Some (1, -1) Matrices

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Some (1, -1) Matrices

Abstract
We define an n-type (1, -1) matrix $N = 1 + R$ of order $n \sim 2 \pmod{4}$ to have $R$ symmetric and $R^2 = (n - 1)/n$. These matrices are analogous to skewtype matrices $M = 1 + W$ which have $W$ skew-symmetric.

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Some \((1, -1)\) Matrices

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ABSTRACT

We define an \(n\)-type \((1, -1)\) matrix \(N = I + R\) of order \(n = 2 \pmod{4}\) to have \(R\) symmetric and \(R^t = (n - 1)I_n\). These matrices are analogous to skew-type matrices \(M = I + W\) which have \(W\) skew-symmetric.

If \(n\) is the order of an \(n\)-type matrix, \(h\) and \(h^2\) the orders of Hadamard matrices, \(h\) the order of a skew-Hadamard matrix, and \(p^r = 1 \pmod{4}\) is a prime power then we show there are:

- \(n\)-type matrices of orders \(p^r + 1\), \((h - 1)^3 + 1\), \((n - 1)^3 + 1\);
- symmetric Hadamard matrices of orders \(2n\), \(2n(n - 1)\), \(2p^r(p^r + 1)\);
- Hadamard matrices of orders \(hn\), \(h_2p(n - 1)\), \(h_hh_2p(n - 3)\)

(this latter with \(n + 4\) also the order of an \(n\)-type matrix);

- Hadamard matrices of orders \(452\), \(612\), \(2452\) and \(3044\), all "new."

We also give existence conditions for many other classes of Hadamard matrices and another formulation for Goldberg's skew-Hadamard matrix of order \((h - 1)^3 + 1\).

INTRODUCTION

An Hadamard matrix \(H\) is a matrix of order \(n\), all of whose elements are \(+1\) or \(-1\) and which satisfies \(HH^T = nI_n\). It is conjectured that an Hadamard matrix exists for \(n = 2\) and for \(n = 4t\), where \(t\) is any positive integer. Many classes of Hadamard matrices are known; most of these can be found by reference to [2], [3], and [4]. Hadamard matrices are known for all orders less than 188.

An Hadamard matrix \(H = S + I\) is called skew-Hadamard if \(S^T = -S\). It is conjectured that, whenever there exists an Hadamard matrix of order \(n\), there exists a skew-Hadamard matrix of the same order. As the existence of skew-Hadamard matrices is needed for some of my results I list the classes of orders for which skew-Hadamard matrices are known to exist:
I. \[2^t k_i\] \(t, r_i\) all positive integers, \(k_i = p_i^{k_i} + 1 \equiv 0 \pmod{4}\), a prime; from [5].

II. \((p - 1)^3 + 1\) \(p\) the order of a skew Hadamard matrix; from [1].

III. \(2^t(q + 1)\) \(t \geq 1\) an integer, \(q\) (prime power) \(\equiv 5 \pmod{8}\); from [7].

IV. 52 From [6].

V. 36 From [8].

VI. \(p^t(p^t + 1)(m - 1)\) \((m - 1)(p^t + 1)/m\) the order of a skew-Hadamard matrix, \(m\) of class 1 and \(p^t = 3 \pmod{4}\) a prime power; from [4].

VII. \(p^t(p^t - 3)(m - 1)\) \((m - 1)(p^t + 3)/m\) the order of a skew-Hadamard matrix, \(m\) and \(p^t\) as in class VI; from [4].

VIII. \(h(p^t + 1)\) \(h\) the order of a skew-Hadamard matrix, \(p^t = 3 \pmod{4}\) a prime power; from [4].

IX. \(2h\) \(h\) the order of a skew-Hadamard matrix.

The only orders up to 200 for which skew-Hadamard matrices have not yet been discovered are 92, 100, 116, 148, 156, 172, 184, 188, and 196.

