$ will be called amicable weighing matrices if

$$w^t = -w, \quad n^t = n$$

$$WN^t = NW^t$$

If $I + W$ and $N$ are Hadamard matrices satisfying these equations then they are called amicable Hadamard matrices (see [28]).


§2 THE CONJECTURES

It is conjectured that:

(I) for \( n \equiv 2 \pmod{4} \) there is a weighing matrix of weight \( k \) and order \( n \) for every \( k \leq n-1 \) which is the sum of two integer squares;

(II) for \( n \equiv 2 \pmod{4} \) there is a skew-symmetric weighing matrix of order \( n \) for every \( k < n-1 \) such that \( k \) is an integer square; or equivalently

(IIA) for \( n \equiv 2 \pmod{4} \) there is an orthogonal design of type \((1,k)\) in order \( n \) for every \( k < n-1 \) such that \( k \) is an integer square;

(III) for \( n \equiv 0 \pmod{4} \) there is a weighing matrix of weight \( k \) and order \( n \) for every \( k \leq n \);

(IV) for \( n \equiv 4 \pmod{8} \) there is a skew-symmetric weighing matrix of order \( n \) for every \( k < n \), where \( k \) is the sum of \( \leq 3 \) squares of integers; or equivalently

(IVA) there is an orthogonal design of type \((1,k)\) in order \( n \equiv 4 \pmod{8} \) for every \( k < n \) which is the sum of \( \leq 3 \) squares of integers;

(V) for \( n \equiv 0 \pmod{8} \) there is a skew-symmetric weighing matrix of order \( n \) for every \( k < n \); or equivalently

(VA) there is an orthogonal design of type \((1,k)\) in order \( n \equiv 0 \pmod{8} \) for every \( k < n \)
Conjecture III above is an extension of the Hadamard conjecture (i.e., for every \( n \equiv 0 \pmod{4} \) there is a \((1,-1)\) matrix, \( H \), of order \( n \) satisfying \( HH^T = nI_n \)) while (IV) and (V) generalize the conjecture that for every \( n \equiv 0 \pmod{4} \) there is a Hadamard, \( H \), matrix of order \( n \), with the property that \( H = I_n + S \) where \( S = -S^T \).

Conjecture (III) was established in [29] for \( n \in \{4, 8, 12, \ldots, 32, 40\} \) and in [10] for \( n = 2^t \). Conjecture (IV) was established in [7, Theorem 17] for \( n = 2^t \) (\( t \geq 3 \)). In [11] conjectures (IV) and (V) (and as a consequence conjecture (III)) were established for \( n = 2^{t+1} \cdot 3 \), \( n = 2^{t+1} \cdot 5 \), \( t \) a positive integer. Also it is only necessary to find a design \((1,47)\) in order 56 and the conjectures (IV) and (IVA) (and hence (III)) will be validated for \( n = 2^t \cdot 7 \) (\( t \geq 3 \)).

Some conjectures that come to mind on orthogonal designs are:

(i) A necessary and sufficient condition that there exist a design of type \((a,b)\) in order \( 2t \), \( t \) odd, is that \( \frac{b}{a} \) is a rational square and that \( a + b < 2t \). (We doubt this is true.)

(ii) A necessary and sufficient condition that there exist a design of type \((a,a,b)\) in order \( n=4t \), \( t \) odd, is that \( \frac{b}{a} \) be a sum of two rational squares;

(iii) A necessary and sufficient condition that there exist a design of type \((a,a,a,b)\) in order \( n=4t \), \( t \) odd, is that \( \frac{b}{a} \) be a rational square;

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A necessary and sufficient condition that there exist a design of type \((a,b)\) in order \(n=4t\), \(t\) odd, is that \(b\) be a sum of three rational squares.

§3 EXISTENCE THEOREMS: NECESSARY CONDITIONS

Lemma 1 [7]. The maximum number of variables in an orthogonal design of order \(n\) is \(\rho(n)\) where \(\rho(n)\) is the Radon number.

Lemma 2. (Raghavarao-van Lint-Seidel). Let \(n = 2\ (mod\ 4)\) and let \(A\) be a matrix of order \(n\) with entries in \(Q\) satisfying \(AA^t = kI_n\). Then \(k = q_1^2 + q_2^2\) where \(q_1, q_2 \in Q\). Moreover, if \(k \in Z\) then \(k = a^2 + b^2\), \(a, b \in Z\).

Corollary [7]. If \(n = 2\ (mod\ 4)\) and there is an orthogonal design of order \(n\) and type \((a, b)\), then \(a, b\) and \(a + b\) must be the sum of \(\leq 2\) integer squares and \(a + b < n\).

Lemma 3 [13]. Let \(X\) be a matrix of order \(n = 2\ (mod\ 4)\) with entries in the field \(Q(i)\) \((i^2 = -1)\). Suppose

\(i)\ \ X = -X^t\)

and

\(ii)\ \ XX^t = kI_n\).

Then \(k = q_1^2 + q_2^2\) where \(q_1, q_2 \in Q\). If, in addition \(k \in Z\) then \(q_1\) and \(q_2\) may be chosen in \(Z\).

Lemma 4 [13]. Let \(X, Y\) be matrices of order \(n = 2s\) \((s\ odd)\) with entries in \(Q(i)\). Suppose

\(i)\ \ X = -X^t, \ Y = -Y^t,\)

\(ii)\ \ XX^t = Y Y^t = kI_{n}\).
(iii) \(XX^* = I_n^t, YY^* = kI_n^t\) (where \(^t\) is the conjugate transpose.)

then \(k = a^2\) for some \(a \in \mathbb{Q}\). If, in addition, \(k \in \mathbb{Z}\) then
"a" may be chosen in \(\mathbb{Z}\).

Lemma 5 [13]. If \(n \equiv 3 \, (\text{mod} \, 4)\) and there is an orthogonal
design of order \(n\) and type \((a,b)\) then \(\frac{b}{a}\) is a rational
square.

Lemma 6 [7]. Let \(n \equiv 4 \, (\text{mod} \, 8)\) and let \(X\) be an orthogonal
design of order \(n\) and type \((a,b)\). Then \(\frac{b}{a}\) must be a sum
of \(\leq 2\) rational squares.

Lemma 7 [13]. Let \(n \equiv 4 \, (\text{mod} \, 8)\) and let \(X\) be an orthogonal
design of order \(n\) and type \((a,a,b)\). Then \(\frac{b}{a}\) must be a sum
of \(\leq 2\) rational squares.

Lemma 8 [13]. Let \(n \equiv 4 \, (\text{mod} \, 8)\) and let \(X\) be an orthogonal
design of order \(n\) and type \((a,a,a,b)\) then \(\frac{b}{a^2} = \frac{t^2}{s}, t \in \mathbb{Q}\).

