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# Purely infinite C\*-algebras arising from crossed products

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## **Abstract**

We study conditions that will ensure that a crossed product of a C-algebra by a discrete exact group is purely infinite (simple or non-simple). We are particularly interested in the case of a discrete non-amenable exact group acting on a commutative C-algebra, where our sufficient conditions can be phrased in terms of paradoxicality of subsets of the spectrum of the abelian C-algebra. As an application of our results we show that every discrete countable non-amenable exact group admits a free amenable minimal action on the Cantor set such that the corresponding crossed product C-algebra is a Kirchberg algebra in the UCT class.

## **Keywords**

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# Purely infinite $C^*$ -algebras arising from crossed products

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*Abstract.* We study conditions that will ensure that a crossed product of a  $C^*$ -algebra by a discrete exact group is purely infinite (simple or non-simple). We are particularly interested in the case of a discrete non-amenable exact group acting on a commutative  $C^*$ -algebra, where our sufficient conditions can be phrased in terms of paradoxicality of subsets of the spectrum of the abelian  $C^*$ -algebra. As an application of our results we show that every discrete countable non-amenable exact group admits a free amenable minimal action on the Cantor set such that the corresponding crossed product  $C^*$ -algebra is a Kirchberg algebra in the UCT class.

## 1. Introduction

Operator algebras and dynamical systems are closely related. Dynamical systems give rise to operator algebras and they can be studied in terms of these algebras and their invariant. It is a challenging task to decipher information about the dynamical system from its corresponding operator algebra and vice versa.

One motivating example for this article is the particular dynamical system where an arbitrary (discrete) group  $G$  acts on  $\ell^\infty(G)$  by left-translation. The associated (reduced) crossed product  $C^*$ -algebra  $\ell^\infty(G) \rtimes_r G$ , also known as the *Roe algebra*, plays an important role in the study of  $K$ -theory of groups, but it also harbors information related to paradoxical sets and Banach–Tarski’s paradox. For example, if a subset  $E$  of  $G$  is  $G$ -paradoxical, then  $1_E \in \ell^\infty(G)$  is a properly infinite projection in the Roe algebra. (We show in this paper that the converse also holds.) In particular, the Roe algebra itself is properly infinite if and only if  $G$  is  $G$ -paradoxical, which happens if and only if  $G$  is non-amenable.

The special class of  $C^*$ -algebras, now called *Kirchberg algebras*, that are purely infinite simple separable and nuclear, are of particular interest because of their classification (by  $K$ - or  $KK$ -theory) obtained by Kirchberg and Phillips in the mid 1990s. Many of the naturally occurring examples of Kirchberg algebras arise from dynamical systems. The Cuntz algebras  $\mathcal{O}_n$ , for example, are stably isomorphic to the crossed product of a stabilized UHF-algebra with an action of the group of integers that scales the trace.

Several classes of examples of Kirchberg algebras arising as crossed products of abelian  $C^*$ -algebras (often times with spectrum the Cantor set) by hyperbolic groups have appeared in the literature. Prompted by Choi's embedding of  $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  into  $\mathcal{O}_2$ , Archbold and Kumjian (independently) proved that there is an action of  $\mathbb{Z}_2 * \mathbb{Z}_3$  on the Cantor set so that the corresponding crossed product  $C^*$ -algebra is isomorphic to  $\mathcal{O}_2$ .

It is well known that if  $G$  acts topologically freely and minimally on a compact Hausdorff space  $X$ , then the crossed product  $C(X) \rtimes_r G$  is simple. If  $G$  acts amenably on a (general)  $C^*$ -algebra  $A$ , then  $(A, G)$  is automatically *regular* (meaning that the full and the reduced crossed products are the same), and  $A \rtimes G$  is nuclear if and only if  $A$  is nuclear; see [8, Theorem 4.3.4]. Conversely, if  $A$  is abelian and unital and  $A \rtimes_r G$  is nuclear, then the action of  $G$  on  $A$  is amenable; see [8, Theorem 4.4.3]. For abelian  $C^*$ -algebras  $A$ , the full crossed product  $A \rtimes G$  is simple if and only if the action is minimal, topologically free and regular [4, Corollary, p. 124]. These results can be reformulated as follows, assuming  $A$  to be unital and abelian: the reduced crossed product  $A \rtimes_r G$  is simple *and* nuclear if and only if the action is minimal, topologically free and amenable. See [1, 3, 4, 8] for details.

There are several known partial results containing sufficient conditions, in terms of the geometry of the dynamical system, for the crossed product to be purely infinite (and hence a Kirchberg algebra, provided that the conditions for nuclearity, separability and simplicity are satisfied). Laca and Spielberg showed in [21] that pure infiniteness of the crossed product is ensured by requiring that the action is a *strong boundary action*, meaning that  $X$  is infinite and that any two non-empty open subsets of  $X$  can be translated by group elements to cover the entire space  $X$ .

Jolissaint and Robertson generalized this result and showed that it is enough to require that the action is *n-filling*, a similar property but with  $n$  subsets instead of two subsets of  $X$ . Their results also provide sufficient conditions for pure infiniteness of crossed products  $A \rtimes_r G$  when  $A$  is non-abelian; see [15, Theorem 1.2]. For other results on when crossed product  $C^*$ -algebras are Kirchberg algebras we refer to [2, 20, 21].

In part guided by the motivating example of the Roe algebra we shall in this paper be interested in conditions on a dynamical system  $(A, G)$  that will ensure that the corresponding reduced crossed product  $C^*$ -algebra  $A \rtimes_r G$  is purely infinite, but not necessarily simple. We refer to [17] for more about non-simple purely infinite  $C^*$ -algebras.

We show that the crossed product  $A \rtimes_r G$  is purely infinite if and only if every non-zero positive element in  $A$  is properly infinite in  $A \rtimes_r G$  provided that  $G$  is discrete and exact,  $A$  is separable and has the so-called ideal property (IP) (projections separate ideals), and the action of  $G$  on  $\widehat{A}$  is essentially free (Theorem 3.3). In the case where  $A$  is the continuous functions on the Cantor set, we show that  $A \rtimes_r G$  is purely infinite if and only if every non-zero projection in  $A$  is properly infinite in  $A \rtimes_r G$ , here assuming that  $G$  is discrete and exact and that the action of  $G$  on  $\widehat{A}$  is essentially free; cf. Theorem 4.1.

In §5 we consider dynamical systems  $(C(X), G)$ , where  $X$  is totally disconnected, and we show that if a clopen subset  $E$  of  $X$  is  $G$ -paradoxical (in a way that respects clopen sets), then  $1_E$  is a properly infinite projection in the crossed product  $C(X) \rtimes_r G$ . Partial converse results are also obtained. In the case of the Roe algebra  $\ell^\infty(G) \rtimes_r G$  the situation is more clear: if  $E$  is a subset of  $G$ , then  $1_E$  is properly infinite in  $\ell^\infty(G) \rtimes_r G$  if and only if  $E$  is  $G$ -paradoxical, which again happens if and only if there is no (unbounded) trace on  $\ell^\infty(G) \rtimes_r G$  that is non-zero and finite on  $1_E$ . In §6 we use this result to show that each countable, discrete, non-amenable, exact group admits an action on the Cantor set such that the crossed product is a Kirchberg algebra in the UCT class.

2. Notation and a preliminary result

Given a  $C^*$ -algebra dynamical system  $(A, G)$  with  $G$  a discrete group, let  $A \rtimes_r G$  and  $A \rtimes G$  denote the reduced and the full crossed product  $C^*$ -algebras, respectively. In general we have a surjective, and not necessarily injective,  $*$ -homomorphism  $A \rtimes G \rightarrow A \rtimes_r G$ . If  $G$  is amenable, or if  $G$  acts amenably on  $A$ , then this  $*$ -homomorphism is injective, whence the reduced and the full crossed products are equal. In this case the dynamical system is said to be regular.

Let  $C_c(G, A)$  be the common subalgebra of both crossed products consisting of finite sums  $\sum_{t \in G} a_t u_t$ , where  $a_t \in A$  (only finitely many non-zero), and  $t \mapsto u_t$ ,  $t \in G$ , is the unitary representation of  $G$  that implements the action of  $G$  on  $A$ . (If  $A$  is unital, then each  $u_t$  belongs to the crossed product, and in general  $u_t$  belongs to the multiplier algebra of the crossed product.) We suppress the canonical inclusion map  $A \rightarrow A \rtimes_r G$  and view  $A$  as being a sub- $C^*$ -algebra of  $A \rtimes_r G$ .

We have a (faithful) conditional expectation  $E : A \rtimes_r G \rightarrow A$  which on  $C_c(G, A)$  is given by  $\sum_{t \in G} a_t u_t \mapsto a_e$ , where  $e \in G$  denotes the neutral element.

For every  $G$ -invariant ideal  $I$  in  $A$ , the natural maps in the short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\rho} A/I \longrightarrow 0 \tag{2.1}$$

extend canonically to maps at the level of reduced crossed products, giving rise to the possibly non-exact sequence

$$0 \longrightarrow I \rtimes_r G \xrightarrow{\iota \rtimes_r \text{id}} A \rtimes_r G \xrightarrow{\rho \rtimes_r \text{id}} A/I \rtimes_r G \longrightarrow 0 \tag{2.2}$$

(cf. [33, Remark 7.14]). In general, the kernel of  $\rho \rtimes_r \text{id}$ , contains—but need not be contained in—the image of  $\iota \rtimes_r \text{id}$ . The sequence (2.2) is exact precisely when  $\ker(\rho \rtimes_r \text{id}) \subseteq \text{im}(\iota \rtimes_r \text{id})$ . By the definition of Kirchberg and Wassermann [19],  $G$  is exact if and only if (2.2) is exact for all exact sequences (2.1) of  $C^*$ -algebras with compatible  $G$ -actions.

