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Combinatorial matrices

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Abstract
We investigate the existence of integer matrices $B$ satisfying the equation $BB^T = rI + sJ$ where $T$ denotes transpose, $r$ and $s$ are integers, $I$ is the identity matrix and $J$ is the matrix with every element $+1$.

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We investigate the existence of integer matrices $B$ satisfying the equation

\[(1) \quad BB^T = rI + sJ,\]

where $T$ denotes transpose, $r$ and $s$ are integers, $I$ is the identity matrix and $J$ is the matrix with every element $+1$.

Hadamard matrices are $(1, -1)$ matrices of order $n = 2$ or $4t$ which have $r = n$ and $s = 0$ in (1). We discuss equivalence of Hadamard matrices over the integers and show that all Hadamard matrices of order $4t$, where $t$ is odd and square-free are equivalent over the integers. Further, if $t$ is even and square-free and there is a Hadamard matrix of order $2t$, then there is a Hadamard matrix of order $4t$ which is equivalent over the integers to the diagonal matrix

$$\text{diag}(1, 2, \ldots, 2m, \ldots, 2m, 4m).$$

We now develop many methods for constructing Hadamard matrices. Many of these constructions use skew-Hadamard matrices, that is Hadamard matrices $H = I + R$ where $R^T = -R$, or $\pi$-type matrices, that is $(1, -1)$ matrices $H = I + P$ of order $n$ where $P^T = P$ and $P^2 = (n-1)I$. We first develop some theory on the Williamson method for constructing skew-Hadamard matrices and show if $h$ is the order of a skew-Hadamard matrix ($\pi$-type matrix) then there exists a skew-Hadamard ($\pi$-type) matrix of order $(h-1)^u + 1$ where $u = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f \cdot 17^g \cdot 19^h \cdot 23^i \cdot 29^j$ with $a < b < c < d < e < f < g < h < i < j$ where $a$ is a positive (non-negative) integer.

The concept of supplementary difference sets, that is, a set of subsets such that when we take all the differences in each subset and collect them, each difference occurs a fixed number of times in the totality, is introduced and an example given. Hadamard designs on \( n \) distinct letters are shown to exist for \( n = 2, 4 \) and \( 8 \).

\((\nu, k, \lambda)\)-configurations are considered, that is, \((0, 1)\)-matrices \( B \) of order \( \nu \) such that \( r = k - \lambda \) and \( s = \lambda \) in (1). We show two similar but distinct methods for proving there exists a \( (q^2(q+2), q(q+1), q) \) configuration whenever \( q \) is prime or \( q = 2^2, 2^3, 2^4, 3^2, 3^3 \) or \( 7^2 \). We prove that whenever a \( (q, k, \lambda)\)-configuration exists, \( q \) a prime power, then a \( (q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)\)-configuration exists.

We consider integer matrices satisfying
\[
BB^T = \nu I - J, \quad Bj = 0 = JB \quad \text{and} \quad B^T = -B
\]
and find that either the greatest common divisor of the elements of \( B \) is 1 or \( B \) has zero diagonal and +1 or -1 elsewhere. Also we show that if \( B \) is an integer matrix of order \( b \) satisfying
\[
BB^T = (p-q)I + qJ
\]
\[
Bj = dJ
\]
where \( p, q \) and \( d > 0 \) are constants then if \( z \), the least element of \( B \), satisfies
\[
z \leq \frac{d}{b} \quad \text{and} \quad z \leq \frac{|d|\nu}{\nu d + \omega d}
\]
where \( \omega \) is the greatest element of \( B \), then
\[
B = \frac{d}{b} J
\]

We give tables of the orders < 4000 of known Hadamard, skew-Hadamard and \( n \)-type matrices at the date of submission as well as lists of known classes of these matrices.