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Comparison of the Performance of a Time-Dependent Short-Interest Rate Model with Time-Independent Models

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Running Title: Comparison of Interest Rate Models
Abstract

The coefficients in the stochastic differential equation that the short interest rate follows are of vital importance in the subsequent modelling of bond prices and other interest rate products. Empirical tests have previously been performed by various authors which compare a variety of popular short rate models. Most recently, Ahn and Gao compare their model with affine-drift models and show that their model with a non-linear drift function outperforms each one. In this paper we compare the model developed by Goard, which is a time-dependent generalisation of the Ahn-Gao model, with the Ahn-Gao model itself. We find that the time-dependent model using a second order Fourier series in time, outperforms the Ahn-Gao model for all data sets considered.

Keywords: Short-rate, interest rate models

1. Introduction

The short term riskless interest rate (or spot rate) is a variable of paramount importance in modern finance. During recent years it has attracted a high level of interest from academics and practitioners within the financial services industry. Various attempts to explain its behaviour have resulted in the construction of a library of mathematical and econometric models. A popular and simple group of these models is known as single-factor spot rate models. These models take the form of a stochastic differential equation

\[ dr = u(r,t)dt + w(r,t)dX \]  

where \( dt \) is an infinitesimal change in time, \( dr \) is a corresponding change in the spot rate and \( dX \) is an increment in a Wiener process. It can be shown (see e.g Wilmott (1998)) that when the short-term interest rate follows the process in (1), the price of a zero-coupon bond \( V(r, t; T) \) with expiry at \( t = T \) will satisfy the partial differential equation (PDE)

\[ \frac{\partial V}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V}{\partial r^2} + (u \square [\nu]) \frac{\partial V}{\partial r} \square rV = 0, \]  

where \( \square (r,t) \) is the market price of risk. The latest developments in comparing interest rate models include those of Chan et al. (1992) and Ahn and Gao (1999). Chan et al. (1992) performed a comprehensive empirical analysis on models of the type \( dr = (\square + \square r)dt + \square [r]dX \) with an affine form for the drift. They found that their unconstrained estimate for the volatility showed a \( r^{3/2} \) dependence on the interest rate. They also found only weak evidence of a long-run level of mean reversion, suggesting that the short rate may revert to a
short-run mean which may be time-dependent. Ahn and Gao (1999) however show that the linearity in the drift appeared to be the main source of misspecification. They presented a one-factor model

\[ dr = (\alpha t + \beta r) dt + \sigma r^{3/2} dX, \] (3)

which included a quadratic non-linearity in the drift term, and produced empirical evidence that it outperformed all of the current popular affine models studied by Chan et al., including the Vasicek (1977) and Cox, Ingersoll and Ross (CIR) (1985) models. They also gave the solution for the zero-coupon bond based on their short rate model (3). Then Goard (2000) used the classical Lie Symmetry method to obtain an explicit solution for a zero-coupon bond based on a more general version of (3), namely

\[ dr = [c^2 r(a(t) q r)] dt + cr^{3/2} dX, \] (4)

where \( c \) and \( q \) are constants and \( a \) is an arbitrary function of time. Since \( q \) and \( c \) are independent, this means that the volatility and reversion rate are independent. Here, the drift function included a free function of time as a moving target. This resulted in the following solution for the bond price which expires at time \( t = T \).

\[ V(r,t) = (r_0^{1/k}) \exp \left( \frac{2}{c^2 r} \frac{\sigma^2}{r} \frac{c^2}{2} \frac{1}{k+2q+2} \left( k+2q+2 \right) \right) \frac{R}{M} \left( k+2q+2, 2k+2+2q, \frac{2\sigma^2}{c^2 r} \right) \] (5)

where

\[ R(t) = \frac{1}{R} + \frac{\sigma^2}{R} \frac{c^2}{k+2q+2} \frac{1}{k+2q+2} \left( k+2q+2 \right) \frac{R}{M} \left( k+2q+2, 2k+2+2q, \frac{2\sigma^2}{c^2 r} \right) \]

and \( k \) satisfies \( k^2 + (1 + 2q) k \frac{2\sigma^2}{c^2} = 0. \)

In Equation (5), \( M \) represents the Kummer-M function (see e.g. Abramowitz and Stegun (1965)). Both Chan et al. (1992) and Ahn and Gao (1999) used the Generalised Method of Moments technique (GMM) of Hansen (1982) to test nested models. In this paper GMM is used to compare the performance of (3) with the more general model (4), using a particular form for \( a(t) \), on a number of different data sets.