The action of  $G$  on  $A$  is said to be *exact* if every  $G$ -invariant ideal in  $A$  induces a short exact sequence at the level of reduced crossed products, i.e. if (2.2) is exact for all  $G$ -invariant closed two-sided ideals in  $A$  (i.e., whenever (2.1) is exact with  $A$  fixed); cf. [29, Definition 1.2]. Recall that  $A$  is said to *separate the ideals in  $A \rtimes_r G$*  if the (surjective) map  $J \mapsto J \cap A$ , from the ideals in  $A \rtimes_r G$  into the  $G$ -invariant ideals in  $A$ , is injective. If the  $C^*$ -algebra  $A$  separates the ideals in  $A \rtimes_r G$ , then the action of  $G$  on  $A$  must be exact; cf. [29, Theorem 1.10].

Recall that the action of  $G$  on  $\widehat{A}$  is said to be *essentially free* provided that, for every closed  $G$ -invariant subset  $Y \subseteq \widehat{A}$ , the subset of points in  $Y$  with trivial isotropy is dense in  $Y$ ; cf. [26]. It was shown in [29] that  $A$  separates the ideals in  $A \rtimes_r G$  if  $G$  is a discrete group, the action of  $G$  on  $A$  is exact, and the action of  $G$  on  $\widehat{A}$  is essentially free. In particular, if a discrete exact group  $G$  acts (essentially) freely on a compact Hausdorff space  $X$ , then  $C(X)$  separates ideals in  $C(X) \rtimes_r G$ .

We remind the reader of some basic definitions involving Cuntz comparison and infiniteness of positive elements in a  $C^*$ -algebra. Let  $a, b$  be positive elements in a  $C^*$ -algebra  $A$ . Write  $a \preceq b$  if there exists a sequence  $(r_n)$  in  $A$  such that  $r_n^* b r_n \rightarrow a$ . More generally for  $a \in M_n(A)^+$  and  $b \in M_m(A)^+$  write  $a \preceq b$  if there exists a sequence  $(r_n)$  in  $M_{m,n}(A)$  with  $r_n^* b r_n \rightarrow a$ . For  $a \in M_n(A)$  and  $b \in M_m(A)$  let  $a \oplus b$  denote the element  $\text{diag}(a, b) \in M_{n+m}(A)$ .

A positive element  $a$  in a  $C^*$ -algebra  $A$  is said to be *infinite* if there exists a non-zero positive element  $b$  in  $A$  such that  $a \oplus b \preceq a$ . If  $a$  is non-zero and if  $a \oplus a \preceq a$ , then  $a$  is said to be *properly infinite*. This extends the usual concepts of infinite and properly infinite projections; cf. [17, pp. 642–643].

A  $C^*$ -algebra  $A$  is *purely infinite* if there are no characters on  $A$  and if, for every pair of positive elements  $a, b$  in  $A$  such that  $b$  belongs to the ideal in  $A$  generated by  $a$ , one has  $b \preceq a$ . Equivalently, a  $C^*$ -algebra  $A$  is purely infinite if every non-zero positive element  $a$  in  $A$  is properly infinite; cf. [17, Theorem 4.16].

We state below a general, but also rather restrictive, condition that implies pure infiniteness of a crossed product.

**PROPOSITION 2.1.** *Let  $(A, G)$  be a  $C^*$ -dynamical system with  $G$  discrete. Suppose that  $A$  separates the ideals in  $A \rtimes_r G$ . Then  $A \rtimes_r G$  is purely infinite if and only if all non-zero positive elements in  $A$  are properly infinite in  $A \rtimes_r G$  and  $E(a) \preceq a$  for all positive elements  $a$  in  $A \rtimes_r G$ .*

*Proof.* Suppose that  $A \rtimes_r G$  is purely infinite. Fix a non-zero positive element  $a$  in  $A \rtimes_r G$ . By [29, Theorem 1.10] we have that  $E(a)$  belongs to the ideal in  $A \rtimes_r G$  generated by  $a$ , whence  $E(a) \preceq a$  by proper infiniteness of  $a$ ; cf. [17, Theorem 4.16].

Now suppose that every non-zero positive element in  $A$  is properly infinite in  $A \rtimes_r G$  and that  $E(a) \preceq a$  for every positive element  $a$  in  $A \rtimes_r G$ . Since the action of  $G$  on  $A$  is exact it follows that  $a$  belongs to the ideal in  $A \rtimes_r G$  generated by  $E(a)$  for all positive elements  $a$  in  $A \rtimes_r G$ ; cf. [29, Proposition 1.3]. This implies that  $a \preceq E(a)$  using that  $E(a)$  is properly infinite; cf. [17, Proposition 3.5(ii)]. By the relation

$$a \oplus a \preceq E(a) \oplus E(a) \preceq E(a) \preceq a$$

in  $A \rtimes_r G$ , we conclude that  $a$  is properly infinite. This shows that  $A \rtimes_r G$  is purely infinite; cf. [17, Theorem 4.16].  $\square$

*Remark 2.2.* The condition in Proposition 2.1 that  $E(a) \preceq a$  for all positive elements  $a$  in  $A \rtimes_r G$  is automatically satisfied in many cases of interest, for example in Theorem 3.3.

*Remark 2.3.* In the proof of Proposition 2.1 we used exactness and proper infiniteness of positive elements in  $A$  to conclude that  $a \preceq E(a)$  for all positive elements  $a$  in  $A \rtimes_r G$ .

It is shown in [29, Proposition 1.3] that  $a$  belongs to the closed two-sided ideal in  $A \rtimes_r G$  generated by  $E(a)$  for all positive  $a$  in  $A \rtimes_r G$  if and only if the action of  $G$  on  $A$  is exact. It follows in particular that  $a \preceq E(a)$  does not hold for all  $a$  when the action of  $G$  on  $A$  is not exact.

Each (positive) element  $a$  in  $A \rtimes_r G$  can be written as a formal sum  $\sum_{t \in G} a_t u_t$ , where  $t \mapsto u_t$  is the unitary representation of  $G$  in (the multiplier algebra of)  $A \rtimes_r G$ , and where  $a_t = E(a u_t^*) \in A$ . Upon writing  $a = x x^*$  and using Lemma 5.2 below, one can show that each term,  $a_t u_t$ , in this formal sum belongs to the hereditary sub- $C^*$ -algebra of  $A \rtimes_r G$  generated by  $E(a)$ . It therefore follows that  $a$  belongs to the hereditary sub- $C^*$ -algebra of  $A \rtimes_r G$  generated by  $E(a)$ , and hence that  $a \preceq E(a)$ , whenever  $a$  belongs to  $C_c(G, A)$ , or, slightly more generally, when the formal sum,  $a = \sum_{t \in G} a_t u_t$ , is norm-convergent.

It remains a curious open problem if the relation  $a \preceq E(a)$  holds for all positive  $a$  in  $A \rtimes_r G$ , when the action of  $G$  on  $A$  is exact.

### 3. Pure infiniteness of crossed products with the ideal property

We provide a necessary condition for a crossed product  $A \rtimes_r G$  to be purely infinite assuming that  $A$  has the so-called ideal property (projections in  $A$  separate ideals in  $A$ ), and under the assumption that the action of  $G$  on  $A$  is exact and essentially free. (The latter notion is defined in §2.)

LEMMA 3.1. *Let  $(A, G)$  be a  $C^*$ -dynamical system with  $G$  discrete. Assume that the action of  $G$  on  $A$  is exact and that the action of  $G$  on  $\widehat{A}$  is essentially free. If  $A$  has the ideal property, then so does  $A \rtimes_r G$ .*

*Proof.* The assumptions on the action imply that  $A$  separates ideals in  $A \rtimes_r G$ ; cf. [29]. As projections in  $A$  separate ideals in  $A$ , it easily follows that projections in  $A$  (and hence also projections in  $A \rtimes_r G$ ) separate ideals in  $A \rtimes_r G$ . □

LEMMA 3.2. *Let  $(A, G)$  be a  $C^*$ -dynamical system with  $A$  separable and  $G$  countable and discrete. Assume that the action of  $G$  on  $\widehat{A}$  is essentially free. Then for every  $G$ -invariant ideal  $I$  in  $A$  and every non-zero positive element  $b$  in  $A/I \rtimes_r G$  there exists a non-zero positive element  $a$  in  $A/I$  such that  $a \preceq b$ .*

*Proof.* Let  $I$  and  $b$  be as above. We can without loss of generality assume that  $\|E(b)\| = 1$ , where  $E : A/I \rtimes_r G \rightarrow A/I$  is the canonical (faithful) conditional expectation. Essential freeness of the action on  $\widehat{A}$  implies that the induced action of  $G$  on  $A/I$  is topologically free and hence properly outer; see [4]. The existence of an element  $x \in (A/I)^+$  satisfying

$$\|x\| = 1, \quad \|xE(b)x - xbx\| \leq 1/4, \quad \|xE(b)x\| \geq \|E(b)\| - 1/4 = 3/4$$

is ensured by [22, Lemma 7.1]. With  $a = (xE(b)x - 1/2)_+$  we claim that  $0 \neq a \preceq xbx \preceq b$ . Indeed, the element  $a$  is non-zero because  $\|xE(b)x\| > 1/2$ , and  $a \preceq xbx$  holds since  $\|xE(b)x - xbx\| < 1/2$ ; cf. [27, Proposition 2.2]. □

The conclusion of Lemma 3.2 also holds if  $A$  is abelian and not necessarily separable (a case that shall have our interest). The existence of the element  $x$  in the proof above was ensured by [22, Lemma 7.1] and the assumption that  $A/I$  is separable. In the case where  $A$

is abelian and not necessarily separable the existence of  $x$  with the desired properties was established in [10, Proposition 2.4].

**THEOREM 3.3.** *Let  $(A, G)$  be a  $C^*$ -dynamical system with  $G$  discrete. Suppose that the action of  $G$  on  $A$  is exact and that the action of  $G$  on  $\widehat{A}$  is essentially free. Suppose also that  $A$  is separable and has the ideal property. Then the following statements are equivalent.*

- (i) *Every non-zero positive element in  $A$  is properly infinite in  $A \rtimes_r G$ .*
- (ii) *The  $C^*$ -algebra  $A \rtimes_r G$  is purely infinite.*
- (iii) *Every non-zero hereditary sub- $C^*$ -algebra in any quotient of  $A \rtimes_r G$  contains an infinite projection.*

*Proof.* (ii)  $\Leftrightarrow$  (iii). By Lemma 3.1 the reduced crossed product  $A \rtimes_r G$  has the ideal property. The equivalence is obtained from [24, Proposition 2.1].