§4 THE MOST POWERFUL CONSTRUCTIONS

Baumert-Hall Arrays:

Let \(n = 4t, t\ \text{odd}, \) then \(\rho(n) = 4\). The Baumert-
Hall Arrays are orthogonal designs of order \(n\) and type
\((t,t,t,t,t)\). These exist for a large number of odd \(t\) (see [1],
[3], [5], [18], [25], [27]). and it has been conjectured they
exist for all odd \(t\).
For a discussion of these arrays see [33; Part 4, chapter VII].

**Designs of type (1,1,...,1):**

Let $n$ be any integer. In [9] it was shown that there is an orthogonal design of order $n$ and type $(1,1,...,1)$ on the variables $x_1,...,x_p(n)$.

**Plotkin's Array:**

Let $n = 8t$, $t$ odd, then $p(n) = 8$. A Plotkin array is an orthogonal design of order $n$ and type $(t,t,t,t,t,t,t,t)$. If $t = 1$ such an array is a classical one derived from the multiplication of the Cayley numbers.

In his paper Plotkin shows:

**Lemma 1 [21].** Let $n = 4t$ be the order of an Hadamard matrix then there are orthogonal designs of type

(i) $(2t,2t)$ in order $4t$;

(ii) $(2t,2t,2t,2t)$ in order $8t$;

(iii) $(2t,2t,2t,2t,2t,2t,2t,2t)$ in order $16t$.

We now give some of the most powerful constructions for orthogonal designs:

**Lemma 2 [7].** If $A$ is an orthogonal design of order $n$ and type $(e_1,e_2,...,e_k)$ on the variables $x_1,x_2,...,x_k$ then there is an orthogonal design of order $n$ and type $(e_1+e_j,...,e_j,...,e_k)$ on the $t-1$ variables $x_1,x_2,...,x_j,...,x_k$.

**Example.** An orthogonal design of type $(1,2,3)$ on the variables $x_1,x_2,x_3$ in order $n$ means there are also orthogonal designs $(1,5)$, $(2,4)$, $(3,3)$ on the variables $x_1,x_2$ in order $n$.

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Lemma 3 [7]. If there exists an orthogonal design of type $(e_1, \ldots, e_k)$ on the variables $x_1, \ldots, x_k$ in order $n$ then there exists an orthogonal design of type $(e_1, \ldots, e_k)$ on the variables $x_1, \ldots, x_k$ in order $mn$ for any integer $m \geq 1$.

Lemma 4 [7]. If there exists an orthogonal design of type $(e_1, \ldots, e_k)$ in order $n$ then there exists an orthogonal design of type $(e_1, e_2, e_3, \ldots, e_k)$ in order $2n$, $e_1 = 1$ or 2.

Lemma 5 [7]. Suppose there exist amicable weighing matrices of weights $k$ and $m$ and order $p$ then there exists an orthogonal design of order $np$ and type $(1, k, m, \ldots, m)$ on the variables $x_1, x_2, x_3, \ldots, x_p(n)$.

Lemma 6 [7]. Suppose there exist two circulant matrices $A, B$ of order $n$ satisfying

$$AA^t + BB^t = fI_n.$$ 

Further suppose the $R$ is the back diagonal matrix. Then

$$G = \begin{bmatrix} A & BR \\ -BR & A \end{bmatrix}$$

is a $W(2n, f)$ when $A, B$ are $(0, 1, -1)$-matrices and an orthogonal design of order $2n$ and type $(e_1, e_2, \ldots, e_k)$ on $x_1, \ldots, x_k$ when

$$f = \frac{1}{k} \sum_{i=1}^{k} e_i^2.$$ 

Further $G$ is skew or skew-type if $A$ is skew or skew-type.
Lemma 7 (Goethals and Seidel [15]).

Suppose there exist four circulant matrices

\[ A, B, C, D \text{ of order } n \text{ satisfying} \]

\[ AA^t + BB^t + CC^t + DD^t = fI_n. \]

Let \( R \) be the back-diagonal matrix. Then

\[
GS = \begin{pmatrix}
A & BR & CR & DR \\
-DR & A & D^tR & -C^tR \\
-CR & -D^tR & A & B^tR \\
-DR & C^tR & -B^tR & A
\end{pmatrix}
\]

is a \( W(4n, f) \) when \( A, B, C, D \) are \((0,1,-1)\) matrices, and an orthogonal design of order \( 4n \) and type \((s_1, s_2, \ldots, s_{2k})\) on \( x_1, x_2, \ldots, x_k \) when

\[ f = \sum_{i=1}^{k} s_i x_i^2. \]

Further \( GS \) is skew or skew-type if \( A \) is skew or skew-type.

Lemma 8 [7]. If there exists an orthogonal design of type \((a,b)\) in order \( n \) there exists an orthogonal design of type \((a,a,b,b)\) in order \( 2n \).

Lemma 9 [7]. If there exists an orthogonal design of type \((a,b)\) in order \( n \) there exists an orthogonal design of type \((a,a,2a,b,b,2b)\) in order \( 4n \).

Lemma 10 [7]. If there exists an orthogonal design of type \((s_1,s_2,\ldots,s_{2k})\) in order \( n_1 \) and \( n_2 \) then there exists an orthogonal design of type \((s_1,s_2,\ldots,s_{2k})\) in order \( n_1 + n_2 \).
Lemma 11 [11], [16]. There exist orthogonal designs \((1,k)\) in orders 
\(2^tm, \ t \geq 3\) an integer and \(m=1,3,5\) or 9.

Part (i) of the following lemma was also discovered by Joan

Murphy Germaine.

Lemma 12 [7], [14]. If there exists an orthogonal design of type
\(\{s_1, s_2, \ldots, s_\ell\}\) in order \(n\) there exists orthogonal designs
\((i)\) \((s_1s_2, s_3, \ldots, s_\ell)\) and
\((ii)\) \((s_2s_3, s_1, s_4, \ldots, s_\ell)\)
in order \(2n\),

§5 APPENDICES

APPENDIX A: The weighing matrix problem; known results.

Lemma 1 [7]. If a \(W(n,e)\) exists and

(i) \(n\) is odd then \(e\) is a square;
(ii) \(n \equiv 1 \mod 4\) then \(e\) is the sum of two squares;
(iii) \((n-e)^2 - (n-e) + 2 > n\) for odd \(n\);
(iv) \(W(m^5 + m + 1, m^5)\) exist only if \(m\) is the order
    of a projective plane.

Lemma 2 [7]. If there exists a \(W(n,k)\) and \(n \equiv 2 \mod 4\) then

(i) \(k\) is the sum of two integer squares;
(ii) \(k \leq n - 1\).

Given these two results we have the conjectures:

(II) for \(n \equiv 2 \mod 4\) there is a weighing matrix of weight
    \(k\) and order \(n\) for every \(k \leq n - 1\) which is the sum
    of two integer squares;
(III) for \( n \equiv 0 \pmod{4} \) there is a weighing matrix of weight \( k \) and order \( n \) for every \( k \leq n \).

As noted before conjecture III is an extension of the Hadamard conjecture.

