Some results on configurations

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Some results on configurations

Abstract
A (v, k, lambda) configuration is conjectured to exist for every v, k and lambda satisfying lambda(v-l) = k(k-l) and k - lambda is a square if v is even, x2 = (k - lambda)y2+(-1)(v-1)/2lambdaZ2 has a solution in integers x,y and z not all zero for v odd.

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SOME RESULTS ON CONFIGURATIONS

BY

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SOME RESULTS ON CONFIGURATIONS

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A \((v, k, \lambda)\) configuration is conjectured to exist for every \(v, k\) and \(\lambda\) satisfying
\[ \lambda(v-1) = k(k-1) \]
and
\[ k-\lambda \text{ is a square if } v \text{ is even}, \]
\[ x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}zx^2 \]
has a solution in integers \(x, y, z\) not all zero for \(v\) odd.

See Ryser [5, p. 111] for further discussion.

Necessary conditions for the existence of \((b, v, r, k, \lambda)\) configurations are that
\[ bk = vr \]
\[ r(k-1) = \lambda(v-1). \]

We write \(I\) for the identity matrix and \(J\) for the matrix with every element +1.
In the case of block matrices, \((X)_{ij}\) means the matrix whose \((i,j)\)th block is \(X\); for example, \((T^* T)_{ij}\) is the matrix whose \((i,j)\)th block is \(T^* T\). We define the Kronecker product of two matrices \(A = (a_{ij})\) of order \(m \times n\) and \(B\) of any order as the \(m \times n\) block matrix
\[ A \times B = (a_{ij}B)_{ij}. \]

**Theorem 1.** There exists a \((q(q^2+2), q(q+1), q)\) configuration whenever \(q\) is a prime.

Takeuchi [7] and Ahrens and Szekeres [1] have proven that Theorem 1 holds for all prime powers \(q\). Our method can be extended to \(q = 2^2, 2^3, 2^4, 3^2, 3^3\) or \(7^3\). We include Theorem 1 as our method is entirely different to the others' and closely connected to the proof of Theorem 2.

**Theorem 2.** A \((q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)\) configuration exists whenever a \((q, k, \lambda)\) configuration exists and \(q\) is a prime power.

**Theorem 3.** If there exists a matrix \(N\) of odd order \(v-1\) with zero diagonal and every other element +1 or -1, such that \(NJ = JN = 0\) and
\[ NN^T = (v-1)I_{v-1} - J_{v-1}, \]
then there is a \((2(v-1), v, v-1, \frac{v-1}{2}, \frac{v}{2}(v-2))\) configuration.
Corollary 4: If \( v \) is the order of a skew-Hadamard or \( n \)-type matrix (see [8] for definitions) then there is a \( (2(v-1), v, v-1, \frac{1}{2}v, \frac{1}{2}(v-2)) \) configuration.

1. Preliminary remark

We require that there exist \((0,1)\) matrices \( R_i, 0 \leq i \leq q-1 \), \( Q \) of order \( q^2 \) and \( \overline{Q} \) which is \( kq \times q^2 \), \( k \) an integer less than \( q \), which together with \( P \) (defined in (iv) below) satisfy the following conditions

\[
\begin{align*}
(i) & \quad PR_i^T = J \times J \\
(ii) & \quad R_i R_j^T = J \times J \quad i \neq j \\
(iii) & \quad \sum_{i=0}^{q-1} R_i R_i^T = q^3 I \times I + q(J-I) \times J \\
(iv) & \quad P = I \times J, \quad PP^T = qI \times J \\
(v) & \quad QQ^T = qI \times I + (J-I) \times J \\
(vi) & \quad Q\overline{Q}^T = qI_k + (J_k - I_k) \times J \\
(vii) & \quad J_k \overline{Q} = kJ \\
(viii) & \quad \overline{Q}J_{q^2} = q\overline{J}.
\end{align*}
\]

In formula (1), unless subscripted otherwise, \( I \) and \( J \) are of order \( q \) and \( J \) is the \( kq \times q^2 \) matrix with every element \( +1 \).

We will show in § 3 some cases where these conditions are satisfied.

2. Constructions

Lemma 5. If \( P \), a \((0,1)\) matrix, is defined as in (1, iv), and if \((0,1)\) matrices \( R_i, 0 \leq i \leq q-1 \) satisfying conditions (1, i, ii, iii) exist then there exists a \((q^2(q+2), q(q+1), q)\) configuration.

Proof. It is easily seen that this triplet satisfies the necessary conditions for \((v, k, \lambda)\) configurations.

Let \( S \) be the \( q^2(q+2) \) block matrix given by

\[
S = \begin{bmatrix}
0 & P & R_0 & R_1 & R_2 & \cdots & R_{q-3} & R_{q-2} & R_{q-1} \\
R_{q-1} & 0 & P & R_0 & R_4 & \cdots & R_{q-4} & R_{q-3} & R_{q-2} \\
\vdots & & & & & \ddots & & & \\
R_0 & R_1 & R_2 & R_3 & \cdots & R_{q-1} & 0 & P \\
P & R_0 & R_1 & R_2 & \cdots & R_{q-2} & R_{q-1} & 0
\end{bmatrix}
\]
then
\[
SS^T = I_{q+2} \times \left( PP^T + \sum_{i=0}^{q-1} R_i R_i^T \right) \times (J_{q+2} - I_{q+2}) \times qJ \times J
\]
\[
= q^2 I_r + qJ_r,
\]
where \( r = q^2(q+2) \).