(ii)  $\Rightarrow$  (i). This is contained in Proposition 2.1 (or in [17, Theorem 4.16]).

(i)  $\Rightarrow$  (iii). Fix an ideal  $J$  in  $A \rtimes_r G$  and a non-zero hereditary sub- $C^*$ -algebra  $B$  in the quotient  $(A \rtimes_r G)/J$ . We will show that  $B$  contains an infinite projection.

By the assumptions on the action of  $G$  on  $A$  and on  $\widehat{A}$  it follows from [29, Theorem 1.16] that  $A$  separates ideals in  $A \rtimes_r G$ . Hence,

$$(A \rtimes_r G)/J = (A/I) \rtimes_r G,$$

with  $I = J \cap A$ . Fix a non-zero positive element  $b$  in  $B$ . By Lemma 3.2 there exists a non-zero positive element  $a$  in  $A/I$  such that  $a \lesssim b$  relative to  $A/I \rtimes_r G$ . By the assumption that  $A$  has the ideal property we can find a projection  $q \in A$  that belongs to the ideal in  $A$  generated by the preimage of  $a$  in  $A$  but not to  $I$ . Then  $q + I$  belongs to the ideal in  $A/I$  generated by  $a$ , whence  $q + I \lesssim a \lesssim b$  in  $A/I \rtimes_r G$  (because  $a$  is properly infinite by the assumption in (i)); see [17, Proposition 3.5(ii)].

From the comment after [17, Proposition 2.6] we can find  $z \in A/I \rtimes_r G$  such that  $q + I = z^*bz$ . With  $v = b^{1/2}z$  it follows that  $v^*v = q + I$ , whence  $p := vv^* = b^{1/2}zz^*b^{1/2}$  is a projection in  $B$ , which is equivalent to  $q$ . By the assumption in (i),  $q$  is properly infinite, and hence so is  $p$ .  $\square$

#### 4. Group actions on the Cantor set and paradoxical sets

Recall that a commutative  $C^*$ -algebra  $C_0(X)$ , with  $X$  a locally compact Hausdorff space, is of real rank zero if and only if  $X$  is totally disconnected (which again happens if and only if  $C_0(X)$  has the ideal property). We shall in this section study group actions on totally disconnected spaces. First we give a sharpening of Theorem 3.3 in the case where  $A$  is abelian, of real rank zero, but not necessarily separable.

**THEOREM 4.1.** *Let  $(A, G)$  be a  $C^*$ -dynamical system with  $A = C_0(X)$  and with  $G$  discrete and exact. Suppose that the action of  $G$  on  $X$  is essentially free and that  $X$  is totally disconnected. Then the following statements are equivalent.*

- (i) *Every non-zero projection in  $A/I$  is infinite in  $A/I \rtimes_r G$  for every  $G$ -invariant ideal  $I$  in  $A$ .*
- (ii) *Every non-zero projection in  $A$  is properly infinite in  $A \rtimes_r G$ .*
- (iii) *The  $C^*$ -algebra  $A \rtimes_r G$  is purely infinite.*



*Proof.* (iii)  $\Rightarrow$  (ii). Every non-zero projection in any purely infinite  $C^*$ -algebra is properly infinite.

(ii)  $\Rightarrow$  (i). Use that  $A$  has real rank zero to lift a projection in  $A/I$  to a projection in  $A$ ; cf. [7, Theorem 3.14].

(i)  $\Rightarrow$  (iii). Fix a non-zero hereditary sub- $C^*$ -algebra  $B$  in the quotient of  $A \rtimes_r G$  by some ideal  $J$  in  $A \rtimes_r G$ . By [17, Proposition 4.7] we need only show that  $B$  contains an infinite projection.

By essential freeness of the action of  $G$  on  $X$  and exactness of  $G$  we have the identification

$$(A \rtimes_r G)/J = (A/I) \rtimes_r G,$$

for  $I := J \cap A$ ; cf. [29, Theorem 1.16]. Fix a non-zero positive element  $b$  in  $B$ . By Lemma 3.2 (and the remark below that lemma) there exists a non-zero positive element  $a$  in  $A/I$  such that  $a \lesssim b$  relative to  $A/I \rtimes_r G$ . The hereditary sub- $C^*$ -algebra  $\overline{a(A/I)a}$  contains a non-zero projection  $q$  by the assumption on  $A$ , and  $q$  is infinite in  $A/I \rtimes_r G$  by the assumption in (i). As  $q \lesssim b$  (and because  $q$  is a projection) we can find  $z \in A/I \rtimes_r G$  such that  $q = z^*bz$ . It follows that  $p := b^{1/2}z z^* b^{1/2}$  is a projection in  $B$ , which is equivalent to  $q$ , and hence is infinite.  $\square$

In lieu of the previous result we are interested in knowing for which clopen subsets  $V$  of  $X$  the projection  $1_V$  in  $C(X)$  is properly infinite in  $C(X) \rtimes_r G$ . A sufficient, and sometimes also necessary, condition for this to happen is that  $V$  is  $G$ -paradoxical in  $X$  with respect to the open subsets of  $X$ . We give a formal definition of this phenomenon below.

*Definition 4.2.* Given a discrete group  $G$  acting on a topological space  $X$  and a family  $\mathbb{E}$  of subsets of  $X$ , a non-empty set  $U \subseteq X$  is called  $(G, \mathbb{E})$ -paradoxical if there exist non-empty sets  $V_1, V_2, \dots, V_{n+m} \in \mathbb{E}$  and elements  $t_1, t_2, \dots, t_{n+m}$  in  $G$  such that

$$\bigcup_{i=1}^n V_i = \bigcup_{i=n+1}^{n+m} V_i = U$$

and such that  $(t_k \cdot V_k)_{k=1}^{n+m}$  are pairwise disjoint subsets of  $U$ .

A projection  $p$  in a  $C^*$ -algebra  $A$  is properly infinite if and only if there exist partial isometries  $x$  and  $y$  in  $A$  such that  $x^*x = y^*y = p$  and  $xx^* + yy^* \leq p$ . In the following let  $\tau_X$  denote the family of open subsets of the topological space  $X$ .

**PROPOSITION 4.3.** *Let  $X$  be a locally compact Hausdorff space, and let  $G$  be a discrete group acting on  $X$ . Suppose that  $U$  is a compact open subset of  $X$ . Then  $U$  is  $(G, \tau_X)$ -paradoxical if and only if there exist elements  $x, y$  in  $C_c(G, C_0(X))^+$  such that  $x^*x = y^*y = 1_U$  and  $xx^* + yy^* \leq 1_U$ . In this case  $1_U$  is properly infinite in  $C_0(X) \rtimes_r G$ .*

*Proof.* Suppose first that  $U$  is  $(G, \tau_X)$ -paradoxical, and let  $(V_i, t_i)_{i=1}^{n+m}$  be a system of subsets of  $U$  and elements in  $G$  that witnesses the paradoxicality. Find partitions of unity  $(h_i)_{i=1}^n$  and  $(h_i)_{i=n+1}^{n+m}$  for  $U$  relative to the open covers  $(V_i)_{i=1}^n$  and  $(V_i)_{i=n+1}^{n+m}$ , respectively. As  $U$  is compact and open, we can assume that each  $h_i$  has support contained in  $U$

so that  $\sum_{i=1}^n h_i = \sum_{i=n+1}^{n+m} h_i = 1_U$ . Set

$$x = \sum_{i=1}^n u_i h_i^{1/2}, \quad y = \sum_{i=n+1}^{n+m} u_i h_i^{1/2}.$$

Notice that

$$h_i^{1/2} u_i^* u_j h_j^{1/2} = u_i^* (t_i \cdot h_i^{1/2}) (t_j \cdot h_j^{1/2}) u_j = 0$$

when  $i \neq j$ , since  $\text{supp}(t_k \cdot h_k^{1/2}) \subseteq t_k \cdot V_k$ . It follows that

$$x^* x = \sum_{i,j=1}^n h_i^{1/2} u_i^* u_j h_j^{1/2} = \sum_{i=1}^n h_i^{1/2} u_i^* u_i h_i^{1/2} = \sum_{i=1}^n h_i = 1_U.$$

In a similar way we see that  $y^* y = 1_U$ ,  $y^* x = 0$ , and hence  $xx^* + yy^* \leq 1_U$ .

Assume, conversely, that we are given two elements

$$x = \sum_{t \in F} u_t h_t^{1/2} \quad \text{and} \quad y = \sum_{s \in F'} u_s g_s^{1/2}$$

in  $C_c(G, C_0(X)^+)$  satisfying  $x^* x = y^* y = 1_U$  and  $xx^* + yy^* \leq 1_U$ , where  $F, F' \subseteq G$  are finite and  $h_t, g_s$  belong to  $C_0(X)^+$ . Put

$$V_t = \{\xi \in X : h_t(\xi) \neq 0\}, \quad W_s = \{\xi \in X : g_s(\xi) \neq 0\}.$$

We show that the system  $\{(V_t, t) : t \in F\} \cup \{(W_s, s) : s \in F'\}$  witnesses the  $(G, \tau_X)$ -paradoxicality of  $U$ . To this end, first note that

$$1_U = x^* x = E(x^* x) = \sum_{t,s \in F} E(h_t^{1/2} u_t^* u_s h_s^{1/2}) = \sum_{t \in F} E(h_t^{1/2} u_t^* u_t h_t^{1/2}) = \sum_{t \in F} h_t.$$

This implies that  $\bigcup_{t \in F} V_t = U$ . In a similar way we see that  $\bigcup_{s \in F'} W_s = U$ .

