A note on BIBDS

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A note on BIBDS

Abstract
A balanced incomplete block design or BIBD is defined as an arrangement of v objects in b blocks, each block containing k objects all different, so that there are r blocks containing a given object and lambda blocks containing any two given objects.

In this note we shall extend a method of Sprott [2, 3] to obtain several new families of BIBD's. The method is based on the first Module Theorem of Bose [1] for pure differences.

We shall frequently be concerned with collections in which repeated elements are counted multiply, rather than with sets. If T₁ and T₂ are two such collections then T₁ & T₂ will denote the result of adjoining the elements of T₁ to T₂, with total multiplicities retained. We use the brackets, {}, to denote sets and square brackets, [ ], to denote collections of elements which may have repetitions. See [5] for results using these concepts.

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A NOTE ON BIBD'S

Dedicated to the memory of Hanna Neumann

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A balanced incomplete block design or BIBD is defined as an arrangement of \( v \) objects in \( b \) blocks, each block containing \( k \) objects all different, so that there are \( r \) blocks containing a given object and \( \lambda \) blocks containing any two given objects.

In this note we shall extend a method of Sprott [2, 3] to obtain several new families of BIBD's. The method is based on the first Module Theorem of Bose [1] for pure differences.

We shall frequently be concerned with collections in which repeated elements are counted multiply, rather than with sets. If \( T_1 \) and \( T_2 \) are two such collections then \( T_1 \& T_2 \) will denote the result of adjoining the elements of \( T_1 \) to \( T_2 \), with total multiplicities retained. We use the brackets, \{\}, to denote sets and square brackets, [ ], to denote collections of elements which may have repetitions. See [5] for results using these concepts.

1. Preliminaries

Let \( v = mh + 1 = p^s \), where \( p \) is a prime. Let \( x \) be a primitive element of \( GF(v) \) and write \( G \) for the group generated by \( x \). Define \( H_0 \) a subgroup of \( G \) and \( H_i, i \neq 0 \), its cosets by

\[
H_i = \{x^{h_i+j}: 0 \leq j \leq m-1\} \quad i = 0, 1, \ldots, h-1,
\]

Now consider the collection of differences between elements of \( H_i \),

\[
[x^{h_i+j} - x^{h_i+l}: l \neq j, 1 \leq j, l \leq m-1] = [x^{h_i+j}(x^{m-1}) - 1: l \neq j, 1 \leq j, l \leq m-1]
\]

\[
= a_0H_0 \& a_1H_1 \& \cdots \& a_{h-1}H_{h-1}
\]

\[
= \sum_{s=0}^{h-1} a_sH_s
\]

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This follows because $H_i = \{ x^{hi+1} \cdot 1 \leq i \leq m-1 \}$ is a coset and whenever it is multiplied by some element $x'$ of the group we have $H_i'$. Now there are $m(m-1)$ differences between elements of $H_i$, so

$$\sum_{s=0}^{h-1} a_s = m-1,$$

where the $a_s$ are non-negative integers.

The differences from $H_i \cup H_j$ where $i \neq j$ are (differences from $H_i$) & (differences from $H_j$) & (elements of $H_i - H_j$) & -(elements of $H_j - H_i$)

$$= \left( \sum_{s=0}^{h-1} a_s H_i \right) \& \left( \sum_{s=0}^{h-1} b_s H_j \right) \& \left( \sum_{s=0}^{h-1} c_s H_i \right) \& - \left( \sum_{s=0}^{h-1} c_s H_j \right)$$

$$= \sum_{s=0}^{h-1} d_s H_s,$$

where

$$\sum_{s=0}^{h-1} a_s = \sum_{s=0}^{h-1} b_s = m-1, \sum_{s=0}^{h-1} c_s = m, \text{ and } \sum_{s=0}^{h-1} d_s = 2(2m-1).$$

Note that if we had started by considering the differences between elements of $H_{i+1}$ we would have

$$\sum_{s=0}^{h-1} a_s H_{i+1},$$

and for $H_{i+1} \cup H_{j+1}$

$$\sum_{s=0}^{h-1} d_s H_{i+1}.$$

So we have, by considering, the totality of differences from the sets $H_i, H_{i+1}, \ldots, H_{i+h-1}$,

$$\left( \sum_{s=0}^{h-1} a_s \right) H_i = (m-1)G,$$

and for the totality of differences from the sets

$$H_i \cup H_j, H_{i+1} \cup H_{j+1}, \ldots, H_{i+h-1} \cup H_{j+h-1}$$

we have

$$\left( \sum_{s=0}^{h-1} d_s \right) H_i = 2(2m-1)G.$$

Similarly, by considering the totality of differences from the sets $H_i, H_{i+1}, \ldots, H_h$, where $i_1 = 0, 1, \ldots, h-1; i_j = i_1 + s_j$ for positive integers $s_j, 0 = s_1 < s_2 < \ldots < s_h < h$, we will have

$$i(mt - 1)G.$$
2. Results

It follows from the preceding observation that the blocks formed by the elements of the sets

\[ B_i = B_i(s_2, \ldots, s_h) = H_i \cup H_i \cup \cdots \cup H_i \]

\[ = \{ x^{i_1}, x^{i_1+i_2}, \ldots, x^{(m-1)i+h+i_1}, x^{i_1+i_2}, \ldots, \\ x^{(m-1)i+i_2}, \ldots, x^{i_1}, x^{i_1+i_2}, \ldots, x^{(m-1)i+h+i_1} \}, \]