The remainder of this paper will be structured in the following manner. In Section 2 we introduce the particular form of the model developed by Goard, given in Equation (4), that we use in this paper, and also our time-variant unrestricted interest rate model which will be used to test nested models. In Section 3 we will outline the estimation technique of GMM used to estimate the parameters for our unrestricted model and test the restrictions for the Chan et al.
model (CKLS), Ahn and Gao model (AG) and the proposed model labelled (GH) here, for convenience. The various data sets to which our model is applied to are listed in Section 4 and the empirical findings for each of these data sets are presented in Section 5. In Section 6 we apply our model to the pricing of zero-coupon bonds based on the model given in Equation (4), for interest rate movements. Lastly in Section 7 we present our conclusions.

2. The Unrestricted Model

The specific form of (4) that we test in this paper is

\[ dr = \{D(t)r + t^2\}dt + t^{3/2}dX \]  \hspace{1cm} (6)

where \( D(t) = d_1 + d_2 \sin(h \tau) + d_3 \cos(h \tau) + d_4 \sin(2h \tau) + d_5 \cos(2h \tau) \),

\( d_1, d_2, d_3, d_4 \) and \( d_5 \) are the parameters to be estimated and \( h \) is a set constant. This model will be referred to as the GH model. Like the AG model, the drift term is non-linear (unlike the model of Chan et al. (1992)) but, unlike the AG model, the threshold level of the interest rate at which the drift is zero is now a function of time, \( D(t) \). We have chosen \( D(t) \) as a second order, periodic Fourier Series, as it is well known that the spot rate contains some approximately cyclical behaviour.

As in the AG model, the drift term in our model is a quadratic function of the interest rate, \( r \). This is consistent with the findings of Ait-Sahalia (1996) and Stanton (1997). They reasoned that a sharp decline in the drift function for high interest rates was necessary to prevent the interest rate exploding. The diffusion in both models (3) and (4) i.e. \( cr^{3/2} \), is the same as that estimated by Chan et al. (1992) to be the best power-law volatility. It implies that the volatility of variation in the spot rate is highly sensitive to the actual spot rate. Hence we will take the CKLS model to also have the volatility of \( cr^{3/2} \). Chan et al. stated that The models that best describe the dynamics of interest rates over time are those that allow the conditional volatility of interest rate changes to be highly dependent on the level of the interest rate.

In this paper we consider the following more general single factor model (the 'unrestricted' model) for the time series of the interest rate:

\[ dr = \{d_1 + D(t)r + d_2 r^2\}dt + d_3 r^{3/2}dX \]  \hspace{1cm} (7)

with \( D(t) \) as in Equation (6) and within which the CKLS, AG and GH models are nested. The three nested models are constructed by placing certain restrictions on the parameters as displayed in Table 1.
Parameter restrictions imposed for CKLS, AG and GH models

The unrestricted model is
\[ dr = \{ a_1 + [b_1 + b_2 \sin(h \rho t) + b_3 \cos(h \rho t) + b_4 \sin(2h \rho t) + b_5 \cos(2h \rho t)]r + b_2 r^2 \} dt + \sigma dX. \]

Table 1

<table>
<thead>
<tr>
<th>Model</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
<th>( b_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CKLS</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AG</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>GH</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the following empirical analysis we take the first point in our data set to be at \( t = 0 \). However the choice of initial time value is unimportant. A translation in time from \( t \) to \( t_0 \) would simply correspond to the change in coefficients of \( \pi(t) \) as follows:

\[
\begin{align*}
\hat{b}_2 & = \hat{b}_2 \cos(h \rho t_0) + \hat{b}_3 \sin(h \rho t_0), \\
\hat{b}_3 & = \hat{b}_2 \sin(h \rho t_0) + \hat{b}_3 \cos(h \rho t_0), \\
\hat{b}_4 & = \hat{b}_4 \cos(2h \rho t_0) + \hat{b}_5 \sin(2h \rho t_0), \\
\hat{b}_5 & = \hat{b}_4 \sin(2h \rho t_0) + \hat{b}_5 \cos(2h \rho t_0).
\end{align*}
\]

3. The Estimation Technique

The technique used to estimate the parameters in the models and compare the models is the Generalised Method of Moments (GMM) of Hansen (1982). Our analysis follows the simple step by step procedure:

**Step 1**
Estimate the parameters for the unrestricted model given in (7).

**Step 2**
Estimate the parameters for the nested AG, CKLS and GH models. It is important to use the same weighting matrix (see below) as that found in step 1.

**Step 3**
Test if the parametric restrictions imposed by each nested model are over-identifying. If this is the case, the model is misspecified.