Lemma 3 [29] [10]. Conjecture III is true for

(i) \( n \in \mathbb{N} = \{4,8,12,\ldots,32,40\} \);

(ii) \( n=2^m \) if \( t = 2 \) for \( m = 1, 5, \) or \( 9 \), and

\[ t \geq 3 \text{ for } m = 1, 5, 9, \text{ or } 9. \]

We note that the existence of an orthogonal design \((1,k)\) of order \( n \) implies the existence of a \( W(n,k) \) and a \( W(n,k+1) \). Thus the appendix on designs \((1,k)\) should be consulted for numerical results.

APPENDIX B: The skew weighing matrix problem: known results.

Lemma 1 [7]. Let \( n \equiv 2 \pmod{4} \). If \( X \) is a skew-symmetric rational matrix of order \( n \) satisfying

\[ XX^* = kI_n \]

then \( k \) is a rational square. If \( k \) is an integer it is an integer square.

Lemma 2 [7]. Let \( n \equiv 4 \pmod{8} \). If \( X \) is a skew-symmetric rational matrix of order \( n \) satisfying

\[ XX^* = kI_n \]

then \( k \) is the sum of three rational squares. If \( k \) is integer it is the sum of three integer squares.

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Lemma 3 [11]. Let \( t \) be an integer \( \geq 3 \).

(i) Conjecture IV is true for \( 4m, m=1,3,5, \) or 7;
(ii) Conjecture V is true for \( 5^m, m=1,3,5 \) or 8.

Now the existence of a skew-weighing matrix \( W(n,k) \) is equivalent to the existence of an orthogonal design \( (1,k) \) of order \( n \) so the appendix on the designs \( (1,k) \) should be consulted for numerical results on this conjecture.

APPENDIX C: The design \( (1,k) \) problem: known results.

Most of these results are cited from [11], [12], and [14]. We are concerned with conjectures

(II) for \( n \equiv 2 \pmod{4} \) there is an orthogonal design of type \( (1,k) \) in order \( n \) for every \( k < n-1 \) such that \( k \) is an integer square;

(IVA) for \( n \equiv 4 \pmod{8} \) there is an orthogonal design of type \( (1,k) \) in order \( n \) for every \( k < n \) which is the sum of \( \leq 3 \) squares of integers.

(VA) for \( n \equiv 0 \pmod{8} \) there is an orthogonal design of type \( (1,k) \) in order \( n \) for every \( k < n \).

We now cite the theorems relevant to this problem

Lemma 1 [7, Corollary to Construction 22]. If there is an orthogonal design of type \( (1,k) \) in order \( n \) then there is an orthogonal design of type \( (1,1,k) \) in order \( 2n \) and of type \( (1,1,3,k,6,4t) \) in order \( 4n \).
Corollary 1. If there are orthogonal designs of type \((1,k)\), 
\[1 \leq k \leq t\] in order \(n\), then there are orthogonal designs of
type \((1,m), \ 1 \leq m \leq 2t + 1\) in order \(2n\).

Corollary 2. If there are orthogonal designs of type \((1,k)\), 
\[1 \leq k \leq n - 1\] in order \(n\), then there are orthogonal designs of
type \((1,m), \ 1 \leq m \leq 2^n - 1\) in order \(2^n, t\) a positive integer.

Lemma 2. Let \(n\) be any number of the form \(2^t.3, 2^t.5\) or 
\(2^t.9\), with \(t\) a positive integer.

(a) If \(t = 1\), then conjecture I is true (except for \(2.9\)).

(b) If \(t = 2\), then conjecture IVA is true (except for \(2^3.9\)).

(c) If \(t \geq 3\), then conjecture V (and consequently conjecture III) is true.

Lemma 3.

(a) Conjecture I is true for \(n = 2.7\).

(b) For \(n = 4.7\), conjecture IVA is true.

(c) For \(n = 8.7\), there is an orthogonal design of type \((1,k)\) for \(1 \leq k \leq 5\), except possibly for \(k = 47\).

Lemma 4. Let \(r\) be any number of the form \(2^a.10^b.36^c\), 
a, b, c non-negative integers, and let \(n\) be any integer > \(r\). Then

(i) There are orthogonal designs of order \(4n\) and types \((1,1,2r)\) and \((1,1,r)\).

If, in addition \(n\) is odd, then

(ii) there are orthogonal designs of order \(4n\) and types \((1,4,r)\) and \((1,4,2r)\).
Lemma 5. The following orthogonal designs exist in the order $2n$ as indicated:

(i) $(1,1)$ in orders $2n \geq 2$;
(ii) $(1,4)$ in orders $2n \geq 6$;
(iii) $(1,9)$ in orders $10, 14, 16, 20, 2n \geq 34$;

Lemma 6. The following orthogonal designs exist in the order $4n$ as indicated:

(i) $(1,1,13)$ in orders $4n$, $n($odd$) \geq 11$;
(ii) $(1,1,16)$ in orders $4n$, $n($odd$) \geq 6$;
(iii) $(1,1,18)$ in orders $20, 24, 28, 32$ and all $4n \geq 40$;
(iv) $(1,1,20)$ in orders $24, 28, 32$ and all $4n \geq 40$;
(v) $(1,1,26)$ in orders $4n$, $n($odd$) \geq 15$;
(vi) $(1,1,32)$ in orders $4n$, $n($odd$) \geq 9$;
(vii) $(1,1,34)$ in order $36$ (from [30]);
(viii) $(1,1,40)$ in orders $4n$, $n($odd$) \geq 11$;
(ix) $(1,1,1,4)$ in orders $4n \geq 8$;
(x) $(1,1,1,9)$ in orders $4n \geq 12$;
(xi) $(1,1,1,16)$ in orders $4n$, $n($odd$) \geq 7$;

(xii) $(1,1,1,32)$ in order $32$;
(xiii) $(1,1,2,8)$ in orders $4n \geq 12$;
(xiv) $(1,1,5,5)$ in orders $4n \geq 12$;
(xv) $(1,2,2,4)$ in orders $4n \geq 12$;
(xvi) $(1,2,3,6)$ in orders $4n \geq 12$.
Using all means available we have (at least):

**Lemma 7.** If $n$ is odd there are orthogonal designs of order $4n$ and type $(1,k)$ when

- (i) $n \geq 5$, $k \in \{1, \ldots, 6, 8, \ldots, 11\}$;
- (ii) $n \geq 5$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17, 18\}$;
- (iii) $n \geq 7$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17, 18, 20, 21, 22, 24, \ldots, 27\}$;
- (iv) $n \geq 9$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17, 18, 20, 21, 22, 24, \ldots, 27, 29, 32, 33, 34\}$;
- (v) $n \geq 11$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, \ldots, 22, 24, \ldots, 27, 29, 32, 33, 34, 40, 41\}$;
- (vi) $n \geq 13$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, \ldots, 22, 24, \ldots, 27, 29, 30, 32, 33, 34, 40, 41\}$;

**Lemma 8.** There are orthogonal designs of order $8n$ and type $(1,k)$ where

- (i) $n = 3, 4, 5$ or $6$, $k \in \{1, \ldots, 8n - 1\}$;
- (ii) $n \geq 7$, $k \in \{1, \ldots, 46\}$.