For  $r \neq e$  in  $G$  we have that

$$0 = E(1_U u_r) = E(x^* x u_r) = \sum_{t,s \in F} E(h_t^{1/2} u_t^* u_s h_s^{1/2} u_r).$$

Each term in the sum on the right-hand side of the equation above is a positive element in  $C_0(X)$ , and must hence be zero. We can therefore conclude that  $h_t^{1/2} u_t^* u_s h_s^{1/2} u_r = 0$  for all  $s, t \in F$  with  $t \neq s$ . This shows that  $t \cdot h_t \perp s \cdot h_s$  for  $t \neq s \in F$ , which again implies that  $t \cdot V_t$  and  $s \cdot V_s$  are disjoint, when  $s, t \in F$  and  $s \neq t$ . In a similar way we obtain that  $t \cdot W_t \cap s \cdot W_s = \emptyset$  for  $t \neq s \in F'$  and that  $t \cdot V_t \cap s \cdot W_s = \emptyset$  for  $t \in F$  and  $s \in F'$  (the last property is obtained from the fact that  $0 = E(x^* y u_r)$  for every  $r \in G$ ).

Finally we show that  $t \cdot V_t \subseteq U$  for every  $t \in F$ . Since  $1_U E(xx^*) = E(1_U xx^*)$  and  $E(xx^*) = \sum_{t \in F} t \cdot h_t$ , we can conclude that  $t \cdot h_t = 1_U(t \cdot h_t)$ , which yields the desired inclusion. In a similar way we see that  $s \cdot W_s \subseteq U$  for all  $s \in F'$ . □

Theorem 4.1 and Proposition 4.3 imply the following corollary.

**COROLLARY 4.4.** *Let  $(A, G)$  be a  $C^*$ -dynamical system with  $A = C(X)$  and  $G$  discrete and exact. Suppose that the action of  $G$  on  $X$  is essentially free and  $X$  has a basis of clopen  $(G, \tau_X)$ -paradoxical sets. Then  $C(X) \rtimes_r G$  is purely infinite.*

*Remark 4.5.* If an action of a discrete group  $G$  on a compact Hausdorff space  $X$  is  $n$ -filling in the sense of Jolissaint and Robertson [15] (see also the introduction), then every non-empty open subset of  $X$  is  $(G, \tau_X)$ -paradoxical and the action is minimal. Minimality is not required in Corollary 4.4, thus giving a new characterization of pure infiniteness for this class of non-simple crossed products  $C^*$ -algebras.

*Remark 4.6.* Let  $G$  be a countable group acting on the Cantor set  $X$ . Assume that the full crossed product  $C(X) \rtimes G$  is simple and that all non-zero projections in  $C(X)$  are properly infinite in  $C(X) \rtimes G$ . Then  $C(X) \rtimes G$  is purely infinite. Indeed, simplicity of  $C(X) \rtimes G$  trivially implies that the action of  $G$  on  $C(X)$  is exact and regular (the canonical surjection  $C(X) \rtimes G \rightarrow C(X) \rtimes_r G$  is injective). We can therefore infer pure infiniteness of  $C(X) \rtimes G$  from [30, Remark 4.3.7].

5. The type semigroup  $S(X, G, \mathbb{E})$

We shall here study pure infiniteness of the crossed product  $C^*$ -algebra  $C(X) \rtimes_r G$  coming from an action of a discrete group  $G$  on a compact Hausdorff space  $X$  (typically the Cantor set) by studying the associated type semigroup considered in [32].

A state on a preordered monoid  $(S, +, \leq)$  is a map  $f : S \rightarrow [0, \infty]$  which respects  $+$  and  $\leq$  and fulfills that  $f(0) = 0$ . A state is said to be *non-trivial* if it takes a value different from 0 and  $\infty$ . An element  $x \in S$  is said to be *properly infinite* if  $2x \leq x$ ; and the monoid  $S$  is said to be *purely infinite* if every  $x \in S$  is properly infinite. The monoid  $S$  is *almost unperforated* if, whenever  $x, y \in S$  and  $n, m \in \mathbb{N}$  are such that  $nx \leq my$  and  $n > m$ , then  $x \leq y$ .

Let  $\mathbb{E}$  be a  $G$ -invariant subalgebra of  $\mathcal{P}(X)$ , the power set of  $X$ . We write  $S(X, G, \mathbb{E})$  for the type semigroup of the induced action of  $G$  on  $\mathbb{E}$ , where only sets from  $\mathbb{E}$  can be used to witness the equidecomposability of sets in  $\mathbb{E}$ ; cf. [32, p. 116]. In more detail, the set  $S(X, G, \mathbb{E})$  is defined to be

$$\left\{ \bigcup_{i=1}^n A_i \times \{i\} : A_i \in \mathbb{E}, n \in \mathbb{N} \right\} / \sim_S,$$

where the equivalence relation  $\sim_S$  is defined as follows: two sets  $A = \bigcup_{i=1}^n A_i \times \{i\}$  and  $B = \bigcup_{j=1}^m B_j \times \{j\}$  are equivalent, denoted  $A \sim_S B$ , if there exist  $l \in \mathbb{N}$ ,  $C_k \in \mathbb{E}$ ,  $t_k \in G$ , and natural numbers  $n_k, m_k$  (for  $k = 1, 2, \dots, l$ ) such that

$$A = \bigsqcup_{k=1}^l C_k \times \{n_k\}, \quad B = \bigsqcup_{k=1}^l t_k \cdot C_k \times \{m_k\},$$

(we use the symbol  $\bigsqcup$  to emphasize the union being disjoint). The equivalence class containing  $A$  is denoted by  $[A]$ , and addition is defined by

$$\left[ \bigcup_{i=1}^n A_i \times \{i\} \right] + \left[ \bigcup_{j=1}^m B_j \times \{j\} \right] = \left[ \bigcup_{i=1}^n A_i \times \{i\} \cup \bigcup_{j=1}^m B_j \times \{n+j\} \right].$$

For  $E \in \mathbb{E}$ , we shall often write  $[E]$  instead of  $[E \times \{1\}]$ . The semigroup  $S(X, G, \mathbb{E})$  has neutral element  $0 = [\emptyset]$ , and it is equipped with the algebraic preorder ( $x \leq y$  if  $y = x + z$  for some  $z$ ). This makes  $S(X, G, \mathbb{E})$  into a preordered monoid.

LEMMA 5.1. *Let  $G$  be a discrete group acting on a compact Hausdorff space  $X$ . Let  $\mathbb{E}$  denote the family of clopen subsets of  $X$ . Suppose that  $\sigma(\mathbb{E}) = \mathbb{B}(X)$  (the Borel  $\sigma$ -algebra on  $X$ ). Then every non-trivial state  $f$  on  $S(X, G, \mathbb{E})$  lifts to a  $G$ -invariant measure  $\mu$  on  $(X, \mathbb{B}(X))$  such that  $0 < \mu(F) < \infty$  for some  $F \in \mathbb{E}$ .*

*Proof.* Define  $\mu_0 : \mathbb{E} \rightarrow [0, \infty]$  by  $\mu_0(F) = f([F])$  for  $F \in \mathbb{E}$ . We show that  $\mu_0$  is a premeasure in the sense of [11, p. 30]. We trivially have that  $\mu_0(\emptyset) = f(0) = 0$ . Suppose that  $(F_i)_{i=1}^\infty$  is a sequence of pairwise disjoint sets in  $\mathbb{E}$  such that  $F = \bigcup_{i=1}^\infty F_i$  belongs to  $\mathbb{E}$ . Then  $F = \bigcup_{i=1}^n F_i$  for some  $n$  by compactness of  $F$  (so  $F_i = \emptyset$  for all  $i > n$ ). Additivity of  $f$  now implies that  $\mu_0(F) = \sum_{i=1}^\infty \mu_0(F_i)$ . This shows that  $\mu_0$  is a premeasure. By [11, Theorem 1.14],  $\mu_0$  extends (uniquely) to a measure  $\mu$  on  $\mathbb{B}(X)$ . As  $\mu(F) = \mu_0(F) = f([F])$  for all  $F \in \mathbb{E}$ , the existence of a clopen subset  $F$  of  $X$  such that  $0 < \mu(F) < \infty$  follows from the fact that  $f$  is non-trivial.

Let  $t \in G$  be given, and let  $t \cdot \mu$  be the Borel measure given by  $(t \cdot \mu)(E) = \mu(t^{-1} \cdot E)$ . Then, for every  $E \in \mathbb{E}$ ,

$$(t \cdot \mu)(E) = \mu(t^{-1} \cdot E) = \mu_0(t^{-1} \cdot E) = f([t^{-1} \cdot E]) = f([E]) = \mu_0(E).$$

By uniqueness of  $\mu$  we must have  $t \cdot \mu = \mu$ , so  $\mu$  is  $G$ -invariant. □

It is well known that a bounded invariant trace on a  $C^*$ -algebra, with a discrete group acting on it, extends to a (bounded) trace on the crossed product. We shall need to extend unbounded invariant tracial weights to the crossed product, cf. Lemma 5.3 below. Whereas this result may be known to experts, we failed to find a suitable reference for it, so we include the following two lemmas. The second named author thanks George Elliott and Uffe Haagerup for explaining the trick below (how to use Dini’s theorem to obtain norm-convergence from convergence on states) and its application to Lemma 5.2.

Let  $A$  be a  $C^*$ -algebra, and denote by  $S(A)$  its state space (i.e., the set of all positive linear functionals on  $A$  of norm 1). For each self-adjoint element  $a$  in  $A$  let  $\hat{a} \in C(S(A), \mathbb{R})$  denote the function given by  $\hat{a}(\rho) = \rho(a)$ ,  $\rho \in S(A)$ . Note that  $\|a\| = \|\hat{a}\|_\infty$ . Suppose that  $a$  and  $(a_n)$  are positive elements in  $A$  such that  $\rho(a) = \sum_{n=1}^\infty \rho(a_n)$  for all  $\rho \in S(A)$ . Then  $\hat{a} = \sum_{n=1}^\infty \hat{a}_n$ , and the sum is pointwise convergent, and hence uniformly convergent by Dini’s theorem. It follows that  $a = \sum_n a_n$ , and that the sum is norm-convergent.