Following Chan et al. (1992), we use a discrete time econometric specification

\[
\begin{align*}
\Delta r_{t+1} r_t &= \{ a_1 + a_2 + \Delta_1 \sin(h \rho t) + \Delta_2 \cos(h \rho t) + \Delta_3 \sin(2h \rho t) + \Delta_4 \cos(2h \rho t) \} r_t \\
&\quad + a_2 r_t^2 + \sigma_{t+1},
\end{align*}
\]
\[ E[\theta_{+1}] = 0, \quad E[\theta_{+1}^2] = 3 \cdot \sigma^2. \] (9)

We let \( \theta \) be the parameter vector with elements \[ \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7 \] and define the vector

\[ f_t(\theta) = [1, r_t, r_t \sin(h_t), r_t \cos(h_t), r_t, r_t \sin(2h_t), r_t \cos(2h_t)]^T. \]

Under the null hypothesis that (8) and (9) are true, the orthogonality conditions, \( E[f_t(\theta)] = 0 \), hold. With the GMM technique, \( E[f_t(\theta)] \) is replaced with its sample counterpart, \( g_t(\theta) \), using \( T \) observations where

\[ g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(\theta), \]

and then the parameters in the vector \( \theta \) are estimated so that the quadratic form

\[ J_t(\theta) = g_T(\theta)^T W g_T(\theta) \]

is minimised. In this formula for \( J_t(\theta) \), \( W \) is a positive definite, symmetric, weighting matrix

\[ W = [E[f_t(\theta)f_t(\theta)^T]]^{-1} \] (as given in Hansen (1982)), with the sample estimate adjusted for serial correlation and heteroscedasticity using the method of Newey and West (1987) with Bartlett weights. For the unrestricted model, the number of unknowns is exactly equal to the number of orthogonality conditions and so the model is exactly identified, and so \( J_T(\theta) = 0 \).

For each nested model, we conduct the hypothesis test of \( H_0 \) versus \( H_1 \), where

- \( H_0 \): The nested model does not impose overidentifying restrictions and is hence not misspecified i.e. \( \theta_k = 0 \) where \( \theta_k \) is a vector of order \( k \) of restrictions for the appropriate nested model.
- \( H_1 \): The nested model does impose overidentifying restrictions and is hence misspecified.

The appropriate test statistic, developed by Newey and West (1987) is

\[ TS = T(J_T^R(\theta) \otimes J_T^U(\theta)), \]

where \( J_T^R(\theta) \) is the criterion function for the appropriate restricted model, and \( J_T^U(\theta) \) is the criterion function for the unrestricted model, both of which are calculated using the same weighting matrix from the unrestricted model. If \( H_0 \) is true, the test statistic is asymptotically distributed \( \chi^2 \) with \( k \) degrees of freedom. The value \( k \) is the number of restrictions imposed on the general model to obtain the nested model.
4. Data

The interest rate model (7) is applied to the six data sets listed in Table 2. The data sets, US Series 1, US Series 2, Aus. Series 1, Aus. Series 2 and UK Series, record 1 month treasury bill yields, in the US, Australia and UK, while the Thai Series records the 28 day treasury bill yield in Thailand.

<table>
<thead>
<tr>
<th>Series name</th>
<th>Country</th>
<th>Duration</th>
<th>Frequency</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thai Series</td>
<td>Thailand</td>
<td>01/03/2001 - 21/01/2003</td>
<td>Daily</td>
<td>Bank of Thailand (2003)</td>
</tr>
</tbody>
</table>

Below, each series is plotted and the means and standard deviations of the interest rates and interest rate changes are recorded.

**Figure 1**

US Series 1

1-month T-bill yields.

\[
\hat{r}_t : \text{Mean} = 0.0482, \quad SD = 0.0319 \\
\Delta r_{t+1} \quad \hat{r}_t : \text{Mean} = 0.0001, \quad SD = 0.0061
\]

12/1946 - 02/1991
**Figure 2**

**Australian Series 1**

Australian 1 month T-bill yields.

\[ r_t : \text{Mean} = 0.055765, \text{ SD } = 0.010227 \]

\[ r_{t+1} - r_t : \text{Mean} = -0.00012, \text{ SD } = 0.001799 \]


---

**Figure 3**

**US Series 2**

US 1 month T-bill yields.

\[ r_t : \text{Mean} = 0.018464, \text{ SD } = 0.005648 \]

\[ r_{t+1} - r_t : \text{Mean} = -0.00007, \text{ SD } = 0.000529 \]

31/07/2001 - 23/01/2003
Figure 4

Australian Series 2

Australian 1 month T-bill yields.
\[ r_t : \text{Mean} = 0.050261, \ \text{SD} = 0.005836 \]
\[ r_{t+1} - r_t : \text{Mean} = -2.22 \times 10^{-6}, \ \text{SD} = 0.000257 \]
02/01/1998 - 19/06/2002

Figure 5

Thai Series

Thai 28 day T-bill yields.
\[ r_t : \text{Mean} = 0.020029, \ \text{SD} = 0.003096 \]
\[ r_{t+1} - r_t : \text{Mean} = -3.84 \times 10^{-6}, \ \text{SD} = 0.000286 \]
01/03/2001 - 21/01/2003
Using Sample Autocorrelation Functions, we found that the first differenced time series for each data set was stationary.