A \((v, k, \lambda)\)-configuration is an arrangement of \(v\) elements \(x_1, x_2, ..., x_v\) into \(v\) sets \(S_1, S_2, ..., S_v\) such that every set contains exactly \(k\) elements, every pair of sets has exactly \(\lambda\) elements in common. A \((v, k, \lambda)\)-configuration can be characterized by its incidence matrix \(A = (a_{ij})\) defined by \(a_{ij} = 1\) if \(x_j \in S_i\) and \(a_{ij} = -1\) if \(x_j \notin S_i\). This matrix \(A\), of order \(v\), consists entirely of 1's and -1's, and it can be seen that \(A\) satisfies the incidence equation

\[AA^T = 4(k - \lambda)I + v - 4(k - \lambda)J\]

where \(I\) is the identity matrix of order \(v\) and \(J\) is the matrix of order \(v\) with every element +1.

A set of elements \(D = \{x_1, x_2, ..., x_k\}\) will be said to generate a circulant \((1, -1)\) matrix \(A = (a_{ij})\) if \(a_{ij} = a_{i,j-i+1} = 1\) when \(j - i + 1 \in D\) (all numbers modulo \(v\)) and \(-1\) otherwise. A back-circulant matrix \(A = (a_{ij})\) of order \(v\) has \(a_{ij} = a_{1+j,i-j}\) where \(1 + j\) and \(i - j\) are reduced to modulo \(v\).
DEFINITION. A $(1, -1)$ matrix $N = I + R$ of order $n \equiv 2 \pmod{4}$ will be called $n$-type if $N$ can be written as
\[
\begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & \ddots & & \\
\vdots & & D & \\
1 & & & 
\end{bmatrix} + I_n,
\]
(1)
where $D^T = D$ and $D^2 = (n - 1) I_{n-1} - J_{n-1}$ or equivalently if $R^T = R$ and $R^2 = (n - 1) I_n$.

We now investigate the existence of $n$-type matrices and we will show three classes of these matrices exist:

I. $p^r + 1$ \quad $p^r \equiv 1 \pmod{4}$ is a prime power.

II. $(n - 1)^2 + 1$ \quad Where $n$ is either the order of a skew-Hadamard matrix or the order of an $n$-type matrix.

III. $(n - 1)^3 + 1$ \quad Where $n$ is the order of an $n$-type matrix.

PROOF OF CLASS I. The observation that matrices of order $p^r + 1 \equiv 2 \pmod{4}$, where $q = p^r$ is a prime power, can be found satisfying the requirements for $n$-type matrices is due to Williamson [5, p. 66]. As in [5] we define $D = (d_{ij})$ of order $q$ by
\[
d_{ii} = 0, \\
d_{ij} = \chi(a_i - a_j), \quad i \neq j,
\]
where $a_1, a_2, \ldots, a_q$ are the elements of a Galois field in some fixed order and $\chi(a) = 1$ or $-1$ according as $a$ is or is not the square of an element of the $GF(p^r)$. Since $q \equiv 1 \pmod{4}$, $\chi(x) = \chi(-x)$, so $D$ is symmetric and $D^2 = p^r I - J$.

Then
\[
N = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & \ddots & & \\
\vdots & & D & \\
1 & & & 
\end{bmatrix} + I_{p^r+1},
\]
is an $n$-type matrix of order $p^r + 1$.

PROOF OF CLASS II. If $N$ is an $n$-type matrix of order $n$ then $N$ may be written as
\[
\begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & \ddots & & \\
\vdots & & D & \\
1 & & & 
\end{bmatrix} + I_{n-1}
\]
(3)
where $D^T = D$, $DD^T = (n - 1) I_{n-1} - J_{n-1}$ and $DJ_{n-1} = 0$. 
If $V$ is a skew-Hadamard matrix of order $n$ then $V$ may be written as

$$
\begin{bmatrix}
0 & 1 & \cdots & 1 \\
-1 & \ddots & & \vdots \\
\vdots & & \ddots & \ddots \\
-1 & & \cdots & 0 \\
\end{bmatrix} + I_{n-1},
$$

(4)

where $W^T = -W$, $WW^T = (n-1)I_{n-1} - J_{n-1}$ and $WJ_{n-1} = 0$.