\( i_1 = 0, 1, \ldots, h-1 \) can be taken as "initial blocks" in Bose's first Module Theorem [1]. That is, the collection of all blocks \( B_i, \theta \in GF(v) \), obtained from \( B_i \) by adding an arbitrary element \( \theta \) of \( GF(v) \) to each member of \( B_i \), form a BIBD with parameters

\[ v = mh + 1 = p^s, b = hv, r = tmh, k = tm, \lambda = t(m-1). \]

So we obtain

**Theorem 1.** (Series \( Z_1 \)). If \( v = mh + 1 = p^s \) where \( p \) is a prime, and \( t \) is a positive integer \( \leq h \), then a design with parameters

\[ v = mh + 1, b = hv, r = tmh, k = tm, \lambda = t(m-1) \]

can be constructed via the initial blocks

\[ B_i(s_2, \ldots, s_h) = H_i \cup H_i \cup \cdots \cup H_i, \quad i_1 = 0, 1, \ldots, h-1, \]

where \( i_j = i_1 + s_j \) for fixed positive integers \( s_j \),

\[ 0 = s_1 < s_2 < \cdots < s_i < h. \]

If instead of considering the previous sets we consider the differences from

\[ 0 \cup H_i \cup H_i \cup \cdots \cup H_i, \quad i_1 = 0, 1, \ldots, h-1, t \leq h, \]

then the totality of differences from these sets is

\[ t(mh + 1)G, \]

and hence we have

**Theorem 2.** (Series \( Z_2 \)). If \( v = mh + 1 = p^s \) where \( p \) is a prime, and \( t \) is a positive integer \( \leq h \), then the design with parameters

\[ v = mh + 1, b = hv, r = (tm + 1)h, k = tm + 1, \lambda = (tm + 1)t \]

can be constructed via the initial blocks

\[ B_i(s_1, \ldots, s_h) = 0 \cup H_i \cup H_i \cup \cdots \cup H_i, \quad i_1 = 0, 1, \ldots, h-1, \]

where \( i_j = i + s_j \) for fixed positive integers \( s_j \), \( 0 = s_1 < s_2 < \cdots < s_i < h. \)
THEOREM 3. (Series \( Z_3 \)). If \( v = (2\mu + 1)2h + 1 = p^s \), where \( p \) is a prime, and \( t \) is a positive integer \( \leq h \), then the design with parameters
\[ v = (2\mu + 1)2h + 1, \ b = vh, \ r = (2\mu + 1)ht, \ k = (2\mu + 1)t, \ \lambda = \frac{t((2\mu + 1)t - 1)}{4} \]
can be constructed via the initial blocks
\[ B_i(s_2, \cdots, s_t) = H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t}, \ i_1 = 0, 1, \cdots, h-1, \]
i_j = i_1 + s_j for fixed positive integers \( s_j, 0 = s_1 < s_2 < \cdots < s_t < h \).

THEOREM 4. (Series \( Z_4 \)). If \( v = (2\mu + 1)2h + 1 = p^s \), where \( p \) is a prime, and \( t \) is a positive integer \( \leq h \), then the design with parameters
\[ v = (2\mu + 1)2h + 1, \ b = vh, \ r = h[(2\mu + 1)t + 1], \ k = (2\mu + 1)t + 1, \ \lambda = t[(2\mu + 1)t + 1] \]
can be constructed via the initial blocks
\[ B_i(s_2, \cdots, s_t) = 0 \cup H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t}, \ i_1 = 0, 1, \cdots, h-1, \]
where \( i_j = i_1 + s_j \) for fixed positive integers \( s_j, 0 = s_1 < s_2 < \cdots < s_t < h \).

PROOF OF THEOREM 3 AND 4. In our previous discussion we have replaced \( m \) by \( 2\mu + 1 \) and \( h \) by \( 2h \). Now \(-1 \in H_8\) so the totality of differences from \( H_1 \) becomes
\[ a_0H_0 \cup a_1H_1 \cup \cdots \cup a_{h-1}H_{h-1} \cup a_0H_8 \cup a_1H_{h+1} \cup \cdots \cup a_{h-1}H_{2h-1} \]
because if \( x^{h+i} - x^{h+1} \in H_1 \) then \( x^{h+i} - x^{h+i} \in H_{1+h} \).

We may then proceed as before while noting the dependence of the coefficients of \( H_i \) and \( H_{i+h} \) in the collection of sums of differences.

By observing that our series are extensions of those of Sprott we can also show

THEOREM 5. (Series \( Z_5 \)). If \( v = (4\mu + 1)4h + 1 = p^s \), where \( p \) is a prime and if the collection of differences from the initial block
\[ B_i(s_2, s_3, \cdots, s_t) = H_{2i_1} \cup H_{2i_2} \cup \cdots \cup H_{2i_t}, \ i_1 = 0, 1, \cdots, h-1. \]
are written as
\[ a_s(x^{4h})^j: 0 \leq j \leq 4\mu \]
where we may pair the coefficients \( a_s \) such that \( a_{2i} = a_{2i+1} \) for all \( i = 0, 1, \cdots, 2h(4\mu + 1) - 1 \), then the design with parameters
\[ v = 4h(4\mu + 1) + 1, \ b = hv, \ r = ht(4\mu + 1), \ k = (4\mu + 1)t, \ \lambda = \frac{t[(4\mu + 1)t - 1]}{4} \]
can be constructed via these initial blocks where \( \frac{1}{2}t[(4\mu + 1)t - 1] \) is a positive integer, \( t_j = t_1 + s_j \) for fixed positive integers \( s_j \), \( 0 = s_1 < s_2 < \cdots < s_t < h \).

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References

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