5. **Empirical Results**

5.1 **US Series 1**

We begin our empirical analysis by analysing the US 1-month treasury bill yields for 12/1946 to 02/1991 of McCulloch and Kwon (1993), which is the same data as that used by Ahn and Gao (1999) in their empirical analysis. The parameters of the unrestricted model and the parameters of the three nested models are estimated. The three nested models were tested for evidence of imposing overidentifying restrictions. Two different values of $h$ were chosen, $h = 1/20$ and $h = 1/25$ corresponding to periods for $\{k(t)\}$ of 40 years and 50 years respectively. The results are displayed in Tables 3 and 4. The p-values are in parentheses.
Table 3
Parameter estimates for the (US Series 1) US 1 month T-Bill yields 12/46 – 02/91 for h=1/20:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coefficient</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unrest.</td>
<td>-0.0042</td>
<td>(0.541)</td>
</tr>
<tr>
<td>GH</td>
<td>0</td>
<td>(0.000)</td>
</tr>
<tr>
<td>AG</td>
<td>0</td>
<td>(0.012)</td>
</tr>
<tr>
<td>CKLS</td>
<td>0.0084</td>
<td>(0.069)</td>
</tr>
</tbody>
</table>

Table 4
Parameter estimates for the (US Series 1) US 1 month T-Bill yields 12/46 – 02/91 for h=1/25:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coefficient</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unrest.</td>
<td>0.00387</td>
<td>(0.659)</td>
</tr>
<tr>
<td>GH</td>
<td>0</td>
<td>(0.000)</td>
</tr>
<tr>
<td>AG</td>
<td>0</td>
<td>(0.018)</td>
</tr>
<tr>
<td>CKLS</td>
<td>0.00789</td>
<td>(0.017)</td>
</tr>
</tbody>
</table>

For $h = 1/20$ the $c^2$ values for the AG and CKLS models imply that both of these models are rejected at the 5% level of significance, while for $h = 1/25$ the $c^2$ values for the AG and CKLS models imply that they both are rejected at the 1% level of significance. Therefore when $h = 1/20$ and $h = 1/25$, the AG and CKLS models are misspecified (at the 5% level of significance) in terms of their overidentifying restrictions. On the other hand, our model is not even rejected at the 20% level of significance, with $c^2$ values of 0.3744 when $h = 1/20$ and 0.1943 when $h = 1/25$. A point worth noting is that few of the trigonometric coefficients of $\beta(t)$ are individually, statistically significantly different from zero, however jointly, they are statistically significantly different from zero. This fact provides evidence that some variation in the short term riskless rate must be explained by an explicit function of time. Further evidence can be seen in the simulations of the US interest rate using CKLS, AG and GH models in Figure 7 below. Comparisons with Figure 1 show how the GH model most closely resembles US series 1.
Figure 7
Simulations of the US interest rate using the GH, AG and CKLS models with parameters in Table 3.
5.2 Australian Series 1

Presented in Tables 5, 6, 7 and 8 are the empirical results for the analysis of the Australian 1-month Treasury Note yields for \( h = 1, 1/2, 1/5 \) and \( 1/10 \) respectively. The AG model and the GH model have been tested for evidence of overidentifying restrictions.

<table>
<thead>
<tr>
<th>Parameter estimates for the (Aus Series 1) Aus 1 month T-Note yields 1992 – 2002 for h=1:</th>
</tr>
</thead>
</table>
| \( \begin{array}{cccccccc}
\langle 1 & 1 & \langle 2 & \langle 3 & \langle 4 & \langle 5 & \langle 2 & \langle 3 & \langle (j) \\
\text{Unrest.} & 0.0891 & -2.9855 & -0.0563 & 0.1428 & 0.0656 & -0.0056 & 23.7499 & 0.1608 \\
& (0.259) & (0.280) & (0.007) & (0.192) & (0.908) & (0.307) & (0.000) & \\
\text{GH} & 0 & 0.1289 & -0.0687 & 0.1197 & 0.0577 & -0.0219 & -2.3866 & 0.1761 \\
& (0.317) & (0.114) & (0.013) & (0.247) & (0.640) & (0.286) & (0.000) & 1.2751 \\
\text{AG} & 0 & 0.1962 & 0 & 0 & 0 & 0 & -3.5811 & 0.1378 \\
& (0.110) & & & & & (0.074) & (0.000) & 17.5864 \\
\end{array} \) |

For \( h = 1 \), the AG model is rejected at the 1% level of significance while our model is not rejected even at the 20% level of significance. The coefficient of \( \cos (j) \), \( \langle 3 \), is statistically significantly different from zero at the 1% level of significance. This fact in itself provides evidence of the importance of the explicit time function when explaining variation in the short term riskless rate.