Now where $X$ is either $W$ or $D$ define $A$ by

$$
A = J_{n-1} \times -I_{n-1} + I_{n-1} \times J_{n-1} + X \times X;
$$

then

$$
AA^T = (n-1)J_{n-1} \times I_{n-1} + I_{n-1} \times (n-1)J_{n-1}
$$

$$
+ [(n-1)I_{n-1} - J_{n-1}] \times [(n-1)I_{n-1} - J_{n-1}]
$$

$$
- 2J_{n-1} \times J_{n-1}
$$

$$
= (n-1)^2 I_{(n-1)^2} - J_{(n-1)^2}.
$$

Now $A^T = A$ if $X \times X = X^T \times X^T$, but this is true for both $W$ and $D$ and so

$$
M = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & \ddots & & \vdots \\
\vdots & & \ddots & \ddots \\
1 & & \cdots & 0 \\
\end{bmatrix} + I_{(n-1)^2 + 1}
$$

is an $n$-type matrix of order $(n-1)^2 + 1$.

**Proof of Class III.** With $D$ as in (3) and $D$, $J$, and $I$ all of order $n-1$ we define

$$
A = I \times D \times J + J \times I \times D + D \times J \times I + D \times D \times D.
$$

This is similar to Goldberg's construction for Hadamard matrices as his matrix may be written as

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
-1 & \ddots & & \vdots \\
\vdots & & \ddots & B \\
-1 & & \cdots & 0 \\
\end{bmatrix},
$$
where $B$ is given by
\[ B = I \times I \times I + I \times W \times J + J \times I \times W + W \times J \times I + W \times W \times W \]
with $W$ as defined in (4).

Now $DJ = 0$ so
\[
AA^T = I \times D^2 \times J^2 + J^2 \times I \times D^2 + D^2 \times J^2 \times I + D^2 \times D^2 \times D^2 \\
= I \times [(n - 1) I - J] \times (n - 1) J + (n - 1) J \times I \times [(n - 1) I - J] \\
+ [(n - 1) I - J] \times (n - 1) J \times I + [(n - 1) I - J] \times [(n - 1) I - J] \\
\times [(n - 1) I - J] \\
= (n - 1)^3 I_{(n-1)^3} - J_{(n-1)^3},
\]
and clearly $A^T = A$.

So
\[
M = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cdots & 1 & A \\
\end{bmatrix} + I_{(n-1)^3+1}
\]
is an $n$-type matrix of order $(n - 1)^3 + 1$.

**Lemma 1.** If there is an $n$-type matrix of order $n$, then there is a symmetric Hadamard matrix of order $2n$.

**Proof:** Let $N = I + P$ be the $n$-type matrix of order $n$. Then $P^2 = (n - 1) I_n$ and $P^T = P$. Now choose
\[
X = \begin{bmatrix}
-P & I & P & -I \\
-P & -I & P & I \\
\end{bmatrix};
\]
then $X^T = X$ and $XX^T = 2nI_{2n}$. So $X$ is the required symmetric Hadamard matrix.

Since there are skew-Hadamard matrices of orders 16, 36, and 40 using class II we see there are $n$-type matrices of order 226, 1226, and 1522. Then using the above lemma there are symmetric Hadamard matrices of orders 452, 2452, and 3044. These orders are new.

**Lemma 2.** If there is an $n$-type matrix of order $n$ and an Hadamard matrix of order $h$, then there is an Hadamard matrix order $hn$.
PROOF: Let $H$ be the Hadamard matrix and define $K = HS$ where

$$S = \begin{bmatrix}
-1 & \ldots & -1 \\
1 & \ldots & -1 \\
-1 & \ldots & 1
\end{bmatrix}$$

is of order $h$ then $SS^T = I_h$ and $S^T = -S$, so

$$HK^T = HST^T = -HS^T = -KH^T \quad \text{and} \quad hl_h = HH^T = HSSH^T = HS(HS)^T = KK^T.$$

Then if $N = I + R$ is an $n$-type matrix of order $n$, $R^2 = (n - 1) I_n$ and $R^T = R$, so, if

$$L = I_n \times H \div R \times K,$$

then

$$LL^T = I_n \times HH^T + R \times KH^T + R^T \times HK^T + R^T \times KK^T$$

$$= I_n \times hI_h + (n - 1) I_n \times hI_n$$

$$= nhI_{nh}.$$

So $L$ is the required Hadamard matrix of order $nh$.