LEMMA 5.2. *Let  $A$  be a  $C^*$ -algebra, and let  $G$  be a countable discrete group acting on  $A$ . For each element  $x$  in the reduced crossed product  $A \rtimes_r G$  and for each  $t \in G$  let  $x_t = E(xu_t^*) \in A$ . It follows that*

$$E(xx^*) = \sum_{t \in G} x_t x_t^*, \quad E(x^*x) = \sum_{t \in G} t \cdot (x_{t^{-1}}^* x_{t^{-1}}),$$

*and the sums are norm-convergent.*

*Proof.* Let  $\pi : A \rightarrow B(H)$  be the universal representation of  $A$ , and let

$$\pi \times \lambda : A \rtimes_r G \rightarrow B(\ell^2(G, H))$$

be the corresponding left regular representation of the reduced crossed product. For each  $s \in G$ , let  $V_s : H \rightarrow \ell^2(G, H)$  be the isometry given by

$$V_s(\xi)(t) = \begin{cases} \xi, & t = s^{-1}, \\ 0, & t \neq s^{-1}, \end{cases} \quad \xi \in H, t \in G.$$

It is straightforward to check that  $\sum_{t \in G} V_t V_t^* = I$  (the sum is strongly convergent), and that  $\pi(E(a u_s^*)) = V_e^*(\pi \times \lambda)(a) V_s$  for all  $a$  in  $A \rtimes_r G$  and all  $s \in G$ . It follows that

$$\pi(E(x x^*)) = V_e^*(\pi \times \lambda)(x) \left( \sum_{t \in G} V_t V_t^* \right) (\pi \times \lambda)(x^*) V_e = \sum_{t \in G} \pi(x_t x_t^*), \tag{5.1}$$

where the sum is strongly convergent, and similarly

$$\pi(E(x^* x)) = \sum_{t \in G} \pi(t \cdot (x_{t^{-1}}^* x_{t^{-1}})). \tag{5.2}$$

For each  $\rho \in S(A)$  there is a vector state  $\varphi$  on  $B(H)$  such that  $\rho = \varphi \circ \pi$  (because  $\pi$  is the universal representation). As vector states are strongly continuous, we can apply  $\varphi$  to equations (5.1) and (5.2) above to obtain that

$$\rho(E(x x^*)) = \sum_{t \in G} \rho(x_t x_t^*), \quad \rho(E(x^* x)) = \sum_{t \in G} \rho(t \cdot (x_{t^{-1}}^* x_{t^{-1}})),$$

for all  $\rho \in S(A)$ .

The conclusion now follows from the comment above the lemma. □

We shall refer to a map  $\varphi$  from the positive cone,  $A^+$ , of a  $C^*$ -algebra  $A$  to  $[0, \infty]$  as being a tracial weight if it satisfies  $\varphi(\alpha a + \beta b) = \alpha \varphi(a) + \beta \varphi(b)$  and  $\varphi(x^* x) = \varphi(x x^*)$  for all  $a, b \in A^+$ , all  $\alpha, \beta \in \mathbb{R}^+$ , and all  $x \in A$ . (A tracial weight as above extends to an unbounded linear trace on the algebraic ideal of  $A$  generated by the set of positive elements  $a \in A$  with  $\varphi(a) < \infty$ .)

LEMMA 5.3. *Let  $G$  be a countable discrete group acting on a  $C^*$ -algebra  $A$ , and let  $\varphi_0 : A^+ \rightarrow [0, \infty]$  be a  $G$ -invariant, lower semi-continuous tracial weight. It follows that the mapping  $\varphi = \varphi_0 \circ E : (A \rtimes_r G)^+ \rightarrow [0, \infty]$  is a lower semi-continuous tracial weight that extends  $\varphi_0$ .*

*Proof.* It is trivial to verify that  $\varphi$  is  $\mathbb{R}^+$ -linear and lower semi-continuous (the latter because  $E$  is continuous). It is also clear that  $\varphi$  extends  $\varphi_0$ . We must show that  $\varphi(x^* x) = \varphi(x x^*)$  for all  $x \in A \rtimes_r G$ . By Lemma 5.2, by the assumption that  $\varphi_0$  is lower semi-continuous (and additive), which entails that it respects norm-convergent sums of positive elements, and by  $G$ -invariance and the trace property of  $\varphi_0$ , it follows that

$$\begin{aligned} \varphi(x^* x) &= \varphi_0(E(x^* x)) = \varphi_0 \left( \sum_{t \in G} t \cdot (x_{t^{-1}}^* x_{t^{-1}}) \right) \\ &= \sum_{t \in G} \varphi_0(t \cdot (x_{t^{-1}}^* x_{t^{-1}})) = \sum_{t \in G} \varphi_0(x_t x_t^*) \\ &= \varphi_0 \left( \sum_{t \in G} x_t x_t^* \right) = \varphi_0(E(x x^*)) = \varphi(x x^*), \end{aligned}$$

as desired. □

THEOREM 5.4. *Let  $(C(X), G)$  be a  $C^*$ -dynamical system with  $G$  countable, discrete, exact, and  $X$  the Cantor set. Let  $\mathbb{E}$  denote the family of clopen subsets of  $X$ . Suppose the action of  $G$  on  $X$  is essentially free. Consider the following properties.*

- (i) *The semigroup  $S(X, G, \mathbb{E})$  is purely infinite.*
- (ii) *Every clopen subset of  $X$  is  $(G, \tau_X)$ -paradoxical.*

- (iii) The  $C^*$ -algebra  $C(X) \rtimes_r G$  is purely infinite.
- (iv) The  $C^*$ -algebra  $C(X) \rtimes_r G$  is traceless<sup>†</sup>.
- (v) There are no non-trivial states on  $S(X, G, \mathbb{E})$ .

Then (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v). Moreover, if  $S(X, G, \mathbb{E})$  is almost unperforated, then (v)  $\Rightarrow$  (i), whence all five properties are equivalent.

*Proof.* (i)  $\Rightarrow$  (ii). Let  $E \in \mathbb{E}$  be given. We show that  $E$  is  $(G, \tau_X)$ -paradoxical if  $2[E] \leq [E]$ . Suppose that  $2[E] \leq [E]$ , and find  $F \in \mathbb{E}$  such that  $2[E] + [F] = [E]$ . Then

$$E \times \{1\} \sim_S (E \times \{1, 2\}) \cup (F \times \{3\}).$$

It follows that there exist  $\ell \in \mathbb{N}$ ,  $A_j \in \mathbb{E}$ ,  $t_j \in G$  and  $m_j \in \{1, 2, 3\}$  for  $j = 1, 2, \dots, \ell$  such that

$$E \times \{1\} = \bigsqcup_{j=1}^{\ell} A_j \times \{1\}, \quad (E \times \{1, 2\}) \cup (F \times \{3\}) = \bigsqcup_{j=1}^{\ell} t_j \cdot A_j \times \{m_j\}.$$

This entails that the sets  $A_j$  are pairwise disjoint subsets of  $E$ . Renumbering the  $A_j$  so that  $m_j = 1$  for  $1 \leq j \leq n$ ,  $m_j = 2$  for  $n < j \leq n + m$ , and  $m_j = 3$  for  $n + m < j \leq \ell$ , we obtain

$$\bigsqcup_{j=1}^n t_j \cdot A_j = \bigsqcup_{j=n+1}^{n+m} t_j \cdot A_j = E.$$

This shows that  $E$  is  $(G, \tau_X)$ -paradoxical.

(ii)  $\Rightarrow$  (iii). This is Corollary 4.4.

(iii)  $\Rightarrow$  (iv). We refer to [18].

(iv)  $\Rightarrow$  (v). Suppose  $f$  is a non-trivial state on the preordered monoid  $S(X, G, \mathbb{E})$ . Since  $X$  is totally disconnected and second countable, we have  $\mathbb{B}(X) = \sigma(\mathbb{E})$ . By Lemma 5.1 there exists a  $G$ -invariant Borel measure  $\mu$  on  $X$  and a clopen subset  $F$  of  $X$  such that  $0 < \mu(F) < \infty$ . By Lemma 5.3, the (tracial) lower semi-continuous weight

$$\varphi_0(f) = \int_X f \, d\mu, \quad f \in C(X)^+,$$

extends to a lower semi-continuous tracial weight  $\varphi : (C(X) \rtimes_r G)^+ \rightarrow [0, \infty]$ . Restrict (and extend)  $\varphi$  to a (linear, unbounded, lower semi-continuous) trace  $\tau$  defined on the algebraic ideal generated by all positive elements  $a$  in  $C(X) \rtimes_r G$  with  $\varphi(a) < \infty$ . As  $0 < \varphi(1_F) < \infty$ , the element  $1_F$  belongs to the domain of  $\tau$  and  $\tau(1_F) = \varphi(1_F) > 0$ , whence  $\tau$  is non-zero. Hence  $C(X) \rtimes_r G$  is not traceless.

(v)  $\Rightarrow$  (i). This implication holds for any preordered almost unperforated abelian semigroup  $(S, +, \leq)$ . Take any (non-zero)  $x$  in  $S$ . By the assumption that there is no state on  $f$  normalized at  $x$ , it follows that a multiple of  $x$  is properly infinite. This is a consequence of the Goodearl–Handelman theorem [12], and does not use the assumption that  $S$  is almost unperforated. (See for example [23, Proposition 2.1], where it is shown that absence of states on  $S$  normalized at  $x$  implies that  $2x <_S x$  and hence that  $2(k + 1)x \leq kx$  for some natural number  $k$ .) Now, if  $kx$  is properly infinite, then  $\ell x \leq kx$  for all natural numbers  $\ell$ . This implies in particular that  $2(k + 1)x \leq kx$ , and so  $2x \leq x$  since  $S$  is almost unperforated. This shows that  $S$  is purely infinite. □

<sup>†</sup> A  $C^*$ -algebra is in [18] said to be traceless if it admits no non-zero lower semi-continuous (possibly unbounded) 2-quasitraces defined on an algebraic ideal of the given  $C^*$ -algebra.