<table>
<thead>
<tr>
<th>Parameter estimates for the (Aus Series 1) Aus 1 month T-Note yields 1992 – 2002 for h=1/2:</th>
</tr>
</thead>
</table>
| \( \begin{array}{cccccccc}
\langle 1 & \langle 1 & \langle 2 & \langle 3 & \langle 4 & \langle 5 & \langle 2 & \langle 3 & \langle (j) \\
\text{Unrest.} & -0.2824 & 10.0000 & -0.2568 & -0.2323 & -0.1125 & 0.1259 & -85.2863 & 0.0964 \\
& (0.020) & (0.015) & (0.000) & (0.004) & (0.003) & (0.003) & (0.011) & (0.000) \\
\text{GH} & 0 & 0.4034 & -0.2190 & -0.0785 & -0.0741 & 0.1390 & -7.2878 & 0.0912 \\
& (0.025) & (0.000) & (0.109) & (0.027) & (0.002) & (0.020) & (0.000) & 5.4371 \\
\text{AG} & 0 & 0.0610 & 0 & 0 & 0 & 0 & -1.0517 & 0.1169 \\
& (0.633) & & & & & (0.599) & (0.000) & 43.6911 \\
\end{array} \) |

For \( h = 1/2 \), the AG model is strongly rejected at the 0.1% level of significance \( (\langle 5^2 = 43.6911; \ p < 0.001) \). Every single parameter in the unrestricted model is statistically significantly different from zero at the 5% level of significance, including \( \langle 1 \). This leads to the rejection of our model \( (\langle 1^2 = 5.4371; \ p = 0.02) \).
Table 7
Parameter estimates for the (Aus Series 1) Aus 1 month T-Note yields 1992 – 2002 for h=1/5:

|  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|
| Unrest. | -0.1114 (0.256) | 4.8748 (0.177) | 0.2115 (0.009) | -0.0392 (0.451) | -0.0167 (0.798) | -0.3198 (0.000) | -50.8280 (0.116) | 0.1108 (0.000) |
| GH | 0 | 0.7898 (0.015) | 0.1506 (0.015) | 0.0058 (0.597) | -0.2706 (0.000) | -14.7020 (0.010) | 1.2910 (0.256) |
| AG | 0 | 0.1389 (0.276) | 0 | 0 | 0 | -2.4843 (0.254) | 0.0920 (0.002) | 32.6455 (0.000) |

For h = 1/5, the AG model is rejected at the lowest level of significance, while the GH model is not rejected even at the 25% level of significance.

Table 8
Parameter estimates for the (Aus Series 1) Aus 1 month T-Note yields 1992 – 2002 for h=1/10:

|  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|
| Unrest. | -0.0065 (0.959) | -0.5505 (0.901) | 1.9903 (0.002) | 0.0405 (0.792) | 0.1050 (0.563) | 0.8499 (0.002) | -11.0362 (0.780) | 0.1359 (0.000) |
| GH | 0 | -0.7761 (0.051) | 1.9779 (0.001) | 0.0457 (0.694) | 0.0976 (0.378) | 0.8443 (0.001) | -9.0176 (0.013) | 0.1355 (0.000) | 0.0026 (0.959) |
| AG | 0 | 0.0889 (0.501) | 0 | 0 | 0 | 0 | -1.7069 (0.446) | 0.1102 (0.000) | 15.2936 (0.009) |

For h = 1/10, while the AG model is rejected at the 1% level of significance, our model is not rejected at any conventional level of significance \( (\hat{\beta}^2 = 0.0026 ; \ p = 0.959) \).