**Lemma 9 [14].** Given a square integer $k$ there exists an integer $N(k)$ such that

- (i) $(1,k)$ exists in every order $8n$ for $n > N(k)$,
- (ii) $(1,1,k)$ exists in every order $4n$ for $n > N(k)$.

Finally we tabulate the unresolved cases in the conjectures.
Unresolved Cases on the conjecture that weighing matrices exist for all \( k < n \) and Unresolved Cases of appropriate conjecture on the existence of \((l,k)\):

<table>
<thead>
<tr>
<th>Order</th>
<th>True/False</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>8</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>12</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>16</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>20</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>24</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>28</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>32</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>36</td>
<td>true, 31</td>
<td>31, 19, 30</td>
</tr>
<tr>
<td>40</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>44</td>
<td>true, 31</td>
<td>30, 42</td>
</tr>
<tr>
<td>48</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>52</td>
<td>true, 37, 46, 47, 49</td>
<td>35-38, 42-46, 48-50</td>
</tr>
<tr>
<td>56</td>
<td>true</td>
<td>47</td>
</tr>
<tr>
<td>60</td>
<td>true, 51, 53</td>
<td>38, 42, 43, 44, 48, 50-54, 56-58</td>
</tr>
<tr>
<td>64</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>72</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>80</td>
<td>true</td>
<td>true</td>
</tr>
</tbody>
</table>

Table 1

True signifies the conjecture is verified.
Unresolved Cases of conjecture the weighing matrices exist for all $k < n$ which are the sum of two squares

<table>
<thead>
<tr>
<th>Order</th>
<th>Unresolved Cases of conjecture that $(1,k)$ exists for all $k &lt; n$ which are squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>true</td>
</tr>
<tr>
<td>6</td>
<td>true</td>
</tr>
<tr>
<td>10</td>
<td>true</td>
</tr>
<tr>
<td>14</td>
<td>true</td>
</tr>
<tr>
<td>18</td>
<td>9</td>
</tr>
<tr>
<td>22</td>
<td>18</td>
</tr>
<tr>
<td>26</td>
<td>17</td>
</tr>
<tr>
<td>30</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 2

True signifies the conjecture is verified.

APPENDIX D: The two variables problem.

The Radon number indicates that designs in two variables can exist only in even orders. The appropriate conjectures are:

IVA  there is an orthogonal design of type $(1,k)$ in order $n \equiv 4 \pmod{8}$ for every $k < n$ which is the sum of $\leq 3$ squares of integers;

VA  there is an orthogonal design of type $(1,k)$ in order $n \equiv 0 \pmod{8}$ for every $k < n$;
(i) For \( n \equiv 2 \pmod{4} \) a necessary and sufficient condition that there exist a design of type \((a, b)\) is that \( \frac{b}{a} \) is a rational square and that \( a + b < n; \)

(iv) For \( n \equiv 4 \pmod{8} \) a necessary and sufficient condition that there exist a design of type \((a, b)\) is that \( \frac{b}{a} \) is a sum of three rational squares.

Some relevant results are Lemmas 2, 3, 4, 5 and 6 of §3. We note that Lemmas 5 and 6 of §3 establish the necessary conditions of conjectures (i) and (iv).

For the known results on designs of order \( n \equiv 2 \pmod{4} \) we refer the reader to table 2 of Appendix C.

**Lemma 1.** If there exist all orthogonal designs \((1, k), k < n,\) in order \( n \) then there exist all orthogonal designs \((1, \ell), \ell < 2n,\) in order \( 2n.\)

**Corollary.** There exist all orthogonal designs \((1, k)\) in order \( 2^t, 3_3, 5_3 \) and \( 7_3, \) \( t \geq 2 \) and integer.

**Lemma 2.** The existence of all designs \((1, i, j)\) in order \( n \) implies the existence of all designs \((k, \ell)\) in order \( 2n.\)

**Lemma 3.** Let \( n=12. \) Then \( \frac{b}{a} \) is the sum of three rational squares a necessary and sufficient condition for the existence of designs \((a, b)\) in order 12.

We give the results on two variables for a few other orders.

**Order 16:** all two variable designs exist.

**Order 20:** the cases \((3,16), (6,13), (7,10)\) and \((7,12)\) remain unsolved.
order 24: all two variable designs exist.

order 28: the status of the two variable problem is that \((j,k)\)
exists for each \(j\) except for those \(k\) explicitly
excluded in the following list.

\((2,k), k \in \{x : 0 \leq x \leq 26, x \neq 22, 23, 24\}\)

\((3,k), k \in \{x : 0 \leq x \leq 25, x \neq 7,10,12,14,15,22,23,24\}\)

\((4,k), k \in \{x : 0 \leq x \leq 24, x \neq 19,20,21,22,24\}\)

\((5,k), k \in \{x : 0 \leq x \leq 23, x \neq 20,21,22,23\}\)

\((6,k), k \in \{x : 0 \leq x \leq 22, x \neq 7,11,13,15,17,18,19,20,21,22\}\)

\((7,k), k \in \{x : 0 \leq x \leq 21, x \neq 10,11,12,14,15,16,18,19,20,21\}\)

\((8,k), k \in \{x : 0 \leq x \leq 20, x \neq 17\}\)

\((9,k), k \in \{x : 0 \leq x \leq 19, x \neq 12,16,17\}\)

\((10,k), k \in \{x : 0 \leq x \leq 18, x \neq 14,15,16,17\}\)

\((11,k), k \in \{x : 0 \leq x \leq 17, x \neq 12,14,15,16,17\}\)

\((12,k), k \in \{x : 0 \leq x \leq 16, x \neq 12,13,14,15,16\}\)

\((13,k)\) all exist

\((14,k)\) all exist
order 32: all two variable designs exist.

order 36: all \((l,k)\) designs except \((1,19)\) and \((1,30)\) are known. The other two variable designs have not yet been studied.

order 40: the following two variable designs are not known:
- \((5,32)\)
- \((6,29)\)
- \((6,31)\)
- \((6,33)\)
- \((7,25)\)
- \((7,32)\)
- \((8,29)\)
- \((8,31)\)
- \((8,33)\)

order 44: all \((l,k)\) designs except \((1,30)\) and \((1,42)\) are known. The other two variable designs have not yet been studied.

order 48: all two variable designs are known.

orders 52, 56, 60, 64, 72, 80: the status of the \((l,k)\) designs is given in table 1 of Appendix C. Other two variable designs have not been studied.

APPENDIX E: The four variables problem.

The Radon number indicates that designs in four variables can only exist in orders \(n \equiv 0 \pmod{4}\). As the number of variables increase the solving of the existence problem becomes correspondingly more difficult. The Goethals-Seidel method is extremely useful but not all designs can be obtained by this method.