We do not know to what extent  $S(X, G, \mathbb{E})$  is almost unperforated (or purely infinite), when  $X, G$  and  $\mathbb{E}$  are as in Theorem 5.4. We know of no examples where  $S(X, G, \mathbb{E})$  is not almost unperforated (in the case where  $X$  is the Cantor set), but we suspect that such examples may exist. In many cases of dynamical systems  $(X, G)$ , with  $X$  totally disconnected, where it is known that  $C(X) \rtimes_r G$  is purely infinite (and simple), it is also known (or easy to see) that the type semigroup  $S(X, G, \mathbb{E})$  is purely infinite. This is for example the case for the strong boundary actions considered by Laca and Spielberg and the  $n$ -filling actions considered by Jolissaint and Robertson (mentioned in the introduction). In §6 we give examples of group actions on the Cantor set such that  $C(X) \rtimes_r G$  is purely infinite, where  $G$  can be any countable discrete non-amenable exact group. This does not immediately imply that  $S(X, G, \mathbb{E})$  is purely infinite, but one can design the action of  $G$  on  $X$  such that  $S(X, G, \mathbb{E})$  is purely infinite (by invoking a strengthened version of Proposition 6.8).

We do not know if every clopen subset  $E$  of  $X$  for which  $1_E$  is properly infinite in  $C(X) \rtimes_r G$  is necessarily  $G$ -paradoxical (not to mention  $(G, \tau_X)$ -paradoxical). However, we have the following result that relies on Tarski’s theorem. (The term ‘tracial weight’ is defined above Lemma 5.3.)

**PROPOSITION 5.5.** *Let  $G$  be a countable discrete group that acts on a discrete set  $X$ , and let  $E \subseteq X$  be given. Then the following conditions are equivalent.*

- (i)  $E$  is  $G$ -paradoxical.
- (ii) For every tracial weight  $\varphi : (\ell^\infty(X) \rtimes_r G)^+ \rightarrow [0, \infty]$ , the value of  $\varphi(1_E)$  is either 0 or  $\infty$ .
- (iii)  $1_E$  is properly infinite in  $\ell^\infty(X) \rtimes_r G$ .
- (iv) The  $n$ -fold direct sum  $1_E \oplus 1_E \oplus \dots \oplus 1_E$  is properly infinite in  $M_n(\ell^\infty(X) \rtimes_r G)$  for some  $n$ .

*Proof.* (ii)  $\Rightarrow$  (i). We note first that, if  $\nu$  is a finitely additive measure on  $\mathcal{P}(X)$  with  $0 < \nu(X) < \infty$ , then there is a bounded positive linear functional  $\psi$  on  $\ell^\infty(X)$  such that  $\psi(1_E) = \nu(E)$  for all  $E \subseteq X$ . Indeed, first define  $\psi$  on simple functions by

$$\psi\left(\sum_{j=1}^n \alpha_j 1_{E_j}\right) = \sum_{j=1}^n \alpha_j \nu(E_j), \quad \alpha_j \in \mathbb{C}, E_j \in \mathcal{P}(X).$$

Noting that  $\psi$  is positive, linear and bounded (with bound  $\nu(X)$ ) on the uniformly dense subspace of  $\ell^\infty(X)$  consisting of simple functions, we can by continuity extend  $\psi$  to  $\ell^\infty(X)$  to the desired functional.

Assume that  $E$  is not  $G$ -paradoxical. It then follows from Tarski’s theorem (cf. [25, p. 116]) that there is a finitely additive  $G$ -invariant measure  $\mu$  on  $\mathcal{P}(X)$  satisfying  $\mu(E) = 1$ . Let  $\mathcal{P}_{\text{fin}}(X)$  be the (upwards directed) set of all subsets  $F$  of  $X$  with  $\mu(F) < \infty$ . For each  $F \in \mathcal{P}_{\text{fin}}(X)$  let  $\mu_F$  be the finitely additive bounded measure on  $\mathcal{P}(X)$  given by  $\mu_F(A) = \mu(A \cap F)$ , and let  $\psi_F$  be the associated bounded linear functional on  $\ell^\infty(X)$  constructed above. Notice that  $\psi_F \leq \psi_{F'}$  if  $F \subseteq F'$ . Put

$$\varphi_0(f) = \sup_{F \in \mathcal{P}_{\text{fin}}(X)} \psi_F(f), \quad f \in \ell^\infty(X)^+.$$

Then  $\varphi_0$  is  $\mathbb{R}^+$ -linear (being the supremum of an upwards directed family of  $\mathbb{R}^+$ -linear functionals) and lower semicontinuous (being a supremum of a family of continuous functionals). We have  $\psi_F(1_E) = \mu_F(E) = \mu(E)$  for all  $F \in \mathcal{P}_{\text{fin}}(X)$  with  $E \subseteq F$ . This shows that  $\varphi_0(1_E) = \mu(E) = 1$ . To prove  $G$ -invariance note first that  $\psi_F(t \cdot f) = \psi_{t^{-1}F}(f)$  for all  $t \in G$  and all  $f \in \ell^\infty(X)$ . Indeed, by linearity and continuity it suffices to verify this for characteristic functions. Use here that  $t \cdot 1_E = 1_{tE}$  and that

$$\mu_F(tE) = \mu(F \cap tE) = \mu(t^{-1}F \cap E) = \mu_{t^{-1}F}(E).$$

Thus

$$\varphi_0(t \cdot f) = \sup_{F \in \mathcal{P}_{\text{fin}}(X)} \psi_F(t \cdot f) = \sup_{F \in \mathcal{P}_{\text{fin}}(X)} \psi_{t^{-1}F}(f) = \sup_{F \in \mathcal{P}_{\text{fin}}(X)} \psi_F(f) = \varphi_0(f)$$

for all  $f$  in  $\ell^\infty(X)^+$ , thus showing that  $\varphi_0$  is  $G$ -invariant.

To complete the proof, use Lemma 5.3 to extend  $\varphi_0$  to a tracial weight  $\varphi : (\ell^\infty(X) \rtimes_r G)^+ \rightarrow [0, \infty]$ . As  $\varphi(1_E) = \varphi_0(1_E) = 1$  we conclude that (ii) does not hold.

(i)  $\Rightarrow$  (ii). Suppose that (ii) does not hold and that  $\varphi : (\ell^\infty(X) \rtimes_r G)^+ \rightarrow [0, \infty]$  is a tracial weight with  $\varphi(1_E) = 1$ . Define a finitely additive measure  $\mu$  on  $\mathcal{P}(X)$  by  $\mu(F) = \varphi(1_F)$ . Then

$$\mu(tF) = \varphi(t \cdot 1_F) = \varphi(u_t 1_F u_t^*) = \varphi(1_F) = \mu(F)$$

by the tracial property of  $\varphi$ , which shows that  $\mu$  is  $G$ -invariant. By (the easy part of) Tarski's theorem, the existence of a  $G$ -invariant finitely additive measure  $\mu$  on  $\mathcal{P}(X)$  with  $\mu(E) = 1$  implies that  $E$  is not  $G$ -paradoxical.

(i)  $\Rightarrow$  (iii). This is contained in Proposition 4.3.

(iii)  $\Rightarrow$  (iv). This is trivial.

(iv)  $\Rightarrow$  (ii). Assume that the  $n$ -fold direct sum

$$P := 1_E \oplus 1_E \oplus \cdots \oplus 1_E \in M_n(\ell^\infty(X) \rtimes_r G)$$

is properly infinite. Suppose that  $\varphi : (\ell^\infty(X) \rtimes_r G)^+ \rightarrow [0, \infty]$  is a tracial weight. Extend  $\varphi$  to the positive cone over any matrix algebra over  $\ell^\infty(X) \rtimes_r G$  in the usual way by the formula  $\varphi((x_{ij})) = \sum_i \varphi(x_{ii})$ . Then  $\varphi(P) = n\varphi(1_E)$ . As  $P$  is properly infinite there exist projections  $P_1, P_2, Q$  in  $M_n(\ell^\infty(X) \rtimes_r G)$  such that  $P = P_1 + P_2 + Q$  and  $P \sim P_1 \sim P_2$ . Thus

$$\varphi(P) = \varphi(P_1) + \varphi(P_2) + \varphi(Q) = 2\varphi(P) + \varphi(Q),$$

whence  $\varphi(P)$  is either 0 or  $\infty$ . It follows that we cannot have  $0 < \varphi(1_E) < \infty$ , thus showing that (ii) holds. □

**COROLLARY 5.6.** *Let  $G$  be a countable discrete non-amenable group and let  $p$  be a projection in  $\ell^\infty(G)$  that is full in  $\ell^\infty(G) \rtimes_r G$  (i.e., is not contained in a proper ideal in  $\ell^\infty(G) \rtimes_r G$ ). Then  $p$  is properly infinite.*

*Proof.* The assumption that  $p$  is full in  $\ell^\infty(G) \rtimes_r G$  implies that the  $n$ -fold direct sum  $p \oplus p \oplus \cdots \oplus p$  dominates the unit 1 of  $\ell^\infty(G) \rtimes_r G$  for some natural number  $n$ . As  $G$



is non-amenable,  $G$  is  $G$ -paradoxical, whence  $1 = 1_G$  is properly infinite in  $\ell^\infty(G) \rtimes_r G$ ; cf. Proposition 5.5. It follows that

$$1 \preceq p \oplus p \oplus \cdots \oplus p \preceq 1 \oplus 1 \oplus \cdots \oplus 1 \preceq 1$$

(the latter because  $1$  is properly infinite). This entails that the  $n$ -fold direct sum  $p \oplus p \oplus \cdots \oplus p$  is properly infinite, whence  $p$  is properly infinite by Proposition 5.5.  $\square$

The  $C^*$ -algebra  $\ell^\infty(G) \rtimes_r G$  is also known as the *Roe algebra*. There is more about this algebra in the next section.

### 6. Crossed products with exact, non-amenable groups

It is well known that the Roe algebra  $\ell^\infty(G) \rtimes_r G$  is *properly infinite* precisely when  $G$  is non-amenable, and *nuclear* precisely when  $G$  is exact. (The action of  $G$  on  $\ell^\infty(G)$  is induced by left-translation.) The former is due to the fact that non-amenable groups are paradoxical (see e.g. [32]). See [8, Theorem 5.1.6] for the latter. In other words, each exact non-amenable discrete group  $G$  admits an amenable action on the compact (non-metrizable) Hausdorff space  $\beta G$  so that the corresponding crossed product is properly infinite and nuclear.