5.3 Shorter time series

For each of the daily interest rate time series for the US, Australia, Thailand and the UK, as found on the website of the Reserve Bank for each country, we present in Tables 9 through 12 the empirical results using GMM. The data sets were unfortunately shorter than what might be expected in order to detect a time dependence in the interest rate dynamics. However, the results generally suggested otherwise. Parameter estimates are given for the unrestricted model in each case.
As we can see from Table 9, our model is not misspecified for any value of $h$ at the 5% level of significance. This is due to the insignificance of the constant term $a_1$. For $h = 1, 1/2, 1/5$ the AG model is misspecified. None of the parameters $a_1, a_2, a_3, a_4, a_5$ are individually statistically significantly different from zero; however they are jointly significantly different from zero and this suggests evidence of a time-variant coefficient in the modelling of interest rate movements.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$\bar{r}_1$</th>
<th>$\bar{r}_2$</th>
<th>$\bar{r}_3$</th>
<th>$\chi^2_{GH}(1)$</th>
<th>$\chi^2_{AG}(5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.0460 (0.559)</td>
<td>-2.6331 (0.734)</td>
<td>0.0299 (0.934)</td>
<td>1.0904 (0.273)</td>
<td>0.5496 (0.358)</td>
<td>-1.5561 (0.048)</td>
<td>-39.6436 (0.830)</td>
<td>7.6745 (0.120)</td>
<td>0.3416 (0.559)</td>
<td>6.6005 (0.252)</td>
</tr>
<tr>
<td>4</td>
<td>0.0035 (0.967)</td>
<td>2.0391 (0.817)</td>
<td>-0.0157 (0.973)</td>
<td>1.0931 (0.029)</td>
<td>0.1031 (0.785)</td>
<td>1.1163 (0.249)</td>
<td>157.0514 (0.474)</td>
<td>7.9684 (0.129)</td>
<td>0.0017 (0.967)</td>
<td>9.7126 (0.084)</td>
</tr>
<tr>
<td>1</td>
<td>0.2544 (0.412)</td>
<td>-17.8260 (0.623)</td>
<td>1.5821 (0.700)</td>
<td>1.6243 (0.684)</td>
<td>-1.4560 (0.596)</td>
<td>1.9256 (0.065)</td>
<td>142.1728 (0.871)</td>
<td>8.2241 (0.143)</td>
<td>0.6722 (0.412)</td>
<td>15.1722 (0.010)</td>
</tr>
<tr>
<td>1/2</td>
<td>0.3878 (0.069)</td>
<td>1.4984 (0.983)</td>
<td>-41.9851 (0.506)</td>
<td>12.0075 (0.419)</td>
<td>-1.6370 (0.853)</td>
<td>-19.6677 (0.280)</td>
<td>-108.2272 (0.907)</td>
<td>7.8775 (0.122)</td>
<td>0.069 (0.069)</td>
<td>16.0866 (0.007)</td>
</tr>
<tr>
<td>1/5</td>
<td>0.3441 (0.119)</td>
<td>1074.69 (0.462)</td>
<td>-975.291 (0.332)</td>
<td>-1168.57 (0.480)</td>
<td>418.868 (0.343)</td>
<td>92.2540 (0.692)</td>
<td>-185.4050 (0.841)</td>
<td>7.8715 (0.044)</td>
<td>2.5220 (0.112)</td>
<td>14.6088 (0.012)</td>
</tr>
</tbody>
</table>

For the Australian Series 2, the GH model is not rejected for any values of $h$, due to the lack of statistical significance of $a_1$. The coefficients of $\sin(\bar{r})$ and $\cos(\bar{r})$ are statistically significantly different from zero when $h = 1$, and so the AG model is misspecified in this case.

For $h = 1/2$, the coefficients of $\cos(\bar{r}/2)$ and $\sin(\bar{r})$ are statistically significantly different from zero and the AG model is again misspecified. Similarly for $h = 1/5$, the AG model is
misspecified. For \( h = 1/10 \) and \( 1/20 \), all of the coefficients \( \theta_2, \theta_3, \theta_4, \theta_5 \) are statistically significantly different from zero; however the parameter estimates are excessively large, due to the long wavelength implied by the small value of \( h \), over a data set that is defined for a relatively short duration of only a couple of years. An interesting point to note is that even though the coefficient of \( r^{3/2} dX \) is very small (around 0.12), it is however significantly different from zero. By comparison, the estimate of the coefficient of \( r^{3/2} dX \) is high for variations in the Thailand 28-day Treasury bill yield. With the time periods for the US Series 2 and the Thai Series similar, it is interesting to compare the corresponding volatility terms for these series. The volatility for the US Series is even higher than that for Thailand (the coefficient of \( r^{3/2} dX \) is around 7 to 8) however it is only statistically significantly different from zero for \( h = 1/5 \). By observing Table 11 one can see that the time-variant parameter of \( r \) is extremely important. Our model is not rejected for any value of \( h \), however at the 5% level of significance the AG model is misspecified for all values of \( h \).