The appropriate conjectures are:
(iii) A necessary and sufficient condition that there exist a design of type \((a,a,a,b)\) in orders \(n \equiv 4 \mod 8\) is that \(\frac{b}{a}\) be a rational square;

(v) All four variable designs not ruled out by conjectures (ii), (iii), (iv) exist in orders \(n \equiv 4 \mod 8\).

(vi) All four variable designs exist in orders \(n \equiv 0 \mod 8\).

Lemmas 6, 7, 8 of §3 apply and we recall the essence of these

Lemma 1. Let \(n \equiv 4 \mod 8\) and let \(X\) be an orthogonal design of order \(n\) and type

(i) \((a,b)\). Then \(\frac{b}{a}\) must be a sum of \(\leq 3\) rational squares.

(ii) \((a,a,b)\). The \(\frac{b}{a}\) must be a sum of two rational squares.

(iii) \((a,a,a,b)\). Then \(\frac{b}{a}\) must be a rational square.

\(\text{order 8: all four variable designs exist.}\)

\(\text{order 12: all possible four variable designs exist i.e. Conjecture (V) is true in order 12.}\)

\(\text{order 16: all four variable designs exist except (1,5,5,5).}\)

\(\text{order 20: the following designs in order 20 are not eliminated by the necessary conditions but are not known either.}\)
There are 369 four variable designs which have not yet been found.

**Higher orders** The situation deteriorates rapidly with little being known.

The most useful constructions are given in the following results:

**Lemma 2** [7]. If there exists an orthogonal design of type $(a,b)$ in order $n$ then there exists an orthogonal design of type

(i) $(a,a,b,b)$ in order $2n$;

(ii) $(a,a,2a,b,b)$ in order $4n$.

**Lemma 3.** If there exists an orthogonal design of type $(s_1,s_2,\ldots,s_e)$ in order $n$ then there exists an orthogonal design of type

(i) $(a_1,a_2,\ldots,a_es_{e+1},\ldots,a_es_e)$ in order $n$;

(ii) $(s_1,a_2,\ldots,s_e\overline{a},\ldots,\overline{s_e})$ with $a = 1$ or 2 in order $3n$;

(iii) $(a_1,a_2,\ldots,\overline{s_1},\overline{a},\ldots,\overline{s_e})$ with $a = 1$ or 2 in order $5n$;

(iv) $(s_1,s_2,\ldots,s_e)$ in order $n + m$ whenever a design of type $(s_1,s_2,\ldots,s_e)$ also exists in order $m$.

The known results on a few orders follow.
APPENDIX F: Unsolved Questions and Known Results for $n$ odd.

**Lemma 1 [7].** An orthogonal design of odd order, $n$, has the following properties

(i) there is only one variable so we have a weighing matrix;

(ii) the weight of the variable, $k$, is a square;

(iii) $(n - k)^2 + (n - k) + 2 > n$;

(iv) a $W(\sqrt{m^2 + m + 1}, m^2)$ exists only if a projective plane of order $m$ exists.

**Lemma 2 [13].** Given a square $k$ there exists an integer $N(k)$ such that an orthogonal design of weight $k$ exists for every order $n > N(k)$.

**Lemma 3 [13] [32] [37].** With $N(k)$ as in the previous lemma $N(4) = 10$, $N(9) = 22$, $N(16) = 36$, $N(25) = 82$, $N(36) = 198$.

In the table we give the known results for some orders. These results are taken from [7], [13], and [32].
Problems:  
(i) Find new relations between n and k.  
(ii) Lower the bounds $N(k)$ by finding the unsolved weights.
APPENDIX G: Unsolved Questions and Known Results for
\( n = 2 \pmod{4} \).

We recall that for \( n = 2 \pmod{4} \) there can be
orthogonal designs on only one or two variables. Further

**Lemma 1 [7].** If there exists an orthogonal design of order
\( n \) and type \((a,b)\) then

(i) \( a, b \) and \( a + b \) must all be the sum of two
integer squares;

(ii) \( \frac{b}{a} = \frac{ab}{a} \) is a rational square (i.e. \( ab \) is an
integer square);

(iii) \( a + b \leq n - 1 \).

**Corollary.** If there exists a \( W(n,k) \) then

(i) \( k \) is the sum of two integer squares;

(ii) \( k \leq n - 1 \).

**Lemma 2 [12].** There exists a \( W(n,k) \), constructed using two
circulant matrices for \( k \in \{0,1,2,4,5\} \) in every order \( n \geq 6 \).

**Lemma 3.** There exists a \( W(n,0) \) for every \( n \geq 22 \).

**Lemma 4 [32].** There exists a \( W(n,9) \) for every \( n \geq 18 \).

**Lemma 5.** There exists a \( (1,16) \) in every order \( n \geq 26 \).

**Lemma 6 [32].** There exists a \( (1,9) \) in every order \( n \geq 42 \).

**Lemma 7.** Given a square \( k \) there exists an integer \( M(k) \)
such that an orthogonal design \( (1,k) \) exists for every \( n > M(k) \).

**Lemma 8 [13], [32], [37].** With \( M(k) \) as is the previous lemma,
for \( n = 2 \pmod{4} \), \( M(4) = 6 \), \( M(9) = 18 \), \( M(16) = 18 \),
\( M(25) = 54 \), \( M(36) = 106 \).
We now consider the different orders with respect to:

**The weighing matrix conjecture:** there exists a \( W(n,k) \) for every \( k < n \) which is the sum of two squares.

**The skew weighing matrix conjecture:** there exists an orthogonal design \((l,k)\) for every \( k < n - 1 \) which is an integer square.

**The orthogonal design conjecture:** Lemma 1 gives necessary and sufficient conditions for the existence of orthogonal designs of type \((a,b)\).

The known results are summarized in the table. The entry "nil" signifies the conjecture is true for this order.

See [7], [12], [29], [32], [37], [39] for details.
Integers which are the sums of two squares: 0, 1, 2, 4, 5, 8, 9,
10, 13, 16, 17, 18, 20, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50, ....

<table>
<thead>
<tr>
<th>Order</th>
<th>Unsettled Cases for weighing matrix conjecture</th>
<th>Unsettled Cases for skew weighing matrix conjecture</th>
<th>Unsettled Cases for orthogonal design conjecture</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>nil</td>
<td>nil</td>
<td>nil</td>
</tr>
<tr>
<td>6</td>
<td>nil</td>
<td>nil</td>
<td>nil</td>
</tr>
<tr>
<td>10</td>
<td>nil</td>
<td>nil</td>
<td>nil</td>
</tr>
<tr>
<td>14</td>
<td>nil</td>
<td>nil</td>
<td>nil</td>
</tr>
<tr>
<td>18</td>
<td>nil</td>
<td>9, 16</td>
<td>nil</td>
</tr>
<tr>
<td>22</td>
<td>18</td>
<td>9</td>
<td>nil</td>
</tr>
<tr>
<td>26</td>
<td>17</td>
<td>16</td>
<td>nil</td>
</tr>
<tr>
<td>30</td>
<td>18</td>
<td>16, 25</td>
<td>nil</td>
</tr>
<tr>
<td>34</td>
<td></td>
<td>16, 25</td>
<td>nil</td>
</tr>
<tr>
<td>38</td>
<td></td>
<td>16, 25, 36</td>
<td>nil</td>
</tr>
<tr>
<td>42</td>
<td>18, 25, 29, 36</td>
<td>25, 36</td>
<td>nil</td>
</tr>
</tbody>
</table>

Problems: 
(i) study the existence of orthogonal designs in orders ≥ 14.
(ii) look for weighing (skew-weighing) matrices in the unsettled cases.
(iii) lower the bounds on M(9), M(25), M(36) and find M(k) for other k.
APPENDIX H: Unsolved questions and known results for \( n \equiv 4 \pmod{8} \).