Every element of \( s \) is 0 or 1 so \( s \) is the incidence matrix of a \((q^2(q+2), q(q+1), q)\) configuration.

**Lemma 6.** If there exists a \((0, 1)\) matrix \( \overline{Q} \) satisfying the conditions (1, vi, vii, viii) and a \((q, k, \lambda)\) configuration exists then there exists a \((q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)\) configuration.

**Proof.** A \((q, k, \lambda)\) configuration exists, so
\[
\lambda(q-1) = k(k-1);
\]
hence it is easily verified that the five numbers satisfy the necessary conditions for \((b, v, r, k, \lambda)\) configurations.

Let \( V \) be the incidence matrix of the \((q, k, \lambda)\) configuration. Then \( A \) defined by
\[
A^T = [I_k \times V, \overline{Q}, \overline{Q}, \cdots, \overline{Q}]
\]
(\( \overline{Q} \) occurring \( \lambda \) times), has \( k \) non-zero elements in every row and \( \lambda q + k = k^2 + \lambda \) non-zero elements in each column. Now
\[
A^T A = I_k \times VV^T + \lambda \overline{Q} \overline{Q}^T
\]
\[
= (k - \lambda + \lambda q)I_{kq} + \lambda J_{kq}
\]
\[
= k^2 I_{kq} + \lambda J_{kq};
\]
so \( A \) is the incidence matrix of the required configuration.

**Proof of Theorem 3.** Since \( N \) has zero diagonal and every other element \(+1\) or \(-1\), \( C \) and \( D \) defined (with \( I \) and \( J \) of order \( v-1 \)) by
\[
C = \frac{1}{2}(N+I+J)
\]
\[
D = \frac{1}{2}(N-I+J)
\]
are \((0, 1)\) matrices. Now
\[
CC^T + DD^T = \frac{1}{2}(NN^T + I + (v-1)J) = \frac{1}{2}vI + \frac{1}{2}(v-2)J
\]
and
\[
JC = \frac{1}{2}vJ = CJ
\]
\[
JD = \frac{1}{2}(v-2)J = DJ.
\]
We define $\omega_v$, $\omega_b$ and $e$ to be the vectors of $v$, $b$ and $(v-1)$'s respectively and $A^T$ by

$$A^T = \begin{bmatrix} D & C \\ e & 0 \end{bmatrix}.$$  

$A$ is $2(v-1) \times v$, and

$$\omega_v A^T = \frac{1}{2} v \omega_b, \quad A^T \omega_b^T = (v-1) \omega_v^T,$$

$$A^T A = \begin{bmatrix} D & C \\ e & 0 \end{bmatrix} \begin{bmatrix} D^T & e^T \\ C^T & 0 \end{bmatrix} = \begin{bmatrix} DD^T + CC^T & \frac{1}{2} (v-2) e^T \\ \frac{1}{2} (v-2) e & v-1 \end{bmatrix} = \frac{v}{2} I_v + \frac{v-2}{2} J_v.$$  

So $A$ is the incidence matrix of a $(2(v-1), v, v-1, \frac{1}{2} v, \frac{1}{2} (v-2))$ configuration.

3. Matrices satisfying condition (1)

We shall show that (1) can be satisfied for all primes $q$ and that matrices $Q$ and $\bar{Q}$ can be found for $q$ any prime power. These facts together with lemmas 5 and 6 complete the proofs of Theorems 1 and 2.

In this section $T$ will be used for the circulant matrix of order $q$ given by

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$  

(2)

3.1 The case of $q$ prime

Choose $q$ block matrices $R_i$ of order $q^2$, $0 \leq i \leq q-1$, thus

$$R_i = \begin{bmatrix} I & T^i & T^{2i} & \cdots & T^{(q-1)i} \\ T^{(q-1)i} & I & T^i & \cdots & T^{(q-2)i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^i & T^{2i} & T^{3i} & \cdots & I \end{bmatrix} = (T^{(m-i)s})_{mn}$$  

and let

$$Q = \begin{bmatrix} I & I & \cdots & I \\ I & T & T^2 & \cdots & T^{q-1} \\ I & T^2 & T^{2\cdot 2} & \cdots & T^{(q-1)2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & T^{q-1} & T^{2(q-1)} & \cdots & T^{(q-1)(q-1)} \end{bmatrix} = (T^{(l-1)(j-1)})_{ij}.$$
and

$$\bar{Q} = \begin{bmatrix}
I & I & I & \cdots & I \\
I & T & T^2 & T^{q-1} \\
I & T^2 & T^{q-2} & \cdots & T^{(q-1)2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
I & T^{k-1} & T^{2(k-1)} & \cdots & T^{(q-1)(k-1)}
\end{bmatrix}.$$ 

We now verify that these matrices satisfy the conditions (1). Note that $JT^i = J$ for all $i$, so (i), (vii) and (viii) are immediate.