### Table 11

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
<th>( \theta_4 )</th>
<th>( \theta_5 )</th>
<th>( \sigma^2_{GH}(1) )</th>
<th>( \sigma^2_{AG}(5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>-0.0813 (0.713)</td>
<td>8.4727 (0.692)</td>
<td>-0.5604 (0.014)</td>
<td>1.0196 (0.000)</td>
<td>0.1451 (0.559)</td>
<td>-21.0727 (0.669)</td>
<td>2.2345 (0.000)</td>
</tr>
<tr>
<td>4</td>
<td>-0.1060 (0.627)</td>
<td>10.9720 (0.603)</td>
<td>-0.1690 (0.381)</td>
<td>1.0360 (0.706)</td>
<td>-0.5646 (0.013)</td>
<td>-27.8393 (0.578)</td>
<td>2.2335 (0.000)</td>
</tr>
<tr>
<td>1</td>
<td>-0.5220 (0.137)</td>
<td>57.4561 (0.098)</td>
<td>1.4362 (0.039)</td>
<td>0.3589 (0.429)</td>
<td>0.7358 (0.002)</td>
<td>-15.4446 (0.068)</td>
<td>2.2263 (0.000)</td>
</tr>
<tr>
<td>1/2</td>
<td>-0.4783 (0.203)</td>
<td>43.3411 (0.231)</td>
<td>15.1459 (0.005)</td>
<td>1.6278 (0.015)</td>
<td>0.0953 (0.886)</td>
<td>-14.4150 (0.107)</td>
<td>2.2356 (0.000)</td>
</tr>
<tr>
<td>1/5</td>
<td>-0.4869 (0.200)</td>
<td>-419.884 (0.005)</td>
<td>385.324 (0.004)</td>
<td>520.775 (0.003)</td>
<td>-168.265 (0.086)</td>
<td>-14.4322 (0.109)</td>
<td>2.2359 (0.000)</td>
</tr>
</tbody>
</table>

### Table 12

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
<th>( \theta_4 )</th>
<th>( \theta_5 )</th>
<th>( \sigma^2_{GH}(1) )</th>
<th>( \sigma^2_{AG}(5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.7623 (0.182)</td>
<td>-175.217 (0.186)</td>
<td>0.1990 (0.353)</td>
<td>-0.5434 (0.341)</td>
<td>0.4352 (0.167)</td>
<td>-0.4906 (0.345)</td>
<td>1541.3 (0.191)</td>
</tr>
<tr>
<td>4</td>
<td>4.7514 (0.184)</td>
<td>-147.758 (0.187)</td>
<td>-0.0832 (0.639)</td>
<td>0.4087 (0.355)</td>
<td>0.1945 (0.335)</td>
<td>-0.5418 (0.343)</td>
<td>1536.83 (0.193)</td>
</tr>
<tr>
<td>1</td>
<td>5.9060 (0.144)</td>
<td>-221.360 (0.146)</td>
<td>0.5093 (0.458)</td>
<td>3.3578 (0.138)</td>
<td>0.7377 (0.203)</td>
<td>1.2511 (0.152)</td>
<td>1985.41 (0.256)</td>
</tr>
<tr>
<td>1/2</td>
<td>13.1060 (0.000)</td>
<td>-503.180 (0.000)</td>
<td>-12.3988 (0.000)</td>
<td>1.1686 (0.000)</td>
<td>1.9141 (0.008)</td>
<td>6.2005 (0.000)</td>
<td>4658.84 (0.000)</td>
</tr>
<tr>
<td>1/5</td>
<td>12.6641 (0.001)</td>
<td>-450.074 (0.001)</td>
<td>2.3933 (0.220)</td>
<td>15.5072 (0.000)</td>
<td>-15.9399 (0.000)</td>
<td>-5.8981 (0.001)</td>
<td>3840.37 (0.001)</td>
</tr>
</tbody>
</table>
The results for the UK Series are interesting. For large values of $h$ (8, 4, 1), none of the parameter estimates are statistically significantly different from zero at the 10% level of significance. For $h = 1/2$, all of the parameter estimates except for $\theta_3$ are statistically significantly different from zero at the 1% level of significance. Consequently, both the AG and the GH models are misspecified. There is a similar result for $h = 1/5$.

Hence to summarise our findings very briefly, even with only relatively short data sets, there is evidence of time variation in the drift term.

6. Bond and Yield Plots

As stated previously in the Introduction, when we assume that the short-rate follows the process as defined in (4), we are able to explicitly find the value of zero-coupon bonds using the formula given in (5). From these solutions we can then provide models of the yield curve via the relation

$$y = \frac{\log V}{T - t}.$$  

In this section, using our longest, most current data set, i.e Aus Series 1, we plot the 10-year bond prices and yield curves for Australia as found for the time of our last data point in the series i.e 7/02 (taken to be at $t = 0$). At this time the risk free rate was $r = 0.0469$. Using our model as the unrestricted model, we found the appropriate parameter estimates for the interest rate model and produced graphs for the bond prices and yields shown in Figure 8. We also compared the prices for bonds computed from Equation (5) with the real prices quoted in the market. These are listed in Table 13. As can be seen from this Table, the percentage errors generally increase with time, but remain very small, with our calculated price for a 10-year bond only having a 4.13% error on the true price.
Figure 8

a) Bond Price curve and b) corresponding yield curve, as predicted for 7/02 using Aus series 1.