Here we are concerned with the following conjectures:

(III) for \( n \equiv 4 \pmod{8} \) there is a weighing matrix of weight \( k \) and order \( n \) for every \( k \leq n \);

(IV) for \( n \equiv 4 \pmod{8} \) there is a skew-symmetric weighing matrix of order \( n \) for every \( k < n \), where \( k \) is the sum of \( \leq \) three integer squares;

(IVA) there is an orthogonal design of type \((1,k)\) in order \( n \equiv 4 \pmod{8} \) for every \( k < n \) which is the sum of \( \leq \) three integer squares;

(ii) a necessary and sufficient condition that there exist a design of type \((a,a,b)\) in order \( n \equiv 4 \pmod{8} \) is that \( \frac{b}{a} \) be a sum of two rational squares;

(iii) a necessary and sufficient condition that there exist a design of type \((a,a,a,b)\) in order \( n \equiv 4 \pmod{8} \) is that \( \frac{b}{a} \) be a rational square;

(iv) a necessary and sufficient condition that there exist a design of type \((a,b)\) in order \( n \equiv 4 \pmod{8} \) is that \( \frac{b}{a} \) be a sum of three rational squares.

The following theorems apply

Lemma 1 [7]. Let \( n \equiv 4 \pmod{8} \). The maximum number of variables in an orthogonal design of order \( n \) is 4.

Lemma 2. [7]. Let \( n \equiv 4 \pmod{8} \) and let \( X \) be an orthogonal design of order \( n \) and type \((a,b)\). Then \( \frac{b}{a} \) must be a sum of \( \leq 3 \) rational squares.
Lemma 3 [13]. Let $n \equiv 3 \pmod{8}$ and let $X$ be an orthogonal design of order $n$ and type $(a,a,b)$. Then $\frac{b}{a}$ must be a sum of two rational squares.

Lemma 4 [13]. Let $n \equiv 4 \pmod{8}$ and let $X$ be an orthogonal design of order $n$ and type $(a,a,a,b)$ then $\frac{b}{a}$ must be a rational square.

Order 12: In this case $\rho(n) = 4$, so we need only consider designs on $\leq 4$ variables. We note that of all the numbers $\leq 12$, seven is the only one which is not the sum of $\leq 3$ squares.

one variable: From [29] all designs exist.

two variables: Conjecture (iv) is true.

three variables: Conjecture (ii) is true. All designs not ruled out by Lemmas 1, 2 or 3 exist.

four variables: Conjecture (iii) is true. All designs not ruled out by Lemmas 1, 2, 3 or 4 exist.

Order 20: The following designs in order 20 are not eliminated by the necessary conditions but are not known either.

\begin{tabular}{cccc}
\hline
(1,1,1,1,1) & (1,2,2,8) & (1,4,5,5) \\
(1,1,2,1,1) & (1,2,2,9) & (1,5,5,6) \\
(1,1,5,8) & (1,2,6,11) & (1,5,5,9) \\
(1,1,5,13) & (1,2,6,9) & (2,2,5,6) \\
(1,1,8,1) & (1,3,6,8) & (2,3,7,6) \\
(1,1,6,10) & (1,4,5,9) & (3,3,6,6) \\
\hline
\end{tabular}
2 variables

(1,1,17) (1,5,13) (2,7,11)
(1,2,10) (1,6,12) (2,8,9)
(1,2,16) (1,6,13) (3,3,12)
(1,3,9) (1,8,10) (3,4,10)
(1,3,10) (2,3,8) (3,4,11)
(1,3,11) (2,3,10) (3,6,8)
(1,3,14) (2,5,6) (3,7,8)
(1,3,16) (2,5,7) (3,7,10)
(1,4,6) (2,6,11) (5,5,9)
(1,4,13) (2,7,8) (5,6,7)
(1,5,8) (2,7,10) (5,6,8)

3 variables

Conjecture (IVA) is true.

(3,16) (6,13) (7,10) (7,12)

these designs are not ruled out by any lemma but are not yet known.

1 variable

all exist and Conjecture (III) is true.

order 28: 1 variable

all exist and Conjecture (III) is true.

2 variables

all \((1,k)\) where \(k\) is the sum of \(\leq 3\) squares exist so Conjecture (IVA) is true.

See Appendix D for a list of the unknown designs.

orders 36, 44, 52 and 60:

the known results are summarized in table 1 of Appendix C.
APPENDIX I: Unsolved questions and known results for orders \( n \equiv 8 \pmod{16} \).

No necessary conditions, other than there is a maximum of 8 variables, are known for these orders. Part (i) of the following lemma was also noted by Joan Murphy Geramita.

**Lemma 1 [14].** If there exists an orthogonal design of type \((e_1,e_2,\ldots,e_8)\) in order \( n \) there exist orthogonal designs

(i) \((e_1,e_1,e_2,\ldots,e_8)\)

(ii) \((e_1,e_1,2e_2,2e_3,\ldots,2e_8)\)

in order \( 2n \).

**Lemma 2 [14].** Suppose there exist 8 circulant matrices of order \( n, A_1, A_2, \ldots, A_8 \), of variables \( 0 \equiv x_1, \ldots, x_8 \), satisfying

\[
\sum_{i=1}^{8} A_i A_i^t = \frac{k}{8} \sum_{j=1}^{8} x_j x_j^t I_n.
\]

Further suppose

(i) \( A_1, A_2, \ldots, A_8 \) are all symmetric; or

(ii) \( A_1, A_2, \ldots, A_8 \) are all skew; or

(iii) \( A_1 = A_2 = \ldots = A_4 \) and \( A_5 = A_6 = \ldots = A_8 \); or

(iv) \( A_1 = A_2 = \ldots = A_4 \) and \( A_5 = A_6 = \ldots = A_8 \); or

symmetric or all skew; or

(v) \( A_1 = A_2 = A_3 = A_4 \) are all skew (symmetric) and \( A_5, A_6, A_7, A_8 \) are all symmetric (skew); or

(vi) \( A_1 = A_2, A_3 = A_4, A_5 = A_6 \), and \( A_7, A_8 \) are all symmetric.