**Theorem 3.** If $N = I + R$ is an $n$-type matrix of order $n$, then there is a symmetric Hadamard matrix of order $2n(n - 1)$.

**Proof:** Let $D$ be as defined in (1) above, then $D^T = D$ and

$$D^2 = (n - 1) I_{n-1} - J_{n-1}.$$

Further $R^T = R$, $R^2 = (n - 1) I_n$ and $JD = 0$.

So, if $I$ is of order $n - 1$, let

$$X = \begin{bmatrix} J & J \\ J & -J \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} D + I & D - I \\ D - I & D + I \end{bmatrix}:$$
then
\[ X^T = X, \quad Y^T = Y, \quad XX^T = 2(n-1)J_{n-1} \times I_n, \quad YY^T = (2nJ_{n-1} - 2J_{n-1}) \times I_n, \]
and \[ XY^T + YX^T = 0. \]

Then, if
\[ H = I_n \times X + R \times Y, \]
\(H\) is symmetric and, since
\[ HH^T = I_n \times XX^T + R \times (YY^T + XY^T) + RR^T \times YY^T \]
\[ = I_n \times 2(n-1)J_{n-1} \times I_n + (n-1)I_n \times (2nJ_{n-1} - 2J_{n-1}) \times I_n \]
\[ = 2n(n-1)I_{2n(n-1)}, \]
\(H\) is Hadamard.

**Corollary 4.** If \( p^r = 1 \pmod{4} \) is a prime power, then there is a symmetric Hadamard matrix of order \( 2p^r(p^r + 1) \).

**Proof:** This follows by putting \( n \) equal to orders of class 1.

This corollary is important as it means class VII of Marshall Hall [2, p. 207] may be partially rewritten. The construction gives an Hadamard matrix of order 612 which was previously not known.

**Theorem 5.** If \( N = I + R \) is an \( n \)-type matrix of order \( m \), then, if there are symmetric \((1, -1)\) matrices \( D \) and \( M \) satisfying \( D^2 = nI_n - J_n \), \( M^2 = (n-1-m)I_n + (m-1)J_n \), and \( MD = DM \), where \( D \) has zero diagonal, there is an Hadamard matrix of order \( 4mn \).

**Proof:** Choose
\[ X = \begin{bmatrix} M & M & M & M \\ M & -M & -M & M \\ M & -M & M & -M \\ M & M & -M & -M \end{bmatrix} \]
and
\[ Y = \begin{bmatrix} D + I & -D - I & D - I & -D + I \\ -D + I & D - I & D + I & -D - I \\ -D + I & -D + I & D - I & -D - I \\ D - I & D - I & D + I & -D + I \end{bmatrix} \]
then \(XY^T + YX^T = 0, X^T = [4(n + 1 - m) I_n + 4(m - 1) J_n] \times I_4\) and \(YY^T = [4(n + 1) I_n - 4J_n] \times I_4\), so, if

\[H = I_m \times X + R \times Y\]

then

\[
HH^T = I_m \times XX^T + R \times (XY^T + YX^T) + R^2 \times YY^T
\]

\[
= I_m \times [4(n + 1 - m) I_n + 4(m - 1) J_n] \times I_4
\]

\[+ (m - 1) I_m \times [4(n - 1) I_n - 4J_n] \times I_4
\]

\[= 4nmI_{4nm}.
\]

So \(H\) is an Hadamard matrix of order \(4nm\).

**Corollary 6.** If \(m\) is the order of an \(n\)-type matrix, then, if there is a back-circulant \((1, -1)\) matrix \(M\) of order \(p = 1 \pmod{4}\), a prime, satisfying \(M^2 = (p + 1 - m) I + (m - 1) J\), then there is an Hadamard matrix of order \(4pm\).