(ii) $R_i R_j^T = \left( \sum_{m=0}^{q-1} T^{(m-i)(n-m)} \right)_{s,n}$

$$= \left( \sum_{m=0}^{q-1} T^{m(i-j)+n-j-s} \right)_{s,n}$$

$$= \left( \sum_{r=0}^{q-1} T^r \right)_{s,n} = (J)_{s,n} = J \times J \quad \text{for } i \neq j.$$

(iii) $R_i R_i^T = \left( \sum_{m=0}^{q-1} T^{(m-i)(n-m)} \right)_{s,n}$

$$= (qT(n-S)i)_{s,n}$$

$$= qR_i;$$

$$\sum_{i=0}^{q-1} R_i = \begin{bmatrix}
qI & J & \cdots & J \\
J & qI & \cdots & J \\
\vdots & \vdots & \ddots & \vdots \\
J & J & \cdots & qI
\end{bmatrix} = qI \times I + (J-1) \times J,$$

so the result follows.

(v) $Q Q^T = \left( \sum_{m=1}^{q} T^{(m-1)(n-1)} \right)_{ij}$

$$= \left( \sum_{m=1}^{q} T^{(m-1)(i-j)} \right)_{ij}$$

then if $i = j$ we have $\sum_{m=1}^{q} I = qI$, and if $i \neq j$, we have $\sum_{m=1}^{q} T^{(m-1)(i-j)} = J$, which gives the result.

(vi) This follows since we have chosen $\bar{Q}$ as the first $kq$ rows of $Q$.

3.2 The case of $q$ a prime power

In this case, unless stated otherwise, $I, J$ are of order $q$.

It is known that a $(q^2+q+1, q+1, 1)$ configuration exists whenever $q$ is a
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prime power. If we form the incidence matrix of this configuration then we may rearrange its rows and columns until the following matrix is obtained:

\[
A = \begin{bmatrix}
1 & e & 0 \\
e^T & 0 & I \times e \\
0 & I \times e^T & N
\end{bmatrix}
\]

where \( e = [1, 1, \ldots, 1] \) is of size \( 1 \times q \) and \( N \) is of size \( p^2 \).

Now \( AA^T = pl^r + J_r \), where \( r = p^2 + p + 1 \), and

\[
AA^T = \begin{bmatrix}
q + 1 & e & e \times e \\
e^T & qI + J & (I \times e)N^T \\
e^T \times e^T & N(I \times e^T) & I \times J + NN^T
\end{bmatrix}
\]

so

(a) \( N \) is of order \( q^2 \);

(b) \( NN^T = qI \times I + J \times J - I \times J = qI \times I + (J-I) \times J \);

(c) \( N(I \times e^T) = J' \) where \( J' \) is of size \( q^2 \times q \).

This last condition implies that if \( N \) is partitioned into \( q^2 \) block matrices \( N_i \), then each block matrix \( N_i \) has exactly one element in each row and column. Now rearrange the columns of \( N \) keeping the first \( q + 1 \) rows of \( A \) unaltered until the first row of block matrices in the partitioned \( N \) are all \( I_q \) and similarly alter the rows of \( N \) keeping the first \( q + 1 \) columns of \( A \) unaltered until the first column of block matrices in the partitioned \( N \) are all \( I_q \). Then this new matrix obtained from \( N \) satisfies all the conditions for the matrix \( Q \). We again choose \( Q \) to consist of the first \( kq \) rows of \( Q \).

3.3 The case of \( q \) certain prime powers

We have not been able to derive enough information from the matrix \( N \) to ensure the existence of the matrices \( R_i \) when \( q \) is a general prime power. However, as noted in the introduction, we can construct these matrices for the following value of \( q \):

\[
2^2, 2^3, 2^4, 3^2, 3^3, 7^2.
\]

The methods used do not generalize.

4. Remarks on numerical results

The block designs given by Theorem 2 with \( k > 4 \) all have \( r > 20 \), and are outside the range of the tables in [2], [3], [4] and [6]. Consequently it is hard to check whether individual designs are new. We observe, however, that the existence of a \( (16,6,2) \) configuration yields a design with parameters \( (608, 96, 38, 6, 2) \); this is the multiple by 2 of the design \( (304, 96, 19, 6, 1) \) which is listed as unknown.
by Sprott [6]. Also the (11, 6, 3) configuration yields a (429, 66, 39, 6, 3) configuration, which is a multiple by 3 of a (143, 66, 13, 6, 1) design. The solution of the latter design in [4] does not appear to have arisen as one of a series of designs. We note in passing that Hall [3] mistakenly lists (143, 66, 13, 6, 1) as 'solution unknown'.

Theorem 3 yields a (34, 18, 17, 9, 8) configuration, which was previously unknown according to [6]. It also gives a (26, 14, 13, 7, 6) configuration, which was already known but was completely omitted from Hall's list, as well as a number of apparently new configurations with $r > 20$.

References


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