- 154 -
(vi) \( A_3 \) is zero, \( A_3 = A_4 \) and \( A_5, A_6, A_7, A_8 \) are all symmetric.

Then there exists an orthogonal design of order \( 8n \) and type \( (e_1, e_2, \ldots, e_k) \).

**Lemma 3** [14]. If there exist all designs \((1, i, j)\) in order \( n \) then there exist all designs \((g, h)\) in order \( 8n \).

**Lemma 4** [11], [14]. Orthogonal designs \((1, k)\) exist for \( 1 \leq k \leq n-1 \) when \( n = 2^t, 2^t \cdot 3, 2^t \cdot 5, 2^t \cdot 9, t \geq 3 \) a positive integer.

**Order 24**: No necessary conditions are known ruling out any designs except that the number of variables is limited to \( p(24) = 8 \).

1 variable

- all one variable designs exist.

2 variables

- all two variable designs exist.

3 variables

- the following 3 variable designs have not yet been found

\[
\begin{align*}
(1,1,21) & \quad (2,8,11) & \quad (4,4,15) \\
(1,3,27) & \quad (3,3,11) & \quad (4,5,14) \\
(1,3,19) & \quad (3,3,17) & \quad (4,8,11) \\
(1,5,27) & \quad (3,6,11) & \quad (5,7,11) \\
(1,7,15) & \quad (3,7,11) & \quad (5,8,8) \\
(1,8,14) & \quad (3,8,10) & \quad (6,7,9) \\
(1,9,13) & \quad (3,8,12) & \quad (7,7,7) \\
(2,7,11) & \quad (4,4,13) & \quad (7,7,9) \\
(2,7,14) & \quad & \quad (7,8,8)
\end{align*}
\]
4 variables
there are 369 four variable designs which have
not yet been found
5 and more variables
these cases have virtually been untouched. Most
of those known have been found using Lemmas 1, 2
and 3.

order 40: As for order 24 no necessary conditions are known
except that the number of variables is limited
to $p(40) = 8$.

1 variable
all one variable designs exist.

2 variables
The following 2 variable designs have not yet
been found:
(5,32)  (7,25)  (8,32)
(6,29)  (7,32)  (9,30)
(6,31)  (8,29)  (14,23)
(6,33)

3 and more variables
virtually untouched.

order 56: Again $p(56) = 8$ so we have at most 8 variables.

1 variable
All one variable designs are known.

2 variables
All $(l,k)$ for $k \in \{x: 0 \leq x \leq 55, \ x \neq 47\}$
are known but no other results.
order 72: Again at most 8 variables.

1 variable
All one variable designs are known.

2 variables
All designs (1,k) in order 72 are known.

Other designs have not yet been studied.

APPENDIX J: Unsolved Questions and Known Results for orders
n a power of 2.

Lemma 1. For orders n = 1,2,4,8 every possible
orthogonal design exists.

Lemma 2 [10], [11]. For every order n a power of 2
(i) W(n,k) exists for every integer k, 0 ≤ k ≤ n;
(ii) an orthogonal design (1,k) exists for every
integer k, 0 ≤ k ≤ n - 1.

order 16: p(16) = 9 so orthogonal designs must have
≤ 9 variables.

By Lemma 2 all 1 variable designs exist and
all designs (1,k), 1 ≤ k ≤ 15.

In [7] it is shown all 2 variable designs exist
Also in [7] it is shown all 3 variable designs
exist.

The only unresolved case for 4 variables is
(1,5,5,5). Unresolved cases for 5 variables:
(1,1,1,1,11) (1,1,1,1,8) (1,1,4,5,5)
(1,1,1,1,12) (1,1,2,2,9) (1,2,2,5,5)
(1,1,1,2,11) (1,1,2,5,7) (1,2,3,5,5)
There are 37 cases which are unresolved for the designs on 6 variables:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1,1,1,1,1,8</td>
<td>1,1,1,1,5,5</td>
<td>1,1,2,2,2,7</td>
</tr>
<tr>
<td>1,1,1,1,1,9</td>
<td>1,1,1,1,5,6</td>
<td>1,1,2,2,2,6</td>
</tr>
<tr>
<td>1,1,1,1,1,10</td>
<td>1,1,1,1,5,7</td>
<td>1,1,2,2,3,7</td>
</tr>
<tr>
<td>1,1,1,1,1,11</td>
<td>1,1,1,1,6,6</td>
<td>1,1,2,2,4,5</td>
</tr>
<tr>
<td>1,1,1,1,2,7</td>
<td>1,1,1,2,2,8</td>
<td>1,1,2,2,5,5</td>
</tr>
<tr>
<td>1,1,1,1,2,8</td>
<td>1,1,1,2,2,9</td>
<td>1,1,2,3,4,5</td>
</tr>
<tr>
<td>1,1,1,1,2,9</td>
<td>1,1,1,2,3,8</td>
<td>1,2,2,2,3,5</td>
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<tr>
<td>1,1,1,1,2,10</td>
<td>1,1,1,2,4,7</td>
<td>1,2,2,3,3,5</td>
</tr>
<tr>
<td>1,1,1,1,3,8</td>
<td>1,1,1,2,5,5</td>
<td>1,3,3,3,3</td>
</tr>
<tr>
<td>1,1,1,1,3,9</td>
<td>1,1,1,2,5,6</td>
<td>2,2,2,3,3</td>
</tr>
<tr>
<td>1,1,1,1,4,5</td>
<td>1,1,1,3,5,5</td>
<td>2,2,3,3,3</td>
</tr>
<tr>
<td>1,1,1,1,4,7</td>
<td>1,1,1,4,4,4</td>
<td></td>
</tr>
<tr>
<td>1,1,1,1,4,8</td>
<td>1,1,1,4,4,5</td>
<td></td>
</tr>
</tbody>
</table>
There are 57 unresolved cases for the designs on 7 variables, they are:

$$\begin{align*}
1,1,1,1,1,1,5 &\quad 1,1,1,1,1,5,5 &\quad 1,1,1,1,4,4,4 \\
1,1,1,1,1,1,6 &\quad 1,1,1,1,1,5,6 &\quad 1,1,1,2,2,2,5 \\
1,1,1,1,1,1,7 &\quad 1,1,1,1,2,2,5 &\quad 1,1,1,2,2,2,6 \\
1,1,1,1,1,1,8 &\quad 1,1,1,1,2,2,6 &\quad 1,1,1,2,2,2,7 \\
1,1,1,1,1,1,9 &\quad 1,1,1,1,2,2,7 &\quad 1,1,1,2,2,3,5 \\
1,1,1,1,1,1,10 &\quad 1,1,1,1,2,2,8 &\quad 1,1,1,2,3,2,6 \\
1,1,1,1,1,2,5 &\quad 1,1,1,2,2,3,3 &\quad 1,1,1,2,4,4,4 \\
1,1,1,1,1,2,6 &\quad 1,1,1,2,2,3,4 &\quad 1,1,1,2,4,4,5 \\
1,1,1,1,1,2,7 &\quad 1,1,1,2,2,3,5 &\quad 1,1,1,2,3,3,5 \\
1,1,1,1,1,2,8 &\quad 1,1,1,2,2,3,6 &\quad 1,1,1,2,3,4,4 \\
1,1,1,1,1,2,9 &\quad 1,1,1,2,2,3,7 &\quad 1,1,2,2,2,5 \\
1,1,1,1,1,3,5 &\quad 1,1,1,2,3,3,6 &\quad 1,1,2,2,3,3 \\
1,1,1,1,1,3,6 &\quad 1,1,1,2,4,4,5 &\quad 1,1,2,3,3,3 \\
1,1,1,1,1,3,7 &\quad 1,1,1,2,4,5,5 &\quad 1,1,2,3,3,3 \\
1,1,1,1,1,3,8 &\quad 1,1,1,2,4,5,6 &\quad 1,1,2,3,3,3 \\
1,1,1,1,1,4,4 &\quad 1,1,1,3,3,3,5 &\quad 1,1,2,3,3,3,4 \\
1,1,1,1,1,4,5 &\quad 1,1,1,3,3,3,6 &\quad 1,1,2,3,3,3,3 \\
1,1,1,1,1,4,6 &\quad 1,1,1,3,3,4,4 &\quad 1,2,2,2,3,3 \\
1,1,1,1,1,4,7 &\quad 1,1,1,3,3,4,5 &\quad 1,2,2,3,3,3
\end{align*}$$
Of 67 possible 8-tuples only 11 are known, viz:

1,1,1,1,1,1,1,1 1,1,2,2,2,2,2,2
1,1,1,1,1,1,1,2 1,1,2,2,2,2,2,2
1,1,1,1,1,1,2,2 1,2,2,2,2,2,2,2
1,1,1,1,1,2,2,2 1,2,2,2,2,2,2,3
1,1,1,1,2,2,2,2 2,2,2,2,2,2,2,2
1,1,1,2,2,2,2,2
1,1,1,2,2,2,2,4
1,1,2,2,2,2,2,2
1,1,2,2,2,2,2,4
1,2,2,2,2,2,2,2
1,2,2,2,2,2,2,3
2,2,2,2,2,2,2,2
2,2,2,2,2,2,2,4

We can only verify that 2 of the possible 9-tuples exist!
They are

1,1,1,1,1,1,1,1,1
1,1,2,2,2,2,2,2,2

order 32: Except for the results of Lemma 2 this case has not yet been studied.

Lemma 3 [7, Theorem 17]. If \( n = 2^t \) then there is an \( L \)-family of order \( n \) having \( n \) members; or equivalently

If \( n = 2^t \) then there exist amicable weighing matrices of weights \( i \) and \( n \) for every \( 1 \leq i \leq n-1 \).
§6 UNSOLVED PROBLEMS

1. Two weighing matrices \( W \) and \( N \) of order \( n \) and weights 1 and \( j \) are called *amicable weighing matrices* if

\[
W^t = -W, \quad N^t = N
\]

\[
WN^t = NW^t
\]

Their existence is known in many cases for \( i = n - 1 \), \( j = n \equiv 0 \pmod{4} \). They have also been studied for \( n = 2 \) and 4.

Find such matrices for \( i, j < n \).

2. Prove that "If all weighing matrices exist in order \( n \) then all weighing matrices exist in order \( 2n \) (or \( 4n \))".

3. Find new Baumert-Hall arrays i.e. orthogonal designs \((t,t,t,t)\) in orders \( 4t \) (see §4).

4. Find new Plotkin arrays i.e. orthogonal designs \((t,t,t,t,t,t,t,t)\) in orders \( 4t \) (see §4).

5. Find a construction, similar to the Goethals-Seidel array, using 8 circulant matrices.

6. Find a design \((1,47)\) in order 56 and hence prove there exist all orthogonal designs \((1,k)\) in orders \( 2^t, \ t \geq 3 \) an integer.

7. Find more results of the type indicated in Lemmas 5, 6, 7, 8 of Appendix C.

8. Fill in the unknown cases of Tables 1 and 2 of Appendix C.

9. Find the two variable designs indicated as unknown in Appendix D.
10. Find the four variable designs indicated as unknown in Appendix E.
11. Find new relations between the order $n$ and weight $k$ of a weighing matrix (see Appendix F).
12. Lower the bounds $N(k)$ by finding the unsolved weights as indicated in Appendix F.
13. Find more results of the type indicated in Lemmas 2, 3, 4, 5, 6, 7 of Appendix G.
14. Study the existence of orthogonal designs in orders $n \equiv 2 \pmod{4}$, $n \geq 14$.
15. Find weighing (skew-weighing) matrices in the table in Appendix G.
16. Lower the bounds on $M(9), M(25), M(36)$, and find $M(k)$ for other $k$ (see Appendix G).
17. Find the unknown designs in order 20 listed in Appendix H.
18. Study the case of 3 and 4 variable designs in orders 28, 36, 44, 52, 60.
19. Find the unknown design in order 24 listed in Appendix I.
20. Study the case of 3 and 4 variable designs in orders 40 (and thus 80) and 48.
21. Find the unknown 2, 3 and 4 variable designs in orders 56 and 72.
22. Find more results in orders 16, 32.
23. There is a theorem: "The existence of all designs
(1, 1, j) in order n implies the existence of all
designs (k, 1) in order 2n." Do there exist similar
theorems of the type "the knowledge of all t variable
designs in order n implies the knowledge of all t - 1
variable designs in 2n."

24. Show that if \( n = 0 \pmod{8} \) all orthogonal designs on
1, 2, 3, 4 or 5 variables must exist.

25. Verify the following three conjectures for \( n = 0 \pmod{8} \).
   i) A necessary sufficient condition that there exist
      an orthogonal design of type \((a, a, a, a, a, b)\) in order
      \( n \) is that \( \frac{b}{a} \) be a sum of \( \leq 3 \) rational squares.
   ii) A necessary and sufficient condition that there
      exist an orthogonal design of type \((a, a, a, a, a, b)\)
      is that \( \frac{b}{a} \) be a sum of \( \leq 2 \) rational squares.
   iii) A necessary and sufficient condition that there
      exist an orthogonal design of type \((a, a, a, a, a, a, b)\)
      is that \( \frac{b}{a} \) be a rational square.

26. If \( n = 0 \pmod{16} \) all orthogonal designs exist! (We have
    no evidence for this statement to be true. If this state-
    ment is false then there might be some exceedingly interest-
    ing number theoretic conditions involved.)

27. The Baumert-Hall arrays have been much used to construct
    Hadamard matrices. We have found several designs in order \( n \)
of type \((s_1, \ldots, s_r)\) where \( \sum_{i=1}^{r} s_i = n \). Find matrices to
    use in these designs to construct new Hadamard matrices.

28. Find the missing cases in order 36.
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