**Proof:** This follows with \(n\) of the theorem of class I and \(D\) defined as in (2). By Theorem 1 of [3], a circulant \(D\) and a back-circulant \(M\) satisfy \(MD^T = DM^T\) and since \(M\) and \(D\) are both symmetric the conditions of the theorem are satisfied.

**Corollary 7.** If there is a \(n\)-type matrix of order:

(i) \(p' = 3\);

(ii) \(p - 4q^{n-1} + 1, \) where \(p = q^n + q^{n-1} + \cdots + 1, \) \(n\) integer;

(iii) \(q^2 - 3q + 2, \) where \(p = q^2 + q + 1;\)

(iv) \(x^2 + 1, \) where \(p = 4x^2 + 1;\)

(v) \(x^2 + 1, \) where \(p = 4x^2 + 9;\)

(vi) \(36b^2 + 6, \) where \(p = 8a^2 + 1 = 64b^2 + 9, \) \(b\) odd;

(vii) \(36b^2 + 250, \) where \(p = 8a^2 + 49 = 64b^2 + 441, \) \(b\) even;

with \(p\) (prime) \(= 1 \pmod{4}, \) \(q\) prime power, and \(x\) and \(a\) odd, then there are Hadamard matrices of orders:

(i) \(4p'(p' - 3);\)

(ii) \(4(q^n + q^{n-1} + \cdots + 1)(q^n - 3q^{n-1} + q^{n-2} + \cdots + q^2 + q + 2);\)

(iii) \(4(q - 1)(q - 2)(q^2 + q + 1);\)

(iv) \(4(x^2 + 1)(4x^2 + 1);\)

(v) \(4(x^2 + 1)(4x^2 + 9);\)
SOME \((1, -1)\) MATRICES

(vi) \(24(6b^2 + 1)(64b^2 + 9)\);
(vii) \(8(18b^2 + 125)(64b^2 + 441)\);
respectively.

PROOF: In each case but (i) \(M\) of the corollary is a back-circulant matrix generated by a difference set: the notation we use for these difference sets is that of Marshall Hall [2, p. 141]. The proof follows with \(M\) given by (i) \(J-2I\); (ii) \(S\); (iii) \(S\) with \(n = 3\); (iv) \(B\); (v) \(B_0\); (vi) \(O\); (vii) \(O_0\); respectively.

**Theorem 8.** If \(h > 1\) is the order of an Hadamard matrix, \(n\) is the order of an \(n\)-type matrix and \(v \equiv 1 \pmod{4}\) the order of three \((1, -1)\) matrices \(A, B,\) and \(D + I\) which satisfy \(DD^T = vl - J, AA^T = aI + (v - a) J, BB^T = [2(v - n + 1) - a] I + [-v + 2(n - 1) + a] J,\) and \(AB^T, BD^T,\) and \(AD^T\) all symmetric, where \(D\) has zero diagonal, then there is an Hadamard matrix of order \(2nh.\)

PROOF: Let \(I_n + R\) be the \(n\)-type matrix, \(H\) the Hadamard matrix, and \(K = HS\) be as defined in the proof of Lemma 2. Now choose

\[
X = \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} D + I & D - I \\ -D + I & D + I \end{bmatrix};
\]

then \(XY^T = YX^T.\) So

\[W = I_n \times H \times X + R \times K \times Y\]

is the required matrix.

In the constructions of Williamson shown in Marshall Hall [2, pp. 214–216], \(m\) and \(n\) are defined as being of the form \(p^r + 1,\) where \(p^r = 1 \pmod{4}\) is a prime power; if we read instead that \(m\) and \(n\) are the orders of \(n\)-type matrices, then the same proofs give us the theorem

**Theorem 9.** If Hadamard matrices of orders \(h_1\) and \(h_2, \) exist, \(h_1 > 1,\)
\(h_2 > 1\) and (i) \(n;\) (ii) \(n\) and \(n \div 4,\) are the orders of \(n\)-type matrices; there exist Hadamard matrices of orders

(i) \(h_1h_2(n - 1);\)
(ii) \(h_1h_2(n + 3);\)
respectively.