Table 13
Australian Bond Prices at 07/02

<table>
<thead>
<tr>
<th>Time to maturity</th>
<th>True Price</th>
<th>Calculated Price</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.988</td>
<td>0.988</td>
<td>0%</td>
</tr>
<tr>
<td>0.5</td>
<td>0.975</td>
<td>0.976</td>
<td>0.103%</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
<td>0.91</td>
<td>1.11%</td>
</tr>
<tr>
<td>5</td>
<td>0.753</td>
<td>0.75</td>
<td>0.39%</td>
</tr>
<tr>
<td>10</td>
<td>0.557</td>
<td>0.58</td>
<td>4.13%</td>
</tr>
</tbody>
</table>

7. Conclusion

There exists at the moment, many popular single-factor models of the short interest rate, including the Vasicek (1977) and the CIR (1985) models. Most are appealing because of their tractability i.e. they lead to analytic solutions to the bond pricing equation (2). However as Chan et al. found, many of the short-rate models perform poorly in their ability to capture
the actual behaviour of the spot rate. In this paper we have demonstrated how a particular form of the model of Goard (2000), with a free function of time, outperforms the models which assume a long term reversion to a fixed mean, including the model of Ahn and Gao, and those with an affine drift. We have used a second-order Fourier series to model the free function of time, and called this the GH model. This function of time can obviously be extended to include higher order terms in the Fourier series for even better results. Our model includes the realistic \( cr^{3/2} \) dependence in the volatility.

When fitting a time series on interest rates, the more accurate and realistic results will occur on data sets over a longer period of time. The longer the data set, the more obvious are the short and long term trends in the interest rate behaviour caused by many factors such as income, inflation, money supply, price levels and general tastes. However as we have shown in Section 5.3, even for the shorter data sets, our model has picked up the time dependence in the mean reversion level. It is then obvious that a short-rate model with a time-dependent moving target is a better match to the term structure of real interest rate data than non-time-dependent models.

Using parameter estimates from GMM and Equation (5) we are able to find the value of zero-coupon bonds. Then from exact solutions to the bond pricing equation we can construct the yield curve which gives the investment return as a function of waiting time to expiry. From Figure 8b) presented in Section 6, we can see how choosing a time-dependent periodic mean reversion level reflects the periodicity in the yield curve.

**Acknowledgements**

The authors would like to thank Associate Professor Ken Russell and Professor David Griffiths for useful comments.

**8. References**


Bank of Thailand (2003): website www.bot.or.th/BOThomepage/index/index_e.asp


Clarification to
“Comparison of the performance of a time-dependent short-interest rate model with time-
independent models” by Joanna Goard and Noel Hansen in Applied Math Finance 11, 147-164 (June, 2004).

The solution (5) to the bond-pricing equation (2) is based on the ‘risk-neutral’ short-rate
process (4) so that (4) corresponds to the process
$$dr = (u(r,t) - \lambda(r,t)w(r,t))dt + w(r,t)dX,$$
where $\lambda(r,t)$ is the market price of risk. Thus in estimating the parameters in (6) with real
data we assume that the form of the ‘real’ spot rate process is the same as that for the
‘risk-neutral’ process, so that for a purely interest-rate-dependent market price of risk,
$\lambda(r)$, it would take the form $\lambda(r) = a_1 r^{-1/2} + a_2 r^{1/2}$. Once the parameters are estimated for
the GH model (Equation (7) with $\alpha_1 = 0$), $a_1$ and $a_2$ can be approximated using an
estimated market price of risk curve such as in Stanton (1997), (in which his market price
of risk curve corresponds to $\lambda(r)w(r) = \lambda(r)\alpha_3 r^{3/2}$ in our notation, and for which a good
approximation is $a_1 = -0.308/\alpha_3$ and $a_2 = 2.117/\alpha_3$ for $r \leq 0.13$). Of course as suggested
in Wilmott (1998), we can also estimate $a_1$ and $a_2$ to use for best/worse $\lambda(r)$ to bound
bond prices.

In pricing zero-coupon bonds using equation (5) the parameters then correspond to

$$c = \alpha_3, \quad q = \frac{a_2 \alpha_3 - \alpha_2}{\alpha_3^2}, a(t) = \frac{\bar{\beta}(t)}{\alpha_3^2} \text{ where}$$

$$\bar{\beta}(t) = (\beta_1 - a_1 \alpha_3) + \beta_2 \sin(h \pi t) + \beta_3 \cos(h \pi t) + \beta_4 \sin(2h \pi t) + \beta_5 \cos(2h \pi t).$$