We show in this section—using these facts—that every countable discrete exact non-amenable group  $G$  admits a free, minimal, amenable action on the Cantor set  $X$  such that the crossed product  $C^*$ -algebra  $C(X) \rtimes_r G$  is simple, nuclear and purely infinite. A related result was obtained by Hjorth and Molberg [13], who proved that any countable discrete group has a free minimal action on the Cantor set admitting an invariant Borel probability measure. The actions constructed by Hjorth and Molberg give rise to crossed product  $C^*$ -algebras with a tracial state (coming from the invariant measure). Our construction goes in the opposite direction of producing crossed product  $C^*$ -algebras that are traceless, and even purely infinite.

We note first that the action of  $G$  on  $\beta G$  given by left-translation is free. We shall need a stronger result, which will allow us to show that  $G$  acts freely also on the spectrum of certain (separable) sub- $C^*$ -algebras of  $\ell^\infty(G)$ . The following three (easy) lemmas are devoted to this. It was pointed out to us by David Kerr that our Lemma 6.1 below (and its application to proving freeness) was used back in 1960 by Ellis in [9], where he proved freeness (strong effectiveness) of any group acting on its universal minimal set. Nonetheless, for the convenience of the reader we have kept the short arguments leading to Lemma 6.4 below.

LEMMA 6.1. *Let  $G$  be a group and let  $t \in G$ ,  $t \neq e$ , be given. Then  $G$  can be partitioned into pairwise disjoint subsets  $G = G_1 \cup G_2 \cup G_3$  such that  $G_j \cap tG_j = \emptyset$  for all  $j = 1, 2, 3$ .*

*Proof.* Let  $H$  be a maximal subset of  $G$  such that  $H \cap tH = \emptyset$ . Put

$$G_1 = H, \quad G_2 = tH, \quad G_3 = G \setminus (G_1 \cup G_2).$$

The fact that  $H$  and  $tH$  are disjoint implies that  $t^k H \cap t^{k+1} H = \emptyset$  for every integer  $k$ , and hence in particular that  $tH \cap t^2 H = \emptyset$  and  $t^{-1} H \cap H = \emptyset$ . It remains to prove

that  $G_3 \cap tG_3 = \emptyset$ . For this it suffices to show that  $G_3 \subseteq t^{-1}H$ , or, equivalently, that  $G = H \cup tH \cup t^{-1}H$ . Suppose that  $s$  belongs to  $G$  and not to  $H$ . Then, by maximality of  $H$ ,

$$(H \cup \{s\}) \cap t(H \cup \{s\}) \neq \emptyset.$$

As  $ts \neq s$ , this entails that  $s \in tH$  or  $ts \in H$  (or both). Thus either  $s \in tH$  or  $s \in t^{-1}H$ .  $\square$

One can give a perhaps more natural proof of the lemma above by selecting a transversal in  $G$  to the right-cosets of the cyclic group generated by  $t$ . In this way one can show that if  $t$  does not have odd order, then  $G$  can be partitioned into two disjoint subsets  $G = G_1 \cup G_2$  such that  $G_2 = tG_1$ . If  $t$  has finite odd order, then we need three sets in the partition of  $G$  to reach the conclusion of Lemma 6.1.

**COROLLARY 6.2.** *Let  $G$  be a discrete group and let  $t \in G$ ,  $t \neq e$ , be given. Then there are projections  $f_1, f_2, f_3$  in  $\ell^\infty(G)$  such that  $1 = f_1 + f_2 + f_3$  and such that  $t \cdot f_j \perp f_j$  for  $j = 1, 2, 3$ .*

*Proof.* Let  $f_j = 1_{G_j}$  where  $G_1, G_2, G_3$  are as in the previous lemma.  $\square$

The following lemma is trivial, and we omit the proof.

**LEMMA 6.3.** *Let  $G$  be a discrete group acting on a compact Hausdorff space  $X$ . Suppose that for each  $t \in G$ ,  $t \neq e$ , one can find projections  $f_1, f_2, f_3$  in  $C(X)$  such that  $1 = f_1 + f_2 + f_3$  and  $t \cdot f_j \perp f_j$  for  $j = 1, 2, 3$ . Then the action of  $G$  on  $X$  is free.*

**LEMMA 6.4.** *Let  $G$  be a countable discrete group. There is a countable subset  $M$  of  $\ell^\infty(G)$  such that if  $A$  is any  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains  $M$ , then the action of  $G$  on  $\widehat{A}$  is free.*

*Proof.* For each  $t \in G$ ,  $t \neq e$ , use Corollary 6.2 to choose projections  $f_{1,t}, f_{2,t}$  and  $f_{3,t}$  in  $\ell^\infty(G)$  such that  $f_{1,t} + f_{2,t} + f_{3,t} = 1$  and  $t \cdot f_{j,t} \perp f_{j,t}$  for  $j = 1, 2, 3$ . It follows from Lemma 6.3 that the action of  $G$  on  $\widehat{A}$  is free whenever  $A$  is a  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains the countable set  $M = \{f_{j,t} \mid t \in G \setminus \{e\}, j = 1, 2, 3\}$ .  $\square$

The next lemma is contained in the book by Brown and Ozawa [8].

**LEMMA 6.5.** (Brown and Ozawa) *Suppose that  $G$  is a countable discrete exact group. Then there is a countable subset  $M'$  of  $\ell^\infty(G)$  such that whenever  $A$  is a  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains  $M'$ , then the action of  $G$  on  $\widehat{A}$  is amenable.*

*Proof.* It is well known (see for example [8, Theorem 5.1.6]) that the action of an exact group  $G$  on its Stone–Cech compactification  $\beta G$  induced by left-multiplication on  $G \subseteq \beta G$  is amenable. Use the definition of amenable actions given in [8, Definition 4.3.1], and let  $T_i : G \rightarrow \ell^\infty(G)$ ,  $i \in \mathbb{N}$ , be finitely supported positive functions satisfying the conditions (1), (2) and (3) of [8, Definition 4.3.1]. Then  $M' = \{T_i(t) \mid i \in \mathbb{N}, t \in G\}$  is a countable set with the required properties.  $\square$

**LEMMA 6.6.** *Let  $G$  be a discrete non-amenable group, and let  $p$  be a projection in  $\ell^\infty(G)$  that is properly infinite in  $\ell^\infty(G) \rtimes_r G$ . Then there is a countable subset  $M''$  of  $\ell^\infty(G)$*

such that whenever  $A$  is a  $G$ -invariant unital sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains  $\{p\} \cup M''$ , then  $p$  is properly infinite in  $A \rtimes_r G$ .

*Proof.* There exist partial isometries  $x$  and  $y$  in  $\ell^\infty(G) \rtimes_r G$  satisfying  $x^*x = y^*y = p$  and  $xx^* + yy^* \leq p$ . For each  $n$  choose  $x_n, y_n$  in  $C_c(G, \ell^\infty(G))$  such that  $\|x - x_n\| < 1/n$  and  $\|y - y_n\| < 1/n$ . Let  $F_n$  be the finite subset of  $\ell^\infty(G)$  consisting of all elements  $E(x_n u_t^*)$  and  $E(y_n u_t^*)$  with  $t \in G$  (only finitely many of these are non-zero by the assumption that  $x_n$  and  $y_n$  belong to  $C_c(G, \ell^\infty(G))$ ). Then  $x_n$  and  $y_n$  belong to  $A \rtimes_r G$  whenever  $A$  is a  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains  $F_n$ . Put  $M'' = \bigcup_{n=1}^\infty F_n$ . It follows that whenever  $A$  is a  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains  $\{p\} \cup M''$ , then  $x$  and  $y$  belong to  $A \rtimes_r G$ , whence  $p$  is properly infinite in  $A \rtimes_r G$ .  $\square$

LEMMA 6.7. *Let  $G$  be a countable discrete group and let  $T$  be a countable subset of  $\ell^\infty(G)$ . Then there is a countable and  $G$ -invariant set  $P$  consisting of projections in  $\ell^\infty(G)$  such that  $T \subseteq C^*(P)$ .*

*Proof.* Each element in  $\ell^\infty(G)$  can be approximated in norm by elements with finite spectrum, and the  $C^*$ -algebra generated by any normal element with finite spectrum is equal to the  $C^*$ -algebra generated by finitely many projections (as many projections as there are elements in the spectrum). It follows that each element in  $\ell^\infty(G)$  belongs to the  $C^*$ -algebra generated by a countable set of projections in  $\ell^\infty(G)$ . Hence the countable set  $T$  is contained in the  $C^*$ -algebra generated by a countable set  $P_0$  of projections from  $\ell^\infty(G)$ . The set  $P = \bigcup_{t \in G} t \cdot P_0$  has the desired properties.  $\square$

PROPOSITION 6.8. *Let  $G$  be a countable discrete group, and let  $N$  be a countable subset of  $\ell^\infty(G)$ . Then there exists a separable  $G$ -invariant sub- $C^*$ -algebra  $A$  of  $\ell^\infty(G)$  that is generated by projections and contains  $N$ , and that has the following property: for every projection  $p$  in  $A$ , if  $p$  is properly infinite in  $\ell^\infty(G) \rtimes_r G$ , then  $p$  is properly infinite in  $A \rtimes_r G$ .*

*Proof.* Let  $\mathcal{P}_{\text{inf}}$  denote the set of properly infinite projections in  $\ell^\infty(G) \rtimes_r G$ . Use Lemma 6.7 to find a countable  $G$ -invariant set of projections  $P_0 \subseteq \ell^\infty(G)$  such that  $N \subseteq C^*(P_0)$ . Let  $Q_0$  be the set of projections in  $C^*(P_0)$ . The set  $Q_0$  is countable because  $C^*(P_0)$  is separable and abelian. For each  $p \in Q_0 \cap \mathcal{P}_{\text{inf}}$  use Lemma 6.6 to find a countable subset  $M(p)$  of  $\ell^\infty(G)$  such that  $p$  is properly infinite in  $A \rtimes_r G$  whenever  $A$  is a  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  that contains  $\{p\} \cup M(p)$ . Put

$$N_1 = Q_0 \cup \bigcup_{p \in Q_0 \cap \mathcal{P}_{\text{inf}}} M(p).$$

Use Lemma 6.7 to find a countable  $G$ -invariant set of projections  $P_1 \subseteq \ell^\infty(G)$  such that  $N_1 \subseteq C^*(P_1)$ .