**Theorem 10.** If there is an \(n\)-type matrix of order \(n\) and \(k\) is the order of three \((1, -1)\) matrices \(A, C,\) and \(D + I\) satisfying

\[
AA^T = aI + (k - a) J, CC^T = (2k - 2n + 2 - a) I + (-k + 2n - 2 + a) J,
\]

...
\(DD^T = kI - J\), and \(ACT, CD^T\), and \(AD^T\) all symmetric where \(D\) has zero diagonal, then there is an Hadamard matrix of order \(4kn\).

**Proof:** Define

\[
M = \begin{bmatrix}
A & A & C & C \\
A & -A & C & -C \\
C & C & -A & -A \\
C & -C & -A & A
\end{bmatrix}
\]

and

\[
P = \begin{bmatrix}
-D - I & D + I & D - I & -D + I \\
D + I & D + I & -D + I & -D - I \\
-D - I & -D - I & D - I & D + I \\
D - I & D - I & D + I & D + I
\end{bmatrix}
\]

then, since \(ACT, CD^T\), and \(AD^T\) are all symmetric, \(MP^T + PM^T = 0\). Also

\[
MM^T = \{4(k - n + 1)I_k + 4(n - 1)J_k\} \times I_4,
\]

\[
PP^T = \{4(k + 1)I_k - 4J_k\} \times I_4.
\]

So, if \(N = I + R\) is the \(n\)-type matrix, \(R^T = R\) and \(R^2 = (n - 1)I_n\). Now consider

\[
H = I_n \times M + R \times P,
\]

then

\[
HH^T = I_n \times MM^T + R \times (MP^T + PM^T) + R^2 \times PP^T
\]

\[
= 4knI_{4kn}.
\]

So \(H\) is the required Hadamard matrix.

**Corollary 11.** If there is an \(n\)-type matrix of order

- (i) \(q^n - q^{n-1} + q^{n-2} + \cdots + q^2 + q + 2\), where \(p = q^n + q^{n-1} + \cdots + q + 1\), \(n\) integer;
- (ii) \(q^n - q^{n-1} + q^{n-2} + \cdots + q^2 + q\), \(p\) as in (i);
- (iii) \((5x^2 - 1)/2\), where \(p = 4x^2 + 1\);
- (iv) \((5x^2 + 7)/2\), where \(p = 4x^2 + 9\);
- (v) \(50b^2 + 8\), where \(p = 8a^2 + 1 = 64b^2 + 9\), \(b\) odd;
- (vi) \(50b^2 + 346\), where \(p = 8a^2 + 49 = 64b^2 + 441\), \(b\) even;

with \(p\) (prime) \(\equiv 1\) (mod 4), \(q\) prime power, and \(x\) and \(a\) odd, then there are Hadamard matrices of orders
(i) \[ 4(q^n + q^{n-1} + \cdots + 1)(q^n - q^{n-3} + q^{n-2} + \cdots + q^2 + q + 2); \]
(ii) \[ 4(q^n + q^{n-1} + \cdots + 1)(q^n - q^{n-2} + q^{n-2} + \cdots + q^2 + q); \]
(iii) \[ 2(5x^2 - 1)(4x^2 + 1); \]
(iv) \[ 2(5x^2 + 7)(4x^2 + 9); \]
(v) \[ 8(25b^2 + 4)(64b^2 + 9); \]
(vi) \[ 8(25b^2 + 173)(64b^2 + 441). \]

**Proof.** In each case \( C \) of the theorem is a back-circulant matrix generated by a difference set; we again use the notation of Marshall Hall [2, p. 141]. The proof follows with

(i) \( A = J, C = S, q \) powers of 2;
(ii) \( A = J - 2I, C = S, q \equiv 3 \pmod{4}; \)
(iii) \( A = J - 2I, C = B; \)
(iv) \( A = J - 2I, C = B_0; \)
(v) \( A = J, C = O; \)
(vi) \( A = J, C = O_0. \)

*Note Added in Proof.* Dr. J. M. Goethals has pointed out to me that some of the results of this paper overlap those of [9].

**References**