Continue in this way to find countable subsets  $N_0 = N, N_1, N_2, \dots$  of  $\ell^\infty(G)$  and countable  $G$ -invariant subsets  $P_0, P_1, P_2, \dots$  consisting of projections in  $\ell^\infty(G)$  such that if  $Q_j$  is the (countable) set of projections in  $C^*(P_j)$ , then  $Q_j \subseteq N_{j+1} \subseteq C^*(P_{j+1})$  and every  $p \in Q_j \cap \mathcal{P}_{\text{inf}}$  is properly infinite in  $C^*(P_{j+1}) \rtimes_r G$ .

Put  $P = \bigcup_{n=0}^{\infty} P_n$  and put  $A = C^*(P)$ . Notice that

$$A = \overline{\bigcup_{n=1}^{\infty} C^*(P_n)}.$$

Then  $P$  is a countable  $G$ -invariant subset of  $\ell^\infty(G)$  consisting of projections, and  $N \subseteq C^*(P)$ . Moreover, if  $p$  is a projection in  $A$  that is properly infinite in  $\ell^\infty(G) \rtimes_r G$ , then  $p$  is equivalent (and hence equal) to a projection in  $C^*(P_n)$  for some  $n$ , whence  $p$  belongs to  $\mathcal{Q}_n \cap \mathcal{P}_{\text{inf}}$ , which by construction implies that  $p$  is properly infinite in  $C^*(P_{n+1}) \rtimes_r G$  and hence also in  $A \rtimes_r G$ .  $\square$

An element in a  $C^*$ -algebra is said to be *full* if it is not contained in a proper ideal of the  $C^*$ -algebra.

LEMMA 6.9. *Let  $A$  be as in Proposition 6.8, and suppose that the group  $G$  is non-amenable. Then every projection in  $A$ , which is full in  $A \rtimes_r G$ , is properly infinite in  $A \rtimes_r G$ .*

*Proof.* If  $p \in A \subseteq \ell^\infty(G)$  is a projection that is full in  $A \rtimes_r G$ , then  $p$  is full in  $\ell^\infty(G) \rtimes_r G$ , which by Corollary 5.6 implies that  $p$  is properly infinite in  $\ell^\infty(G) \rtimes_r G$ , and hence also properly infinite in  $A \rtimes_r G$  by construction of  $A$ .  $\square$

Let  $G$  be a countable discrete group acting on a compact Hausdorff space  $X$ . An open subset  $U$  of  $X$  is said to be  *$G$ -full* in  $X$  if  $\bigcup_{t \in G} t \cdot U = X$ . If  $U$  is a clopen subset of  $X$ , then  $U$  is  $G$ -full in  $X$  if and only if  $1_U$  is full in  $C(X) \rtimes_r G$ .

LEMMA 6.10. *Let  $G$  be a countable discrete group acting on a compact, metrizable, totally disconnected Hausdorff space  $Y$ . Let  $X$  be a closed  $G$ -invariant subset of  $Y$ , and let  $U$  be a subset of  $X$  that is clopen and  $G$ -full relative to  $X$ . Then there exists a clopen  $G$ -full subset  $V$  of  $Y$  such that  $V \cap X = U$ .*

*Proof.* Let  $V_0$  be any clopen subset of  $Y$  such that  $V_0 \cap X = U$ . Let  $(Z_n)$  be an increasing sequence of clopen subsets of  $Y$  such that  $\bigcup_{n=1}^{\infty} Z_n = Y \setminus X$ , and put  $V_n = V_0 \cup Z_n$ . Then each  $V_n$  is clopen and  $V_n \cap X = U$ .

Since  $U$  is a relatively open and  $G$ -full subset of the compact set  $X$  there are finitely many elements  $t_1, t_2, \dots, t_k \in G$  such that  $\bigcup_{j=1}^k t_j \cdot U = X$ . Put  $t_0 = e$  and put  $R_n = \bigcup_{j=0}^k t_j \cdot V_n$ . Then each  $R_n$  is clopen and  $\bigcup_{n=1}^{\infty} R_n = Y$ . By compactness of  $Y$  there is  $n_0$  such that  $R_{n_0} = Y$ . Hence  $V = V_{n_0}$  is  $G$ -full in  $Y$ , and we have already noted that  $V$  is clopen and  $V \cap X = U$ .  $\square$

A compact Hausdorff space  $X$  is homeomorphic to the Cantor set if and only if it is separable, metrizable, totally disconnected and without isolated points. Equivalently, the spectrum,  $\widehat{A}$ , of an abelian unital  $C^*$ -algebra  $A$  is the Cantor set if and only if  $A$  is separable, is generated as a  $C^*$ -algebra by its projections, and has no minimal projections.

THEOREM 6.11. *Let  $G$  be a countable discrete group. Then  $G$  admits a free, amenable, minimal action on the Cantor set  $X$  such that  $C(X) \rtimes_r G$  is a Kirchberg algebra $\dagger$  in the UCT class if and only if  $G$  is exact and non-amenable.*

$\dagger$  A Kirchberg algebra is a simple, separable, nuclear, purely infinite  $C^*$ -algebra.

We remind the reader that the full and the reduced crossed products coincide for amenable actions (see e.g. [8, Theorem 4.3.4]), so it does not matter which crossed product we choose in the theorem above.

*Proof.* The ‘only if’ part is well known. If  $G$  is an amenable group acting on a unital  $C^*$ -algebra  $A$  that admits a tracial state, then the crossed product  $A \rtimes G$  also admits a tracial state (and hence is not purely infinite). The  $C^*$ -algebra  $C_r^*(G)$  can be embedded into  $C(X) \rtimes_r G$  and is therefore exact if  $C(X) \rtimes_r G$  is nuclear. Hence  $G$  is exact.

The ‘if’ part goes as follows. Let  $M$  and  $M'$  be as in Lemmas 6.4 and 6.5, respectively. By Proposition 6.8 and Lemma 6.9 there is a separable  $G$ -invariant sub- $C^*$ -algebra  $A$  of  $\ell^\infty(G)$ , which is generated by projections, which contains  $M \cup M' \cup \{1\}$ , and which moreover has the property that every projection in  $A$  that is full in  $A \rtimes_r G$  is properly infinite in  $A \rtimes_r G$ . It follows from Lemmas 6.4 and 6.5 that the action of  $G$  on  $A$  is amenable and the action of  $G$  on  $Y := \widehat{A}$  is free.

The set  $Y$  is compact, Hausdorff, metrizable and totally disconnected (the latter because  $A$  is generated by projections). The action of  $G$  on  $Y$  is free and amenable. However,  $Y$  may have isolated points, the action of  $G$  on  $Y$  may not be minimal, and  $A \rtimes_r G$  need not be purely infinite.

It follows from [29, Theorem 1.16] that the correspondence  $I \mapsto I \cap A$  from the set of (closed two-sided) ideals in  $A \rtimes_r G$  to the set of  $G$ -invariant ideals in  $A$  is bijective. Let  $I$  be a maximal proper ideal in  $A \rtimes_r G$  and put  $J = A \cap I$ . Then

$$(A \rtimes_r G)/I \cong (A/J) \rtimes_r G \cong C(X) \rtimes_r G,$$

where  $X = \widehat{A/J}$ . We can identify  $X$  with a closed  $G$ -invariant subset of  $Y$ , and the action of  $G$  on  $X$  is the restriction to  $X$  of the action of  $G$  on  $Y$ .

As  $G$  acts freely and amenably on  $Y$  it also acts freely and amenably on  $X$ . (To see that  $G$  acts amenably on  $X$ , or on  $A/J = C(X)$ , use the characterization of amenable actions given in [8, Definition 4.3.1]. If  $T_i : G \rightarrow A$  satisfy (1), (2) and (3) in that definition, then  $\tilde{T}_i : G \rightarrow A/J$  also satisfy these conditions, when  $\tilde{T}_i = \pi \circ T_i$  and  $\pi : A \rightarrow A/J$  is the quotient mapping.)

We have now established a free and amenable action of  $G$  on a separable metrizable compact Hausdorff space  $X$ , and  $C(X) = A/J$  is generated by its projections (because  $A$  is generated by its projections). The crossed product  $C(X) \rtimes_r G$  is simple by construction, which entails that  $G$  acts minimally on  $X$ .

We show next that  $X$  has no isolated points (which will imply that  $X$  is the Cantor set). Assume that  $x_0$  is an isolated point in  $X$ . Then  $G \cdot x_0$  is an open and  $G$ -invariant subset of  $X$ . By minimality of the action of  $G$  on  $X$  this forces  $X = G \cdot x_0$ . As  $X$  also is compact, this can only happen if  $X$  is finite, which would entail that  $G$  is finite ( $G$  acts freely), but  $G$  is non-amenable.

Each non-zero projection in  $C(X)$  is full in  $C(X) \rtimes_r G$  (by simplicity) and hence lifts to a projection in  $A$  which is full in  $A \rtimes_r G$  by Lemma 6.10. The lifted projection is properly infinite in  $A \rtimes_r G$  by construction (Lemma 6.9), whence the given projection in  $C(X)$  is properly infinite in  $C(X) \rtimes_r G$ . Theorem 4.1 now implies that  $C(X) \rtimes_r G$  is purely infinite.

The crossed product  $C(X) \rtimes_r G$  is the reduced  $C^*$ -algebra of the amenable groupoid  $X \rtimes G$ , and those have been proved to belong to the UCT class by Tu in [31].  $\square$

We have not calculated the  $K$ -theory of the crossed product  $C^*$ -algebra in Theorem 6.11. It ought to depend on which separable sub- $C^*$ -algebra  $A$  of  $\ell^\infty(G)$  we choose and of which maximal ideal in  $A \rtimes_r G$  we divide out by in the construction. It seems plausible that one should be able to design the construction such that the  $K$ -theory of the crossed product in Theorem 6.11 becomes trivial, in which case it would be isomorphic to  $\mathcal{O}_2$  by the Kirchberg–Phillips classification theorem.

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